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On the Nonhamiltonian Character of Shocks in 2-D Pressureless Gas.

YU. G. RYKOV

Sunto. – *Si considera un sistema bidimensionale della dinamica dei gas introdotto nel 1970 da Ya. Zeldovich per descrivere la formazione della struttura di grande scala dell'universo. Il sistema si rivela come qualcosa di intermedio tra un sistema di equazioni differenziali ordinarie e un sistema iperbolico di equazioni alle derivate parziali. La caratteristica principale è la nascita di singolarità: discontinuità della velocità e funzioni delta di vario tipo per la densità. Si dà una descrizione rigorosa delle soluzioni generalzzate in termini di misure di Radon e si ottiene una generalizzazione delle condizione di Rankine-Hugoniot. Sulla base di tali condizioni si mostra che la rappresentazione variazionale delle soluzioni generalizzate, valida nel caso unidimensionale, non vale in generale nel caso bidimensionale. Si ottiene anche un sistema unidimensionale non banale non strettamente iperbolico per la descrizione dell'evoluzione all'interno dell'urto.*

Summary. – *The paper deals with the* 2*-D system of gas dynamics without pressure which was introduced in 1970 by Ua. Zeldovich to describe the formation of large-scale structure of the Universe. Such system occurs to be an intermediate object between the systems of ordinary differential equations and hyperbolic systems of PDE. The main its feature is the arising of singularities: discontinuities for velocity and δ-functions of various types for density. The rigorous notion of generalized solutions in terms of Radon measures is introduced and the generalization of Rankine-Hugoniot conditions is obtained. On the basis of such conditions it is shown that the variational representation for the generalized solutions, which is valid for* 1*-D case, in* 2*-D case generally speaking does not take place. A nontrivial* 1*-D system of nonstrictly hyperbolic type is also obtained to describe the evolution inside the shock.*

1. – Introduction. Basic definitions.

This paper studies the shock waves for the system of 2-D pressureless gas dynamics

$$\left\{\begin{array}{l} \dfrac{\partial \varrho}{\partial t}+\dfrac{\partial(\varrho u)}{\partial x}+\dfrac{\partial(\varrho v)}{\partial y}=0 \\ \dfrac{\partial(\varrho u)}{\partial t}+\dfrac{\partial(\varrho u^2)}{\partial x}+\dfrac{\partial(\varrho u v)}{\partial y}=0 \\ \dfrac{\partial(\varrho v)}{\partial t}+\dfrac{\partial(\varrho u v)}{\partial x}+\dfrac{\partial(\varrho v^2)}{\partial y}=0, \qquad (t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^2, \end{array}\right. \tag{1}$$

where (u, v) has the physical meaning of velocity vector, ϱ has the physical meaning of density. The system (1) is nonstrictly hyperbolic system of conservation laws which has three coinciding characteristic fields and incomplete set of eigenvectors. Due to these properties the shock waves develop strong singularities in the density which are of type of δ-functions on the surface. So (1) is relevant to describe some process of concentration of matter. We will consider the Cauchy problem for (1)

$$\begin{array}{l} \varrho(0, x, y)=\varrho_0(x, y)>0 \\ u(0, x, y)=u_0(x, y) \\ v(0, x, y)=v_0(x, y), \end{array} \tag{2}$$

where ϱ_0, u_0, v_0 are piecewise $C^1(\mathbb{R}^2)$ functions and will be taken in the special form (see below) to produce locally the single shock front.

For the smooth functions the system (1) is equivalent to the following system of equations

$$\left\{\begin{array}{l} \dfrac{\partial \varrho}{\partial t}+\dfrac{\partial(\varrho u)}{\partial x}+\dfrac{\partial(\varrho v)}{\partial y}=0 \\ \dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}=0 \\ \dfrac{\partial v}{\partial t}+u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}=0, \qquad (t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^2. \end{array}\right. \tag{3}$$

The last two equations in the system (3) constitute the so-called inviscid 2-D Burgers equation, which was proposed (but in 3-D case) by Ya. Zeldovich [24] to describe the formation of the large scale structure of the Universe. Further this approach, in particular the consideration of the whole system (3), was developed in the consequent papers (see, for example, [21], [12], [22], [23] and the references therein) from the physical point of view. But if we are dealing with the laws of conservation of mass and momentum it would be more convenient from the mathematical and physical also points of view to investigate the system in divergent form (1) especially in the case when one has the developing of such singularities as shock waves. For 1-D variant of the system (1) the particular generalized solutions were constructed in [2] (there also were proposed some numerical schemes), and the existence theorem in the sense of *Radon measures* for wide class of initial data was proved in [9], [10], [11], see also [3], [4] for the uniqueness results in the framework of *duality* solutions. In [9], [10] the generalized solutions were constructed by the aid of variational principle for appropriate Hamilton-Jacobi equation (see [13], [14], [15], [17] for the variational principle to one quasilinear equation of the first order).

There were number of attempts to construct the generalized solutions to system (1) by analogy with 1-D case and with the aid of variational principle, which can be written for 2-D inviscid Burgers equation and then generalized through corresponding Hamilton-Jacobi equation. Further in this paper we will show that in general this is impossible. The problems for multi-D which involve different types of singularities can expose rather complicated behavior, see, for example, [16], [7], [25]. Here we investigate the «simplest» degenerate case of 2-D system when the characteristics can be calculated explicitly. Nevertheless the generalized solutions to system (1) (see Definition 1 below) exhibit the different behavior compared to ones for the single first order 2-D equation of Hamilton-Jacobi type. The main ideas of the present paper were outlined in [19], [20].

As it has been already said for smooth functions the system (1) is equivalent to (3) and for small enough values of time and smooth initial data the characteristics method can be applied. So the solution to the problem (1), (2) reads

$$\begin{aligned} &\varrho(t, x, y) = \varrho_0(a, b)\,\frac{\partial(a, b)}{\partial(x, y)} \\ &u(t, x, y) = u(0, a, b); \quad v(t, x, y) = v(0, a, b), \end{aligned} \tag{4}$$

where functions $a(t, x, y)$, $b(t, x, y)$ are determined by the equations

$$x = a + tu(0, a, b); \quad y = b + tv(0, a, b). \tag{5}$$

But it is well known that the characteristics of the system (3) can intersect themselves for finite time even for infinitely smooth initial data. Hence one finds the formation of singularities: discontinuities for the velocity (u, v) and δ-functions for the density ϱ which correspond to concentration process in some points or along some curves. So it is necessary to introduce the notion of generalized solution to the problem (1), (2) which is natural to formulate in terms of Radon measures because of the type of singularities.

DEFINITION 1. – *Suppose* $(P_t(dx, dy), I_t(dx, dy), J_t(dx, dy))$ *are the families of Radon measures defined on Borel subsets of* $\mathbb{R}^2$, *weakly continuous with respect to* t *and such that* $P_t \geqslant 0$, *and* I_t, J_t *are absolutely continuous with respect to* P_t *for almost every fixed* $t > 0$. *Let us define the vector function* $(u(t, x, y), v(t, x, y))$ *as Radon-Nykodim derivatives*

$$u(t, x, y) = \frac{dI_t}{dP_t}; \quad v(t, x, y) = \frac{dJ_t}{dP_t}. \tag{6}$$

