BOLLETTINO
UNIONE MATEMATICA ITALIANA

Ulderico Dardano, Clara Franchi

On group automorphisms fixing subnormal subgroups setwise

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **3-B** (2000), n.3, p. 811–820.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2000_8_3B_3_811_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2000.

Bollettino U. M. I.
(8) 3-B (2000), 811-820

On Group Automorphisms Fixing Subnormal Subgroups Setwise (*)

ULDERICO DARDANO - CLARA FRANCHI

Dedicated to Mario Curzio on his 70th birthday

Sunto. – *In questo lavoro si studiano i gruppi* $\mathrm{Aut}_{sn}(G)$, $\mathrm{Aut}_d(G)$, $\mathrm{Aut}_\chi(G)$ *degli automorfismi di un gruppo G che fissano — come insiemi — tutti i sottogruppi di G che risultano essere rispettivamente subnormali, subnormali di difetto al più d, oppure che sono compresi tra un sottogruppo caratteristico ed il suo derivato. Si danno condizioni sufficienti affinché tali gruppi siano parasolubili di para-altezza al più* 2 *o* 3*. Si generalizzano così risultati da* [4], [7], [8], [10].

1. – Introduction and statement of main results.

The group $\mathrm{Aut}_{sn}(G)$ of all automorphisms of a group G fixing every subnormal subgroup of G setwise featured recently in a few papers. It has been shown that there are restrictions on its structure, when G is either finite or soluble. In particular, D. J. S. Robinson [10] has shown that $\mathrm{Aut}_{sn}(G)/\mathrm{Inn}(G) \cap \mathrm{Aut}_{sn}(G)$ is always soluble with derived length at most 4, if G is finite. Concerning the soluble case, S. Franciosi and F. de Giovanni [8] proved that if G is any soluble group then $\mathrm{Aut}_{sn}(G)$ is metabelian and that it is either abelian or finite, if G is polycyclic. Moreover, M. Dalle Molle [4] has shown that if G is soluble then $\mathrm{Aut}_{sn}(G)$ normalizes each subgroup of its derived subgroup (whence it is locally supersoluble), provided that G is a Chernikov group or the Fitting subgroup $\mathrm{Fit}(w(G))$ of the Wielandt subgroup of G either has finite exponent or is non-periodic (recall that $w(G)$ is the subgroup of the elements of G normalizing all subnormal subgroups). We improve these results to:

THEOREM 1. – *Let G be a soluble group. Then the metabelian group $\mathrm{Aut}_{sn}(G)$ acts by means of power automorphisms on its derived subgroup, provided either G is nilpotent-by-(finitely generated) or* $\mathrm{Fit}(G)$ *is non-periodic.*

(*) AMS Subject Classification: 20F28, E36, D45, D35.

We show that similar restrictions actually hold for some groups of automorphisms larger then the above one.

For each $d \geqslant 2$, let $\mathrm{Aut}_d(G)$ be the group of all automorphisms of G fixing subnormal subgroups with defect at most d setwise. Similarly, let $\mathrm{Aut}_\chi(G)$ be the group of all automorphisms of G setwise fixing all subgroups which lie between a characteristic subgroup of G and its derived subgroup. Clearly, $\mathrm{Aut}_d(G)$ and $\mathrm{Aut}_\chi(G)$ are normal subgroups of $\mathrm{Aut}(G)$ and

$$\mathrm{Aut}_\chi(G) \geqslant \mathrm{Aut}_2(G) \geqslant \mathrm{Aut}_d(G) \geqslant \mathrm{Aut}_{d+1}(G) \geqslant \bigcap_d \mathrm{Aut}_d(G) = \mathrm{Aut}_{sn}(G).$$

Note that $\mathrm{Aut}_\chi(G)$ induces power automorphisms on the factors of the derived series of G. Thus $\mathrm{Aut}_\chi(G)$ is locally supersoluble, if G is soluble and G' has finite exponent (see [5]). However, more can be said as in the next theorem below.

