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Eigenfunctions of the Laplace Operators for Buildings of type $\widetilde{B}_2$ (*)

A. M. MANTERO - A. ZAPPA

Sunto. – *Si considera per un palazzo affine di tipo $\widetilde{B}_2$ la congettura di Helgason relativa a operatori di Laplace definiti su diversi tipi di vertici. Si prova che ci sono casi in cui la congettura non è verificata, in quanto esistono autofunzioni che non sono la trasformata di Poisson di misure finitamente additive sulla frontiera massimale del palazzo.*

Summary. – *We consider for an affine building of type $\widetilde{B}_2$ Helgason's conjecture with respect to Laplace operators defined over different types of vertices. We prove that there are cases in which the conjecture fails, since there exist eigenfunctions which are not the Poisson transform of finitely additive measures at the maximal boundary of the building.*

1. – Introduction.

In his paper [4] S. Kato proved a p-adic analogue of the classical Helgason's conjecture, giving a characterization of the eigenspaces of the Hecke algebra $\mathcal{H}$ of a p-adic reductive group G with respect to a maximal compact subgroup K. One can give a definition of this algebra which uses only the geometry of the affine building associated with the group G and which therefore applies also to buildings not arising from linear groups. According to this definition, the algebra $\mathcal{H}$ is generated by averaging operators, called Laplacians.

Since all buildings of rank greater than 2 are linear, we focus our attention on buildings of rank 2. In [5] we considered for a type $\widetilde{A}_2$ building two Laplace operators defined on all vertices, whereas in [6] we dealt with Laplacians defined on just one special type of vertices of a type $\widetilde{G}_2$ building. In both cases we have had success in characterizing the eigenfunctions of these operators in terms of the Poisson transform of a unique finitely additive measure on the maximal boundary Ω of the building, making use of only its combinatorial structure, never considering the linear p-adic group eventually acting on it.

In this paper we treat a building Δ of type $\widetilde{B}_2$. We recognize three possible

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ways to define the Laplace operators on Δ, by considering firstly averages on just one special type of vertices, then on all special vertices and finally on non-special vertices. Unlike the buildings of types $\widetilde{A}_2$ and $\widetilde{G}_2$, for building of type $\widetilde{B}_2$ we prove the existence of singular cases in which the conjecture fails. Actually either for special vertices of one type or for non-special vertices, particular choices of the valencies of the edges of Δ lead to the existence of eigenfunctions of the Laplace operators that cannot be expressed in terms of a finitely additive measure.

Also for buildings of type $\widetilde{A}_2$ it is possible to define Laplace operators acting only on one type of vertices. We didn't consider this possibility in [5], but in a forthcoming paper we show that, as in the case $\widetilde{B}_2$, Helgason's conjecture sometimes fails in this case as well. Therefore the Helgason's conjecture is satisfied for all buildings of rank 2, apart from their type, only if the Laplace operators are two operators averaging on all special vertices of Δ.

According to the strategy adopted for the other types of buildings of rank 2, a fundamental step in solving the conjecture is to determine the dimension of the joint eigenspace of the operators obtained by retracting on the abstract apartment $\mathbb{A}$ (with respect to a chamber) the Laplacians and then to construct a basis for this space by retracting the Poisson kernel for suitable boundary points. To this end we have to define, in each case, a fundamental region on $\mathbb{A}$ and to evaluate on its vertices the retraction of the Poisson kernel. This computation may be realized using the algorithm described in [6].

In Section 2 we describe the features of buildings of type $\widetilde{B}_2$ which we need here.

In Section 3 we analyze the case of the special vertices of type 0 (resp. of type 2) and we identify all possible eigenvalues for which Helgason's conjecture fails, for particular choices of the valencies.

In Section 4 we solve Helgason's conjecture for all special vertices.

Finally in Section 5 we explicitly describe the cases where the conjecture fails for non-special vertices.

We would like to thank Tim Steger for some useful comments and suggestions about the content of this paper.

2. – The building.

A building Δ of type $\widetilde{B}_2$ is a simplicial] complex of rank 2, consisting of vertices, edges and triangles (the «chambers»), which contains a family of subcomplexes (the «apartments»), each of which is isomorphic to the Coxeter complex $\mathbb{A}$ (the «abstract apartment») of a Coxeter group

$$W = \langle \{r_0, r_1, r_2\} : r_0^2 = r_1^2 = r_2^2 = 1, (r_0 r_1)^4 = (r_0 r_2)^2 = (r_1 r_2)^4 = 1 \rangle .$$

The associated Coxeter graph is

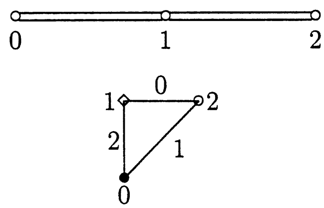


Figure 1

and W acts as an affine reflection group on the Euclidean plane E^2, in such a way that the associated Coxeter complex $\mathbb{A}$ is isomorphic to a Euclidean plane tessellated by isosceles right triangles. We refer to [7] and [1] for formal definition and further details.

We denote by $\mathcal{C}$ the set of chambers of Δ. We assign the «type» to the vertices and to the edges of any chamber according to the notation of the Coxeter graph, as shown in Figure 1; thus the angle between the i-edge and the j-edge of any chamber is π/m_{ij}, where $m_{01} = m_{12} = 4$ and $m_{02} = 2$. Two chambers are said to be «i-adjacent» if they share an edge of type i. Any two chambers c, c' (resp. any two vertices x, x') may be joined by a «minimal» gallery $[c, c']$ (resp. $[x, x']$) of two by two adjacent and distinct chambers $c_0, \ldots, c_l$, such that $c_0 = c$ and $c_l = c'$ (resp. $x \in c_0$ and $x' \in c_l$). If c_{k-1} and c_k are i_k $-$adjacent, then $\pi = (i_1, \ldots, i_l)$ is the «type» of the gallery, denoted by $\pi(c, c')$ (resp. $\pi(x, x')$) and the number $l + 1$ is its «length». If we denote by d the usual graph-theoretic distance on the set of vertices of Δ, then $d(x, x')$ is not greater than the length of $[x, x']$. The residue $St(x)$ of any vertex x is a spherical building, whose type depends on $\tau(x)$. Actually $St(x)$ has type B_2 if $\tau(x) = 0$ or $\tau(x) = 2$ and type $A_1 \times A_1$ if $\tau(x) = 1$ (see [7] or [1]). Therefore if $\tau(x) = 0$ or $\tau(x) = 2$ the finite Coxeter group W_0 associated to $St(x)$ is the dihedral group of order 8

$$D_4 = \langle \{r_1, r_2\} : r_1^2 = r_2^2 = (r_1 r_2)^4 = 1 \rangle$$

and $W = \mathbb{Z}^2 \rtimes W_0$. Thus the vertices of both type 0 and 2 are «special».

Since the building Δ is assumed to be locally finite, any edge belongs to finitely many chambers. We call the «valency» of an edge the number of chambers sharing it, and we denote by $q_i + 1$ the valency of any edge of type i. For ease of notation we set $p = q_0$, $q = q_1$ and $r = q_2$. There are restrictions on p, q and on r; actually in [2] and in [3] W. Feit and G. Higmann proved that the following relations must be satisfied:

$$\frac{pq(pq+1)}{p+q}, \qquad \frac{qr(qr+1)}{q+r} \in \mathbb{Z},$$

$$p \leq q^2,\ q \leq p^2 \qquad \text{and} \qquad q \leq r^2,\ r \leq q^2.$$

If $\tau(x)=0$, the residue $St(x)$ contains $(q+1)(qr+1)$ vertices of type 1, $(r+1)(qr+1)$ vertices of type 2 and $(q+1)(r+1)(qr+1)$ chambers. If $\tau(x)=2$, we obtain the number of vertices of type 0 and 1 and the number of chambers in $St(x)$ simply replacing r by p. If $\tau(x)=1$, the residue $St(x)$ contains $(p+1)$ vertices of type 0, $(r+1)$ vertices of type 2 and $(p+1)(r+1)$ chambers.

For any special vertex x, a sector Q_x based at x is a simplicial cone of vertex x determined, in any apartment containing x, by a chamber (the «base chamber» of the sector) having x as one of its vertices [7]. We call the «i-wall» of Q_x, for $i=1,2$, the wall containing the edge of type $\tau(x)+i$ emanating from x. Two sectors based at a same vertex are said to be «i-adjacent» if they share an i-wall.

The maximal boundary Ω of Δ is defined as the set of equivalence classes ω of parallel sectors, two sectors Q_x, Q_y being equivalent, or parallel, $Q_x\sim Q_y$, if they contain a common subsector. We denote by $Q_x(\omega)$ the sector based at x associated with ω. If we fix a special vertex e (the «fundamental vertex» of the building), Ω may be endowed with a totally disconnected compact Hausdorff topology, generated by the family $\mathcal{B}$ consisting of the sets

$$\Omega(c)=\{\omega\in\Omega : c\subset Q_e(\omega)\}, \qquad \forall\, c\in\mathcal{C}.$$

The definition of the retraction r_c of Δ to $\mathbb{A}$ with respect to a chamber c and that of the retraction $r_\omega^{x_0}$ of Δ to $\mathbb{A}$ with respect to a boundary point ω (of initial vertex any special vertex x_0) generalize those given in [5].

3. – The case of special vertices of type 0.

3.1. *Coordinates on an apartment.*

In this section we only consider vertices of type 0 and we denote by $\mathcal{U}_0$ (resp. $\overline{\mathcal{U}}_0$) the set of such vertices of Δ (resp. of $\mathbb{A}$).

Given an apartment $\mathcal{A}$ and a sector Q_{x_0} on it, with $x_0\in\mathcal{U}_0$, each type 0 vertex in $\mathcal{A}$ may be assigned a pair of integer coordinates (m, n) (with respect to Q_{x_0}), in the same manner we did for a building of type $\widetilde{G}_2$ (see [6]). We select the line H_1 (resp. H_2) passing through x_0 and containing the 2-wall of Q_{x_0} (resp. the 2-wall of the sector Q'_{x_0} 1-adjacent to Q_{x_0}). The type 0 vertices of H_1 (resp. of H_2) are assigned coordinates $(m, 0)$, $m\in\mathbb{Z}$, (resp. $(0, n)$, $n\in\mathbb{Z}$,) assuming that $(1, 0)$ (resp. $(0, 1)$) are the coordinates of the vertex of $Q_{x_0}\cap H_1$ (resp. $Q'_{x_0}\cap H_2$) at distance 2 from x_0. With this assumption, the type 0 vertices of Q_{x_0} are characterized by coordinates (m, n) with $0\leqslant n\leqslant m$ (see Figure 2).

As usual, these coordinates are independent of the apartment containing the vertex and the sector; moreover vertices lying on two different sectors based at the same vertex of $\mathcal{U}_0$ have the same coordinates with respect to both sectors.

On the abstract apartment $\mathbb{A}$ we set $X=X_{m,\,n}$ if the vertex X has coordinates (m, n) (with respect to a sector $\overline{Q}$).

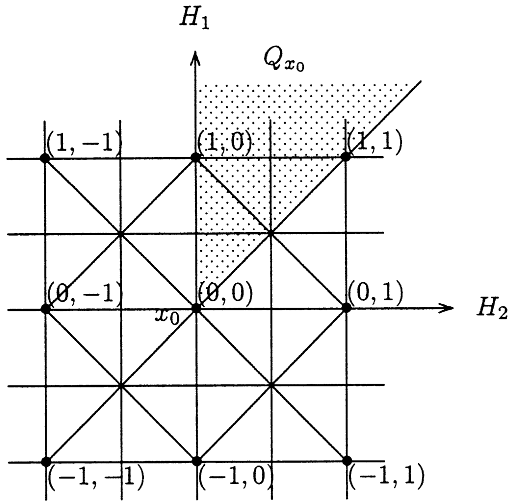


Figure 2

As for a building of type $\widetilde{G}_2$, each chamber c of $\mathcal{C}$ may be assigned a triple of integer coordinates (k, m, n) (with respect to Q_{x_0}), where $(m, n) \in \mathbb{Z}^2$ are the coordinates of the vertex $x \in \mathcal{U}_0$ of c (with respect to Q_{x_0}) and $k \in \{1, \dots, 8\}$, characterizes (among the chambers of $\mathcal{C}$ sharing the vertex x) the position of c with respect to the sector $Q_x \sim Q_{x_0}$. In Figure 3 we exhibit the chosen numbering.

