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Some Properties of Neutral Differential Systems Equations (*).

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Sunto. – *Nel lavoro vengono esaminate le caratteristiche oscillatoriche delle soluzioni dei sistemi di equazioni differenziali di tipo neutro.*

Summary. – *We study oscillatory properties of solutions of the system of differential equations of neutral type.*

In this paper we consider systems of neutral differential equations of the form:

$$
\begin{aligned}
&(x_1(t) - p_1 x_1(t-\tau_1))' = a_1(t)\, f_1(x_2(g_2(t))) \\
&(x_2(t) - p_2 x_2(t-\tau_2))' = a_2(t)\, f_2(x_1(g_1(t)))
\end{aligned}
\tag{1}
$$

The following conditions are assumed to hold without further mention:

(a) $p_i,\ \tau_i,\ i=1,2$ are positive numbers, $0 < p_i \leqslant 1,\ i=1,2$;

(b) $a_i \in C(R_+, R_+),\ i=1,2$ are not identically zero on any subinterval $[T, \infty) \subset [0, \infty)$ and

$$\int^{\infty} a_1(s)\, ds = \infty\,;$$

(c) $g_i \in C(R_+, R_+)$ is a nondecreasing function, $\lim_{t \to \infty} g_i(t) = \infty,\ i=1,2$ and $g_1(t) \leqslant t,\ g_2(t) + \tau_2 < t$;

(d) $f_i \in C(R, R)$ is a nondecreasing function, $f_i(u)\, u > 0$ for $u \neq 0$ and

$$\liminf_{u \to 0} \frac{f_i(u)}{u} > 0, \qquad i=1,2.$$

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For any $x_i(t)$ we define $u_i(t)$ by

$$u_i(t) = x_i(t) - p_i x_i(t - \tau_i), \qquad i = 1, 2. \tag{2}$$

For $t_0 \geqslant 0$ we define

$$t_1 = min\left\{t - \tau_i,\ \inf_{t \geqslant t_0} g_i(t),\ i = 1, 2\right\}.$$

A vector function $\boldsymbol{x} = (x_1, x_2)$ is defined to be a solution of the system (1) if there exists a $t_0 \geqslant 0$ such that $\boldsymbol{x}$ is continuous on $[t_1, \infty)$, $u_i(t)$, $i = 1, 2$ are continuously differentiable on $[t_0, \infty)$ and $\boldsymbol{x}$ satisfies system (1) on $[t_0, \infty)$.

As it is customery, we restrict our attention to those solution $\boldsymbol{x} = (x_1, x_2)$ of the system (1) which satisfy

$$\sup\{|x_1(t)| + |x_2(t)| : t \geqslant T\} > 0 \qquad \text{for any } T \geqslant 0$$

(the so-called proper solution). Such a solution is defined to be nonoscillatory if there exists a $T_0 \geqslant 0$ such that its every component is different from zero for all $t \geqslant T_0$. Otherwise a solution $\boldsymbol{x}$ is defined to be oscillatory.

Surveying the rapidly expanding literature devoted to the study of oscillatory and asymptotic properties of neutral differential equations, one finds that few papers concern systems of neutral equations (for example [4]). In this paper we give theorems on the relation between boundedness and oscillation of components of solutions for the system (1). These criteria extend those introduced in [5, 6].

We start with a preliminary analysis of the asymptotic behavior of possible nonoscillatory solution of (1).

Let $\boldsymbol{x} = (x_1, x_2)$ be a nonoscillatory solution of the system (1). Then from (1), $u_i(t)$,$i = 1, 2$ are eventually monotone, so that $u_i(t)$ have to be of constant sign. Therefore, either

$$x_i(t)\, u_i(t) > 0, \qquad i \in \{1, 2\} \tag{3}$$

or

$$x_i(t)\, u_i(t) < 0, \qquad i \in \{1, 2\} \tag{4}$$

for all sufficiently large t. Denote by N^+ [or N^-] the set of components of all nonoscillatory solutions (x_1, x_2) of system (1) such that (3) [or (4)] is satisfied.

