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$C^{1,\beta}$-Partial Regularity of p-Harmonic Maps at the Free Boundary.

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Sunto. – *Dimostriamo la $C^{1,\beta}$-regolarità parziale fino alla frontiera libera delle mappe p-armoniche che minimizzano la p-energia $\int |Du|^p dx$.*

Summary. – *We prove the partial $C^{1,\beta}$-regolarity up to the free boundary of the p-harmonic maps which minimize the p-energy $\int |Du|^p dx$.*

1. – Introduction.

Let B be the open unit ball in $\mathbb{R}^n$ and $S \subset \mathbb{R}^N$ a $(k-1)$-dimensional submanifold of $\mathbb{R}^N$ without boundary of class $C^{1,\sigma}$, $0 < \sigma < 1$, $1 \leq k \leq N+1$. Assume $u \in H^{1,p}(B, \mathbb{R}^N)$ is a minimizer for the functional

$$F(u) = \int_B \left(g_{ij} \circ u \ \gamma^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} \right)^{p/2} dx\,, \qquad \text{with } p > 2 \tag{1}$$

in the class of all maps $v \in H^{1,p}(B, \mathbb{R}^N)$ with $v(\partial B) \subset S$, (g_{ij}) and $(\gamma_{\alpha\beta})$ are uniformly elliptic, g is in $C^{0,\sigma}(\mathbb{R}^N)$ and bounded, and γ is in $C^{0,\sigma}(\overline{B})$. Generally we call a map u which is a critical point of F p-harmonic. Note that this definition is slightly different from the usual one, but it agrees in the case $g = id$, $\gamma = id$. In Theorem 8.2 we show that $u \in C^{1,\beta}_{loc}(\Omega_0)$ for some relatively open subset $\Omega_0 \subset \overline{B}$ and some $0 < \beta < \sigma$, if we assume that S is a linear subspace V. We can assume this if we can localize in the image, i.e. for some $q' > p$, for $\mathcal{H}^{n-q'}$-almost all points $x \in \partial B$ there exists a neighborhood $U(x)$ such that $u(U(x))$ lies completely in a N-dimensional coordinate neighborhood of S. Furthermore we show that $\mathcal{H}^{n-q}(\overline{B} \setminus \Omega_0) = 0$ for some $q' > q > p$. In particular we therefore show that the first derivatives of u are Hölder continuous in $\mathcal{H}^{n-q}$-almost all boundary points $x \in \partial B$. In the special case $p = n$ we therefore get everywhere regularity. Partial $C^{1,\beta}$-regularity in the interior has been proven independently by FUSCO&HUTCHINSON in [3] and GIAQUINTA&MODICA in [5]. These

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authors use a result of UHLENBECK [8] and get partial regularity for the first derivatives of minima of (1). Crucial for the methods in [3] is an estimate from [8] for solutions of

$$\operatorname{div}(|Du|^{p-2}Du) = 0\,. \tag{2}$$

This estimate says for $B_{2r} \subset \Omega$ and $x, y \in B_r$:

$$|Du(x) - Du(y)| < c \cdot \sup_{B_{2r}} |Du| \frac{|x-y|^{\alpha}}{r^{\alpha}}\,. \tag{3}$$

Now we describe the methods we use.

Since all estimates are local, it is enough to prove estimates on $B^+ = \{x \in B : x_n > 0\}$. Furthermore we assume that we can localize in the image, and therefore we can assume that S is a $(k-1)$-dimensional subspace of $\mathbb{R}^N$. By reflection we get an estimate similar to (3) at the free boundary. Therefore, we are able to adapt the technique of FUSCO&HUTCHINSON to the situation at the free boundary: first we derive a Caccioppoli inequality at the free boundary for solutions of (2). This Caccioppoli inequality leads to the study of a suitable decay function, similarly as in FUSCO&HUTCHINSON. More precisely, by comparison with the solution of a suitable linear free boundary value problem, we get a decay relation at the free boundary for

$$\varphi(r) = |\xi_r|^{p-2} \fint_{B_r^+} |Du - \xi_r|^2\, dx + \fint_{B_r^+} |Du - \xi_r|^p\, dx\,,$$

where ξ_r is the matrix which arises from $Du_r := \fint_{B_r^+} Du dx$ by projecting the tangential derivatives onto V and by projecting the normal derivative onto $V^{\perp}$.

Using only the estimates (3), we get by a local comparison principle (cp. [4] and [3]) partial α-Hölder continuity at the free boundary for all $\alpha < 1$ and a decay estimate for

$$r^p \fint_{B_r^+} |Du|^p\, dx\,.$$

Again by a comparison principle we finally get partial Hölder continuity of the first derivatives at the free boundary. Since we are arguing with the functional F and not with the Euler equations, we can only treat minimizers (cp. [3]).

In the special cases $k = 1$ and $k = N + 1$ the above boundary value problem (1) reduces to a Dirichlet problem with zero boundary values and the Neumann problem resp. Among other things these cases have already been treated by HAMBURGER in [7]. However, in the case of the general free boundary value problem as above, the choice of test functions is restricted further, so we need different methods. In [2], DUZAAR&GASTEL show partial Hölder continuity up to the free boundary for p-harmonic maps between manifolds, which are solutions to a free boundary value problem as above and which are locally minimizing. This result justifies our assumption that S is a linear subspace.

Our results are of interest especially in the case $p = n$, if we look at the geometric free boundary value problem for n-harmonic maps. For example, if we try to find «nice» maps between $B \subset \mathbb{R}^N$ and suitable domains in $\mathbb{R}^N$ by minimizing the energy, we would get regularity up to the boundary. Secondly, it is a natural continuation of the results of DUZAAR&GASTEL.

We make the following general agreements about the notation we use: c is always a universal constant which depends only on the data and which may change from line to line. Dependence on any other quantities will be noted explicitly.

The Greek indices $\alpha, \beta, \gamma, \ldots$ have values in $\{1, \ldots, n\}$, the Latin indices $i, j, k, \ldots$ have values in $\{1, \ldots, N\}$. We use the convention of summation. Finally ω_n is the volume of the n-dimensional unit ball.

I would like to thank Professor Kuwert for calling my attention to this problem, and for continual support and various helpful ideas.

2. – Linear free boundary value problems.

In this section we will prove some fundamental properties of linear free boundary value problems. First we give some Poincaré inequalities, which we will use later. Always let $p > 1$.

Notation:

$$\begin{aligned}
V &:= \{x \in \mathbb{R}^N : x_k = \ldots = x_N = 0\}, \\
B_r &:= \{x \in \mathbb{R}^n : \|x\| < r\}, \\
A^+ &:= \{x \in A : x_n > 0\} \qquad \text{for } A \subset\subset \mathbb{R}^n, \\
B &:= B_1, \\
I_r &:= B_r \cap \{x_n = 0\}, \\
I &:= I_1 .
\end{aligned}$$

Furthermore let

$$u_{x_0, r} := \fint_{B_r^+(x_0)} u\,dx ,$$

$$u_{x_0, r}^{\top} := pr_V(u_{x_0, r}) ,$$

$$u_{x_0, r}^{\perp} := pr_{V^\perp}(u_{x_0, r}) ,$$

where pr_V and $pr_{V^\perp}$ are the orthogonal projections onto V and $V^\perp$ respectively.

With Theorem 5.15 from [1], we get

LEMMA 2.1. – *Let $u \in H^{1,p}(B_R^+, \mathbb{R}^N)$ with $u(I_R) \subset V$. Then we have*

$$R^{-p} \fint_{B_R^+} |u(x) - u_{0,R}^\top|^p \, dx \leqslant c \fint_{B_R^+} |Du|^p \, dx . \qquad \blacksquare$$

Let

$$H := \{\varphi|_{B^+} : \varphi \in H_0^{1,2}(B, \mathbb{R}^N),\ \varphi(I) \subset V\} .$$

Furthermore let $u \in H^{1,2}(B^+)$ be a solution of the free boundary problem

$$\int_{B^+} A_{ij}^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial \varphi^j}{\partial x^\beta} dx = 0 \qquad \forall \varphi \in H \tag{4}$$

$$u(I) \subset V , \tag{5}$$

where $A_{ij}^{\alpha\beta}$ is elliptic in the sense that $\sum_{\alpha, \beta, i, j} A_{ij}^{\alpha\beta} \xi_i^\alpha \xi_j^\beta \geqslant \lambda \|\xi\|^2$. Furthermore, $|A_{ij}^{\alpha\beta}| \leqslant M$.

