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Some Generic Properties of Concentration Dimension of Measure.

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Sunto. – *Sia K un sottoinsieme quasi similare compatto di uno spazio metrico completo. Sia $\mathcal{M}_1(K)$ lo spazio delle misure di probabilità su K munito della metrica di Fortet - Mourier. Si dimostra che per una misura $\mu \in \mathcal{M}_1(K)$ tipica (nel senzo della categoria di Baire) la dimensione inferiore di concentrazione è uguale a zero, invece la dimensione superiore di concentrazione è uguale alla dimensione di Hausdorff dell'insieme K.*

Summary. – *Let K be a compact quasi self-similar set in a complete metric space X and let $\mathcal{M}_1(K)$ denote the space of all probability measures on K, endowed with the Fortet-Mourier metric. We will show that for a typical (in the sense of Baire category) measure in $\mathcal{M}_1(K)$ the lower concentration dimension is equal to* 0, *while the upper concentration dimension is equal to the Hausdorff dimension of K.*

1. – Introduction.

The concept of dimension of measure plays a great role in diverse branches of mathematics, among other in the theory of dynamical system, the geometric theory of measure and the theory of fractals. Various definitions of dimension as box dimension, packing dimension, correlation dimension, informatic dimension, entropy, has be proposed, but undoubtedly the most widely investigated and used is Hausdorff dimension. Unfortunately, the Hausdorff dimension as well as all mentioned above alternative dimensions of even relatively simple measures or sets, can be hard to calculated.

Recently A. Lasota (see [6]) suggested to study a new concept of dimension of measure which is defined by mean of the Lévy concentration function. This dimension, called concentration dimension, has advantage — at least in the case of fractal measures — to be relatively easy calculable. It is also important that the concentration dimension is strongly connected with the Hausdorff dimension. In particular, the Hausdorff dimension of the self-similar set K is equal to the supremum of lower concentration dimensions of probability measures with the support contained in K (see variational principle in [6]). Moreover, likewise as Hausdorff dimension, the concentration dimension is also related with the topological dimension. Namely, in [11] it is proved that if X is a Menger metric space (i.e. such that closed bounded sets are compact), then the supremum of the lower concentration dimensions of the probability measures on X is equal to the topological dimension of X.

In this Note we prove that for a typical probability measure defined on a compact quasi self-similar set K, the lower concentration dimension is equal to zero, while the upper concentration dimension is equal to the Hausdorff dimension of K. In reality we need on K the weaker hypotheses, namely it is sufficient that K is compact and quasi self-similar from below.

For other results concerning typical properties of measures see [2, 3, 8, 9, 10, 13, 14].

2. – Preliminaries.

Let (X, ϱ) be a Polish space i.e. complete and separable metric space. By $B(x, r)$ we denote the closed ball in X with centre at x and radius $r > 0$. For $C \subset X$ and $r > 0$ we denote

$$B(C, r) = \{x \in X : \varrho(x, C) \leq r\},$$

where

$$\varrho(x, C) = \min\{\varrho(x, y) : y \in C\},$$

Moreover, for $A, B \subset X$ we set

$$\operatorname{dist}(A, B) = \inf\{\varrho(x, y) : x \in A, y \in B\}.$$

A subset K of X is called *quasi self similar from below* if there exist $a > 0$ and $r_0 > 0$ such that for every ball $B(x, r)$ with center at $x \in K$ and radius $r \in (0, r_0)$ there is a mapping $\varphi : K \to K \cap B(x, r)$ such that

$$ar\varrho(x, y) \leq \varrho(\varphi(x), \varphi(y)) \quad \text{for all} \quad x, y \in K.$$

These sets are a natural generalization of self-similar sets which appears in the theory of iterated function systems. Namely, let $S_i : X \to X$, $i = 1, \ldots, N$ be strictly contracting similarity transformations. It is well known (see [5]) that there is a unique compact set K such that $K = \bigcup_{k=1}^{N} S_i(K)$. Such set, called self-similar fractal, is obviously quasi self-similar from below. For details see Example 2 in [1]. In fact in this case the function φ satisfies also the quasi self-similarity condition from above, and the corresponding set K is call quasi self-similar. Note that quasi self-smilar sets appear in the theory of dynamical systems and were intensively studied by several authors (see [1, 7, 12]).