3.2. *Poisson kernel.*

Fix a sector $\overline{Q}$ on the fundamental apartment $\mathbb{A}$.

DEFINITION 3.2.1. – For every $(\alpha, \beta) \in (\mathbb{C}^\times)^2$, let $\phi_{\alpha, \beta}: \mathcal{U}_0 \to \mathbb{C}$ be the multiplicative function

$$\phi_{\alpha, \beta}(X_{m, n}) = \alpha^m \beta^n, \qquad \forall (m, n) \in \mathbb{Z}^2,$$

with respect to the coordinate system associated to $\overline{Q}$.

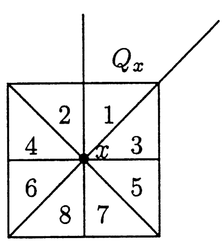


Figure 3

For $x_0 \in \mathcal{U}_0$, the function defined by

$$P^{x_0}_{\alpha,\beta}(x, \omega) = \phi_{\alpha,\beta}(r^{x_0}_{\omega}(x)), \qquad \forall x \in \mathcal{U}_0, \ \forall \omega \in \Omega ,$$

is called the Poisson kernel of initial point x_0 and of parameters α, β.

We simply write $P(x, \omega) = P^{x_0}_{\alpha,\beta}(x, \omega)$, whenever there is no ambiguity.

This definition extends Definition 2.3.1 of [6]. The Poisson kernel depends on the initial point according to the following formula:

$$P^{y_0}(x, \omega) = P^{x_0}(x, \omega)(P^{x_0}(y_0, \omega))^{-1}, \qquad \forall x \in \mathcal{U}_0, \ \forall \omega \in \Omega . \tag{1}$$

We refer the reader to [5, Lemma 2.8] for the proof of (1).

Given a chamber c_0, we denote by $\tilde{f}_{c_0}$ the retraction of a function f with respect to c_0, defined by

$$\tilde{f}_{c_0}(X) = \frac{1}{|r_{c_0}^{-1}(X)|} \sum_{x \in r_{c_0}^{-1}(X)} f(x) .$$

The method described in [6, Section 3] to determine the retraction of the Poisson kernel with respect to a chamber applies also in the present case. Actually, if c_0 is a fixed chamber of Δ and $r_{c_0}(c_0) = C_0 = r^{x_0}_{\omega}(c_0)$, the following theorem holds.

THEOREM 3.2.2. – *Let (k, m, n) be the coordinates of the chamber C_0. For every $X \in \overline{\mathcal{U}_0}$, let C be the chamber containing X in a minimal gallery connecting C_0 to X. Let $\pi = (i_1, \dots, i_l)$ be the type of $[C_0, C]$; then*

$$\tilde{P}(X, \omega) = \frac{1}{|r_{c_0}^{-1}(C)|} \alpha^m \beta^n \mathbb{V}_0 \mathbb{M}_\pi e_k,$$

where $\mathbb{V}_0$ is the 1×8-matrix such that $V_{0,h} = 1$, e_k is the 8×1-matrix such that $e_{h,k} = \delta_{hk}$ and $\mathbb{M}_\pi = \mathbb{M}_{i_l} \dots \mathbb{M}_{i_1}$, where $\mathbb{M}_0$, $\mathbb{M}_1$ and $\mathbb{M}_2$ are the 8×8-matrices

$$\mathbb{M}_0 = \mathbb{M}_0(\alpha,\beta) = \begin{pmatrix} p-1 & 0 & 0 & 0 & 0 & 0 & p\alpha^{-1} & 0 \\ 0 & p-1 & 0 & 0 & 0 & 0 & 0 & p\alpha^{-1} \\ 0 & 0 & p-1 & p\beta^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p-1 & p\beta^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbb{M}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ q & 0 & q-1 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & q-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & q & 0 & q-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 0 & q-1 \end{pmatrix},$$

$$\mathbb{M}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ r & r-1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & r & 0 & r-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 0 & r-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & r & r-1 \end{pmatrix}.$$

REMARK 3.2.3. – Let F be any complex valued function on $\overline{\mathcal{U}}_0$; for every $\omega \in \Omega$ and $x_0 \in \mathcal{U}_0$ consider $F \cdot r_\omega^{x_0}$. If $c_0 \in \mathcal{C}$ and $r_{c_0}(c_0) = C_0 = r_\omega^{x_0}(c_0)$, then

$$\sum_{x \in r_{c_0}^{-1}(X)} F(r_\omega^{x_0}(x)) = \sum_{X' \in \widetilde{R}(X)} h(X')F(X'),$$

where

$$\widetilde{R}(X) = \{X' = r_\omega^{x_0}(x),\ \forall x \in r_{c_0}^{-1}(X)\}$$

and

$$h(X') = |\{x \in r_{c_0}^{-1}(X) : r_\omega^{x_0}(x) = X'\}|,$$

for every $X \in \overline{\mathcal{U}}_0$, $X' \in \widetilde{R}(X)$.

Since $\widetilde{R}(X)$ and $h(X')$ do not depend on the function F, we may evaluate them by considering $F = \phi_{\alpha,\beta}$ and by using Theorem 3.2.2.

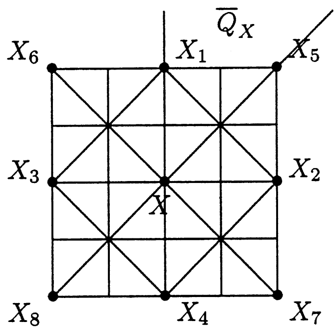


Figure 4

3.3. *Laplace operators on the abstract apartment* $\mathbb{A}$.

For every $X \in \overline{\mathcal{U}}_0$, we define (see Figure 4):

$$\overline{S}_1(X) = \{Y \in \overline{\mathcal{U}}_0 \colon \pi(X, Y) = (0)\} = \{X_1, X_2, X_3, X_4\},$$

$$\overline{S}_2(X) = \{Y \in \overline{\mathcal{U}}_0 \colon \pi(X, Y) = (0, 1, 0)\} = \{X_5, X_6, X_7, X_8\}.$$

If $X = X_{m,n}$, then

$$X_1 = X_{m+1,n}, \quad X_2 = X_{m,n+1}, \quad X_3 = X_{m,n-1}, \quad X_4 = X_{m-1,n},$$

$$X_5 = X_{m+1,n+1}, \quad X_6 = X_{m+1,n-1}, \quad X_7 = X_{m-1,n+1}, \quad X_8 = X_{m-1,n-1}.$$

DEFINITION 3.3.1. – The linear operators

$$L_i F(X) = \sum_{Y \in \overline{S}_i(X)} F(Y), \qquad X \in \overline{\mathcal{U}}_0, \quad i = 1, 2,$$

acting on the space of the complex valued functions F on $\overline{\mathcal{U}}_0$ are called «Laplace operators» on $\mathbb{A}$.

The linear operators

$$\Lambda_i F(X) = \sum_{Y \in \overline{S}_i(X)} h_i(X, Y)\, F(Y), \qquad X \in \overline{\mathcal{U}}_0, \quad i = 1, 2,$$

with coefficients $h_i(X, Y) \in \mathbb{C}^\times$, are called «generalized» Laplace operators on $\mathbb{A}$.

The operators Λ_1, Λ_2 are called «homogeneous» if $h_i(X, Y) = h_i(Y)$, for every $X \in \overline{\mathcal{U}}_0$; they are called «symmetric homogeneous» if

$$h_i(X, Y) = h_i, \qquad \forall X \in \overline{\mathcal{U}}_0, \qquad \forall Y \in \overline{S}_i(X), \quad i = 1, 2.$$

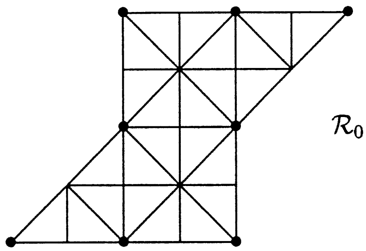


Figure 5

For every pair $(\lambda_1, \lambda_2) \in \mathbb{C}^2$, we denote

$$\mathcal{S}_{\Lambda_1, \Lambda_2}(\lambda_1, \lambda_2) = \{F : \overline{\mathcal{U}}_0 \to \mathbb{C} : \Lambda_i F = \lambda_i F,\ i = 1, 2\}.$$

In particular we denote by $S(\lambda_1, \lambda_2)$ the joint eigenspace of the Laplace operators L_1, L_2 associated with eigenvalues (λ_1, λ_2).

We may choose on $\mathbb{A}$ a particular region $\mathcal{R}_0$ characterized by the property that knowing the values of a function of $\mathcal{S}_{\Lambda_1, \Lambda_2}(\lambda_1, \lambda_2)$ on the vertices of this region allows one to reconstruct the whole function on $\overline{\mathcal{U}}_0$.

DEFINITION 3.3.2. – We call «fundamental region» any region $\mathcal{R}_0$ of $\mathbb{A}$ obtained applying any element $w \in W$ to the region pictured in Figure 5.

By the same argument used in [6, Proposition 4.4.2] we prove:

PROPOSITION 3.3.3. – *Let $F \in \mathcal{S}_{\Lambda_1, \Lambda_2}(\lambda_1, \lambda_2)$; then F is uniquely determined by its values on the type 0 vertices of $\mathcal{R}_0$.*

As an evident consequence of this proposition we obtain:

COROLLARY 3.3.4. – *For every pair $(\lambda_1, \lambda_2) \in \mathbb{C}^2$,* $\dim \mathcal{S}_{\Lambda_1, \Lambda_2}(\lambda_1, \lambda_2) \le 8$.

Proposition 3.3.3 and Corollary 3.3.4 easily extend to the operators

$$\Lambda_i^\sharp F(X) = \sum_{Y \in \overline{S}_i^\sharp(X)} h_i(X, Y) F(Y), \quad X \in \overline{\mathcal{U}}_0, \quad i = 1, 2,$$

where

$$\overline{S}_1^\sharp(X) = \overline{S}_1(X) \cup \{X\},$$

$$\overline{S}_2^\sharp(X) = \overline{S}_2(X) \cup \overline{S}_1(X) \cup \{X\},$$

provided $h_i(X, Y) \neq 0$, for $Y \in \overline{S}_i(X)$.

We restrict now to the symmetric homogeneous operators L_1, L_2; in this case we can explicitly determine their joint eigenvalues.

For every $(\xi, \eta) \in (\mathbb{C}^\times)^2$, let $\phi_{\xi, \eta}$ be the multiplicative function of parameters ξ, η, with respect to the coordinate system associated with a sector $\overline{Q}$.