THEOREM 2. – *Let G be a soluble group. Then the derived subgroup of* $\mathrm{Aut}_\chi(G)$ *is nilpotent of class at most* 2. *Moreover* $\mathrm{Aut}_\chi(G)$ *induces power automorphisms on the factors of a series of length at most* 2 *in the torsion-subgroup of its derived subgroup.*

Furthermore, $\mathrm{Aut}_3(G)$ *is metabelian and normalizes each periodic subgroup of its derived subgroup.*

We will see that the bound 3 for the derived length of $\mathrm{Aut}_2(G)$ is best possible (see Proposition 2). Moreover, the consideration of the torsion-subgroup in the above statement cannot be avoided (see Proposition 3). However, sufficient conditions for the groups $\mathrm{Aut}_d(G)$ to be locally supersoluble will be given in Theorem 1$'$.

Apart from its intrinsic interest, the group $\mathrm{Aut}_\chi(G)$ will be also a tool to prove the following result, concerning the non-soluble case.

THEOREM 3. – *If the group G is finite, then* $\mathrm{Aut}_2(G)$ *is abelian-by-*(*completely reducible*)*-by-*(*soluble with derived length at most* 3).

Finally, we consider the group $\mathrm{Aut}_\gamma(G)$ of all automorphisms of a nilpotent group G which induce power automorphisms on the factors of the lower central series of G. By Proposition 5 we generalize results in [7] on the group $\mathrm{Aut}_n(G)$ of all automorphisms fixing each normal subgroup of a nilpotent group G setwise. Clearly, $\mathrm{Aut}_n(G) \leqslant \mathrm{Aut}_\gamma(G)$, but the inclusion may be strict.

For notation and terminology we refer mainly to [9]. By *a dihedral group over an abelian group A* we will mean a group isomorphic to the split extension of A by the *inversion map*, i.e. the automorphism

$x \mapsto x^{-1}$. For each subgroup H of G we denote by $\overline{H}$ the group of inner automorphisms of G induced by H.

2. – Proofs and related results.

Recall that power automorphisms of a group G form an abelian normal subgroup $\mathrm{PAut}(G)$ of $\mathrm{Aut}(G)$. Moreover $\mathrm{PAut}(G) \leq Z(\mathrm{Aut}(G))$, if G is abelian. Furthermore, if G is abelian and has finite exponent then its power automorphisms are even *universal*, i.e. of type $x \mapsto x^n$ where n is independent of x. The same holds, with $n = \pm 1$, if G is a non-periodic nilpotent group (see [2]).

Denote by $u_d(G)$ the subgroup of all elements of G determining an inner automorphism in $\mathrm{Aut}_d(G)$ (as in [1]) and by $u_\chi(G)$ the normalizer in G of all subgroups lying between a characteristic subgroup and its derived subgroup. From now on letter d either denotes a natural number greater than 1 or stands for χ, where we set $\chi < 2$. Clearly $[G, \mathrm{Aut}_d(G)] \leq u_d(G)$, that is $\mathrm{Aut}_d(G)$ *acts trivially on* $G/u_d(G)$. The next statement concernes solubility in $\mathrm{Aut}_d(G)$.

PROPOSITION 1. – *Let G be a group.*

(*i*) *If Γ is an $Inn(G)$-subgroup of* $\mathrm{Aut}_3(G)$ *such that $[G, \Gamma]$ is hyperabelian, then Γ and $[G, \Gamma]$ are metabelian and $[G, \Gamma]$ is a Dedekind group.*

(*ii*) *If Δ is an $Inn(G)$-subgroup of* $\mathrm{Aut}_2(G)$ (*resp.* $\mathrm{Aut}(G)$*-subgroup of* $\mathrm{Aut}_\chi(G)$), *such that $[G, \Delta]$ is hypoabelian, then Δ' and $[G, \Delta]$ are nilpotent of class at most* 2.

(*iii*) $\mathrm{Aut}_d(G)$ *is soluble* (*resp. soluble-by-finite*) *if and only if $u_d(G)$ is soluble* (*resp. soluble-by-finite*).