Let $\mathcal{M}_1(X)$ and $\mathcal{M}_1(K)$ be the space of all probability Borel measures on X and K, respectively. For $\mu_1, \mu_2 \in \mathcal{M}_1(X)$ we consider the *Fortet-Mourier* *norm* given by the formula

$$\|\mu_1 - \mu_2\| = \sup\left\{\left|\int_X f d\mu_1 - \int_X f d\mu_2\right| : f \in \mathcal{L}\right\},$$

where $\mathcal{L}$ is the subset of $C(X)$ which contains all the functions f such that

$$|f(x)| \leq 1 \quad \text{and} \quad |f(x) - f(y)| \leq \varrho(x, y) \quad \text{for } x, y \in X.$$

It can be proved that the convergence in the Fortet-Mourier norm is equivalent to the weak convergence. In the sequel we assume that the space $\mathcal{M}_1(X)$ is endowed with the metric generated by the norm $\|\cdot\|$. It is well known that such space is complete.

For $A \subset X$, $s \geq 0$ and $\delta > 0$ we define

$$\mathcal{H}^s_\delta(A) = \inf \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s,$$

where the infimum is taken over all countable covers $\{U_i\}$ of A such that $0 < \operatorname{diam} U_i \leq \delta$. Then

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A)$$

is the *s-dimensional Hausdorff measure*. Clearly for s sufficiently large $\mathcal{H}^s(A) = 0$. The *Hausdorff dimension* of A is defined by the formula

$$\dim_H A = \inf\{s > 0 : \mathcal{H}^s(A) < \infty\}.$$

For $\mu \in \mathcal{M}_1(X)$ the function $\Phi_\mu : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\Phi_\mu(r) = \sup_{x \in X} \mu(B(x, r))$$

is called the Lévy concentration function. This function was successfully used to study the convergence of sequences of random variables. For the properties and other applications of concentration function see [4].

Given $\mu \in \mathcal{M}_1(X)$ we define the *lower* and *upper concentration dimension* of μ by the formulas

$$\underline{\dim}_L \mu = \liminf_{r \to 0} \frac{\ln \Phi_\mu(r)}{\ln r}$$

and

$$\overline{\dim}_L \mu = \limsup_{r \to 0} \frac{\ln \Phi_\mu(r)}{\ln r}.$$

Clearly, both these dimension are well defined and nonnegative, however they can be infinite. If $\overline{\dim}_L \mu = \dim_L \mu$, this common valued is called *concentration dimension*.

Recall that a set in a metric space $\mathcal{X}$ is called nowhere dense, if its closure has empty interior. A countable union of nowhere dense sets is said to be of the first Baire category. A subset A of a complete metric space $\mathcal{X}$ is said to be residual in $\mathcal{X}$ if its complement is of the first Baire category. If the set of all elements of $\mathcal{X}$ satisfying some property P is residual in $\mathcal{X}$, then the property P is called typical or generic. We also say that a typical element of $\mathcal{X}$ has property P. In this paper we prove that for a typical measure μ in $\mathcal{M}_1(K)$ we have $\underline{\dim}_L \mu = 0$ and $\overline{\dim}_L \mu = \dim_H K$, where $\dim_H K$ stands for the Hausdorff dimension of K.

3. – Auxiliary results.

LEMMA 1. – *Let $\mu_1, \mu_2 \in \mathcal{M}_1(X)$ and $\varepsilon > 0$. If $\|\mu_1 - \mu_2\| \leq \varepsilon^k$, where $k > 1$, then*

$$\mu_1(B(C, \varepsilon)) \geq \mu_2(C) - \varepsilon^{k-1},$$

for every Borel set $C \subset K$.

