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Geometric Probabilities for non Convex Lattices (II).

ANDREI DUMA - MARIUS STOKA (*)

Sunto. – *Si risolvono problemi di tipo Buffon per un reticolo avente per cellula fondamentale un poligono non convesso, utilizzando come corpo test un segmento ed un cerchio.*

Summary. – *We solve problems of Buffon type for a lattice with elementary tile a non-convex polygon, using as test bodies a line sigment and a circle.*

In a previous work [3] we dealt with a lattice having as elementary tile a non–convex polygon. In this paper we examine another lattice.

Let be given, in the Euclidean plane E_2, a lattice $\mathcal{R}$ whose elementary tile is a concave polygon, formed by five squares of side a, as in figure 1.

At first we want to determine the probability p that a segment of constant length l and random position in E_2 intersects one of the sides of a fundamental cell of the lattice $\mathcal{R}$. We assume that the segment is uniformly distributed in a bounded region of the plane.

We denote by $\mathcal{M}$ the family of segments s, of length l, whose midpoints lie inside a fixed tile $\mathcal{C}_0$ of the lattice $\mathcal{R}$ and by $\mathcal{N}$ the set of segments s, of length l, which are completely contained in $\mathcal{C}_0$. With these notations we have [6], p. 53

$$p_l = 1 - \frac{\mu(\mathcal{N})}{\mu(\mathcal{M})} \tag{1}$$

where μ is the Lebesgue measure.

The measures $\mu(\mathcal{N})$ and $\mu(\mathcal{M})$ are computed by means of the kinematic measure in the Euclidean plane [4], p. 126

$$dK = dx \wedge dy \wedge d\varphi \ ,$$

where x and y are the coordinates of the midpoint of the segment s and φ is an angle of rotation.

Assuming the segment s «small» compared to the lattice $\mathcal{R}$, i.e. $l < a$, the

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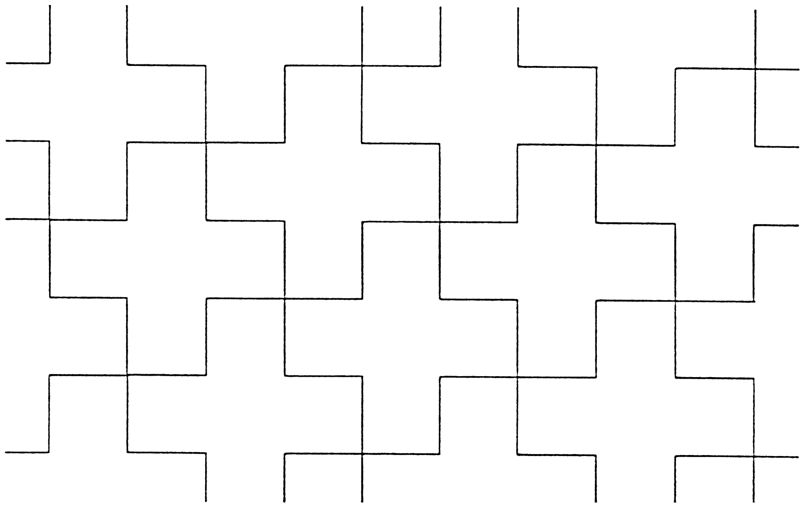

Figure 1

probability of intersection is

$$p_l = \frac{12}{5\pi}\frac{l}{a} - \frac{2}{5\pi}\left(\frac{l}{a}\right)^2. \tag{2}$$

This result was found by S. Rizzo [5], p. 14.

The notion of «non-small» segment and «non-small» circle with respect to a planar lattice with elementary tile a convex polygon was first defined in the works [1] and [2]. Subsequently it was generalized to a convex test body $\mathcal{K}$ and a lattice having as fundamental cell a concave polygon (we refer to the quoted work [3] for more details). By applying this latter definition to the lattice $\mathcal{R}$ in figure 1, we have that the segment s is «non–small» with respect to the lattice in the following cases:

1) $a < l < a\sqrt{2}$; 2) $a\sqrt{2} < l < 2a$; 3) $2a < l < \sqrt{5}a$;
4) $\sqrt{5}a < l < 2\sqrt{2}a$; 5) $2\sqrt{2}a < l < 3a$; 6) $3a < l < \sqrt{10}a$.

