Bollettino
Unione Matematica Italiana

Marina Ghisi

Analytic solutions to nonlocal abstract equations

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **6-B** (2003), n.1, p. 181–198.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2003_8_6B_1_181_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2003.

Bollettino U. M. I.
(8) **6**-B (2003), 181-198

Analytic Solutions to Nonlocal Abstract Equations.

GHISI MARINA

Sunto. – *Si considera il problema dell'esistenza di soluzioni globali analitiche per equazioni astratte, in spazi di Hilbert, di tipo Klein-Gordon corrette con termini non locali, del tipo:*

$$u'' + m(\|u\|_H^2, \langle Au, u\rangle)\, Au + n(\|u\|_H^2, \langle Au, u\rangle)\, u = 0 .$$

In particolare si individuano classi di condizioni sulle funzioni m ed n (sia in presenza che in assenza di energie conservate) che garantiscono l'esistenza di tali soluzioni.

Summary. – *In this paper we study the problem of existence of global solutions for some classes of abstract equations, that generalize some type of Klein-Gordon equations, with nonlinear nonlocal terms of Kirchhoff type. We find some conditions that guarantee the existence of such solutions whether in presence or in absence of a conserved energy.*

1. – Introduction.

Let V be an Hilbert space, which is imbedded in this antidual space V' by a symmetric continuous compact map, and let H be the Hilbert completion of V with respect to the product $(u, v)_H = \langle u, v\rangle$, where $\langle u, v\rangle$ is the antiduality between V' and V.

Let $A : V \to V'$ be a symmetric positive definite isomorphism, i.e.

$$\langle Au, v\rangle = \langle Av, u\rangle \quad \text{and} \quad \langle Au, u\rangle \geq c\|u\|_V^2 \quad \text{with} \quad c > 0 . \tag{1.1}$$

In this framework, we consider the following abstract Cauchy problem:

$$\begin{cases} u'' + m(\|u\|_H^2, \langle Au, u\rangle)\, Au + n(\|u\|_H^2, \langle Au, u\rangle)\, u = 0 \\ u(0) = u_0 \in V,\ u'(0) = u_1 \in V \end{cases} \tag{1.2}$$

while $m, n : [0, +\infty[\times [0, +\infty[\to \boldsymbol{R}$ are continuous functions and:

$$m(r, s) \geq 0 \quad \text{on} \quad [0, +\infty[\times [0, +\infty[.$$

Since the operator A is symmetric and coercive, and m is nonnegative, equation in (1.2) is of weakly hyperbolic type.

In the case $n = 0$ and $m(r, s) = m(s)$, a concrete version of (1.2) is the Kirchhoff equation (introduced by [8]):

$$u_{tt} - m\left(\int_{\Omega} |\nabla u|^2\right) \Delta u = 0 \qquad x \in \Omega \tag{1.3}$$

where $\Omega = [0, 2\pi]^h$ (and we look for solutions u which are 2π-periodic functions in the space variables). The problem of existence of local-global solutions for (1.3) has been studied by a lot of authors (both in Sobolev spaces and in the analytic case); we refer to [1] and [10] for a complete bibliography. Only we recall some authors who studied the problem of analytic global solutions.

Bernstein [3] proved that equation (1.3) with analytic periodic data has a global solution in one space dimension, assuming that

$$m \quad \text{Lipschitz continuous and} \quad m \geq \nu > 0\,. \tag{1.4}$$

Pohozaev [9] extended this result to several space dimensions. Later on Arosio & Spagnolo [2] relaxed hypothesis (1.4) by assuming merely that m is continuous and:

$$m \text{ is bounded} \quad \text{or} \quad \int_0^{+\infty} m(s)\, ds = +\infty\,. \tag{1.5}$$

Condition (1.5) was later removed by D'Ancona & Spagnolo [5]-[6], indeed they supposed only m continuous and $m \geq 0$. We remark that in [6] it was considered the abstract generalization of (1.3), i.e. $u'' + m(\langle Au, u\rangle)\, Au = 0$. Later on in [7] it was proved the existence of global in time, periodic in x, analytic solutions for some system of the form:

$$U_t = \sum_{i=1}^{h} B_i(\|u_1\|^2, \ldots, \|u_m\|^2)\, U_{x_i} \tag{1.6}$$

where $U = (u_1, \ldots, u_m)$, matrices B_i are continuous, $\sum_{i=1}^{h} B_i(r_1, \ldots, r_m)\, \xi_i$ has real eigenvalues for all $\xi = (\xi_1, \ldots, \xi_h) \in \boldsymbol{R}^h \setminus \{0\}$ and $\|f\|$ denote the L^2-norm. Moreover they assumed that:

THEOREM 1. – The matrices $B_i(r_1, \ldots, r_m)$ are bounded.

or

THEOREM 2. – System (1.6) has a conserved coercive energy, i.e. there exists some function $L(r_1, \ldots, r_m)$ (with $r_1, \ldots, r_m \geq 0$) such that if $U =$

$(u_1, \ldots, u_m)$ is a solution of (1.6) then

$$L(\|u_1(t)\|^2, \ldots, \|u_m(t)\|^2) = L(\|u_1(0)\|^2, \ldots, \|u_m(0)\|^2). \tag{1.7}$$

Moreover

$$\lim_{r_1 + \ldots + r_m \to +\infty} L(r_1, \ldots, r_m) = +\infty .$$

or

THEOREM 3. – System (1.6) is 2×2 in one space variable, with a conserved energy (see (1.7)). Moreover, denoted by $\phi_{i,j}$ $i, j = 1, 2$ the coefficients of the matrix B, one has:

- $\phi_{1,2}, \phi_{2,1} \geq 0$
- $|\phi_{2,1}(r, s)| \leq \Lambda(r)$ (Λ continuous function)
- $\inf\limits_{s \geq 0} L(r, s) \to +\infty$ as $r \to +\infty$
- $|\phi_{1,1}(r, s) - \phi_{2,2}(r, s)|^2 \leq C\phi_{1,2}(r, s)$ for some constant C.

