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A New Metrization Theorem.

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Sunto. – *Presentiamo un nuovo teorema di metrizzazione, utilizzando una nuova struttura introdotta dagli autori in* [2] *detta struttura frattale. Come corollario otteniamo i teoremi di metrizzazione di Nagata-Smirnov e di Uryshon.*

Summary. – *We give a new metrization theorem on terms of a new structure introduced by the authors in* [2] *and called fractal structure. As a Corollary we obtain Nagata-Smirnov's and Uryshon's metrization Theorems.*

1. – Introduction.

Metrization is, from the beginnings of General Topology, one of the most important fields in it, and still is. There are many metrization theorems in the literature. Although the thesis is always the same, the hypotheses to ensure metrizability are very different from one metrization theorem to another. Moreover, not only the proofs are very different, but there is no easy way to deduce one metrization theorem from another one, too.

On the other hand, looking for a generalization of symbolic self-similar sets for compact metric spaces, we developed in [2] the concept of GF-space (or generalized fractal space) and found that it is a common framework for the study of self-similar sets (the most important class of fractals, and the importance of fractals nowadays needs no emphasis) and non-archimedeanly quasimetrizable spaces. In that paper we introduced GF-spaces and we used them to characterize non-archimedeanly quasimetrizable spaces in several ways (including some relations with inverse limits of partially ordered sets). And non-archimedeanly quasimetrizable spaces are the starting point in our study of metrizability.

In [3] we found a new metrization theorem and we used it to relate many metrization theorems among them. In the present paper we improve that

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metrization theorem to include more different aspects of the theory of GF-spaces, for example, metrizabilty in terms of inverse limits of posets. As a cosequence we can obtain Urysohn's and Nagata-Smirnov's metrization theorems.

The paper is organized as follows. In section 2 we introduce all the relevant information about GF-spaces (definitions and some useful results), quasiuniformities, quasimetrics, posets and certain kind of coverings. Section 3 is devoted to the metrization theorem, and includes some technical results about how to obtain from a covering another one that has the same properties and is also a tiling. Nagata-Smirnov's metrization theorem is one of the corollaries at the end of the section. Finally, section 4 gives a characterization of second countable spaces in terms of the fractal structure and allows to obtain Urysohn's metrization theorem.

2. – GF-spaces.

In order to obtain our metrization theorem, we need to develop the theory of GF-spaces, started in [2] by the authors; our metrization theorem characterizes metrizability in terms of certain conditions over the so called «fractal structure» that a topological space may have.

Now, we recall some definitions and introduce some notations that will be useful in this paper.

Let $\boldsymbol{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a countable family of coverings. Recall that $St(x, \Gamma_n) = \bigcup_{x \in A_n,\, A_n \in \Gamma_n} A_n$; we also define $U^{\boldsymbol{\Gamma}}_{xn} = \mathrm{St}(x, \Gamma_n) \setminus \bigcup_{x \notin A_n,\, A_n \in \Gamma_n} A_n$ which will be noted also by U_{xn} if there is no doubt about the family. We also denote by $St(x, \boldsymbol{\Gamma}) = \{St(x, \Gamma_n) : n \in \mathbb{N}\}$ and $\mathcal{U}_x = \{U_{xn} : n \in \mathbb{N}\}$.

A (base $\mathcal{B}$ of a) quasiuniformity $\mathcal{U}$ on a set X is a (base $\mathcal{B}$ of a) filter $\mathcal{U}$ of binary relations (called entourages) on X such that (a) each element of $\mathcal{U}$ contains the diagonal Δ_X of $X \times X$ and (b) for any $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ satisfying $V \circ V \subseteq U$. A base $\mathcal{B}$ of a quasiuniformity is called transitive if $B \circ B = B$ for all $B \in \mathcal{B}$. The theory of quasiuniform spaces is covered in [6].

If $\mathcal{U}$ is a quasiuniformity on X, then so is $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$, where $U^{-1} = \{(y, x) : (x, y) \in U\}$. The generated uniformity on X is denoted by $\mathcal{U}^*$. A base is given by the entourages $U^* = U \cap U^{-1}$. The topology $\tau(\mathcal{U})$ induced by the quasiuniformity $\mathcal{U}$ is that in which the sets $U(x) = \{y \in X : (x, y) \in U\}$, where $U \in \mathcal{U}$, form a neighbourhood base for each $x \in X$. There is also the topology $\tau(\mathcal{U}^{-1})$ induced by the inverse quasiuniformity. In this paper, we consider only spaces where $\tau(\mathcal{U})$ is T_0.

A quasipseudometric on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x, x) = 0$, and (ii) $d(x, y) \leq d(x, z) + d(z, y)$. If in addition d satisfies the condition (iii) $d(x, y) = 0$ iff $x = y$, then d

is called a quasi-metric. A non-archimedean quasipseudometric is a quasipseudometric that verifies $d(x, y) \leqslant \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$.

Each quasipseudometric d on X generates a quasiuniformity $\mathcal{U}_d$ on X which has as a base the family of sets of the form $\{(x, y) \in X \times X : d(x, y) < 2^{-n}\}$, $n \in \mathbb{N}$. Then the topology $\tau(\mathcal{U}_d)$ induced by $\mathcal{U}_d$, will be denoted simply by $\tau(d)$.

A space (X, τ) is said to be (non-archimedeanly) quasipseudometrizable if there is a (non-archimedean) quasipseudometric d on X such that $\tau = \tau(d)$.

A relation $\leqslant$ on a set G is called a partial order on G if it is a transitive antisymmetric reflexive relation on G. If $\leqslant$ is a partial order on a set G, then $(G, \leqslant)$ is called a partially ordered set.

