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Generation of Finite Groups by Nilpotent Subgroups.

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Sunto. – *Si studia la generazione di gruppi finiti tramite sottogruppi nilpotenti, in particolare viene esaminata la struttura di gruppi finiti non generabili con n sottogruppi nilpotenti e tali che ogni quoziente proprio sia generabile con n sottogruppi nilpotenti. Si ottengono alcuni risultati di struttura per questi gruppi e un limite inferiore per il loro ordine.*

Summary. – *We study the generation of finite groups by nilpotent subgroups and in particular we investigate the structure of groups which cannot be generated by n nilpotent subgroups and such that every proper quotient can be generated by n nilpotent subgroups. We obtain some results about the structure of these groups and a lower bound for their orders.*

1. – Introduction.

As a consequence of the Classification Theorem, every finite nonabelian simple group can be generated by two cyclic groups and indeed, by a result of Aschbacher and Guralnick in [1], every finite group can be generated by two conjugate solvable subgroups. By contrast, it has been shown by Cossey and Hawkes in [3] that there exists no bound for the number of nilpotent subgroups necessary to generate a finite group. In fact, for every natural number n, they construct a finite solvable group G_n which cannot be generated by fewer than n nilpotent subgroups. Although the actual orders of the groups G_n are not relevant for the result, it may be noticed that they are extremely big, even for small values of n. So an independent problem, which is interesting in itself, is to find some results about the structure of the groups of minimal order with respect to the property of not being generated by n nilpotent subgroups. The

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analysis of this situation leads us to study a larger class of groups: we say that a finite group G is in $\mathcal{C}_n$ if it cannot be generated by n nilpotent subgroups and G/N can be generated by n nilpotent subgroups for every nontrivial normal subgroup N.

If we consider the generation of finite solvable groups by nilpotent subgroups, as in [3], we find out that the complexity of a minimal group not generated by n nilpotent subgroups increases with n. The first result we can obtain in this case is about the Fitting length, that is the minimal length of a normal series with nilpotent factors.

THEOREM 1. – *Let G be a finite solvable group with Fitting length at most r, then G can be generated by r nilpotent subgroups.*

This implies that a finite solvable group that cannot be generated by n nilpotent subgroups has Fitting length at least $n+1$, and this shows how the group structure is complex.

In section 2 we obtain some results about the structure of the groups in $\mathcal{C}_n$ and a lower bound for their orders. We can observe that the groups of minimal order with respect to the property of not being generated by n nilpotent subgroups are groups of minimal order in $\mathcal{C}_n$; to determine these groups is a problem we can solve for $n=1$, $n=2$. As groups of minimal order in $\mathcal{C}_1$ are minimal nonnilpotent groups we can use Theorem 9.1.9 in [9] to have a complete characterization of their structure and order. In [1] there is an example of a group of order $5^6 * 12$ which cannot be generated by two nilpotent subgroups; using the results of section 2 we show in section 3 that up to isomorphism this is the group of minimal order in $\mathcal{C}_2$. In particular we get:

THEOREM 2. – *Let G be a finite group of $\mathcal{C}_2$, then G is either isomorphic to the group described in* [1] *or* $|G| > 5^6 * 12$.

2. – The structure of groups in $\mathcal{C}_n$.

PROPOSITION 2.1. – *Let G be a finite group and N a nilpotent normal subgroup of G. If G/N is generated by n nilpotent subgroups, then G can be generated by $n+1$ nilpotent subgroups.*

PROOF. – By hypothesis there exist n nilpotent subgroups, $H_1/N, \ldots, H_n/N$, of G/N such that $G/N = \langle H_1/N, \ldots, H_n/N \rangle$. By Th. 12.2 in [7], the solvable group H_i contains a Carter subgroup K_i. It can be easily seen that $K_i N = H_i$. Hence we can say that $\langle K_1, \ldots, K_n, N \rangle = \langle H_1, \ldots, H_n, N \rangle = G$. ■

As a consequence we have the following results.