Taking into account the symmetries of the polygon $\mathcal{C}_0$ we can confine ourselves to consider $\varphi \in [0, \pi/4]$. Thus

$$\mu(\mathcal{M}) = \int_0^{\pi/4} d\varphi \int\int_{\{(x,\, y) \in \mathcal{C}_0\}} dx\, dy = \int_0^{\pi/4} \text{area}\,(\mathcal{C}_0)\, d\varphi = \frac{5\pi}{4} a^2. \tag{3}$$

For any fixed value of the angle φ, we denote by $\mathcal{C}_0(\varphi)$ the (convex or concave) domain determined by the midpoints of the segments s entirely contained in $\mathcal{C}_0$ in each one of the limit positions, so that

$$\mu(\mathcal{N}) = \int_0^{\pi/4} d\varphi \int \int_{\{(x,\, y) \in \mathcal{C}_0(\varphi)\}} dx\, dy = \int_0^{\pi/4} \text{area}\,(\mathcal{C}_0(\varphi))\, d\varphi\,. \tag{4}$$

By formulas (1), (3) and (4) we get

$$p_l = 1 - \frac{4}{5\pi a^2} \int_0^{\pi/4} \text{area}\,(\mathcal{C}_0(\varphi))\, d\varphi\,. \tag{5}$$

THEOREM 1. – *If $a < l < a\sqrt{2}$, the probability that a segment s, of length l, intersects a side of one of the tiles of the lattice $\mathcal{R}$ is*

$$p_l = \frac{4}{5\pi}\left(2 \arccos \frac{a}{l} + 1\right) + \frac{4l}{5\pi a} - \frac{8\sqrt{l^2 - a^2}}{5\pi a} + \frac{2l^2}{5\pi a^2}\,. \tag{6}$$

PROOF. – Let $\varphi_0 := \arccos \dfrac{a}{l}$. We have to consider the cases $0 < \varphi < \varphi_0$ and $\varphi_0 < \varphi < \dfrac{\pi}{4}$.

If $0 < \varphi < \varphi_0$, we get (see figure 2)

$$\begin{aligned}\text{area } \mathcal{C}_0(\varphi) &= 2\, \frac{(a - a\tan\varphi) + (a - l\sin\varphi + a\tan\varphi)}{2}(2a - l\cos\varphi)\\ &\quad + (a - l\sin\varphi)(l\cos\varphi - a)\\ &= 3a^2 - al(\sin\varphi + \cos\varphi)\,.\end{aligned}$$

Whereas when $\varphi_0 < \varphi < \dfrac{\pi}{4}$ we find (see figure 3)

$$\begin{aligned}\text{area } \mathcal{C}_0(\varphi) &= 2a(a - l\sin\varphi) + 2a(a - l\cos\varphi)\\ &\quad + (a - l\sin\varphi)(a - l\cos\varphi) + 2\,\frac{l\sin\varphi \cdot l\cos\varphi}{2}\\ &= 5a^2 - 3al(\sin\varphi + \cos\varphi) + l^2 \sin 2\varphi\,.\end{aligned}$$

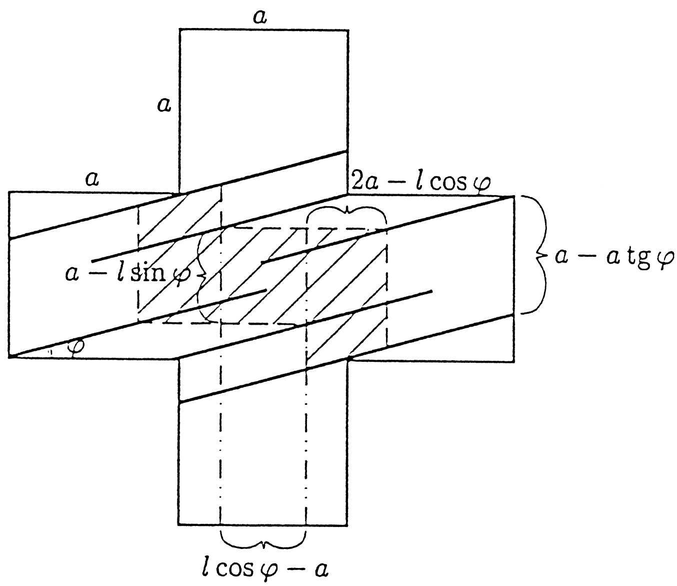

Figure 2

Formula (4) then yields

$$\mu(\mathcal{N}) = \int_0^{\varphi_0} [3a^2 - al(\sin\varphi + \cos\varphi)]\, d\varphi$$