By following [7], the purpose of this paper is to study the problem of existence of A-analytic solutions (see Definition 2.1) for (1.2). We observe that, in contrast with the cases considered in the literature, in our situation we have not necessarily a positive conserved energy and the functions m and n in (1.2) in general are not bounded.

We remark that (1.2) is an abstract equation modeling the Klein-Gordon nonlocal equation:

$$u_{tt} - m(\|u\|^2, \|\nabla u\|^2)\, \Delta u + n(\|u\|^2, \|\nabla u\|^2)\, u = 0 . \tag{1.8}$$

In fact we treat (1.2) if there exists a conserved energy (see Theorem 3.1-3.3) or a *semi*-conserved energy (see Theorem 3.5). In particular we prove the global well-posedness in the class of analytic 2π-periodic functions for the Cauchy problem to (see example 3.7):

$$u_{tt} - m(\|\nabla u\|^2)\, \Delta u + n(\|u\|^2)\, u = 0$$

where $m \geq 0$ and $\int_0^{+\infty} n(s)\, ds \in \boldsymbol{R}$. Another equation to which our results apply is (see example 3.11):

$$u_{tt} - \|\nabla u\|^4 \Delta u + \|\nabla u\|^2 u = 0 .$$

In Section 2 we give some definitions and a result of extension of solutions of the linear equation $u'' + m(t)\, \Delta u + n(t)\, u = 0$.

In Section 3 we state the main results and give some applications.

In Section 4 we give the proofs.

2. – Preliminaries-Linear case.

2.1. *Preliminaries.*

Let V, H, V', A be as in the Introduction. We give the following (see [9]):

DEFINITION 2.1. – *A vector $v \in V$ is called A-analytic if there exist constants K, Λ such that:*

$$A^j v \in V \quad \textit{and} \quad |\langle A^j v, v\rangle|^{1/2} \leqslant K\Lambda^j j! \quad \textit{for each} \quad j = 0, 1, \dots$$

In the following we denote the class of A-analytic vectors by $\boldsymbol{A}$.

Since the embedding $V \hookrightarrow V'$ is compact, the Hilbert space H has a orthonormal basis $(v_k) \subseteq V$ such that for each $k = 1, 2, \dots$

$$Av_k = \lambda_k^2 v_k, \quad \lambda_k > 0 \quad \text{and} \quad \lambda_k \to +\infty \quad \text{as} \quad k \to +\infty. \tag{2.1}$$

Let us remark that we can assume that (λ_k) is a nondecreasing sequence. Now let us give the following (see [2], Proposition 1)

PROPOSITION 2.2. – *A vector $u = \sum_k u_k v_k$ is in $\boldsymbol{A}$ if and only if there exists some $\delta > 0$ such that:*

$$\sum_k |u_k|^2 e^{\delta \lambda_k} < +\infty.$$

At this point we recall some examples of A-analytic vectors, when $A = -\Delta$ (see [2], p. 3).

Let $H^1_{\alpha\text{-per}}(\boldsymbol{R}^h)$ be the space of the functions $u \in H^1_{\text{loc}}(\boldsymbol{R}^h)$, α-periodic in each variable ($\alpha > 0$).

1. Let us set $V = H^1_{\alpha\text{-per}}(\boldsymbol{R}^h)$, and $V' = H^{-1}_{\alpha\text{-per}}(\boldsymbol{R}^h)$; then $A : V \to V'$ and if $u \in V$ is analytic, then it is A-analytic.

2. Let $\Omega \subseteq \boldsymbol{R}^h$ be a bounded open subset. Let us set $V = H^1_0(\Omega)$ and $V' = H^{-1}(\Omega)$, then $A : V \to V'$. Moreover if u is analytic in some neighborhood of Ω and

$$\Delta^k u = 0 \quad \text{on} \quad \partial\Omega \quad \text{for each} \quad \mathrm{k} = 0, 1 \dots$$

then $u \in V$ and u is A-analytic.

2.2. *Linear equation.*

Let us consider the Cauchy problem

$$\begin{cases} u'' + m(t)\, Au + n(t)\, u = 0 \\ u_0,\, u_1 \in \boldsymbol{A} \end{cases} \tag{2.2}$$

where the coefficients m, n satisfy the following conditions:

$$m \geqslant 0, \quad \int_0^T m(s)\, ds < +\infty, \quad \int_0^T |n(s)|\, ds < +\infty. \tag{2.3}$$

The following lemma is proved by using the method of perturbed energy of infinite order, firstly introduced by [4] and already used by [2], [6], [7]. For the convenience of the reader we sketch the proof.

LEMMA 2.3. – *Let us suppose that* m, n *satisfy* (2.3) *and let* $u \in C^2([0, T[, V)$ *be a solution of* (2.2).
Then u *and* u' *can be extended as A-analytic functions on* $[0, T]$.