PROOF. – To prove the first part of the statement set $K := [G, \Gamma]$. Then $\overline{K} = [\overline{G}, \Gamma] \leq \Gamma$ induces a group of power automorphisms on each abelian section S/T of G, where S, T are subnormal subgroups of G with defect at most 2. Therefore K' stabilizes S/T. Hence each soluble normal subgroup N of K' stabilizes its own derived series and therefore it is nilpotent of class at most 2, as $\gamma_3(N) = [N, N'] \leq N'' \leq \gamma_4(N)$. Since K' is clearly hyperabelian, it follows that $\gamma_3(K') = 1$ and K is a soluble normal subgroup of G. Then K stabilizes its own derived series and therefore it is nilpotent of class at most 2. Hence each subgroup of K is subnormal in G with defect at most 3 and Γ induces power automorphisms on K. Thus K is a Dedekind group and Γ' is abelian, since it stabilizes the series $G \geq K \geq 1$. The second part of the statement can be proved in a similar way.

If $\mathrm{Aut}_d(G)$ is soluble (resp. soluble-by-finite), then it is clear that $u_d(G)$ is soluble (resp. soluble-by-finite) since $\overline{u_d(G)} \leq \overline{G} \cap \mathrm{Aut}_d(G)$. Conversely, if the soluble radical R of $K := [G, \mathrm{Aut}_d(G)] \leqslant u_d(G)$ has finite index and is soluble, then $\Gamma := C_{\mathrm{Aut}_d(G)}(K/R)$ has finite index in $\mathrm{Aut}_d(G)$ and Γ' stabilizes the characteristic finite series $G \geqslant K \geqslant R \geqslant R' \geqslant R'' \geqslant \ldots \geqslant 1$. Hence Γ' is nilpotent by the Hall-Kaloujnine theorem, and $\mathrm{Aut}_d(G)$ is soluble-by-finite. ■

COROLLARY *If* $\mathrm{Aut}_d(G)$ *is soluble, then its derived subgroup is nilpotent of class at most* 2 *or even abelian if* $d \geqslant 3$.

The difference between the structure of $\mathrm{Aut}_2(G)$ and $\mathrm{Aut}_3(G)$ is a genuine one, as we are going to show that actually the derived subgroup of a soluble $\mathrm{Aut}_2(G)$ need not be even a Dedekind group.

PROPOSITION 2. – *Let p be any prime and let U be either the extra-special group of order p^3 and exponent p if $p > 2$ or the quaternion group with order 8 if $p = 2$. Then U has an automorphism σ with order $p^2 - 1$ such that the group $G := U \rtimes \langle \sigma \rangle$ has the following properties:*

(i) $\mathrm{Aut}_2(G)$ *has derived length exactly* 3 *and its derived subgroup has nilpotency class exactly* 2, *if* $p \neq 2$;

(ii) $[G, \mathrm{Aut}_2(G)]$ *has nilpotency class exactly* 2;

(iii) $[G, \mathrm{Aut}_{sn}(G)]$ *is the quaternion group with order* 8, *if* $p = 2$.

PROOF. – Recall that $\mathrm{Out}(U)$ is isomorphic to the general linear group $\mathrm{GL}(2, p)$ (see [6]) and pick $\sigma \in \mathrm{Aut}(U)$ such that $\langle \sigma \rangle \,\mathrm{Inn}(U)$ corresponds to a Singer cycle of $\mathrm{GL}(2, p)$ in the above isomorphism.

Then each element of $G \setminus U$ induces on U/U' a fixed-point-free automorphism, since it acts as a non-trivial element of $\langle \sigma \rangle$. It follows that $[U, x] = U$ for each $x \in G \setminus U$. Thus each subnormal subgroup of G either contains U or is contained in U and $\mathrm{Fit}(G) = U$. Moreover $\langle \sigma \rangle$ is irreducible as a group of linear transformations of U/U'. Hence U' is the only non-trivial normal subgroup of G properly contained in U and the subnormal subgroups of G with defect exactly 2 are the maximal subgroups of U.

If $p > 2$, then $Z(G) = 1$ and $u_2(G) \geqslant U\langle \sigma^{(p^2-1)/2} \rangle$, since $\sigma^{(p^2-1)/2}$ acts on U/U' as the inversion map. Thus $\mathrm{Aut}_2(G)' \geqslant u_2(G)' = U$. The remaining part of statement follows now in all cases from the relation

$$U \geqslant [G, \mathrm{Aut}_2(G)] \geqslant [G, \overline{u_2(G)}] = [G, u_2(G)] = U\,. \qquad ■$$

For the proofs of Theorems 1 and 2 we need the following technical lemmas. We omit the easy proof of the second one.