Then (P_t, I_t, J_t) *will be called the generalized solution of the problem* (1), (2), *iff:* 1) *for an arbitrary functions* $f, g, h \in C_0^1(\mathbb{R}^2)$ (*the space of continuously differentiable functions with compact support*) *and* $0 < t_1 < t_2 < +\infty$

$$\iint f(x, y)\, P_{t_2}(dx, dy) - \iint f(x, y)\, P_{t_1}(dx, dy) = \int_{t_1}^{t_2} \left\{ \iint \frac{\partial f}{\partial x}(x, y)\, I_\tau(dx, dy) + \iint \frac{\partial f}{\partial y}(x, y)\, J_\tau(dx, dy) \right\} d\tau \tag{7}$$

$$\iint g(x, y)\, I_{t_2}(dx, dy) - \iint g(x, y)\, I_{t_1}(dx, dy) = \int_{t_1}^{t_2} \left\{ \iint \frac{\partial g}{\partial x}(x, y)\, u(\tau, x, y)\, I_\tau(dx, dy) + \iint \frac{\partial g}{\partial y}(x, y)\, v(\tau, x, y)\, I_\tau(dx, dy) \right\} d\tau \tag{8}$$

$$\iint h(x, y)\, J_{t_2}(dx, dy) - \iint h(x, y)\, J_{t_1}(dx, dy) = \int_{t_1}^{t_2} \left\{ \iint \frac{\partial h}{\partial x}(x, y)\, u(\tau, x, y)\, J_\tau(dx, dy) + \iint \frac{\partial h}{\partial y}(x, y)\, v(\tau, x, y)\, J_\tau(dx, dy) \right\} d\tau, \tag{9}$$

where $\iint$ *stands for the integration with respect to whole* $\mathbb{R}^2$;

2) *in the weak sense as* $t \to +0$

$$P_t \to \varrho(0, a, b)\, da\, db; \quad I_t \to \varrho(0, a, b)\, u(0, a, b)\, da\, db;$$
$$J_t \to \varrho(0, a, b)\, v(0, a, b)\, da\, db.$$

Concerning the system (3) there is well-known Hopf-Cole's representation [13] for the solution with potential initial velocity. This representation can be derived with the aid of infinitesimal viscosity method (i.e. the adding of the second order operators with small parameter ε to the right-hand side of the last two equations of (3) and then letting ε tend to zero) and allows to determine the location of singularities. Namely

$$(u, v) = \nabla_{(x, y)} \Psi(t, x, y), \tag{10}$$

where

$$\Psi(t, x, y) = \min_{a, b} \left\{ S_0(a, b) + \frac{(x-a)^2}{2t} + \frac{(y-b)^2}{2t} \right\},$$

S_0 is the potential of initial velocity. The representation (10) is valid in domains of smoothness of Ψ and gives the location of the set where singularities arise. That is the singular points of Ψ will be such points (t, x, y) that the global minimum of the expression in the braces with respect to (a, b) is attained more than in one point. For 1-D case the representation (10) in case of constant initial density gives us the way to construct the generalized solution to the 1-D analogue of the problem (1), (2), see [9], [10] (in these papers one can also find the generalization of (10) to nonconstant initial density). Unfortunately in 2-D case the generalized solution in the sense of Definition 1 to the problem (1), (2) for constant initial density can not be constructed via (10).

Our study is organized as follows. In § 2 we formulate the conditions on the initial data (2) which allow to conjecture the development of shock and then assuming that such shock exists and satisfies the problem (1), (2) in the sense of Definition 1 give the formulas for the velocity along the shock. These formulas are the generalization of Rankine-Hugoniot conditions for hyperbolic conservation laws. In § 3 it is shown that in the case of potential initial velocity these formulas determine the singularity surface which in general does not coincide with one determined by the representation (10). As it is shown in § 4 one can obtain certain 1-D quasilinear hyperbolic system of PDEs for the motion *inside* the shock. But this system does not satisfy Friedrichs' symmetrizability condition, so it creates difficulties to prove the global existence theorem. An interesting heuristic procedure to guess the system is deferred to Appendix. Finally in § 5 one investigates some simplest solutions for derived system in case when ϱ_0, u_0, v_0 are piecewise constant but the data along the initial shock front are rather arbitrary. Even with such trivial external field one encounters with complicate behavior of the generalized solutions. It is shown that there are nontrivial cases when the solution can be defined explicitly and other cases when the corresponding Cauchy problem is reduced to equation of type $P_{tt} = \mathrm{const} \cdot \mathrm{P}_{\mathrm{x}}$. Such equations are ill-posed in the spaces of functions of finite smoothness.

Further the letter subscripts (except «i», «j» and «t») will denote appropriate derivatives.

2. – The propagation of the shock front.

Let us denote the coordinates in $\mathbb{R}^2 \times \{t=0\}$ as (a, b). Suppose there is the C^1 curve $\mathcal{G} = (A(l), B(l))$ in the plane (a, b), l is a parameter, such that, for definiteness, $A_l \geqslant 0$, $B_l \geqslant 0$; $A_l^2 + B_l^2 \neq 0$. Suppose that $\mathcal{G}$ can also be found as the solution of the equation $G(a, b) = 0$, $G \in C^1(\mathbb{R}^2)$. Consider the following initial data (2)

$$\begin{aligned} \varrho_0(a, b) &= \varrho_- + (\varrho_+ - \varrho_-)\, H(G(a, b)) \\ u_0(a, b) &= u_- + (u_+ - u_-)\, H(G(a, b)) \\ v_0(a, b) &= v_- + (v_+ - v_-)\, H(G(a, b)), \end{aligned} \tag{11}$$

where H is the Heaviside function, i.e. $H(\theta) = 0$ for $\theta < 0$, $H(\theta) = 1$ for $\theta > 0$; and the following conditions hold

I) $u_-(a, b), v_-(a, b) \in C^1(\overline{\mathcal{G}^-})$; $u_+(a, b), v_+(a, b) \in C^1(\overline{\mathcal{G}^+})$, here $\mathcal{G}^- \equiv \{(a, b)\colon G(a, b) < 0\}$, $\mathcal{G}^+ \equiv \{(a, b)\colon G(a, b) > 0\}$.

II) $A_l v_- - B_l u_- < 0$ and $A_l v_+ - B_l u_+ > 0$ on $\mathcal{G}$.

REMARK 2.1. – *Let us note that the condition* I) *reads that for every point* $(a^*, b^*) = (A(l^*), B(l^*)) \in \mathcal{G}$ *there exists some domain* $Q \subset \mathbb{R}^2 \times [0, T(l^*)]$, $T(l^*) > 0$, $(a^*, b^*) \in Q \cap \{t = 0\}$ *where the transformations* $(j = +, -)$

$$\begin{cases} x = a + tu_j(a, b) \\ y = b + tv_j(a, b) \end{cases}$$

are nondegenerate for every $0 < t < T(l^*)$, *the characteristic lines* (5) *with* $u(0, a, b) = u_-$, $v(0, a, b) = v_-$ *never cross themselves, and the characteristic lines* (5) *with* $u(0, a, b) = u_+$, $v(0, a, b) = v_+$ *never cross themselves.*

But the condition II) *reads that for* $0 < t < T(l^*)$ *there will be another domain* $Q_1 \subset Q$ *where exactly two characteristic lines issued from different sides of* $\mathcal{G}$ *will come to the same point (among these points the shock surface will form itself). So these conditions are natural for the propagating shock front to exist locally up to time moment* $T(l^*) > 0$.

The following definition is essential for our analysis to elicit shock fronts with «good» behavior. Let us denote through $\widehat{Q}_1$ the maximal Q_1-domain from Remark 2.1 which is valid for $l = l^*$.

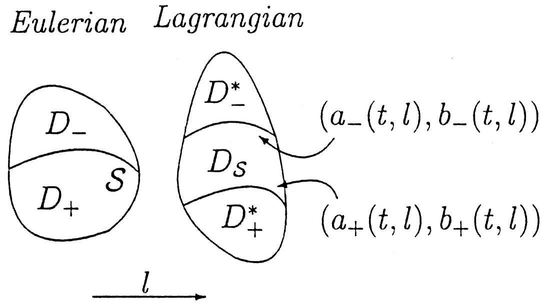

Figure 1. – The map $\mathcal{L}_t$ for fixed t; l is a parameter along $\mathcal{S}$.