PROOF. – Let $\varrho_\varepsilon(t)$ be a family of Friedrics mollifiers and let us define the positive function:

$$m_\varepsilon(t) = \widetilde{m} * \varrho_\varepsilon(t) + \varepsilon + \|\widetilde{m} * \varrho_\varepsilon - m\|_{L^1(0,T)}$$

where $\widetilde{m}$ denote the hull extension of m on the whole real axis $\boldsymbol{R}$.
We have (see [2]):

$$\left\| \frac{m_\varepsilon - m}{\sqrt{m_\varepsilon}} \right\|_{L^1(0,T)} \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{2.4}$$

Now let us denote, by using the Fourier's expansion, the considered solution of (2.2) by $u(t) = \sum_{k=1}^{+\infty} u_k(t)\, v_k$, then u_k satisfies the Cauchy problem:

$$\begin{cases} u_k'' + m(t)\, \lambda_k^2 u_k + n(t)\, u_k = 0 \\ u_k(0) = u_{0,k}, \quad u_k'(0) = u_{1,k}, \end{cases}$$

where $u_0 = \sum_{k=1}^{+\infty} u_{0,k} v_k$ and $u_1 = \sum_{k=1}^{+\infty} u_{1,k} v_k$.
If we define

$$E_{\varepsilon,k}(t) = |u_k'(t)|^2 + m_\varepsilon(t)\, |\lambda_k u_k|^2,$$

we find easily:

$$E'_{\varepsilon,k} \leqslant \left| \frac{m_\varepsilon - m}{\sqrt{m_\varepsilon}} \right| \lambda_k E_{\varepsilon,k} + \left| \frac{m_\varepsilon'}{m_\varepsilon} \right| E_{\varepsilon,k} + |n| |u_k| |u_k'|$$

$$\leqslant \left(\left| \frac{m_\varepsilon - m}{\sqrt{m_\varepsilon}} \right| \lambda_k + C_\varepsilon \left(1 + \frac{|n|}{\lambda_k}\right)\right) E_{\varepsilon,k}.$$

Hence, by (2.1)-(2.3), we obtain:

$$E_{\varepsilon,k}(t) \leqslant C_{\varepsilon,T} E_{\varepsilon,k}(0) \exp\left(\lambda_k \int_0^T \left| \frac{m_\varepsilon(s) - m(s)}{\sqrt{m_\varepsilon(s)}} \right| ds\right).$$

Let δ (see Proposition 2.2) be such that:

$$\sum_{k=1}^{+\infty} e^{\delta\lambda_k}(|u_{1,k}|^2 + |\lambda_k u_{0,k}|^2) < +\infty,$$

then, by (2.4) there exists $\bar{\varepsilon} > 0$ such that

$$\sum_{k=1}^{+\infty} E_{\bar{\varepsilon},k}(t)\, e^{\frac{1}{2}\delta\lambda_k} \leqslant K_{\bar{\varepsilon},T} \sum_k e^{\delta\lambda_k} E_{\varepsilon,k}(0) < +\infty.$$

Therefore as in [2] u and u' can by extended as A-analytic functions on $[0, T]$.

3. – Results-Applications.

3.1. *Principal results.*

Let $L : [0, +\infty[\to [0, +\infty[$ be a continuous function. We say L *admissible function* if for all $y_0 \geqslant 0$ the greatest solution of

$$\begin{cases} y' = L(y) \\ y(0) = y_0 \end{cases}$$

is bounded from above on the bounded subsets of $[0, +\infty[$.

In the following we call *conserved energy* for (1.2) a continuous function $E(w, r, s) = w + M(r, s)$ defined for $w, r, s \geqslant 0$ such that for all solution $u \in C^2([0, T[, V)$ of (1.2):

$$E(\|u'\|_H^2(t), \|u\|_H^2(t), \langle Au, u\rangle(t)) = E(\|u_1\|_H^2, \|u_0\|_H^2, \langle Au_0, u_0\rangle).$$

Let us recall that we indicate by c the constant in (1.1).

At this point we can state:

THEOREM 3.1. – *Let us suppose that the initial data u_0, $u_1 \in \boldsymbol{A}$ and that at least one of the following is verified:*

1. *the functions m, n are bounded;*
2. *E is a conserved energy for* (1.2), *moreover:*

(*a*) $M(r, s) = M_0(r, s) + K(r)$, *with* $K \leqslant 0$ *and*

$$\inf_{r,\, s \geqslant 0,\, r \leqslant (1/c)\, s} M_0(r, s) \in \boldsymbol{R}\ ; \tag{3.1}$$

(*b*) *for all* $\beta \geqslant 0$, *the function* $L(y) = y + \beta - K(y)$ *is an admissible function;*

(*c*) *for each* $I = [0, z] \subseteq [0, +\infty[$

$$\lim_{w + s \to +\infty} \ \min_{r \in I,\, r \leqslant (1/c)\, s} E(w, r, s) = +\infty\,. \tag{3.2}$$

Then problem (1.2) *has a global A-analytic solution* $u \in C^2([0, +\infty[, V)$.

An immediate consequence of Theorem 3.1 is the following:

COROLLARY 3.2. – *Let us suppose that E is a conserved energy for* (1.2) *and:*

$$\lim_{r + w + s \to +\infty} E(w, r, s) = +\infty\,.$$

Then problem (1.2) *has a global A-analytic solution* $u \in C^2([0, +\infty[, V)$ *if* $u_0, u_1 \in \boldsymbol{A}$.

Let us remark that the result of [6] is not contained in the previous theorem, since in that case there exists a conserved energy, but not verifies necessary (3.2). Now we give a generalization of such result.

THEOREM 3.3. – *Let us suppose that E is a conserved energy for* (1.2) *such that* $M(r, s) = M_0(r, s) + K(r)$, *with* $K \leqslant 0$ *and:*

$$\inf_{r,\, s \geqslant 0,\, r \leqslant (1/c)\, s} M_0(r, s) \in \boldsymbol{R}\,. \tag{3.3}$$

Moreover let us assume that for all $\beta \geqslant 0$, *the function* $L(y) = y + \beta - K(y)$ *is an admissible function and that for some continuous function* ψ *and* $r \leqslant c^{-1} s$:

$$|n(r, s)| \leqslant \psi(r, M(r, s))\,. \tag{3.4}$$

Then the Cauchy problem (1.2) *with* $u_0, u_1 \in \boldsymbol{A}$ *has a global A-analytic solution* $u \in C^2([0, +\infty[, V)$.

