Soo Hwan Kim, Yangkok Kim

θ-curves inducing two different knots with the same 2-fold branched covering spaces

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **6-B** (2003), n.1, p. 199–209.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2003_8_6B_1_199_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2003.

Bollettino U. M. I.
(8) **6**-B (2003), 199-209

θ-Curves Inducing two Different Knots with the Same 2-fold Branched Covering Spaces.

SOO HWAN KIM (*) - YANGKOK KIM

Sunto. – *Per un nodo K con un'inversione forte i indotta da un tunnel di scioglimento abbiamo una proiezione $\Pi : S^3 \to S^3/i$ che è un ricoprimento doppio ramificato sopra un nodo banale $\Pi(\mathrm{fix}(i))$, dove $\mathrm{fix}(i)$ è l'asse i. Allora un insieme $\Pi(\mathrm{fix}(i) \cup K)$ è chiamato θ-curva. Costruiamo θ-curve e i ricoprimenti $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ ciclici ramificati sopra θ-curve, che hanno due decomposizioni di Heegaard non isotopiche che sono uno stabilmente equivalenti.*

Summary. – *For a knot K with a strong inversion i induced by an unknotting tunnel, we have a double covering projection $\Pi : S^3 \to S^3/i$ branched over a trivial knot $\Pi(\mathrm{fix}(i))$, where $\mathrm{fix}(i)$ is the axis of i. Then a set $\Pi(\mathrm{fix}(i) \cup K)$ is called a θ-curve. We construct θ-curves and the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ cyclic branched coverings over θ-curves, having two non-isotopic Heegaard decompositions which are one stable equivalent.*

1. – Introduction.

Every closed orientable 3-manifold M has a Heegaard splitting which is a decomposition of M into two handlebodies. There is a canonical process, called stabilizations, which transforms a Heegaard splitting of genus g into one of genus $g+1$. It is known that for many manifolds, there are more than one isotopy class of minimal genus Heegaard splittings. However for non-minimal genus Heegaard splittings, very little is known, and the only manifolds for which non-isotopic Heegaard splittings of non-minimal genus have been exhibited are obtained by surgery on pretzel knots [3] and by Kogayashi for torus sum of pretzel link complements with 2-bridge link complements [6]. In both cases the manifold is shown to contain irreducible Heegaard splittings of arbitrary large genus. Examples of arbitrarily high genus Heegaard splittings which are isotopic after one stabilization were found by Sedgwick [9]. However, it is not known whether the Heegaard splittings in those examples are non-isotopic before the stabilizations, nor whether that they are stabilizations

(*) The first author was supported by KOSEF 1999.

of a common low genus Heegaard splittings. We now are interested in the problem what kind of knots have homeomorphic branched coverings. In [5] they consider two different knots K_q and a torus knot $T(3, q)$ which have the Brieskorn homology spheres $\sum(2, 3, q)$ as 2-fold cyclic branched coverings, where $(2, 3, q)$ are relatively prime. Also, it is shown that two inequivalent Heegaard splittings of $\sum(2, 3, q)$ of genus 2 associated with $T(3, q)$ and K_q are one stable equivalent.

In $(1, 1)$-decomposition and tunnel theory, we have θ-curves constructed as follows: For a knot K with a strong inversion i, we have a double covering projection $\Pi : S^3 \to S^3/i$ branched over a trivial knot $\Pi(\text{fix}(i))$, where $\text{fix}(i)$ is the axis of a strong inversion i. Then the set $\Pi(\text{fix}(i) \cup K)$ is called a spatial θ-curve associated with (K, i), and more restrictively, a spatial θ-curve of a strong inversion induced by an unknotting tunnel is called a θ-curve. One overall approach to the knot theory of graphs is to seek knots associated with a graph G so that questions about G can be translated into questions about knots. For example, we recall that a θ-curve G is locally unknotted if every constituent knots are trivial. For each of the three trivial simple closed curves $J_i (i = 1, 2, 3)$ in a locally unknotted θ-curve G, the arc $\overline{G - J_i}$ lifts to a knot K_i in the two-fold cyclic branched covering of S^3 branched over J_i. The unordered triple (K_1, K_2, K_3) of knots in S^3 is an isotopy invariant of a θ-curve. In [12] it is shown that a locally unknotted θ-curve is planar if and only if one of the three knots K_i is trivial. Thus the problem of deciding whether or not a θ-curve is planar reduces to determining whether or not its three constituent knots and one knot in a branched cover are trivial. In [7] they show that a θ-curve of a strong inversion induced by an unknotting tunnel has the following characteristic: The set of the constituent knots of a θ-curve consists of two trivial knots and a knot with a 2-bridge decomposition.