LEMMA 1. – *Let $G \geqslant N \geqslant 1$ be a series in a group G and let $\sigma, \gamma \in \mathrm{Aut}(G)$. Assume that σ stabilizes the series, N and $[G, \sigma]$ are γ-invariant, γ^{-1}*

acts as a universal power n_1 on G/N and γ acts as a universal power n_2 on $[G, \sigma]$. If either $n_1 = 1$ or $[G, \sigma] \leqslant Z(G)$, then $\sigma^\gamma = \sigma^{n_1 n_2}$.

PROOF. – For each $g \in G$, there exists $a \in N$ such that $g^{\gamma^{-1}} = ag^{n_1}$. Thus $[g, \sigma^\gamma] = [g^{\gamma^{-1}}, \sigma]^\gamma = [ag^{n_1}, \sigma]^\gamma = [g^{n_1}, \sigma]^\gamma = [g, \sigma]^{n_1\gamma} = [g, \sigma^{n_1}]^\gamma = [g, \sigma^{n_1}]^{n_2} = [g, \sigma^{n_1 n_2}]$. ■

LEMMA 2. – *Let $\Gamma \leqslant \mathrm{Aut}(G)$ stabilize a series $G = G_0 \geqslant G_1 \geqslant \ldots \geqslant G_m = 1$. If $m = 2$, for each $\gamma \in \Gamma$ the subgroup $[G, \gamma]$ has exponent s if and only if γ has order s. If G_1 has finite exponent e, then Γ has finite exponent dividing e^{m-1}, for each $m \geqslant 1$.*

PROOF OF THEOREM 2. – Denote $\Gamma := \mathrm{Aut}_\chi(G)$. By Proposition 1, the subgroups Γ' and $K := [G, \Gamma]$ are nilpotent with class at most 2. Set $\Sigma := C_\Gamma(K)$. Then Γ acts by means of power automorphisms on $[G, \Sigma] \leqslant Z(K)$ and therefore Γ induces power automorphisms on the torsion-subgroup of Σ, by Lemmas 1 and 2. Now regard Γ/Σ as a group of automorphisms of K. Clearly, Γ/Σ induces power automorphisms on each factor of the lower central series of K and $\Gamma'\Sigma/\Sigma$ stabilizes this series. Thus if $\sigma \in \Gamma'\Sigma/\Sigma$, then $C_K(\sigma) \geqslant K' \geqslant [K, \sigma]$. Furthermore if σ has order s, then $K/C_K(\sigma)$ and $[K, \sigma]$ both have finite exponent, since $[k^s, \sigma] = [k, \sigma]^s = [k, \sigma^s]$, for each $k \in K$. Hence applying Lemma 1 to the series $K \geqslant C_K(\sigma) \geqslant 1$ we get that Γ/Σ induces power automorphisms on the torsion-subgroup of $\Gamma'\Sigma/\Sigma$, and the statement follows. If $\Gamma = \mathrm{Aut}_3(G)$, then it induces power automorphisms on K and so $\Gamma' \leqslant \Sigma$. ■

PROPOSITION 3. – *There exists a metabelian group G of rank 2 and trivial centre such that $\Gamma := \mathrm{Aut}_{sn}(G)$ is not locally supersoluble and does not normalize any non-trivial subgroup of $\Gamma'/tor(\Gamma')$.*

PROOF. – Let $G = \mathrm{Dr}_n G_n$, where G_n is the subgroup of order $p_n q_n$ of the holomorph of the cyclic group of order p_n and $q_1, p_1, \ldots, q_n, p_n, \ldots$ is an increasing sequence of distinct primes such that q_n divides $p_n - 1$, for each n. Recall that such a sequence exists by Dirichlet Theorem. Then $\Gamma = \mathrm{Aut}_{sn}(G) = \mathrm{Cr}_n \mathrm{Aut}_{sn}(G_n) = \mathrm{Cr}_n \mathrm{Aut}(G_n)$. Clearly $\mathrm{Aut}(G_n) = A_n \rtimes B_n$, with A_n and B_n cyclic subgroups. Then $\Gamma = \Gamma' \rtimes B$, where $\Gamma' = \mathrm{Cr}_n A_n$ and $B = \mathrm{Cr}_n B_n$.