$$+ \int_{\varphi_0}^{\pi/4} [5a^2 - 3al(\sin\varphi + \cos\varphi) + l^2 \sin 2\varphi]\, d\varphi$$

$$= \left(\frac{5}{4}\pi - 2\varphi_0 - 1\right) a^2 - al + 2a\sqrt{l^2 - a^2} - \frac{1}{2} l^2$$

and formula (5) gives the probability (6). ■

REMARK 1. – When $l = a$ formulas (2) and (7) give the probability $p_l = \dfrac{2}{\pi}$.

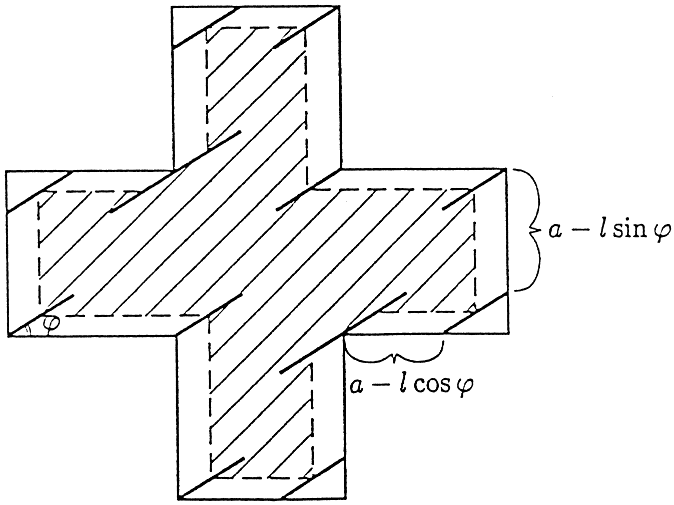

Figure 3

THEOREM 2. – *If* $a\sqrt{2} < l < 2a$, *the probability that a segment s, of length l, intersects a side of one of the tiles of the lattice* $\mathcal{R}$ *is*

$$p_l = \frac{1}{5} + \frac{2}{5\pi}\left(2\ \arcsin\frac{a}{l} - 1\right) + \frac{4\sqrt{l^2 - a^2}}{5\pi a} + \frac{4l}{5\pi a} - \frac{l^2}{5\pi a^2}\,. \tag{7}$$

PROOF. – Let $\varphi_1 := \arcsin\frac{a}{l}$. We have to examine the cases $0 < \varphi < \varphi_1$ and $\varphi_1 < \varphi < \frac{\pi}{4}$.

If $0 < \varphi < \varphi_1$, we have (see figure 2)

$$\text{area } \mathcal{C}_0(\varphi) = 3a^2 - al(\sin\varphi + \cos\varphi)\,.$$

If $\varphi_1 < \varphi < \frac{\pi}{4}$, we get

$$\text{area } \mathcal{C}_0(\varphi) = 2\,\frac{(a - a\tan\varphi) + (a - l\sin\varphi + a\tan\varphi)}{2}(2a - l\cos\varphi)$$

$$= 4a^2 - 2al(\sin\varphi + \cos\varphi) + l^2\sin\varphi\cos\varphi\,.$$

Hence formula (4) yields

$$\mu(\mathcal{N}) = \int_0^{\varphi_1} [3a^2 - al(\sin\varphi + \cos\varphi)]\, d\varphi$$

$$+ \int_{\varphi_1}^{\pi/4} [4a^2 - 2al(\sin\varphi + \cos\varphi) + l^2 \sin\varphi \cos\varphi]\, d\varphi$$

$$= a^2\left(\pi - \varphi_1 + \frac{1}{2}\right) - a\sqrt{l^2 - a^2} - al + \frac{1}{4} l^2$$

and formula (5) gives the probability (7). ■

REMARK 2. – When $l = a\sqrt{2}$ formulas (6) and (7) coincide, having the same value $p_l = \frac{2}{5} + \frac{4\sqrt{2}}{5\pi}$.