In this paper we construct a locally unknotted non-planar θ-curve and show that the unordered triple (K_1, K_2, K_3) of knots in a branched cover of S^3 have two different knots, torus knot $T(3, |6\beta + 1|)$ and montesinos knot $m(-1 : (2, 1)(3, 1)(|6\beta + 1|, |\beta|))$, $\beta \neq -1$. We then show that $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ covering over a θ-curve have two equivalence classes of genus 2 Heegaard splittings which are one stable equivalent. This construction comes from montesinos knots or Dunwoody Heegaard decompositions, and produces infinitely many pairs of different prime knots whose 2-fold branched coverings are homeomorphic.

2. – The θ-Curves of Montesinos knots and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ coverings.

In [1] montesinos knots were considered to construct homeomorphic 2-fold cyclic branched covering space associate to the seifert fiber space.

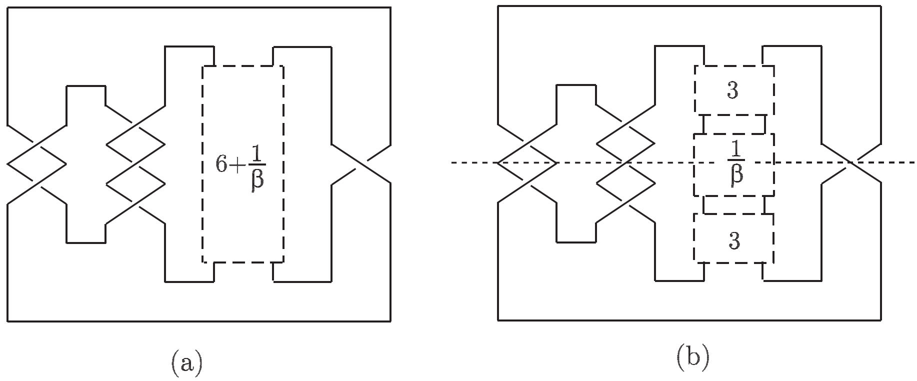


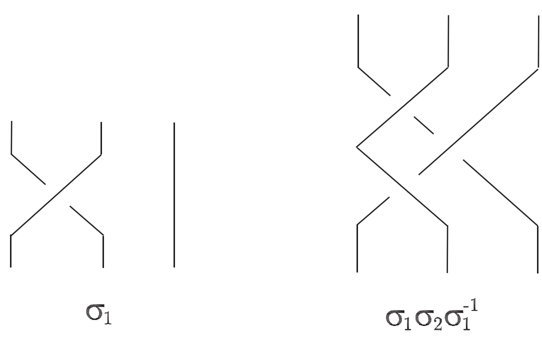
Fig. 1. – $m(-1:(2,1)(3,1)(|6\beta+1|,|\beta|))$

Indeed $\Sigma(2,3,q=|6\beta+1|)$ is a 2-fold cyclic branched covering of two different knots, the torus knot $T(3,q)$ and montesinos knot $m(-1:(2,1)(3,1)(|6\beta+1|,|\beta|))$, where β is integer such that

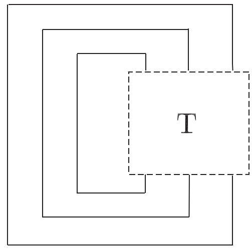
$$2(-1)(3)(|6\beta+1|)+(3)(|6\beta+1|)+2(1)(|6\beta+1|)+2(|\beta|)(3)=\pm 1\,.$$

The manifold admits at least two equivalence classes of genus 2 Heegaard splittings.

For $i=1,2$ and an integer n, we denote by σ_i^n the n-tangle whose incoming arcs are i-th and $(i+1)$-th strings. For example,



For a product T of tangles, we define a T-tangle by the following link or knot.



For example, $(\sigma_1\sigma_2)^{|6\beta+1|}$-tangle denotes the torus knot $T(3, |6\beta+1|)$.