A standard difference-quotient method adapted to the free boundary situation gives

LEMMA 2.2. – *Let u be a solution of* (4) *and* (5), $x_0 \in I$, $B_{2r}(x_0) \subset\subset B$. *Then there exists a $c = c(\lambda, M, n, N, k, r)$ such that the estimate*

$$\|D^k u\|_{L^2(B_r^+)} \leqslant c \|Du\|_{L^2(B_{2r}^+)}$$

holds. $\blacksquare$

LEMMA 2.3. – *Let u be a solution of* (4) *and* (5) *and $\varphi \in H$ with $A = \operatorname{supp}(\varphi) \subset\subset B$. Then the following natural boundary condition holds:*

$$\int_I A_{ij}^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \nu_\beta \varphi^j dx = -\int_I A_{ij}^{\alpha n} \frac{\partial u^i}{\partial x^\alpha} \varphi^j dx = 0 .$$

Here ν is the exterior unit normal vector to B^+.

PROOF. – By Lemma 2.2 and a covering argument we see that $u \in H^{2,2}(A^+)$. Therefore we can integrate by parts:

$$\int_{B^+} A_{ij}^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial \varphi^j}{\partial x^\beta} dx = -\int_{B^+} A_{ij}^{\alpha\beta} \frac{\partial^2 u^i}{\partial x^\alpha \partial x^\beta} \varphi^j dx$$

$$+ \int_I A_{ij}^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \nu_\beta \varphi^j dx .$$

Because of the equation, the first term on the right hand side vanishes. ■

LEMMA 2.4. – *Let u be a solution of* (4) *and* (5), $x_0 \in I$, $B_{2r}(x_0) \subset\subset B$ *und* $v \in V$. *Then there exists a constant* $c = c(\lambda, M)$ *such that the Caccioppoli inequality*

$$\int_{B_r^+} |Du|^2 \, dx \leqslant \frac{c}{r^2} \int_{B_{2r}^+} |u-v|^2 \, dx$$

holds.

PROOF. – We use in (4) the test function $\varphi := (u-v)\,\eta^2$, where $\operatorname{supp}(\eta) \subset B_{2r}(x_0)$, $\eta \equiv 1$ on $B_r(x_0)$. We get

$$0 = \int_{B^+} A_{ij}^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} \eta^2 \, dx + 2 \int_{B^+} A_{ij}^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} (u-v)^j \frac{\partial \eta}{\partial x^\beta} \eta dx \, ,$$

and therefore

$$\lambda \int_{B^+} \eta^2 \, |Du|^2 \, dx \leqslant c \int_{B^+} |D\eta|^2 \, |u-v|^2 \, dx \, . \qquad \blacksquare$$

Using Lemma 2.2 and arguing similarly as in the proof of Theorem 2.1 in [4] we get

LEMMA 2.5. – *Let u be a solution of* (4) *and* (5), $x_0 \in I$, $B_r(x_0) \subset\subset B$. *Then there exists a constant c which depends only on the data, such that the estimate*

$$\fint_{B_\varrho^+(x_0)} |Du|^2 \, dx \leqslant c \fint_{B_r^+(x_0)} |Du|^2 \, dx \qquad \forall \varrho \leqslant r$$

holds. ■

Using Caccioppoli inequality (2.4) and Poincaré inequality we get

LEMMA 2.6. – *Let u be a solution of* (4) *and* (5), $x_0 \in I$, $B_r(x_0) \subset\subset B$. *Then there exists a constant c which depends only on the data, such that*

$$\fint_{B_\varrho^+(x_0)} |u - u_{x_0,\varrho}^{\top}|^2 \, dx \leqslant c \left(\frac{\varrho}{r} \right)^2 \fint_{B_r^+(x_0)} |u - u_{x_0,r}^{\top}|^2 \, dx \qquad \forall \varrho \leqslant r \, .$$

PROOF. – First let $\varrho \leqslant r/2$. Then

$$\begin{aligned}\varrho^{-2} \fint_{B_\varrho^+(x_0)} |u - u_{x_0,\varrho}^\top|^2\,dx &\leqslant c \fint_{B_\varrho^+(x_0)} |Du|^2\,dx \\ &\leqslant c\left(\frac{\varrho}{r}\right)^n \fint_{B_{r/2}^+(x_0)} |Du|^2\,dx \\ &\leqslant c\left(\frac{\varrho}{r}\right)^n r^{-2} \fint_{B_r^+(x_0)} |u - u_{x_0,r}^\top|^2\,dx\,.\end{aligned}$$

The first inequality follows from Lemma 2.1, the second from Lemma 2.5, and the third one from lemma 2.4 if we set $v = u_{x_0,r}^\top$. This proves the claim for $\varrho \leqslant r/2$. The claim for $\varrho > r/2$ follows trivially. ■

All tangential derivatives, i.e. $\dfrac{\partial u}{\partial x^\alpha}$, $\alpha < n$, also solve the free boundary value problem (4) and (5). Therefore we infer immediately

COROLLARY 2.1. – *Let u be a solution of* (4) *and* (5), $x_0 \in I$, $B_r(x_0) \subset\subset B$. *Then we have for all $\varrho \leqslant r$ with a constant c which depends only on the data*

$$\fint_{B_\varrho^+(x_0)} \sum_{\alpha<n} \left|\frac{\partial u}{\partial x^\alpha} - \left(\frac{\partial u}{\partial x^\alpha}\right)_{x_0,\varrho}^\top\right|^2 dx \leqslant c\left(\frac{\varrho}{r}\right)^2 \fint_{B_r^+(x_0)} \sum_{\alpha<n} \left|\frac{\partial u}{\partial x^\alpha} - \left(\frac{\partial u}{\partial x^\alpha}\right)_{x_0,r}^\top\right|^2 dx\,. \quad ■$$

In the following, we consider only $A_{ij}^{\alpha\beta}$ with a special structure, namely

$$A_{ij}^{\alpha\beta} = \xi_\alpha^i \xi_\beta^j + id \equiv B_{ij}^{\alpha\beta},$$

and

$$\xi_n \in V^\perp\,.$$

A short calculation together with Lemma 2.3 shows

LEMMA 2.7. – *Let u be a solution of* (4) *and* (5), *and $\varphi \in H$ with* $\operatorname{supp}(\varphi) \subset\subset B$. *Then*

$$\fint_I \left\langle \frac{\partial u}{\partial x^n}, \varphi \right\rangle dx = 0\,. \quad ■$$

Therefore we can show:

THEOREM 2.1. – *Let u be a solution of* (4) *and* (5) *with $A_{ij}^{\alpha\beta} = B_{ij}^{\alpha\beta}$, and $x_0 \in I$, $B_r(x_0) \subset\subset B$. Then for all $\varrho \leqslant r$*

$$\fint_{B_\varrho^+(x_0)} \sum_{\alpha<n} \left| \frac{\partial u}{\partial x^\alpha} - \left(\frac{\partial u}{\partial x^\alpha} \right)_{x_0,\varrho}^{\top} \right|^2 + \left| \frac{\partial u}{\partial x^n} - \left(\frac{\partial u}{\partial x^n} \right)_{x_0,\varrho}^{\perp} \right|^2 dx \leqslant$$

$$c\left(\frac{\varrho}{r}\right)^2 \fint_{B_r^+(x_0)} \sum_{\alpha<n} \left| \frac{\partial u}{\partial x^\alpha} - \left(\frac{\partial u}{\partial x^\alpha} \right)_{x_0,r}^{\top} \right|^2 + \left| \frac{\partial u}{\partial x^n} - \left(\frac{\partial u}{\partial x^n} \right)_{x_0,r}^{\perp} \right|^2 dx\,,$$

with a constant c which depends only on the data.

PROOF. – First let $\varrho \leqslant r/2$. We use Lemma 2.1 and get

$$\varrho^{-2} \int_{B_\varrho^+(x_0)} \left| \frac{\partial u}{\partial x^n} - \left(\frac{\partial u}{\partial x^n} \right)_{x_0,\varrho}^{\perp} \right|^2 dx \leqslant c \int_{B_\varrho^+(x_0)} \left| \frac{\partial}{\partial x^n} Du \right|^2 dx$$

$$= c \int_{B_\varrho^+(x_0)} \left| \frac{\partial}{\partial x^n} \frac{\partial u}{\partial x^n} \right|^2 + \sum_{\alpha<n} \left| \frac{\partial}{\partial x^n} \frac{\partial u}{\partial x^\alpha} \right|^2 dx\,.$$

Since $\dfrac{\partial u}{\partial x^\alpha}$, for $\alpha < n$, solves (4) and (5) we get with Lemma 2.5 and the Caccioppoli inequality Lemma 2.4

$$\int_{B_\varrho^+(x_0)} \sum_{\alpha<n} \left| \frac{\partial}{\partial x^n} \frac{\partial u}{\partial x^\alpha} \right|^2 dx \leqslant c\left(\frac{\varrho}{r}\right)^n \int_{B_{r/2}^+(x_0)} \sum_{\alpha<n} \left| D\frac{\partial u}{\partial x^\alpha} \right|^2 dx$$

$$\leqslant c\left(\frac{\varrho}{r}\right)^n r^{-2} \int_{B_r^+(x_0)} \sum_{\alpha<n} \left| \frac{\partial u}{\partial x^\alpha} - \left(\frac{\partial u}{\partial x^\alpha} \right)_{x_0,r}^{\top} \right|^2 dx\,.$$

Using the equality it follows for all $\varrho \leqslant r/2$

$$\varrho^{-2} \int_{B_\varrho^+(x_0)} \left| \frac{\partial u}{\partial x^n} - \left(\frac{\partial u}{\partial x^n} \right)_{x_0,\varrho}^{\perp} \right|^2 dx \leqslant c\left(\frac{\varrho}{r}\right)^n r^{-2} \int_{B_r^+(x_0)} \sum_{\alpha<n} \left| \frac{\partial u}{\partial x^\alpha} - \left(\frac{\partial u}{\partial x^\alpha} \right)_{x_0,r}^{\top} \right|^2 dx\,.$$

Together with Corollary 2.1 this shows the assertion for $\varrho \leqslant r/2$. For $\varrho \geqslant r/2$ the assertion follows trivially. ■

3. – Reflection of the Solution.