THEOREM 3. – *If* $2a < l < \sqrt{5}a$, *the probability that a segment s, of length l, intersects a side of one of the tiles of the lattice* $\mathcal{R}$ *is*

$$p_l = \frac{1}{5} - \frac{2}{\pi} + \frac{4}{5\pi} \arcsin\frac{a}{l} + \frac{4\sqrt{l^2 - a^2}}{5\pi a} + \frac{12l}{5\pi a} - \frac{3l^2}{5\pi a^2}. \tag{8}$$

PROOF. – Let $\varphi_2 := \arccos\frac{2a}{l}$. As $l < \sqrt{5}a$ we have $\varphi_2 < \varphi_1$, so we must consider the following three cases:

$$0 < \varphi < \varphi_2, \quad \varphi_2 < \varphi < \varphi_1, \quad \varphi_1 < \varphi < \frac{\pi}{4}.$$

When $0 < \varphi < \varphi_2$ we have

$$\begin{aligned} \text{area } \mathcal{C}_0(\varphi) &= (3a - l\cos\varphi)(a - l\sin\varphi) \\ &= 3a^2 - al(3\sin\varphi + \cos\varphi) + l^2 \sin\varphi\cos\varphi\,. \end{aligned}$$

When $\varphi_2 < \varphi < \varphi_1$ we find (see figure 2)

$$\textit{area } \mathcal{C}_0(\varphi) = 3a^2 - al(\sin\varphi + \cos\varphi).$$

Finally when $\varphi_1 < \varphi < \frac{\pi}{4}$ we get (see figure 4)

$$\textit{area } \mathcal{C}_0(\varphi) = 4a^2 - 2al(\sin\varphi + \cos\varphi) + l^2 \sin\varphi\cos\varphi\,.$$

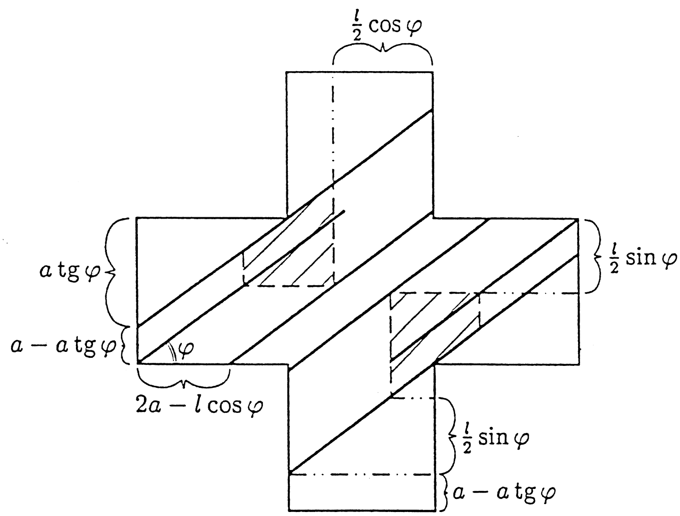

Figure 4

Hence formula (4) yields

$$\mu(\mathcal{N}) = \int_0^{\varphi_2} [3a^2 - al(3\sin\varphi + \cos\varphi) + l^2 \sin\varphi\cos\varphi]\, d\varphi$$

$$+ \int_{\varphi_2}^{\varphi_1} [3a^2 - al(\sin\varphi + \cos\varphi)]\, d\varphi$$

$$+ \int_{\varphi_1}^{\pi/4} [4a^2 - 2al(\sin\varphi + \cos\varphi) + l^2\sin\varphi\cos\varphi]\, d\varphi$$

$$= a^2\left(\frac{5}{2} - \varphi_1 + \pi\right) - a\sqrt{l^2 - a^2} - 3al + \frac{3}{4}l^2.$$

and, taking into account (1) and (3) we immediately find the stated probability (8). ■

Remark 3. – It is easily checked that (7) and (8) give the same probabiltiy $p_l = \frac{1}{3} + \frac{2}{5\pi} + \frac{4\sqrt{3}}{5\pi}$ when $l = 2a$.

