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Finite Groups with Primitive Sylow Normalizers.

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Sunto. – *Si prova che sono primitivi i gruppi finiti nei quali siano primitivi i normalizzanti dei sottogruppi di Sylow. Si classificano i gruppi di tale classe, denotata con* $\mathrm{N}\mathcal{P}$, *e si studiano le classi di Schunck il cui bordo sia contenuto in* $\mathrm{N}\mathcal{P}$, *fornendo, tra l'altro, condizioni necessarie e sufficienti affinchè i proiettori siano subnormalmente immersi.*

Summary. – *We prove that are primitive the finite groups whose normalizers of the Sylow subgroups are primitive. We classify the groups of such class, denoted by* $\mathrm{N}\mathcal{P}$, *and we study the Schunck classes whose boundary is contained in* $\mathrm{N}\mathcal{P}$. *We give also necessary and sufficient conditions in order that the projectors be subnormally embedded.*

Let $\mathcal{X}$ be a non empty class of finite groups and denote by $\mathrm{N}\mathcal{X}$ the class of finite groups G such that, for any prime p which divides $|G|$, the normalizer of a Sylow p-subgroup of G belongs to $\mathcal{X}$.

A well known result of Glauberman [9] asserts that, if $\mathcal{X} = \bigcup_{p \in \mathbb{P}} \mathcal{X}_p$, where $\mathbb{P}$ is the set of primes and $\mathcal{X}_p$ is the class of finite p-groups, then $G \in \mathrm{N}\mathcal{X}$ if and only if $G \in \mathcal{X}$. In 1986 Bianchi, Berta Mauri and Hauck [3] generalized that result by proving that, if $\mathcal{N}$ is the class of nilpotent finite groups, then $\mathrm{N}\mathcal{N} = \mathcal{N}$. Recently Ballester-Bolinches and Shemetkov [2] proved that if normalizers of Sylow p-subgroups are p-nilpotent for every prime p then the group is nilpotent.

If the class $\mathcal{X}$ is subgroup-closed it is always true that $\mathcal{X} \subseteq \mathrm{N}\mathcal{X}$, nevertheless equality does not hold in general, even if $\mathcal{X}$ satisfies strong closure properties. In fact, in 1988 Fedri and Serena [7] showed that, if $\mathcal{U}$ is the class of supersoluble finite groups, then the class $\mathrm{N}\mathcal{U}$ is much larger than $\mathcal{U}$; in 1991 the same authors and Bryce [5] throughly studied the class $\mathrm{N}\mathcal{U}$.

Observe that $\mathrm{N}\mathcal{X}$ need not coincide with $\mathcal{X}$, even if $\mathcal{X}$ is a S-closed saturated Fitting formation; for instance, if $\mathcal{X}$ is the class of finite groups with ordered Sylow tower, $S_4 \in \mathrm{N}\mathcal{X} - \mathcal{X}$.

In this paper we prove that, if $\mathcal{P}$ is the class of finite primitive groups, then $\mathrm{N}\mathcal{P}$ is a homomorph strictly contained in $\mathcal{P}$; this means that the class $\mathcal{P}$, which has a very few closure properties, is N-closed and that $\mathcal{X} \subseteq \mathrm{N}\mathcal{X}$ is no more true if the class is not S-closed. Then we classify the groups in $\mathrm{N}\mathcal{P}$.

As $\mathrm{N}\mathcal{P}$ is contained in $\mathcal{P}$, a homomorph $\mathcal{X}$ whose Q-boundary is contained in

$\mathrm{N}\mathcal{P}$ is an $\mathcal{F}$-Schunck class, where $\mathcal{F}$ is the class of finite groups; equivalently, $\mathcal{X}$ is an $\mathcal{F}$-projecive class (see [6], III, (2.7) and (3.10)). We consider $\mathcal{F}$-Schunck classes whose boundary is contained in $\mathrm{N}\mathcal{P}$ and characterize that ones which are Gaschütz classes. Recall that a Gaschütz class is a class $\mathcal{X}$ of (finite) groups such that the set of $\mathcal{X}$-covering subgroups is non-empty, for all groups G (see [6], III, (3.5)). Moreover we study the behaviour of $\mathcal{X}$-projectors and find conditions for them to be subnormally embedded. A subgroup H is subnormally embedded in a group G if, for all primes, a Sylow subgroup of H is a Sylow subgroup of some subnormal subgroup of G. Most of our notation is standard and can for instance be found in [6]. «Group» will stand for «finite group».

1. – N-closure property of $\mathcal{P}$.

1.1. LEMMA. – *Let $G \in \mathrm{N}\mathcal{P}$. If p is a prime dividing $|G|$ and $G_p \in Syl_p(G)$, then G_p is the unique minimal normal subgroup of $N_G(G_p)$. In particular, $C_G(G_p) = G_p$.*

PROOF. – This follows immediately from the hypothesis that $N_G(G_p)$ is primitive.

1.2. LEMMA. – *Let $G \in \mathrm{N}\mathcal{P}$ and let N be a normal subgroup of G. Then N is a Hall subgroup of G.*

PROOF. – Let p be a prime dividing $|N|$ and G_p be a Sylow p-subgroup of G. Since $N_G(G_p) \cap N \neq 1$, we get by Lemma 1.1, that $G_p \leq N_G(G_p) \cap N$, hence $G_p \leq N$.