For $u \in H^{1,p}(\Omega)$, $2 < p \leqslant n$ we consider the functional

$$F^0(u,\Omega) := \int_\Omega \left(\sum_{i,\alpha} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^i}{\partial x^\alpha} \right)^{p/2} dx\,.$$

The Euler equations of F^0 are

$$0 = \int_\Omega \sum_{i,\alpha} |Du|^{p-2} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial \varphi^i}{\partial x^\alpha} dx \qquad \forall \varphi \in H_0^{1,p}(\Omega).$$

Now we consider the following free boundary problem for F^0:

Let

$$H_V := \{\varphi|_{B^+} : \varphi \in H_0^{1,p}(B), \varphi(I) \subset V\}$$

and let u be a solution of

$$\int_{B^+} \sum_{i,\alpha} |Du|^{p-2} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial \varphi^i}{\partial x^\alpha} dx = 0 \qquad \forall \varphi \in H_V \tag{6}$$

$$u(I) \subset V. \tag{7}$$

We want to extend this solution by reflection to a solution of

$$\int_B \sum_{i,\alpha} |Du|^{p-2} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial \varphi^i}{\partial x^\alpha} dx = 0 \qquad \forall \varphi \in H_0^{1,p}(\Omega).$$

We define the reflection $\sigma_V : \mathbb{R}^N \to \mathbb{R}^N$ at V by

$$\sigma_V := 2 \cdot \sum_{j=1}^{k-1} \langle z, e_j \rangle e_j - z .$$

Arguing similarly as in the proof of Lemma 9.12 in [6], we can show

LEMMA 3.1. – *Let $u \in H^{1,p}(B^+)$ be a solution of* (6) *and* (7). *Then*

$$U(\bar{x}, x_n) := \begin{cases} u(\bar{x}, x_n), & x_n \geqslant 0, (\bar{x}, x_n) \in B \\ \sigma_V u(\bar{x}, -x_n), & x_n \leqslant 0, (\bar{x}, x_n) \in B . \end{cases}$$

solves the equation

$$\int_B \sum_{i,\alpha} |DU|^{p-2} \frac{\partial U^i}{\partial x^\alpha} \frac{\partial \varphi^i}{\partial x^\alpha} dx = 0 \qquad \forall \varphi \in H_0^{1,p}(B). \qquad \blacksquare$$

Now let $g \in \mathbb{R}^{N \times N}$ and $\gamma \in \mathbb{R}^{n \times n}$ symmetric matrices such that for some $0 < \Lambda \in \mathbb{R}$

$$|\xi|^2 \leqslant g_{ij} \xi^i \xi^j \leqslant \Lambda |\xi|^2 \qquad \forall \xi \in \mathbb{R}^N, \tag{8}$$

$$|\eta|^2 \leqslant \gamma^{\alpha\beta} \eta_\alpha \eta_\beta \leqslant \Lambda |\eta|^2 \qquad \forall \eta \in \mathbb{R}^n. \tag{9}$$

Then we look at the free boundary value problem

$$\int_{B^+} \sum_{i,\alpha} (|Du|^2_{g,\gamma})^{p/2-1} g_{ij} \gamma^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial \varphi^j}{\partial x^\beta} dx = 0 \qquad \forall \varphi \in H_V$$

$$u(I) \subset V .$$

Here for $\xi \in \mathbb{R}^{nN}$ we write $|\xi|^2_{g,\gamma} = \xi^i_\alpha \xi^j_\beta \gamma^{\alpha\beta} g_{ij}$. Moreover let $G := (g_{ij})$ and $\Gamma := (\gamma_{\alpha\beta})$.

We define

$$\tilde{u} := \sqrt{G} u ,$$

$$\tilde{\varphi} := \sqrt{G} \varphi ,$$

$$\tilde{V} := \sqrt{G} V .$$

$\tilde{V}$ is a $(k-1)$-dimensional subspace of $\mathbb{R}^N$ and $\tilde{u}(I) = \sqrt{G} u(I) \subset \tilde{V}$. Let O be an orthogonal transformation with $O\tilde{V} = V$.

We define

$$P_g := (O \circ G^{1/2}) , \tag{10}$$

$$v := O\tilde{u} ;$$

Finally let

$$w(y) := v(\Gamma^{-1/2} y) \tag{11}$$

on $\tilde{B}^+ := \Gamma^{1/2} B^+$ and $\tilde{I} := \Gamma^{1/2} I$.

Then w solves the free boundary value problem

$$\fint_{\tilde{B}^+} \sum_{i,\alpha} (|Dw|^2)^{p/2-1} \delta_{ij} \delta^{\alpha\beta} \frac{\partial w^i}{\partial x^\alpha} \frac{\partial \varphi^j}{\partial x^\beta} dx = 0 \qquad \forall \varphi \in \tilde{H}_V$$

$$w(\tilde{I}) \subset V ,$$

with $\tilde{H}_V := \{\varphi|_{\tilde{B}^+} : \varphi(\tilde{I}) \subset V,\ \varphi \in H^{1,p}_0(\tilde{B}, \mathbb{R}^N)\}$.

Now we can apply Lemma 3.1 together with Theorem 2.1 from [3]. We conclude

LEMMA 3.2. – *Let w as in* (11). *Then for all $B_r(x_0) \subset\subset \widetilde{B}$, $x_0 \in I$ we have*

$$(12)\qquad \begin{aligned} &(i)\ \sup_{B^+_{r/2}} |Dw|^p \leqslant c \fint_{B_r^+} |Dw|^p\,dx\,, \\ &(ii)\ |Dw(x) - Dw(y)| \leqslant c \cdot \sup_{B_r^+} |Dw| \frac{|x-y|^\gamma}{r^\gamma} \qquad \forall x, y \in B^+_{r/2}, \end{aligned}$$

where $c = c(n, N, p)$ and $\gamma = \gamma(n, N, p)$. ■

4. – Higher integrability.

In this section we prove higher integrability for solutions of

$$(13)\qquad \int_{B^+} \sum_{i,\alpha} |Du|^{p-2} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial \varphi^i}{\partial x^\alpha}\,dx = 0 \qquad \forall \varphi \in H_V$$

$$(14)\qquad u(I) \subset V\,.$$

Actually the following proof of Theorem 4.1 contains also the proof of a Caccioppoli type estimate. But we need only higher integrability, so we don't note explicitly this Caccioppoli inequality.

We need some further definitions:

Let

$$\zeta^u_R := \fint_{B_R^+} Du\,dx$$

and ξ^u_R be the matrix

$$(\xi^u_\alpha)_R = (\zeta^u_\alpha)^\top_R \qquad \text{for } \alpha < n\,,$$

$$(\xi^u_n)_R = (\zeta^u_n)^\perp_R\,.$$

Generally let ξ be a matrix with

$$(\xi_\alpha) \in V \qquad \text{for } \alpha < n\,,$$

$$(\xi_n) \in V^\perp\,.$$

A calculation then shows

LEMMA 4.1. – *If* $B_R(x_0) \subset\subset B$, $x_0 \in I$ *and* $\varphi \in H_V$ *with* $\operatorname{supp}(\varphi) \subset B_R$ *then, with* $\xi = (\xi)_R^u$ *from above, it holds*

$$\int_{B_R^+} |\xi|^{p-2} \xi D\varphi \, dx = 0 . \qquad \blacksquare$$

THEOREM 4.1. – *Let* $B_R = B_R(x_0) \subset\subset B$, $x_0 \in I$, u *a solution of* (13) *and* (14) *and* $r < R$. *Then there exist constants* $c = c(n, N, p)$ *and* $\kappa = \kappa(n, N, p) > 0$ *such that, with* ξ *as above, the estimate*

$$\fint_{B_{R/2}^+} (|\xi|^{p-2}|Du-\xi|^2+|Du-\xi|^p)^{1+\kappa} dx \leq c \left(\fint_{B_R^+} |\xi|^{p-2}|Du-\xi|^2+|Du-\xi|^p dx \right)^{1+\kappa}$$

holds.