$(G, \leqslant, \tau)$ will be called a poset (partially ordered set) or T_0-Alexandroff space if $(G, \leqslant)$ is a partially ordered set and τ is that in which the sets $[g, \rightarrow[= \{h \in G: g \leqslant h\}$ form a neighborhood base for each $g \in G$ (we say that the topology τ is induced by $\leqslant$). Note that then $\overline{\{g\}} =]\leftarrow, g]$ for all $g \in G$.

Let us remark that a map $f : G \rightarrow H$ between two posets G and H is continuous if and only if it is order preserving, i.e. $g_1 \leqslant g_2$ implies $f(g_1) \leqslant f(g_2)$.

Let Γ be a covering of X. Γ is said to be locally finite if for all $x \in X$ there exists a neighborhood of x which meets only a finite number of element of Γ. Γ is said to be a tiling, if all elements of Γ are regularly closed an they have disjoint interiors (see [1]). We say that Γ is quasi-disjoint if $A^\circ \cap B = \emptyset$ or $A \cap B^\circ = \emptyset$ holds for all $A \neq B \in \Gamma$. Note that if Γ is a tiling, then it is quasi-disjoint.

DEFINITION 2.1. – *Let X be a topological space. A pre-fractal structure over X is a family of coverings $\boldsymbol{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ such that $\mathcal{U}_x$ is an open neighborhood base of x for all $x \in X$.*

Furthermore, if Γ_n is a closed covering and for all n, Γ_{n+1} is a refinement of Γ_n, such that for all $x \in A_n$, with $A_n \in \Gamma_n$, there is $A_{n+1} \in \Gamma_{n+1}$: $x \in A_{n+1} \subseteq A_n$, we will say that $\boldsymbol{\Gamma}$ is a fractal structure over X.

If $\boldsymbol{\Gamma}$ is a (pre-) fractal structure over X, we will say that $(X, \boldsymbol{\Gamma})$ is a generalized (pre-) fractal space or simply a (pre-) GF-space. If there is no doubt about $\boldsymbol{\Gamma}$, then we will say that X is a (pre-) GF-space.

If $\boldsymbol{\Gamma}$ is a fractal structure over X, and $\mathrm{St}(x, \boldsymbol{\Gamma})$ is a neighbourhood base of x for all $x \in X$, we will call $(X, \boldsymbol{\Gamma})$ a starbase GF-space.

If Γ_n has the property P for all $n \in \mathbb{N}$, and $\boldsymbol{\Gamma}$ is a fractal structure over X, we will say that $\boldsymbol{\Gamma}$ is a fractal structure over X with the property P, and that X is a GF-space with the property P. For example, if Γ_n is locally finite for all natural number n, and $\boldsymbol{\Gamma}$ is a fractal structure over X, we will say that $\boldsymbol{\Gamma}$ is a locally finite fractal structure over X, and that $(X, \boldsymbol{\Gamma})$ is a locally finite GF-space.

Call $U_n = \{(x, y) \in X \times X : y \in U_{xn}\}$, $U_{xn}^{-1} = U_n^{-1}(x)$ and $\mathcal{U}_x^{-1} = \{U_{xn}^{-1} : n \in \mathbb{N}\}$.

The following proposition is proved in [2], Prop. 3.2, though we state here with the proof.

PROPOSITION 2.2. – *Let X be a pre-GF-space. Then $U_{xn}^{-1} = \bigcap_{x \in A_n} A_n$.*

PROOF. – $y \in U_{xn}^{-1}$ if and only if $x \in U_{yn}$. Now, if $x \in A_n$ then $y \in A_n$ (since $x \in U_{yn} = X \setminus \bigcup_{y \notin A_n} A_n$). ■

In [2], the authors introduce the following construction. Let $\boldsymbol{\Gamma}$ be a fractal structure, and let define $G_n = \{U_{xn}^* : x \in X\}$, and define in G_n the following order relation $U_n^*(x) \leqslant_n U_n^*(y)$ if $y \in U_n(x)$. It holds that G_n is a poset with this order relation and its associated topology.

Let ϱ_n be the quotient map from X onto G_n which carries x in X to $U_n^*(x)$ in G_n. It holds that ϱ_n is continuous.

We also consider the map $\phi_n : G_n \to G_{n-1}$ defined by $\phi_n(\varrho_n(x)) = \varrho_{n-1}(x)$. It holds that ϕ_n is continuous.

Let ϱ be the map from X to $\varprojlim G_n$ which carries x in X to $(\varrho_n(x))_n$ in $\varprojlim G_n$. Note that ϱ is well defined and continuous (by definition of ϕ_n and the continuity of ϱ_n and ϕ_n for all n). It holds that ϱ is an embedding of X into $\varprojlim G_n$.

REMARK 2.3. – *Note that if $\boldsymbol{\Gamma}$ is a tiling pre-GF-space, then if $x \in A_n^\circ$, we have that $U_{xn}^{-1} = A_n$, since $A_n^\circ \cap B_n = \emptyset$.*

PROPOSITION 2.4. – *Let $\boldsymbol{\Gamma}$ be a pre-fractal structure over X. Then Γ_n is closure-preserving for each $n \in \mathbb{N}$. Moreover, A_n is closed for all $A_n \in \Gamma_n$ and for all $n \in \mathbb{N}$.*

PROOF. – Let $x \in \overline{\bigcup_{\lambda \in \Lambda} A_n^\lambda}$. Then $U_{xn} \cap \bigcup_{\lambda \in \Lambda} A_n^\lambda$ is non empty, so there exists $\lambda \in \Lambda$ and $y \in A_n^\lambda$ such that $y \in U_{xn} \cap A_n^\lambda$, but then $x \in U_{yn}^{-1} = \bigcap_{y \in A_n} A_n$, and hence $x \in A_n^\lambda$. Therefore $\bigcup_{\lambda \in \Lambda} A_n^\lambda$ is closed. ■

Let $\boldsymbol{\Gamma}$ be a pre-fractal structure, we define $\mathrm{fs}(\boldsymbol{\Gamma}) = \{\mathrm{fs}(\Gamma_n) : n \in \mathbb{N}\}$, where $\mathrm{fs}(\Gamma_n) = \left\{\bigcap_{i \leqslant n} A_i : A_i \in \Gamma_i\right\}$ and we call it the fractalization of the pre-fractal structure $\boldsymbol{\Gamma}$.