1.3. LEMMA. – *The groups in $\mathrm{N}\mathcal{P}$ are monolithic.*

PROOF. – Suppose that G is a group in $\mathrm{N}\mathcal{P}$. Assume that N and L are two different minimal normal subgroups. Then N and L have trivial intersection and by Lemma 1.2 they are Hall subgroups of G. Let p be a prime dividing the order of N. Then L centralizes the Sylow p-subgroups of N which are Sylow p-subgroups of G. This contradicts Lemma 1.1. Therefore G is monolithic.

1.4. PROPOSITION. – *The class $\mathrm{N}\mathcal{P}$ is (strictly) contained in $\mathcal{P}$.*

PROOF. – Let G be a group belonging to $\mathrm{N}\mathcal{P}$. If there is a prime p such that $O_p(G) \neq 1$, then, by Lemma 1.1, $O_p(G) = G_p \in Syl_p(G)$, and $G = N_G(G_p)$ is primitive. If instead $Fit(G)=1$ then by Lemma 1.3, G has a unique minimal normal subgroup and it is not abelian, therefore G is primitive.

The inclusion $\mathrm{N}\mathcal{P} \subseteq \mathcal{P}$ is strict; the smallest order of a group $G \in \mathcal{P} - \mathrm{N}\mathcal{P}$ is 20.

1.5. PROPOSITION. – *The class $\mathrm{N}\mathcal{P}$ is a homomorph.*

PROOF. – Let G be a group in $\mathrm{N}\mathcal{P}$ and let N be a normal subgroup of G such that $G/N \notin \mathrm{N}\mathcal{P}$. By Lemma 1.2 N is a Hall subgroup of G and therefore G splits over N (Schur-Zassenhaus Theorem). Let $G = K \ltimes N$. Since $K \notin \mathrm{N}\mathcal{P}$ and K is a Hall subgroup of G, it is easy to see that there exists a Sylow p-subgroup G_p of G such that $G_p \leqslant K$ and $N_G(G_p) = N_K(G_p) \notin \mathcal{P}$: a contradiction.

2. – Classification of the groups in $\mathrm{N}\mathcal{P}$.

As it is well known there exist 3 distinct types of primitive groups

$$\mathcal{P} = \mathcal{P}_1 \dot{\cup} \mathcal{P}_2 \dot{\cup} \mathcal{P}_3$$

where $\mathcal{P}_1$ is the class of primitive groups of affine type, i.e. primitive groups whose socle is abelian; $\mathcal{P}_2$ is the class of (primitive) monolithic groups, whose socle is not abelian; $\mathcal{P}_3$ is the class of primitive non-monolithic groups (see, for instance, [6]). As an easy consequence of Lemma 1.3 we get that

2.1. PROPOSITION. – *The class* $\mathrm{N}\mathcal{P}$ *contains no primitive group of type* $\mathcal{P}_3$.

A fundamental theorem of Walter on simple groups is used in the proof of the following auxiliary result. We were not able to decide whether or not it can be avoided.

2.2. LEMMA. – *If G is a non abelian simple group with elementary abelian Sylow subgroups, for all primes, then one of the following holds:*

(i) *G is isomorphic to the Janko group J_1;*

(ii) *G is isomorphic to $PSL(2, p^f)$, where either $p^f > 3$, $p^f \equiv 3, 5 \pmod 8$, $(p^f \pm 1)/2$ square-free or $p = 2$, $f > 1$, $2^f \pm 1$ square-free.*

PROOF. – Walter (see, for instance, [10] p. 485) proved that if G is a non-abelian simple group with abelian Sylow 2-subgroups, then one of the following holds:

(i) G is isomorphic to the Janko group J_1;

(ii) G is isomorphic to $PSL(2, p^f)$, where either $p^f > 3$, $p^f \equiv 3, 5 \pmod 8$ or $p = 2$, $f > 1$;

(iii) G is of Ree type.

By hypothesis all Sylow subgroups of G are elementary abelian. Then G is not of Ree-type, because a Sylow 3-subgroup of a group of Ree type has class 3 (see, for instance, [10], p. 483) . If G is isomorphic to $PSL(2, p^f)$ for an odd prime q dividing $(p^f \pm 1)/k$ ($k = (2, p^f \pm 1)$), the Sylow q-subgroups of G are cyclic and so they have order q; it follows that $(p^f \pm 1)/k$ is square-free.

2.3. LEMMA. – *The alternating group A_5 is, up to isomorphism, the only non abelian simple group in* $\mathrm{N}\mathcal{P}$.

PROOF. – Let G be a non abelian simple group in $\mathrm{N}\mathcal{P}$. Then all Sylow subgroups of G are elementary abelian (see Lemma 1.1), hence by Lemma 2.2 G is either isomorphic to the Janko group J_1 or to $PSL(2, p^f)$, with conditions in the statement. On the other hand, Lemma 1.1 says that Sylow subgroups of G are self-centralizing. The group G is not isomorphic to J_1, because the centralizer of a Sylow 3-subgroup of J_1 is cyclic of order 6 (see, for instance, [10], p. 482). Thus G is isomorphic to $PSL(2, p^f)$. Then G possesses cyclic subgroups of order $(p^f \pm 1)/k$, $(k = (2, p^f \pm 1))$, whose Sylow q-subgroups are Sylow q-subgroups of G, if $q \neq 2$; therefore $(p^f \pm 1)/k$ is a prime, because the Sylow subgroups of G are selfcentralizing; it follows $p^f = 5$ or $p^f = 4$, that is $G \simeq PSL(2, 4) \simeq PSL(2, 5) \simeq A_5$. It is easy to see that $A_5 \in \mathrm{N}\mathcal{P}$.