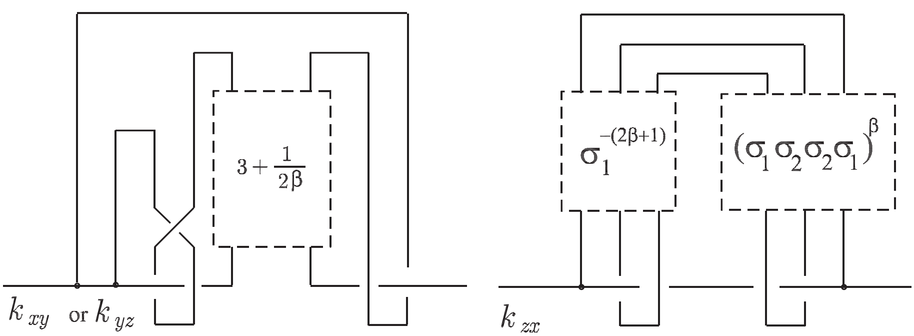
We now construct a θ-curve from $m(-1:(2, 1)(3, 1)(|6\beta+1|, |\beta|))$ as shown in Fig. 1, where the dotted line is the axis of the strong inversion i induced by an unknotting tunnel. We obtain a θ-curve, denoted by $\theta(x, y, z)$, as the quotient of (S^3, m) by a strong inversion i as shown in Fig. 2(a), where $\{x, y, z\}$ is the set of three edges, each of which joins the two vertices. The $x\cup y$, $y\cup z$ and $z\cup x$ are called the constituent knots of $\theta(x, y, z)$, and denoted by k_{xy}, k_{yz} and k_{zx} respectively. First, we construct the 2-fold covering $M_2(k_{xy})$ of S^3 branched along k_{xy}. Let $\tilde{z}$ be the lift of the edge z in $M_2(k_{xy})$. Then $\tilde{z}$ is a knot which bounds an orientable surface in $M_2(k_{xy})$. Hence we can construct the 2-fold covering of $M_2(k_{xy})$ branched along $\tilde{z}$. This is the $\mathbb{Z}_2\oplus\mathbb{Z}_2$ branched covering $M_\phi(\theta)$ of $\theta(x, y, z)$. By the same way, we can also construct $M_\phi(\theta)$ by choosing $M_2(k_{yz})$ or $M_2(k_{zx})$ instead of $M_2(k_{xy})$. Indeed $M_\phi(\theta)$ is independent of the choice of the constituent knots, when we first construct the 2-fold branched covering of one of constituent knots of $\theta(x, y, z)$.

LEMMA 1. – *Let σ_1, σ_2 be tangles defined as above. Then*

(1) $\sigma_1^{\varepsilon}\sigma_2^{\varepsilon}\sigma_1^{\eta}=\sigma_2^{\eta}\sigma_1^{\varepsilon}\sigma_2^{\varepsilon}$, $\sigma_2^{\varepsilon}\sigma_1^{\varepsilon}\sigma_2^{\eta}=\sigma_1^{\eta}\sigma_2^{\varepsilon}\sigma_1^{\varepsilon}$,

(2) $(\sigma_1\sigma_2\sigma_2\sigma_1)^{\varepsilon}$ *is commutative with* σ_2^{η}, *and*

(3) $(\sigma_1\sigma_2\sigma_2\sigma_1)^{\varepsilon}\sigma_2^{2\varepsilon}=(\sigma_1\sigma_2)^{3\varepsilon}=(\sigma_2\sigma_1)^{3\varepsilon}$ *where* $\varepsilon, \eta=\pm 1$.



(a) (b)

Fig. 2. – A θ-curve $\theta(x, y, z)$

PROOF. – (1) It is clear.

(2) Note that

$$\sigma_1^{\varepsilon}\sigma_2^{\varepsilon}\sigma_2^{\varepsilon}\sigma_1^{\varepsilon}\sigma_2^{\eta} = \sigma_1^{\varepsilon}\sigma_2^{\varepsilon}\sigma_1^{\eta}\sigma_2^{\varepsilon}\sigma_1^{\varepsilon} = \sigma_2^{\eta}\sigma_1^{\varepsilon}\sigma_2^{\varepsilon}\sigma_2^{\varepsilon}\sigma_1^{\varepsilon}.$$

(3) Note that

$$\sigma_1^{\varepsilon}\sigma_2^{\varepsilon}\sigma_2^{\varepsilon}\sigma_1^{\varepsilon}\sigma_2^{2\varepsilon} = \underline{\sigma_2^{\varepsilon}\sigma_1^{\varepsilon}\sigma_2^{\varepsilon}}\sigma_2^{\varepsilon}\sigma_1^{\varepsilon}\sigma_2^{\varepsilon} = \underline{\sigma_1^{\varepsilon}\sigma_2^{\varepsilon}\sigma_1^{\varepsilon}}\sigma_2^{\varepsilon}\sigma_1^{\varepsilon}\sigma_2^{\varepsilon} \quad \text{and}$$

$$\sigma_1^{\varepsilon}\sigma_2^{\varepsilon}\sigma_2^{\varepsilon}\sigma_1^{\varepsilon}\sigma_2^{2\varepsilon} = \sigma_2^{\varepsilon}\sigma_1^{\varepsilon}\sigma_2^{\varepsilon}\underline{\sigma_2^{\varepsilon}\sigma_1^{\varepsilon}\sigma_2^{\varepsilon}} = \sigma_2^{\varepsilon}\sigma_1^{\varepsilon}\sigma_2^{\varepsilon}\sigma_1^{\varepsilon}\sigma_2^{\varepsilon}\sigma_1^{\varepsilon}. \qquad \blacksquare$$