PROOF. – Using $u - a - \xi(x - x_0)$ (with $a \in V$) as a test function, and using Lemma 4.1 and Lemma 2.1, by a similar calculation as in Lemma 3.1 in [3] we get

$$\text{(15)} \qquad \fint_{B_r^+} |\xi|^{p-2} |Du - \xi|^2 + |Du - \xi|^p \, dx \leq$$

$$c \left(\fint_{B_R^+} (|\xi|^{p-2} |Du - \xi|^2 + |Du - \xi|^p)^{\frac{n}{n+2}} dx \right)^{\frac{n+2}{n}} .$$

Now let

$$g := (|\xi|^{p-2} |Du - \xi|^2 + |Du - \xi|^p)^{\frac{n}{n+2}} .$$

Then it follows because of (15)

$$\text{(16)} \qquad \fint_{B_{R/2}^+} g^{\frac{n+2}{n}} dx \leq c \left(\fint_{B_R^+} g dx \right)^{\frac{n+2}{n}} .$$

Now define

$$\bar{g}(x) = \begin{cases} g(x), & \text{if } x \in B^+ \\ 0, & \text{if } x \notin B^+ . \end{cases}$$

Let $x \in B$ be given and $R < \operatorname{dist}(x, \partial B)$. We distinguish three cases:

Case 1: $B_R(x) \cap B^+ = \emptyset$.

In this case $\bar{g} = 0$ on $B_R(x)$.

Case 2: $B_R(x) \subset B^+$.

We can apply Lemma 3.1 from [3]:

$$\fint_{B_{R/2}(x)} \overline{g}^{\frac{n+2}{n}}\, dx \leqslant c \Bigg(\fint_{B_R(x)} \overline{g}\, dx \Bigg)^{\frac{n+2}{n}}.$$

Case 3: $B_R(x) \cap (\mathbb{R}^n \setminus B^+) \neq \emptyset$.

(I) If $|x_n| < R/4$, then $B_{R/8}(x) \subset B_{|x_n| + R/4}((\overline{x}, 0)) \subset B_{(3/4)R}((\overline{x}, 0)) \subset B_R(x)$. Therefore, we get

$$\begin{aligned} \fint_{B_{R/8}(x) \cap B^+} \overline{g}^{\frac{n+2}{n}}\, dx &\leqslant c \fint_{B_{|x_n| + R/4}((\overline{x}, 0)) \cap B^+} \overline{g}^{\frac{n+2}{n}}\, dx \\ &\leqslant c \Bigg(\fint_{B_{(3/4)R}((\overline{x}, 0)) \cap B^+} \overline{g}\, dx \Bigg)^{\frac{n+2}{n}} \quad \text{by (16)} \\ &\leqslant c \Bigg(\fint_{B_R(x) \cap B^+} \overline{g}\, dx \Bigg)^{\frac{n+2}{n}}. \end{aligned}$$

(II) If $|x_n| \geqslant R/4$, it follows from Lemma 3.1 in [3] that

$$\begin{aligned} \fint_{B_{R/8}(x)} \overline{g}^{\frac{n+2}{n}}\, dx &\leqslant c \Bigg(\fint_{B_{|x_n|}(x)} \overline{g}\, dx \Bigg)^{\frac{n+2}{n}} \\ &\leqslant c \Bigg(\fint_{B_R(x)} \overline{g}\, dx \Bigg)^{\frac{n+2}{n}}, \end{aligned}$$

since $|x_n| \leqslant R$, $(x = (\overline{x}, x_n))$.

In each case we therefore get

$$\fint_{B_{R/8}(x) \cap B^+} \overline{g}^{\frac{n+2}{n}}\, dx \leqslant c \Bigg(\fint_{B_R(x) \cap B^+} \overline{g}\, dx \Bigg)^{\frac{n+2}{n}}.$$

A covering argument shows that actually

$$\fint_{B_{R/2}(x) \cap B^+} \overline{g}^{\frac{n+2}{n}}\, dx \leqslant c \Bigg(\fint_{B_R(x) \cap B^+} \overline{g}\, dx \Bigg)^{\frac{n+2}{n}}.$$

With Proposition V.1.1 in [4], we get for all $q = \dfrac{n+2}{n} + \varepsilon$ with $\varepsilon \leqslant \varepsilon_0$ and ε_0 sufficiently small

$$\left(\fint_{B^+_{R/2}} \bar{g}^{\,q}\, dx \right)^{\frac{1}{q}} \leqslant c \left(\fint_{B^+_R} \bar{g}^{\,\frac{n+2}{n}}\, dx \right)^{\frac{n}{n+2}}.$$

If we set $\kappa = \dfrac{n\varepsilon}{n+2}$ the assertion follows. ■

5. – Decay estimate.

Similarly as in [3], we will derive a decay estimate for

$$\psi(x_0, r) := |\xi^u_r|^{p-2} \fint_{B^+_r} |Du - \xi^u_r|^2\, dx + \fint_{B^+_r} |Du - \xi^u_r|^p\, dx\,,$$

where $x_0 \in I$ and $B_r = B_r(x_0) \subset\subset B$. More precisely, we show

LEMMA 5.1. – *Let $B_R = B_R(x_0) \subset\subset B$ and $0 < \tau < 1/2$. If u is a solution of* (13) *and* (14), *then there exists a constant $c = c(n, N, p)$ with*

$$(i)\ \ \psi(R) > \tau^{\frac{n+2}{\kappa}} |\xi^u_R|^p \ \Rightarrow\ \psi(\tau^k R) \leqslant c(\tau^k)^{2\gamma - \frac{n+2}{\kappa}}\, \psi(R)\,,$$

$$(ii)\ \ \psi(R) \leqslant \tau^{\frac{n+2}{\kappa}} |\xi^u_R|^p \ \Rightarrow\ \psi(\tau R) \leqslant c\tau^2\, \psi(R)\,.$$

PROOF. – We first prove (i):
For $\tau < 1/2$ we have

$$\begin{aligned}\psi(\tau^k R) &\leqslant |\xi^u_{\tau^k R}|^{p-2} \fint_{(B_{\tau^k R})^+} |Du - \xi^u_{\tau^k R}|^2\, dx + \fint_{(B_{\tau^k R})^+} |Du - \xi^u_{\tau^k R}|^p\, dx \\ &\leqslant c \sup_{B^+_{R/2}} |Du|^{p-2} \fint_{(B_{\tau^k R})^+} |Du - \xi^u_{\tau^k R}|^2\, dx \\ &\leqslant c \sup_{B^+_{R/2}} |Du|^p (\tau^k)^{2\gamma},\end{aligned}$$

because

$$\begin{aligned}\fint_{(B_{\tau^k R})^+} |Du - \xi^u_{\tau^k R}|^2\, dx \leqslant &\fint_{(B_{\tau^k R})^+} \fint_{(B_{\tau^k R})^+} \sum_{\alpha < n} \left| \frac{\partial u}{\partial x^\alpha}(x) - \left(\frac{\partial u}{\partial x^\alpha}(y) \right)^{\top} \right|^2 \\ &+ \fint_{(B_{\tau^k R})^+} \fint_{(B_{\tau^k R})^+} \left| \frac{\partial u}{\partial x^n}(x) - \left(\frac{\partial u}{\partial x^n}(y) \right)^{\perp} \right|^2 dx\, dy\,,\end{aligned}$$

$\left(\text{because } \left(\dfrac{\partial u}{\partial x^\alpha}\right)^{\perp} = 0 = \left(\dfrac{\partial u}{\partial x^n}\right)^{\top} \text{ on } I\right)$, and the terms on the right hand side may be estimated by

$$c \sup_{B^+_{R/2}} |Du|^2 (\tau^k)^{2\gamma};$$

cf. (12) and the assumptions. From (12) it furthermore follows that

$$\begin{aligned}\psi(\tau^k R) &\leqslant c \fint_{B^+_R} |Du|^p \, dx \cdot (\tau^k)^{2\gamma} \\ &\leqslant c(\tau^k)^{2\gamma} \left(\fint_{B^+_R} |Du - \xi^u_R|^p \, dx + |\xi^u_R|^p \right) \\ &\leqslant c(\tau^k)^{2\gamma - \frac{n+2}{k\kappa}} \psi(R).\end{aligned}$$

Now we prove (ii):

A short calculation shows that we can assume that $|\xi^u_R| \neq 0$. Now let $w := u - \xi^u_R (x - x_0)$. Let v be the solution of

$$\begin{cases} \displaystyle\int_{B^+_{R/2}} \left(Id + (p-2) \frac{\xi^u_R \xi^u_R}{|\xi^u_R|^2} \right) Dv \, D\varphi \, dx = 0 \qquad \forall \varphi \in H \\ v \in w + H, \end{cases}$$

where

$$H := \{\varphi|_{B^+_{R/2}} \colon \varphi \in H^{1,2}_0(B_{R/2}), \ \varphi(B_{R/2} \cap I) \subset V\}.$$