COROLLARY 2.1. – *Let* $\{1_G\} = G_r \trianglelefteq G_{r-1} \trianglelefteq \ldots \trianglelefteq G_0 = G$ *be a normal series with nilpotent factors of a finite group* G. *Then* G *can be generated by* r *nilpotent subgroups.*

COROLLARY 2.2. – *Let* G *be a finite solvable group with derived length at most* r, *then* G *can be generated by* r *nilpotent subgroups.*

THEOREM 2.1. – *Let* G *be a finite group in* $\mathcal{C}_n$. *Then* $\mathrm{Fit}(G)$ *is the product of all the abelian minimal normal subgroups of* G *and it has a complement in* G. *Moreover* $Z(G) = 1_G$.

PROOF. – We remark that $\mathrm{Frat}(G) = 1_G$ otherwise $G/\mathrm{Frat}(G)$ could be generated by n nilpotent subgroups $H_1/\mathrm{Frat}(G), \ldots, H_n/\mathrm{Frat}(G)$ and, taking K_i to be a Carter subgroup of H_i, we have $H_i = K_i \mathrm{Frat}(G)$ hence $G = \langle K_1, \ldots K_n, \mathrm{Frat}(G)\rangle = \langle K_1, \ldots, K_n\rangle$, which is false.

As $\mathrm{Frat}(G) = 1_G$, by [9] Th. 5.2.15, we can say that $\mathrm{Fit}(G)$ is the product of all the abelian minimal normal subgroups of G, hence it is abelian and, by [9] Th. 5.2.13, it has a complement in G.

If $Z = Z(G) \neq 1_G$, then $G/Z = \langle H_1/Z, \ldots, H_n/Z\rangle$ with H_i/Z nilpotent for every $i = 1, \ldots, n$. This means that each H_i is nilpotent and G can be generated by n nilpotent subgroups, in contrast with the hypothesis. ■

PROPOSITION 2.2. – *Let* G *be a finite group in* $\mathcal{C}_n$ *and let* N *be an abelian minimal normal subgroup of* G. *Then* N *has complements in* G *and the number of its different complements is at least* $|N|^n$.

PROOF. – As G is in $\mathcal{C}_n$, G/N is generated by n nilpotent subgroups, $H_1/N, \ldots H_n/N$, thus $G = \langle H_1, \ldots, H_n\rangle N$. If K_i is a Carter subgroup of H_i, then $K_i N = H_i$ hence $G = \langle K_1, \ldots, K_n\rangle N$. Since G cannot be generated by n nilpotent subgroups, $G \neq \langle K_1, \ldots, K_n\rangle$ and $N \nleqslant \langle K_1, \ldots, K_n\rangle$; in addition, as $G = \langle K_1, \ldots, K_n\rangle N$ and N is abelian, $\langle K_1, \ldots, K_n\rangle \cap N$ is a normal subgroup of G, so, by the minimality of N, we have $\langle K_1, \ldots, K_n\rangle \cap N = 1$. It follows that $\langle K_1, \ldots, K_n\rangle$ is a complement of N in G. In the same way we can show that for every $s_1, \ldots, s_n$ in N also $\langle K_1^{s_1}, \ldots, K_n^{s_n}\rangle$ is a complement of N in G. From $N_N(K_i) \leqslant N \cap N_{H_i}(K_i) \leqslant N \cap K_i = 1$, it follows $C_N(K_i) = N_N(K_i) = 1_G$. So, given $m_1, \ldots, m_n$ in N such that $(s_1, \ldots, s_n) \neq (m_1, \ldots m_n)$, we claim $\langle K_1^{s_1}, \ldots, K_n^{s_n}\rangle \neq \langle K_1^{m_1}, \ldots, K_n^{m_n}\rangle$. For this purpose we first note that $(s_1, \ldots, s_n) \neq (m_1, \ldots m_n)$ implies $s_i \neq m_i$, for some i, then $s_i m_i^{-1} \notin N_N(K_i)$ hence $K_i^{s_i} \neq K_i^{m_i}$. Next assume that $X = \langle K_1^{s_1}, \ldots, K_n^{s_n}\rangle = \langle K_1^{m_1}, \ldots, K_n^{m_n}\rangle$. It follows $K_i^{s_i} < \langle K_i^{s_i}, K_i^{m_i}\rangle \leqslant K_i^{s_i} N \cap X = K_i^{s_i}(N \cap X)$, hence $X \cap N \neq 1_G$, against the hypothesis that X is a complement of N in G. We reach the conclusion that N has at least $|N|^n$ different complements. ■

We need the following formula due to Aschbacher and Guralnick, [2].