2.4. PROPOSITION. – *The only group in* $\mathrm{N}\mathcal{P}$ *whose socle is not abelian is, up to isomorphism,* A_5.

PROOF. – Let $G \in \mathrm{N}\mathcal{P} \cap \mathcal{P}_2$ and let R be the socle of G. Since R is a Hall subgroup of G (see Lemma 1.2), R is a direct power U^n of a non abelian simple group U, whose Sylow subgroups are elementary abelian and self-centralizing (see Lemma 1.1); it follows $U \simeq A_5$ (see proof of Lemma 2.3) and therefore $G \simeq A_5 \wr_{nat} K$ (natural wreath product), where K is a transitive subgroup of S_n and 2, 3, 5 do not divide $|K|$. If $n > 1$, since $C_R(K) \neq 1$, we have that a Sylow p-subgroup of G, with p dividing $|K|$, is not self-centralizing, a contradiction. Thus $n = 1$, that is $K = 1$ and $G \simeq A_5$.

Next propositions provide the classification of groups in $\mathrm{N}\mathcal{P} \cap \mathcal{P}_1$.

2.5. PROPOSITION. – *Every group in* $\mathrm{N}\mathcal{P} \cap \mathcal{P}_1$ *is soluble.*

PROOF. – Assume that the theorem is false and let a counterexample G of smallest order be chosen. Let S be the soluble radical of G. By Lemma 1.2 S is a Hall subgroup of G, therefore by minimality of $|G|$ we have that $S = \mathrm{Soc}(G)$ is a Sylow subgroup of G. On the other hand, since $G/S \in \mathrm{N}\mathcal{P} \cap \mathcal{P}_2$, we get $G/S \simeq A_5$ (Proposition 2.4). It follows that G is a holomorph of an elementary abelian p-group S by an irreducible subgroup K of $GL(n, p) \simeq \mathrm{Aut}\ S$, with $K \simeq A_5$, where $|S| = p^n$ $(p > 5)$. Now it is well known that an irreducible $GF(p)\,A_5$-module V on which A_5 acts faithfully is, up to isomorphism, one of the following.

1st case: $3 \mid p - 1$ and V is the $GF(p)\,A_5$-module induced from a linear non-principal $GF(p)\,A_4$-module, where A_4 is a point stabilizer ($\dim_{GF(p)} V = 5$).

2nd case: V is a non-principal direct summand of the $GF(p)\,A_5$-module induced from the principal $GF(p)\,A_4$-module, A_4 as above ($\dim_{GF(p)} V = 4$).

3rd case: $p \neq 2$, $\sqrt{5} \in GF(p)$ and any point stabilizer acts irreducibly on V ($\dim_{GF(p)} V = 3$).

In any case it is easy to see that there exists a Sylow subgroup Q of A_5 such that $C_V(Q) \neq 0$: in the first case $Q \in \mathrm{Syl}_5(A_5)$, in 2nd and 3rd cases $Q \in \mathrm{Syl}_3(A_5)$.

Therefore $C_S(Q) \neq 1$ and so $C_G(Q) > Q$, which is false by Lemma 1.1.

Let p and q be different primes and put $r = exp(p, q)$, i.e. the (multiplicative) order of p modulo q. We will denote by $\Lambda(p, q)$ the holomorph of the additive group of $GF(p^r)$ by the subgroup of order q in the Singer cycle of $GL(r, p)$ (see, for instance, [11] p. 187).

2.6. PROPOSITION. – *For any pair (p, q) of distinct primes the groups $\Lambda(p, q)$ and $\Lambda(q, p)$ are, up to isomorphism, the only two groups G belonging to* $\mathrm{N}\mathcal{P}$ *such that* $\pi(G) = \{p, q\}$.

PROOF. – It is easy to see that $\Lambda(p, q)$ and $\Lambda(q, p)$ are in $\mathrm{N}\mathcal{P}$. Conversely, let G be a group in $\mathrm{N}\mathcal{P}$ with $\pi(G) = \{p, q\}$. Since G is soluble (Burnside's theorem) we can assume, by Lemma 1.2, that $\mathrm{Soc}(G) = G_p \in \mathrm{Syl}_p(G)$. On the other hand Lemma 1.1 ensures that a Sylow q-subgroup G_q of G is elementary abelian; therefore, since G_q acts faithfully and irreducibly on G_p, G_q has order q and so $G \simeq \Lambda(p, q)$.