LEMMA 1. – *Let $x_i \in N^+$, $i \in \{1, 2\}$. Then for every $T \geqslant t_0$ and every integer $n > 0$ there exists $T_n \geqslant T$ such that $t - n\tau_i \geqslant T$, $i = 1, 2$ and*

$$|x_i(t)| \geqslant \sum_{j=0}^{n} p_i^j\, |u_i(t - j\tau_i)|. \tag{5}$$

PROOF. – Using the relation

$$x_i(t) = u_i(t) + p_i x_i(t - \tau_i)$$

repeatedly, we find

$$|x_i(t)| = \sum_{j=0}^{n} p_i^j |u_i(t - j\tau_i)| + p_i^{n+1} |x_i(t - (n+1)\tau_i)| \geqslant \sum_{j=0}^{n} p_i^j |u_i(t - j\tau_i)|$$

for $t \geqslant T_n$ as claimed. ■

LEMMA 2. – *Let $x_i \in N^-$, $i \in \{1, 2\}$. Then for every $T \geqslant t_0$ and every integer $n > 0$ there exists $T_n \geqslant T$ such that*

$$|x_i(t)| \geqslant \sum_{j=1}^{n} \frac{|u_i(t + j\tau_i|)}{p_i^j}. \tag{6}$$

PROOF. – Using the relation

$$x_i(t) = \frac{x_i(t + \tau_i) - u_i(t + \tau_i)}{p_i}$$

repeatedly, we find

$$|x_i(t)| = \frac{|x_i(t + (n+1)\tau_i)|}{p_i^{n+1}} + \sum_{j=1}^{n} \frac{|u_i(t + j\tau_i)|}{p_i^j} \geqslant \sum_{j=1}^{n} \frac{|u_i(t + j\tau_i)|}{p_i^j}. \quad ■$$

LEMMA 3. – *Let $x_i \in N^-$, $i \in \{1, 2\}$ and $0 < p_i < 1$. Then $\lim_{t \to \infty} u_i(t) = 0$, $\lim_{t \to \infty} x_i(t) = 0$, $i \in \{1, 2\}$.*

PROOF. – Without loss of generality we assume that $x_i(t) > 0$, $u_i(t) < 0$ for $t \geqslant t_0$. Using (2) we obtain

$$x_i(t) < p_i x_i(t - \tau_i) < p_i^2 x_i(t - 2\tau_i) < \ldots < p_i^n x_i(t - n\tau_i)$$

for $t \geqslant t_0 + n\tau_i$, $i = 1, 2$, which implies that $\lim_{t \to \infty} x_i(t) = 0$. Consequently we get $\lim_{t \to \infty} u_i(t) = 0$. ■

THEOREM 1. – *Assume that (a)–(d) hold and $0 < p_i < 1$, $i = 1, 2$. Let there exist integer numbers $n \geqslant 0$, $m \geqslant 0$ such that*

$$\limsup_{t \to \infty} \int_{g_1(t)}^{t} a_2(s) \int_{g_1(t)}^{s} a_1(v)\, dv\, ds > \frac{1 - p_1}{1 - p_1^{n+1}} \frac{1 - p_2}{1 - p_2^{m+1}} \limsup_{u \to 0} \frac{u}{f_1(u)} \limsup_{v \to 0} \frac{v}{f_2(v)}. \tag{7}$$

Then every solution of (1) *with a bounded first component is oscillatory or* $x_i(t)$, $i=1,2$ *tend monotonically to zero as* $t\to\infty$.

PROOF. – Let $\boldsymbol{x}=(x_1, x_2)$ be a nonoscillatory solution of (1) with the bounded first component. Without loss of generality we may assume that $x_1(t)>0$, $x_1(g_1(t))>0$ for $t\geqslant T$.