THEOREM 1. – *Let k_{xy}, k_{yz} and k_{zx} be three constituent knots of $\theta(x, y, z)$, $M_2(k_{xy})$, $M_2(k_{yz})$ and $M_2(k_{zx})$ be the 2-fold branched coverings of S^3 branched along k_{xy}, k_{yz} and k_{zx}, respectively. Then*

(1) *The lifts $\tilde{x}$ and $\tilde{z}$ of x and z in $M_2(k_{yz})$ and $M_2(k_{xy})$ are the knot $m(-1:(2,1)(3,1)(|6\beta+1|, |\beta|))$.*

(2) *The lift $\tilde{y}$ of y in $M_2(k_{zx})$ is the torus knot $T(3, |6\beta+1|)$.*

PROOF. – (1) It is clear that $\tilde{x}$ and $\tilde{z}$ represent $m(-1:(2,1)(3,1)(|6\beta+1|, |\beta|))$ as the lifts of arcs x and z, respectively.

(2) We note that the lift $\tilde{y}$ of y in $M_2(k_{zx})$ is a T-tangle as shown Fig. 3, where

$$T = \sigma_2^{-(2\beta+1)}(\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1})^{\beta}\sigma_1^{-1}(\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1})^{\beta}\sigma_2^{-(2\beta+1)}\sigma_1.$$

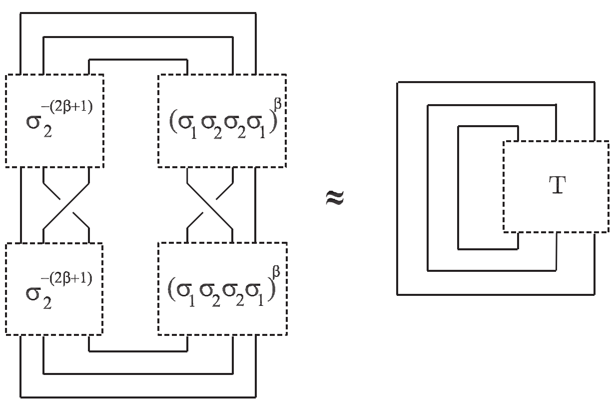


Fig. 3. – The lift $\tilde{y}$ of y in $M_2(k_{zx})$

Thus it is enough to show that the T-tangle is $(\sigma_1^{-1}\sigma_2^{-1})^{6\beta+1}$-tangle$(\beta > 0)$ or $(\sigma_1^{-1}\sigma_2^{-1})^{6\beta+1}$-tangle$(\beta < 0)$ which is the torus knot $T(3, |6\beta+1|)$.

Case i) $\beta = -\gamma > 0$. Since $\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1}$ and σ_2^{-1} are commutative, we have

$$\begin{aligned}
&\sigma_1^{-1}(\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1})\sigma_2^{-2} \\
&\qquad = \underline{\sigma_1^{-1}\sigma_2^{-1}(\sigma_1^{-1}}\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1})\,\sigma_2^{-1} = \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}\underline{\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}} \\
&\qquad = \sigma_2^{-1}(\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1})\sigma_2^{-1}\sigma_1^{-1} = \sigma_2^{-2}(\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1})\,\sigma_1^{-1}
\end{aligned}$$

and so σ_1^{-1} and $(\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1})\,\sigma_2^{-2}$ are commutative. Hence we have

$$\begin{aligned}
&\sigma_2^{2\gamma-1}(\sigma_1\sigma_2\sigma_2\sigma_1)^{\gamma}\underline{\sigma_1^{-1}(\sigma_1\sigma_2\sigma_2\sigma_1)^{\gamma}\sigma_2^{2\gamma}}\sigma_2^{-1}\sigma_1 \\
&= \sigma_2^{4\gamma-1}(\sigma_1\sigma_2\sigma_2\sigma_1)^{2\gamma}\sigma_1^{-1}\sigma_2^{-1}\sigma_1 \\
&= \sigma_2^{4\gamma-1}(\sigma_1\sigma_2\sigma_2\sigma_1)^{2\gamma}\sigma_2\sigma_1^{-1}\sigma_2^{-1} \\
&= \sigma_2^{4\gamma}(\sigma_1\sigma_2\sigma_2\sigma_1)^{2\gamma}\sigma_1^{-1}\sigma_2^{-1} = \sigma_2^{-4\beta}(\sigma_1\sigma_2\sigma_2\sigma_1)^{-2\beta}\sigma_1^{-1}\sigma_2^{-1} \\
&= \{(\sigma_1\sigma_2)^{-3}\}^{2\beta}(\sigma_1^{-1}\sigma_2^{-1}) = (\sigma_1^{-1}\sigma_2^{-1})^{6\beta+1}.
\end{aligned}$$