Now let $\varphi = v - w$; then by similar calculations as in [3]

$$\fint_{B^+_{R/2}} |Dv - Dw|^2 \, dx \leqslant c|\xi|^{-2} \fint_{B^+_{R/2}} |Dw|^4 \left(1 + \left(\frac{|Dw|}{|\xi|} \right)^{2p-6} \right) dx. \tag{17}$$

Because of the assumption, we can estimate the right hand side:

$$\begin{aligned}\sup_{B^+_{R/2}} |Dw| &\leqslant \sup_{B^+_{R/2}} |Du| + |\xi| \\ &\leqslant c \left(\fint_{B^+_R} |Du - \xi|^p \, dx \right)^{1/p} + c|\xi| \qquad \text{by (12)} \\ &\leqslant c|\xi|.\end{aligned}$$

Therefore, it follows from (17)

$$\begin{aligned}\fint_{B^+_{R/2}} |Dv - Dw|^2\,dx &\leqslant c|\xi|^{-2\kappa} \fint_{B^+_{R/2}} |Dw|^{2+2\kappa}\,dx \\ &\leqslant c|\xi|^{-2\kappa-(p-2)(1+\kappa)}\psi^{1+\kappa}(R) \qquad \text{by Theorem 4.1.}\end{aligned}$$

A comparison of u and v now gives the estimate: Let $\tau < 1/2$. Using the assumption in (ii) and (12), as in [3] we get

$$\begin{aligned}\psi(\tau R) &\leqslant c|\xi^u_R|^{p-2} \fint_{B^+_{\tau R}} |Du - \xi^u_{\tau R}|^2\,dx \\ &\leqslant c|\xi^u_R|^{p-2}\left(\fint_{B^+_{\tau R}} |Dw-Dv|^2 dx + \fint_{B^+_{\tau R}} |Dv-\xi^v_{\tau R}|^2 dx + \fint_{B^+_{\tau R}} |\xi^v_{\tau R} - \xi^w_{\tau R}|^2\,dx\right).\end{aligned}$$

Now

$$\begin{aligned}|\xi^v_{\tau R} - \xi^w_{\tau R}|^2 &\leqslant \left|\fint_{B^+_{\tau R}} Dw - \xi^v_{\tau R}\,dx\right|^2 \\ &\leqslant c\fint_{B^+_{\tau R}} |Dw - Dv|^2\,dx + c\fint_{B^+_{\tau R}} |Dv - \xi^v_{\tau R}|^2\,dx\,.\end{aligned}$$

Therefore

$$\begin{aligned}\psi(\tau R) \quad &\leqslant \quad c|\xi^u_R|^{p-2}\tau^{-n}\fint_{B^+_{R/2}} |Dw - Dv|^2\,dx + c|\xi^u_R|^{p-2}\fint_{B^+_{\tau R}} |Dv - \xi^v_{\tau R}|^2\,dx \\ &\overset{\text{Thm. 2.1}}{\leqslant} c|\xi^u_R|^{p-2}\tau^{-n}\fint_{B^+_{R/2}} |Dw - Dv|^2\,dx + c|\xi^u_R|^{p-2}\tau^2\fint_{B^+_{R/2}} |Dv - \xi^v_{\tau R}|^2\,dx\,, \\ &\leqslant \quad c\tau^2\psi(R)\end{aligned}$$

by the same computations as in the proof of Lemma 4.1 in [3]. ■

With slight modifications of the proof of Theorem 4.2 in [3], we then get

THEOREM 5.1. – *Let* $B_R \subset\subset B$, $0 < \alpha < 2\gamma$ *and* u *a solution of* (13) *and* (14). *Then there exists a constant* $c = c(n, N, p, \alpha)$ *such that*

$$\psi(\varrho) \leqslant c\left(\frac{\varrho}{R}\right)^{\alpha}\psi(R) \qquad \forall\ 0 < \varrho \leqslant R\,. \qquad ■$$

Let now $\xi_r^{u,g}$ defined by

$$(\xi_r^{u,g})_\alpha := pr_V\left(P_g\left(\frac{\partial u}{\partial x^\alpha}\right)_r\right) \qquad \text{if } \alpha < n$$

and

$$(\xi_r^{u,g})_n := pr_{V^\perp}\left(P_g\left(\frac{\partial u}{\partial x^n}\right)_r\right),$$

with P_g from (10).

We will show a decay estimate for

$$\psi^g(\varrho) := |\xi_\varrho^{u,g}|^{p-2} \fint_{B_\varrho^+} |P_g Du - \xi_\varrho^{u,g}|^2 dx + \fint_{B_\varrho^+} |P_g Du - \xi_\varrho^{u,g}|^p dx .$$

COROLLARY 5.1. – *Let $B_R \subset\subset B$, $0 < \alpha < 2\gamma$ and let u be a solution of*

$$\int_{B^+} \sum_{i,\alpha} (|Du|^2_{g,\gamma})^{p/2-1} g_{ij} \gamma^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial \varphi^j}{\partial x^\beta} dx = 0 \qquad \forall \varphi \in H_V$$

$$u(I) \subset V$$

Then there exists a constant $c = c(n, N, p, \Lambda, \alpha)$ such that

$$\psi^g(\varrho) \leq c(\alpha)\left(\frac{\varrho}{R}\right)^\alpha \psi^g(R) \qquad \forall\, 0 < \varrho \leq R .$$

PROOF. – We can apply Theorem 5.1 to w from (11). Then we get (with the obvious notation)

$$\psi_w(\varrho) \leq c\left(\frac{\varrho}{R}\right)^\alpha \psi_w(R) \qquad \forall\, 0 < \varrho \leq R$$

with

$$\psi_w(\varrho) := |\xi_\varrho^w|^{p-2} \fint_{B_\varrho^+} |Dw - \xi_\varrho^w|^2 dx + \fint_{B_\varrho^+} |Dw - \xi_\varrho^w|^p dx .$$

By definition of w, it follows for

$$\psi_u^*(\varrho) := |\xi_{E_\varrho^+}^{u,g}|^{p-2} \fint_{E_\varrho^+} |P_g Du - \xi_{E_\varrho^+}^{u,g}|^2 dx + \fint_{E_\varrho^+} |P_g Du - \xi_{E_\varrho^+}^{u,g}|^p dx$$

$$\psi_u^*(\varrho) \leq c\left(\frac{\varrho}{R}\right)^\alpha \psi_u^*(R) \qquad \forall\, 0 < \varrho \leq R .$$

As above, $E_\varrho^+ = \Gamma^{1/2} B_\varrho^+$, and c depends also on Λ. ■

6. – A comparison lemma.

Let g and γ satisfy the conditions

$$|\xi|^2 \leqslant g_{ij}\xi^i\xi^j \leqslant \Lambda|\xi|^2 \qquad \forall \xi \in \mathbb{R}^N, \tag{18}$$

$$|\eta|^2 \leqslant \gamma^{\alpha\beta}\eta_\alpha\eta_\beta \leqslant \Lambda|\eta|^2 \quad \forall \eta \in \mathbb{R}^n. \tag{19}$$

Furthermore, let g and γ be continuous, bounded functions on $\mathbb{R}^N$ and $\mathbb{R}^n$ resp.

For minima of F, we have

LEMMA 6.1. – *Let $u \in H^{1,p}(B^+, \mathbb{R}^N)$ be a minimizer for F in $u + H_V$ with $u(I) \subset V$. Then there exists a $q > p$, which depends only on the data, such that $u \in H^{1,q}(B_r^+, \mathbb{R}^N)$ for all $B_r(x_0) \subset\subset B$ with $x_0 \in I$. More precisely, we have the estimate*

$$\left(\fint_{B_{r/2}^+(x_0)} |Du|^q\, dx \right)^{1/q} \leqslant c \left(\fint_{B_r^+(x_0)} |Du|^p\, dx \right)^{1/p}.$$

PROOF. – We can assume $x_0 = 0$. Let $\eta \in C_c^\infty(B_s)$ be a cutoff function with $\eta \equiv 1$ on $B_t(0)$ and $|D\eta| \leqslant \dfrac{c}{s-t}$, $t < s < r$. Then

$$v = u - \eta(u - u_r^\top)$$

is an admissible comparison function. As in the proof of Theorem V.3.1 in [4], we get

$$\int_{B_t^+} |Du|^p\, dx \leqslant c\frac{1}{(r-t)^p} \int_{B_r^+} |u - u_r^\top|^p\, dx\,.$$

Using Poincaré inequality Lemma 2.1, we can apply Sobolev-Poincaré inequality to the right hand side:

$$\fint_{B_{r/2}^+} |Du|^p\, dx \leqslant c \cdot \left(\fint_{B_r^+} |Du|^{\frac{np}{n+p}}\, dx \right)^{\frac{n+p}{n}}.$$

Let $g := |Du|^{\frac{np}{n+p}}$. The assertion follows as in the proof of Theorem 4.1 if we note that there exists an analogous interior estimate as above. ■

Now we make the following additional assumptions:

There exists a function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ with the following properties:

(i) $|g(z)\,\gamma(x) - g(w)\,\gamma(y)| \leqslant \omega(|x-y|^p + |z-w|^p)$,

(ii) ω is bounded, continuous and $\omega(0) = 0$,

(iii) ω is nondecreasing,

(iv) ω is concave.