Let us observe that in case of a completely general M we can not assure the existence of a global analytic solution. In fact we have:

EXAMPLE 3.4. – *Let* $V = H^1_{2\pi\text{-per}}(\boldsymbol{R})$, $V' = H^{-1}_{2\pi\text{-per}}(\boldsymbol{R})$, $\|w\|^2 = \int_0^{2\pi} w(x)^2\,dx$ *and* $A = -\Delta$. *Then there exist some* $u_0, u_1 \in \boldsymbol{A}$ *such that the Cauchy problem*

$$\begin{cases} u_{tt} - \dfrac{1}{1+\|\nabla u\|^4}\Delta u - \|u\|^4 u = 0\,, \\ u(0, x) = u_0,\ u_t(0, x) = u_1, \end{cases}$$

has not a global analytic solution.

Let us point out that in the case of Example 3.4 the hypotheses of Theorem 3.1-3.3 are not verified. Indeed if E is a conserved energy, then

$$E(w, r, s) = \frac{1}{2}(w + \arctan s) - \frac{r^3}{3} + \text{constant}\,.$$

Therefore, if we want satisfy (3.1) (resp (3.3)) then must be $K(r) \leqslant -\frac{r^3}{3}$ for large r, then L is not an admissible function.

Let us consider now the case in which do not exists a conserved energy.

Let $E(w, r, s) = w + M(r, s)$, $w, r, s \geqslant 0$ be a continuous function. We call E *semi-conserved energy* for (1.2) if there exists a continuous function $n_0(r, s)$ such that, if $u \in C^2([0, T[, V)$ is a solution of (1.2) then

$$\frac{d}{dt}E(\|u'\|_H^2, \|u\|_H^2, \langle Au, u\rangle) = n_0(\|u\|_H^2, \langle Au, u\rangle)\frac{d}{dt}\|u\|_H^2.$$

We can therefore state:

THEOREM 3.5. – *Let us suppose that E is a semi-conserved energy for* (1.2) *with* $M(r, s) \geqslant 0$. *Moreover let us suppose that:*

1. $n_0^2(r, s)r \leqslant K(M(r, s))$, *for* $r \leqslant c^{-1}s$, *where K is a nondecreasing function, and* $L(y) = y + K(y)$ *is an admissible function.*

2. *At least one of the following conditions is verified:*

(*a*) *for each* $I = [0, z] \subseteq [0, +\infty[$

$$\lim_{s \to +\infty} \inf_{r \in I,\, r \leqslant (1/c)s} M(r, s) = +\infty\,; \tag{3.5}$$

(*b*) *for some continuous function* γ *and* $r \leqslant c^{-1}s$:

$$|n(r, s)| \leqslant \gamma(r, M(r, s)) \tag{3.6}$$

(*c*) *for some continuous functions* ϕ, χ *with* $\lim_{s \to +\infty} \phi(s) = +\infty$:

$$|n_0(r, s)| \phi(s) \leqslant \chi(r, M(r, s))(r \leqslant c^{-1} s) \tag{3.7}$$

and for some continuous function $\gamma(\cdot, \cdot, \cdot)$, *nondecreasing in each variable*:

$$|n(r, s)| \leqslant \gamma(r, M(r, s), n_0(r, s)) \ (r \leqslant c^{-1} s). \tag{3.8}$$

Then Problem (1.2) *has a global A-analytic solution* $u \in C^2([0, +\infty[, V)$ *as soon as* $u_0, u_1 \in \boldsymbol{A}$.

An immediate consequence of Theorem 3.5 is the following:

COROLLARY 3.6. – *Let us suppose that E is a semi-conserved energy for* (1.2) *such that*

$$n_0^2(r, s) r \leqslant c_1 + c_2 M(r, s) \quad \textit{and} \quad \lim_{r+s \to +\infty} M(r, s) = +\infty .$$

Then Problem (1.2) *has a global A-analytic solution* $u \in C^2([0, +\infty[, V)$ *as soon as* $u_0, u_1 \in \boldsymbol{A}$.

3.2. *Applications.*

Now we get some examples in which we can apply Theorem 3.1-3.5.

In these examples, we assume $V = H^1_{\alpha\text{-per}}(\boldsymbol{R}^h)$, $V' = H^{-1}_{\alpha\text{-per}}(\boldsymbol{R}^h)$, and $A = -\Delta$.

Moreover, in all the considered case, we suppose that the initial data $u_0, u_1 \in V$ are A-analytic, and $\|\cdot\|$ denotes the usual L^2 norm.

EXAMPLE 3.7. – *Let us suppose that* $m, n : [0, +\infty[\to \boldsymbol{R}$ *are continuous functions and that:*

$$m \geqslant 0 \quad \textit{and} \quad \inf_{r \geqslant 0} \int_0^r n(\sigma)\, d\sigma \in \boldsymbol{R} . \tag{3.9}$$

Then the Cauchy problem

$$\begin{cases} u_{tt} - m(\|\nabla u\|^2)\, \Delta u + n(\|u\|^2)\, u = 0 \\ u(0, x) = u_0, \ u_t(0, x) = u_1 \end{cases} \tag{3.10}$$

has a global analytic solution $u \in C^2([0, +\infty[, V)$.

EXAMPLE 3.8. – *Let* c_0 *be a constant for which* (1.1) *is verified. Then the Cauchy problem:*

$$\left\{\begin{array}{l} u_{tt} - \|\nabla u\|^2 \Delta u - (c_0^2 \|u\|^2 + 1)\, u = 0 \\ u(0, x) = u_0,\ u_t(0, x) = u_1 \end{array}\right. \tag{3.11}$$

has a global analytic solution $u \in C^2([0, +\infty[, V)$.