The next Proposition is proved in [3].

PROPOSITION 2.5. – *Let $\boldsymbol{\Gamma}$ be a pre-fractal structure over a topological space X. Then* $\mathrm{fs}(\boldsymbol{\Gamma})$ *is a fractal structure over X. If $\boldsymbol{\Gamma}$ is starbase (resp. locally finite, finite) then so is* $\mathrm{fs}(\boldsymbol{\Gamma})$.

3. – Metrization.

The next two results relate the local symmetry of quasi-uniformities and quasimetrics with the starbaseness of fractal structures.

PROPOSITION 3.1. – *Let X be a starbase pre-GF-space. Then $\{U_n\}$ is base of a locally symmetric transitive quasi-uniformity for X.*

PROOF. – It is proved in [2], Prop. 3.5 that it is a transitive base of quasi-uniformity over X. Now, we will see that it is a locally symmetric one.

Given $x \in X$ and n a natural number, let m be such that $\mathrm{St}(x, \Gamma_m) \subseteq U_{xn}$. Now, we are going to see that $U_m^{-1}(U_m(x)) \subseteq U_n(x)$.

To see this, let $y \in U_m^{-1}(U_m(x))$, then there exists $z \in X$ such that $x, y \in U_{zm}^{-1}$. Let $A_m \in \Gamma_m$ be such that $z \in A_m$; then, since $x, y \in U_{zm}^{-1} = \bigcap_{z \in B_m} B_m$, we have that $x, y \in A_m$, what means $y \in St(x, \Gamma_m) \subseteq U_{xn}$, and hence $(x, y) \in U_n$. Therefore $\{U_n\}$ is locally symmetric. ■

COROLLARY 3.2. – *Let X be a starbase pre-GF-space. Then X admits a locally symmetric quasimetric.*

PROOF. – It follows from [6], Lemma 1.5 and Proposition 3.1. ■

Now comes the first part of our metrization theorem.

THEOREM 3.3. – *Let X be a starbase pre-GF-space. Then X is metrizable.*

PROOF. – X admits a locally symmetric quasi-metric by Corollary 3.2, and then we apply [6], Th. 2.32.

REMARK 3.4. – *This Theorem can be proved using many different metrization Theorems as is shown in* [3].

Now we are looking for a converse of Theorem 3.3. The first step is the following, which is proved in [3].

THEOREM 3.5. – *Let X be a metrizable space. Then there exists a locally finite starbase fractal structure over X.*

The proof of the following lemma is straightforward, so we omit it.

LEMMA 3.6. – *Let X be a topological space, and let $\{F_i : i \in I\}$ be a finite family of closed set in X. Suppose that $\left(\bigcup_{i \in I} F_i\right)^\circ$ is non empty. Then there exists $i_0 \in I$, such that F_i° is nonempty.*

Now, we construct the regularization of a fractal structure and study how properties are induced from the fractal structure to its regularization.

THEOREM 3.7. – *Let $\boldsymbol{\Gamma}$ be a locally finite fractal structure over a regular space X and let $\Gamma'_n = \operatorname{reg}(\Gamma_n) = \{A'_n = \operatorname{Cl}(A_n^\circ) : A_n \in \Gamma_n\}$. Then $\boldsymbol{\Gamma}' = \{\Gamma'_n : n \in \mathbb{N}\}$ (called the regularization $\operatorname{reg}(\boldsymbol{\Gamma})$ of $\boldsymbol{\Gamma}$) is a locally finite fractal structure over X. Moreover, if $\boldsymbol{\Gamma}$ is starbase, then $\boldsymbol{\Gamma}'$ also is.*

PROOF. – (1) Γ'_n is a closed covering.

It is obvious that $\operatorname{Cl} A_n^\circ$ is closed. Let see that Γ'_n is a covering.

Suppose, there exists $n \in \mathbb{N}$ and $x \in X$ such that

$$x \notin \bigcup_{x \in A_n} \operatorname{Cl} A_n^\circ = \operatorname{Cl}\Big(\bigcup_{x \in A_n} A_n^\circ\Big)$$

(since the union is finite). Then there exists a natural number m such that $U_{xm} \cap \Big(\bigcup_{x \in A_n} A_n^\circ\Big) = \emptyset$. That is, $U_{xm} \cap A_n^\circ = \emptyset$ for all $A_n \in \Gamma_n$ with $x \in A_n$.

Let $A_m \in \Gamma_m$ with $x \in A_m$, then there exists $A_n \in \Gamma_n$ with $x \in A_m \subseteq A_n$. But then $A_m^\circ \cap U_{xm} \subseteq U_{xm} \cap A_n^\circ = \emptyset$. Therefore $A_m^\circ \cap U_{xm} = \emptyset$ for all $A_m \in \Gamma_m$ with $x \in A_m$.

On the other hand, $x \in U_{xm} = U_{xm} \cap (St(x, \Gamma_m))^\circ = U_{xm} \cap \Big(\bigcup_{x \in A_m} A_m\Big)^\circ = \Big(\bigcup_{x \in A_m} (U_{xm} \cap A_m)\Big)^\circ$ and since Γ_m is locally finite, the union is finite and $U_{xm} \cap A_m$ is closed in U_{xm} and by the previous lemma, there exists $A_m \in \Gamma_m$ with $x \in A_m$, such that the interior of $A_m \cap U_{xm}$ in U_{xm} is nonempty, but then $A_m^\circ \cap U_{xm} \neq \emptyset$ and the contradiction shows that Γ'_n is a covering.