THEOREM 4. – *If* $\sqrt{5}a < l < 2\sqrt{2}a$, *the probability that a segment* s, *of length* l, *intersects a side of one of the tiles of the lattice* $\mathcal{R}$ *is*

$$p_l = \frac{1}{5} + \frac{2}{\pi} + \frac{16}{5\pi} \arccos \frac{2a}{l} - \frac{12}{5\pi} \arcsin \frac{a}{l} + \frac{12l}{5\pi a} - \frac{12}{5\pi a}\sqrt{l^2 - a^2} - \frac{8}{5\pi a}\sqrt{l^2 - 4a^2} + \frac{l^2}{5\pi a^2}. \tag{9}$$

PROOF. – This time the angle φ_1 is less than φ_2 because we are now assuming $l > a\sqrt{5}$, thus we have to consider three possible ranges of variation for the angle φ:

$$0 < \varphi < \varphi_1, \quad \varphi_1 < \varphi < \varphi_2, \quad \varphi_2 < \varphi < \frac{\pi}{4}.$$

In the first case, $0 < \varphi < \varphi_1$, we have (see figure 5)

$$\text{area } \mathcal{C}_0(\varphi) = 3a^2 - al(3 \sin\varphi + \cos\varphi) + l^2 \sin\varphi \cos\varphi .$$

In the second case, $\varphi_1 < \varphi < \varphi_2$, we find area $\mathcal{C}_0(\varphi) = 0$ because $\mathcal{C}_0(\varphi) = \emptyset$.

In the third case, $\varphi_2 < \varphi < \frac{\pi}{4}$, we get (see figure 4)

$$\text{area } \mathcal{C}_0(\varphi) = 4a^2 - 2al(\sin\varphi + \cos\varphi) + l^2 \sin\varphi \cos\varphi .$$

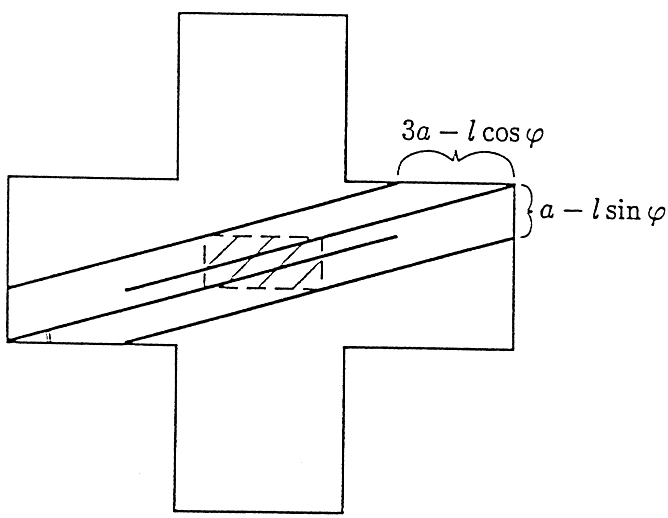

Figure 5

By applying formula (4) to obtain the value of the measure of the set $\mathcal{N}$ we find

$$\mu(\mathcal{N}) = \int_0^{\varphi_1} [3a^2 - al(3\sin\varphi + \cos\varphi) + l^2 \sin\varphi\cos\varphi]\, d\varphi$$

$$+ \int_{\varphi_2}^{\pi/4} [4a^2 - 2al(\sin\varphi + \cos\varphi) + l^2 \sin\varphi\cos\varphi]\, d\varphi$$

$$= a^2\left(\pi - \frac{5}{2} - 4\varphi_2 + 3\varphi_1\right) - 3al + 3a\sqrt{l^2 - a^2} + 2a\sqrt{l^2 - 4a^2} - \frac{l^2}{4}$$

and formula (5) gives the probability (9).

REMARK 4. – When $l = a\sqrt{5}$ both formula (8) and (9) become

$$p_l = \frac{1}{5} + \frac{12\sqrt{5} - 17}{5\pi} + \frac{4}{5\pi}\arcsin\frac{1}{\sqrt{5}}\,.$$

THEOREM 5. – *If* $2\sqrt{2}a < l < 3a$, *the probability that a segment s, of length l, intersects a side of one of the tiles of the lattice* $\mathcal{R}$ *is*

$$p_l = 1 - \frac{12}{5\pi}\arcsin\frac{a}{l} + \frac{2}{5\pi} + \frac{12l}{5\pi a} - \frac{12}{5\pi a}\sqrt{l^2 - a^2}. \tag{10}$$

PROOF. – If $0 < \varphi < \varphi_1$, we find (see figure 5)

$$\text{area } \mathcal{C}_0(\varphi) = 3a^2 - al(3\sin\varphi + \cos\varphi) + l^2\sin\varphi\cos\varphi\,.$$