Moreover let

$$H_V^R := \{\varphi|_{B_R^+} : \varphi \in H_0^{1,p}(B_R),\ \varphi(I \cap B_R) \subset V\}.$$

Using Lemma 6.1 and global L^p-estimates (cf. [4], Remark 3.5, p. 163) together with the reflection principle, the same arguments as in the proof of Lemma 5.1 in [3] give

LEMMA 6.2. – *Let $B_{2R} = B_{2R}(x_0) \subset\subset B$, $x_0 \in I$ and $u \in H^{1,p}(B^+, \mathbb{R}^N)$ minimizing for F in $u + H_V^R$ with $u(I) \subset V$. Furthermore let v be the solution of*

$$v \in u + H_V^R$$
$$F^0(v) \equiv \fint_{B_R^+} \left(g_{ij}(u_{x_0,R})\, \gamma^{\alpha\beta}(x_0) \frac{\partial v^i}{\partial x^\alpha} \frac{\partial v^j}{\partial x^\beta} \right)^{p/2} dx \text{ is minimal.}$$

Then there exists a $0 < \sigma < 1$ with

$$\fint_{B_R^+} |Du - Dv|^p\, dx \leq c \fint_{B_{2R}^+} |Du|^p\, dx \cdot \omega^\sigma \left(cR^p \fint_{B_R^+} 1 + |Du|^p\, dx \right). \qquad \blacksquare$$

7. – Hölder continuity.

For $x_0 \in I$ with $B_r = B_r(x_0) \subset\subset B$, we will derive a decay estimate for

$$\varphi(x_0, r) := \varphi(r) := r^p \fint_{B_r^+(x_0)} |Du|^p\, dx.$$

This together with the interior estimates of [3] gives Hölder continuity near the boundary. Firstly we show

LEMMA 7.1. – *Let u be minimizing for*

$$\int_{B^+} \left(g_{ij} \circ u\ \gamma^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} \right)^{p/2} dx$$

in $u + H_V$ with $u(I) \subset V$ and $0 < \alpha < 1$. If $x_0 \in I$ is given, then there exists an ε_0 with:

If there exists $R_0 = R_0(\varepsilon_0) > 0$ *with* $\operatorname{dist}(x_0, \partial B) > R_0$ *and*

$$\varphi(x_0, R_0) + R_0^p = R_0^p \fint_{B^+_{R_0}(x_0)} |Du|^p \, dx + R_0^p < \varepsilon_0,$$

then

$$\varphi(x_0, \varrho) \leqslant c(\alpha) \left(\frac{\varrho}{R_0} \right)^{p\alpha} \varphi(x_0, R_0) \qquad \forall \varrho \leqslant R_0.$$

Here without loss of generality $c(\alpha) \geqslant 1$.

PROOF. – We apply Lemma 6.2 with $R/2$ instead of R. Then as in [3] with $0 < \tau < 1/4$, we get

$$\varphi(\tau R) \leqslant c_1 \tau^p \varphi(R)(1 + \tau^{-n} \omega^{1-p/q}(c_2 R^p + c_2 \varphi(R))) .$$

We choose $0 < \tau < 1/4$ such that $2c_1 \tau^p < \tau^{p\alpha}$. Now

$$\tau^{-n} \omega^{1-p/q}(c_2 R^p + c_2 \varphi(R)) < 1 ,$$

if

$$R^p + \varphi(R) \leqslant \varepsilon_0 = \varepsilon_0(\alpha)$$

and ε_0 sufficiently small.

We now assume that there exists a $R_0 = R_0(\alpha)$ with $B_{R_0}(x_0) \subset\subset B$ such that

$$R_0^p + \varphi(R_0) \leqslant \varepsilon_0.$$

Then we have

$$\varphi(\tau R_0) \leqslant \tau^{p\alpha} \varphi(R_0).$$

Iteration gives the result. ■

Of course we have an analogous interior estimate from [3]. When we write interior estimate it is understood that we refer to this interior estimate.

For $y \in B^+$ with $B_\varrho(y) \subset\subset B$ we define

$$\Omega(y, \varrho) := B_\varrho(y) \cap B^+$$

and

$$\Phi(y, \varrho) := \varrho^p \fint_{\Omega(y, \varrho)} |Du|^p \, dx .$$

Similarly as Theorem IV.2.2 in [4], we get the following boundary version of this theorem.

THEOREM 7.1. – *Let $\Omega \subset \mathbb{R}^n$ open and bounded, $v \in L^1(\Omega)$ and $0 \leq \alpha < n$. Define*

$$E_\alpha := \left\{ x \in \overline{\Omega} : \limsup_{\varrho \to 0} \varrho^{-\alpha} \int_{B_\varrho(x) \cap \Omega} |v(y)| \, dy > 0 \right\}.$$

Then

$$\mathcal{H}^\alpha(E_\alpha) = 0 . \qquad \blacksquare$$

Now we show a decay estimate for Φ.

THEOREM 7.2. – *Let u be minimizing for*

$$\int_{B^+} \left(g_{ij} \circ u \ \gamma^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} \right)^{p/2} dx$$

in $u + H_V$ with $u(I) \subset V$, $0 < \alpha < 1$ and $0 < d < 1/4$.

Then for $\mathcal{H}^{n-p}$-almost all $x_0 \in I$ with $\operatorname{dist}(x_0, \partial B) > d$, *there exist radii $r < d/4$, $R_0 < d/4$ and a constant $K = K(\alpha, u)$ such that for all $y \in B_r^+(x_0)$*

$$\Phi(y, \varrho) \leq K \left(\frac{\varrho}{R_0} \right)^{p\alpha} \qquad \forall \varrho \leq R_0 .$$

PROOF. – Because of Lemma 7.1, for $\mathcal{H}^{n-p}$-almost all $x_0 \in I$

$$\liminf_{\varrho \to 0} \varrho^{p-n} \int_{B^+ \cap B_\varrho(x_0)} |Du|^p \, dx = 0 .$$

Therefore for $\mathcal{H}^{n-p}$-almost all $x_0 \in I$, there exists $0 < R_0 < d/4$ for given $\tilde{\varepsilon}_0$ such that

$$R_0^p \fint_{B^+ \cap B_{R_0}(x_0)} |Du|^p \, dx < \tilde{\varepsilon}_0$$

and

$$R_0^p < \tilde{\varepsilon}_0 .$$

We choose $\tilde{\varepsilon}_0 := c(\alpha)^{-1} 2^{p-n-p\alpha-1} \varepsilon_0/2$, where ε_0 and $c(\alpha)$ are the constants from Lemma 7.1. Since

$$R_0^p \fint_{B^+ \cap B_{R_0}(y)} |Du|^p \, dx$$

depends continuously on y, there exists $0 < r < d/4$ such that

$$R_0^p \fint_{B^+ \cap B_{R_0}(y)} |Du|^p \, dx < \tilde{\varepsilon}_0$$

for all $y \in B_r(x_0) \cap B^+$.

Let $x_0 \in I$ with $\operatorname{dist}(x_0, \partial B) > d$. We will show that there exists a uniform decay estimate for

$$\Phi(y, \varrho) := \varrho^p \fint_{\Omega(y, \varrho)} |Du|^p \, dx \qquad \forall \varrho \leqslant R_0$$

if $y \in B_r(x_0) \cap B^+$.

We have to distinguish three cases (we write $y = (\overline{y}, y_n)$):

Case 1: $0 < \varrho \leqslant R_0 \leqslant y_n$.

In this case $B_{R_0}(y)$ lies completely in B^+, and we can apply the interior estimates of [3]:

$$\begin{aligned}\Phi(y, \varrho) &\leqslant c(\alpha) \left(\frac{\varrho}{R_0} \right)^{p\alpha} \Phi(y, R_0) \\ &\leqslant c(\alpha) \left(\frac{\varrho}{R_0} \right)^{p\alpha} \tilde{\varepsilon}_0.\end{aligned}$$

Case 2: $0 < \varrho \leqslant y_n \leqslant R_0$.