Case ii) $\beta < 0$. Since $\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1}$ and σ_2^{-1} are commutative, we have

$$\begin{aligned}
&(\sigma_2\sigma_2\sigma_1)(\sigma_1\sigma_2\sigma_2\sigma_1)\,\sigma_2^2 = \sigma_2\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\underline{\sigma_2\sigma_1\sigma_2} = \sigma_2\underline{\sigma_2\sigma_1\sigma_2}\,\underline{\sigma_1\sigma_2\sigma_1}\sigma_2\sigma_1 \\
&\qquad = \sigma_2\sigma_1\sigma_2\underline{\sigma_1\sigma_2\sigma_1}\sigma_2\sigma_2\sigma_1 = \underline{\sigma_2\sigma_1\sigma_2}\sigma_2\sigma_1\sigma_2\sigma_2\sigma_2\sigma_1 \\
&\qquad = \sigma_1\sigma_2\underline{\sigma_1\sigma_2\sigma_1}\sigma_2\sigma_2\sigma_2\sigma_1 = (\sigma_1\sigma_2\sigma_2\sigma_1)(\sigma_2\sigma_2)(\sigma_2\sigma_2\sigma_1)
\end{aligned}$$

and so $\sigma_2\sigma_2\sigma_1$ and $(\sigma_1\sigma_2\sigma_2\sigma_1)\,\sigma_2^2$ are commutative. Hence we have

$$\begin{aligned}
&\sigma_2^{-2\beta-1}(\sigma_1\sigma_2\sigma_2\sigma_1)^{-\beta}\underline{\sigma_1^{-1}(\sigma_1\sigma_2\sigma_2\sigma_1)^{-\beta}\sigma_2^{-2\beta-1}}\sigma_1 \\
&= \sigma_2^{-2\beta-1}(\sigma_1\sigma_2\sigma_2\sigma_1)^{-\beta}\underline{\sigma_2\sigma_2\sigma_1(\sigma_1\sigma_2\sigma_2\sigma_1)^{-\beta-1}\sigma_2^{-2\beta-1}}\sigma_1 \\
&= \sigma_2^{-4\beta-3}(\sigma_1\sigma_2\sigma_2\sigma_1)^{-2\beta-1}\sigma_2\sigma_2\sigma_1\sigma_2\sigma_1 \\
&= \sigma_2^{-4\beta-2}(\sigma_1\sigma_2\sigma_2\sigma_1)^{-2\beta-1}\sigma_2\sigma_1\sigma_2\sigma_1 \\
&= (\sigma_2\sigma_1)^{-6\beta-3}(\sigma_2\sigma_1)^2 = (\sigma_2^{-1}\sigma_1^{-1})^{6\beta+1}. \qquad \blacksquare
\end{aligned}$$

The three constituent knots k_{xy}, k_{yz} and k_{zx} of $\theta(x, y, z)$ are trivial. Moreover the lifts $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ are nontrivial as shown in Theorem 1. Hence $\theta(x, y, z)$ is locally unknotted and non-planar by [12].

LEMMA 2 ([8]). – *Let $M_\phi(\theta)$ be the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ branched covering of $\theta(x, y, z)$. Then we have*

$$H_1(M_\phi(\theta), \mathbb{Z}) = H_1(M_2(k_{12}), \mathbb{Z}) \oplus H_1(M_2(k_{23}), \mathbb{Z}) \oplus H_1(M_2(k_{31}), \mathbb{Z}).$$

COROLLARY 1. – *Let $M_\phi(\theta)$ be the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ branched covering of $\theta(x, y, z)$. Then $M_\phi(\theta)$ is a Brieskorn homology sphere.*

LEMMA 3 ([2]). – *A link type is presented in an m-bridge presentation if and only if it is represented as a 2m-plat.*

THEOREM 2. – *Let $M_\phi(\theta)$ be the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ branched covering of $\theta(x, y, z)$. Then $M_\phi(\theta)$ admits two inequivalent Heegaard splittings of genus 2. Moreover two inequivalent Heegaard splittings are one stable equivalent.*