PROPOSITION 2.3. – *Let G be a finite group, M an irreducible G-module and $q = |\mathrm{End}_G(M)|$, then*

$$|H^1(G, M)| = q^{\delta_G(M)} \left| H^1\left(\frac{G}{C_G(M)}, M\right) \right|,$$

where $\delta_G(M)$ is the number of the complemented factors G-isomorphic to M in a chief series of G. In addition, if M is faithful then $|H^1(G, M)| < |M|$ and $|H^1(G, M)| = 0$ if G is solvable.

A consequence of the last two results is the following proposition.

PROPOSITION 2.4. – *Let G be a finite group in $\mathcal{C}_n$ and let N be an abelian minimal normal subgroup of G, then*

$$\delta_G(N) \geq r(n-1) - t + 1,$$

where $E = \mathrm{End}_G(N)$, $r = \dim_E N$, $t = \dim_E H^1(G/C_G(N), N)$. Moreover $t < r$ and if $G/C_G(N)$ is solvable, $t = 0$ and $\delta_G(N) \geq r(n-1) + 1$.

PROOF. – The number of complements of N in G is $|\mathrm{Der}(G/N, N)|$, since N is a minimal normal subgroup of G and $Z(G) = 1$ we get $|\mathrm{Der}(G/N, N)| = |N| |H^1(G/N, N)|$. As it comes from proposition 2.2 and proposition 2.3:

$$|N| \left| H^1\left(\frac{G}{C_G(N)}, N\right) \right| |\mathrm{End}_G(N)|^{\delta_G(N) - 1} \geq |\mathrm{Der}(G/N, N)| \geq |N|^n.$$

Observe that N and $H^1(G/C_G(N), N)$ are E-vector spaces so $|E|^{rn} \leq |E|^t |E|^r |E|^{\delta_G(N) - 1}$ and $\delta_G(N) \geq r(n-1) - t + 1$. ■

LEMMA 2.1. – *Let G be a finite group and let N be an abelian minimal normal subgroup of G. If $\dim_{\mathrm{End}_G(N)}(N) = 1$, then $G/C_G(N)$ is a cyclic group.*

PROOF. – Let $|N| = p^n$ and $E = \mathrm{End}_G(N)$; $\dim_E(N) = 1$ implies $|E| = |N| = p^n$ and $E \simeq F_q$, the field with $q = p^n$ elements. It follows $G/C_G(N) \leq \mathrm{Aut}(N) \simeq \mathrm{GL}(n, p)$, where $\mathrm{GL}(n, p)$ is the group of the invertible matrices of order $n \times n$ over a field with p elements. In addition, $x^{ge} = x^{eg}$ for all $x \in N$, $g \in G$, $e \in E$, so $G/C_G(N) \leq C_{\mathrm{GL}(n, p)}(E)$. We know that $C_{\mathrm{GL}(n, p)}(F_q) \simeq F_q^\star$, [7], Satz 7.3, pag. 187, so $G/C_G(N) \leq F_q^\star$ is a cyclic group. ■

PROPOSITION 2.5. – *Let G be a finite group in $\mathcal{C}_n$, assume $F := \mathrm{Fit}(G) \neq 1$ and $F = C_G(F)$. Then there exist $n+1$ distinct primes, $p_1, \ldots, p_{n+1}$ such that $p_1^{2(2n-2)} * p_2 * \ldots * p_{n+1}$ divides the order of G.*

PROOF. – By Theorem 2.1 $G = HF$, where $F = N_1^{a_1} \times \ldots \times N_t^{a_t}$, each N_i is an elementary abelian p_i-group and for $i \neq j$, N_i and N_j are not isomorphic as G-modules, moreover $H \leqslant \prod_{i=1}^{t} G/C_G(N_i)$. Note that if $\dim_{\mathrm{End}_G(N_i)}(N_i) = 1$ for all i, then, by Lemma 2.1, H is abelian and G is generated by two nilpotent subgroups. Thus we may assume that $\dim_{\mathrm{End}_G(N_1)}(N_1) \geqslant 2$. This of course implies that p_1^2 divides the order of N_1, moreover, using Proposition 2.4 we get $\delta_G(N_1) \geqslant 2(n-2)+2$. Hence $p_1^{2(2n-2)}$ divides the order of G. Now, let $p_1, \ldots, p_r$ be all the distinct primes dividing $|G|$ and, for all $1 \leqslant i \leqslant r$, let P_i be a Sylow p_i-subgroup of G. Since G can be generated by r nilpotent subgroups, $P_1, \ldots, P_r$, it must be $r \geqslant n+1$ and $p_1^{2(2n-2)} * p_2 * \ldots * p_{n+1}$ divides the order of G. ■

