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Non-Markovian Quadratic Forms Obtained by Homogenization.

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Sunto. – *Questo articolo riguarda il comportamento asintotico delle forme quadratiche definite in L^2. Più precisamente consideriamo la Γ-convergenza di questi funzionali per la topologia debole di L^2. Noi diamo un esempio in cui certe forme limite non sono Markoviane e quindi la formula di Beurling-Deny non si applica. Questo esempio è ottenuto tramite l'omogeneizzazione di un materiale stratificato composto da strati sottili isolanti.*

Summary. – *This paper is devoted to the asymptotic behaviour of quadratic forms defined on L^2. More precisely we consider the Γ-convergence of these functionals for the L^2-weak topology. We give an example in which some limit forms are not Markovian and hence the Beurling-Deny representation formula does not hold. This example is obtained by the homogenization of a stratified medium composed of insulating thin-layers.*

1. – Introduction.

In this paper we study the asymptotic behaviour of some quadratic forms on $L^2(\Omega)$ (where Ω is a bounded open subset of $\mathbb{R}^2$) of type

$$
\begin{cases} F_\varepsilon(u) := \int_\Omega a_\varepsilon |\nabla u|^2 & \text{if } u \in H_0^1(\Omega) \\ F_\varepsilon(u) := +\infty & \text{if } u \in L^2(\Omega) \setminus H_0^1(\Omega), \end{cases} \tag{1.1}
$$

in particular when a_ε is a sequence of positive measurable functions, which is uniformly bounded from above but not from below. This kind of homogenization problem has been already studied and leads to non-classical limit problems: the double porosity effect in [1] and more generally coupled systems in [11], [8], [6] and [3].

An important question is to know if there exists a general (integral) representation of the asymptotic quadratic forms of the sequences (1.1). In fact that strongly depends on the topology which defines the limit process.

Thanks to the theory of the Dirichlet forms introduced by Beurling, Deny [4], [5] and extended in [9], [12], Mosco [13] proved that such a representation is available by considering the Γ-convergence of the forms for the $L^2(\Omega)$-strong topology – recall that the sequence of functionals F_ε Γ-converges to F for the $L^2(\Omega)$-strong topology if

$$(1.2)\qquad \begin{cases} F(u) \leqslant \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) & \text{for any } \quad u_\varepsilon \to u \text{ strongly in } L^2(\Omega), \\ F(u) = \limsup_{\varepsilon\to 0} F_\varepsilon(\overline{u}_\varepsilon) & \text{for some } \overline{u}_\varepsilon \to u \text{ strongly in } L^2(\Omega). \end{cases}$$

More precisely in the particular case of quadratic forms of type (1.1) the Mosco result is the following. Let F_ε be a sequence of forms (1.1) which is asymptotically regular, *i. e.* for any $u \in C_0^1(\Omega)$,

$$(1.3)\qquad \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) < +\infty \qquad \text{for some } u_\varepsilon \to u \text{ strongly in } L^2(\Omega).$$

Then F_ε Γ-converges for the $L^2(\Omega)$-strong topology, up to a subsequence, to a densily defined Dirichlet form F. This means that F is a quadratic form on $L^2(\Omega)$ which satisfies the three properties:

- F is densely defined, *i. e.* the domain of F

$$(1.4)\qquad D_F := \{u \in L^2(\Omega)/F(u) < +\infty\} \text{ is dense into } L^2(\Omega);$$

- F is closed, *i.e.*

$$(1.5)\qquad D_F \text{ is complete provided with the norm } \|u\|_F := \left(F(u) + \|u\|^2_{L^2(\Omega)}\right)^{1/2};$$

- F is Markovian, *i.e.* for any contraction $T : \mathbb{R} \to \mathbb{R}$ such that

$$T(0) = 0 \quad \text{and} \quad \forall x, y \in \mathbb{R}, \quad |T(x) - T(y)| \leqslant |x - y|,$$

one has

$$(1.6)\qquad \forall u \in D_F, \qquad T(u) \in D_F \quad \text{and} \quad F(T(u)) \leqslant F(u).$$

In fact the Markovian property is less restrictive than (1.6) but it is equivalent to (1.6) in the case of a densely form (see [9]). Such a Dirichlet form F then satisfies the Beurling-Deny representation formula, *i. e.* for any $u \in C_0^1(\Omega)$,

$$(1.7)\qquad F(u) = \int_\Omega A(dx)\, \nabla u \cdot \nabla u + \int_\Omega u^2 k(dx) + \int_{\Omega\times\Omega\setminus\mathrm{diag}} (u(x) - u(y))^2 j(dx, dy),$$

where A is a symmetric positive matrix-valued measure on Ω, k a positive mesure on Ω and j a positive measure on $\Omega \times \Omega\setminus$diag. Moreover the Dirichlet form F is said to be regular if

$$(1.8)\qquad C_0^1(\Omega) \text{ is dense into } (D_F, \|\cdot\|).$$

The natural question is now to know if the Beurling-Deny representation (1.7) still holds true if we replace the $L^2(\Omega)$-strong topology by the $L^2(\Omega)$-weak topology in the Γ-convergence of the forms. Note that the weak topology is the most adapted for studying the degenerate problems, in particular for the forms (1.1) in which the sequence a_ε is not uniformly elliptic. Indeed the loss of ellipticity implies a loss of compactness in $L^2(\Omega)$. In this framework Bellieud and Bouchitté [3] gave examples of limit forms which satisfy the representation formula (1.7) with explicit measures $j \neq 0$.