PROOF. – Let $\mu_1, \mu_2 \in \mathcal{M}_1$ and $\varepsilon > 0$ be as in the statement of the Lemma 1. Let C be a Borel subset of K. Consider a function $f : X \to [0, \varepsilon]$ given by the formula

$$f(x) = \max\{\varepsilon - \varrho(C, x), 0\}.$$

Since $f \in \mathcal{L}$ and $f(x) = 0$ for $x \notin B(C, \varepsilon)$, $f(x) = \varepsilon$ for $x \in C$, we have

$$\varepsilon\mu_2(C) - \varepsilon\mu_1(B(C, \varepsilon)) \leq \int_X f d\mu_2 - \int_X f d\mu_1 \leq \|\mu_2 - \mu_1\| \leq \varepsilon^k,$$

whence the statement of Lemma 1 follows. This completes the proof. ■

LEMMA 2. – *Let K be a compact quasi self-similar from below subset of X. Then the set*

$$\mathcal{F} = \{\mu \in \mathcal{M}_1(K) : \underline{\dim}_L \mu = 0\}$$

is dense in $\mathcal{M}_1(K)$.

PROOF. – Obviously $\underline{\dim}_L \delta_x = 0$ for $x \in K$, where δ_x is a δ-Dirac measure supported at the point x. From this and the fact that the linear combinations of point Dirac measures are dense in the space $\mathcal{M}_1(K)$, the statement of Lemma 2 follows. ■

LEMMA 3. – *Let K be a compact quasi self-similar from below subset of X and let $0 < s < \dim_H K$. Then for every ball $B(z, \varepsilon)$ with center at $z \in K$ and radius $\varepsilon > 0$ there exists a measure $\mu \in \mathcal{M}_1(K)$ with* $\operatorname{supp} \mu \subset B(z, \varepsilon) \cap K$ *such that* $\underline{\dim}_L \mu \geq s$.

PROOF. – By Theorem 2.1 from [6] there exists a measure $\tilde{\mu} \in \mathcal{M}_1(K)$ such that $\dim_L \tilde{\mu} \geq s$. Since K is a quasi self-similar set, there exist a mapping $\varphi : K \to B(z, \varepsilon) \cap K$ and the constant $l > 0$ such that

$$l\varrho(x, y) \leq \varrho(\varphi(x), \varphi(y)) \quad \text{for all} \quad x, y \in K .$$

Clearly φ is invertible and the inverse function φ^{-1} (from $\varphi(K)$ into K) is Lipschitzian with Lipschitz constant equal to $1/l$.

Define

$$\mu = \tilde{\mu} \circ \varphi^{-1}.$$

Obviously $\operatorname{supp} \mu \subset B(z, \varepsilon) \cap K$. From inclusion $\varphi^{-1}(B(x, r)) \subset B(\varphi^{-1}(x), r/l)$ it follows that $\Phi_\mu(r) \leq \Phi_{\tilde{\mu}}(r/l)$. Using the last inequality one can easely verify that $\underline{\dim}_L \mu \geq \underline{\dim}_L \tilde{\mu} \geq s$. This completes the proof. ■

LEMMA 4. – *Let K be a compact quasi self-similar from below subset of X and let $0 < s < \dim_H K$. Then the set*

$$\{\mu \in \mathcal{M}_1(K) : \underline{\dim}_L \mu \geq s\}$$

is dense in $\mathcal{M}_1(K)$.

PROOF. – Let $0 < s < \dim_H K$. Let $\mu \in \mathcal{M}_1(K)$ be an arbitrary linear combination of δ-Dirac point measures, i.e.

$$\mu = \sum_{i=1}^{N} a_i \delta_{x_i},$$

where $x_i \in K$, $i = 1, \ldots, N$. Let

$$0 < r_1 < \min \left\{ \frac{1}{3} \varrho(x_i, x_j) : i, j \in \{1, \ldots, N\}, \, i \neq j \right\}.$$

Without any loss of generality we can assume that $r_1 \leq r_0$, where r_0 is that from definition of quasi self-similarity from below. Fix an arbitrary $\varepsilon \in (0, r_1)$. Obviously

$$\operatorname{dist}(B(x_i, \varepsilon), B(x_j, \varepsilon)) \geq \varepsilon \quad \text{for} \quad i, j \in \{1, \ldots, N\}, \; i \neq j .$$