PROPOSITION 3.3.5. – *For every $(\xi, \eta) \in (\mathbb{C}^\times)^2$, the function $\phi_{\xi, \eta}$ belongs to the eigenspace $S(\lambda_1, \lambda_2)$, associated with eigenvalues $\lambda_i = \lambda_i(\xi, \eta)$, where*

$$\lambda_1(\xi, \eta) = \xi + \eta + \xi^{-1} + \eta^{-1},$$

$$\lambda_2(\xi, \eta) = (\xi + \xi^{-1})(\eta + \eta^{-1}).$$

PROOF. – Since $\phi_{\xi, \eta}$ is multiplicative, $(\phi_{\xi, \eta}(X))^{-1} \sum_{Y \in \overline{S}_i(X)} \phi_{\xi, \eta}(Y)$ does not depend on the choice of the vertex X. An explicit computation gives the result. ∎

REMARK 3.3.6. – Let us assume that the finite Coxeter group W_0 stabilizes the base vertex $\overline{X}$ of $\overline{Q}$. Thus for every $(\xi, \eta) \in (\mathbb{C}^\times)^2$, and for every $\sigma \in W_0$, the function

$$\sigma \cdot \phi_{\xi, \eta}(X) = \phi_{\xi, \eta}(\sigma(X)), \qquad X \in \overline{\mathcal{U}}_0,$$

is multiplicative; we denote by $(\xi_\sigma, \eta_\sigma)$ the pair of $(\mathbb{C}^\times)^2$ such that $\sigma \cdot \phi_{\xi, \eta} = \phi_{\xi_\sigma, \eta_\sigma}$. Moreover

$$\lambda_1(\xi, \eta) = \sum_{\sigma \in W_0} \phi_{\xi, \eta}(\sigma(X_{1,0})) = \sum_{\sigma \in W_0} \phi_{\xi_\sigma, \eta_\sigma}(X_{1,0}),$$

$$\lambda_2(\xi, \eta) = \sum_{\sigma \in W_0} \phi_{\xi, \eta}(\sigma(X_{1,1})) = \sum_{\sigma \in W_0} \phi_{\xi_\sigma, \eta_\sigma}(X_{1,1}).$$

Therefore $\lambda_1(\xi, \eta)$ and $\lambda_2(\xi, \eta)$ are W_0-invariant.

COROLLARY 3.3.7.. – *For every pair $(\lambda_1, \lambda_2) \in \mathbb{C}^2$, there exists $(\xi, \eta) \in (\mathbb{C}^\times)^2$ such that $\lambda_i = \lambda_i(\xi, \eta)$, $i = 1, 2$. Moreover $\lambda_i(\xi, \eta) = \lambda_i(\xi', \eta')$ implies $(\xi', \eta') = (\xi_\sigma, \eta_\sigma)$, for some $\sigma \in W_0$.*

PROOF. – The statement easily follows by setting $a_1 = \xi + \xi^{-1}$ and $a_2 = \eta + \eta^{-1}$. ∎

REMARK 3.3.8. – We give an useful characterization of the Laplace operators on $\mathbb{A}$. We first remark that $\overline{\mathcal{U}}_0$ may be seen as a W_0-invariant lattice of the Euclidean plane, if we set $X + Y = X_{m+j, n+k}$, for $X = X_{m,n}$ and $Y = X_{j,k}$. Then, for every $X \in \overline{\mathcal{U}}_0$, we define a difference operator T_X (acting on the complex valued functions on $\overline{\mathcal{U}}_0$) by setting $T_X(F)(Y) = F(X + Y)$. Thus

$$L_1 = \sum_{\sigma \in W_0} T_{\sigma(X_{1,0})}, \qquad L_2 = \sum_{\sigma \in W_0} T_{\sigma(X_{1,1})}$$

and L_1, L_2 generate the algebra of W_0-invariant difference operators with constant coefficients.

Moreover, for every $(\xi, \eta) \in (\mathbb{C}^\times)^2$, the eigenspace $S(\lambda_1, \lambda_2)$ associated with eigenvalues $\lambda_i = \lambda_i(\xi, \eta)$ coincides with the space of the solutions of the system of difference equations:

$$\Sigma_{\phi_{\xi,\eta}}: \sum_{\sigma \in W_0} T_{\sigma(X)}(F) = \Big(\sum_{\sigma \in W_0} \phi_{\xi,\eta}(\sigma(X))\Big) F, \qquad \forall X \in \overline{\mathcal{U}}_0.$$

In view of the previous remark, we may improve, for the Laplace operators, the result of Corollary 3.3.4.

PROPOSITION 3.3.9. – *For every pair* $(\lambda_1, \lambda_2) \in \mathbb{C}^2$, $\dim S(\lambda_1, \lambda_2) = 8$.

PROOF. – The characterization of the Laplace operators given in Remark 3.3.8 allows us to apply to these operators Proposition 1.1 of [4]. Therefore $\dim S(\lambda_1, \lambda_2) \geqslant 8$. ■

REMARK 3.3.10. – The equality in Proposition 3.3.9 corresponds to the fact that the stabilizer in W_0 of the multiplicative function $\phi_{\xi,\eta}$ is a reflection group (see [4, Proposition 1.1]).

3.4. *Laplace operators on Δ.*

For every $x \in \mathcal{U}_0$ we define

$$S_1(x) = \{y \in \mathcal{U}: \pi(x, y) = (0)\},$$

$$S_2(x) = \{y \in \mathcal{U}: \pi(x, y) = (0, 1, 0)\}.$$

The cardinality of $S_i(x)$ does not depend on x. Actually, if we consider the retraction of Δ on $\mathbb{A}$ with respect to a chamber c containing x, then

$$S_i(x) = \bigcup_{Y \in \overline{S}_i(X)} r_c^{-1}(Y), \qquad i = 1, 2,$$

where $X = r_c(x)$. This implies, denoting $K_i = |S_i(x)|$,

$$K_1 = p(q+1)(qr+1) \qquad K_2 = p^2 q(r+1)(qr+1).$$

Starting from this definition of $S_1(x)$ and $S_2(x)$, we extend, to a building of type $\widetilde{B}_2$, the definition of Laplace operators on Δ given in [6] for the case $\widetilde{G}_2$.

DEFINITION 3.4.1. – The linear operators

$$\mathcal{L}_i f(x) = K_i^{-1} \sum_{y \in S_i(x)} f(y), \qquad \forall x \in \mathcal{U}_0, \quad i = 1, 2,$$

acting on the space of complex valued functions f on $\mathcal{U}_0$, are called Laplace operators on Δ.

For every pair $(\gamma_1, \gamma_2) \in \mathbb{C}^2$, we denote

$$\mathcal{S}(\gamma_1, \gamma_2) = \{f : \mathcal{U}_0 \to \mathbb{C} : \mathcal{L}_i f = \gamma_i f,\ i = 1, 2\}.$$

PROPOSITION 3.4.2. – *For every $\omega \in \Omega$ and $x_0 \in \mathcal{U}_0$, the function $P^{x_0}(\cdot, \omega)$ belongs to the eigenspace $\mathcal{S}(\gamma_1, \gamma_2)$, associated with the eigenvalues $\gamma_i = \gamma_i(\alpha, \beta)$ given by*

$$\gamma_1(\alpha, \beta) = K_1^{-1}(\alpha + q\beta + pqr\beta^{-1} + pq^2 r\alpha^{-1}) + c_1,$$

$$\gamma_2(\alpha, \beta) = K_2^{-1}(p-1)(\alpha + q\beta + pq^2 r\alpha^{-1} + pqr\beta^{-1})$$

$$+ K_2^{-1}(\alpha\beta + pr\alpha\beta^{-1} + pq^2 r\alpha^{-1}\beta + p^2 q^2 r^2 \alpha^{-1}\beta^{-1}) + c_2,$$

where $c_1 = K_1^{-1}(p-1)(q+1)$, $c_2 = K_2^{-1}[q(p-1)^2 + p(q-1)(r+1)]$.

PROOF. – Let c be the base chamber of the sector $Q_x(\omega)$ and $C = r_c(c) = r_\omega^x(c)$. If $X = r_c(x)$, we evaluate $\sum_{y \in r_c^{-1}(X_i)} P^x(y, \omega))$, for every $i = 1, \ldots, 8$, by using Theorem 3.3.2. We conclude by the same argument as in Proposition 4.2.1 of [6]. ■

REMARK 3.4.3. – For every $(\alpha, \beta) \in (\mathbb{C}^\times)^2$, let $\xi = \alpha/q\sqrt{pr}$ and $\eta = \beta/\sqrt{pr}$. If $\lambda_1(\xi, \eta)$, $\lambda_2(\xi, \eta)$ are the eigenvalues of the Laplace operators on the fundamental apartment corresponding to parameters (ξ, η), then

$$\gamma_1(\alpha, \beta) = K_1^{-1} q\sqrt{pr}(\xi + \xi^{-1} + \eta + \eta^{-1}) + c_1$$

$$= K_1^{-1} q\sqrt{pr}\lambda_1(\xi, \eta) + c_1,$$

$$\gamma_2(\alpha, \beta) = K_2^{-1} q\sqrt{pr}(p-1)(\xi + \xi^{-1} + \eta + \eta^{-1}) + K_2^{-1} pqr(\xi + \xi^{-1})(\eta + \eta^{-1}) + c_2$$

$$= K_2^{-1} q\sqrt{pr}(p-1)\,\lambda_1(\xi, \eta) + K_2^{-1} pqr\lambda_2(\xi, \eta) + c_2.$$

LEMMA 3.4.4. – *For every pair $(\gamma_1, \gamma_2) \in \mathbb{C}^2$, there exists $(\alpha, \beta) \in (\mathbb{C}^\times)^2$ such that $\gamma_i = \gamma_i(\alpha, \beta)$, $i = 1, 2$. Moreover $\gamma_i(\alpha, \beta) = \gamma_i(\alpha', \beta')$, $i = 1, 2$ if and only if $(\alpha', \beta') = \sigma(\alpha, \beta)$, for some $\sigma \in W_0$, where*

$$\sigma(\alpha, \beta) = (q\sqrt{pr}\,\xi_\sigma, \sqrt{pr}\,\eta_\sigma)$$

and $\xi = \alpha/q\sqrt{pr}$ and $\eta = \beta/\sqrt{pr}$.

PROOF. – Let $\Gamma_i(\xi, \eta) = \gamma_i(q\sqrt{pr}\,\xi, \sqrt{pr}\,\eta)$, $i = 1, 2$. Keeping in mind Remark 3.4.3, Corollary 3.3.7 shows that there exists a pair (ξ, η) such that $\Gamma_i =$

$\Gamma_i(\xi, \eta)$; moreover it proves that $\Gamma_i(\xi, \eta) = \Gamma_i(\xi', \eta')$ if and only if $(\xi', \eta') = (\xi_\sigma, \eta_\sigma)$, for some $\sigma \in W_0$. Thus the lemma is proved. ■

COROLLARY 3.4.5. – *Let $\gamma_i = \gamma_i(\alpha, \beta)$, $i = 1, 2$. For every $\sigma \in W_0$, the function $P^{x_0}_{\sigma(\alpha, \beta)}(\cdot, \omega)$ belongs to $\mathcal{S}(\gamma_1, \gamma_2)$.*

Later on it will be useful the following definition.

DEFINITION 3.4.6. – The pairs (α, β) and (α', β') are said to be «equivalent», and we write $(\alpha, \beta) \sim (\alpha', \beta')$, if $\gamma_i(\alpha, \beta) = \gamma_i(\alpha', \beta')$, $i = 1, 2$.

3.5. *Retraction on $\mathbb{A}$ of the Laplace operators $\mathcal{L}_1, \mathcal{L}_2$.*

As for buildings of type $\widetilde{A}_2$ and $\widetilde{G}_2$, we prove that the Laplace operators on Δ retract (with respect to a chamber) to a pair of linear operators on $\mathbb{A}$.

LEMMA 3.5.1. – *Let $c_0 \in \mathcal{C}$; then, for every function f on $\mathcal{U}_0$,*

$$(\widetilde{\mathcal{L}_i f})_{c_0}(X) = \sum_{Y \in \overline{S}_i^\sharp(X)} \chi_i(X, Y)\, \tilde{f}_{c_0}(Y), \qquad i = 1, 2,$$

for suitable non negative $\chi_i = \chi_i(X, Y)$. Moreover $\chi_i(X, Y) > 0$ if $Y \in \overline{S}_i(X)$.