DEFINITION 2. – *The shock front $\mathcal{S}$ will be called stable iff*

$$\mathcal{S} \cap \{t = \tau\} \subset \widehat{Q}_1 \cap \{t = \tau\} \quad as \quad 0 < \tau < T(l^*).$$

It is natural to seek the solution to the problem (1), (11) in the form (see Fig. 1, *Eulerian* representation)

$$\begin{aligned} \varrho(t, x, y) &= \varrho_- + (\varrho_+ - \varrho_-) H(S) + \widetilde{P}_t \delta(S) \\ \varrho u(t, x, y) &= \varrho_- u_- + (\varrho_+ u_+ - \varrho_- u_-) H(S) + \widetilde{I}_t \delta(S) \\ \varrho v(t, x, y) &= \varrho_- v_- + (\varrho_+ v_+ - \varrho_- v_-) H(S) + \widetilde{J}_t \delta(S), \end{aligned} \tag{12}$$

where $\varrho_j(t, x, y)$, $u_j(t, x, y)$, $v_j(t, x, y)$, $S(t, x, y) \in C^1(\mathbb{R}_+ \times \mathbb{R}^2)$, $(j = +, -)$ and satisfy (1) in classical sense; $S(t, x, y) = 0$ represents some surface $\mathcal{S}$ in $\mathbb{R}_+ \times \mathbb{R}^2$ and $\mathcal{S} \cap (\{0\} \times \mathbb{R}^2) = \mathcal{G}$; $\widetilde{P}_t, \widetilde{I}_t, \widetilde{J}_t \in C^1(\mathcal{S})$; H is the Heaviside function mentioned above, δ is standard Dirac measure.

If the solution to the problem (1), (11) exists in the sense of Definition 1 then by formulas (6) there is defined the velocity vector $\widetilde{U} = (\tilde{u}, \tilde{v})$ on the surface $\mathcal{S}$ and from each point of $\mathcal{G}$ one can draw the integral curve of $\widetilde{U}$ on $\mathcal{S}$. Suppose $(x^s(t, l), y^s(t, l))$ is the corresponding parametrization of $\mathcal{S}$ (l is a parameter along $\mathcal{G}$) and $d(x^s)/dt = \tilde{u}$, $d(y^s)/dt = \tilde{v}$. Then according to formulas (6) and conditions I), II) one can define the functions $a_j(t, l), b_j(t, l) \in C^1(\mathcal{S})$, $(j = +, -)$ which correspond to the initial positions of characteristics which come to the same points of the surface $\mathcal{S}$ from left and right.

Further let us define the map (see Fig. 1)

$$\mathcal{L}_t \colon (a, b) \to (x, y) \tag{13}$$

in the following way. Let us issue the characteristic line (keeping in mind (11)) from the point (a, b)

$$x(\tau) = a + \tau u_0(a, b); \quad y(\tau) = b + \tau v_0(a, b) \tag{14}$$

and consider the time τ_0 of the intersection of line (14) and $\mathcal{S}$. Then let us take $\mathcal{L}_t(a, b) = (x(t), y(t))$ if $\tau_0 > t$ and $\mathcal{L}_t(a, b) = (x^s(t, l_0), y^s(t, l_0))$ if $\tau_0 \leqslant t$, where l_0 is defined from the condition $(x(\tau_0), y(\tau_0)) = (x^s(\tau_0, l_0), y^s(\tau_0, l_0))$.

Now we are able to formulate the generalization of Rankine-Hugoniot conditions.

THEOREM 2.1. – *The generalized solution in the sense of Definition* 1 *to the problem* (1), (11) *in the form* (12) *exists iff the following formulas are true*

$$\tilde{u} = \tilde{I}_t/\tilde{P}_t\,; \quad \tilde{v} = \tilde{J}_t/\tilde{P}_t \tag{15}$$

$$\tilde{P}_t = \int_0^t [\varrho_0^+((a_+)_\tau(b_+)_l - (b_+)_\tau(a_+)_l) - \varrho_0^-((a_-)_\tau(b_-)_l - (b_-)_\tau(a_-)_l)]\,d\tau$$

$$\tilde{I}_t = \int_0^t [\varrho_0^+ u_0^+((a_+)_\tau(b_+)_l - (b_+)_\tau(a_+)_l) - \varrho_0^- u_0^-((a_-)_\tau(b_-)_l - (b_-)_\tau(a_-)_l)]\,d\tau \tag{16}$$

$$\tilde{J}_t = \int_0^t [\varrho_0^+ v_0^+((a_+)_\tau(b_+)_l - (b_+)_\tau(a_+)_l) - \varrho_0^- v_0^-((a_-)_\tau(b_-)_l - (b_-)_\tau(a_-)_l)]\,d\tau\,,$$

where $\varrho_0^j \equiv \varrho_0(a_j, b_j)$, $u_0^j \equiv u_0(a_j, b_j)$, $v_0^j \equiv v_0(a_j, b_j)$, $(j = +, -)$.

PROOF. – Suppose there are some family of domains $D(\tau)$ with oriented boundaries $\partial D(\tau)$ in the plane (a, b), $D(\tau_1) \subset D(\tau_2)$ for $0 < \tau_1 < \tau_2 < T$. Suppose $\partial D(\tau)$ is closed curve $(a(\tau, l), b(\tau, l))$, l is a parameter along the curve, a, $b \in C^1([\tau_1, \tau_2] \times \mathbb{R})$. Then it is easy to check that the following formula is valid

$$\frac{d}{d\tau} \iint_{D(\tau)} \varphi\, da\, db = \iint_{D(\tau)} \frac{\partial \varphi}{\partial \tau}\, da\, db + \oint_{\partial D(\tau)} \varphi(a_\tau b_l - b_\tau a_l)\, dl\,, \tag{17}$$

where $\varphi \in C^1([\tau_1, \tau_2] \times \overline{\bigcup_{\tau \in [\tau_1, \tau_2]} D(\tau)})$.

Denote through Π some circle in the plane (a, b) such that $\Pi \times \{t\} \cap \mathcal{S}_t \neq \emptyset$, $\mathcal{S}_t \equiv \mathcal{S} \cap \{t\} \times \mathbb{R}^2$. Denote through $D_-(t)$, $D_+(t)$ (Consult with the Fig. 1.) the domains in the plain (x, y) to which at time t the characteristics come from (a_-, b_-), (a_+, b_+) respectively (note that $D_-(t) \cup D_+(t) = \Pi \times \{t\}$ and $D_-(t) \cup D_+(t) = \mathcal{S}_t$). Further let us consider the right hand side of integral identity (7). Take $f \in C_0^1(\mathbb{R}^2)$ such that $\operatorname{supp} f \subset \Pi$. Suppose $D_j^*(t) = \mathcal{L}_t^{-1}(D_j(t))$, $(j = +, -)$, $D_{\mathcal{S}}(t) = \mathcal{L}_t^{-1}(\mathcal{S}_t)$. Define the function $f_t^*(a, b)$ such that $f_t^*(a, b) = f(\mathfrak{L}_t(a, b))$. Then one has

$$\int_{t_1}^{t_2} dt \left\{ \sum_{j=+,-} \iint_{D_j(t)} (f_x u + f_y v)\, P_t(dx, dy) + \int_{S_t} (f_x u + f_y v)\, P_t(dl) \right\} =$$

$$\int_{t_1}^{t_2} dt \left\{ \sum_{j=+,-} \iint_{D_j^*(t)} [f_x(a + tu_0, b + tv_0)\, \varrho_0(a, b)\, u_0(a, b) + \right.$$