By virtue of Lemma 3 for every $i \in \{1, \ldots, N\}$ there exists a measure $\mu_i \in$

$\mathcal{M}_1(K)$ with $\operatorname{supp}\mu_i \subset B(x_i, \varepsilon) \cap K$ such that $\underline{\dim}_L \mu_i \geq s$. Consider

$$\tilde{\mu} = \sum_{i=1}^{N} a_i \mu_i$$

and observe that

$$\|\mu - \tilde{\mu}\| \leq \sum_{i=1}^{N} a_i \|\mu_i - \delta_{x_i}\| \leq \sum_{i=1}^{N} a_i \varepsilon = \varepsilon .$$

Since for every $r \in (0, r_1)$ and $x \in B(x_i, \varepsilon)$ the ball $B(x, r)$ intersects only the ball $B(x_i, \varepsilon)$ we have

$$\tilde{\mu}(B(x, r)) = a_i \mu_i (B(x, r)) .$$

From this and the inclusion $\operatorname{supp} \tilde{\mu} \subset \bigcup_{i=1}^{N} B(x_i, \varepsilon)$ it follows that

$$\Phi_{\tilde{\mu}}(r) = \max_{1 \leq i \leq N} a_i \Phi_{\mu_i}(r) \qquad \text{for} \qquad r \in (0, r_1).$$

Consequently

$$\frac{\ln \Phi_{\tilde{\mu}}(r)}{\ln r} = \min_{1 \leq i \leq N} \frac{\ln a_i \Phi_{\mu_i}(r)}{\ln r} \qquad \text{for} \qquad r \in (0, r_1).$$

From the last equality and inequalities $\underline{\dim}_L \mu_i \geq s$ for $i = 1, \ldots, N$, it follows that $\underline{\dim}_L \tilde{\mu} \geq s$. From this and the fact that the linear combinations of Dirac point measures are dense in the space $\mathcal{M}_1(K)$, the statement of Lemma 4 follows. ■

4. – Main results.

THEOREM 1. – *Let K be a compact quasi self-similar from below subset of X. Then the set*

$$\{\mu \in \mathcal{M}_1(K) : \underline{\dim}_L \mu = 0\}$$

is residual in the space $\mathcal{M}_1(K)$.

PROOF. – Let $\mathcal{F}$ be the set defined in Lemma 2. For $\mu \in \mathcal{F}$ and $n \in \mathbb{N}$ we define

$$\mathcal{O}_n(\mu) = \{\nu \in \mathcal{M}_1(K) : \|\mu - \nu\| < (r_{\mu, n})^{(n+1)/n}\},$$

where $r_{\mu, n} < 1/n$ and such that $\Phi_\mu(r_{\mu, n}) \geq 2(r_{\mu, n})^{1/n}$. The existence of such $r_{\mu_n, n}$ follows from condition $\underline{\dim}_L \mu = 0$.

Define

$$\mathcal{O} = \bigcap_{n=1}^{\infty} \mathcal{O}_n \quad \text{where} \quad \mathcal{O}_n = \bigcup_{\mu \in \mathcal{F}} \mathcal{O}_n(\mu).$$

Since $\mathcal{O}_n$, $n \in \mathbb{N}$, are open and dense in $\mathcal{M}_1(K)$, the set $\mathcal{O}$ is residual. We claim that $\underline{\dim}_L \nu = 0$ for every $\nu \in \mathcal{O}$. Indeed, let $\nu \in \mathcal{O}$ and let $\{\mu_n\} \subset \mathcal{F}$ be such that

$$\|\mu_n - \nu\| < (r_{\mu_n, n})^{(n+1)/n}.$$

By virtue of Lemma 1 for arbitrary $x \in K$ we have

$$\nu(B(x, 2r_{\mu_n, n})) \geq \mu_n(B(x, r_{\mu_n, n})) - (r_{\mu_n, n})^{1/n}.$$

From the last inequality and the definition of $r_{\mu, n}$ we have

$$\Phi_\nu(2r_{\mu_n, n}) \geq \Phi_{\mu_n}(r_{\mu_n, n}) - (r_{\mu_n, n})^{1/n} \geq (r_{\mu_n, n})^{1/n}.$$