EXAMPLE 3.9. – *The Cauchy problem:*

$$\left\{\begin{array}{l} u_{tt} - \dfrac{\|\nabla u\|^2}{1 + \|u\|^4} \Delta u - \dfrac{\|\nabla u\|^4 \|u\|^2}{(1 + \|u\|^4)^2} u = 0 \\ u(0, x) = u_0,\ u_t(0, x) = u_1 \end{array}\right. \tag{3.12}$$

has a global analytic solution $u \in C^2([0, +\infty[, V)$.

EXAMPLE 3.10. – *The Cauchy problem:*

$$\left\{\begin{array}{l} u_{tt} - \dfrac{\|u\|^2}{1 + \|\nabla u\|^2} \Delta u + \arctan(\|\nabla u\|^2)\, u = 0 \\ u(0, x) = u_0,\ u_t(0, x) = u_1 \end{array}\right. \tag{3.13}$$

has a global analytic solution $u \in C^2([0, +\infty[, V)$.

EXAMPLE 3.11. – *The Cauchy problem:*

$$\left\{\begin{array}{l} u_{tt} - \|\nabla u\|^4 \Delta u + \|\nabla u\|^2 u = 0 \\ u(0, x) = u_0,\ u_t(0, x) = u_1 \end{array}\right. \tag{3.14}$$

has a global analytic solution $u \in C^2([0, +\infty[, V)$.

4. – Proofs.

We fix a notation that we use in the following proofs, i.e.:

$$m_v(t) := m(\|v(t)\|_H^2, \langle Av(t), v(t)\rangle), \qquad n_v(t) := n(\|v(t)\|_H^2, \langle Av(t), v(t)\rangle).$$

Firstly we prove:

LEMMA 4.1. – *For every* $u_0, u_1 \in \boldsymbol{A}$ *there exists a time* $T = T(u_0, u_1)$ *such that problem* (1.2) *has a solution* $u \in C^2([0, T], V)$ *with* $Au \in C^0([0, T], V)$. *Moreover* u, u' *are* $\boldsymbol{A}$*-analytic.*

PROOF. – (we follow the outline of [2])

Let V_h be the linear space spanned by $v_1, \ldots, v_h$ (the first h-eigenvectors) and let $P_h : H \to V_h$ be defined by:

$$P_h u := \sum_{k=1}^{h} (u, v_k)_H v_k .$$

Let us consider the Cauchy problem in V_h:

$$(CP_h) \quad \begin{cases} u_h'' + m(\|u_h\|_H^2, \langle Au_h, u_h\rangle)\, Au_h + n(\|u_h\|_H^2, \langle Au_h, u_h\rangle)\, u_h = 0 \\ u_h(0) = P_h u_0,\ u_h'(0) = P_h u_1 . \end{cases}$$

Since V_h is finite dimensional, by the Peano's Theorem, problem (CP_h) has a local solution, which can be extended to a maximal solution $u_h : [0, T_h[\to V_h$.

Now let us prove that $T_h \geqslant T > 0$ for all $h \in \boldsymbol{N}$.

If we set $y_k(t) := (u_h(t), v_k)_H$, then we can define:

$$e_k(u_h, t) := \frac{1}{2}(\lambda_k^2 |y_k(t)|^2 + |y_k(t)|^2 + |y_k'(t)|^2) .$$

It is easy to prove that:

$$e_k(u_h, t) \leqslant e_k(u_h, 0) \exp\left(\int_0^t \lambda_k |1 - m_{u_h}(s)| ds + \int_0^t |1 - n_{u_h}(s)| ds \right)$$

$$=: e_k(u_h, 0)\, \gamma_k(t) ,$$

therefore one has

$$\|u_h\|_H^2 + \langle Au_h, u_h\rangle \leqslant 2 \sum_{k=1}^{h} e_k(u_h, 0)\, \gamma_k(t) . \tag{4.1}$$

On the other part, by the A-analyticity of u_0, u_1 (see Proposition 2.2), there exists some $\delta > 0$ such that:

$$2 \sum_{k=1}^{+\infty} e_k(u_h, 0)\, e^{2\delta\lambda_k} < C^{-1} \beta , \tag{4.2}$$

where we have set, for $C := e^{\delta}$

$$\beta := 1 + C \sum_{k=1}^{+\infty} e^{2\delta\lambda_k} (|(u_0, v_k)_H|^2 \lambda_k^2 + |(u_0, v_k)_H|^2 + |(u_1, v_k)_H|^2) .$$

Now let us define:

$$T := \left(1 + \sup_{0 \leqslant r, s \leqslant \beta} |1 - m(r, s)| + \sup_{0 \leqslant r, s \leqslant \beta} |1 - n(r, s)|\right)^{-1} \delta .$$

Let us prove that $T_h > T$ for all $h \in \boldsymbol{N}$, and

$$\|u_h\|_H^2, \langle Au_h, u_h\rangle \leqslant \beta \quad \text{on} \quad [0, T]. \tag{4.3}$$

Let us set

$$T_h^* = \sup\{t \in [0, T_h[: \|u_h\|_H^2, \langle Au_h, u_h\rangle \leqslant \beta \quad \text{on} \quad [0, t]\}.$$

We shall prove that $T_h^* > T$. Let us suppose by contradiction that $T_h^* \leqslant T$. In this case, by the definition of T:

$$\int_0^{T_h^*} |1 - m_{u_h}(s)|\,ds + \int_0^{T_h^*} |1 - n_{u_h}(s)|\,ds \leqslant \delta\,.$$

Now let us observe also that $T_h^* = T_h$ is not admissible (since in this situation m_{u_h} and n_{u_h} are bounded and then the solution, using Lemma 2.3, can be extended on $[0, T_h]$), then must be $T_h^* < T_h$. Therefore, by (4.1)-(4.2):

$$\|u_h\|_H^2(T_h^*) + \langle Au_h(T_h^*), u_h(T_h^*)\rangle \leqslant 2C \sum_{k=1}^{h} e_k(u_h, 0)\, e^{\delta\lambda_k} < \beta\,,$$

whereas, by the definition of T_h^* one obtains

$$\|u_h\|_H^2(T_h^*) + \langle Au_h(T_h^*), u_h(T_h^*)\rangle \geqslant \beta\,.$$

Hence we have a contradiction. So we have achieved (4.3).