We refer the reader to [6, Proposition 4.3.1] for the proof of this lemma.

From now on we denote by $\widetilde{\mathcal{L}}_1$ and $\widetilde{\mathcal{L}}_2$ the linear operators on $\mathbb{A}$ obtained by retracting $\mathcal{L}_1$ and $\mathcal{L}_2$ with respect to any chamber:

$$\widetilde{\mathcal{L}}_i F(X) = \sum_{Y \in \overline{S}_i^\sharp(X)} \chi_i(X, Y)\, F(Y), \qquad \forall X \in \overline{\mathcal{U}_0}, \quad i = 1, 2.$$

The following lemma exhibits two homogeneous operators $\mathcal{L}_1^*, \mathcal{L}_2^*$ on $\mathbb{A}$ associated with the Laplace operators $\mathcal{L}_1, \mathcal{L}_2$ on Δ.

LEMMA 3.5.2. – *Consider, for any F on $\mathbb{A}$, the function $f = F \cdot r_\omega^{x_0}$. For every $x \in \mathcal{U}_0$, let $X = r_\omega^{x_0}(x)$. Then*

$$\begin{aligned} \mathcal{L}_1 f(x) &= K_1^{-1} \mathcal{L}_1^* F(X) + c_1 F(X), \\ \mathcal{L}_2 f(x) &= K_2^{-1} [\mathcal{L}_2^* F(X) + (p-1)\, \mathcal{L}_1^* F(X)] + c_2 F(X) \end{aligned}$$

where

$$\mathcal{L}_1^* F(X) = F(X_1) + qF(X_2) + pqrF(X_3) + pq^2 rF(X_4),$$

$$\mathcal{L}_2^* F(X) = F(X_5) + prF(X_6) + pq^2 rF(X_7) + p^2 q^2 r^2 F(X_8).$$

PROOF. – Let c be the base chamber of the sector $Q_x(\omega)$ and $C = r_\omega^{x_0}(c)$; then, assuming $r_c(c) = C$, we have

$$\mathcal{L}_i f(x) = K_i^{-1} \sum_{Y \in \overline{\mathcal{S}}_i(X)} \left(\sum_{y \in r_c^{-1}(Y)} F(r_\omega^{x_0}(y)) \right).$$

By using Remark 3.2.3 we obtain the requested identities. ■

REMARK 3.5.3. – For every chamber c, we set $\widetilde{F} = (\widetilde{F \cdot r_\omega^{x_0}})_c$. If $\overline{Q}^\vee$ is the sector of $\mathbb{A}$ opposite to $\overline{Q}$, then

$$\widetilde{F}(X) = F(X), \qquad \forall X \in \overline{Q}^\vee,$$

since $r_\omega^{x_0}(x) = X$, for every $X \in \overline{Q}^\vee$ and $x \in r_c^{-1}(X)$. This implies that in a subsector of $\overline{Q}^\vee$ we have

$$\widetilde{\mathcal{L}}_1 = K_1^{-1} \mathcal{L}_1^* + c_1 I,$$

$$\widetilde{\mathcal{L}}_2 = K_2^{-1} [\mathcal{L}_2^* + (p-1)\mathcal{L}_1^*] + c_2 I.$$

We also remark that a suitable change of variables turns the homogeneous operators $\mathcal{L}_1^*, \mathcal{L}_2^*$ into the Laplace operators L_1, L_2. Actually, a direct computation shows that, by setting

$$G(X_{m,n}) = (q\sqrt{pr})^{-m} (\sqrt{pr})^{-n} F(X_{m,n}),$$

then

$$\mathcal{L}_1^* F(X_{m,n}) = q\sqrt{pr} (q\sqrt{pr})^m (\sqrt{pr})^n L_1 G(X_{m,n})$$

$$\mathcal{L}_2^* F(X_{m,n}) = pqr (q\sqrt{pr})^m (\sqrt{pr})^n L_2 G(X_{m,n}).$$

Let us define

$$\widetilde{\mathcal{S}}(\gamma_1, \gamma_2) = \{ F : \overline{\mathcal{U}}_0 \to \mathbb{C} : \widetilde{\mathcal{L}}_i F = \gamma_i F, \ i = 1, 2 \}.$$

As an evident consequence of the definition, for every chamber c_0, the function $\tilde{f}_{c_0}$ belongs to $\widetilde{\mathcal{S}}(\gamma_1, \gamma_2)$, if $f \in \mathcal{S}(\gamma_1, \gamma_2)$. Moreover if $F \in \widetilde{\mathcal{S}}(\gamma_1, \gamma_2)$, then the function $f(x) = F(X)$, for $x \in r_{c_0}^{-1}(X)$, belongs to $\mathcal{S}(\gamma_1, \gamma_2)$; actually it is easy to prove that, for such a function, $\mathcal{L}_i f$ is constant on the r_{c_0}-fibers and then the identity $(\widetilde{\mathcal{L}_i f})_{c_0}(X) = \gamma_i \tilde{f}_{c_0}(X)$ implies $\mathcal{L}_i f(x) = \gamma_i f(x)$.

Let us denote by $\mathcal{S}^*(\varrho_1, \varrho_2)$ the joint eigenspace of the operators $\mathcal{L}_1^*, \mathcal{L}_2^*$ corresponding to eigenvalues (ϱ_1, ϱ_2). Remark 3.5.3 implies that there is a bijection among $\widetilde{\mathcal{S}}(\gamma_1, \gamma_2)$, $\mathcal{S}^*(\varrho_1, \varrho_2)$ and $S(\lambda_1, \lambda_2)$, for suitable choices of the pairs (γ_1, γ_2), (ϱ_1, ϱ_2) and (λ_1, λ_2).

PROPOSITION 3.5.4. – *Let* $(\gamma_1, \gamma_2) \in \mathbb{C}^2$. *If*

$$\gamma_1 = K_1^{-1} \varrho_1 + c_1, \qquad \gamma_2 = K_2^{-1}((p-1)\varrho_1 + \varrho_2) + c_2 \tag{2}$$

and

$$\lambda_1 = \frac{1}{q\sqrt{pr}}\varrho_1, \qquad \lambda_2 = \frac{1}{pqr}\varrho_2,$$

then the eigenspaces $\tilde{\mathcal{S}}(\gamma_1, \gamma_2)$, $\mathcal{S}^*(\varrho_1, \varrho_2)$ *and* $S(\lambda_1, \lambda_2)$ *are isomorphic.*

PROOF. – For every function F, we consider the function G defined in Remark 3.5.3. It is evident that $F \in \mathcal{S}^*(\varrho_1, \varrho_2)$ if and only if $G \in S(\lambda_1, \lambda_2)$; hence the map $F \to G$ is a bijection from $\mathcal{S}^*(\varrho_1, \varrho_2)$ onto $S(\lambda_1, \lambda_2)$. We observe now that, for any choice of (γ_1, γ_2), there exists a unique pair (ϱ_1, ϱ_2) such that (2) holds. It is then evident that, for every $F \in \mathcal{S}^*(\varrho_1, \varrho_2)$, then $F \cdot r_\omega^{x_0}$ belongs to $\mathcal{S}(\gamma_1, \gamma_2)$ and hence, for every chamber c, the function $\tilde{F} = \widetilde{(F \cdot r_\omega^{x_0})}_c$ belongs to $\tilde{\mathcal{S}}(\gamma_1, \gamma_2)$. As we observed in Remark 3.5.3,

$$\tilde{F}(X) = F(X), \qquad \forall X \in \overline{Q}^\vee .$$

Choose a fundamental region $\mathcal{R}_0$ in $\overline{Q}^\vee$; since each function of $\tilde{\mathcal{S}}(\gamma_1, \gamma_2)$ (resp. $\mathcal{S}^*(\varrho_1, \varrho_2)$) is uniquely determined by its values on the vertices of $\mathcal{R}_0$, we conclude that the map $F \to \tilde{F}$ is a bijection from $\mathcal{S}^*(\varrho_1, \varrho_2)$ onto $\tilde{\mathcal{S}}(\gamma_1, \gamma_2)$. ■

As an evident consequence of Proposition 3.5.4, we have the following result.

COROLLARY 3.5.5. – *For every pair* $(\gamma_1, \gamma_2) \in \mathbb{C}^2$, $\dim \tilde{\mathcal{S}}(\gamma_1, \gamma_2) = 8$.

3.6. *Bijectivity of the Poisson transform.*

Let $H(\Omega)$ be the linear space of all locally constant functions on Ω and let $H'(\Omega)$ be its dual (consisting of all finitely additive measures defined on the algebra generated by the open sets of Ω). For every $x_0, x \in \mathcal{U}_0$ the function $P^{x_0}(x, \cdot)$ belongs to $H(\Omega)$.

For every pair (α, β) and every $x_0 \in \mathcal{U}_0$, the Poisson transform (having initial point x_0 and parameters α, β) of any $\nu \in H'(\Omega)$ is defined as the function

$$\mathcal{P}^{x_0}_{\alpha,\beta}\nu(x) = \int_\Omega P^{x_0}_{\alpha,\beta}(x, \omega)\, d\nu(x), \qquad x \in \mathcal{U}.$$

For ease of notation, we simply denote this function by $\mathcal{P}^{x_0}\nu$, when (α, β) is fixed.

As a direct consequence of Proposition 3.4.2, $\mathcal{P}^{x_0}\nu$ belongs to the eigenspace $\mathcal{S}(\gamma_1, \gamma_2)$, if $\gamma_i = \gamma_i(\alpha, \beta)$.

In order to investigate the bijectivity of the Poisson transform, according to the machinery used in cases $\tilde{A}_2$ and $\tilde{G}_2$, a fundamental step is to prove that

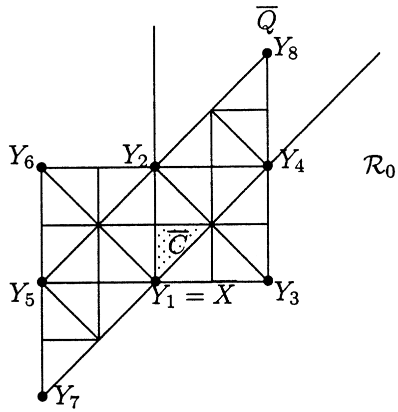


Figure 6

there exists a basis for the eigenspace $\tilde{\mathcal{S}}(\gamma_1, \gamma_2)$ consisting of functions obtained by retracting the Poisson kernel for a suitable choice of boundary points.

Fix a chamber c_0 and assume $x_0 \in c_0$.

DEFINITION 3.6.1. – We denote by $\Omega_k = \Omega_k(c_0)$ the set of all boundary points ω such that the base chamber of the sector $Q_{x_0}(\omega)$ has coordinates $(k, 0, 0)$ with respect to the sector based at c_0 in any apartment containing both sectors.

We pick a point ω_k in Ω_k, for each $k = 1, \ldots, 8$, and we consider $\tilde{P}(\cdot, \omega_k)$. In order to investigate the linear independence of these functions, we fix, in the abstract apartment $\mathbb{A}$, a fundamental region $\mathcal{R}_0$ as in Figure 6 and we denote by $Y_1, \ldots, Y_8$ its vertices (of type 0). Moreover, we assume that $r_\omega^{x_0}$ maps $Q_{x_0}(\omega)$ onto the sector Q_k based at the chamber C_k of coordinates $(k, 0, 0)$ (with respect to $\overline{Q}$). Then we construct the 8×8-matrix $\mathbb{P} = (P_{j,k})$, where $P_{j,k} = \tilde{P}(Y_j, \omega_k)$, for every j, k, according to the following proposition, which extends Proposition 4.5.3 of [6].