As the set $\mathcal{C}_0(\varphi)$ is empty when $\varphi_1 < \varphi < \frac{\pi}{4}$ we simply have

$$\mu(\mathcal{N}) = \int_0^{\varphi_1} [3a^2 - al(3\sin\varphi + \cos\varphi) + l^2\sin\varphi\cos\varphi]\, d\varphi$$

$$= 3a^2 \arcsin\frac{a}{l} - 3al + 3a\sqrt{l^2 - a^2} - \frac{5a^2}{2}$$

and this value immediately yields the probability (10). ■

REMARK 5. – When $l = 2\sqrt{2}a$, formulas (9) and (10) give the same probability

$$p_l = 1 - \frac{12}{5\pi} \arcsin \frac{1}{2\sqrt{2}} + \frac{2(1 + 12\sqrt{2} - 6\sqrt{7})}{5\pi} .$$

THEOREM 6. – *If $3a < l < \sqrt{10}a$, the probability that a segment s, of length l, intersects a side of one of the tiles of the lattice $\mathcal{R}$ is*

$$p_l = 1 + \frac{4}{\pi} - \frac{12}{5\pi}\left(\arcsin \frac{a}{l} - \arccos \frac{3a}{l}\right)$$

$$- \frac{4(\sqrt{l^2 - 9a^2} + \sqrt{l^2 - a^2})}{5\pi a} + \frac{2l^2}{5\pi a^2} .$$

PROOF. – Let $\varphi_3 := \arccos \frac{3a}{l}$. As we are assuming $l < \sqrt{10}a$, we have $\varphi_3 < \varphi_1$ and therefore the different ranges of variation of the angle φ which we have to consider are

$$0 < \varphi < \varphi_3, \quad \varphi_3 < \varphi < \varphi_1, \quad \varphi_1 < \varphi < \frac{\pi}{4} .$$

In the first and in the latter case the set $\mathcal{C}_0(\varphi)$ is empty and consequently area $\mathcal{C}_0(\varphi) = 0$. When $\varphi_3 < \varphi < \varphi_1$ we find (see figure 5)

$$\text{area } \mathcal{C}_0(\varphi) = 3a^2 - al(3 \sin\varphi + \cos\varphi) + l^2 \sin\varphi \cos\varphi .$$

Formula (4) gives

$$\mu(\mathcal{N}) = \int_{\varphi_3}^{\varphi_1} [3a^2 - al(3 \sin\varphi + \cos\varphi) + l^2 \sin\varphi \cos\varphi]\, d\varphi$$

$$= 3a^2\left(\arcsin \frac{a}{l} - \arccos \frac{3a}{l}\right) - 5a^2$$

$$+ a(\sqrt{l^2 - 9a^2} + 3\sqrt{l^2 - a^2}) - \frac{l^2}{2}$$

and (5) yields the probability (11). ■

REMARK 6. – If we substitute $l = 3a$ in (10) and (11) we find the same probability

$$p_l = 1 - \frac{12}{5\pi} \arcsin \frac{1}{3} + \frac{2(19 - 12\sqrt{2})}{5\pi} .$$

Next we choose as test body a circle γ of constant radius r. We denote by $\mathcal{C}_0(r)$ the set of points $P \in \mathcal{C}_0$ with the property that the circle of center P and radius r is completely contained in the elementary tile $\mathcal{C}_0$. Taking into account formula (1), the probability p_r of intersection between a random circle (of constant radius r), uniformly distributed in a bounded region of the plane, and one of the sides of an elementary tile of the lattice $\mathcal{R}$ is

$$p_r = 1 - \frac{\text{area } \mathcal{C}_0(r)}{\text{area } \mathcal{C}_0} = 1 - \frac{\text{area } \mathcal{C}_0(r)}{5a^2} . \tag{12}$$

The circle γ is «small» with respect to the lattice $\mathcal{R}$ if and only if

$$2r < a . \tag{13}$$

THEOREM 7. – *If r satisfies condition* (13), *the probability that a circle γ intersects one of the sides of an elementary tile of the lattice $\mathcal{R}$ is*

$$p_r = \frac{12r}{5a} - \frac{(8-\pi)\, r^2}{5a^2} . \tag{14}$$

PROOF. – From figure 6 we get

$$\text{area } \mathcal{C}_0(r) = 2(3a - 2r)(a - 2r) - (a - 2r)^2 + 4r^2 - \pi r^2$$

$$= 5a^2 - 12ar + (8 - \pi)\, r^2.$$

Using this value of area $\mathcal{C}_0(r)$ in formula (12) we obtain the probability (14). ■