Let us observe that the hypothesis $C_G(F) = F \neq 1$ is satisfied in particular if G is a solvable group in $\mathcal{C}_n$. Moreover, in this case, by proposition 2.4 we get $\delta_G(N_1) \geqslant 2(n-1)+1$ and it follows that $p_1^{2(2n-1)} * p_2 * \ldots p_{n+1}$ divides the order of G.

In the proof of the main theorem of the next section we will use the following results.

PROPOSITION 2.6 (Guralnick, [6]). – *Let G be a finite simple group, then it can be generated by two Sylow 2-subgroups of G.*

PROPOSITION 2.7 (Gaschütz, [5]). – *Let N be a normal subgroup of G and let $g_1, \ldots, g_d \in G$ be such that $G/N = \langle g_1N, \ldots, g_dN\rangle$. If G can be generated with d elements then there exists $u_1, \ldots, u_d \in N$ such that $G = \langle g_1u_1, \ldots, g_du_d\rangle$.*

PROPOSITION 2.8 (Lucchini, [8]). – *If G is a finite group, N a minimal normal subgroup of G and $d(G)$ the minimal number of elements needed to generate G, then $d(G) \leqslant \max(2, d(G/N)+1)$.*

PROPOSITION 2.9 (DallaVolta, Lucchini, [4]). – Let S be a non abelian simple group. If G is an automorphism group of S with $S \leqslant G \leqslant \mathrm{Aut}(S)$, then $d(G) = \max(2, d(G/S))$.

3. – The group of minimal order in $\mathcal{C}_2$.

Let $W = \langle w_1, w_2\rangle$ be a 2-dimensional vector space over a field with 5 elements and let H be the subgroup of order 12 of $GL(W)$ generated by

$$a : w_1 \mapsto 2w_2,\ w_2 \mapsto 2w_1, \qquad b : w_1 \mapsto -w_1 - w_2,\ w_2 \mapsto w_1.$$

Consider the H-module $V = W^3$ and let M be the semidirect product of V by H. In [1] the authors notice that M cannot be generated by two nilpotent subgroups. In this section we show that, up to isomorphism, M is the group of minimal order in $\mathcal{C}_2$.

In the following table we heighlight some useful tips about non abelian finite simple groups of order less than 62500 as they will be largely employed in the proof of the next theorem.

TABLE 1. – *Simple groups of order* ≤ 62500.

S	|S|	|**Out** (S)|
A_5	60	2
A_6	360	4
A_7	2520	2
$A_8 = L_4(2)$	20160	2
M_{11}	7920	1
$L_3(2) = L_2(7)$	168	2
$L_2(8)$	504	3
$L_2(11)$	660	2
$L_2(13)$	1092	2
$L_2(16)$	4080	4
$L_2(17)$	2448	2
$L_2(19)$	3420	2
$L_2(23)$	6072	2
$L_2(25)$	7800	4
$L_2(27)$	9828	6
$L_2(29)$	12180	2
$L_2(31)$	14880	2
$L_2(32)$	32736	5
$L_2(49)$	58800	4
$L_3(3)$	5616	2
$L_3(4)$	20160	12
$U_3(3)$	6048	2
$U_3(4)$	62400	4
$U_4(2)$	25920	2
$Sz(8)$	29120	3

THEOREM 3.1. – *Let G be a finite group in $\mathcal{C}_2$ then G is either isomorphic to M or $|G| > 5^6 * 12$.*

PROOF. – Let us suppose that $|G| \leq 5^6 * 12$ and $G \neq M$.

Assume that $F := \mathrm{Fit}(G) = 1$.

Under this assumption the minimal normal subgroups of G, $N_1, \ldots, N_r$, are non abelian and $G \simeq G/\bigcap_{i=1}^{r} C_G(N_i) \leq \prod_{i=1}^{r} \mathrm{Aut}(N_i)$.