PROPOSITION 3.6.2. – *For every* $j = 1, \ldots, 8$, *let* π_j *be the type of a minimal gallery connecting* $\overline{C}$ *to* Y_j; *then*

$$\tilde{P}(Y_j, \omega_k) = \frac{1}{|r_{c_0}^{-1}(Y_j)|} \mathbb{V}_0 \mathbb{M}_{\pi_j} e_k, \qquad \textit{for } k \neq 4, 5\,,$$

$$\tilde{P}(Y_j, \omega_4) = \frac{1}{|r_{c_0}^{-1}(Y_j)|} \mathbb{V}_0 \mathbb{M}_{\pi_j} e_5,$$

$$\tilde{P}(Y_j, \omega_5) = \frac{1}{|r_{c_0}^{-1}(Y_j)|} \mathbb{V}_0 \mathbb{M}_{\pi_j} e_4,$$

PROOF. – If $k \neq 4, 5$, the chamber $\overline{C}$ has coordinates $(k, 0, 0)$ with respect to Q_k; moreover $\overline{C}$ has coordinates $(5, 0, 0)$ with respect to Q_4 and coordinates $(4, 0, 0)$ with respect to Q_5. Applying Theorem 3.2.2, we conclude. ■

The following definition extends that given in [6, Section 4.5].

DEFINITION 3.6.3. – A pair (α, β) is called «singular» if $\det \mathbb{P} = 0$ and «ultrasingular» if $\sigma(\alpha, \beta)$ is singular for every $\sigma \in W_0$. We call the eigenvalues (γ_1, γ_2) «singular» if they correspond to ultrasingular pairs (α, β).

THEOREM 3.6.4. – *Let* $(\gamma_1, \gamma_2) \in \mathbb{C}^2$ *and* $\gamma_i = \gamma_i(\alpha, \beta)$, $i = 1, 2$.

(1) *The Poisson transform* $\mathcal{P}^{x_0}_{\alpha, \beta}$ *is injective.*

(2) *If* (α, β) *is non singular, then for every* $f \in \mathcal{S}(\gamma_1, \gamma_2)$ *there exists a unique* $\nu \in H'(\Omega)$ *such that* $f = \mathcal{P}_{\alpha, \beta}\nu$.

(3) *If* (γ_1, γ_2) *are singular, then* $\mathcal{P}^{x_0}_{\alpha, \beta}(H'(\Omega))$ *is properly contained in* $\mathcal{S}(\gamma_1, \gamma_2)$.

PROOF. – We assume that the initial point x_0 is the fundamental vertex e. The injectivity follows as in [5, Proposition 4.1].

Let (α, β) be non singular; the functions $\widetilde{P}(\cdot, \omega_1), \ldots, P(\cdot, \omega_8)$ are linearly independent and so they can be chosen as a basis for $\widetilde{\mathcal{S}}(\gamma_1, \gamma_2)$. Starting from this property we obtain the required characterization of the eigenfunctions of the Laplacians, as in the case $\widetilde{G}_2$ (see [5]).

Let (γ_1, γ_2) be singular. By Lemma 3.5.3 $\dim \widetilde{\mathcal{S}}(\gamma_1, \gamma_2) = 8$, but the functions $\widetilde{P}(\cdot, \omega_k)$, $k = 1, \ldots, 8$ are not linearly independent. Thus there exists a function $F \in \widetilde{\mathcal{S}}(\gamma_1, \gamma_2)$ which does not belong to their linear span.

Let us consider the function $f \in \mathcal{S}(\gamma_1, \gamma_2)$ defined by

$$f(x) = F(X), \qquad \forall x \in r_c^{-1}(X), \quad \forall X \in \overline{\mathcal{U}}_0.$$

If $f = \mathcal{P}_{\alpha, \beta}\nu$, for some $\nu \in H'(\Omega)$, then we can write (see [5, Proposition 4.1])

$$F(X) = \tilde{f}_c(X) = \sum_{k=1}^{8} \mu_k \widetilde{P}(X, \omega_k), \qquad \forall X \in \overline{\mathcal{U}}_0,$$

for suitable $\mu_1, \ldots, \mu_8$, but this is absurd because of the choice of F. ■

In the following proposition we exhibit the singular pairs.

PROPOSITION 3.6.5. – *The pair* (α, β) *is singular in the following cases*:

$$\alpha = q, \quad \alpha = -pq, \quad \beta = 1, \quad \beta = -p, \quad \beta = \alpha, \quad \alpha\beta = pr.$$

PROOF. – We consider the 8×8 matrix $\mathbb{P} = (P_{jk})$, where $P_{jk} = \widetilde{P}(X_j, \omega_k)$. The entries of this matrix are polynomials in the five variables p, q, r, α, β. Therefore for computing its determinant we give step by step commands to the mathematical software «Mathematica 2» in order to operate appropriate simplifications on the columns of the matrix, so reducing the computation of its determinant to that of a 5×5-matrix. We get, for every (α, β),

$$\det \mathbb{P} = K\alpha^{-8}\beta^{-8}(\alpha - q)^4(\alpha + pq)^4(\beta - 1)^4(\beta + p)^4(\alpha - \beta)^4(\alpha\beta - pr)^4,$$

where K is a constant depending on p, q and r. ■

While for buildings of both types $\widetilde{A}_2$ and $\widetilde{G}_2$ (if the Laplace operators are defined on all vertices in the case $\widetilde{A}_2$ and only on the vertices of type 0 in the case $\widetilde{G}_2$) no pair (α, β) is ultrasingular, for a building $\widetilde{B}_2$ (considering only the vertices of type 0) there exist ultrasingular pairs for particular choices of the valencies p, q, r.

PROPOSITION 3.6.6. – *There exist ultrasingular pairs if and only if $p = r$, qr or $q^2 r$. Moreover the ultrasingular pairs (α, β) are the following*:

(1) $(\alpha, \beta) = \sigma(-pq, t)$ *or* $(\alpha, \beta) = \sigma(t, -p)$, *for* $\sigma \in W_0$ *and* $t \in \mathbb{C}$, *if* $p = r$,

(2) $(\alpha, \beta) = (-pq, -p) \sim (-p, -r) \sim (-pq, -r) \sim (-pq, -pq)$, *if* $p = qr$,

(3) $(\alpha, \beta) = (-p, -p) \sim (-pq, -qr) \sim (-p, -r) \sim (-qr, -qr)$, *if* $p = q^2 r$.

PROOF. – Setting $\alpha = q\sqrt{pr}\,\xi$, $\beta = \sqrt{pr}\,\eta$, the pair (ξ, η) is singular if and only if at least one of the following relations is satisfied:

$$\begin{aligned} &\xi = 1/\sqrt{pr}, \qquad \xi = -\sqrt{p/r}, \\ &h = 1/\sqrt{pr}, \qquad \eta = -\sqrt{p/r}, \\ &\eta = q\xi, \quad \xi\eta = 1/q. \end{aligned} \tag{3}$$

Replacing, if necessary, (ξ, η) by an equivalent pair, we may assume $1 \leqslant |\eta| \leqslant |\xi|$. Let us denote

$$\mathcal{F} = \{(\xi, \eta) : 1 \leqslant |\eta| \leqslant |\xi|\}.$$

If $p < r$, none of the conditions (3) may be satisfied by a pair of $\mathcal{F}$; so in this case there are no ultrasingular pairs.

When $p = r$, the only singular pairs in $\mathcal{F}$ are

$$(-1, \eta), \qquad \text{for } |\eta| = 1\,; \qquad (\xi, -1), \qquad \text{for } |\xi| \geqslant 1\,.$$

It is easy to check that each of these pairs is ultrasingular.

Let assume $p > r$. In this case the singular pairs of $\mathcal{T}$ are the following:

$$(-\sqrt{p/r}, \eta), \quad \text{for } 1 \leqslant |\eta| \leqslant \sqrt{p/r}; \qquad (\xi, -\sqrt{p/r}), \quad \text{for } |\xi| \geqslant \sqrt{p/r}.$$

If η (resp. ξ) is not real, then $(-\sqrt{r/p}, \eta)$ (resp. $(\xi, -\sqrt{r/p})$) is a non-singular pair which is equivalent to $(-\sqrt{p/r}, \eta)$ (resp. $(\xi, -\sqrt{p/r})$); so we restrict to $\xi, \eta \in \mathbb{R}$.

If $1 \leqslant \eta \leqslant \sqrt{p/r}$, the pair $(\xi, \eta) = (-\sqrt{p/r}, \eta)$ is not ultrasingular because the equivalent pair $(\xi', \eta') = (\xi^{-1}, \eta)$ is not singular. Analogously, if $\xi \geqslant \sqrt{p/r}$ the pair $(\xi, \eta) = (\xi, -\sqrt{p/r})$ is not ultrasingular because $(\xi', \eta') = (\xi, \eta^{-1})$ is not singular.

Moreover a direct computation shows that all pairs $(-\sqrt{p/r}, \eta)$, corresponding to $-\sqrt{p/r} < \eta < -1$, and all pairs $(\xi, -\sqrt{p/r})$, corresponding to $\xi < -\sqrt{p/r}$, are not ultrasingular.

Finally we check that the pair $(-\sqrt{p/r}, -\sqrt{p/r})$ (resp. the pair $(-\sqrt{p/r}, -1)$) has an equivalent pair which is not singular if and only if $p \neq qr$ (resp. $p \neq q^2 r$).

Keeping in mind the relation between (ξ, η) and (α, β), the proposition is proved. ∎

COROLLARY 3.6.7. – *There exist singular eigenvalues (γ_1, γ_2) if and only if $p = r$, qr or $q^2 r$. Moreover*

(1) *if $p = r$, then (γ_1, γ_2) are singular if and only if $p(q+1)\gamma_1 + p^2 q \gamma_2 + 1 = 0$;*

(2) *if $p = qr$, or $p = q^2 r$, then $(\gamma_1, \gamma_2) = (-1/p, 1/p^2)$ is the only pair of singular eigenvalues.*

PROOF. – By a direct computation we show that if $p = qr$ (resp. $p = q^2 r$) then $\gamma_1(-pq, -p) = -1/p$ and $\gamma_2(-pq, -p) = 1/p^2$ (resp. $\gamma_1(-p, -p) = -1/p$ and $\gamma_2(-p, -p) = 1/p^2$). Moreover, if $p = r$, then

$$\gamma_1(-pq, t) = pqK_1^{-1} T + (-pq + p - q - 1) K_1^{-1},$$

$$\gamma_2(-pq, t) = -pq(p+1) K_2^{-1} T + (p+1)(q-p) K_2^{-1},$$

where $T = p^{-1} t + pt^{-1}$, and

$$\gamma_1(t, -p) = pqK_1^{-1} U + (-pq + p - q - 1) K_1^{-1},$$

$$\gamma_2(t, -p) = -pq(p+1) K_2^{-1} U + (p+1)(q-p) K_2^{-1},$$

where $U = (pq)^{-1} t + pqt^{-1}$. ∎

REMARK 3.6.8. – Keeping in mind the relation between the valencies p, q and r stated in Section 2, if $p = q^2 r$ then $p = q = r = 1$ and hence the building reduces to an apartment.

REMARK 3.6.9. – If we consider the set $\mathcal{U}_2$ of all vertices of type 2, all results of Section 3 hold, simply replacing p, r by r, p.

4. – The case of all special vertices.

4.1. *Coordinates on an apartment.*

We consider now all special vertices of both types 0 and 2; we denote $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_2$ and $\overline{\mathcal{U}} = \overline{\mathcal{U}}_0 \cup \overline{\mathcal{U}}_2$.

Given an apartment $\mathcal{A}$ and a sector Q_{x_0} on it, with $x_0 \in \mathcal{U}$, each special vertex in $\mathcal{A}$ may be assigned a pair of integer coordinates (M, N) (with respect to Q_{x_0}), as we did in Section 3 for type 0 vertices. Now H_1 is the line passing through x_0 and containing the 1-wall of Q_{x_0} and H_2 is that containing the 1-wall of the sector Q''_{x_0} 2-adjacent to Q_{x_0}. With this assumption the special vertices of Q_{x_0} are characterized by coordinates (M, N) with $0 \leqslant N \leqslant M$ (see Figure 7).