However it seems difficult to obtain the same general representation for the weak topology of the forms. Indeed contrary to the strong topology the weak topology does not commute with the contractions. So the Markovian property (1.6) is not clearly satisfied by any limit form obtained by weak convergence. In this paper we prove that in general there is no Beurling-Deny representation of the Γ-limits of Dirichlet forms for the $L^2(\Omega)$-weak topology. We give an example of a sequence of forms (1.1) which Γ-converges for the $L^2(\Omega)$-weak topology to a closed (1.5), densely defined (1.4) and regular (1.8) quadratic form, but this form is not Markovian (1.6).

First we prove a homogenization result for an insulating thin-layered medium. Then we prove that some of the limit forms are not Markovian.

2. – A homogenization result.

2.1. *Statement of the result.*

We consider a two-dimensional stratified medium composed of layers of constant conductivity separated by thinner insulating layers.

More precisely let $Y_{1,\varepsilon} \cup Y_{2,\varepsilon} \cup Q_\varepsilon$ be the partition of the torus Y (identified to $[0, 1[^2$) defined by

$$
\text{(2.1)} \qquad \begin{cases} Y_{1,\varepsilon} := [0, 1[\times \left] r_\varepsilon, \dfrac{1 - r_\varepsilon}{2} \right[\\[2ex] Y_{2,\varepsilon} := [0, 1[\times \left] \dfrac{1 + r_\varepsilon}{2}, 1 \right[\\[2ex] Q_\varepsilon := [0, 1[\times \left([0, r_\varepsilon] \cup \left[\dfrac{1 - r_\varepsilon}{2}, \dfrac{1 + r_\varepsilon}{2} \right] \right) \end{cases} \qquad 0 < r_\varepsilon \ll 1 .
$$

Let $A_\varepsilon(x, y)$ be the positive function which is periodic Y-periodic with respect to y and defined, for any $x \in \Omega$ and $y \in Y$, by

$$
\text{(2.2)} \qquad A_\varepsilon(x, y) := a_1(x)\, \mathbf{1}_{Y_{1,\varepsilon}}(y) + a_2(x)\, \mathbf{1}_{Y_{2,\varepsilon}}(y) + \alpha_\varepsilon\, \mathbf{1}_{Q_\varepsilon}(y),
$$

where a_1, a_2 are two positive functions from $C^0(\overline{\Omega})$ and $0 < \alpha_\varepsilon \ll 1$. The conductivity of the stratified medium is defined by rescaling A_ε

$$a_\varepsilon(x) := A_\varepsilon\left(x, \frac{x}{\varepsilon}\right), \qquad x \in \Omega . \tag{2.3}$$

Let $\Omega :=]0, 1[^2$, we study the asymptotic behaviour of the conduction problem

$$\begin{cases} -\operatorname{div}(a_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega , \end{cases} \tag{2.4}$$

where $f \in L^2(\Omega)$. The asymptotic behaviour of problem (2.4) is given by the following homogenization result.

THEOREM 2.1. – *Assume that the parameters of the insulating thin-layers medium satisfy the condition*

$$\frac{4\alpha_\varepsilon}{\varepsilon^2 r_\varepsilon} \xrightarrow[\varepsilon \to 0]{} \delta \in \mathbb{R}_+^* . \tag{2.5}$$

Then the quadratic form defined by (1.1) *and* (2.3) *is equicoercive in* $L^2(\Omega)$, *i.e. there exists a non-negative constant* C_δ *such that*

$$\forall u \in H_0^1(\Omega), \qquad \int_\Omega a_\varepsilon \, |\nabla u|^2 \geq C_\delta \|u\|^2_{L^2(\Omega)}. \tag{2.6}$$

Moreover, let H *be the Hilbert space defined by*

$$H := L^2(]0, 1[_{x_2}; H_0^1(]0, 1[_{x_1})) \tag{2.7}$$

$$\text{provided with the norm} \quad \|u\|_H := \left\| \frac{\partial u}{\partial x_1} \right\|_{L^2(\Omega)}.$$

Then the solution u_ε *of the Dirichlet problem* (2.4) *weakly converges in* $L^2(\Omega)$ *to the function* $u_1 + u_2$ *where* u_1, u_2 *are the solutions from* H *of the coupled system*

$$\begin{cases} -\dfrac{\partial}{\partial x_1}\left(a_1 \dfrac{\partial u_1}{\partial x_1}\right) + \delta(u_1 - u_2) = \dfrac{1}{2} f & \text{in } \Omega \\[2ex] -\dfrac{\partial}{\partial x_1}\left(a_2 \dfrac{\partial u_2}{\partial x_1}\right) + \delta(u_2 - u_1) = \dfrac{1}{2} f & \text{in } \Omega . \end{cases} \tag{2.8}$$