2.7. PROPOSITION. – *Let* $G \in N\mathcal{P} \cap \mathcal{P}_1$. *Then* $|\pi(G)| \leq 2$.

PROOF. – Let G be a minimal counterexample. By Proposition 2.5 G is soluble; therefore, since $\mathrm{N}\mathcal{P}$ is a homomorph, minimality of $|G|$ and Lemma 1.2 ensure that G is a holomorph of an elementary abelian p-group V by an irreducible p'-subgroup K of $GL(n, p) \simeq \mathrm{Aut}\, V$, with $|\pi(K)| = 2$ ($|V| = p^n$). On the other hand, by Proposition 2.6, we get $K \simeq \Lambda(q, t)$, for some pair (q, t) of distinct primes (and different from p). If K_t is a Sylow t-subgroup of K (of G), we have $C_V(K_t) \neq 1$, because $O_t(K) = 1$ (see, for instance, [12] IX. 6.2 Theorem). Thus K_t is not self-centralizing in G, which contradicts Lemma 1.1.

2.8. THEOREM. – *The class* $\mathrm{N}\mathcal{P}$ *is the following homomorph* (*contained in* $\mathcal{P}$): $\mathrm{N}\mathcal{P} = (1) \dot{\cup} (A_5) \dot{\cup} (C_p \mid p \in \mathbb{P}) \dot{\cup} (\Lambda(p, q) \mid (p, q)$ pair of different primes).

PROOF. – This follows immediately from Propositions 2.4, 2.6 and 2.7.

2.9. COROLLARY. – $\mathrm{N}\mathcal{P}$ *is a* N-*closed homomorph, that is* $\mathrm{N}(\mathrm{N}\mathcal{P}) = \mathrm{N}\mathcal{P}$.

PROOF. – This follows easily from Theorem 2.8.

3. – Schunck classes with boundary in $\mathrm{N}\mathcal{P}$.

In this section we examine the Schunck classes in title.

The word «Schunck class» will stand for «$\mathcal{F}$-Schunck class», where $\mathcal{F}$ is the class of finite groups.

We notice that, since $(C_p \mid p \in \mathbb{P}) \subseteq \mathrm{N}\mathcal{P}$, among Schunck classes whose boundary is in $\mathrm{N}\mathcal{P}$ there are all Schunck classes $\mathcal{X}$ with the property that $\mathcal{X}$-projectors of G are normal in G for all soluble groups G (see [4]). On the other hand it is easy to see that a Schunck class, whose boundary is in $\mathrm{N}\mathcal{P}$, is not, in general, a NE-class, i.e. a Schunck class with normally embedded projectors. For example, let $\mathcal{O}=\mathrm{b}\mathcal{X}$ be a Q-boundary contained in $\mathrm{N}\mathcal{P}$ and containing $\Lambda(p, q)$ but not C_p, with $q \mid p-1$. Let G be the holomorph of an elementary abelian group N of order p^2 by a cyclic group U of order pq, that contains a power automorphism of N of order q. It is easy to see that $Proj\,(G)$ is the class of conjugates of U and U is not normally embedded in G.

More meaningful is the behaviour of these classes for what concerning SNE-property: we say that a Schunck class has the SNE-property if projectors are subnormally embedded.

3.1. PROPOSITION. – *Let $\mathcal{X}$ be a Schunck class whose boundary $\mathrm{b}\mathcal{X}$ is contained in $\mathrm{N}\mathcal{P}$.*

(i) *If $A_5 \notin \mathrm{b}\mathcal{X}$, then $\mathcal{X}$ is a Gaschütz class and $\mathrm{Cov}_{\mathcal{X}}(G)$ is a conjugacy class for all groups G.*

(ii) *If $A_5 \in \mathrm{b}\mathcal{X}$ and one of the maximal subgroups of A_5 does not belong to $\mathrm{b}\mathcal{X}$, then $\mathcal{X}$ is a Gaschütz class.*

PROOF. – We prove that $\mathrm{Cov}_{\mathcal{X}}(G)$ is not empty and that, if $A_5 \notin \mathrm{b}\mathcal{X}$, then $\mathrm{Cov}_{\mathcal{X}}(G)$ is a conjugacy class, for all $G \in \mathrm{b}\mathcal{X}$ (see, for instance, [6], III, (3.8), (3.13)).