(2) If we prove that $U'_{xn} \subseteq \overline{U_{xn}}$ then, since X is regular and U'_{xn} is open (since Γ'_n is locally finite), we have that $\{U'_{xn} : n \in \mathbb{N}\}$ is a neighborhood base of x for all $x \in X$.

So, let $y \in U'_{xn}$ and suppose that $y \notin \overline{U_{xn}}$. Then there exists a natural number m such that $U_{ym} \cap U_{xn} = \emptyset$ and since $y \in U'_{xn}$ and U'_{xn} is open (since Γ'_n is locally finite), we can get m such that $U_{ym} \subseteq U'_{xn}$. Hence $U_{ym} \subseteq (U'_{xn} \setminus U_{xn}) \cap U_{yn}$.

Let us see that $(U'_{xn} \setminus U_{xn}) \cap U_{yn} \subseteq \bigcup_{x \notin A_n,\, y \in A_n} BdA_n$.

Let $z \in (U'_{xn} \setminus U_{xn}) \cap U_{yn}$. Since $z \notin U_{xn}$, then there exists $A_n \in \Gamma_n$ such that $z \in A_n$ but $x \notin A_n$, and since $z \in U_{yn}$ then $y \in U_{zn}^{-1} = \bigcap_{z \in B_n} B_n$, and hence $y \in A_n$. Since $z \in U'_{xn}$ and $x \notin \operatorname{Cl} A_n^\circ = A'_n$ then $z \notin A'_n$. Therefore $U_{ym} \subseteq (U'_{xn} \setminus U_{xn}) \cap U_{yn} \subseteq \bigcup_{x \notin A_n,\, y \in A_n} (A_n \setminus A'_n) \subseteq \bigcup_{x \notin A_n,\, y \in A_n} BdA_n$, and since Γ_n is locally finite and the previous lemma $\left(\text{note that since } U_{ym} \subseteq \bigcup_{x \notin A_n,\, y \in A_n} BdA_n \text{ then } \left(\bigcup_{x \notin A_n,\, y \in A_n} BdA_n\right)^\circ \neq \emptyset\right)$, there exists $A_n \in \Gamma_n$ with $x \notin A_n$ and $y \in A_n$ such that $(BdA_n)^\circ \neq \emptyset$, but this cannot be true, since the interior of the boundary of a closed set is empty.

(3) $\boldsymbol{\Gamma}'$ is a fractal structure.

Let $x \in \mathrm{Cl}\, A_n^\circ = A_n'$. Suppose that $x \notin \mathrm{Cl}\, A_{n+1}^\circ = A_{n+1}'$ for all $A_{n+1} \in \Gamma_{n+1}$ with $x \in A_{n+1} \subseteq A_n$. Then there exists a natural number m (greater than n) such that $U_{xm} \cap A_{n+1}^\circ = \emptyset$ for all $A_{n+1} \in \Gamma_{n+1}$ with $x \in A_{n+1} \subseteq A_n$ (note that this is possible since Γ_{n+1} is locally finite). Then, for all $y \in U_{xm} \cap A_n^\circ$ (note that this intersection is nonempty since $x \in \mathrm{Cl}\, A_n^\circ$) there exists $A_{n+1}(y) \in \Gamma_{n+1}$ with $x \in A_{n+1}(y)$ (since $x \in U_{y(n+1)}^{-1}$) such that $y \in A_{n+1}(y) \subseteq A_n$.

Now, $U_{xm} \cap A_n^\circ = \bigcup_{y \in U_{xm} \cap A_n^\circ} (A_{n+1}(y) \cap (U_{xm} \cap A_n^\circ))$ and the union is finite (since $x \in A_{n+1}(y)$ for all $y \in U_{xm} \cap A_n^\circ$ and Γ_{n+1} is locally finite). Then $U_{xm} \cap A_n^\circ$ is the finite union of closed (in $U_{xm} \cap A_n^\circ$) sets, and then by the previous lemma, there exists $y_0 \in U_{xm} \cap A_n^\circ$, such that the interior of $(A_{n+1}(y_0) \cap (U_{xm} \cap A_n^\circ))$ in $U_{xm} \cap A_n^\circ$ is nonempty, but then, since $U_{xm} \cap A_n^\circ$ is open in X, $(A_{n+1}(y_0))^\circ \cap U_{xm} \cap A_n^\circ$ is non empty, which is a contradiction with the fact that $U_{xm} \cap A_{n+1}^\circ = \emptyset$ for all $A_{n+1} \in \Gamma_{n+1}$ with $x \in A_{n+1} \subseteq A_n$.

(4) If $\boldsymbol{\Gamma}$ is starbase, then $\boldsymbol{\Gamma}'$ is.

This is clear, since $\mathrm{St}(x, \Gamma_n') \subseteq \mathrm{St}(x, \Gamma_n)$ (because Γ_n' is a refinement of Γ_n). ■

The following definition is in a setting more general than the one we really need, but we include it here, since it requires the same effort.

DEFINITION 3.8. – *Let $\Gamma = \{A^\lambda : \lambda \in \Lambda\}$ be a covering of a topological space X. For each $w \in P(\Lambda)$ (the set of nonempty subsets of Λ) we define $A^w = \mathrm{Cl}\left(\bigcap_{\lambda \in w} A^\lambda\right) \setminus \left(\bigcup_{\lambda \notin w} A^\lambda\right)$. We note by* $\mathrm{qdi}(\Gamma) = \{A^w : w \in P(\Lambda)\} \setminus \{\emptyset\}$, *and called it the quasi-disjointification of Γ.*

Let $U_x = U_x^\Gamma = X \setminus \bigcup_{x \notin A^\lambda} A^\lambda$, $U_x^{-1} = \{y \in X : x \in U_y\}$ and $U_x^* = U_x \cap U_x^{-1}$. Then note that $U_x^{-1} = \bigcap_{x \in A^\lambda} A^\lambda$ (see Lemma 2.2).