$$f_y(a + tu_0, b + tv_0) \varrho_0(a, b) v_0(a, b)]\, da\, db + \int_{S_t} [f_x(x^s(t, l), y^s(t, l)) u(t, l) +$$

$$\left. f_y(x^s(t, l), y^s(t, l))\, v(t, l)]\, P_t(dl) \right\} = \int_{t_1}^{t_2} dt \left\{ \sum_{j=+,-} \iint_{D_j^*(t)} \frac{\partial f_t^*}{\partial t} \varrho_0(a, b)\, da\, db + \right.$$

$$\left. \int_{S_t} \frac{\partial f_t^*}{\partial t} (x^s(t, l), y^s(t, l))\, P_t(dl) \right\} = \int_{t_1}^{t_2} dt \left\{ \sum_{j=+,-} \frac{d}{dt} \iint_{D_j^*(t)} f_t^* \varrho_0(a, b)\, da\, db - \right.$$

$$\int_{S_t} f(x^s(t, l), y^s(t, l)) \left[\varrho_0(a_-, b_-) \left(\frac{\partial(a_-)}{\partial t} \frac{\partial(b_-)}{\partial l} - \frac{\partial(b_-)}{\partial t} \frac{\partial(a_-)}{\partial l} \right) - \right.$$

$$\left. \varrho_0(a_+, b_+) \left(\frac{\partial(a_+)}{\partial t} \frac{\partial(b_+)}{\partial l} - \frac{\partial(b_+)}{\partial t} \frac{\partial(a_+)}{\partial l} \right) \right] dl +$$

$$\left. \int_{S_t} \frac{df_t^*}{dt} (x^s(t, l), y^s(t, l))\, P_t(dl) \right\},$$

using formula (17).

Consider the right hand side of integral identity (8), here g is playing the role of f,

$$\int_{t_1}^{t_2} dt \left\{ \sum_{j=+,-} \iint_{D_j(t)} (g_x u + g_y v)\, I_t(dx, dy) + \int_{S_t} (g_x u + g_y v)\, I_t(dl) \right\} =$$

$$\int_{t_1}^{t_2} dt \left\{ \sum_{j=+,-} \iint_{D_j^*(t)} [g_x(a + tu_0, b + tv_0)\, \varrho_0(a, b)\, u_0^2(a, b) + \right.$$

$$g_y(a + tu_0, b + tv_0)\, \varrho_0(a, b)\, u_0(a, b)\, v_0(a, b)]\, da\, db +$$

$$\left. \int_{S_t} [g_x(x^s(t, l), y^s(t, l))\, u(t, l) + g_y(x^s(t, l), y^s(t, l))\, v(t, l)]\, I_t(dl) \right\} =$$

$$\int_{t_1}^{t_2} dt \left\{ \sum_{j=+,-} \iint_{D_j^*(t)} \frac{\partial g_t^*}{\partial t} \varrho_0(a,b)\, u_0(a,b)\, da\, db + \int_{S_t} \frac{\partial g_t^*}{\partial t} (x^s(t,l), y^s(t,l))\, I_t(dl) \right\} =$$

$$\int_{t_1}^{t_2} dt \left\{ \sum_{j=+,-} \frac{d}{dt} \iint_{D_j^*(t)} g_t^* \varrho_0(a,\, b)\, u_0(a,\, b)\, da\, db - \right.$$

$$\int_{S_t} g(x^s(t,\, l),\, y^s(t,\, l)) \left[\varrho_0^- u_0^- \left(\frac{\partial(a_-)}{\partial t} \frac{\partial(b_-)}{\partial l} - \frac{\partial(b_-)}{\partial t} \frac{\partial(a_-)}{\partial l} \right) - \right.$$

$$\left. \left. \varrho_0^+ u_0^+ \left(\frac{\partial(a_+)}{\partial t} \frac{\partial(b_+)}{\partial l} - \frac{\partial(b_+)}{\partial t} \frac{\partial(a_+)}{\partial l} \right) \right] dl + \int_{S_t} \frac{\partial g_t^*}{\partial t} (x^s(t,\, l),\, y^s(t,\, l))\, I_t(dl) \right\},$$

using formula (17).

Consider the right hand side of integral identity (9), here h is playing the role of f,

$$\int_{t_1}^{t_2} dt \left\{ \sum_{j=+,-} \iint_{D_j(t)} (h_x u + h_y v)\, J_t(dx,\, dy) + \int_{S_t} (h_x u + h_y v)\, J_t(dl) \right\} =$$

$$\int_{t_1}^{t_2} dt \left\{ \sum_{j=+,-} \iint_{D_j^*(t)} [h_x(a + tu_0,\, b + tv_0)\, \varrho_0(a,\, b)\, u_0(a,\, b)\, v_0(a,\, b) + \right.$$

$$h_y(a + tu_0,\, b + tv_0)\, \varrho_0(a,\, b) v_0^2(a,\, b)]\, da\, db +$$

$$\left. \int_{S_t} [h_x(x^s(t,\, l),\, y^s(t,\, l))\, u(t,\, l) + h_y(x^s(t,\, l),\, y^s(t,\, l))\, v(t,\, l)]\, J_t(dl) \right\} =$$

$$\int_{t_1}^{t_2} dt \left\{ \sum_{j=+,-} \iint_{D_j^*(t)} \frac{\partial h_t^*}{\partial t} \varrho_0(a,b)\, v_0(a,b)\, da\, db + \int_{S_t} \frac{\partial h_t^*}{\partial t} (x^s(t,l), y^s(t,l))\, J_t(dl) \right\} =$$

$$\int_{t_1}^{t_2} dt \left\{ \sum_{j=+,-} \frac{d}{dt} \iint_{D_j^*(t)} h_t^* \varrho_0(a,\, b)\, v_0(a,\, b)\, da\, db - \right.$$

$$\int_{S_t} h(x^s(t,\, l),\, y^s(t,\, l)) \left[\varrho_0^- v_0^- \left(\frac{\partial(a_-)}{\partial t} \frac{\partial(b_-)}{\partial l} - \frac{\partial(b_-)}{\partial t} \frac{\partial(a_-)}{\partial l} \right) - \right.$$

$$\left. \left. \varrho_0^+ v_0^+ \left(\frac{\partial(a_+)}{\partial t} \frac{\partial(b_+)}{\partial l} - \frac{\partial(b_+)}{\partial t} \frac{\partial(a_+)}{\partial l} \right) \right] dl + \int_{S_t} \frac{\partial h_t^*}{\partial t} (x^s(t,\, l),\, y^s(t,\, l))\, J_t(dl) \right\},$$

again using formula (17). ■

THEOREM 2.2. – *For every l and $0 < t < T(l)$ the following formulas are true*

$$\int_0^t [x^s(t,l) - a_+(\tau,l) - tu_0^+(\tau,l)]\,\varrho_0(a_+,b_+)((a_+)_\tau(b_+)_l - (b_+)_\tau(a_+)_l)\,d\tau =$$
$$\int_0^t [x^s(t,l) - a_-(\tau,l) - tu_0^-(\tau,l)]\,\varrho_0(a_-,b_-)((a_-)_\tau(b_-)_l - (b_-)_\tau(a_-)_l)\,d\tau\,;$$
$$(18)\qquad \int_0^t [y^s(t,l) - b_+(\tau,l) - tv_0^+(\tau,l)]\,\varrho_0(a_+,b_+)((a_+)_\tau(b_+)_l - (b_+)_\tau(a_+)_l)\,d\tau =$$
$$\int_0^t [y^s(t,l) - b_-(\tau,l) - tv_0^-(\tau,l)]\,\varrho_0(a_-,b_-)((a_-)_\tau(b_-)_l - (b_-)_\tau(a_-)_l)\,d\tau\,.$$