It is easy to see that, if $C_p \in \mathrm{b}\mathcal{X}$, then $\mathrm{Cov}_{\mathcal{X}}(C_p) = \{1\}$. Now let $\Lambda(p, q) \in \mathrm{b}\mathcal{X}$, for a pair (p, q) of distinct primes (see Theorem 2.8). Since $\mathrm{b}\mathcal{X}$ is a Q-boundary, we get $C_q \notin \mathrm{b}\mathcal{X}$ and therefore $C_q \in \mathcal{X}$; it follows $\mathrm{Cov}_{\mathcal{X}}(\Lambda(p, q)) = \mathrm{Syl}_q(\Lambda(p, q))$, hence $\mathrm{Cov}_{\mathcal{X}}(\Lambda(p, q))$ is a conjugacy class of $\Lambda(p, q)$. Thus we proved (i). Now let $A_5 \in \mathrm{b}\mathcal{X}$ and let M be a maximal subgroup of A_5 which not belongs to $\mathrm{b}\mathcal{X}$. It remains to show that $\mathrm{Cov}_{\mathcal{X}}(A_5)$ is not empty (see Theorem 2.8). If $M_{ab} \notin \mathrm{b}\mathcal{X}$ we have $M \in \mathcal{X}$ and therefore $M \in \mathrm{Cov}_{\mathcal{X}}(A_5)$. Then let $M_{ab} \in \mathrm{b}\mathcal{X}$, that is $C_3 \in \mathrm{b}\mathcal{X}$, if $M \simeq A_4$, $C_2 \in \mathrm{b}\mathcal{X}$, if $M \not\simeq A_4$. Let firstly $M \simeq A_4$. If $C_2 \notin \mathrm{b}\mathcal{X}$, the Sylow 2-subgroups of A_5 belong to $\mathcal{X}$ and they are $\mathcal{X}$-covering subgroups of A_5, because $A_4 \notin \mathcal{X}$ and $C_3 \notin \mathcal{X}$. If instead $C_2 \in \mathrm{b}\mathcal{X}$, then either all non-trivial subgroups of A_5 do not belong to $\mathcal{X}$ or the Sylow 5-subgroups of A_5 belong to $\mathcal{X}$ (if $C_5 \notin \mathrm{b}\mathcal{X}$); in any case $\mathrm{Cov}_{\mathcal{X}}(A_5)=\{1\}$ or $\mathrm{Cov}_{\mathcal{X}}(A_5) = \mathrm{Syl}_5(A_5)$ and so $\mathrm{Cov}_{\mathcal{X}}(A_5) \neq \emptyset$. Now let $M \not\simeq A_4$. Since $\mathrm{b}\mathcal{X}$ is a Q-boundary, we can assume $A_4 \in \mathrm{b}\mathcal{X}$ and so $C_3 \notin \mathrm{b}\mathcal{X}$. Therefore $C_3 \in \mathcal{X}$ and, by hypothesis, $C_2 \in \mathrm{b}\mathcal{X}$. It follows that the Sylow 3-subgroups of A_5 are $\mathcal{X}$-covering subgroups of A_5.

REMARK. – If $\mathcal{X}$ is a Schunck class whose boundary contains A_5 and all its maximal subgroups, then $\mathcal{X}$ is not a Gaschütz class; thus condition in (ii) of Proposition 3.1 is necessary for $\mathcal{X}$ is a Gaschütz class. Indeed, if $(A_5, A_4, S_3, D_5) \subseteq \mathrm{b}\mathcal{X}$, we get $C_2, C_3 \in \mathcal{X}$, because $\mathrm{b}\mathcal{X}$ is a Q-boundary; it follows obviously that $\mathrm{Cov}_{\mathcal{X}}(A_5) = \emptyset$. If instead $\mathcal{X}$ is a Schunck class whose boundary is contained in $\mathrm{N}\mathcal{P}$, the condition $A_5 \notin \mathrm{b}\mathcal{X}$ (Proposition 3.1, (i)) is not necessary for $\mathrm{Cov}_{\mathcal{X}}(G)$ is a conjugacy class, for all groups G. For instance, let $\mathrm{b}\mathcal{X} = \mathcal{B}_r = (A_5, N_s, N_t)$, where $\{r, s, t\} = \{2, 3, 5\}$ and $N_p = N_{A_5}(S_p)$, with $S_p \in \mathrm{Syl}_p(A_5)$, $p \in \{2, 3, 5\}$. It is easy to see that $\mathrm{Cov}_{\mathcal{X}}(A_5) = \{N_r^{A_5}\}$ (conjugacy class of N_r); therefore $\mathrm{Cov}_{\mathcal{X}}(G)$ is a conjugacy class for all groups G (see proof Proposition 3.1). We mention that the classes $\mathrm{h}\mathcal{B}_r$, $r = 2, 3, 5$, provide an example, due to Förster [8], of three Gaschütz classes whose intersection is not a Gaschütz class.

Finally, it is easy to see that, if $\mathcal{X}$ is a Gaschütz class whose boundary is in $\mathrm{N}\mathcal{P}$, then $\mathrm{Cov}_{\mathcal{X}}(G)$ need not be a conjugacy class for all groups G; for instance $\mathrm{b}\mathcal{X} = (A_5, M)$, with M a maximal subgroup of A_5.

3.2. PROPOSITION. – *Let $\mathcal{X}$ be a Schunck class whose boundary $b\mathcal{X}$ is contained in $\mathrm{N}\mathcal{P}$. If $b\mathcal{X}$ satisfies the following two conditions*:

(α) $A_5 \notin b\mathcal{X}$,

(β) $\Lambda(p, q) \in b\mathcal{X}$ *implies* $\Lambda(t, p) \notin b\mathcal{X}$, *for every prime* t $(\neq p)$,
then $\mathcal{X}$ is a SNE-class.

PROOF. – Let G be a group whose $\mathcal{X}$-projectors are not subnormally embedded (sne), and let G be of minimal order with respect to this property. Let U be an $\mathcal{X}$-projector of G, p be a prime, and U_p a Sylow p-subgroup of U which is not a Sylow p-subgroup of any subnormal subgroup of G. Since $\mathcal{X}$ is a Gaschütz class (see Proposition 3.1, (i)), minimality of G ensures that $\mathrm{Core}_G(U) = 1$ and $\mathrm{Soc}(G) \leq \mathrm{Fit}(G) = O_p(G)$. Let $H = UO_p(G)$. Since $\mathrm{Core}_G(U) = 1$, we have $U < H$; therefore H has a quotient H/K in $\mathrm{b}\mathcal{X}$ and so H/K is either isomorphic to a $\Lambda(p, q)$ or to $C_p \in \mathrm{b}\mathcal{X}$. On the other hand G has a quotient G/L in $\mathrm{b}\mathcal{X}$. If G/L is of prime order, then $UL/L = 1$ $(U \leq L)$. By minimality of G we get that U is *sne* in L and so in G, a contradiction. Consequently G/L is isomorphic to a $\Lambda(p_1, p_2)$. If $p_2 \neq p$, we get $U_p \leq L$; on the other hand minimality of G implies that there exists a subnormal subgroup S of UL such that $U_p \in \mathrm{Syl}_p(S)$; it follows $U_p \in \mathrm{Syl}_p(S \cap L)$ and we get a contradiction, because $S \cap L$ sn G. Thus $p_2 = p$ and so $C_p \in \mathcal{X}$. Therefore $H/K \simeq \Lambda(p, q)$, which contradicts condition (β).