The notation $\mathrm{qdi}(\Gamma)$ is due to the fact that $\mathrm{qdi}(\Gamma)$ is quasi-disjoint (see 4 below).

PROPOSITION 3.9. – *Let $\Gamma = \{A^\lambda : \lambda \in \Lambda\}$ be a closed covering of a topological space X. Then:*

(1) *For all* $w \in P(\Lambda)$, *there exists* $x \in X$ *such that* $A^w = \mathrm{Cl}(U_x^*)$ *or* $A^w = \emptyset$, *and for all* $x \in X$, *there exists* $w \in P(\Lambda)$ *such that* $A^w = \mathrm{Cl}(U_x^*)$.

(2) $A^\lambda = \bigcup \{A^w : \lambda \in w;\ w \in P(\Lambda)\}$.

(3) $U_x^{\mathrm{qdi}(\Gamma)} \subseteq U_x$.

(4) $\mathrm{qdi}(\Gamma)$ *is quasi-disjoint.*

(5) $\mathrm{qdi}(\Gamma)$ *is a closed covering and if* Γ *is locally finite, so is* $\mathrm{qdi}(\Gamma)$.

(6) *If* Γ_2 *is a refinement of* Γ_1 *such that* $A_1^\mu = \bigcup_{A_2^\lambda \subseteq A_1^\mu} A_2^\lambda$ *for each* $A_1^\mu \in \Gamma_1$, *then* $\mathrm{qdi}(\Gamma_2)$ *is a refinement of* $\mathrm{qdi}(\Gamma_1)$ *such that* $A_1^v = \bigcup_{A_2^w \subseteq A_1^v} A_2^w$ *for each* $A_1^v \in \mathrm{qdi}(\Gamma_1)$.

PROOF. – (1) Let $w \in P(\Lambda)$, and suppose that there exists $x \in \bigcap_{\lambda \in w} A^\lambda \setminus \left(\bigcup_{\lambda \notin w} A^\lambda \right)$. Then it is clear that $x \in A^\lambda$ if and only if $\lambda \in w$. Hence

$$A^w = \mathrm{Cl}\left(\bigcap_{\lambda \in w} A^\lambda \setminus \left(\bigcup_{\lambda \notin w} A^\lambda \right) \right) = \mathrm{Cl}\left(\bigcap_{x \in A^\lambda} A^\lambda \setminus \left(\bigcup_{x \notin A^\lambda} A^\lambda \right) \right) = \mathrm{Cl}(U_x^*).$$

Conversely, let $x \in X$, and let $w = \{\lambda \in \Lambda : x \in A^\lambda\}$. Then it is clear that $x \in A^\lambda$ if and only if $\lambda \in w$, and hence $A^w = \mathrm{Cl}(U_x^*)$, analogously to the preceding paragraph.

(2) It is clear that $\bigcup \{A^w : \lambda \in w;\ w \in P(\Lambda)\} \subseteq A^\lambda$, since $A^w \subseteq A^\mu$ for all $\mu \in w$.

Let $x \in A^\lambda$. Then $x \in A^w = \mathrm{Cl}(U_x^*)$ (by the first item and for some $w \in P(\Lambda)$) and since $x \in A^\lambda$ then $\lambda \in w$, and this proves the equality.

(3) Let see that $(U_x^{\mathrm{qdi}(\Gamma)})^{-1} = \bigcap_{x \in A^w} A^w \subseteq U_x^{-1} = \bigcap_{x \in A^\lambda} A^\lambda$.

Let $y \in \bigcap_{x \in A^w} A^w$ and let $\lambda \in \Lambda$ such that $x \in A^\lambda$. Let see that $y \in A^\lambda$. By the second item, there exists $w \in P(\Lambda)$, with $\lambda \in w$ such that $x \in A^w$, but then $y \in A^w \subseteq A^\lambda$. Therefore $y \in \bigcap_{x \in A^\lambda} A^\lambda = U_x^{-1}$.

Since $(U_x^{\mathrm{qdi}(\Gamma)})^{-1} \subseteq U_x^{-1}$ for all $x \in X$, then $U_x^{\mathrm{qdi}(\Gamma)} \subseteq U_x$ for all $x \in X$.

(4) $A^w = \mathrm{Cl}\left(\bigcap_{\lambda \in w} A^\lambda \setminus \left(\bigcup_{\lambda \notin w} A^\lambda \right) \right) \subseteq X \setminus \left(\bigcup_{\lambda \notin w} A^\lambda \right)^\circ$, and hence $A^w \cap \left(\bigcup_{\lambda \notin w} A^\lambda \right)^\circ = \emptyset$.

On the other hand, if there exists $\lambda \in v \setminus w$, then $A^v \subseteq A^\lambda$, and $(A^v)^\circ \subseteq (A^\lambda)^\circ \subseteq \left(\bigcup_{\lambda \notin w} A^\lambda \right)^\circ$. Therefore $A^w \cap (A^v)^\circ = \emptyset$, and hence Γ is quasi-disjoint.

(5) $\mathrm{qdi}(\Gamma)$ is a covering since Γ is and the second item. Obviously each A^w is closed. Suppose that Γ is locally finite and let see that $\mathrm{qdi}(\Gamma)$ also is.

Let $x \in X$, then there exists U an open neighborhood of x and there exists a finite set $\{\lambda_1, \ldots, \lambda_n\}$ such that $U \cap A^{\lambda_i} \neq \emptyset$ for each $i \in \{1, \ldots, n\}$, but $U \cap A^\mu = \emptyset$ for all $\mu \notin \{\lambda_1, \ldots, \lambda_n\}$. Since $A^w \subseteq A^\lambda$ for all $\lambda \in w$, we have that $w \in P(\{\lambda_1, \ldots, \lambda_n\})$ for all w such that $U \cap A^w \neq \emptyset$. Therefore there are only a finite number of w for which $U \cap A^w \neq \emptyset$ and then $\mathrm{qdi}(\Gamma)$ is locally finite.