PROOF. – Let us take formulas (15) and write them in the following way

$$(x^s)_\tau P_{t=\tau} = I_{t=\tau}; \quad (y^s)_\tau P_{t=\tau} = J_{t=\tau},$$

Now let us integrate these equalities with respect to τ from 0 to t and then integrating by parts in both sides one obtains

$$\int_0^t (x^s(t) - x^s(\tau)) \times$$
$$[\varrho_0^+((a_+)_\tau(b_+)_l - (b_+)_\tau(a_+)_l) - \varrho_0^-((a_-)_\tau(b_-)_l - (b_-)_\tau(a_-)_l)]\,d\tau =$$
$$\int_0^t (t-\tau) \times$$
$$[\varrho_0^+ u_0^+((a_+)_\tau(b_+)_l - (b_+)_\tau(a_+)_l) - \varrho_0^- u_0^-((a_-)_\tau(b_-)_l - (b_-)_\tau(a_-)_l)]\,d\tau$$
$$\int_0^t (y^s(t) - y^s(\tau)) \times$$
$$[\varrho_0^+((a_+)_\tau(b_+)_l - (b_+)_\tau(a_+)_l) - \varrho_0^-((a_-)_\tau(b_-)_l - (b_-)_\tau(a_-)_l)]\,d\tau =$$
$$\int_0^t (t-\tau) \times$$
$$[\varrho_0^+ v_0^+((a_+)_\tau(b_+)_l - (b_+)_\tau(a_+)_l) - \varrho_0^- v_0^-((a_-)_\tau(b_-)_l - (b_-)_\tau(a_-)_l)]\,d\tau\,.$$

Then applying formulas (14) we come to the assertion of Theorem 2.2. ∎

REMARK 2.2. – *Let us mention that expressions* (18) *look similar to the expressions for the adhesion principle in* [9], [10]. *So all characteristics initially started at points* $(a_j(\tau, l), b_j(\tau, l))$, $j = +, -$, $\tau < t$, *l is fixed, will «concentrate» in one point at time* t.

3. – Comparison with the variational representation.

In this section we assume that $\varrho_0(a, b) \equiv 1$. Now let us compare the formulas (16) for shocks with the restrictions on singularity surface which can be obtained through the variational representation (10) in case of potential initial data. Suppose $\mathcal{L}=(x^v(t, l), y^v(t, l))$ is the singularity surface obtained from (10). Then $\mathcal{L}$ can be represented as the solution of the equation

$$F(a_+, b_+; t, x, y) - F(a_-, b_-; t, x, y) = 0\,, \tag{19}$$

where

$$F \equiv S_0(a, b) + \frac{(x-a)^2}{2t} + \frac{(y-b)^2}{2t}\,,$$

and $a_j(t, x, y)$, $b_j(t, x, y)$, $(j = +, -)$ are defined from the system of equations

$$F_a(a, b; t, x, y) = 0\,; \quad F_b(a, b; t, x, y) = 0\,.$$

THEOREM 3.1. – *Suppose the initial velocity vector in* (11) *is a potential vector and the potential* $S_0(a, b)$ *has the form*

$$\begin{cases} S_0(a, b) = 0\,, & \text{as } b > -\varepsilon f(a, b) \\ S_0(a, b) = b + \varepsilon f(a, b)\,, & \text{as } b < -\varepsilon f(a, b)\,, \end{cases} \tag{20}$$

where $\varepsilon > 0$ *is sufficiently small and the function* $f \in C^\infty(\mathbb{R}^2)$, $f(0, 0) = 0$, *satisfies the conditions: there exists such neighborhood of the point* $(a=0, b=0)$ *that* $f_{aa}(a, 0) \not\equiv 0$ *and the curve determined by the equation* $b + \varepsilon f(a, b) = 0$ *is monotone for every sufficiently small* $\varepsilon > 0$.

Then the formulas (15) *and* (19) *determine different surfaces.*

PROOF It is easy to infer from (19) that as $\varepsilon > 0$ is sufficiently small in the neighborhood of the point $(t = 0, x_1 = 0, x_2 = 0)$ the singularities surface has the form

$$y = \frac{t}{2} - \varepsilon f\left(x,\ -\frac{t}{2}\right) + o(\varepsilon). \tag{21}$$

Now let us find the equation of singularities surface on the basis of formulas (15). As $\varepsilon = 0$ the solution of (15) looks as follows

$$x(t,\ l) = a_-(t,\ l) = a_+(t,\ l) = l\,; \quad y(t,\ l) = b_-(t,\ l) = -b_+(t,\ l) = t/2\,.$$

As $\varepsilon > 0$ let us seek the solution of the system (15) in the form

$$x(t,\ l) = l + \varepsilon\tilde{x}(t,\ l) + o(\varepsilon)\,; \quad y(t,\ l) = t/2 + \varepsilon\tilde{y}(t,\ l) + o(\varepsilon). \tag{22}$$

Then taking into account the relations

$$\begin{aligned} x(t,\ l) &= a_-(t,\ l) = a_+(t,\ l) + t\varepsilon f_a(a_+,\ b_+) \\ y(t,\ l) &= b_-(t,\ l) = b_+(t,\ l) + t(1 + \varepsilon f_b(a_+,\ b_+)), \end{aligned}$$

one obtains $\tilde{P}_t$, $\tilde{I}_t$, $\tilde{J}_t$ and finds

$$\begin{aligned} \frac{\partial\tilde{x}(t,\ l)}{\partial t} &= \frac{1}{2t}\int_0^t f_a\,d\tau \\ \tilde{y}(t,\ l) &= -f(l,\ 0) - \frac{1}{4}\int_0^t \tau f_{aa}\,d\tau + \frac{1}{2}\int_0^t f_b\,d\tau + \frac{1}{4t}\int_0^t \tau^2 f_{aa}\,d\tau\,, \end{aligned} \tag{23}$$

where the values of the derivatives of the function $f(a,\ b)$ are taken at the point $(l,\ -\tau/2)$. But functions (22) must satisfy (21) which in view of (23) is equivalent to the following identity

$$f(l,\ -t/2) - f(l,\ 0) - \frac{1}{4}\int_0^t\left(\tau - \frac{\tau^2}{t}\right) f_{aa}\,d\tau + \frac{1}{2}\int_0^t f_b\,d\tau \equiv 0\,,$$

where the values of the derivatives of the function $f(a,\ b)$ are taken at the point $(l,\ -\tau/2)$. Further, multiplying by t then three times differentiating with respect to t and substituting $t = 0$ one obtains $f_{aa}(l,\ 0) \equiv 0$. The last identity contradicts to the conditions on function $f(a,\ b)$. ■

Differentiating (19) with respect to t and changing notations from t to τ one gets the following condition on the surface $\mathcal{L}$

$$(a_+ - a_-)(x^v)_\tau + (b_+ - b_-)(y^v)_\tau = S_0(a_+,\ b_+) - S_0(a_-,\ b_-), \tag{24}$$

THEOREM 3.2. – *Suppose one has the potential initial data* (11). *Then the surfaces which are defined by* (15) *and* (24) *coincide iff the following relation is true*

$$(a_+ - a_-)\frac{d}{dl}\int_0^t i^*\,d\tau + (b_+ - b_-)\frac{d}{dl}\int_0^t j^*\,d\tau = (S_+ - S_-)\frac{d}{dl}\int_0^t p^*\,d\tau\,, \tag{25}$$

where

$$p^* \equiv (b_+ - b_-)\frac{(a_+)_\tau + (a_-)_\tau}{2} - (a_+ - a_-)\frac{(b_+)_\tau + (b_-)_\tau}{2}$$

$$i^* \equiv (b_+ - b_-)\frac{(S_+)_\tau + (S_-)_\tau}{2} - (S_+ - S_-)\frac{(b_+)_\tau + (b_-)_\tau}{2}$$

$$j^* \equiv (a_+ - a_-)\frac{(S_+)_\tau + (S_-)_\tau}{2} - (S_+ - S_-)\frac{(a_+)_\tau + (a_-)_\tau}{2}\,.$$

PROOF. – It is easy to see that because of our construction of the shock surface x^s, y^s, keeping in mind the assumption that $x^s = x^v$, $y^s = y^v$, one has

$$\begin{aligned} x^s &= a_- + \tau u_-(a_-, b_-) = a_+ + \tau u_+(a_+, b_+) \\ y^s &= a_- + \tau v_-(a_-, b_-) = a_+ + \tau v_+(a_+, b_+)\,. \end{aligned} \tag{26}$$

and the relations (15) at least locally determine the shock surface (if it exists). So (24) occurs to be an additional relation and it is consistent with (15) only if it follows from them.