Now $B_{R_0}(y)$ does not lie completely in B^+. But because of the boundary estimate Lemma 7.1

(i) $y_n \leqslant R/2$:

$$\begin{aligned}\Phi(y, y_n) &\leqslant 2^{n-p} \varphi((\overline{y}, 0), 2y_n) \\ &\leqslant 2^{n-p} c(\alpha) \left(\frac{2y_n}{R_0} \right)^{p\alpha} \varphi((\overline{y}, 0), R_0) \quad \text{by Lemma 7. 1} \\ &\leqslant c(\alpha) 2^{n-p+p\alpha} \left(\frac{y_n}{R_0} \right)^{p\alpha} \tilde{\varepsilon}_0. \end{aligned} \tag{21}$$

(ii) $y_n > R_0/2$:

$$\Phi(y, y_n) \leqslant c(\alpha) 2^{n-p+1+p\alpha} \left(\frac{y_n}{R_0} \right)^{p\alpha} \tilde{\varepsilon}_0 . \tag{22}$$

Therefore we have shown that in the second case generally

$$\Phi(y, y_n) \leqslant \varepsilon_0/2$$

because of the assumptions on $\tilde{\varepsilon}_0$.

Therefore, the conditions for the interior estimates are satisfied, and we can apply the interior estimates with y_n instead of R_0. We get

$$\Phi(y, \varrho) \leqslant c(\alpha)\left(\frac{\varrho}{y_n}\right)^{p\alpha} \Phi(y, y_n). \tag{23}$$

Again we distinguish the two cases $y_n \leqslant R_0/2$ and $y_n > R_0/2$.

(iii) $y_n \leqslant R_0/2$:

With (23) and (21) follows

$$\Phi(y, \varrho) \leqslant c(\alpha)\, 2^{n-p+p\alpha}\left(\frac{\varrho}{R_0}\right)^{p\alpha} \tilde{\varepsilon}_0.$$

(iv) $y_n > R_0/2$:

With (23) and (22) follows

$$\begin{aligned}\Phi(y, \varrho) &\leqslant c(\alpha)\left(\frac{\varrho}{y_n}\right)^{p\alpha} \Phi(y, y_n)\\ &\leqslant c(\alpha)\, 2^{n-p+p\alpha+1}\left(\frac{\varrho}{R_0}\right)^{p\alpha} \tilde{\varepsilon}_0.\end{aligned}$$

We have thus shown:

For $0 < \varrho \leqslant y_n \leqslant R_0$

$$\Phi(y, \varrho) \leqslant c(\alpha)\left(\frac{\varrho}{R_0}\right)^{p\alpha} \tilde{\varepsilon}_0.$$

Case 3: $0 \leqslant y_n \leqslant \varrho \leqslant R_0$.

In this case, $B_{R_0}(y)$ does not lie completely in B^+ either. But because of the boundary estimate in Lemma 7.1, we have

(i) $\varrho \leqslant R_0/2$:

$$\Phi(y, \varrho) \leqslant c(\alpha)\left(\frac{\varrho}{R_0}\right)^{p\alpha} \tilde{\varepsilon}_0.$$

(ii) $\varrho > R/2$:

$$\Phi(y, \varrho) \leqslant \frac{2}{\omega_n} \varrho^{p-n} \int_{B_\varrho(y) \cap B^+} |Du|^p \, dx$$

$$\leqslant c(\alpha) \, 2^{n-p+1+p\alpha} \left(\frac{\varrho}{R_0} \right)^{p\alpha} \tilde{\varepsilon}_0 .$$

This proves the assertion. ∎

COROLLARY 7.1. – *Under the assumptions of Theorem* 7.2, *u is α-Hölder continuous in* $\mathcal{H}^{n-p}(I)$*-almost all* $x_0 \in I$ *with* $\operatorname{dist}(x_0, \partial B) > d$. *In general the Hölder constant depends on* x_0.

PROOF. – The claim follows from Theorem 7.2 and Theorem 7.19 in [6]. ∎

8. – $C^{1,\alpha}$-Estimates.

Similarly as in section 1.5, we will derive a decay estimate for

$$\psi^g(x_0', R) := |\xi_R^{u,g}|^{p-2} \fint_{B_R^+} |P_g Du - \xi_R^{u,g}|^2 \, dx + \fint_{B_R^+} |P_g Du - \xi_R^{u,g}|^p \, dx ,$$

where $B_R(x_0') \subset\subset B$, $x_0' \in B_r(x_0) \cap I$, $g = g(u_{x_0', R/2})$ and P_g are as in (10). In the following we will always look at a fixed point x_0' and therefore omit the index g.

Furthermore, for $z \in B^+$ we define

$$M(z, R) := \max\left\{1, \fint_{\Omega(z, R)} |Du|^p \, dx\right\}$$

and we make the additional assumption

$$\omega(t) \leqslant c t^{\tilde{\sigma}/p}$$

with some $\tilde{\sigma} < 1$.

LEMMA 8.1. – *Let u with* $u(I) \subset V$ *be minimizing for*

$$\int_{B^+} \left(g_{ij} \circ u \ \gamma^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} \right)^{p/2} dx$$

in $u + H_V$ *and* $0 < d < 1/4$.

Then α from Theorem 7.2 *can be chosen so large that for* $\mathcal{H}^{n-p}$*-almost all*

$x_0 \in I$ with $\operatorname{dist}(x_0, \partial B) > d$, *there exist radii $r < d/4$, $R_0 < d/4$ and $0 < \theta < \alpha < 1$ such that for all $x_0' \in B_r(x_0) \cap I$*

$$\psi(x_0', \varrho) \leq K \left(\frac{\varrho}{R_0} \right)^{\theta} (\psi(x_0', R_0) + R_0^{\theta}) \qquad \forall \varrho \leq R_0$$

with a constant K which depends only on the data, on R_0 and on the constants in Theorem 7.2.

PROOF. – We apply Theorem 6.2 with $R/2$ instead of R.

Let $0 < \tau < 1/4$ and $R \leq R_0$; R_0 will be chosen later. We write B_r for $B_r(x_0')$ and ξ_r^u for $\xi_{x_0', r}^u$ etc. The same calculations as in the proof of Theorem 7.1 in [3] give with (20), Corollary 5.1 and the assumption $\omega \leq c(\cdot)^{\tilde{\sigma}/p}$

$$\psi(x_0', \tau R) \leq c\tau^{\gamma} \psi(x_0', R)$$
$$+ cM(x_0', R)\, \tau^{-n} (R^{2\sigma'/p} + \varphi^{\sigma'/p}(x_0', R/2) + \varphi^{2\sigma'/p^2}(x_0', R/2)) \,.$$

where $\nu < 2\gamma$ and $\sigma' = \sigma\tilde{\sigma}$.

Now let α, x_0, R_0, r as in Theorem 7.2. Then we have for $R \leq R_0$ and $x_0' \in B_r^+(x_0)$:

$$\varphi(x_0', R) \leq K \cdot R^{p\alpha}.$$

Therefore, for all $x_0' \in B_r(x_0) \cap I$

$$\psi(x_0', \tau R) \leq c\tau^{\nu} \psi(x_0', R) + cM(x_0', R)\, \tau^{-n} R^{2\sigma' \alpha/p}. \tag{24}$$

Because of Theorem 7.2, we have for all $0 < \alpha < 1$

$$M(x_0', R) = \fint_{B_R^+} |Du|^p \, dx \leq K(\alpha) \left(\frac{R}{R_0} \right)^{p\alpha} R^{-p}$$
$$= \widetilde{K}(R_0, \alpha)\, R^{-p+p\alpha} \qquad \forall R \leq R_0$$

and therefore

$$\psi(x_0', \tau R) \leq c\tau^{\nu} \psi(x_0', R) + c(R_0, \alpha)\, \tau^{-n} R^{-p+p\alpha+\frac{2\sigma'\alpha}{p}}.$$

Now we choose $\alpha < 1$ so close to 1 that

$$p\alpha + \frac{2\sigma' \alpha}{p} > p\,.$$

We get

$$\psi(x_0', \tau R) \leq c\tau^{\nu} \psi(x_0', R) + c\tau^{-n} R^{\theta} \qquad \forall R \leq R_0$$

with

$$\theta := p\alpha + \frac{2\sigma'\alpha}{p} - p\,.$$

Therefore, for all $x_0' \in B_r(x_0) \cap I$

$$\psi(x_0', \tau^k R) \leqslant c\tau^\nu \psi(x_0', \tau^{k-1}R) + c\tau^{-n}(\tau^{k-1}R)^\theta \qquad \forall R \leqslant R_0.$$

Now we choose $0 < \tau < 1/4$ such that

$$c\tau^\nu = \tau^{\tilde\theta}, \qquad \text{with} \qquad \nu > \tilde\theta > \theta\,.$$

Iteration gives the result. ■

Now we define for $x \in B^+$

$$u_{x,r} := \fint_{\Omega(x,r)} u dx$$

and

$$\Psi(x,r) := |Du_{x,r}|^{p-2} \fint_{\Omega(x,r)} |Du - Du_{x,r}|^2 dx + \fint_{\Omega(x,r)} |Du - Du_{x,r}|^p dx\,.$$

Obviously this definition is consistent with the one given before.