As every N_i is characteristically simple we get $N_i = S_i^{n_i}$, where S_i is a non abelian simple group, and $\mathrm{Aut}(N_i) \simeq \mathrm{Aut}(S_i) \wr \mathrm{Sym}(n_i)$. Considering the bound on the order of G we deduce that it has either two minimal normal simple subgroups, N_1, N_2, or a minimal normal subgroup $N_1 = S_1^{n_1}$ where $n_1 \leqslant 2$. In any case, by Proposition 2.6, there exists a prime $p \geqslant 3$ dividing $|G/\mathrm{soc}(G)|$ so we may assume that p divides the order of $\mathrm{Out}(N_1)$ hence

$$N_1 \in \{L_2(8), L_3(4), Sz(8)\, L_2(27), L_2(32)\}.$$

If $N_1 \in \{L_2(27), L_3(4), Sz(8), L_2(32)\}$ then it is the unique minimal normal subgroup, G is almost simple and it is generated by two cyclic subgroups by proposition 2.9. If $N_1 = L_2(8)$ then it is either the unique minimal normal subgroup or there exists $N_2 = A_5$; in both cases we can generate G by a Sylow 2-subgroup and a Sylow 3-subgroup.

It follows that $1 < F$ and let us suppose that $1 < F = C_G(F)$.

By Theorem 2.1, $G = HF$, F is the product of all the abelian minimal normal subgroups of G, $F = N_1^{a_1} \times \ldots \times N_t^{a_t}$ where, for $i \neq j$, N_i and N_j are not isomorphic as G-modules; moreover $H = G/F = G/C_G(F) \leqslant \prod_{i=1}^{t} \mathrm{Aut}(N_i)$ and $\delta_G(N_i) \geqslant 2$, $i = 1, \ldots, t$. In addition $Z(G) = 1$, thus, in particular, no minimal normal abelian subgroup of G can be isomorphic to $\mathbb{Z}_2$.

Let us suppose that G is a solvable group.

By lemma 2.1 we may assume $\dim_{End_G(N_1)} N_1 \geqslant 2$. As a consequence of the remark at the end of Proposition 2.5 we get that $|G|$ is divisible by $|N_1|^3 * p * q$, where p and q are distinct primes not dividing the order of N_1. Hence the only possible choices for N_1 are: $\mathbb{Z}_2^2$, $\mathbb{Z}_3^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$, $\mathbb{Z}_5^2$.

Let us consider $N_1 = \mathbb{Z}_2^2$.

Note that $\mathrm{Aut}(N_1) = \mathrm{GL}(2,2) \simeq \mathrm{Sym}(3)$ and, since G has order divisible by at least three distinct primes, there exists in F a minimal normal subgroup $N_2 = \mathbb{Z}_p^n$, such that $|N_2|\,|\mathrm{Aut}(N_2)|$ is divisible by a prime different from 2 and 3. Let $d = \dim_{\mathrm{End}_G(\mathrm{N}_2)} N_2$. If $d \geqslant 3$ then $|N_2| \geqslant 8$, $\delta_G(N_2) \geqslant 4$ hence $|G| \geqslant 4^3 * 8^4 > 5^6 * 12$. Let $d = 2$; since 2 and 3 are the only primes dividing $|\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2)|$ and $|\mathrm{Aut}(\mathbb{Z}_3 \times \mathbb{Z}_3)|$ it follows that $|N_2| \geqslant 4^2$, thus $|G| \geqslant 4^3 * 16^3 > 5^6 * 12$. So we can assume $d = 1$; in addition, let N be a minimal normal p-subgroup of G not G-isomorphic to N_1 or to N_2 and let $\dim_{\mathrm{End}_G(\mathrm{N})} N = r > 1$, since $|N_2| \geqslant 5$ we get $|N|^{r+1} \leqslant \frac{5^6 * 12}{2^6 * 5^2} < 118$ so $N = \mathbb{Z}_2^2$. It follows that $N_2 = \mathbb{Z}_5$, $G/C_G(F)$ is a 2-group and G is generated by a Sylow 2-Subgroup and a Sylow 5-subgroup; hence $r = 1$. Let $H_i = G/C_G(N_i)$. From what we have just seen, H_i is a cyclic group when $2 \leqslant i \leqslant t$ while $H_1 \leqslant \mathrm{Sym}(3)$. As $H \leqslant H_1 \times \ldots \times H_t$, we consider π_i the projection of H on H_i. We must have $\pi_1(H) = H_1 = \mathrm{Sym}(3)$, on the contrary H should be abelian. If H' is the derived group of H, then $|H'| = 3$ and $\pi_i(H') = 1$ when $i \geqslant 2$ so, in particular H' centralizes $N_i^{a_i}$ for all $i \geqslant 2$ whence $K_1 = \langle H', N_2^{a_2}, \ldots, N_t^{a_t} \rangle$ is abelian. Let P be a Sylow 2-subgroup of $N_1^{a_1} H$;