Hence the condition (24) can be rewritten as follows

$$(a_+ - a_-)\,\tilde{I}_t + (b_+ - b_-)\,\tilde{J}_t = (S_0(a_+, b_+) - S_0(a_-, b_-))\,\tilde{P}_t, \tag{27}$$

where $\tilde{P}_t$, $\tilde{I}_t$, $\tilde{J}_t$ are taken from (16). Further, it is easy to check that the following relations are true

$$((a_+)_\tau (b_+)_l - (b_+)_\tau (a_+)_l) - ((a_-)_\tau (b_-)_l - (b_-)_\tau (a_-)_l) =$$

$$\frac{d}{dl}\left[(b_+ - b_-)\frac{(a_+)_\tau + (a_-)_\tau}{2} - (a_+ - a_-)\frac{(b_+)_\tau + (b_-)_\tau}{2}\right] -$$

$$\frac{d}{d\tau}\left[(b_+ - b_-)\frac{(a_+)_l + (a_-)_l}{2} - (a_+ - a_-)\frac{(b_+)_l + (b_-)_l}{2}\right] = \frac{d}{dl}p^* - \frac{d}{d\tau}p\,;$$

$$u_+((a_+)_\tau(b_+)_l - (b_+)_\tau(a_+)_l) - u_-((a_-)_\tau(b_-)_l - (b_-)_\tau(a_-)_l) =$$

$$((S_+)_\tau(b_+)_l - (b_+)_\tau(S_+)_l) - ((S_-)_\tau(b_-)_l - (b_-)_\tau(S_-)_l) =$$

$$\frac{d}{dl}\left[(b_+ - b_-)\,\frac{(S_+)_\tau + (S_-)_\tau}{2} - (S_+ - S_-)\,\frac{(b_+)_\tau + (b_-)_\tau}{2}\right] -$$

$$\frac{d}{d\tau}\left[(b_+ - b_-)\,\frac{(S_+)_l + (S_-)_l}{2} - (S_+ - S_-)\,\frac{(b_+)_l + (b_-)_l}{2}\right] = \frac{d}{dl} i^* - \frac{d}{d\tau} i\,;$$

$$-[v_+((a_+)_\tau(b_+)_l - (b_+)_\tau(a_+)_l) - v_-((a_-)_\tau(b_-)_l - (b_-)_\tau(a_-)_l)] =$$

$$((S_+)_\tau(a_+)_l - (a_+)_\tau(S_+)_l) - ((S_-)_\tau(a_-)_l - (a_-)_\tau(S_-)_l) =$$

$$\frac{d}{dl}\left[(a_+ - a_-)\,\frac{(S_+)_\tau + (S_-)_\tau}{2} - (S_+ - S_-)\,\frac{(a_+)_\tau + (a_-)_\tau}{2}\right] -$$

$$\frac{d}{d\tau}\left[(a_+ - a_-)\,\frac{(S_+)_l + (S_-)_l}{2} - (S_+ - S_-)\,\frac{(a_+)_l + (a_-)_l}{2}\right] = \frac{d}{dl} j^* - \frac{d}{d\tau} j\,,$$

where $S_k \equiv S_0(a_k, b_k)$, $(k = +, -)$. Taking into account these formulas relation (27) produce (25) and inversely. ■

REMARK 3.1. – *In case the time interval t is small and initial data are sufficiently smooth all terms in relation* (25) *approximately equal to zero and one recovers formulas suggested in* [18]. *Thus when the time elapsing from the moment of the emerging of a shock front is small the variational representation is approximately valid.*

REMARK 3.2. – *The nontrivial* 2-*D example which is known to the author where the variational representation works rigorously is the* 2-*D Riemann problem for* (1) *with compressive piecewise constant potential velocity vector. The example of corresponding initial potential* $S_0(a, b)$ *is shown below* (A, B, C *are constants*)

$$S_0(a, b) = Ca - \left(B + \frac{A + C}{\sqrt{3}}\right) b\,; \qquad b \geqslant \min(0, \sqrt{3}a)$$

$$S_0(a, b) = -Aa - Bb\,; \qquad a \geqslant 0,\ \sqrt{3}a \geqslant b \geqslant -\sqrt{3}a$$

$$S_0(a, b) = Ca + \left(\frac{A + C}{\sqrt{3}} - B\right) b\,; \qquad b \leqslant \min(0, -\sqrt{3}a),$$

$$C > 0,\ A + \sqrt{3}B > 0,\ A - \sqrt{3}B > 0\,.$$

This result is obtained when the discontinuity lines are straight and between *every two neighboring lines is the same angle:* $90°$ *or* $120°$ *or* $180°$. *The conjecture is that the angle value can be taken arbitrary.*

But there the another type of singularity arises: δ*-function in one point for density.*

The validity of variational principle in this case could be understood in the following way: the geometry of corresponding initial potential was flat and had high degree of symmetry.

4. – The flow description inside the shock.

THEOREM 4.1. – *Suppose there exists the generalized solution to the problem* (1), (11) *in the form* (12), (15), (16) *and the surface* S *can be defined from the parametric equations* $x = x(t, l)$, $y = y(t, l)$. *Then the following system of equations is true*

$$
\begin{aligned}
&(\tilde{P}_t)^{\cdot} + x_l\{V(\varrho_+ - \varrho_-) - (\varrho_+ v_+ - \varrho_- v_-)\} + \\
&\qquad\qquad y_l\{(\varrho_+ u_+ - \varrho_- u_-) - U(\varrho_+ - \varrho_-)\} = 0 \\
&(\tilde{I}_t)^{\cdot} + x_l\{V(\varrho_+ u_+ - \varrho_- u_-) - (\varrho_+ u_+ v_+ - \varrho_- u_- v_-)\} + \\
&\qquad\qquad y_l\{(\varrho_+ u_+^2 - \varrho_- u_-^2) - U(\varrho_+ u_+ - \varrho_- u_-)\} = 0 \\
&(\tilde{J}_t)^{\cdot} + x_l\{V(\varrho_+ v_+ - \varrho_- v_-) - (\varrho_+ v_+^2 - \varrho_- v_-^2)\} + \\
&\qquad\qquad y_l\{(\varrho_+ u_+ v_+ - \varrho_- u_- v_-) - U(\varrho_+ v_+ - \varrho_- v_-)\} = 0 \\
&\dot{y} = V, \quad \dot{x} = U,
\end{aligned}
\tag{28}
$$

where $U = \tilde{I}_t/\tilde{P}_t$, $V = \tilde{J}_t/\tilde{P}_t$ *and «dot» denotes the differentiation with respect to* t.