(6) It is easy to see that $U^*_{x2} \subseteq U^*_{x1}$ for all $x \in X$. Therefore $\mathrm{qdi}(\Gamma_2)$ is a refinement of $\mathrm{qdi}(\Gamma_1)$.

It is also clear that $U^*_{x1} = \bigcup_{y \in U^*_{x1}} U^*_{y2} = \bigcup_{U^*_{y2} \subseteq U^*_{x1}} U^*_{y2}$ (since $y \in U^*_{x1}$ if and only if $U^*_{y2} \subseteq U^*_{x1}$) for all $x \in X$, and this proves the item. ■

REMARK 3.10. – *Note that the requirement of the covering to be by closed sets, is only needed for items* 2, 3 *and* 4.

COROLLARY 3.11. – *Let* $\boldsymbol{\Gamma}$ *be a locally finite fractal structure over* X. *Then* $\mathrm{qdi}(\boldsymbol{\Gamma}) = \{\mathrm{qdi}(\Gamma_n) : n \in \mathbb{N}\}$ *is a locally finite quasi-disjoint fractal structure over* X.

PROOF. – By the fifth item of Proposition 3.9, $\mathrm{qdi}(\Gamma_n)$ is a locally finite closed covering for all n and by the third item $U_{xn}^{\mathrm{qdi}(\Gamma_n)} \subseteq U_{xn}$. Therefore $\{U_{xn}^{\mathrm{qdi}(\Gamma_n)} : n \in \mathbb{N}\}$ is an open neighborhood base of x for all $x \in X$ (note that $U_{xn}^{\mathrm{qdi}(\Gamma_n)}$ is open since $\mathrm{qdi}(\Gamma_n)$ is locally finite).

By the sixth item of Proposition 3.9 and from we have already proved, we have that $\mathrm{qdi}(\boldsymbol{\Gamma})$ is a locally finite fractal structure over X and by the fourth item we have that $\mathrm{qdi}(\boldsymbol{\Gamma})$ is quasi-disjoint. ■

Combining the regularization and the quasi-disjointification of a fractal structure we obtain another one with the same properties that is also a tiling.

DEFINITION 3.12. – *Let* $\boldsymbol{\Gamma}$ *be a locally finite fractal structure over a regular space* X. *We define* $\mathrm{til}(\boldsymbol{\Gamma}) = \{\mathrm{til}(\Gamma_n) : n \in \mathbb{N}\}$, *where* $\mathrm{til}(\Gamma_n) = \mathrm{reg}(\mathrm{qdi}(\Gamma_n))$ *for all* $n \in \mathbb{N}$.

THEOREM 3.13. – *Let* $\boldsymbol{\Gamma}$ *be a locally finite fractal structure over a regular space* X. *Then* $\mathrm{til}(\boldsymbol{\Gamma})$ *is a locally finite tiling fractal structure over* X. *Moreover, if* $\boldsymbol{\Gamma}$ *is starbase, so is* $\mathrm{til}(\boldsymbol{\Gamma})$.

PROOF. – Let $\Gamma'_n = \mathrm{til}(\Gamma_n) = \{\mathrm{Cl}((A^w)^\circ) : A^w \in \mathrm{qdi}(\Gamma_n)\}$. Let see that $\boldsymbol{\Gamma}'$ is a locally finite tiling fractal structure over X.

Since $\mathrm{qdi}(\boldsymbol{\Gamma})$ is a locally finite quasi-disjoint fractal structure over X, by Corollary 3.11, then it is clear from Theorem 3.7 that $\boldsymbol{\Gamma}'$ is a locally finite tiling fractal structure over X.

Suppose that $\boldsymbol{\Gamma}$ is starbase. Then $\mathrm{St}(x, \mathrm{qdi}(\Gamma_n)) \subseteq \mathrm{St}(x, \Gamma_n)$ since $\mathrm{qdi}(\Gamma_n)$ is a refinement of Γ_n by the second item of proposition 3.9. Then by Theorem 3.7 we have that $\boldsymbol{\Gamma}'$ is starbase. ■

For a locally finite tiling fractal structure, we have the following relation between the star and the neighborhood base.

PROPOSITION 3.14. – *Let* $\boldsymbol{\Gamma}$ *be a locally finite tiling fractal structure over* X. *Then* $\mathrm{St}(x, \Gamma_n) = \overline{U_{xn}}$. *Moreover, if* $\boldsymbol{\Gamma}$ *is starbase, then* $\mathrm{St}(x, \Gamma_n) = U_n^{-1}(U_n(x)) = \overline{U_{xn}}$.

PROOF. – Let $\boldsymbol{\Gamma}$ be a locally finite tiling fractal structure over X. It is obvious that $\overline{U_{xn}} \subseteq \mathrm{St}(x, \Gamma_n)$, since $\mathrm{St}(x, \Gamma_n)$ is closed because Γ_n is locally finite.

Let $y \in \mathrm{St}(x, \Gamma_n)$. Suppose $y \notin \overline{U_{xn}}$, then there exists a natural number m, such that $U_{ym} \cap U_{xn} = \emptyset$. Let A_n be such that $x, y \in A_n$, then there exists (since $\boldsymbol{\Gamma}$ is a fractal structure) $A_m \in \Gamma_n$ such that $y \in A_m \subseteq A_n$. Since $A_m^\circ \subseteq U_{ym}$ (if $z \in A_m^\circ$ then, since Γ_m is tiling, $y \in A_m = U_{zm}^{-1}$, that is, $z \in U_{ym}$), then $A_m^\circ \cap U_{xn} = \emptyset$, and hence $\emptyset \neq A_m^\circ \subseteq (St(x, \Gamma_n) \backslash U_{xn})^\circ$ (note that $A_m^\circ \neq \emptyset$, since $\boldsymbol{\Gamma}$ is a tiling).