In terms of Γ*-convergence the previous result implies the following one.*

COROLLARY 2.2. – *Under assumption* (2.5) *the Dirichlet form* F_ε *defined by* (1.1) *and* (2.3) *Γ-converges for the $L^2(\Omega)$-weak topology to the quadratic form*

$$
(2.9)\qquad \begin{cases} F(u) := +\infty & \textit{if } u \in L^2(\Omega)\setminus H \\ F(u) := 2\displaystyle\int_\Omega a_1\left(\frac{\partial u_1}{\partial x_1}\right)^2 + a_2\left(\frac{\partial u_2}{\partial x_1}\right)^2 + \delta(u_1-u_2)^2 & \textit{if } u \in H\,, \end{cases}
$$

where $u_1, u_2 \in H$ are uniquely determined by the system

$$
(2.10)\qquad \begin{cases} u = u_1 + u_2 \\ -\dfrac{\partial}{\partial x_1}\left(a_1\dfrac{\partial u_1}{\partial x_1}\right) + \delta(u_1-u_2) = -\dfrac{\partial}{\partial x_1}\left(a_2\dfrac{\partial u_2}{\partial x_1}\right) + \delta(u_2-u_1)\,. \end{cases}
$$

2.2. *Proof of the homogenization result.*

PROOF OF (2.6). – Let $\Omega_{i,\varepsilon}$, $i=1,2$, be the open subset of Ω obtained by εY-repetition in Ω of $\varepsilon Y_{i,\varepsilon}$ and let ω_ε be the subset of Ω obtained by εY-repetition in Ω of $\varepsilon Q_\varepsilon$; note that $|\omega_\varepsilon| \to 0$. Let $u \in H_0^1(\Omega)$, we have for any $u \in C_0^1(\overline{\Omega})$,

$$
u(x_1, x_2) = \int_0^{x_1} \frac{\partial u}{\partial x_1}(t, x_2)\,dt\,, \qquad x \in \Omega_{1,\varepsilon} \cup \Omega_{2,\varepsilon},
$$

which yields by density the following L^2-estimate

$$
(2.11)\qquad \|u\|_{L^2(\Omega_{1,\varepsilon}\cup\Omega_{2,\varepsilon})} \leqslant \|\nabla u\|_{L^2(\Omega_{1,\varepsilon}\cup\Omega_{2,\varepsilon})}.
$$

On the other side we have for any function $V \in C^1(\overline{Y})$,

$$
V(y_1, y_2) = V(y_1, y_{2,\varepsilon}) + \int_{y_{2,\varepsilon}}^{y_2} \frac{\partial v}{\partial x_2}(y_1, t)\,dt, \qquad y \in Q_\varepsilon \text{ and } y_{2,\varepsilon} \in \left\{ r_\varepsilon, \frac{1 \pm r_\varepsilon}{2} \right\},
$$

whence, since $|y_2 - y_{2,\varepsilon}| \leqslant r_\varepsilon$,

$$
\|V\|_{L^2(Q_\varepsilon)} \leqslant c\|V\|_{L^2(\partial Q_\varepsilon)} + c\sqrt{r_\varepsilon}\|\nabla V\|_{L^2(Q_\varepsilon)}.
$$

By using a density argument and the imbedding from $H^1(Y_{1,\varepsilon}\cup Y_{2,\varepsilon})$ into $L^2(\partial Q_\varepsilon)$, we obtain

$$
\|V\|_{L^2(Q_\varepsilon)} \leqslant c\|V\|_{H^1(Y_{1,\varepsilon}\cup Y_{2,\varepsilon})} + c\sqrt{r_\varepsilon}\|\nabla V\|_{L^2(Q_\varepsilon)},
$$

where c is a constant. Then by ε-rescaling in Ω the previous estimate we obtain

$$\|u\|_{L^2(\omega_\varepsilon)} \leqslant c\|u\|_{H^1(\Omega_{1,\varepsilon} \cup \Omega_{2,\varepsilon})} + c\ \ \varepsilon\sqrt{r_\varepsilon}\|\nabla u\|_{L^2(\omega_\varepsilon)}$$

$$\leqslant c\|u\|_{H^1(\Omega_{1,\varepsilon} \cup \Omega_{2,\varepsilon})} + c_\delta\left\|\sqrt{\alpha_\varepsilon}\nabla u\right\|_{L^2(\omega_\varepsilon)} \ \text{ by (2.5),}$$

which combined to (2.11) yields

$$\|u\|_{L^2(\Omega)} \leqslant (1+2c)\|\nabla u\|_{L^2(\Omega_{1,\varepsilon} \cup \Omega_{2,\varepsilon})} + c_\delta\left\|\sqrt{\alpha_\varepsilon}\nabla u\right\|_{L^2(\omega_\varepsilon)}.$$

By the definition (2.3) of a_ε the previous estimate implies that

$$\|u\|_{L^2(\Omega)} \leqslant C_\delta\left\|\sqrt{a_\varepsilon}\nabla u\right\|_{L^2(\Omega)}$$

where C_δ is a constant, which yields (2.6).

PROOF OF THE HOMOGENIZATION RESULT. – The proof is an adaptation of [6] by replacing thin bridges by insulating thin-layers. We thus give the main steps of the proof without details.

Let u_ε be the solution of the Dirichlet problem (2.4). Thanks to estimate (2.6) it is easy to check that u_ε is bounded in $L^2(\Omega)$ and ∇u_ε is bounded in $L^2(\Omega_{1,\varepsilon} \cup \Omega_{2,\varepsilon})$. Then, since the characteristic function $\mathbf{1}_{\Omega_{i,\varepsilon}}$, $i=1,2$, does not depend on the variable x_1, we have up to a subsequence

$$\mathbf{1}_{\Omega_{i,\varepsilon}} u_\varepsilon \rightharpoonup \xi_i \quad \text{weakly in } H\,, \tag{2.12}$$

where H is defined by (2.7) and

$$\mathbf{1}_{\Omega_{i,\varepsilon}} a_\varepsilon \nabla u_\varepsilon \rightharpoonup \xi_i \quad \text{weakly in } L^2(\Omega)^2. \tag{2.13}$$

Moreover both limits (2.12) and $|\omega_\varepsilon| \to 0$ imply that

$$u_\varepsilon \rightharpoonup u_1 + u_2 \in H \quad \text{weakly in } L^2(\Omega), \tag{2.14}$$

which also holds up to a subsequence.