Hence

$$\frac{\ln \Phi_\nu(2r_{\mu_n, n})}{\ln 2r_{\mu_n, n}} \leq \frac{1}{n} \frac{\ln r_{\mu_n, n}}{\ln 2r_{\mu_n, n}},$$

and consequently

$$\liminf_{r \to 0} \frac{\ln \Phi_\nu(r)}{\ln r} \leq \liminf_{n \to \infty} \frac{1}{n} \frac{\ln r_{\mu_n, n}}{\ln 2r_{\mu_n, n}} = 0,$$

which completes the proof. ■

THEOREM 2. – *Let K be a compact quasi self-similar from below subset of X. Then the set*

$$\{\mu \in \mathcal{M}_1(K) : \overline{\dim}_L \mu = \dim_H K\}$$

is residual in the space $\mathcal{M}_1(K)$.

PROOF. – Set $d = \dim_H K$. For $n \in \mathbb{N}$ define

$$\mathcal{F}_n = \{\mu \in \mathcal{M}_1(K) : \overline{\dim}_L \mu > d - 1/n\}.$$

By Lemma 4 the set $\mathcal{F}_n$ is dense in $\mathcal{M}_1(K)$.

Now, for $\mu \in \mathcal{F}_n$ we define

$$\mathcal{G}_n(\mu) = \{\nu \in \mathcal{M}_1(K) : \|\mu - \nu\| < (r_{\mu, n})^{d + (n-1)/n}\},$$

where $r_{\mu, n} < 1/n$ is such that $\Phi_\mu(2r_{\mu, n}) \leq (r_{\mu, n})^{d - 1/n}$.

Define

$$\mathcal{G} = \bigcap_{n=1}^{\infty} \mathcal{G}_n \quad \text{where} \quad \mathcal{G}_n = \bigcup_{\mu \in \mathcal{F}_n} \mathcal{G}_n(\mu).$$

Since $\mathcal{G}_n$, $n \in \mathbb{N}$, are open and dense in $(\mathcal{M}_1(K), \|\cdot\|)$, the set $\mathcal{G}$ is residual. We claim that $\overline{\dim}_L \nu = d$ for every $\nu \in \mathcal{G}$. For, let $\nu \in \mathcal{G}$ be fixed. Clearly, by the definition of $\mathcal{G}$, there exists a sequance $\{\mu_n\}$ with $\mu_n \in \mathcal{F}_n$ such that

$$\|\mu_n - \nu\| < (r_{\mu_n, n})^{d + (n-1)/n}.$$

By virtue of Lemma 1 we have

$$\mu_n(B(x, 2r_{\mu_n, n})) \geqslant \nu(B(x, r_{\mu_n, n})) - (r_{\mu_n, n})^{d - 1/n} \quad \text{for all} \quad x \in K.$$

From the last inequality and the definition of $r_{\mu, n}$ we have

$$\Phi_\nu(r_{\mu_n, n}) \leqslant 2(r_{\mu_n, n})^{d - 1/n}$$

and consequently

$$\frac{\ln \Phi_\nu(r_{\mu_n, n})}{\ln r_{\mu_n, n}} \geqslant d - \frac{1}{n} + \frac{\ln 2}{\ln r_{\mu_n, n}}.$$

Thus

$$\limsup_{r \to 0} \frac{\ln \Phi_\nu(r)}{\ln r} \geqslant \limsup_{n \to \infty} \left(d - \frac{1}{n} + \frac{\ln 2}{\ln r_{\mu_n, n}} \right) = d.$$

The proof is complete. ■

As an immediate consequence of Theorem 1 and 2 we obtain:

COROLLARY 3. – *Let K be a compact quasi self-similar from below subset of X. Then the set*

$$\{\mu \in \mathcal{M}_1(K) : \underline{\dim}_L \mu = 0 \ \textit{and} \ \overline{\dim}_L \mu = \dim_H K\}$$

is residual in $\mathcal{M}_1(K)$.

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