Then a calculation shows

LEMMA 8.2. – *Let u be minimizing for*

$$\int_{B^+} \left(g_{ij} \circ u \;\gamma^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} \right)^{p/2} dx$$

in $u + H_V$ with $u(I) \subset V$, $0 < d < 1/4$.

Then α from Theorem 7.2 can be chosen so large that for $\mathcal{H}^{n-p}$-almost all $x_0 \in I$ with $\operatorname{dist}(x_0, \partial B) > d$, there exist radii $r < d/4$, $R_0 < d/4$ and $0 < \theta < \alpha < 1$ such that for all $x_0' \in B_r(x_0) \cap I$

$$\Psi(x_0', \varrho) \leqslant K \left(\frac{\varrho}{R_0} \right)^\theta (\psi(x_0', R_0) + R_0^\theta) \qquad \forall \varrho \leqslant R_0, \tag{25}$$

with a constant K which depends only on the data, on R_0 and on the constant in Theorem 7.2. ■

Now we can prove the crucial decay estimate:

THEOREM 8.1. – *Let u with $u(I) \subset V$ be minimizing for*

$$\int_{B^+} \left(g_{ij} \circ u \ \gamma^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} \right)^{p/2} dx$$

in $u + H_V$, $0 < d < 1/4$.

Then α from Theorem 7.2 *can be chosen so large that for $\mathcal{H}^{n-p}$-almost all $x_0 \in I$ with* $\operatorname{dist}(x_0, \partial B) > d$, *there exist radii $r < d/4$, $R_0 < d/4$, a constant K and $0 < \theta < \alpha < 1$ such that for all $x_0' \in B_r^+(x_0)$*

$$\Psi(x_0', \varrho) \leqslant K \left(\frac{\varrho}{R_0} \right)^\theta \qquad \forall \varrho \leqslant R_0. \tag{26}$$

PROOF. – We may assume that

$$\sup_{y \in B_r^+(x_0)} \fint_{\Omega(y, R_0)} |Du|^p \, dx \leqslant L < \infty .$$

Again we distinguish three possible cases. Let $y \in B_r(x_0) \cap B^+$.

Case 1: $0 \leqslant \varrho \leqslant R_0 \leqslant y_n$.

Because of the interior estimates from [3],

$$\Psi(y, \varrho) \leqslant c \left(\frac{\varrho}{R_0} \right)^\theta (\Psi(y, R_0) + R_0^\theta)$$

and

$$\begin{aligned} \Psi(y, R_0) &= |Du_{y, R_0}|^{p-2} \fint_{B_{R_0}(y)} |Du - Du_{y, R_0}|^2 \, dx + \fint_{B_{R_0}(y)} |Du - Du_{y, R_0}|^p \, dx \\ &\leqslant c \left(\fint_{B_{R_0}(y)} |Du| \, dx \right)^{p-2} \cdot \fint_{B_{R_0}(y)} |Du|^2 \, dx + c \fint_{B_{R_0}(y)} |Du|^p \, dx \\ &\leqslant c \fint_{B_{R_0}(y)} |Du|^p \, dx \leqslant cL . \end{aligned}$$

Case 2: $0 < \varrho \leqslant y_n \leqslant R_0$.

Firstly, since $B_\varrho(y) \subset B_{y_n}(y)$, we have

(i) $y_n \leqslant R_0/2$:

$$\begin{aligned}\Psi(y, \varrho) &\leqslant c\left(\frac{\varrho}{y_n}\right)^{\theta}(\Psi(y, y_n) + y_n^{\theta}) \text{ by the interior estim., cp.[3]}\\ &\leqslant c\left(\frac{\varrho}{y_n}\right)^{\theta}\left(\Psi((\overline{y}, 0), 2y_n) + R_0\left(\frac{y_n}{R_0}\right)^{\theta}\right)\\ &\leqslant c\left(\frac{\varrho}{y_n}\right)^{\theta}\left(\left(\frac{2y_n}{R_0}\right)^{\theta}(\psi((\overline{y}, 0), R_0) + R_0^{\theta}) + R_0\left(\frac{y_n}{R_0}\right)^{\theta}\right) \text{ by Lem. 8.2}\\ &\leqslant cL\left(\frac{\varrho}{R_0}\right)^{\theta}.\end{aligned}$$

(ii) $y_n \geqslant R_0/2$:

$$\begin{aligned}\Psi(y, \varrho) &\leqslant c\left(\frac{\varrho}{y_n}\right)^{\theta}(\Psi(y, y_n) + y_n^{\theta})\\ &\leqslant c\left(\frac{\varrho}{R_0}\right)^{\theta}(\Psi(y, y_n) + R_0^{\theta}).\end{aligned}$$

Because of the assumption $y_n \geqslant R_0/2$, it follows with the above argument that

$$\Psi(y, y_n) \leqslant c\Psi(y, R_0) \leqslant cL .$$

Therefore we get in the second case

$$\Psi(y, \varrho) \leqslant cL\left(\frac{\varrho}{R_0}\right)^{\theta}.$$

Case 3: $0 \leqslant y < \varrho \leqslant R_0$.

In this case we have

$$B_{\varrho}(y) \subset B_{2\varrho}((\overline{y}, 0)).$$

(i) $\varrho \leqslant R_0/2$:

$$\begin{aligned}&\Psi(y, \varrho)\,\Psi((\overline{y}, 0), 2\varrho)\\ &\qquad\leqslant c\left(\frac{\varrho}{R_0}\right)^{\theta}(\Psi((\overline{y}, 0), R_0) + R_0^{\theta}) \quad \text{by Lemma 8.2}\\ &\qquad\leqslant cL\left(\frac{\varrho}{R_0}\right)^{\theta}.\end{aligned}$$

(ii) $\varrho > R_0/2$:

$$\Psi(y, \varrho) \leqslant c \fint_{\Omega(y, \varrho)} |Du|^p \, dx$$

$$\leqslant cL \left(\frac{\varrho}{R_0} \right)^{\theta} . \qquad \blacksquare$$

By Theorem 8.1 and Theorem III.1.2 in [4] follows

COROLLARY 8.1. – *Under the assumptions of Theorem* 8.1, *Du is Hölder continuous in* $\mathcal{H}^{n-p}(I)$*-almost all* $x_0 \in I$ *with* $\operatorname{dist}(x_0, \partial B) > d$. *In general, the Hölder constant depends on* x_0. $\blacksquare$

REMARK 8.1. – *Actually we have also proved that u is regular inside a relatively open set* $\Omega_0 \subset \overline{B^+_{1-d}}$ *and*

$$\Omega_0 \subset \left\{ x \in \overline{B^+_{1-d}} : \liminf_{r \to 0} r^q \fint_{B_r(x) \cap B^+} |Du|^q \, dx = 0 \right\} \quad \textit{for some } q > p .$$

Because of Theorem 7.1, *the singular set has* $\mathcal{H}^{n-q}$*-measure zero for some* $q > p$.

REMARK 8.2. – *If* x_0' *is a regular point then* $M(x_0', R)$ *is bounded for all* $R \leqslant R_0$. *We can again iterate* (24) *and see that first derivatives are Hölder continuous for all exponents*

$$\beta < \min\{\nu/p, 2\sigma\tilde{\sigma}/p^2\} .$$

Finally we have

THEOREM 8.2. – *Let* $S \subset \mathbb{R}^N$ *a* $(k-1)$*-dimensional submanifold of class* $C^{1, \tilde{\sigma}}$. *Let* $u \in H^{1,p}(B, \mathbb{R}^N)$ *be a minimizer for the functional*

$$F(u) := \int_B \left(g_{ij} \circ u \;\, \gamma^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} \right)^{p/2} dx$$

which respects the boundary condition

$$u(\partial B) \subset S .$$

Here g is $\in C^{0, \tilde{\sigma}}(\mathbb{R}^N)$ *and bounded, and* $\gamma \in C^{0, \tilde{\sigma}}(B)$. *Furthermore, let* $q' > p$ *and assume that for* $\mathcal{H}^{n-q'}$*-almost all points* $x \in \partial B$, *there exists a neighborhood* $U(x)$ *such that* $u(U(x))$ *lies completely in a N-dimensional coordinate neighborhood of* S. *Then u is locally* α*-Hölder continuous in* $\mathcal{H}^{n-q}$*-almost all points* $x \in \overline{B}$ *for each* $\alpha < 1$, *and the first derivatives are* β*-Hölder continuous with all* $\beta < \min\{2\gamma/p, 2\sigma\tilde{\sigma}/p^2\}$, $(q < q')$. $\blacksquare$

REMARK 8.3. – *Using the main theorem from* [2], *which essentially says that u is continuous on $\overline{B}$ outside a set of $(n - q')$-Hausdorff-measure zero, the assumptions of Theorem* 8.2 *are satisfied.*

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