Now we have to determine the functions u_i and ξ_i, $i=1,2$.

FIRST STEP: construction of two test functions.

On the first side let us introduce a smooth function which allows us to separate both sets $\Omega_{1,\varepsilon}$ and $\Omega_{2,\varepsilon}$. Let $\widehat{V}_\varepsilon$ be the function of $H^1_\#(Y)$ defined by (see (2.1))

$$\text{(2.15)}\qquad \begin{cases} \widehat{V}_\varepsilon(y) := \dfrac{y_2}{r_\varepsilon} & \text{if } y \in [0,1[\times[0,r_\varepsilon] \\ \widehat{V}_\varepsilon(y) := 1 & \text{if } y \in Y_{1,\varepsilon} \\ \widehat{V}_\varepsilon(y) := \dfrac{1+r_\varepsilon-2y_2}{2r_\varepsilon} & \text{if } y \in [0,1[\times\left[\dfrac{1-r_\varepsilon}{2}, \dfrac{1+r_\varepsilon}{2}\right] \\ \widehat{V}_\varepsilon(y) := 0 & \text{otherwise}. \end{cases}$$

Let $V \in H^1(Y)$. By integrating by parts the integral

$$\int_Y \nabla\widehat{V}_\varepsilon\cdot\nabla\left(V-\overline{V}^{Y_{1,\varepsilon}}\widehat{V}_\varepsilon-\overline{V}^{Y_{2,\varepsilon}}(1-\widehat{V}_\varepsilon)\right), \quad \text{where} \quad \overline{V}^{Y_{i,\varepsilon}} := \fint_{Y_{i,\varepsilon}} V,$$

and by using the Poincaré-Wirtinger inequalities

$$\|V-\overline{V}^{Y_{i,\varepsilon}}\|_{L^2(Y_{i,\varepsilon})} \leqslant c_i\|\nabla V\|_{L^2(Y_{i,\varepsilon})}, \qquad i=1,2,$$

we obtain the following estimate

$$\left|\int_Y A_\varepsilon\nabla\widehat{V}_\varepsilon\cdot\nabla V-\widehat{\delta}(\varepsilon)(\overline{V}^{Y_{1,\varepsilon}}-\overline{V}^{Y_{2,\varepsilon}})\right| \leqslant c\widehat{\delta}(\varepsilon)\|\nabla V\|_{L^2(Y_{1,\varepsilon}\cup Y_{2,\varepsilon})}, \qquad \text{where } \widehat{\delta}(\varepsilon) := \frac{2\alpha_\varepsilon}{r_\varepsilon}$$

and c a constant. Then by rescaling the previous estimate and by using limit (2.5) we obtain, for any $\varphi \in \mathcal{D}(\Omega)$, the limit

$$\text{(2.16)}\qquad \int_\Omega a_\varepsilon\nabla\widehat{v}_\varepsilon\cdot\nabla(\varphi u_\varepsilon) \xrightarrow[\varepsilon\to 0]{} \int_\Omega \delta\varphi(u_1-u_2), \quad \text{where} \quad \widehat{v}_\varepsilon(x) := \widehat{V}_\varepsilon\left(\frac{x}{\varepsilon}\right),$$

and u_1, u_2 are defined by (2.10).

On the other side, for any $\lambda \in \mathbb{R}^2$, let w^λ_ε be the function defined by

$$\text{(2.17)}\qquad w^\lambda_\varepsilon(x) := \lambda\cdot x - \varepsilon(\widehat{V}_\varepsilon X^\lambda)\left(\frac{x}{\varepsilon}\right),$$

$$X^\lambda(y) := \lambda_2(y_2-k_2), \qquad y \in (k+Y), \qquad k \in \mathbb{Z}^2.$$

Thanks to the periodicity of $\widehat{V}_\varepsilon X^\lambda$ and to limit (2.5) the function w_ε^λ satisfies

$$\text{(2.18)}\qquad \begin{cases} w_\varepsilon^\lambda \to \lambda\cdot x & \text{strongly in } L^\infty(\Omega) \\ \mathbf{1}_{\omega_\varepsilon} a_\varepsilon\,|\nabla w_\varepsilon^\lambda|^2 \to 0 & \text{strongly in } L^1(\Omega) \end{cases} \quad\text{and}\quad \begin{cases} \nabla w_\varepsilon^\lambda = \lambda_1 e_1 & \text{in } \Omega_{1,\varepsilon} \\ \nabla w_\varepsilon^\lambda = \lambda & \text{in } \Omega_{2,\varepsilon}. \end{cases}$$

SECOND STEP: determination of u_1, u_2.