An argument similar to that in Example 1 of [4] shows that Γ is not locally supersoluble. Moreover, if $\gamma = (\gamma_n)_{n \in \mathbb{N}} \in \Gamma'$ we have $\langle \gamma \rangle^\Gamma = \langle \gamma \rangle^B = \mathrm{Cr}_{n \in X} A_n$, where $X := \{n \in \mathbb{N} \mid \gamma_n \neq 1\}$. It follows that Γ does not normalize any non-trivial subgroup of $\Gamma'/tor(\Gamma')$. ■

We show now that for soluble groups G of finite exponent the picture is rather clear since $\mathrm{Aut}_d(G)$ is in the same condition as G is.

LEMMA 3. – *If* $\alpha \in \mathrm{Aut}(G)$ *acts as a power automorphism on* G/G', *then* α *acts as a power automorphism on each factor of the lower central series of* G. *In particular, if* α *has form* $x \mapsto x^n$ *on* G/G', *then* α *has form* $x \mapsto x^{n^i}$ *on* $\gamma_i(G)/\gamma_{i+1}(G)$, *for each* i.

PROOF. – By a well known argument by D. J. S. Robinson, each element of $\gamma_i(G)/\gamma_{i+1}(G)$ can be regarded as a (finite) sum $t = \sum_j t_j$, where $t_j = a_{j1} \otimes \ldots \otimes a_{ji}$, with $a_{jk} \in G/G'$. Since power automorphisms of abelian groups are locally universal, there is an integer n such that $a_{jk}^\alpha = n a_{jk}$ (in additive notation). Thus $t_j^\alpha = n^i t_j$ and $t^\alpha = n^i t$. ∎

PROPOSITION 4. – *Let* G *be a soluble group such that* $\mathrm{Fit}(u_d(G))$ *has finite exponent* e. *Then* $\mathrm{Aut}_d(G)$ *has finite exponent at most* e^3 (*resp.* e^2) *and the exponent of its derived subgroup divides* e^2 (*resp.* e, *if* $d \geq 3$).

PROOF. – As above, denote $\Gamma := \mathrm{Aut}_d(G)$, $K := [G, \Gamma]$ and $\Sigma_1 := C_\Gamma(K/K')$. By Proposition 1, the subgroup K is nilpotent of class at most 2, so $K \leq \mathrm{Fit}(u_d(G))$. Moreover Σ_1 stabilizes the series $G \geq K \geq K' \geq 1$, by Lemma 3. It follows that $\Gamma' \leq \Sigma_1$ has finite exponent dividing e^2, by Lemma 2. Furthermore Γ/Σ_1 has order at most e as a group of power automorphisms of K/K'.

If $d \geq 3$, then K is a Dedekind group and Γ induces power automorphisms on K. The statement follows as above since the group of power automorphisms of a hamiltonian group with exponent e has exponent at most e. ∎

Note that the consideration of the dihedral group G over a quasicyclic p-group shows that there is no hope for $\mathrm{Aut}_{sn}(G)'$ to have finite exponent, if only $\pi(G)$ is finite.

Next lemma proves Theorem 1 in the case $\mathrm{Fit}(G)$ is non-periodic, while the remaining part of the statement will follow from Theorem 1′.

LEMMA 4. – *If* G *is a soluble group with a nilpotent non-periodic normal* (*resp. characteristic if* $d = \chi$) *subgroup* N *with class at most* $d-1$ *if* $d \geq 3$ *or* 2 *otherwise, then* $\mathrm{Aut}_d(G)$ *has a central subgroup* Δ *such that* $\mathrm{Aut}_d(G)/\Delta$ *is either abelian or dihedral. Moreover, if* $d \geq 3$ *the above holds with* $\Delta = 1$.

PROOF. – Set $\Gamma := \mathrm{Aut}_d(G)$ and $K := [G, \Gamma]$. Let us first prove that NK has class at most $d-1$ if $d \geq 3$ or 2 otherwise. By Proposition 1, $\gamma_3(K) = 1$. Moreover K stabilizes the lower central series of N. Thus, applying the Three Subgroup Lemma to K, $\gamma_i(K)$, N, we get $[\gamma_i(K), N] \leq \gamma_{i+1}(N)$ for each i. Then, by an inductive argument, we have

$$\gamma_{i+1}(NK) = [\gamma_i(NK), NK] = [\gamma_i(N)\gamma_i(K), NK] =$$
$$[\gamma_i(N), N][\gamma_i(N), K][\gamma_i(K), N][\gamma_i(K), K] = \gamma_{i+1}(N)\gamma_{i+1}(K)$$

and NK is nilpotent with either the same class as N if $d \geq 3$, or 2 otherwise.