Therefore on $[0, T]$ we obtain:

$$e_k(u_h, t) \leqslant Ce^{\delta\lambda_k}(|(u_0, v_k)_H|^2\lambda_k^2 + |(u_0, v_k)_H|^2 + |(u_1, v_k)_H|^2)\,,$$

hence, for some $d > 0$:

$$\sum_{k=1}^{+\infty} \lambda_k^8 e_k(u_h, t) \leqslant d \sum_{k=1}^{+\infty} e^{2\delta\lambda_k}(|(u_0, v_k)_H|^2\lambda_k^2 + |(u_0, v_k)_H|^2 + |(u_1, v_k)_H|^2)\,.$$

By this, the sequences $(A^2 u_h')$ and $(A^{5/2} u_h)$ are bounded in $C^0([0, T], V)$.

Now the compactness of $A^{-1}\colon V \to V$ and Ascoli's Theorem ensure that there exists a subsequence (u_{h_k}) and a function u such that $Au \in C^0([0, T], V)$ and $Au_{h_k} \to Au$, $u_{h_k} \to u$, $u_{h_k}' \to u'$ in $C^0([0, T], V)$. Therefore, by letting $k \to +\infty$ in (CP_{h_k}) we see that $u_{h_k}'' \to u''$ in $C^0([0, T], V)$, u solves problem (1.2) and

$$\sum_{k=1}^{+\infty} e^{\delta\lambda_k} e_k(u, t) \leqslant \sum_{k=1}^{+\infty} Ce^{2\delta\lambda_k} e_k(u, 0) < +\infty\,. \qquad \blacksquare$$

We recall that, by (1.1), if u is a solution of (1.2) then we have:

$$\|u\|_H^2 \leqslant \frac{1}{c}\langle Au, u\rangle. \tag{4.4}$$

Let u be a local A-analytic solution (see Lemma 4.1) of (1.2) defined on $[0, T[$, $T > 0$. If we prove that u can be extended on the whole $[0, T]$ as an A-analytic function, then by standard arguments we can easy obtain the global existence of u. In fact we prove Theorem 3.1-3.3-3.5 if we show that we can apply Lemma 2.3.

Proof of Theorem 3.1

- Case m, n *bounded. We can apply directly Lemma* 2.3.
- Case 1)-3) hold true.

By (3.1), there exists θ such that $M_0(r, s) \geqslant \theta$ on the strip $r \leqslant \frac{s}{c}$. Moreover since (4.4) holds true and E is a conserved energy, then, for some $\beta \geqslant 0$:

$$\begin{aligned}\|u'\|_H^2 &= E(\|u_1\|_H^2, \|u_0\|_H^2, \langle Au_0, u_0\rangle) - M_0(\|u\|_H^2, \langle Au, u\rangle) - K(\|u\|_H^2)\\ &\leqslant \beta - K(\|u\|_H^2),\end{aligned}$$

hence:

$$\begin{aligned}(\|u\|_H^2)' = 2(u', u)_H &\leqslant \|u\|_H^2 + \|u'\|_H^2\\ &\leqslant \|u\|_H^2 + \beta - K(\|u\|_H^2).\end{aligned}$$

Now, if we define $y := \|u\|_H^2$ we obtain the ordinary differential inequality $y' \leqslant y + \beta - K(y)$, and since $y + \beta - K(y)$ is an admissible function, by a standard comparison argument y must be bounded on $[0, T[$. Hence $\|u'\|_H^2$ and, by (3.2), $\langle Au, u\rangle$ must be also bounded on $[0, T[$.

Therefore $m_u(t)$, $n_u(t)$ are bounded, and we can apply Lemma 2.3. ■

Proof of Theorem 3.3.

We only have to prove that we can apply Lemma 2.3, that is

$$\int_0^T m_u(s)\, ds + \int_0^T |n_u(s)|\, ds < +\infty . \tag{4.5}$$

As in the second case of the previous theorem, we can prove that $\|u\|_H^2$, and hence $\|u'\|_H^2$ are bounded on $[0, T[$. By this fact, since

$$M(\|u\|_H^2, \langle Au, u\rangle) = E(\|u_1\|_H^2, \|u_0\|_H^2, \langle Au_0, u_0\rangle) - \|u'\|_H^2,$$

then $M(\|u\|_H^2, \langle Au, u\rangle)$ is bounded too.

Let us define

$$E_0(t) := \|u + u'\|_H^2 + \|u\|_H^2 + M(\|u\|_H^2, \langle Au, u\rangle).$$

Then, since E is a conserved energy and $M(\|u\|_H^2, \langle Au, u\rangle)$ is bounded, one can easy see that for some constant C_T:

$$\begin{aligned} E_0' &= -2m_u(t)\langle Au, u\rangle - 2n_u(t)\|u\|_H^2 + 2\|u'\|_H^2 + 4(u', u)_H \\ &= 2(-m_u(t)\langle Au, u\rangle - n_u(t)\|u\|_H^2) + 2(\|u+u'\|_H^2 - \|u\|_H^2) \\ &\leqslant 2(-m_u(t)\langle Au, u\rangle - n_u(t)\|u\|_H^2 + 2E_0 + C_T). \end{aligned}$$