Let $\varphi \in \mathcal{D}(\Omega)$, putting the function $\varphi\,\widehat{v}_\varepsilon\, w_\varepsilon^\lambda$ (defined by (2.15) and (2.17)) in equation (2.4) and passing to the limit thanks to (2.13), (2.16) and (2.18) yields

$$\int_\Omega \xi_1\cdot\nabla\varphi(\lambda\cdot x) + \int_\Omega \xi_1\cdot\lambda_1 e_1\,\varphi + \int_\Omega \delta(u_1-u_2)\,\varphi(\lambda\cdot x) = \int_\Omega \frac{1}{2} f\varphi(\lambda\cdot x)\,.$$

Similarly with the test function $\varphi\,\widehat{v}_\varepsilon(\lambda\cdot x)$ we have

$$\int_\Omega \xi_1\cdot\nabla(\varphi(\lambda\cdot x)) + \int_\Omega \delta(u_1-u_2)\,\varphi(\lambda\cdot x) = \int_\Omega \frac{1}{2} f\varphi(\lambda\cdot x)\,.$$

Then by substracting both previous equalities we obtain

$$\int_\Omega \xi_1\cdot(\lambda-\lambda_1 e_1)\,\varphi = 0 \qquad \text{for any } \varphi\in\mathcal{D}(\Omega)\,,$$

which implies $\xi_1\cdot e_2 = 0$. Moreover by the definition (2.12) of u_1 and since $\mathbf{1}_{\Omega_{1,\varepsilon}}$ does not depend on x_1, we have

$$a_1\,\mathbf{1}_{\Omega_{1,\varepsilon}}\nabla u_\varepsilon\cdot e_1 = a_1\,\frac{\partial}{\partial x_1}(\mathbf{1}_{\Omega_{1,\varepsilon}} u_\varepsilon) \rightharpoonup a_1\,\frac{\partial u_1}{\partial x_1} \quad \text{weakly in } L^2(\Omega)\,.$$

Therefore

$$\xi_1 = a_1\,\frac{\partial u_1}{\partial x_1} \qquad \text{and similarly} \qquad \xi_2 = a_2\,\frac{\partial u_2}{\partial x_1}\,.$$

Finally, for any $\varphi\in\mathcal{D}(\Omega)$, putting the function $\varphi\,\widehat{v}_\varepsilon$ in equation (2.4) and passing to the limit yields

$$\int_\Omega \xi_1\cdot\nabla\varphi + \delta(u_1-u_2)\,\varphi = \int_\Omega a_1\,\frac{\partial u_1}{\partial x_1}\,\frac{\partial\varphi}{\partial x_1} + \delta(u_1-u_2)\,\varphi = \int_\Omega \frac{1}{2} f\varphi\,,$$

and similarly for the index 2

$$\int_\Omega a_2\,\frac{\partial u_2}{\partial x_1}\,\frac{\partial\varphi}{\partial x_1} + \delta(u_1-u_2)\,\varphi = \int_\Omega \frac{1}{2} f\varphi\,.$$

Since $\mathcal{D}(\Omega)$ is dense into H (2.7), both previous equations also hold for any $\varphi \in H$, which gives system (2.8). By the Lax-Milgram Theorem system (2.8) has a unique solution (u_1, u_2) in $H \times H$. Therefore by (2.14) the whole sequence u_ε weakly converges to $u_1 + u_2$ in $L^2(\Omega)$. Theorem 2.1 is proved.

2.3. *Proof of the Γ-convergence result.*

Let us start by the following remark concerning Theorem 2.1.

REMARK 2.3. – Theorem 2.1 can be extended without restriction to the case where the right hand side $f \in L^2(\Omega)$ of the Dirichlet problem (2.4) is replaced by any sequence

$$f_\varepsilon := \mathbf{1}_{\Omega_{1,\varepsilon}} f_1 + \mathbf{1}_{\Omega_{2,\varepsilon}} f_2 - \frac{\partial}{\partial x_1}(\mathbf{1}_{\Omega_{1,\varepsilon}} g_1) - \frac{\partial}{\partial x_1}(\mathbf{1}_{\Omega_{2,\varepsilon}} g_2), \qquad f_i, g_i \in L^2(\Omega).$$

Therefore the solution u_ε of the problem

$$\begin{cases} -\operatorname{div}(a_\varepsilon \nabla u_\varepsilon) = f_\varepsilon & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega , \end{cases} \tag{2.19}$$

weakly converges in $L^2(\Omega)$ to $u_1 + u_2$ where $u_1, u_2 \in H$ are the solutions of the coupled system

$$\begin{cases} -\dfrac{\partial}{\partial x_1}\left(a_1 \dfrac{\partial u_1}{\partial x_1}\right) + \delta(u_1 - u_2) = \dfrac{1}{2}\left(f_1 - \dfrac{\partial g_1}{\partial x_1}\right) & \text{in } \Omega \\ -\dfrac{\partial}{\partial x_1}\left(a_2 \dfrac{\partial u_2}{\partial x_1}\right) + \delta(u_2 - u_1) = \dfrac{1}{2}\left(f_2 - \dfrac{\partial g_2}{\partial x_1}\right) & \text{in } \Omega . \end{cases} \tag{2.20}$$

By (2.6) the quadratic form F_ε defined by (1.1) and (2.3) is equicoercive with respect to the $L^2(\Omega)$-norm. Then by a classical result of Γ-convergence (see Corollary 8.12 p. 95 from [7]), F_ε Γ-converges for the $L^2(\Omega)$-weak topology, up to a subsequence, to a functional $F : L^2(\Omega) \to [0, +\infty]$. We have to determine $F(u)$ for any $u \in L^2(\Omega)$.