Next proposition shows that condition (β) is necessary for *SNE*-property holds.

3.3. PROPOSITION. – *Let p, q, t be primes, with $p \neq q$, t. If $\mathcal{X}$ is a Schunck class whose boundary* b$\mathcal{X}$ *satisfies the following condition*:

$$(\Lambda(p, q), \Lambda(t, p)) \subseteq b\mathcal{X} \subseteq N\mathcal{P},$$

then there exists a soluble group G, whose projectors are not sne *in G.*

PROOF. – A group of desidered type can be constructed as follows. Since $\Lambda(t, p)$ is monolithic and $O_p(\Lambda(t, p)) = 1$, a well known result of Gaschütz (see [6], B. (10.9)) ensures that, if $r = \exp(p, q)$, there exists an irreducible $GF(p^r)\Lambda(t, p)$-module V on which $\Lambda(t, p)$ acts faithfully. On the other hand, if Q is a group of order q, V is a $GF(p^r)Q$-module on which Q acts (faithfully) as a group of scalar transformations. Thus, denoted $\Lambda(t, p) \times Q$ by Γ, V is an irreducible $GF(p^r)\Gamma$-module on which Γ acts faithfully. Let G be the holomorph of the additive group of V by Γ. Let $N = \mathrm{Soc}(G)$, $P \in \mathrm{Syl}_p(\Gamma)$, $U = PQ$, H be a subgroup of UN that contains strictly U and $L = H \cap N$. If L/K is a chief factor of H, since $O_p(UK/K) = PK/K$, we get $PK = C_{UK}(L/K)$ and therefore $H/C_{UK}(L/K) \simeq LQ/K \simeq \Lambda(p, q) \in \mathrm{b}\mathcal{X}$; thus $H \notin \mathcal{X}$ and so U is $\mathcal{X}$-maximal in UN. On the other hand, since P is an $\mathcal{X}$-projector of $\Lambda(t, p)$, we have that $U = PQ$ is an $\mathcal{X}$-projector of G. Now, let S be a subnormal subgroup of G that contains P. Since $O_p(G) = N$, we get $O_p(S) = N \cap S$ and therefore $P \notin \mathrm{Syl}_p(S)$; thus U is not *sne* in G.

3.4. THEOREM. – *Let $\mathcal{X}$ be a Schunck class whose boundary* b$\mathcal{X}$ *is contained in* N$\mathcal{P}$ *and not contains A_5. Then $\mathcal{X}$ is a* SNE-*class if and only if* b$\mathcal{X}$ *satisfies condition* (β).

PROOF. – It follows immediately from Propositions 3.2 and 3.3.

The condition $A_5 \notin \mathrm{b}\mathcal{X}$ is not necessary for a Schunck class $\mathcal{X}$, with $\mathrm{b}\mathcal{X} \subseteq \mathrm{N}\mathcal{P}$, is a *SNE*-class. For example, if $(A_5, C_2, C_3, C_5) \subseteq \mathrm{b}\mathcal{X} \subseteq (A_5, C_p \mid p \in \mathbb{P})$, then $\mathcal{X}$-projectors are normal subgroups (see [6], III (4.2)). On the other hand condition (β) is not sufficient for *SNE*-property. In fact, let $\mathcal{D} = \mathrm{b}\mathcal{X}$ be a Q-boundary contained in N$\mathcal{P}$, containing A_5, satisfying condition (β) and $C_2, S_3, D_5 \notin \mathcal{D}$. The maximal subgroups of A_5, that are not isomorphic to A_4, are $\mathcal{X}$-projectors of A_5 and are not *sne* in A_5. Thus, if $\mathcal{X}$ is a *SNE*-Schunck class, with $A_5 \in \mathrm{b}\mathcal{X} \subseteq \mathrm{N}\mathcal{P}$, then b$\mathcal{X}$ satisfies, besides (β), one of the following two conditions:

(γ) $C_2 \in b\mathcal{X}$ or (δ) $(S_3, D_5) \subseteq b\mathcal{X}$.

Neverthless these conditions are not sufficient for SNE-property, as the following examples show. In all examples $\mathrm{b}\mathcal{X}$ is a Q-boundary contained in $\mathrm{N}\mathcal{P}$, containing A_5 and satisfying condition (β).