Recalling the definition of «non-small» circle with respect to the lattice $\mathcal{R}$, we can say that this case occurs exactly when

$$a < 2r < \sqrt{2}a . \tag{15}$$

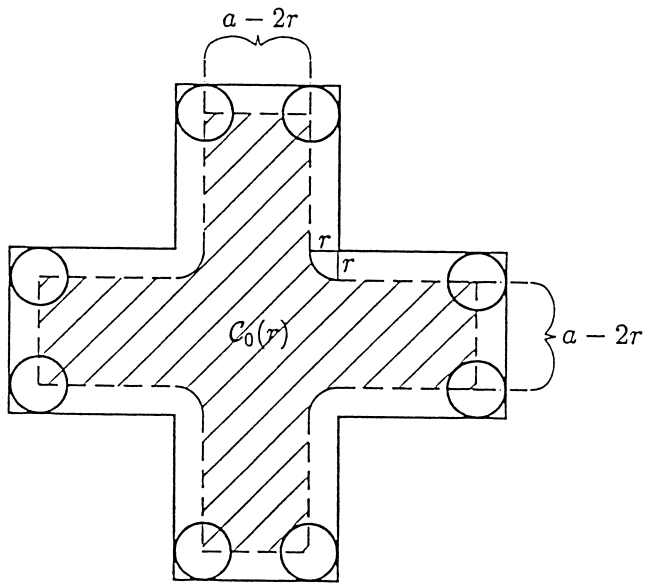

Figure 6

THEOREM 8. – *If* r *satisfies condition* (15), *the probability that a circle* γ *intersects one of the sides of an elementary tile of the lattice* $\mathfrak{R}$ *is*

$$p_r = \frac{4}{5} - \frac{\pi r^2}{5a^2} + \frac{4r^2}{5a^2}\arcsin\frac{a}{2r} + \frac{1}{5a}\sqrt{4r^2-a^2}. \tag{16}$$

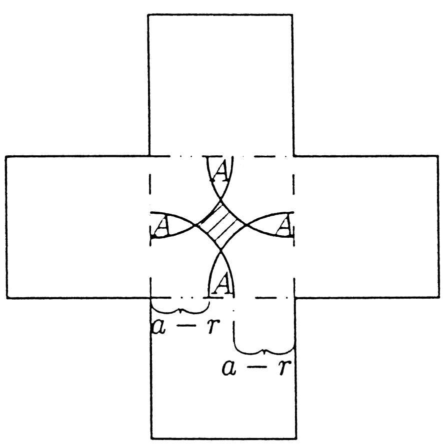

Figure 7

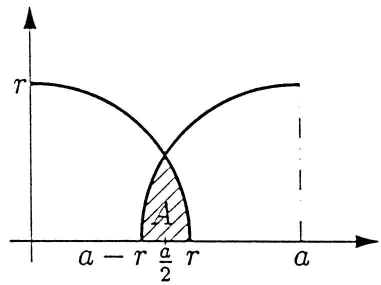

Figure 8

PROOF. – Using the notations of figure 7 we can write

$$4\cdot\frac{\pi r^2}{4} - 4\ \text{area}\ \mathcal{A} + \text{area}\ \mathcal{C}_0(r) = a^2,$$

hence

$$\text{area}\ \mathcal{C}_0(r) = a^2 - \pi r^2 + 4\ \text{area}\ \mathcal{A}\,. \tag{17}$$

From figure 8 we get

$$\text{area}\ \mathcal{A} = 2\int_{a/2}^{r}\sqrt{r^2-x^2}\,dx = r^2\left[\frac{\pi}{2} - \arcsin\frac{a}{2r} - \frac{4}{4r^2}\sqrt{4r^2-a^2}\right].$$

By substituting this value in (17) we obtain

$$\text{area}\ \mathcal{C}_0(r) = a^2 + \pi r^2 - 4r^2\arcsin\frac{a}{2r} - a\sqrt{4r^2-a^2}$$

and formula (12) yields the probability (16). ■

REMARK 7. – When $r = \frac{a}{2}$ formulas (14) and (16) give the same probability $p_r = \frac{4}{5} + \frac{\pi}{20}$.

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