Let $u \in L^2(\Omega) \setminus H$ and let u_ε be a sequence of $L^2(\Omega)$ which weakly converges to u in $L^2(\Omega)$. Assume that

$$\varliminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) < +\infty .$$

Then the function u_ε belongs to $H_0^1(\Omega)$ and ∇u_ε is bounded in $L^2(\Omega_{1,\varepsilon} \cup \Omega_{2,\varepsilon})$. Therefore the weak limits (2.12) and (2.14) hold for the sequence u_ε, which contradicts $u \notin H$. We thus have

$$F(u) = \varliminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) = \varlimsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) = +\infty .$$

Let us now consider a function $u \in H$ and let $u_1 \in H$ be the unique solution of the equation

$$-\frac{\partial}{\partial x_1}\left((a_1+a_2)\frac{\partial u_1}{\partial x_1}\right)+4\delta u_1=-\frac{\partial}{\partial x_1}\left(a_2\frac{\partial u}{\partial x_1}\right)+2\delta u\,.$$

Then u_1 and $u_2 := u-u_1$ are the solutions of system (2.10). Let f_ε be the sequence from H' (the dual of H) defined by

$$f_\varepsilon := -2\frac{\partial}{\partial x_1}\left(\mathbf{1}_{\Omega_{1,\varepsilon}}a_1\frac{\partial u_1}{\partial x_1}\right)-2\frac{\partial}{\partial x_1}\left(\mathbf{1}_{\Omega_{2,\varepsilon}}a_2\frac{\partial u_2}{\partial x_1}\right)+2\delta(u_1-u_2)(\mathbf{1}_{\Omega_{1,\varepsilon}}-\mathbf{1}_{\Omega_{2,\varepsilon}})$$

and let u_ε be the solution of problem (2.19). By Theorem 2.1 and Remark 2.3 the sequences $\mathbf{1}_{\Omega_{i,\varepsilon}}u_\varepsilon$ weakly converge in H to the functions v_i solutions of the system (2.20) with the right hand sides

$$-\frac{\partial}{\partial x_1}\left(a_1\frac{\partial u_1}{\partial x_1}\right)+\delta(u_1-u_2)\quad\text{and}\quad\frac{\partial}{\partial x_1}\left(a_2\frac{\partial u_2}{\partial x_1}\right)+\delta(u_2-u_1)\,,$$

whence $v_i=u_i$ for $i=1,2$. In particular we have the following convergences

$$u_\varepsilon \rightharpoonup u_1+u_2\quad\text{weakly in } L^2(\Omega)\quad\text{and}$$

$$\mathbf{1}_{\Omega_{i,\varepsilon}}a_i\frac{\partial u_\varepsilon}{\partial x_1}=a_i\frac{\partial}{\partial x_1}(\mathbf{1}_{\Omega_{i,\varepsilon}}u_\varepsilon)\rightharpoonup a_i\frac{\partial u_i}{\partial x_1}\quad\text{weakly in } L^2(\Omega),\quad i=1,2\,.$$

Then, denoting by $\langle , \rangle$ the duality H'-H, we have by the definition of u_ε

$$F_\varepsilon(u_\varepsilon)=\langle f_\varepsilon, u_\varepsilon\rangle \underset{\varepsilon\to 0}{\longrightarrow} \overline{F}(u):=2\int_\Omega a_1\left(\frac{\partial u_1}{\partial x_1}\right)^2+a_2\left(\frac{\partial u_2}{\partial x_1}\right)^2+\delta(u_1-u_2)^2, \tag{2.21}$$

whence by the definition of the Γ-convergence

$$F(u)\leqslant \liminf_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon)=\overline{F}(u)<+\infty\,. \tag{2.22}$$

On the other side there exists a sequence $\overline{u}_\varepsilon$ from $H_0^1(\Omega)$ such that

$$\overline{u}_\varepsilon\rightharpoonup u\text{ weakly in } L^2(\Omega)\quad\text{and}\quad F(u)=\lim_{\varepsilon\to 0}F_\varepsilon(\overline{u}_\varepsilon)<+\infty\,.$$

Similarly to u_ε the sequence $\overline{u}_\varepsilon$ satisfies, up to a subsequence, the convergences $\mathbf{1}_{\Omega_{i,\varepsilon}}u_\varepsilon \rightharpoonup \overline{u}_i$ weakly in H, $i=1,2$, whence

$$\langle f_\varepsilon, \overline{u}_\varepsilon\rangle \underset{\varepsilon\to 0}{\longrightarrow} 2\int_\Omega a_1\frac{\partial u_1}{\partial x_1}\frac{\partial \overline{u}_1}{\partial x_1}+a_2\frac{\partial u_2}{\partial x_1}\frac{\partial \overline{u}_2}{\partial x_1}+\delta(u_1-u_2)(\overline{u}_1-\overline{u}_2)=\overline{F}(u) \tag{2.23}$$

by the definition (2.21) of $\overline{F}$ combined with $u=\overline{u}_1+\overline{u}_2=u_1+u_2$ and the system (2.10) satisfied by u_1, u_2. Moreover u_ε being a minimizer of the functional $F_\varepsilon - 2\langle f_\varepsilon, \cdot\rangle$ on $H_0^1(\Omega)$, we have

$$F_\varepsilon(u_\varepsilon) \leqslant F_\varepsilon(\overline{u}_\varepsilon) + 2\langle f_\varepsilon, u_\varepsilon - \overline{u}_\varepsilon\rangle .$$

Then passing to the limit in the previous inequality thanks to limits (2.21) and (2.23) yields

$$\overline{F}(u) = \lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \leqslant F(u),$$

which combined with inequality (2.22) implies $F(u) = \overline{F}(u)$. The functional F thus satisfies formula (2.9), which concludes the proof of Corollary 2.2.