I. EXAMPLE. – Suppose that $\mathrm{b}\mathcal{X}$ satisfies condition (δ) and $C_5 \notin b\mathcal{X}$. Let $G = A_5 \wr_{nat} D_5$, where, for example, D_5 is the normalizer of $\langle(12345)\rangle$ in A_5. Denote by $\mathrm{A}_5^\natural = \times_{i=1}^5 \mathrm{A}_5(i)$ the base group of G. Let $P \in Syl_5(A_5)$, $Q \in Syl_2(D_5)$ and assume, w.l.o.g., that Q is contained in the stabilizer of 5. The subgroup $U = Q(P(5) \times \left(\times_{i=1}^{4} A_5(i)\right)$ is an $\mathcal{X}$-projector of G and is not *sne* in G, because the normal closure of its Sylow 2-subgroup U_2 coincides with G and $U_2 \notin Syl_2(G)$.

II. EXAMPLE. – Suppose that $\mathrm{b}\mathcal{X}$ satisfies condition (δ) and $C_3 \notin b\mathcal{X}$. Let $G = S_3 \wr_{nat} A_5$ and denote by $S_3^\natural = \times_{i=1}^{5} S_3(i)$ the base group of G. Let A be a stabilizer in A_5 (w.l.o.g., A the stabilizer of 5) and $Q \in Syl_2(S_3)$. The subgroup $U = A(Q(5) \times \left(\times_{i=1}^{4} S_3(i)\right)$ is an $\mathcal{X}$-projector of G and is not *sne* in G, because the normal closure of its Sylow 3-subgroup U_3 coincides with G and $U_3 \notin Syl_3(G)$.

III. EXAMPLE. – Suppose that $\mathrm{b}\mathcal{X}$ satisfies condition (γ) and furthermore the following conditions: $\mathrm{A}_4 \in \mathrm{b}\mathcal{X}$, $\mathrm{C}_5 \notin \mathrm{b}\mathcal{X}$. Let $G = \mathrm{A}_5 \wr_{nat} \mathrm{A}_4$ and denote by $\mathrm{A}_5^\natural = \times_{i=1}^{4} \mathrm{A}_5(i)$ the base group of G. Let $P \in Syl_5(\mathrm{A}_5)$, $Q \in Syl_3(\mathrm{A}_4)$ and assume, w.l.o.g., that Q is the stabilizer of 4. The subgroup $U = Q\left(P(4) \times \left(\times_{i=1}^{3} \mathrm{A}_5(\mathrm{i})\right)\right)$ is an $\mathcal{X}$-projector of G and is not *sne* in G, because the normal closure of its Sylow 3-subgroup U_3 coincides with G and $U_3 \notin Syl_3(G)$.

Next propositions provide some sufficient conditions for a Schunck class, whose boundary is contained in $\mathrm{N}\mathcal{P}$ and contains A_5, is a *SNE*-class.

3.5. PROPOSITION. – *Let $\mathcal{X}$ be a Schunck class, whose boundary $b\mathcal{X}$ is contained in* $\mathrm{N}\mathcal{P}$ *and contains A_5. If $b\mathcal{X}$ satisfies (β) and*

(ε) $(C_2, C_3, C_5) \subseteq b\mathcal{X}$,
then $\mathcal{X}$ is a SNE-*class.*

PROOF. – Let G be a group whose $\mathcal{X}$-projectors are not *sne* and let G be of minimal order with respect to this property. Let U be an $\mathcal{X}$-projector of G, p a prime, and U_p a Sylow p-subgroup of U which is not a Sylow p-subgroup of any subnormal subgroup of G. Since $\mathrm{b}\mathcal{X}$ satisfies (β), $\mathcal{X}$ is a Gaschütz class (see, Proposition 3.1, (ii)). Then, arguing as in the proof of Proposition 3.2, we get $Core_G(U) = 1$, $O_{p'}(G) = 1$ and so $Fit(G) = O_p(G)$. If G has a soluble quotient G/L in $\mathrm{b}\mathcal{X}$, then G/L is isomorphic to a $\Lambda(q, p)$ and $O_p(G) = 1$. On the other hand, by hypothesis (ε), $\mathrm{Proj}_{\mathcal{X}}(\mathrm{A}_5)=\{1\}$; it follows that, if G has a quotient G/M isomorphic to A_5, then $UM/M = 1$ and minimality of G ensures that U is *sne* in M and so in G, a contradiction. Thus $Fit(G) = 1$ and the quotients of G in $\mathrm{b}\mathcal{X}$ are isomorphic to groups $\Lambda(q, p)$; therefore $C_p \in \mathcal{X}$ and so $p \neq 2, 3, 5$. Now let R be the centreless CR-radical of G. Since $Core_G(U) = 1$, UR has a quotient UR/K in $\mathrm{b}\mathcal{X}$; since $RK/K \neq 1$, UR/K is isomorphic to A_5; consequently R is a direct power of A_5 and so $R \leqslant O_{p'}(G) = 1$, a contradiction.