If we restrict to the vertices of the same type of x_0 (say 0, for instance), and (m, n) are the coordinates of x defined in Section 3, then

$$M = m + n\,, \qquad N = m - n\,. \tag{4}$$

We observe that if (k, m, n) (resp. (k', m, n)) are the coordinates of a chamber c of $\mathcal{A}$, defined in Section 3, with respect to its vertex of type 0 (resp. of

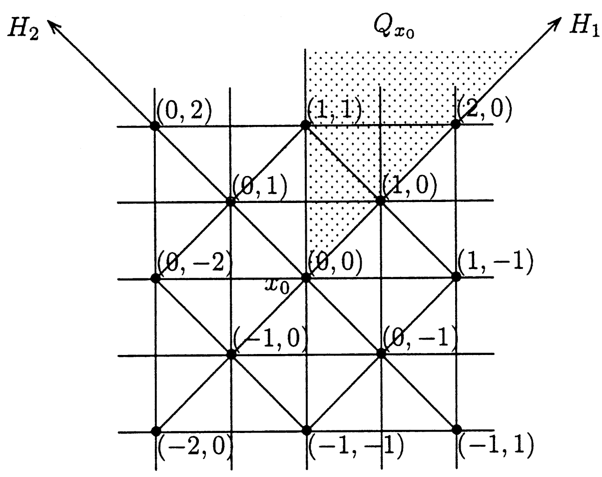


Figure 7

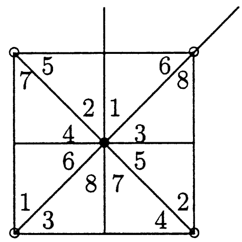


Figure 8

type 2), then k and k' are related as in Figure 8 (in each chamber the number near the type 0 vertex (marked $\bullet$) denotes k and the other one denotes k').

4.2. *Poisson kernel.*

The definition of Poisson kernel is the same as that given in Section 3.2, with respect to the new coordinate system, and the Poisson kernel depends on the initial point, according to the formula (1) of Section 3.2.

The method described in [6, section 3] allows us to determine the retraction (with respect to a chamber) of the Poisson kernel. Indeed, for the vertices of type 0 we apply Proposition 3.2.2, provided we consider, in view of the the relation (4), parameters $\alpha' = \alpha\beta$, $\beta' = \alpha\beta^{-1}$; for the vertices of type 2, we choose an initial point of type 2 and then we apply the previous method and formula (1).

4.3. *Laplace operators on the abstract apartment.*

The definition of Laplace operators L_1, L_2 and generalized Laplace operators Λ_1, Λ_2 given in Section 3.3 extends to the present case, if we define (see Figure 9)

$$\overline{S}_1(X) = \{Y \in \overline{\mathfrak{U}} : d(X, Y) = 1\} = \{X_1, X_2, X_3, X_4\},$$

$$\overline{S}_2(X) = \{Y \in \overline{\mathfrak{U}} : \pi(X, Y) = (0)\} = \{X_5, X_6, X_7, X_8\}.$$

We point out that each $Y \in \overline{S}_1(X)$ has different type from X, while the vertices of $\overline{S}_2(X)$ have the same type of X and coincide with the vertices of the set $\overline{S}_1(X)$ defined in Section 3.3.

A fundamental region $\mathcal{R}_0$ in $\mathbb{A}$ consists now of four type 0 vertices and four type 2 vertices, as shown in Figure 10.

All the results of Section 3.3 extend to the present case. In particular, for

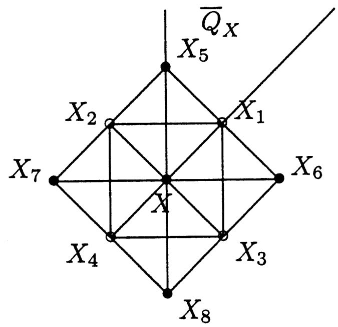


Figure 9

every pair $(\lambda_1, \lambda_2) \in \mathbb{C}^2$, dim $\mathcal{S}_{\Lambda_1^\sharp, \Lambda_2^\sharp}(\lambda_1, \lambda_2) \leqslant 8$, provided $h_i(X, Y) \neq 0$, for $Y \in \overline{S}_i(X)$. The operators L_1, L_2 generate the algebra of W_0-invariant difference operators with constant coefficients acting on the complex valued functions on $\overline{\mathcal{U}}$ and their joint eigenspaces have dimension 8.

4.4. *Laplace operators on the building.*

For every $x \in \mathcal{U}$ we define

$$S_1(x) = \{ y \in \mathcal{U} : d(x, y) = 1 \},$$

$$S_2(x) = \{ y \in \mathcal{U} : \pi(x, y) = (0) \}.$$

In order to make the cardinality of $S_i(x)$ independent of the type of the vertex x, we assume from now on $p = r$. Thus

$$K_1 = |S_1(x)| = (p+1)(pq+1), \qquad K_2 = |S_2(x)| = p(p+1)(pq+1).$$

The Laplace operators are defined as in Section 3.4. The following proposition extends Proposition 3.4.2 to all special vertices.

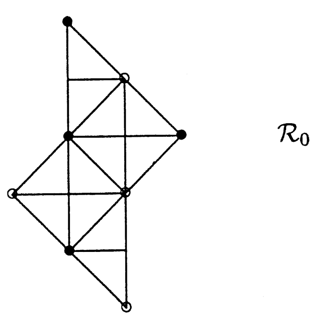


Figure 10

PROPOSITION 4.4.1. – *For every $\omega \in \Omega$ and $x_0 \in \mathcal{U}$, the function $P^{x_0}(\cdot, \omega)$ is a joint eigenfunction of the Laplacians, associated with the eigenvalues*

$$\gamma_1 = \gamma_1(\alpha, \beta) = K_1^{-1}(\alpha + p\beta + pq\beta^{-1} + p^2 q\alpha^{-1}),$$

$$\gamma_2 = \gamma_2(\alpha, \beta) = K_2^{-1}(\alpha\beta + q\alpha\beta^{-1} + p^2 q\alpha^{-1}\beta + p^2 q^2 \alpha^{-1}\beta^{-1}) + K_2^{-1}(p-1)(q+1).$$

PROOF. – Let c be the base chamber of the sector $Q_x(\omega)$ and $C = r_c(c) = r_\omega^x(c)$. If $X = r_c(x)$, we evaluate $\sum_{y \in r_c^{-1}(X_i)} P^x(y, \omega)$, $i = 1, \ldots, 4$, by a direct computation. Then we use Theorem 3.3.2, with parameters $\alpha' = \alpha\beta$, $\beta' = \alpha\beta^{-1}$, to evaluate $\sum_{y \in r_c^{-1}(X_i)} P^x(y, \omega)$, $i = 5, \ldots, 8$. We conclude by the same argument used in Proposition 3.4.2. ■

By the same notation of Section 3.4, we have the following corollary.

COROLLARY 4.4.2. – *For every $(\gamma_1, \gamma_2) \in \mathbb{C}^2$ there exists a pair $(\alpha, \beta) \in (\mathbb{C}^\times)^2$ such that $\gamma_i = \gamma_i(\alpha, \beta)$, $i = 1, 2$; moreover $\gamma_i(\alpha, \beta) = \gamma_i(\alpha', \beta')$ if and only if, for some $\sigma \in W_0$, $(\alpha', \beta') = \sigma(\alpha, \beta) = (p\sqrt{q}\,\xi_\sigma, \sqrt{q}\,\eta_\sigma)$, where $\xi = \alpha/p\sqrt{q}$, $\eta = \beta/\sqrt{q}$. Therefore, for every $\sigma \in W_0$, the function $P^{x_0}_{\sigma(\alpha,\beta)}(\cdot, \omega)$ belongs to $\mathcal{S}(\gamma_1, \gamma_2)$.*

As in Section 3.5 we state that the joint eigenspaces of the operators $\widetilde{\mathcal{L}}_1$, $\widetilde{\mathcal{L}}_2$ have dimension equal to the cardinality of the finite Weyl group W_0.

4.5. *Bijectivity of the Poisson transform.*

We can prove for the Poisson transform on all special vertices an analogous of Theorem 3.6.4 if we choose a fundamental region $\mathcal{R}_0$ as in Figure 11.

The following proposition exhibits all singular pairs (α, β).

PROPOSITION 4.5.1. – *The pair (α, β) is singular in the following cases*:

$$\alpha = p\,, \quad \alpha = -p\,, \quad \beta = 1\,, \quad \beta = -1\,, \quad \beta = \alpha\,, \quad \alpha\beta = q\,.$$

PROOF. – We note that $Z_1 = Y_1$, $Z_2 = Y_2$, $Z_3 = Y_4$, $Z_4 = Y_6$, following notation of Section 3.6; therefore we evaluate $\widetilde{P}(Z_j, \omega_k)$, $j = 1, \ldots, 4$, $k = 1, \ldots, 8$, by using Proposition 3.6.2 with parameters $\alpha' = \alpha\beta$, $\beta' = \alpha\beta^{-1}$ and $p = r$. The row $\widetilde{P}(Z_5, \omega_k)$, $k = 1, \ldots, 8$, is the following vector

$$(\alpha, \beta, \alpha, \beta^{-1}, \beta, \alpha^{-1}, \beta^{-1}, \alpha^{-1}).$$

Finally, for $j = 6, 7, 8$ we fix as initial point the type 2 vertex of c_0, which retracts to Z_5; then we apply (1) and Proposition 3.6.2 with parameters $\alpha' = \alpha\beta$,

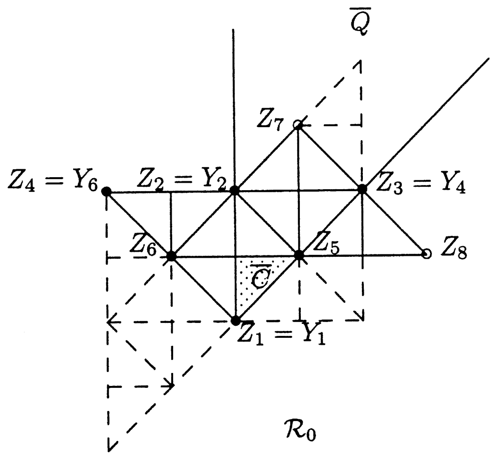


Figure 11

$\beta' = \alpha\beta^{-1}$ and $p = r$. Keeping in mind the relation between the coordinates k and k' of a chamber with respect to its type 0 and type 2 vertex respectively, illustrated in Figure 8, we have

$$\widetilde{P}(Z_j, \omega_1) = \alpha(\mathbb{V}_0 \mathbb{M}'_{\pi_j})\, e_6$$

$$\widetilde{P}(Z_j, \omega_2) = \beta(\mathbb{V}_0 \mathbb{M}'_{\pi_j})\, e_5$$

$$\widetilde{P}(Z_j, \omega_3) = \alpha(\mathbb{V}_0 \mathbb{M}'_{\pi_j})\, e_8$$

$$\widetilde{P}(Z_j, \omega_4) = \beta^{-1}(\mathbb{V}_0 \mathbb{M}'_{\pi_j})\, e_2$$

$$\widetilde{P}(Z_j, \omega_5) = \beta(\mathbb{V}_0 \mathbb{M}'_{\pi_j})\, e_7$$

$$\widetilde{P}(Z_j, \omega_6) = \alpha^{-1}(\mathbb{V}_0 \mathbb{M}'_{\pi_j})\, e_1$$

$$\widetilde{P}(Z_j, \omega_7) = \beta^{-1}(\mathbb{V}_0 \mathbb{M}'_{\pi_j})\, e_4$$

$$\widetilde{P}(Z_j, \omega_8) = \alpha^{-1}(\mathbb{V}_0 \mathbb{M}'_{\pi_j})\, e_3,$$

where $\mathbb{M}'_0 = \mathbb{M}_2$, $\mathbb{M}'_1 = \mathbb{M}_1$, $\mathbb{M}'_2 = \mathbb{M}_0(\alpha', \beta')$, and $\pi_6 = (2)$, $\pi_7 = (0, 1, 2)$, $\pi_8 = (1, 0, 1, 2)$. We can compute the determinant of $\mathbb{P}$ by using, as in Proposition 3.6.5, the mathematical software «Mathematica 2». We get, for every (α, β),

$$\det \mathbb{P} = K\alpha^{-8}\beta^{-8}(\alpha - p)^4(\alpha + p)^4(\beta - 1)^4(\beta + 1)^4(\alpha - \beta)^4(\alpha\beta - q)^4,$$

where K is a constant depending on p and q. ■

Unlike the case of only one type of special vertices, for all special vertices there not exist ultrasingular pairs.