3. – A counter-example.

The main result of the paper is the following.

THEOREM 3.1. – *The quadratic forms defined by* (2.9) *are densely defined, closed and regular. However there exists at least one of these forms which is not Markovian.*

PROOF. – Let F be a form defined by (2.9). The domain of F is the Hilbert space H defined by (2.7) and F is densely defined (1.4) since H is clearly dense into $L^2(\Omega)$.

The form F is closed. Indeed on the first side we have for any $u \in H$,

$$F(u) \leqslant c_\delta \left(\left\| \frac{\partial u_1}{\partial x_1} \right\|^2_{L^2(\Omega)} + \left\| \frac{\partial u_2}{\partial x_1} \right\|^2_{L^2(\Omega)} \right) \leqslant c_\delta' \left\| \frac{\partial u}{\partial x_1} \right\|^2_{L^2(\Omega)};$$

the last estimate holds since by (2.10) $u_{1,2}$ are solutions of the equations

$$- \frac{\partial}{\partial x_1}\left((a_1 + a_2) \frac{\partial u_{1,2}}{\partial x_1} \right) + 4\delta u_{1,2} = - \frac{\partial}{\partial x_1}\left(a_{2,1} \frac{\partial u}{\partial x_1} \right) + 2\delta u . \tag{3.1}$$

On the other side since $u = u_1 + u_2$, we also have

$$F(u) \geqslant 2c \left(\left\| \frac{\partial u_1}{\partial x_1} \right\|^2_{L^2(\Omega)} + \left\| \frac{\partial u_2}{\partial x_1} \right\|^2_{L^2(\Omega)} \right) \geqslant c \left\| \frac{\partial u}{\partial x_1} \right\|^2_{L^2(\Omega)} = c\|u\|^2_H,$$

Therefore $\sqrt{F}$ defines a norm which is equivalent to $\|\cdot\|_H$ in H, whence the closedness (1.5).

The form H is regular since $C_0^1(\Omega)$ is dense into H.

For the last part of Theorem 3.1 we proceed by contradiction. We assume that the quadratic forms defined by (2.9) are Markovian (1.6) for any choice of δ and a_1, a_2. We will obtain a contradiction by making $\delta \to 0$ and by passing to a one-dimensional form whose domain is $H_0^1(]0, 1[)$.

FIRST STEP: reduction to $\delta = 0$.

Let $u \in H$, by choosing $a_1^\delta(x) := x_1 + \delta$ and $a_2^\delta(x) := 1 - x_1 + \delta$ the form (2.9) (multiplied by 1/2) can be written

$$F_\delta(u) := \int_\Omega a_1^\delta \left(\frac{\partial u_1^\delta}{\partial x_1} \right)^2 + a_2^\delta \left(\frac{\partial u_2^\delta}{\partial x_1} \right)^2 + \delta (u_1^\delta - u_2^\delta)^2, \tag{3.2}$$

where by (3.1) $u_{1,2}^\delta$ are solutions of the equations $u = u_1^\delta + u_2^\delta$ and

$$-\frac{\partial}{\partial x_1}\left((1+2\delta) \frac{\partial u_{1,2}^\delta}{\partial x_1} \right) + 4\delta u_{1,2}^\delta = f_{1,2}^\delta := -\frac{\partial}{\partial x_1}\left(a_{2,1}^\delta \frac{\partial u}{\partial x_1} \right) + 2\delta u \,.$$

The distributions f_i^δ, $i = 1, 2$, are clearly compact in H' since u is fixed. Then u_i^δ, $i = 1, 2$, strongly converge in H to the functions u_i solutions of the equations

$$u_1 + u_2 = u \quad \text{and} \quad \begin{cases} \dfrac{\partial^2 u_1}{\partial x_1^2} = \dfrac{\partial}{\partial x_1}\left((1 - x_1) \dfrac{\partial u}{\partial x_1} \right) & \text{in } \Omega \\ \dfrac{\partial^2 u_2}{\partial x_1^2} = \dfrac{\partial}{\partial x_1}\left(x_1 \dfrac{\partial u}{\partial x_1} \right) & \text{in } \Omega \,. \end{cases} \tag{3.3}$$

Therefore the sequence $F_\delta(u)$ converges to

$$F(u) := \int_\Omega x_1 \left(\frac{\partial u_1}{\partial x_1} \right)^2 + (1 - x_1) \left(\frac{\partial u_2}{\partial x_1} \right)^2, \tag{3.4}$$

where u_1, u_2 are defined by (3.3). Formula (3.4) defines a quadratic form on H which is Markovian since any F_δ is assumed to be so. In particular we have

$$\forall u \in H, \qquad F(u^+) \leqslant F(u), \qquad \text{where } u^+ := \max(0, u) \,. \tag{3.5}$$

SECOND STEP: reduction to a one-dimensional form.