as $\ker \pi_1$ centralizes P, $K_2 = \langle \ker \pi_1, P\rangle$ is nilpotent. Hence $G = \langle K_1, K_2\rangle$ so G is generated by two nilpotent subgroups.

Suppose $N_1 = \mathbb{Z}_3^2$.

Having $|\mathrm{Aut}(N_1)| = |\mathrm{GL}(2, 3)| = 48$, as in the previous case it follows that G contains at least a minimal normal subgroup not G-isomorphic to N_1 and $\dim_{\mathrm{End}_G(\mathrm{N_i})} N_i = 1$ when $i \geqslant 2$. Thus $H \leqslant H_1 \times \ldots \times H_t$, with $\pi_1(H) = H_1 \leqslant \mathrm{GL}(2, 3)$ and H_i is cyclic when $i \geqslant 2$. In particular H_1 must be an irreducible, solvable and non nilpotent subgroup of $\mathrm{GL}(2, 3)$. But this implies that 24 divides $|H_1|$ and $|G| \geqslant 24 * 3^6 * 25 > 5^6 * 12$.

Consider $N_1 = \mathbb{Z}_2^3$ or $N_1 = \mathbb{Z}_2^4$.

As the order of G is divisible by three distinct prime numbers and by the fact that $|N_1|^{\delta_G(N_1)} \geqslant 2^{12}$ we can deduce that all the minimal normal subgroups of G are G-isomorphic to $N = N_1$. Hence G is isomorphic to the direct product $N^a H$ where H is an irreducible, solvable and non nilpotent subgroup of $\mathrm{Aut}(N)$. Since $|H|$ must be divisible by at least two distinct primes not dividing $|N|$ and $|H| * |N|^3 \leqslant |G| \leqslant 5^6 * 12$ we deduce that for either choices of N there is a unique possible choice for H.

If $N = \mathbb{Z}_2^3$, then N can be identified with the additive group of the field K, the field of order 8; H has order 21 and is generated by σ and ϱ where for every $x \in N$, we have $x^\varrho = xt$ with $K^* = \langle t\rangle$ and $x^\sigma = x^2$.

When $N = \mathbb{Z}_2^4$, N can be viewed as the additive group of the field K of order 16; H has order 30 and is generated by σ and ϱ where, for every $x \in N$, we have $x^\varrho = xt$, with $K^* = \langle t\rangle$ and $x^\sigma = x^4$.

In both cases $K_1 = \langle \varrho\rangle$ and $K_2 = \langle (C_N(\sigma))^a, \sigma\rangle$ are abelian subgroups of G such that $G = \langle K_1, K_2\rangle$.

Let us consider the case $N_1 = \mathbb{Z}_5^2$.

If there exists a minimal normal subgroup, N_2, not G-isomorphic to N_1 then $5^6 * 12 \geqslant |G| \geqslant 5^6 * |N_2|^2$ and $N_2 = \mathbb{Z}_3$, it follows that $G = \mathrm{Fit}(G)$ which is nilpotent. Hence $\mathrm{Fit}(G) = N_1^3$ and G/F is an irreducible solvable non nilpotent subgroup of $\mathrm{GL}(2, 5)$ such that $|G/F| \leqslant 12$. So we have three possible choices: H is isomorphic to the subgroup of $\mathrm{GL}(2, 5)$ we considered in the example at the beginning of this section and in this case $G \simeq M$; we can describe the other two possibilities in the following way, we may view N_1 as the additive group of the finite field, K, of order 25 and H is a subgroup of $\mathrm{Aut}(K)$ generated by σ and ϱ where, for every $x \in N_1$, $x^\varrho = x^5$ and $x^\sigma = xt$, $t \in K^\star$, and t has either order three or six, in the first case $|H| = 6$, in the other $|H| = 12$. So the group G can be generated by the following nilpotent subgroups: $K_1 := \langle \varrho_i\rangle$, $K_2 := \langle (C_{N_1}(\sigma_i))^3, \sigma_i\rangle$, $i := 1, 2$.