Let see that $(\mathrm{St}(x, \Gamma_n) \backslash U_{xn})^\circ$ is empty, which is a contradiction. Let $z \in (\mathrm{St}(x, \Gamma_n) \backslash U_{xn})^\circ$. Then there exist A_n and B_n in Γ_n such that $z \in A_n \cap B_n$, $x \in A_n$ and $x \notin B_n$. Then $z \in A_n \cap B_n \subseteq BdA_n$ (since Γ_n is a tiling). Then $(\mathrm{St}(x, \Gamma_n) \backslash U_{xn})^\circ \subseteq \left(\bigcup \mathrm{Bd} A_n\right)^\circ$, where the union of the right is finite, since Γ_n is locally finite and x belongs to all of that A_n. Then by Lemma 3.6 there exists A_n such that $(\mathrm{Bd} A_n)^\circ$ is nonempty, which is a contradiction (because the interior of the boundary of a closed set is empty). Therefore, the first statements is proved.

Let us prove the second one. That is, let see that $\mathrm{St}(x, \Gamma_n) = U_n^{-1}(U_n(x))$. First, if $y \in \mathrm{St}(x, \Gamma_n)$, then there exists $A_n \in \Gamma_n$ such that $x, y \in A_n$. Let $z \in A_n^\circ$, then, since Γ_n is a tiling, $U_{zn}^{-1} = A_n$, so $x, y \in U_{zn}^{-1}$, that is, $y \in U_n^{-1} \circ U_n(x)$.

On the other hand, if $y \in U_n^{-1}(U_n(x))$, there exists $z \in X$ such that $y, x \in U_{zn}^{-1}$, but then there exists $A_n \in \Gamma_n$ such that $y, x \in A_n$, that is $y \in \mathrm{St}(x, \Gamma_n)$. ∎

Hence, in regular spaces, starbaseness is obtained as a consequence.

COROLLARY 3.15. – *Let* $\boldsymbol{\Gamma}$ *be a locally finite tiling fractal structure over a regular space. Then* $\boldsymbol{\Gamma}$ *is starbase.*

One can think that if $\boldsymbol{\Gamma}$ is a locally finite fractal structure, then the family $\{U_{xn}: x \in X\}$ should be locally finite. The following example shows that this is not so.

EXAMPLE 3.16. – *Let* $X = [0, 1[$, *and let*

$$\Gamma_n = \left\{ \left[\frac{k}{2^n}, \frac{k+1}{2^n} \right] : 0 \leqslant k \leqslant 2^n - 2 \right\} \cup \left\{ \left[1 - \frac{1}{2^k}, 1 \right[: k \geqslant n \right\}.$$

It is clear that $\boldsymbol{\Gamma}$ *is a locally finite starbase fractal structure over* X, *but* $\{U_{xn}: x \in X\}$ *is not locally finite for any* $n \in \mathbb{N}$ *at* $x_0 = 1 - \frac{1}{2^n}$, *since any neighborhood of* x_0 *meets the infinite family* $\left\{ U_{x_k n}: x_k = 1 - \frac{1}{2^k} \right\}$ *(note that* $U_{x_k n} =]x_0, x_{k+1}[$*).*

The following definition will be needed in order to state our main result.

DEFINITION 3.17. – *Let* G *be a poset. We say that* G *is locally finite if* $\mathrm{Card}(\{h \in G : g \leqslant h\})$ *is finite for all* $g \in G$.

And now, we have our main result: a metrization theorem in terms of fractal structures and inverse sequence of posets. A weaker version of this result (the equivalence between 1 and 6) is [3], Th. 3.7.

THEOREM 3.18. – *Let* X *be a topological space. The following statements are equivalent:*

(1) X *is metrizable.*

(2) *There exists a locally finite tiling starbase fractal structure over* X.

(3) X *is regular and can be embedded into the inverse limit of a sequence of locally finite posets.*

(4) X *is regular and can be embedded into the countable product of locally finite posets.*

(5) X *is regular and there exists a locally finite pre-fractal structure over* X.

(6) *There exists a starbase pre-fractal structure over* X.

PROOF. – 1) implies 2) is by Theorem 3.5 and Theorem 3.13.

2) implies 3) By Theorem 3.3, X is metrizable, so it is regular. Let see that if $\boldsymbol{\Gamma}$ is locally finite then $G_n = G(\Gamma_n)$ is locally finite for all $n \in \mathbb{N}$ (where G_n is as in the Introduction).

Let $n \in \mathbb{N}$, and let $\varrho_n(x) \in G_n$. Since Γ_n is locally finite, there exists a finite number of $A_n \in \Gamma_n$ such that $x \in A_n$. Let $y \in X$ such that $\varrho_n(x) \leqslant \varrho_n(y)$ then $x \in U_{yn}^{-1} = \bigcap_{y \in A_n} A_n$, and this means that $y \in A_n$ only for a subcollection of $\{A_n$:

$x \in A_n\}$. So the number of different $\varrho_n(y)$ with $\varrho_n(x) \leq \varrho_n(y)$ is at most the number of possible combinations of the elements of the family $\{A_n : x \in A_n\}$, which is finite. Therefore G_n is a locally finite poset and X can be embedded into $\varprojlim G_n$ (see [2]).

3) implies 4) Obvious.

4) implies 5) The fractal structure associated with the countable product of posets (see [2]) is locally finite, since the posets are.

5) implies 6) By Proposition 2.5 and Theorem 3.13, there exists a locally finite tiling fractal structure over X, and by Corollary 3.15, it is starbase.