Let w be a function from $L^2(]0, 1[_{x_2})$, $w = w(x_2)$, such that $w > 0$ and $\|w\|_{L^2(]0,1[)} = 1$. For any $v \in H^1(]0, 1[_{x_1})$, $v = v(x_1)$, let v_1, $v_2 \in H^1(]0, 1[_1)$ be the solutions of the equations

$$v_1 + v_2 = v \quad \text{and} \quad \begin{cases} \dfrac{\mathrm{d}^2 v_1}{\mathrm{d}x_1^2} = \dfrac{\mathrm{d}}{\mathrm{d}x_1}\left((1 - x_1) \dfrac{\mathrm{d}v}{\mathrm{d}x_1} \right) & \text{in }]0, 1[\\ \dfrac{\mathrm{d}^2 v_2}{\mathrm{d}x_1^2} = \dfrac{\mathrm{d}}{\mathrm{d}x_1}\left(x_1 \dfrac{\mathrm{d}v}{\mathrm{d}x_1} \right) & \text{in }]0, 1[\,. \end{cases} \tag{3.6}$$

Then the functions $u(x) := v(x_1)\, w(x_2)$ and $u_i(x) := v_i(x_1)\, w(x_2)$ are solutions of (3.3) and by definition (3.4) we have

$$F(u) = G(v) := \int_0^1 x_1 \left(\frac{dv_1}{dx_1} \right)^2 + (1 - x_1) \left(\frac{dv_2}{dx_1} \right)^2, \tag{3.7}$$

where v_1, v_2 are solutions of (3.6).

Moreover the quadratic form G defined by (3.7) also satisfies the inequality

$$\forall v \in H_0^1(]0, 1[), \qquad G(v^+) \leqslant G(v). \tag{3.8}$$

Indeed let $v \in H_0^1(]0, 1[)$ and $u := (v_1 + v_2)\, w$ where v_1, v_2 are the solutions of (3.6). Since $w > 0$, we have $u^+ = (v_1 + v_2)^+ w = (v_1' + v_2')\, w$ where v_1', v_2' are the solutions of (3.6) for the function v^+. Then, since w does not depend on x_1, the functions $u_i := v_i w$ and $u_i' = v_i' w$, $i = 1, 2$, are solutions of (3.3) respectively with u and u^+. Therefore by inequality (3.5) we obtain

$$G(v^+) = F(u^+) \leqslant F(u) = G(v),$$

whence (3.8).

We can also simplify the expression (3.7) of the quadratic form G defined by (3.7). Let $v \in H_0^1(]0, 1[)$, by (3.6) there exists a constant c_v such that

$$\frac{dv_1}{dx_1} = (1 - x_1) \frac{dv}{dx_1} + c_v$$

and by an integration by parts we obtain

$$c_v = -\int_0^1 v \quad \text{and} \quad \frac{dv_1}{dx_1} = (1 - x_1) \frac{dv}{dx_1} - \int_0^1 v\,.$$

Similarly for v_2 we have

$$\frac{dv_2}{dx_1} = (1 - x_1) \frac{dv}{dx_1} + \int_0^1 v\,.$$

Then replacing the derivatives of v_1, v_2 by both previous expressions in the definition (3.7) of G yields

$$G(v) = \left(\int_0^1 v \right)^2 + \int_0^1 x_1 (1 - x_1) \left(\frac{dv}{dx_1} \right)^2. \tag{3.9}$$

THIRD STEP: the contradiction.

Let v be a positive function from $H_0^1(I)$ where I is an interval such that $\bar{I} \subset]0, 1[$. We extend the function $-v$ by $\tilde{v} \in H_0^1(]0, 1[)$ such that

$$\tilde{v}_{|I} = -v, \qquad \tilde{v} \geqslant 0 \text{ in }]0, 1[\setminus \bar{I} \qquad \text{and} \qquad \int_0^1 \tilde{v} = 0\,,$$

whence

$$\tilde{v}^- = v \qquad \text{and} \qquad \int_0^1 \tilde{v}^+ = \int_0^1 v\,.$$

Therefore by putting the functions $\tilde{v}, \tilde{v}^+$ in inequality (3.8) with formula (3.9) of G we obtain the equality

$$\left(\int_0^1 v\right)^2 \leqslant \int_0^1 x_1(1-x_1)\left(\frac{\mathrm{d}v}{\mathrm{d}x_1}\right)^2$$

which holds for any positive function $v \in H_0^1(I)$. In particular, for the sequence v_n, $n \in \mathbb{N}^*$, defined by

$$\begin{cases} v_n(x_1) := 0 & \text{if } x_1 \in \left[0, \dfrac{1}{n^n}\right] \\[2mm] v_n(x_1) := x_1^{1/n} - \dfrac{1}{n} & \text{if } x_1 \in \left[\dfrac{1}{n^n}, \dfrac{1}{2}\right] \\[2mm] v_n(x_1) := v_n(1-x_1) & \text{if } x_1 \in \left[\dfrac{1}{2}, 1\right], \end{cases}$$

the previous inequality implies that

$$1 = \lim_{n \to +\infty} \left(\int_0^1 v_n\right)^2 \leqslant \lim_{n \to +\infty} \int_0^1 x_1(1-x_1)\left(\frac{\mathrm{d}v_n}{\mathrm{d}x_1}\right)^2 = 0\,,$$

which yields the contradiction.

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