1. Assume first that $x_1\in N^+$ for $t\geqslant T$ ($x_1>0, u_1>0$).

a) Let $x_2\in N^+$ and $x_2(t)>0$, $x_2(g_2(t))>0$ for $t\geqslant t_0\geqslant T$. From the second equation of (1) we get that $u_2(t)$ is an increasing function, therefore there exists $t_1\geqslant t_0$ such that $u_2(g_2(t))\geqslant c>0$ for $t\geqslant t_1$. Taking into account Lemma 1 (for $i=2$, $n=0$) we obtain $x_2(g_2(t))\geqslant u_2(g_2(t))>c$, $t\geqslant t_2\geqslant t_1$. Using (d) and integrating the first equation of (1) we have

$$u_1(t)-u_1(t_2)\geqslant f_1(c)\int_{t_2}^{t} a_1(s)\,ds. \tag{8}$$

From (4) and (b) for $t\to\infty$ we obtain $\lim_{t\to\infty} u_1(t)=\infty$ which contradicts the boundedness of x_1.

b) Let $x_2\in N^+$ and $x_2(t)<0$, $x_2(g_2(t))<0$ for $t\geqslant t_0\geqslant T$. From the second equation of (1) we have that $u_2(t)$ is an increasing function, $u_1(t)$ is the decreasing function and therefore exist $\lim_{t\to\infty} u_1(t)=c\geqslant 0$ and $\lim_{t\to\infty} u_2(t)=d\leqslant 0$. We shall show that $c=0$, $d=0$.

Let $d<0$. Then

$$\begin{gathered} x_2(g_2(t))=u_2(g_2(t))+p_2x_2(g_2(t)-\tau_2)<u_2(g_2(t))<d \\ f(x_2(g_2(t))\leqslant f_1(d)<0 \quad \text{for} \quad t\geqslant t_1\geqslant t_0. \end{gathered} \tag{9}$$

Integrating the first equation of (1) taking into account (b), (9) we get $\lim_{t\to\infty} u_1(t)=-\infty$, which contradicts $u_1(t)>0$ for $t\geqslant t_1$.

Let $c>0$. Then there exists a $t_1\geqslant t_0$ such that for $t\geqslant t_1$

$$x_1(t))=u_1(t)+p_1x_1(t-\tau_1)>u_1(t)>c>0$$

and in view of system (1) we have

$$u_2'(t)>f_2(c)\,a_2(t), \qquad t\geqslant t_2\geqslant t_1. \tag{10}$$

Multiplying (10) by $\overline{A}(t)=\int_{t_3}^{t} a_1(s)\,ds$ and integrating from t_3 to t we get

$$\overline{A}(t)\,u_2(t)-\sup_{t_2\leqslant s\leqslant t}\frac{u_2(s)}{f_1(u_2(s))}\cdot(u_1(t)-u_1(t_2))\geqslant f_2(c)\int_{t_2}^{t} a_2(s)\int_{t_2}^{s} a_1(v)\,dv\,ds. \tag{11}$$

We obtain from (11) for $t \to \infty$ that

$$\int_{t_3}^{\infty} a_2(s) \int_{t_3}^{s} a_1(v)\, dv\, ds < \infty ,$$

which implies that

$$\lim_{t \to \infty} \int_{g_1(t)}^{\infty} a_2(s) \int_{g_1(t)}^{s} a_1(v)\, dv\, ds = 0 .$$

This contradicts the assumption (7). Therefore $\lim\limits_{t \to \infty} u_1(t) = 0$.