PROOF. – Since each constituent knots is trivial, each corresponding covering space is a 3-sphere S^3 having lifted knots $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ as Theorem 1. Firstly, the torus knot $T(3, |6\beta + 1|)$ has bridge index 3; hence by Lemma 3 it has plat index 6. Therefore, by Theorem 5 of [2], the 2-fold covering space $\widetilde{M}_2(k_{zx})$ of $M_2(k_{zx})$ branched over $(3, |6\beta + 1|)$ has Heegaard genus $\leqslant 2$. Since $\widetilde{M}_2(k_{zx})$ has a non-cyclic fundamental group, its Heegaard genus cannot be less than 2, which establishes genus two. Secondly, in [2], the 2-fold covering space $\widetilde{M}_2(k_{yz}) = \widetilde{M}_2(k_{xy})$ of $M_2(k_{yz}) = M_2(k_{xy})$ branched over 3-bridge knot $K_q = m(-1 : (2, 1)(3, 1)(|6\beta + 1|, |\beta|))$ admit a genus two Heegaard splitting [5]. Note that the algorithm given in section 5 of [2] allows one to find a Heegaard splitting of genus 2 for the 2 -fold covering space $\widetilde{M}_2(k_{zx})$ (resp. $\widetilde{M}_2(k_{yz})$) of a knot $T(3, |6\beta + 1|)$ (resp. K_q), whenever two knots are presented in 6-plat. Thus we may find Heegaard splittings of genus 2 for $\widetilde{M}_2(k_{zx})$ and $\widetilde{M}_2(k_{yz})$ from 6 -plat presentations of $T(3, |6\beta + 1|)$ and K_q, respectively, using that algorithm. By Theorem 8 of [2], these Heegaard splittings are equivalent only if two knots are equivalent knot types. Since two knots $T(3, |6\beta + 1|)$ and K_q are inequivalent, the first assertion is established.

To prove the second assertion, suppose that $M_\phi(\theta)$ admit two genus 2 Heegaard splittings coming from $T(3, |6\beta + 1|)$ and K_q. Then by assertion above, $M_\phi(\theta)$ admit inequivalent genus 2 Heegaard splittings. Note that by singer-reidemeister all Heegaard representations of a closed orientable 3-manifold are stably equivalent. Using 4-colored graph (or crystallization) for genus two Heegaard splittings as [5], we obtain two crystallizations corresponding to two inequivalent Heegaard splittings of genus 2. Adding a dipole of type 2 that corresponds to a single stabilization of the Heegaard splittings, we obtain two crystallizations corresponding to two Heegaard splittings of genus 3. By the finite sequence of crystallization moves (called LCG moves in [5]) we have two isomorphic crystallizations. As a consequence, we have two equivalent Heegaard splittings of genus 3. (See [5] for detail.) ■

3. – The θ-curves of Dunwoody Heegaard decompositions and $Z_2 \oplus Z_2$ coverings.

In 1994, Dunwoody [4] introduced 6-tuples of non-negative integers (d, a, b, c, r, s), where $d = 2a + b + c$, yielding a family of genus n Heegaard diagrams H_n of closed orientable 3-manifolds $D_n(d, a, b, c, r, s)$. In [10] they proved that $D_n(d, a, b, c, r, s)$ are the n-fold cyclic coverings of S^3 branched over knots K, called Dunwoody $(1, 1)$ knots. Furthermore they showed how to find the unknotting tunnels of Dunwoody $(1,1)$ knots. In [7] a θ-curve of a strong inversion induced by an unknotting tunnel has two trivial knots and a knot with a 2-bridge decomposition as its constituent knots. As a general case of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ covering in Theorem 2, we have the following $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ covering of θ-curves induced by Dunwoody decomposition $D_2(d, a, b, c, r, s)$.

THEOREM 3. – *Let M be a closed orientable 3-manifold with genus two Dunwoody decomposition $D_2(d, a, b, c, r, s)$. Then M is the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ branched covering of a θ-curve G. In particular it is a homology sphere if G is locally unknotted and a lense space if G is not locally unknotted.*

PROOF. – Let $(H, H') = (D_2(a, b, c, r), D_2^*(a, b, c, r))$ be the Heegaard splittings of M, where D_2^* is the dual handlebody of D_2. In other words, D_2 and D_2^* are 1-handle and 2 -handle. Then H is embedded into R^3 so that the π-rotation with respect to x, y and z axes induce three involutions ϱ, ε and σ on H respectively as shown in Fig. 4.