PROPOSITION 4.5.2. – *For every singular pair* (α, β), *there exists* $\sigma \in W_0$ *such that* $\sigma(\alpha, \beta)$ *is non-singular.*

PROOF. – Setting $\alpha = p\sqrt{q}\,\xi$, $\beta = \sqrt{q}\,\eta$, then (ξ, η) is singular if and only if at least one of the following relation is satisfied:

$$\begin{aligned} &\xi = 1/\sqrt{q}\,, \qquad \xi = -1/\sqrt{q}\,, \\ &\eta = 1/\sqrt{q}\,, \qquad \eta = -1/\sqrt{q}\,, \\ &\eta = p\xi\,, \qquad \xi\eta = 1/p\,. \end{aligned} \tag{5}$$

Replacing eventually (ξ, η) by an equivalent pair, we may assume $(\xi, \eta) \in \mathcal{T}$. Since none of conditions (5) may be satisfied in $\mathcal{T}$, there are no ultrasingular pairs. ■

Proposition 4.5.2 implies that, if we consider special vertices of both types 0 and 2, then all joint eigenvalues (γ_1, γ_2) are non singular; therefore, denoted by (α, β) a non singular pair such that $\gamma_i = \gamma_i(\alpha, \beta)$, we prove that for every $f \in \mathcal{S}(\gamma_1, \gamma_2)$ there exists a unique $\nu \in H'(\Omega)$ such that $f = \mathcal{P}_{\alpha, \beta}\nu$.

5. – The case of non-special vertices.

5.1. *Generalities.*

In this section we consider the set $\mathcal{U}_1$ of all type 1 vertices, i.e. the non-special vertices of the building; as usual, we denote by $\overline{\mathcal{U}_1}$ the set of non-special vertices of the abstract apartment $\mathbb{A}$. For every $x \in \mathcal{U}_1$, $S(x)$ contains $(r+1)$ vertices of type 0, $(p+1)$ vertices of type 2 and $(p+1)(r+1)$ chambers.

Given an apartment $\mathcal{A}$ and a sector Q_{x_0} on it, say $x_0 \in \mathcal{U}_0$, each non-special vertex in $\mathcal{A}$ may be assigned a pair of integer coordinates (with respect to Q_{x_0}), as we done in Section 3 and 4 for the special vertices. If z_0 is the non-special vertex of the base chamber of Q_{x_0}, then we choose as H_1 and H_2 the lines passing through z_0 and parallel to the lines containing the 1-wall of Q_{x_0} and the 1-wall of the sector Q''_{x_0} 2-adjacent to Q_{x_0} respectively. With this assumption the

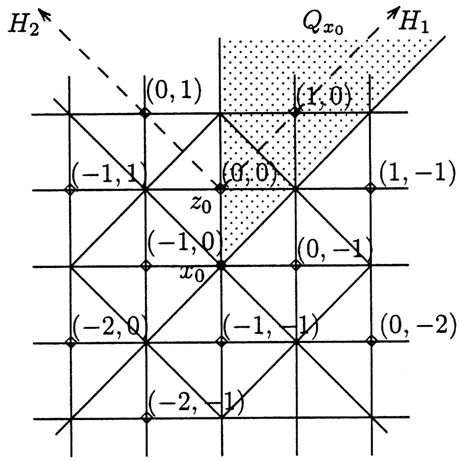


Figure 12

non-special vertices of Q_{x_0} are characterized by coordinates (M, N) with $0 \leqslant N \leqslant M$ (see Figure 12).

We point out that the coordinates defined above are those defined in Section 4 for all special vertices, simply replacing x_0 by z_0. From now on, we denote by $z_{M,N}$, the non-special vertex of $\mathcal{A}$ of coordinates (M, N).

The coordinates of a chamber of $\mathcal{A}$ are defined as in Section 3.

If $\{c_{k,m,n}, k = 1, \ldots, 8\}$ are the chambers of $\mathcal{A}$ sharing a type 0 vertex of coordinates (m, n), then we deduce from Figure 8 that (see Figure 13):

$$(6) \quad \begin{array}{ll} z_{m+n,\, m-n} \in c_{k,m,n}, \quad k = 1, 2\,, & z_{m+n,\, m-n-1} \in c_{k,m,n}, \quad k = 3, 5\,, \\ z_{m+n-1,\, m-n} \in c_{k,m,n}, \quad k = 4, 6\,, & z_{m+n-1,\, m-n-1} \in c_{k,m,n}, \quad k = 7, 8\,. \end{array}$$

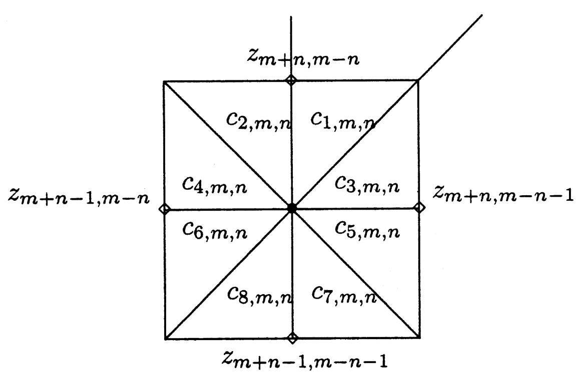


Figure 13

5.2. *Poisson kernel.*

We fix on $\mathbb{A}$ a sector $\overline{Q}$ based at a special vertex $\overline{X}$ and we denote by $\overline{Z}$ the non-special vertex of the base chamber $\overline{C}$ of $\overline{Q}$. The definition of Poisson kernel given in Section 3.2 extends to the non-special vertices, if we refer to the coordinate system defined in Section 5.1 with respect to $\overline{Q}$; for every $x_0 \in \mathcal{U}$,

$$P^{x_0}_{\alpha,\beta}(z,\omega) = \phi_{\alpha,\beta}(r^{x_0}_{\omega}(z)), \qquad \forall z \in \mathcal{U}_1, \ \ \forall \omega \in \Omega .$$

Obviously, for every $\omega \in \Omega$, $P^{x_0}_{\alpha,\beta}(z_0,\omega) = 1$, if z_0 is the non-special vertex of the base chamber of the sector $Q_{x_0}(\omega)$; nevertheless the vertex z_0 depends on the choice of ω (as well as on x_0). We point out that, if we consider a non-special vertex z adjacent to x_0, then $P^{x_0}(z,\omega)$ may only assume the values:

$$1, \qquad \alpha^{-1}, \qquad \beta^{-1}, \qquad \alpha^{-1}\beta^{-1},$$

according to the position of z with respect to $Q_{x_0}(\omega)$ in any apartment containing both them (see Figure 14).

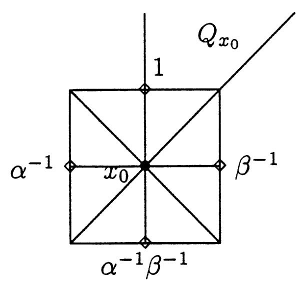


Figure 14

The Poisson kernel depends on the choice of the vertex x_0, in the following way.

LEMMA 5.4. – *Let $\omega \in \Omega$ and let x_0, x_1 be special vertices. Then*

$$P^{x_1}(z,\omega) = P^{x_0}(z,\omega)(P^{x_0}(z_1,\omega))^{-1}, \qquad \forall z \in \mathcal{U}_1,$$

where z_1 is the non-special vertex of the base chamber of $Q_{x_1}(\omega)$.

The retraction of $P^{x_0}(\cdot,\omega)$ with respect to a chamber may be determined using an algorithm analogous to that used in Section 3 for special vertices of a fixed type.

Assume $\tau(x_0) = 0$. Let $\phi_{\alpha,\beta}$ be the multiplicative function on $\mathcal{U}_1$; as usual it may be considered as a function on the chambers of $\mathbb{A}$, which is constant on the four chambers sharing a non-special vertex. If (k, m, n) are the coordinates of a chamber C, then it is easy to check, using (6), that

$$\phi_{\alpha,\beta}(C) = \psi_{\alpha',\beta'}(m, n)\,\chi(k),$$

where $\psi_{\alpha',\beta'}$ is the multiplicative function on $\mathcal{U}_0$ of parameters $\alpha' = \alpha\beta$, $\beta' = \alpha\beta^{-1}$ and the vector

$$\mathbb{V}_1 = (\chi(1), \ldots, \chi(8)) = (1, 1, \beta^{-1}, \alpha^{-1}, \beta^{-1}, \alpha^{-1}, \alpha^{-1}\beta^{-1}, \alpha^{-1}\beta^{-1})$$

plays the role of the vector $\mathbb{V}_0$ defined for the type 0 vertices. Therefore, referring to this vector, we may use the method applied in Section 3 for the type 0 vertices to evaluate in every C the retraction of $P^{x_0}(\cdot, \omega)$ with respect to a chamber c_0, in terms of the coordinates of the chamber $C_0 = r_{c_0}(c_0) = r_\omega^{x_0}(c_0)$, and of the type of a minimal gallery joining C_0 to C. The following theorem holds.

THEOREM 5.2.2. – *Let (k, m, n) be the coordinates of the chamber C_0 (with respect to $\overline{\mathrm{Q}}$). For every $Z \in \overline{\mathcal{U}}_1$, let C be the chamber containing Z in a minimal gallery connecting C_0 to Z. Let $\pi = (i_1, \ldots, i_l)$ be the type of $[C_0, C]$; then*

$$\widetilde{P}(Z, \omega) = \widetilde{P}(C, \omega) = \frac{1}{|r_{c_0}^{-1}(C)|}\,\psi(m, n)\,\mathbb{V}_1 \mathbb{N}_\pi e_k,$$

where e_k is the 8×1-matrix such that $e_{h,k} = \delta_{hk}$ and $\mathbb{N}_\pi = \mathbb{N}_{i_l} \ldots \mathbb{N}_{i_1}$, where $\mathbb{N}_0 = \mathbb{M}_0(\alpha', \beta')$, $\mathbb{N}_1 = \mathbb{M}_1$ and $\mathbb{N}_2 = \mathbb{M}_2$.