To prove now the statement, note that since $C_\Gamma(K)$ is abelian, if $[K, \Gamma] = 1$, there is nothing to prove. Suppose then $[K, \Gamma] \neq 1$, from now on. It follows that if $d \geq 3$, then $H := NK$ is necessarily abelian, since it is a non-periodic nilpotent group with a non-trivial power automorphism (see [2]). Hence H is nilpotent of class at most 2 in all cases.

Since H is non-periodic, then H/H' is non-periodic and therefore Γ acts on H/H' as either the identity or the inversion map. In both cases Γ acts trivially on H', by Lemma 3. Set $\Sigma := C_\Gamma(H/H')$, $\Delta_1 := C_\Gamma(H)$, $\Delta_2 := C_\Gamma(G/H')$ and $\Delta := \Delta_1 \cap \Delta_2$. Then Γ/Σ has order at most 2 as a group of power automorphisms of H/H'. Clearly $\Delta = 1$ if H is abelian. Applying Lemma 1 to the series $G \geq H \geq 1$, since $[G, \Delta] \leq H'$ we get $[\Delta, \Gamma] = 1$. Moreover Σ/Δ_1 and Σ/Δ_2 are abelian, if regarded as groups stabilizing the series $H \geq H' \geq 1$ and $G \geq H \geq H'$ respectively. Lemma 1 applied to the previous series yields that each $\gamma \in \Gamma \backslash \Sigma$ acts on Σ/Δ_1 and Σ/Δ_2 as the inversion map. Thus γ acts as the inversion map on Σ/Δ.

Finally for each $\gamma \in \Gamma\backslash\Sigma$, $g \in G$, we have: $[g, \gamma^2] = [g, \gamma][g, \gamma]^\gamma \equiv [g, \gamma][g, \gamma]^{-1} \equiv 1 \mod H'$. Moreover, if $h \in H$, then $hh^\gamma \in H' \leq Z(H)$ and so $[h, \gamma^2] = h^{-1}h^{\gamma^2} = h^{-1}(h^{-1}hh^\gamma)^\gamma = h^{-1}hh^\gamma(h^\gamma)^{-1} = 1$. Therefore $\gamma^2 \in \Delta$ and the statement follows by considering the series $\Gamma \geq \Sigma \geq \Delta \geq 1$. ■

Note that when G is polycyclic, $\mathrm{Aut}_d(G)$ is either finite or described by Lemma 4.

Recall that a group G is *parasoluble of paraheight at most n* if it has an abelian normal series of length n on whose factors it induces power automorphisms (see [11]).

THEOREM 1′. – *If G is a soluble group, then $\mathrm{Aut}_d(G)$ is parasoluble with paraheight at most 2 if $d \geq 3$ and 3 otherwise, provided one of the following holds*:

(*i*) *either* $\mathrm{Fit}(u_d(G))$ *has finite exponent or* $\mathrm{Fit}(u_3(G))$ *is non-periodic*;

(*ii*) *G/N is finitely generated, where N is a normal nilpotent subgroup of class at most $d-1$ (and $d \neq \chi$)*;

(*ii′*) *G/N is finitely generated, where N is a characteristic abelian subgroup of G (and $d = \chi$).*

PROOF. – Denote $\Gamma := \mathrm{Aut}_d(G)$ and $K := [G, \Gamma]$. Note that since $\mathrm{Fit}(u_3(G))$ is a Dedekind group, if either $\mathrm{Fit}(u_3(G))$ or K is non-periodic, by Lemma 4 there is nothing left to prove. Let us show that in all remaining cases, Γ' is periodic and the statement follows from Theorem 2. If $\mathrm{Fit}(u_d(G))$ has finite exponent, then Γ is periodic, by Proposition 4.