Taking into account the monotonicity of u_1, u_2 and Lemma 1 we get

$$\begin{gathered} x_2(t) < l u_2(t), \qquad x_2(g_2(t)) \leqslant l u_2(g_2(t)) \\ x_1(g_1(t)) \geqslant k u_1(g_1(t)), \end{gathered} \tag{12}$$

for $t \geqslant T$ (T sufficiently large), where $l = \sum\limits_{i=0}^{m} p_2^i$, $k = \sum\limits_{i=0}^{m} p_1^i$. Integrating the first equation of (1) from t to s, $s \geqslant t$ and using monotonicity of f_1 and (12) we obtain

$$\begin{aligned} u_1(t) \geqslant u_1(s) - \inf_{t \leqslant v \leqslant s} \frac{f_1(l u_2(g_2(v)))}{l u_2 g_2(v)}\, l u_2(s) \int_t^s a_1(z)\, dz \\ + \inf_{t \leqslant v \leqslant s} \frac{f_1(l u_2(g_2(v)))}{l u_2(g_2(v))} \int_t^s l u_2'(z) \int_t^z a_1(v)\, dv\, dz \\ \geqslant \inf_{t \leqslant v \leqslant s} \frac{f_1(l u_2(g_2(v)))}{l u_2(g_2(v))} \int_t^s l a_2(z)\, f_2(x_1(g_1(z))) \int_t^z a_1(v)\, dv\, dz \\ \geqslant \inf_{t \leqslant v \leqslant s} \frac{f_1(l u_2(g_2(v)))}{l u_2(g_2(v))} f_2(k u_1(g_1(s)))\, l \int_t^s a_2(z) \int_t^z a_1(v)\, dv\, dz . \end{aligned}$$

Hence for $t = g_1(s) < s$ we get

$$\frac{1}{kl} \geqslant \inf_{g_1(s) < v < s} \frac{f_1(l u_2(g_2(v)))}{l u_2(g_2(v))} \frac{f_2(k u(g_1(s)))}{k u_1(g_2(s))} \int_{g_1(s)}^{s} a_2(z) \int_{g_1(s)}^{z} a_1(v)\, dv\, dz \tag{13}$$

which contradicts (7).

c) Let $x_2 \in N^-$, $x_2(t) < 0$. The system (1) implies that $u_2(t)$ is an increasing function and there exists $\lim\limits_{t \to \infty} u_2(t) = a > 0$, which is contrary Lemma 3.

d) Let $x_2 \in N^-$ and $x_2(t) > 0$ for $t \geqslant t_0$. In view of (c), Lemma 2 ($n = 1$) and monotonicity of u_2 we obtain

$$x_2(g_2(t)) > -\frac{1}{p_2} u_2(g_2(t) + \tau_2) > -u_2(g_2(t) + \tau_2) > -u_2(t), \tag{14}$$

$$\text{for } t \geqslant t_1 \geqslant t_0.$$

Integrating the first equation of (1) from t_1 to t we have

$$u_1(t) - u_1(t_1) \geqslant \inf_{t_1 \leqslant s \leqslant t} \frac{f_1(-u_2(s))}{(-u_2(s))} \int_{t_1}^{t} a_1(s)(-u_2(s))\, ds. \tag{15}$$

We see that $u_1 > 0$, $u_1' > 0$ for $t \geqslant t_1 \geqslant t_0$ so therefore $u_1(g_1(t)) \geqslant c > 0$ and $f_2(u_1(g_1(t))) \geqslant f_2(c)$. Using this fact and integrating the second equation of (1) from s to t, $s \geqslant t_1$ we get

$$-u_2(s) \geqslant u_2(t) - u_2(s) \geqslant f_2(c) \int_{s}^{t} a_2(z)\, dz. \tag{16}$$

Combining (16), (15) we have

$$u_1(t) - u_1(t_1) \geqslant \inf_{t_1 \leqslant s \leqslant t} \frac{f_1(-u_2(s))}{(-u_2(s))} f_2(c) \int_{t_1}^{t} a_2(z) \int_{t_1}^{z} a_1(s))\, ds\, dz.$$

We can proceed analogously as for the case b) the inequality (11) to get a contradiction.