Since $\|u\|_H^2$ and $M(\|u\|_H^2, \langle Au, u\rangle)$ are bounded, then by assumption (3.4), $n(\|u\|_H^2, \langle Au, u\rangle)$ is bounded on $[0, T[$. Hence:

$$\int_0^T |n(\|u\|_H^2, \langle Au, u\rangle)|\, ds < +\infty .$$

Moreover, for some constant c_T:

$$E_0' \leqslant -2m(\|u\|_H^2, \langle Au, u\rangle)\langle Au, u\rangle + c_T + 2E_0.$$

By this, for some constant B_T:

$$\int_0^T 2m(\|u\|_H^2, \langle Au, u\rangle)\langle Au, u\rangle\, ds \leqslant E_0(0)\, e^{2T} + B_T,$$

hence it is also bounded

$$\int_0^T m(\|u\|_H^2, \langle Au, u\rangle)\, ds = \int_{[0, T[\cap\{\langle Au, u\rangle > 1\}} m(\|u\|_H^2, \langle Au, u\rangle)\, ds + \int_{[0, T[\cap\{\langle Au, u\rangle \leqslant 1\}} m(\|u\|_H^2, \langle Au, u\rangle)\, ds . \qquad \blacksquare$$

Proof of Theorem 3.5.

Firstly, we prove that, $\|u'\|_H^2$, and hence $\|u\|_H^2$ are bounded on $[0, T[$. In fact:

$$E' \leqslant |n_0(\|u\|_H^2, \langle Au, u\rangle)|\, \|u\|_H \|u'\|_H \leqslant \frac{1}{2}(n_0^2(\|u\|_H^2, \langle Au, u\rangle)\|u\|_H^2 + \|u'\|_H^2).$$

Hence, since $M \geqslant 0$ and K is nondecreasing $E' \leqslant E + K(E)$. Since $L(y) = y + K(y)$ is an admissible function, then by a standard argument for the ordinary differential inequalities, E must be bounded on $[0, T[$. Then $\|u'\|_H^2$ and M (and hence $\|u\|_H^2$ and $n_0^2(\|u\|_H^2, \langle Au, u\rangle)\|u\|_H^2$) are bounded.

Moreover if (3.5) hold true, then $\langle Au, u\rangle$ is bounded, and hence the functions $m(\|u\|_H^2, \langle Au, u\rangle)$ and $n(\|u\|_H^2, \langle Au, u\rangle)$ are bounded too and we can apply Lemma 2.3.

If it is not the case, let us define, as in proof of Theorem 3.3:

$$E_0(t) := \|u + u'\|_H^2 + \|u\|_H^2 + M(\|u\|_H^2, \langle Au, u\rangle).$$

Then, since E is a semi-conserved energy and $n_0(\|u\|_H^2, \langle Au, u\rangle)\|u\|_H$ is a bounded function, we have, for some constant C_T:

$$\begin{aligned} E_0' &= 2(-m_u(t)\langle Au, u\rangle - n_u(t)\|u\|_H^2 + \|u'\|_H^2) + \\ &\quad + 4(u', u)_H + 2n_0(\|u\|_H^2, \langle Au, u\rangle)(u', u)_H \\ &\leqslant 2(-m_u(t)\langle Au, u\rangle - n_u(t)\|u\|_H^2) + 5\|u'\|_H^2 \\ &\quad + 2\|u\|_H^2 + n_0^2(\|u\|_H^2, \langle Au, u\rangle)\|u\|_H^2 \\ &\leqslant 2(-m_u(t)\langle Au, u\rangle - n_u(t)\|u\|_H^2 + C_T). \end{aligned}$$

- Case (3.6) holds true.

The function $n(\|u\|_H^2, \langle Au, u\rangle)$ is bounded, hence as in the second case of the previous theorem we can prove that

$$\int_0^T m(\|u\|_H^2, \langle Au, u\rangle)\, ds + \int_0^T |n(\|u\|_H^2, \langle Au, u\rangle)|\, ds < +\infty,$$

and apply Lemma 2.3.

- Case (3.7)-(3.8) hold true.

Since γ is nondecreasing in each variable, then there exist two constant a_1, a_2 such that:

$$\begin{aligned} |n(\|u\|_H^2, \langle Au, u\rangle)| &\leqslant \gamma(a_1, a_2, n_0(\|u\|_H^2, \langle Au, u\rangle)) \\ &=: \gamma_0(n_0(\|u\|_H^2, \langle Au, u\rangle)). \end{aligned}$$

Let us set

$$\Gamma_1 = \int_{[0, T[\cap\{\langle Au, u\rangle \leqslant a_3\}} \gamma_0(n_0(\|u\|_H^2, \langle Au, u\rangle))\, d\tau$$

$$\Gamma_2 = \int_{[0, T[\cap\{\langle Au, u\rangle > a_3\}} \gamma_0(n_0(\|u\|_H^2, \langle Au, u\rangle)\, \phi(\langle Au, u\rangle))\, d\tau,$$

where, for $s \geqslant a_3$, we have $\phi(s) \geqslant 1$. Since

$$\int_0^T \gamma_0(n_0(\|u\|_H^2, \langle Au, u\rangle))\, d\tau \leqslant \Gamma_1 + \Gamma_2,$$

we can conclude, by (3.7), that

$$\int_0^T |n(\|u\|_H^2, \langle Au, u\rangle)|\, ds < +\infty .$$

Therefore as in the previous theorem we can prove that

$$\int_0^T m(\|u\|_H^2, \langle Au, u\rangle)\, ds < +\infty$$

and apply Lemma 2.3. ■

Proof of Example 3.4.