3.6. LEMMA. – *Let $\mathcal{X}$ be a Schunck class, whose boundary $\mathrm{b}\mathcal{X}$ is contained in* $\mathrm{N}\mathcal{P}$, *contains A_5 and satisfies (β). If $(S_3, D_5, C_5) \subseteq b\mathcal{X}$, then $\mathcal{X}$-projectors of G contain a Sylow 2-subgroup of G, for all groups G.*

PROOF. – Let G be a group having an $\mathcal{X}$-projector U, whose Sylow 2-subgroups are not Sylow subgroups of G, and let G be of minimal order with respect to this property. Minimality of G provides $O_{2'}(G) = 1$ and $G = \langle G_2, U\rangle$, where $G_2 \in Syl_2(G)$, by recalling that (β) implies that $\mathcal{X}$ is a Gaschütz class. Now let N be a minimal normal subgroup of G such that $N \leqslant O_2(G)$. Since $G = \langle G_2, U\rangle$ we get $G = UN$, by minimality of G. On the other hand G has a quotient G/K in $\mathrm{b}\mathcal{X}$; therefore, since $C_2 \in b\mathcal{X}$, we get G/K isomorphic to a $\Lambda(2, q)$, which contradicts (β). Thus $O_2(G) = 1$ and so $Fit(G) = 1$. Now, arguing as above, we have that $G = UN$, for every minimal normal subgroup N of G, hence $G \in \mathrm{b}\mathcal{X}$ and so $G \simeq \mathrm{A}_5$; consequently either U is a Sylow 2-subgroup of G or the normalizer of a Sylow 2-subgroup of G, a contradiction.

3.7. PROPOSITION. – *Let $\mathcal{X}$ be a Schunck class, whose boundary $\mathrm{b}\mathcal{X}$ is contained in* $\mathrm{N}\mathcal{P}$, *contains A_5 and satisfies (β). If $(S_3, D_5, C_3, C_5) \subseteq b\mathcal{X}$, then $\mathcal{X}$ is a SNE- class.*

PROOF. – Let G be a group having an $\mathcal{X}$-projector U, such that a Sylow p-subgroup U_p of U is not a Sylow subgroup of a subnormal subgroup of G, and let G be of minimal order with respect to this property. If G has a quotient isomorphic to A_5, we get $p = 2$, by minimality of G, and therefore we contradict Lemma 3.6. Thus G has no quotient isomorphic to A_5 and $p \neq 2$. Then, arguing as in the proof of Proposition 3.5, we have that G has a quotient G/L isomorphic to a $\Lambda(q, p) \in \mathrm{b}\mathcal{X}$ and that $Fit(G) = 1$. Hence $C_p \in \mathcal{X}$ and so $p \neq 3, 5$. After that the proof goes over as in the proof of Proposition 3.5.

3.8. THEOREM. – *Let $\mathcal{X}$ be a Schunck class, whose boundary $\mathrm{b}\mathcal{X}$ is contained in* $\mathrm{N}\mathcal{P}$, *contains (A_5, S_3, D_5) and satisfies (β). Then $\mathcal{X}$ is a* SNE-*class if and only if* $\mathrm{b}\mathcal{X}$ *satisfies the following condition*

(η) $(C_3, C_5) \subseteq b\mathcal{X}$.

PROOF. – This follows immediately from Proposition 3.7 and Examples I and II.

REMARK. – Let $\mathcal{X}$ be a Schunck class, whose boundary $\mathrm{b}\mathcal{X}$ is contained in $\mathrm{N}\mathcal{P}$, contains (A_5, C_2) and satisfies (β). Example III ensures that, if $\mathcal{X}$ is a *SNE*-class, then $\mathrm{b}\mathcal{X}$ satisfies one of the following conditions:

$$A_4 \notin \mathrm{b}\mathcal{X} \quad \text{or} \quad C_3 \in \mathrm{b}\mathcal{X} \quad \text{or} \quad C_5 \in \mathrm{b}\mathcal{X}.$$

REFERENCES

[1] A. BALLESTER-BOLINCHES, *Characterizations of Schunck classes of finite soluble groups*, J. Algebra, **202** (1998), 243-249.
[2] A. BALLESTER-BOLINCHES - L. A. SHEMETKOV, *On normalizers of Sylow subgroups in finite groups*, Sib. Math. Journal, to appear.
[3] M. G. BIANCHI - A. GILLIO BERTA MAURI - P. HAUCK, *On finite groups with nilpotent Sylow normalizers*, Arch. Math., **47** (1986), 193-197.
[4] D. BLESSENOHL - W. GASCHÜTZ, *Über normale Schunck und Fittingklassen*, Math. Z., **118** (1970), 1-8.
[5] R. A. BRYCE - V. FEDRI - L. SERENA, *Bounds on the Fitting length of finite soluble groups with supersoluble Sylow normalizers*, Bull. Austral. Math. Soc., **44** (1991), 19-31.
[6] K. DOERK - T. HAWKES, *Finite soluble groups*, W. de Gruyter, Berlin (1992).
[7] V. FEDRI - L. SERENA, *Finite soluble groups with supersoluble Sylow normalizers*, Arch. Math., **50** (1988), 11-18.
[8] P. FÖRSTER, *Projektive Klassen endlicher Gruppen I, Schunck und Gaschützklassen*, Math. Z., **186** (1984), 248-278.
[9] G. GLAUBERMAN, *Prime-power factor groups of finite groups II*, Math. Z., **117** (1970), 46-56.
[10] D. GORENSTEIN, *Finite Groups*, Harper and Row, New York (1968).
[11] B. HUPPERT, *Endliche Gruppen I*, Springer-Verlag, Berlin (1969).
[12] B. HUPPERT - N. BLACKBURN, *Finite groups II*, Springer-Verlag, Berlin (1982).

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