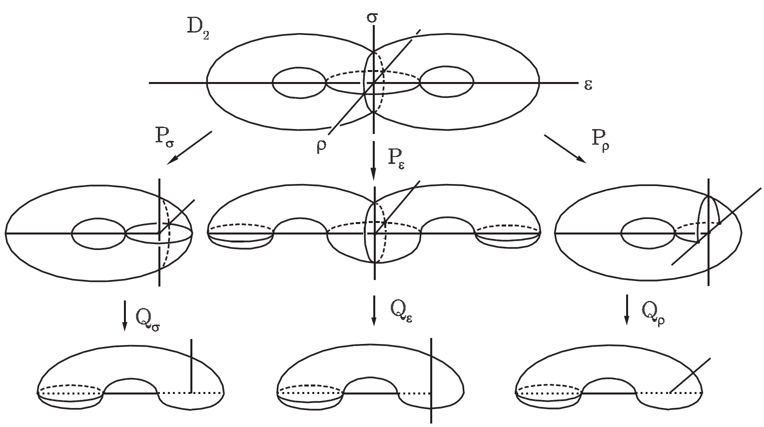


Fig. 4. – The θ-curves of $D_2(d, s, b, c, r, s)$

We now can choose a gluing homeomorphism h of the two handlebodies so that h is compatible with the involution σ, i.e., $\sigma_{|\partial H} \circ h = h \circ \sigma_{|\partial H'}$. Then by the facts of Birman-Hilden's theorem [2] and $\varrho = \varepsilon \circ \sigma = \sigma \circ \varepsilon$, we may get a genus 2 closed orientable 3-manifold M with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ symmetry generated by ε and σ. For each involution i on H, we have a double covering projection $P_i\colon D_2 \to D_2/i$ branched over $K_i = P(A_i)$, where A_i is the axis of an involution i. Note that K_ϱ, K_σ and K_ε are knots in a lens space, S^3 and S^3, respectively. Moreover K_ϱ and K_σ are knots in genus one solid torus with $(1,\ 1)$-decomposition with involutions $t_\varrho = (\sigma/\varrho \cup \varepsilon/\varrho)$ and $t_\sigma = \varrho/\sigma \cup \varepsilon/\sigma$ respectively, and K_ε is a 3-bridge knot with a strong inversion $t_\varepsilon = (\sigma/\varepsilon \cup \varrho/\varepsilon)$ (see fig. 4). Note that the quotient space D_2/σ is Dunwoody 3-sphere $D_1(a,\ b,\ c,\ r)$. Now for each involutions t_σ, t_ε and t_ϱ, we have a double covering projection $Q_i\colon D_2/i \to \dfrac{D_2/i}{t_i} = S^3$ such that

$$Q_{t_\sigma} \circ P_\sigma = Q_{t_\varrho} \circ P_\varrho = Q_{t_\varepsilon} \circ P_\varepsilon .$$

Hence a genus two three manifold M has the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ covering projection $\Pi_{\mathbb{Z}_2 \oplus \mathbb{Z}_2}\colon M \to S^3$ with $\Pi_{\mathbb{Z}_2 \oplus \mathbb{Z}_2}(A_\varepsilon \cup A_\sigma \cup A_\varrho)$ as a θ-curve induced by Dunwoody decomposition $D_2(d,\ a,\ b,\ c,\ r,\ s)$. ■

We now give some examples of two classes of θ-curves induced by Dunwoody decompositions. Let G be a θ-curve induced by Dunwoody decomposition $D_2(1,\ 1,\ 6,\ 6)$ as shown in Fig. 5. Then the three constituent knots of G are trivial and so $M_{\mathbb{Z}_2 \oplus \mathbb{Z}_2}(\theta)$ is a homology sphere. The unordered triple $(K_1,\ K_2,\ K_3)$ of knots in a branched cover of S^3 have two different knots, torus knot $T(3, 7)$ and montesinos knot $m(-1:(2,1)(3,1)(7,1))$.