We shall prove that there exist some initial data such that $\|u\|_H^2$ blows-up in a finite time. In fact we have that:

$$((u_t, u)_H)' = -\frac{\|\nabla u\|^2}{1+\|\nabla u\|^4} + \|u_t\|^2 + \|u\|^6,$$

hence, integrating over $[0, T]$:

$$(u_t, u)_H = (u_1, u_0)_H + \int_0^t \|u_t\|^2 + \|u\|^6\, d\tau - \int_0^t \frac{\|\nabla u\|^2}{1+\|\nabla u\|^4}\, d\tau .$$

Let us assume that $t \leqslant 1$ and $(u_1, u_0)_H > 1$, therefore

$$(u_t, u)_H \geqslant \int_0^t \|u\|^6\, d\tau .$$

If we denote $y = \|u\|^2$, $y_0 = \|u_0\|^2$, we obtain, for $t \leqslant 1$:

$$y' \geqslant 2\int_0^t y^3(\tau)\, d\tau ,$$

hence

$$\left(\frac{y^4}{4}\right)' = y^3 y' \geqslant \left(\left[\int_0^t y^3(\tau)\, d\tau\right]^2\right)'.$$

Then we have proved that:

$$\frac{y^4}{4} \geqslant \frac{y_0^4}{4} + \left(\int_0^t y^3(\tau)\, d\tau\right)^2 . \tag{4.6}$$

Now let us define

$$z := \int_0^t y^3(\tau)\, d\tau .$$

By (4.6) we deduce:

$$(z')^{4/3} \geq y_0^4 + 4z^2,$$

hence by a standard comparison argument, if y_0 is sufficiently big, z blows-up in a time $T_0 < 1$, and therefore y blows-up too. ■

Proof of Example 3.7.
The function

$$E(w, r, s) = w + \int_0^s m(x)\, dx + \int_0^r n(x)\, dx = w + M(r, s)$$

is a conserved energy, that, by (3.9) verifies (3.3) with $M = M_0$ and $K = 0$. Moreover $L(y) = y + \beta$ is obviously an admissible function, and n depends only from r, hence we can apply Theorem 3.3. ■

Proof of Example 3.8.
In this case a conserved energy is the function

$$E(w, r, s) = w + \frac{s^2}{2} - \frac{c_0 r^2}{2} - r = w + M_0(r, s) - r .$$

Moreover $M_0(r, s)$ is nonnegative on the strip $r \leq \frac{s}{c_0}$ and $L(y) = 2y + \beta$ is an admissible function for all $\beta \geq 0$. Then we can apply Theorem 3.3. ■

Proof of Example 3.9.
The function

$$E(w, r, s) = w + \frac{s^2}{2(1 + r^2)} = w + M(r, s)$$

is a conserved energy. Therefore all the hypotheses of Theorem 3.1 as obviously verified, by assuming $M_0(r, s) = M(r, s)$ and $K(r) = 0$. ■

Proof of Example 3.10.
The function

$$E(w, r, s) = w + \arctan(s)\, r = w + M(r, s)$$

is a conserved energy that verifies (3.3) with $M_0(r, s) = M(r, s)$ and $K(r) = 0$. Moreover $L(y) = y + \beta$ is an admissible function, and n is bounded. Then we can apply Theorem 3.3. ■

Proof of Example 3.11.

We can apply Corollary 3.6, since the function

$$E(w, r, s) = w + \frac{s^3}{3} = w + M(r, s)$$

is a semi-conserved energy, with $n_0(r, s) = -s$, and for $r \leqslant c_0^{-1} s$ (where (1.1) is verified with $c = c_0$):

$$n_0^2(r, s)\, r = s^2 r \leqslant c_0^{-1} s^3. \qquad \blacksquare$$

REFERENCES

[1] A. Arosio, *Averaged evolution equations. The Kirchhoff string and its treatment in scales of Banach Spaces*, 2° Workshop on «Functional-analytic methods in complex analysis» (in Proc. Trieste 1993, World Singapore).

[2] A. Arosio - S. Spagnolo, *Global existence for abstract evolution equations of weakly hyperbolic type*, J. Math. Pures et Appl., **65** (1986), 263-305.

[3] S. Bernstein, *Sur une classe d'équations fonctionelles aux dérivées partielles*, Izv. Akad. Nauk. SSSR, Sér Math., **4** (1940), 17-26.

[4] F. Colombini - E. De Giorgi - S. Spagnolo, *sur les équations hyperboliques avec des coefficients qui ne dépendent que du temps*, Ann. Sc. Norm. Sup. Pisa, **6** (1979), 511-559.

[5] P. D'Ancona - S. Spagnolo, *Global solvability for the degenerate Kirchhoff equation with real analytic data*, Invent. Math., **108** (1992), 247-262.

[6] P. D'Ancona - S. Spagnolo, *On an abstract weakly hyperbolic equation modeling the nonlinear vibrating string*, Developments in Partial Differential Equations and Applications to Mathematical Physic, Edited by G. Buttazzo *et. al.* Plenum Press, New-York 1992.

[7] M. Ghisi - S. Spagnolo, *Global analytic solutions to hyperbolic systems*, NODEA, **5** (1998), 245-264.

[8] G. Kirchhoff, «Vorlesurngen über Mechanik» Teubner, Leipzig 1883.

[9] S. I. Pohozaev, *On a class of quasilinear hyperbolic equations*, Mat. Sbornik, **96** (138) (1975) No. 1, 152-166 (Trans. Math. USSR Sbornik, **25** (1975) No. 1).

[10] S. Spagnolo, *The Cauchy problem for the Kirchhoff equations*, Rend. Sem. Fisico Matematico di Milano, **62** (1992), 17-51.

Dipartimento di Matematica, Università di Pisa
Via Buonarroti 2, 56100 Pisa, Italy
e-mail: ghisi@dm.unipi.it

Pervenuta in Redazione
il 26 ottobre 2001