PROOF. – Taking into account (4), (5) one has

$$
\begin{aligned}
x(t, l) &= a_i + t u_0^i(a_i, b_i) \\
y(t, l) &= b_i + t v_0^i(a_i, b_i) \\
\varrho_i &= \varrho_0(a_i, b_i)/D(a_i, b_i),
\end{aligned}
\tag{29}
$$

where $(i = +, -)$ and

$$D(a_i, b_i) \equiv (1 + t(u_0^i)_a)(1 + t(v_0^i)_b) - t^2(u_0^i)_b(v_0^i)_a.$$

Further in this proof for our convenience we will do all calculations independently for indices «+», «−» and so omit indices in the expressions and write a, b, ϱ_0, u, v, D instead of $a_i, b_i, \varrho_0^i, u_0^i, v_0^i, D_i$, $i = +, -$. From (29) one can infer after differentiation the relations for x_l, y_l, $\dot{x}$, $\dot{y}$.

Now consider the right hand side of the first equation from (28) and write all terms concerning the index «+» (for «−» the calculations are analogous). One has

$$\varrho[x_l(V-v)+y_l(u-U)] = \varrho[x_l(\dot y - v) + y_l(u-\dot x)] =$$

$$\varrho[a_l(1+tu_a)+b_l tu_b][\dot b(1+tv_b)+\dot a tv_a] -$$

$$\varrho[a_l tv_a + b_l(1+tv_b)][\dot a(1+tu_a)+\dot b tu_b] =$$

$$-\varrho D(\dot a b_l - \dot b a_l) = -\varrho_0(\dot a b_l - \dot b a_l)\,.$$

Thus the first equation from (28) is equivalent to the first equation from (16). Taking into account the relations

$$\varrho u[x_l(V-v)+y_l(u-U)] = -\varrho_0 u(\dot a b_l - \dot b a_l)$$

and

$$\varrho v[x_l(V-v)+y_l(u-U)] = -\varrho_0 v(\dot a b_l - \dot b a_l)$$

one obtains the other two equations. ■

COROLLARY 4.1. – *The system* (28) *is well defined.*

PROOF. – We have to prove that $\widetilde{P}_t > 0$ under the dynamics of (28). Let us integrate $\widetilde{P}_t$ with respect to l from some l_0 to $l_0+\Delta l$, where Δl is small enough. Note that $\int_{l_0}^{l_0+\Delta l} \widetilde{P}_t\, dl$ is exactly the sum of the areas which are bounded by the curves ($i=+,\,-$):

$$(a_i(\tau,l_0),\, b_i(\tau,l_0))\,,\qquad (a_i(\tau,l_0+\Delta l),\, b_i(\tau,l_0+\Delta l))\,,\qquad 0\leqslant\tau\leqslant t\,;$$

$$(a_i(t,l),\, b_i(t,l))\,,\qquad (a_i(0,l),\, b_i(0,l))\,,\qquad l_0\leqslant l\leqslant l_0+\Delta l\,.$$

Since Δl is arbitrary then $\widetilde{P}_t>0$ as $t>0$. ■

REMARK 4.1. – *The system* (28) *is nonstrictly hyperbolic system with one eigenvalue and three eigenvectors. From the first three equations of system* (28) *it is obvious to find the additional relation between* $(\widetilde{P}_t)^{\cdot}$, $(\tilde{I}_t)^{\cdot}$, $(\tilde{J}_t)^{\cdot}$ *by eliminating* x_l *and* y_l.

REMARK 4.2. – *From the Cauchy-Kovalevskaya theorem one immediately gets the local existence and uniqueness theorem for* (28) *in case of analytic coefficients and initial data.*

5. – Specific type of motion. Constant external state.

The system (28) does not satisfy Friedrichs' symmetrizability condition so to obtain the existence theorem encounters some problems. Nevertheless the internal dynamics of (28) is highly nontrivial which we demonstrate here in the simplest case of constant external density and velocity.

Suppose that in (28) $\varrho_+ \equiv \text{const} \equiv \tilde{\varrho}$, $\varrho_- \equiv \text{const} \equiv \varrho$, $u_+ = v_+ \equiv 0$, $u_- \equiv \text{const} \equiv u$, $v_- \equiv \text{const} \equiv v$; the velocity vector (u, v) satisfies condition II). To simplify the notations we also drop index «t» and «waves» in (28). Then one has

$$\begin{aligned} &\dot{P} + (\varrho - \tilde{\varrho})\{y_l U - x_l V\} - \varrho\{y_l u - x_l v\} = 0 \\ &\dot{I} + \varrho u\{y_l U - x_l V\} - \varrho u\{y_l u - x_l v\} = 0 \\ &\dot{J} + \varrho v\{y_l U - x_l V\} - \varrho v\{y_l u - x_l v\} = 0 \\ &\dot{x} = U \equiv I/P\,, \qquad \dot{y} = V \equiv J/P\,. \end{aligned} \tag{30}$$

From (30) one immediately gets the first integrals

$$uJ - vI = uJ_0 - vI_0 \equiv C(l)\,. \tag{31}$$

and

$$u\dot{y} - v\dot{x} = \frac{C(l)}{P}\,. \tag{32}$$

Now let us take the special initial conditions such that $C(l) \equiv 0$. In other words there exists such $k_0(l)$ that

$$I_0 = k_0 u\,, \qquad J_0 = k_0 v\,. \tag{33}$$

Then taking into account (31) there exists some unknown $k(t, l)$ such that

$$I = ku\,, \qquad J = kv\,.$$

And from (32)

$$uy_l - vx_l = uy_0' - vx_0' \equiv G(l) > 0\,.$$

due to condition II). After rather simple transformations one arrives to the following system

$$
\begin{aligned}
&\dot{P} + G(l)\{\widehat{k}(\varrho - \tilde{\varrho}) - \varrho\} = 0 \\
&(P\widehat{k})^{\cdot} + \varrho G(l)\{\widehat{k} - 1\} = 0 ,
\end{aligned}
\tag{34}
$$

where $\widehat{k} \equiv k/P$.

Now expressing $\widehat{k}$ from the first equation of (34) and substituting it into the second equation one can find the expressions for P and $\widehat{k}$ which read

$$
\begin{aligned}
&P^2 = P_0(l)^2 - 2G(l)\, P_0(l)\, N(l) t + \varrho\tilde{\varrho}\, G(l)^2 t^2 \\
&\widehat{k}(\varrho - \tilde{\varrho}) = \varrho - \frac{\dot{P}}{G(l)} ,
\end{aligned}
\tag{35}
$$

where $N(l) \equiv \widehat{k}_0(l)(\varrho - \tilde{\varrho}) - \varrho$.

From (30) one obtains that $\dot{x} = \widehat{k}u$, $\dot{y} = \widehat{k}v$ and the stability condition (see Definition 2) takes the form

$$
0 < \int_0^t \widehat{k}(\tau, l)\, d\tau < t . \tag{36}
$$

It is easy to find that from (36) taking into account (35) follows the stability condition (see Definition 2)

$$
0 < \widehat{k}_0(l) < 1 . \tag{37}
$$

Then from (37) one can easily see that $N(l) < 0$ and so (35) are well defined. From (35) also follows that $\widehat{k} \to \kappa$ as $t \to \infty$, where

$$
\kappa \equiv \frac{\sqrt{\varrho}}{\sqrt{\varrho} + \sqrt{\tilde{\varrho}}} . \tag{38}
$$

Thus we have proved the following theorem

THEOREM 5.1. – *Suppose that* $\varrho_+ \equiv \text{const} \equiv \tilde{\varrho}$, $\varrho_- \equiv \text{const} \equiv \varrho$, $u_+ = v_+ \equiv 0$, $u_- \equiv \text{const} \equiv u$, $v_- \equiv \text{const} \equiv v$.

Suppose also that $G(l) > 0$ *and* (37) *is true. Than there exists the solution to the problem* (1), (11) *and the shock front tends to the following one as* $t \to +\infty$

$$
\begin{aligned}
x(l, t) &= x_0(l) + \kappa u t \\
y(l, t) &= y_0(l) + \kappa v t ,
\end{aligned}
$$

where κ *is taken from* (38).