2. Assume that $x_1 \in N^-$ for $t \geqslant t_0$. Then by Lemma 3 $\lim_{t \to \infty} x_1(t) = 0$, $\lim_{t \to \infty} u_1(t) = 0$.

a) Let $x_2 \in N^+$, $x_2(t) > 0$, $x_2(g_2(t)) > 0$ for $t \geqslant t_1 \geqslant t_0$. Then we can proceed the same way as for the case 1a) to get (8) which is contradicts the negativity of u_1.

b) Let $x_2(t) < 0$ and $x_2 \in N^+$ or $x_2 \in N^-$. From system (1) we have that u_1 is a decreasing function and therefore exists $\lim_{t \to \infty} u_1(t) = a < 0$, which is contrary to Lemma 3.

c) Let $x_2 \in N^-$, $x_2(t) > 0$. In this case by Lemma 3 $\lim_{t \to \infty} x_2(t) = 0$. The system (1) has a nonoscillatory solution with the property $\lim_{t \to \infty} x_1(t) = 0$, $\lim_{t \to \infty} x_2(t) = 0$. ■

LEMMA 4. – *Assume that* $p_i = 1$, $i \in \{1, 2\}$.

a) *Let* $x_i \in N^-$ *and* $x_i(t)\, u_i'(t) > 0$ *for* $t \geqslant t_0$, $i \in \{1, 2\}$. *Then* $\lim_{t\to\infty} u_i(t) = 0$, $i \in \{1, 2\}$.

b) *The system* (1) *has no solution with components* $x_i \in N^-$, $x_i(t)\, u_i'(t) < 0$, $i \in \{1, 2\}$.

PROOF. – a) We assume that $x_i(t) > 0$, $u_i(t) < 0$ and $u_i'(t) > 0$ for $t \geqslant t_0$. The proof in the case $x_i(t) < 0$ is analogous. Then exists $\lim_{t\to\infty} u_i(t) = c \leqslant 0$. We suppose that $c<0$. So there exists $t_1 \geqslant t_0$ such that $c < u_i(t) < c/2 < 0$. Thus using (2)

$$c < x_i(t) - x_i(t-\tau_i) < \frac{c}{2} \quad \text{for } t \geqslant t_1 .$$

Consequently

$$x_i(t) < x_i(t-\tau_i) + \frac{c}{2} < 2\frac{c}{2} + x_i(t-2\tau_i) < \ldots < n\frac{c}{2} + x_i(t-n\tau_i),$$

for $t \geqslant t_1 + n\tau_i$. Chose a sequence $\{t_n\}$ such that $t_n = t_1 + n\tau_i$. Then

$$x_i(t_1 + n\tau_i) \leqslant n\frac{c}{2} + x_i(t_1)$$

and therefore $\lim_{t\to\infty} x_i(t_n) = -\infty$. This is a contradiction with positivity of $x_i(t)$.

b) We suppose contrary. Let the system have a solution $x_1(t) > 0$, $x_1 \in N^-$ and $u_1'(t) < 0$ for $t \geqslant t_0$. The system (1) implies that $x_2(t) < 0$, $x_2 \in N^-$ and $u_2'(t) > 0$. Then there exist $\lim_{t\to\infty} u_1(t) = a < 0$, $\lim_{t\to\infty} u_2(t) = b > 0$ and there exists $t_1 \geqslant t_0$ such that $a < u_1(t) < a/2 < 0$, $0 < b/2 < u_2(t) < b$. Thus using (2)

$$a < x_1(t) - x_1(t-\tau_1) < \frac{a}{2} < 0$$

$$0 < \frac{b}{2} < x_2(t) - x_2(t-\tau_2) < b .$$

Consequently

$$\begin{aligned} &x_1(t) < n\frac{a}{2} + x_1(t-n\tau_1) \\ &x_2(t) > n\frac{b}{2} + x_2(t-n\tau_2). \end{aligned} \tag{17}$$