Hence G is not a solvable group.

Therefore, by theorem 2.1, $H = G/C_G(F) \leqslant \prod_{i=1}^{t} G/C_G(N_i)$ and H cannot be a solvable group, so we may assume that $G/C_G(N_1)$ is an irreducible, not solvable subgroup of $\mathrm{Aut}(N_1)$. In particular $\dim_E N_1 \geq 2$ and $|N_1|^2\,|G/C_G(N_1)| \leq |G| \leq 5^6 * 12$. Hence the only possible choices for N_1 are: $N_1 = \mathbb{Z}_2^3$, $N_1 = \mathbb{Z}_2^4$, $N_1 = \mathbb{Z}_2^5$, $N_1 = \mathbb{Z}_3^3$, $N_1 = \mathbb{Z}_5^2$, $N_1 = \mathbb{Z}_7^2$. If $N_1 \in \{\mathbb{Z}_7^2, \mathbb{Z}_3^3, \mathbb{Z}_2^5\}$ then $F = N_1^2$ and $\mathrm{SL}(N_1)$ has no proper irreducible solvable subgroups hence $|G| = |\mathrm{Fit}(G)\,\mathrm{SL}(N_1)| > 5^6 * 12$. Let us consider $N_1 = \mathbb{Z}_2^3$. As $\mathrm{SL}(3, 2)$ has no proper irreducible non solvable subgroups we get that $G/C_G(N_1) = \mathrm{SL}(3, 2)$ and it can be generated by an element of order two and one of order three. If N_1 is the only minimal normal subgroup we can generate G by a Sylow 2-subgroup and a Sylow 3-subgroup; the same can be done if there exists another minimal normal subgroup, N_2, as in this case we should have $|N_2|^2 < \dfrac{5^6 * 12}{8^2 * 168}$ so N_2 can be either a 2-group or a 3-group.

Finally we consider the case $N_1 = \mathbb{Z}_2^4$; if there exists another minimal normal subgroup it should be $N_2 = \mathbb{Z}_3$. The irreducible not solvable subgroups of $\mathrm{Aut}(N_1) = \mathrm{GL}(4, 2)$ are: $\mathrm{SL}(4, 2)$, $\mathrm{GL}(2, 4)$, $\mathrm{SL}(2, 4)$, the composition factors of these groups, and consequentely G itself, can be generated by a Sylow 2-subgroup and a Sylow 3-subgroup.

It follows that $1 < F < C_G(F)$.

This implies in particular that $C := C_G(F)$ is not solvable, otherwise $C_C(\mathrm{Fit}(C)) \leq \mathrm{Fit}(C)$ and this, since $\mathrm{Fit}(C) = F$, would imply $C \leq F$. Let K be a complement of F in G; $L = K \cap C$ is a complement of F in C and it cannot be solvable. In particular $S = \mathrm{soc}\, L = S_1 \times \ldots \times S_k$ is a product of k non abelian simple groups and it is a normal subgroup of K.

Observe that, by Theorem 2.1, $F = N_1^{a_1} \times \ldots \times N_t^{a_t}$ and $G/C \leq \prod_{i-1}^{t} \mathrm{Aut}(N_i)$ is solvable, otherwise it should be $|N_i| \geq 8$, for some i, and

$$|G| \geq |G/C|\,|C/F|\,|N_i|^2 \geq 60 \cdot 60 \cdot 8^2 > 5^6 \cdot 12\,.$$

It is easy to see that $|G/C_G(N_1)|\,|N_1|^{\delta_G(N_1)} \geq 18$ and that this number divides $|G|/|S|$ so $|S| \leq \dfrac{5^6 * 12}{18} \leq 10417$. It follows that S is either A_5^2 or a simple group. We can see that $L \leq \mathrm{Aut}(S)$ is generated by a Sylow 2-subgroup and a Sylow 3-subgroup; as a consequence there exists $p \geq 5$ dividing $|G|/|L| = |G/C|\,|F|$. We may assume that p divides $|G/C_G(N_1)|\,|N_1|$ so $|G/C_G(N_1)|\,|N_1|^{\delta_G(N_1)} \geq 50$ and $|S| \leq 3750$.