6) implies 1) by Theorem 3.3. ■

REMARK 3.19. – *Note that in* [8], *Morita proves that a topological space is metrizable if and only if there exists a locally finite starbase pre-fractal structure over it, so our theorem is a direct generalization of this result (of course, Morita does not use our notation).*

As a corollary of Morita's metrization theorem we have the following.

COROLLARY 3.20. – (*Hanai-Morita-Stone Theorem*) *For every closed mapping $f : X \to Y$ of a metrizable space X onto a space Y the following conditions are equivalent:*

(1) *The space Y is metrizable.*

(2) *The space Y is first countable.*

(3) *For every $y \in Y$, the set $Bd(f^{-1}(y))$ is compact.*

PROOF. – See [5], Ex. 5.4.D.b (see also [7], and [10]). ■

Our metrizaton theorem allow us to prove the Nagata-Smirnov metrization one in a way different from the standard one as well as the following Theorem, due to Burke, Engelking and Lutzer in [4] (see [3]).

COROLLARY 3.21. – *Let X be a T_3 space with a σ-hereditarily closure preserving base. Then X is metrizable.*

We can also obtain from Theorem 3.18 one that is equivalent to it (see [3]).

COROLLARY 3.22. – *Let X be a topological space. Then X is metrizable if and only if there exists a locally symmetric non archimedeanly quasimetric d on X.*

4. – Urysohn's metrization theorem and *GF*-spaces.

Urysohn's metrization theorem can also be obtained using our techniques as follows. First, we deal with second countable spaces.

THEOREM 4.1. – *Let X be a finite GF-space. Then X is second countable.*

PROOF. – If X is a finite GF-space, then the set $\{U_{xn} : x \in X\}$ is clearly finite for all $n \in \mathbb{N}$. Therefore $\{U_{xn} : x \in X;\ n \in \mathbb{N}\}$ is a countable base for X. ■

Now, we find a converse of the above result.

THEOREM 4.2. – *Let X be a totally bounded non-archimedeanly quasimetrizable space. Then there exists a finite fractal structure over* X.

PROOF. – Let $\boldsymbol{\Gamma}^d$ be the fractal structure associated to d.

Since d is totally bounded, for each n there exists a finite cover $\boldsymbol{A}_n$ of X such that $A \times A \subseteq U_n$ for all $A \in \boldsymbol{A}_n$. Then if $A \in \boldsymbol{A}$ and $x, y \in A$ we have $\varrho_n(x) = \varrho_n(y)$ (if $(x, y) \in A \times A \subseteq U_n$, then $y \in U_{xn}$ and similarly $x \in U_{yn}$, hence $U_{xn} = U_{yn}$ or what is the same $\varrho_n(x) = \varrho_n(y)$). Then $\{U_{xn} : x \in X\}$ is finite for all $n \in \mathbb{N}$ (since $\boldsymbol{A}_n$ is a finite covering). Therefore $\Gamma_n^d = \{U_{xn}^{-1} : x \in X\}$ is finite for all $n \in \mathbb{N}$ and $\boldsymbol{\Gamma}$ is a finite fractal structure over X. ■

So we can characterize second countable spaces in terms of fractal structures.

THEOREM 4.3. – *Let X be a topological space. The following statements are equivalent:*

(1) *X is second countable.*

(2) *There exists a totally bounded non-archimedean quasimetric over X.*

(3) *There exists a finite fractal structure over X.*

(4) *X can be embedded into the inverse limit of a sequence of finite posets.*

(5) *X can be embedded into a countable product of finite posets.*

PROOF. – The equivalence among 1), 2) and 3), follows from [6], Prop. 7.2, Theorem 4.2 and Theorem 4.1.

The equivalence among 3), 4) and 5) follows from the relation between fractal structures and the embedding of the space into the inverse limit of a sequence of posets (see [2]). ■

Theorem 3.18 and Theorem 4.3 allow us to prove Urysohn's metrization Theorem:

THEOREM 4.4. – *Let X be a second countable T_3 space. Then X is separable metrizable.*

REFERENCES

[1] F. G. ARENAS, *Tilings in topological spaces*, Int. Jour. of Maths. and Math. Sci., **22** (1999), 3, 611-616.

[2] F. G. ARENAS - M. A. SÁNCHEZ-GRANERO, *A characterization of non-archimedeanly quasimetrizable spaces*, Rend. Istit. Mat. Univ. Trieste Suppl., **30** (1999), 21-30.

[3] F. G. ARENAS - M. A. SÁNCHEZ-GRANERO, *A new aproach to metrization*, Topology and its. Appl., to appear.

[4] D. BURKE - R. ENGELKING - D. LUTZER, *Hereditarily closure-preserving collections and metrizability*, Proc.-Amer.-Math.-Soc., **51** (1975), 483-488.

[5] R. ENGELKING, *General Topology*, Heldermann Verlag, Berlin, 1989.

[6] P. FLETCHER - W. F. LINDGREN, *Quasi-Uniform Spaces*, Lecture Notes Pure Appl. Math., **77**, Marcel Dekker, New York, 1982.

[7] S. HANAI - K. MORITA, *Closed mappings and metric spaces*, Proc. Japan Acad., **32** (1956), 10-14.

[8] K. MORITA, *A condition for the metrizability of topological spaces and for n-dimensionality*, Sci. Rep. Tokyo Kyoiku Daigaku Sec. A, **5** (1955), 33-36.

[9] K. MORITA - J. NAGATA, *Topics in general topology*, North Holland, 1989.

[10] A. H. STONE, *Metrizability of decomposition spaces*, Proc. Amer. Math. Soc., **7** (1956), 690-700.

[11] G. ZHI-MIN, *$\aleph$-spaces and g-metrizable spaces and CF family*, Topology Appl., **82** (1998), 153-159.

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