Finally to illustrate the nontrivial character of the problem even with constant external fields let us derive the equation in the case $C(l) \not\equiv 0$, but $\varrho = \tilde{\varrho}$. Then from (30) one infers

$$\begin{aligned} &\dot{P} = \varrho\{y_l u - x_l v\} \\ &\dot{I} + \dot{P}\left(\frac{I}{P} - u\right) = \varrho x_l \frac{C(l)}{P} \\ &\dot{x} = \frac{I}{P}\,. \end{aligned} \tag{39}$$

Differentiating the first equation from (39) with respect to t and taking into account integral (32) one gets the equation for P

$$\ddot{P} = \varrho\left(\frac{C(l)}{P}\right)_l \tag{40}$$

So even in the simplest case our system delivers us rather unusual equations of type (40). To illustrate this let us perform the stability analysis for small perturbations to the model linear equation

$$\ddot{P} = KP_x, \tag{41}$$

where $K = \text{const}$. One have to find partial solutions to (41) in the form

$$P(t, x) = e^{i(\xi x + \lambda t)}, \tag{42}$$

where $i^2 = -1$, $\xi \in \mathbb{R}$, $\lambda = \sigma + i\Delta$, $\sigma \in \mathbb{R}$, $\Delta \in \mathbb{R}$. One immediately gets the restriction to the choice of ξ and λ

$$\lambda^2 = -Ki\xi\,.$$

Further, one has

$$\sigma^2 = \Delta^2; \qquad 2\sigma\Delta = -K\xi\,. \tag{43}$$

From (43) it follows that

$$2\Delta^2 = \pm K\xi\,,$$

so we can choose such signs that $-\Delta \sim \text{const}\sqrt{\xi}$ and $-\Delta$ tends to $+\infty$ as ξ tends to $+\infty$. Thus in (42) one has an arbitrary rapid growth of small perturbations with high frequencies and equation (41) is ill-posed in the class of functions of finite smoothness because it does not satisfy classical Petrovsky condition.

Appendix.

Let us carry out some heuristic calculations to obtain the system (28). The generalized solution (1), (11) can also be written in the form

$$\begin{aligned} u &= u_-(t,x,y) + (u_+(t,x,y) - u_-(t,x,y))\,H(S) \\ v &= v_-(t,x,y) + (v_+(t,x,y) - v_-(t,x,y))\,K(S) \\ \varrho &= \varrho_-(t,x,y) + (\varrho_+(t,x,y) - \varrho_-(t,x,y))\,R(S) + \lambda|_{S=0}\,\delta(S), \end{aligned} \tag{44}$$

where δ is usual Dirac δ-function, but H, K, R are different Heaviside functions which can be distinguished by means of the following heuristic multiplication formulas

$$\delta\cdot H = s_1\cdot\delta\,, \qquad \delta\cdot K = s_2\cdot\delta\,, \tag{45}$$

where s_1, s_2 are some functions on the surface S (in what follows we need not multiplication with R, so it is not included in (45)). In addition the following rather natural formulas are supposed to be true in the sense of distributions

$$H^2 \approx K^2 \approx HK \approx RH \approx RK \approx H\,. \tag{46}$$

Let us note that these formulas can be treated rigorously, for example, with the help of theory of new generalized functions [5], [1] (all basic ideas and lines in application to physics can also be found in [6]) but here we do not need such rigor.

We can write

$$U \equiv u_- + s_1(u_+ - u_-)\,, \qquad V \equiv v_- + s_2(v_+ - v_-)\,.$$

Then taking into account the relations (46) one has the following equalities

$$\begin{aligned} \varrho u &= \varrho_- u_- + (\varrho_+ u_+ - \varrho_- u_-)\,H + \lambda U\delta \\ \varrho v &= \varrho_- v_- + (\varrho_+ v_+ - \varrho_- v_-)\,H + \lambda V\delta \\ \varrho u^2 &= \varrho_- u_-^2 + (\varrho_+ u_+^2 - \varrho_- u_-^2)\,H + \lambda U^2\delta \\ \varrho uv &= \varrho_- u_- v_- + (\varrho_+ u_+ v_+ - \varrho_- u_- v_-)\,H + \lambda UV\delta \\ \varrho v^2 &= \varrho_- v_-^2 + (\varrho_+ v_+^2 - \varrho_- v_-^2)\,H + \lambda V^2\delta\,. \end{aligned}$$

Substituting these equalities in the system (1) and equating to zero expressions with different kind of singularities one obtains

$$
\begin{aligned}
&S_t + US_x + VS_y = 0\\
&\lambda_t + (\lambda U)_x + (\lambda V)_y + (\varrho_+ - \varrho_-) S_t +\\
&(\varrho_+ u_+ - \varrho_- u_-) S_x + (\varrho_+ v_+ - \varrho_- v_-) S_y = 0\\
&(\lambda U)_t + (\lambda U^2)_x + (\lambda UV)_y + (\varrho_+ u_+ - \varrho_- u_-) S_t +\\
&(\varrho_+ u_+^2 - \varrho_- u_-^2) S_x + (\varrho_+ u_+ v_+ - \varrho_- u_- v_-) S_y = 0\\
&(\lambda V)_t + (\lambda UV)_x + (\lambda V^2)_y + (\varrho_+ v_+ - \varrho_- v_-) S_t +\\
&(\varrho_+ u_+ v_+ - \varrho_- u_- v_-) S_x + (\varrho_+ v_+^2 - \varrho_- v_-^2) S_y = 0\,.
\end{aligned}
\tag{47}
$$

Now let us mention that the surface S can be defined from the equation $S \equiv x - X(t, y) = 0$ but the functions λ, U, V depend only on t, y. Introducing the differentiation along the direction (U, V) which will be denoted by «dot» rewrite the system (47) in the form

$$
\begin{aligned}
&\dot{y} = V\,, \qquad \dot{X} = U\,, \qquad y(0, l) = l\\
&\dot{\lambda} + \lambda V_y + X_y(V(\varrho_+ - \varrho_-) - (\varrho_+ v_+ - \varrho_- v_-)) +\\
&(\varrho_+ u_+ - \varrho_- u_-) - U(\varrho_+ - \varrho_-) = 0\\
&\lambda \dot{U} + X_y[V(\varrho_+ u_+ - \varrho_- u_- - U(\varrho_+ - \varrho_-)) +\\
&U(\varrho_+ v_+ - \varrho_- v_-) - (\varrho_+ u_+ v_+ - \varrho_- u_- v_-)] +\\
&U(U(\varrho_+ - \varrho_-) - (\varrho_+ u_+ - \varrho_- u_-)) +\\
&(\varrho_+ u_+^2 - \varrho_- u_-^2) - U(\varrho_+ u_+ - \varrho_- u_-) = 0\\
&\lambda \dot{V} + X_y[V(\varrho_+ v_+ - \varrho_- v_- - V(\varrho_+ - \varrho_-)) +\\
&V(\varrho_+ v_+ - \varrho_- v_-) - (\varrho_+ v_+^2 - \varrho_- v_-^2)] +\\
&V(U(\varrho_+ - \varrho_-) - (\varrho_+ u_+ - \varrho_- u_-)) +\\
&(\varrho_+ u_+ v_+ - \varrho_- u_- v_-) - U(\varrho_+ v_+ - \varrho_- v_-) = 0\,.
\end{aligned}
$$

Now let us transform the system (48) in the following way. Multiply the first equation by U and add to the second equation, then multiply the first equation by V and add to the third equation. Finally multiplying all obtained equations by y_l and using the relation $\dot{y}_l = V_y y_l$ one gets the system (28).

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