Let us observe that K/S cannot be a 2-group as in that case we get $S = \langle P, P^x \rangle$, where P is a Sylow 2-subgroup of S, $x \in S$. As a consequence $G = \langle H_1, H_2 \rangle$ where $H_1 = \langle F, P \rangle$ and H_2 is a Sylow 2-subgroup of K containing P^x. In particular it follows that S cannot be A_5^2.

In addition we cannot have $G = LC_G(L)$ as in that case $L \leq \mathrm{Aut}(S)$ can be generated by two elements, x, y, and $C_G(L) = \langle X, Y \rangle$ where X, Y are two nilpotent subgroups of $C_G(L)$, hence G is generated by $\langle X, x \rangle$ and $\langle Y, y \rangle$.

As a consequence $L \neq \mathrm{Aut}(S)$ and L/S is a cyclic 2-group. It follows that

G/C cannot be a cyclic group as in that case, by Proposition 2.9 and Proposition 2.6, we can generate K with two elements x, y, where $y \in L$, and G is generated by $\langle y, F \rangle$ and $\langle x \rangle$. As K/S is not a 2-group we can conclude that G/C cannot be a 2-group.

So G/C is neither a cyclic group nor a 2-group and $p \geqslant 5$ divides $|G/C_G(N_1)|\,|N_1|$.

As G/C is not a cyclic group then G has either a unique minimal normal subgroup N_1 with $\dim_{\mathrm{End}_G(\mathrm{N}_1)} N_1 \geqslant 2$ or at least two minimal normal subgroups, N_1, N_2, not G-isomorphic.

If p divides $|G/C_G(N_1)|$ then $|N_1| \geqslant 8$ and it cannot be the unique minimal normal subgroup as in this case we get that $|N_1|^3 \leqslant \dfrac{5^6 * 12}{60} = 3125$ so $N_1 = \mathbb{Z}_2^3$ and $|G| \geqslant 8^4 * |L| * |G/C| > 5^6 * 12$. Note that $|N_2|^{\delta_G(N_2)} \leqslant \dfrac{5^6 * 12}{8^2 * 60} < 49$ so G has two minimal normal subgroups not G-isomorphic, N_1, N_2 such that $\dim_{\mathrm{End}_G(N_i)} N_i = 1$, $i = 1, 2$. It follows that every minimal normal subgroup of G is G-isomorphic either to N_1 or to N_2; in addition, as G/C is abelian it has no composition factor G-isomorphic to N_1 or to N_2, hence $|N_1|^2 |N_2|^2 \leqslant |F|$ so $|G| \geqslant 8^2 * 3^2 * |L| * 10 > 5^6 * 12$.

As a consequence the only prime numbers dividing $|G/C|$ are 2 and 3; say $p \geqslant 5$ the greatest prime number dividing $|F|$, we may assume that N_1 is a p-group. Observe that $\dim_{\mathrm{End}_G(\mathrm{N}_1)} N_1 = 1$, otherwise $p^6 * 60 \geqslant 5^6 * 60 \geqslant 5^6 * 12$, as G/C is not a cyclic group we can conclude as above that there are only two minimal normal subgroups of G not G-isomorphic, N_1, N_2. Hence $|N_1|^2 |N_2|^2 \leqslant |F| \leqslant \dfrac{5^6 * 12}{60 * 6}$ so $|N_1|\,|N_2| < 23$ and $(|N_1|, |N_2|) \in \{(5, 3), (5, 4), (7, 3)\}$. In the first case G/C is a 2-group, in the second case it is a cyclic group so the only possibility is the third and $G/C \leqslant \mathbb{Z}_6 \times \mathbb{Z}_2$; as G/C is neither a 2-group nor a cyclic group we get $|G/C| = 12$ but $|G| \geqslant 7^2 * 3^2 * 60 * 12 > 5^6 * 12$.

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