In general, we see that $M_{\mathbb{Z}_2 \oplus \mathbb{Z}_2}(\theta)$ is a homology sphere if θ is induced by Dunwoody decompositions $D_2(1,\ 1,\ 4t-2,\ 4t-2)$, $D_2(1,\ 1,\ 4t,\ 5)$ and $D_2(|\beta|,\ |2\beta+1|,\ 4|\beta|,\ |6\beta+1|)$ with $|6\beta+1| = 3t+1$. Moreover the induced locally unknotted θ-curves are the one in Fig. 2. The unordered triple $(K_1,\ K_2,\ K_3)$ of knots in a branched cover of S^3 consist of two different knots, torus knot $T(3,\ |6\beta+1|)$ and montesinos knot $m(-1:(2,\ 1)(3,\ 1)(|6\beta+$

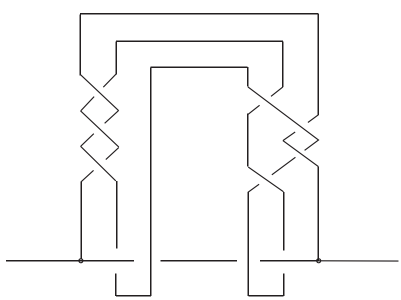


Fig. 5. – A θ-curve of $D_2(1, 1, 6, 6)$

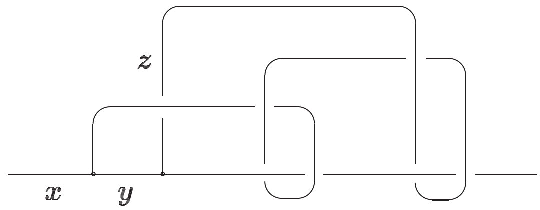


Fig. 6. – A θ-curve of $D_2(9,0,3,8)$

$1|,\ |\beta|)),\ \beta \neq -1$. Hence the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ coverings over θ-curves have two equivalence classes of genus 2 Heegaard splittings which are one stable equivalent.

Let G be a θ-curve induced by Dunwoody decomposition $D_2(9,0,3,8)$ as shown in Fig. 6. The three constituent knots of G are two trivial knots and the knot 6_2. Then the lift $\tilde{z}$ of z in $M_2(k_{xy})$ is the knot 8_5 and so $M_{\mathbb{Z}_2 \oplus \mathbb{Z}_2}(\theta) \approx L(21,8)$.

Similarly we see that $M_{\mathbb{Z}_2 \oplus \mathbb{Z}_2}(\theta)$ is a lens space if θ is induced by $D_2(5,1,2,6)$, $D_2(3,4,3,6)$ and $D_2(2,2,3,5)$.

REFERENCES

[1] J. S. Birman - E. Gonzalez-Acuña - J. M. Montesinos, *Heegaard splittings of prime 3-manifolds are not unique*, Michingan Math. J., **23** (1976), 97-103.

[2] J. S. Birman - H. M. Hilden, *Heegaard splittings of branched coverings of S^3*, Trans. Amer. Math. Soc., **270** (1975), 315-352.

[3] A. Casson - C. McA. Gordon, *Reducing Heegaard splittings*, Topology and its applications, **27** (1987), 273-275.

[4] M. J. Dunwoody, *Cyclic presentations and 3-manifolds*, Proceedings of Groups-Korea 94' (edited by A. C. Kim and D. L. Johnson) (1979), 47-55.

[5] Y. H. Im - S. H. Kim, *Heegaard splittings of the Brieskorn homology spheres that are equivalent after one stabilization*, Note di Mathematica, **20** (1) (2000/20001), 53-63.

[6] T. Kobayashi, *A construction of* 3*-manifolds whose homeomorphism classes of Heegaard splittings have polynomial growth*, Osaka Journal of Math., **29** (1992), 653-674.

[7] K. Morimoto - M. Sakuma - Y. Yokota, *Identifying tunnel number one knots*, J. Math. Soc. Japan, **48** (4) (1996), 667-688.

[8] M. Nakao, *On the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ branched coverings of spatial K_4-graphs*, Knots 90 (by Walter de Gruyter), Berlin New York (1992), 103-116.

[9] E. Sedwick, *An infinite collection of Heegaard splittings that are equivalent after one stabilization*, Mathematisch Annalen, **308** (1997), 65-72.

[10] H. J. SONG - S. H. KIM, *Dunwoody* 3-*manifolds and* (1,1)-*decomposiable knots*, Proc. Workshop in pure math (edited by Jongsu Kim and Sungbok Hong), Geometry and Topology, **19** (2000), 193-211.

[11] M. TAKAHASHI, *Two knots with the same 2-fold branched covering space*, Yokohama Math. J., **25** (1977), 91-99.

[12] K. WOLCOTT, *The knotting of theta curves and other graphs in* S^3, in Geometry and Topology (edited by McCrory and T. Shlfrin) Marcel Dekker, New York (1987) 325-346.

Soo Hwan Kim: Department of Mathematics, Pusan National University
Busan 609-735, Korea. E-mail: soohwankim@hanmir.com

Yangkok Kim: Department of Mathematics, Dongeui University
Busan 614-714, Korea. E-mail: ykkim@dongeui.ac.kr

Pervenuta in Redazione
il 9 aprile 2001