5.3. *Laplace operators on the abstract apartment.*

The Laplace operators L_1, L_2 and the operators $\Lambda_1^\sharp$, $\Lambda_2^\sharp$ are defined as for special vertices if, for $Z \in \overline{\mathcal{U}}_1$, the sets $\overline{S}_1(Z)$ and $\overline{S}_2(Z)$ are the following (see Figure 15):

$$\overline{S}_1(Z) = \{Y \in \overline{\mathcal{U}}_1 \colon \pi(Z, Y) = (1)\} = \{Z_1, \ldots, Z_4\},$$

$$\overline{S}_2(Z) = \{Y \in \overline{\mathcal{U}}_1 \colon \pi(Z, Y) = (1, i, 1), i = 0, 2\} = \{Z_5, \ldots, Z_8\}.$$

We point out that, if $Z = Z_{M,N}$, then

$$Z_1 = Z_{M+1,N}, \qquad Z_2 = Z_{M,N+1}, \qquad Z_3 = Z_{M,N-1}, \qquad Z_4 = Z_{M-1,N},$$

$$Z_5 = Z_{M+1,N+1}, \qquad Z_6 = Z_{M+1,N-1}, \qquad Z_7 = Z_{M-1,N+1}, \qquad Z_8 = Z_{M-1,N-1}.$$

A fundamental region $\mathcal{R}_0$ of $\mathbb{A}$ consists of eight non-special vertices, as shown in Figure 16. Any joint eigenfunction of $\Lambda_1^\sharp$, $\Lambda_2^\sharp$ is uniquely determined by its values on the non-special vertices of this region. Hence the eigenspaces $\mathcal{S}_{\Lambda_1^\sharp, \Lambda_2^\sharp}(\lambda_1, \lambda_2)$ have dimension less than or equal to 8.

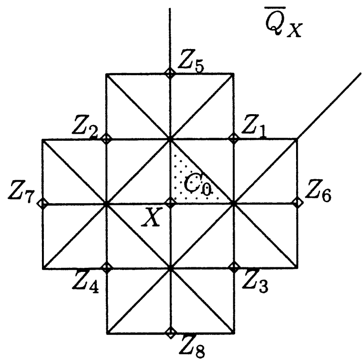


Figure 15

Let $\widetilde{W}_0$ be the dual finite Weyl group of the building; $\widetilde{W}_0$ acts on the abstract apartment as the finite reflection group generated by $\tilde{\sigma}_1, \tilde{\sigma}_2$, where $\tilde{\sigma}_1$ is the reflection with respect to the line containing the edge of type 0 starting from $\overline{Z}$ and $\tilde{\sigma}_2$ is the reflection with respect to the line orthogonal to the edge of type 1 opposite to $\overline{Z}$. All the results stated in Section 3.3 for the Laplace operators L_1, L_2 extend to the present case, if we refer to the group $\widetilde{W}_0$.

5.4. *Laplace Operators on Δ.*

For every $z \in \mathcal{U}_1$ we define

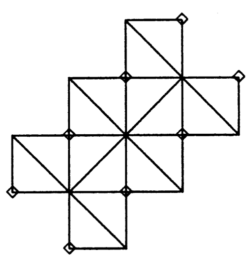
$$S_1(z) = \{ y \in \mathcal{U}_1 \colon \pi(z, y) = (1) \},$$

$$S_2(z) = \{ y \in \mathcal{U}_1 \colon \pi(z, y) = (1, i, 1),\ i = 0, 2 \}.$$

The cardinality of $S_i(z)$ does not depend on the vertex z; actually

$$K_1 = |S_1(z)| = q(p+1)(r+1) \qquad K_2 = |S_2(z)| = q^2(p + r + 2pr).$$

Starting from this definition of $S_1(z)$ and $S_2(z)$, we extend to non-special vertices the definition of Laplace operators $\mathcal{L}_1$ and $\mathcal{L}_2$ on Δ given for special verti-



$\mathcal{R}_0$

Figure 16

ces. As before, we define, for every pair $(\gamma_1, \gamma_2) \in \mathbb{C}^2$, the eigenspaces $\mathcal{S}(\gamma_1, \gamma_2)$ and $\tilde{\mathcal{S}}(\gamma_1, \gamma_2)$.
We point out that, if $p \neq r$, the sum $\sum_{y \in S_i(z)} P^x(y, \omega)$, for a fixed $\omega \in \Omega$, and z in the base chamber of the sector $Q_x(\omega)$, depends on z. Precisely it assumes different values if x has type 0 or 2, because edges of type 0 and 2 have different valencies. To avoid this problem, from now on we assume $p = r$. We state, as for special vertices, the following proposition.

PROPOSITION 5.4.1. – *For every $\omega \in \Omega$ and for every special x_0, $P^{x_0}(\cdot, \omega)$ belongs to the eigenspace $\mathcal{S}(\gamma_1, \gamma_2)$, associated with the eigenvalues*

$$\gamma_1 = \gamma_1(\alpha, \beta) = K_1^{-1}(\alpha + p\beta + pq\beta^{-1} + p^2 q\alpha^{-1}) + c_1,$$

$$\gamma_2 = \gamma_2(\alpha, \beta) = K_2^{-1}(q-1)(\alpha + p\beta + pq\beta^{-1} + p^2 q\alpha^{-1})$$

$$+ K_2^{-1}(\alpha + p^2 q\alpha^{-1})(\beta + q\beta^{-1}) + c_2,$$

where $c_1 = K_1^{-1}(p+1)(q-1)$, $c_2 = K_2^{-1}(pq^2 + p - 2q)$.

Using notation of Section 3.4, we have the following corollary.

COROLLARY 5.4.2. – *For every $(\gamma_1, \gamma_2) \in \mathbb{C}^2$, there exists $(\alpha, \beta) \in (\mathbb{C}^\times)^2$ such that $\gamma_i = \gamma_i(\alpha, \beta)$, $i = 1, 2$; moreover $\gamma_i(\alpha, \beta) = \gamma_i(\alpha', \beta')$ if and only if, for some $\sigma \in W_0$, $(\alpha', \beta') = \sigma(\alpha, \beta) = (p\sqrt{q}\xi_\sigma, \sqrt{q}\eta_\sigma)$, where $\xi = \alpha/p\sqrt{q}$, $\eta = \beta/\sqrt{q}$. Therefore, for every $\sigma \in W_0$, the function $P^{x_0}_{\sigma(\alpha, \beta)}(\cdot, \omega)$ belongs to $\mathcal{S}(\gamma_1, \gamma_2)$.*

As in Section 3.5 and 4.4, we state that the joint eigenspaces of the operators $\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_2$, obtained by retracting the Laplace operators with respect to a chamber, have dimension equal to the cardinality of the finite Weyl group W_0.

5.5. *Bijectivity of the Poisson transform.*

We can prove for the Poisson transform on non-special vertices an analogous of Theorem 3.6.4 if we choose a fundamental region $\mathcal{R}_0$ as in Figure 17. We denote by $Z_1, \ldots, Z_8$ its type 1 vertices; moreover, as in Section 3, we assume that $r^{x_0}_\omega$ maps $Q_{x_0}(\omega)$ onto the sector Q_k based at the chamber C_k of coordinates $(k, 0, 0)$ (with respect to $\overline{Q}$). Then we construct the 8×8-matrix $\mathbb{P} = (P_{j,k})$, where $P_{j,k} = \tilde{P}(Z_j, \omega_k)$, with the aid of the following proposition.

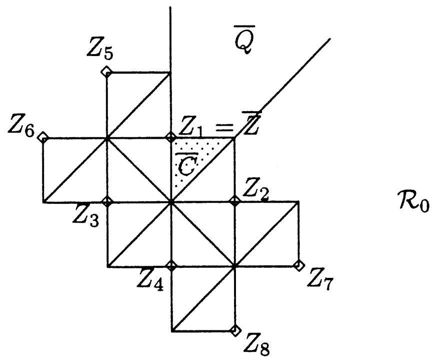


Figure 17

PROPOSITION 5.5.1. – *For every $j = 1, \ldots, 8$, let π_j be the type of a minimal gallery connecting $\overline{C}$ to Z_j; then*

$$\widetilde{P}(Z_j, \omega_k) = \frac{1}{|r_{c_0}^{-1}(Z_j)|}\mathbb{V}_1 \mathbb{N}_{\pi_j} e_k, \qquad \text{for } k \neq 4, 5\,,$$

$$\widetilde{P}(Z_j, \omega_4) = \frac{1}{|r_{c_0}^{-1}(Z_j)|}\mathbb{V}_1 \mathbb{N}_{\pi_j} e_5,$$

$$\widetilde{P}(Z_j, \omega_5) = \frac{1}{|r_{c_0}^{-1}(Z_j)|}\mathbb{V}_1 \mathbb{N}_{\pi_j} e_4.$$

The following proposition exhibits all singular pairs (α, β).

PROPOSITION 5.5.2. – *The pair (α, β) is singular in the following cases*:

$$\alpha = p\,, \quad \alpha = -pq\,, \quad \beta = 1\,, \quad \beta = -q\,, \quad \beta = \alpha\,, \quad \alpha\beta = q\,.$$

PROOF. – We compute the determinant of the 8×8 matrix $\mathbb{P}$ by using the mathematical software «Mathematica 2» as in previous cases. We get, for every (α, β),

$$\det \mathbb{P} = K\alpha^4 \beta^{-12}(\alpha - p)^4(\alpha + pq)^4(\beta - 1)^4(\beta + q)^4(\alpha\beta - q)^4(\alpha - \beta)^4,$$

where K is a constant depending on p and q. ■

As for one type of special vertices, also for non-special vertices there exist ultrasingular pairs; the following proposition exhibits them.

PROPOSITION 5.5.3. – *There exist ultrasingular pairs if and only if $q = p$, or p^2. Moreover the ultrasingular pairs (α, β) are the following*:

$$
(7) \quad \begin{aligned}
&(\alpha,\beta)=(-p^2,-1)\sim(-p,-p)\sim(-p,-1)\sim(-p^2,-p), \quad \text{if } q=p,\\
&(\alpha,\beta)=(-p^3,-p)\sim(-p^2,-1)\sim(-p,-p)\sim(-p^2,-p^2), \quad \text{if } q=p^2.
\end{aligned}
$$

PROOF. – Setting $\alpha = p\sqrt{q}\,\xi$, $\beta = \sqrt{q}\,\eta$, the pair (ξ, η) is singular if and only if

$$\xi = 1/\sqrt{q}\,, \quad \xi = -\sqrt{q}\,,$$

$$\eta = 1/\sqrt{q}\,, \quad \eta = -\sqrt{q}\,,$$

$$\eta = p\xi\,, \quad \xi\eta = 1/p\,.$$

Replacing, if necessary, (ξ, η) by an equivalent pair, we may assume $(\xi, \eta) \in \mathcal{F}$. In this set the only singular pairs are those satisfying

$$\xi = -\sqrt{q}\,, \qquad 1 \leqslant |\eta| \leqslant \sqrt{q}\,,$$

$$\eta = -\sqrt{q}, \qquad |\xi| \geqslant \sqrt{q}\,.$$

If $q \neq p, p^2$, for each of these pairs there exists an equivalent non-singular pair.

If $q = p$ or $q = p^2$, the ultrasingular pairs lying in $\mathcal{F}$ are

$$(\xi, \eta) = (-\sqrt{p}, -\sqrt{p}), \qquad \text{if } q = p,$$

$$(\xi, \eta) = (-p, -1), \qquad \text{if } q = p^2.$$

Keeping in mind the relation between (ξ, η) and (α, β), (7) is proved. ■

COROLLARY 5.5.4. – *There exist singular eigenvalues (γ_1, γ_2) if and only if $q = p$ or p^2. Moreover in both cases $(\gamma_1, \gamma_2) = (-1/q, 1/q^2)$ is the only pair of singular eigenvalues.*

Corollary 5.5.4 implies that, if we consider joint eigenvalues $(\gamma_1, \gamma_2) \neq (-1/q, 1/q^2)$, then, denoting by (α, β) a non singular pair such that $\gamma_i = \gamma_i(\alpha, \beta)$, for every $f \in \mathcal{S}(\gamma_1, \gamma_2)$ there exists a unique $\nu \in H'(\Omega)$ such that $f = \mathcal{P}_{\alpha, \beta}\nu$. On the contrary, if $(\gamma_1, \gamma_2) = (-1/q, 1/q^2)$, then Helgason's conjecture fails.

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