Chose sequences $\{t_n\}$, $\{t_n^*\}$ such that $t_n = t_1 + n\tau_1$, $t_n^* = t_1 + n\tau_2$ we get from the (17) that $x_1(t_n) \to -\infty$, $x_2(t_n^*) \to \infty$, $n \to \infty$, which is a contradiction with positivity of x_1 and negativity of x_2. ■

THEOREM 2. – *Assume that (a)–(d) hold and* $p_i = 1$, $i = 1, 2$. *Let there exist integer numbers* $n > 0$, $m > 0$ *such that*

$$\text{(18)} \quad \limsup_{t \to \infty} \int_{g_1(t)}^{t} a_2(s) \int_{g_1(t)}^{s} a_1(v)\, dv\, ds > \frac{1}{nm} \limsup_{u \to 0} \frac{u}{f_1(u)} \limsup_{v \to 0} \frac{v}{f_2(v)}\,.$$

Then every solution of (1) *with a bounded first component is either oscillatory or* $u_i(t)$, $i = 1, 2$ *tend monotonicaly to zero as* $t \to \infty$.

PROOF. – Assume that $\boldsymbol{x} = (x_1, x_2)$ is a nonoscillatory solution of (1) with the positive bounded first component. We can proceed exactly as in the proof of Theorem 1. We shall discuss the possibilities 1a-d, 2a-c.

The proofs of cases 1a), 1d), 2a) are the same. In the case 1b) we are led to (12) with the constant $k = n$, $l = m$ and so to (13) with these new constants, which contradicts the assumption.

Lemma 4 implies that the cases 1c), 2b) are impossible. In the case 2c) from Lemma 4 we see that that system (1) has a nonoscillatory solution with the property $\lim_{t \to \infty} u_i(t) = 0$, $i = 1, 2$. ■

THEOREM 3. – *Assume that (a)–(d) hold and* $0 < p_i < 1$, $i = 1, 2$. *Further assume that*

$$\text{(19)} \quad \limsup_{t \to \infty} \int_{g_1(t)}^{t} a_2(s) \int_{g_1(t)}^{s} a_1(v)\, dv\, ds > (1 - p_1)(1 - p_2) \limsup_{u \to 0} \frac{u}{f_1(u)} \limsup_{v \to 0} \frac{v}{f_2(v)}\,.$$

Then every solution of (1) *with a bounded first component is either oscillatory or* $x_i(t)$,, $i = 1, 2$ *tend monotonicaly to zero for* $t \to \infty$.

PROOF. – Denote $a = \limsup_{t \to \infty} \int_{g_1(t)}^{t} a_2(s) \int_{g_1(t)}^{s} a_1(v)\, dv\, ds$. Let integers n, m be chosen such that

$$a > \frac{1 - p_1}{1 - p_1^{n+1}} \frac{1 - p_2}{1 - p_2^{m+1}} \limsup_{u \to 0} \frac{u}{f_1(u)} \limsup_{v \to 0} \frac{v}{f_2(v)}\,.$$

Then the assertion of this theorem follows immediately from Theorem 1. ■

THEOREM 4. – *Assume that (a)–(d), hold and* $p_i = 1$, $i = 1, 2$. *Further assume that*

$$\limsup \int_{g_1(t)}^{t} a_2(s) \int_{g_1(t)}^{s} a_1(v)\, dv\, ds > 0.$$

Thus every solution of (1) *with a bounded first component is either oscillatory or* $u_i(t)$, $i = 1, 2$ *tend monotonicaly to zero as* $t \to \infty$.

REMARK 1. – For the case $p_i = 0$, $i = 1, 2$ in the paper [6] and for the case $p_2 = 0$ in the paper [5] it is shown that condition (19) is sufficiently for the oscillation all solutions with a bounded first component.

REMARK 2. – If $a_1(t) = 1, f_i(u) = u$, $i = 1, 2$, $p_2 = 0$ the system (1) is equivalent to the differential equation

$$(x(t)) - px(t - \tau))' - a(t)\, x(g(t)) = 0.$$

Theorem 1,2 give results for this equation in paper [2].

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