Let now K be periodic and (*ii*) or (*ii'*) hold. Since G/N is finitely generated, there is a subgroup $X = \langle x_1, \ldots, x_n \rangle$ such that $G = NX$. On the other hand Γ' acts trivially on N, thus for each $\sigma \in \Gamma'$ the group $[G, \sigma] = [X, \sigma] \leqslant K$ is a periodic nilpotent group generated by conjugates of the $[x_i, \sigma]$'s and the $[x_i^{-1}, \sigma]$'s. Therefore $[G, \sigma]$ has finite exponent. Thus if $\Sigma := C_\Gamma(K)$, then $\Gamma' \cap \Sigma$ is periodic, by Lemma 2. Then, in case $d \geqslant 3$ the proof is already achieved, since $\Gamma' \leqslant \Sigma$. Otherwise regard Γ/Σ as a group of automorphisms of K and observe that Γ' stabilizes the series $K \geqslant K' \geqslant 1$. Then, by arguing as above, $\Gamma'\Sigma/\Sigma$ is periodic, and so Γ' is periodic, as we claimed. ■

REMARK. – Note that Theorem 1′ and Theorem 2 in case $d \geqslant 3$ can be proved also for any group G and the soluble radical of $\mathrm{Aut}_d(G)$ instead of the whole $\mathrm{Aut}_d(G)$.

We show now a result which implies Theorem 3.

THEOREM 3′. – *Let G be a soluble-by-finite group with soluble radical S. Then the 3rd term $\Gamma^{(3)}$ of the derived series of $\Gamma := \mathrm{Aut}_2(G)$ has got an abelian normal subgroup A such that $\Gamma^{(3)}/A$ is isomorphic to the completely reducible radical of G/S. Moreover, $\Gamma^{(5)}$ is finite.*

Furthermore, $\mathrm{Aut}_2(G)/\mathrm{Inn}(G) \cap \mathrm{Aut}_2(G)$ *is soluble with derived length at most* 4, *or even* 3 *if* $H^1(G/S, Z(S)) = 0$.

PROOF. – Let G be a finite semisimple group with completely reducible radical R. Then $R = S_1 \times \ldots \times S_k$, where the S_i's are non-abelian simple groups. Via the Three Subgroup Lemma applied to G, R and $C_{\mathrm{Aut}(G)}(R)$ from equality $C_G(R) = 1$ we deduce $C_{\mathrm{Aut}(G)}(R) = 1$. Thus $\mathrm{Aut}_2(G)$ can be regarded as a subgroup of $\mathrm{Aut}(R)$ fixing all S_i's setwise. Therefore $\mathrm{Aut}_2(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(S_1) \times \ldots \times \mathrm{Aut}(S_k)$. Because of Schreier Conjecture, each $\mathrm{Out}(S_i)$ is soluble with derived length at most 3. Therefore $\mathrm{Aut}_2(G)^{(3)} \leqslant \overline{R} \simeq R$ and, by a theorem by Wielandt (see [9, Th. 13.3.2, p. 383]), $S_i \leqslant w(G)$ for each i. Thus $\overline{R} \leqslant \mathrm{Aut}_{sn}(G) \leqslant \mathrm{Aut}_2(G)$ and so $\mathrm{Aut}_2(G)^{(3)} = \overline{R}$.

From now on let G be any group as in the statement, S its soluble radical, $\Gamma := \mathrm{Aut}_2(G)$ and $\Delta := C_\Gamma(S)$. Clearly all characteristic subgroups of S are characteristic in G, since S in turn is characteristic. Then Γ/Δ can be regarded as a subgroup of $\mathrm{Aut}_\chi(S)$ and we have $\Gamma^{(3)} \leqslant \Delta$, by Theorem 2. On the other hand G/S is semisimple and the argument of the previous paragraph applies. Therefore, denoting by Σ the subgroup of Γ stabilizing the series $G \geqslant S \geqslant 1$ we have that Σ is abelian and that the 3rd term of the derived series of Γ/Σ is isomorphic to the completely reducible radical R/S of G/S.

Moreover, by the same argument used in [10], we have that $\Gamma^{(4)} \leqslant \overline{C_R(S)}$, and $\Gamma^{(3)} \leqslant \overline{G}$ if $H^1(G/S, Z(S)) = 0$. Then, since $C_R(S)$ is central-by-finite, a well known theorem by Schur implies that $\Gamma^{(5)}$ is finite. ■

Recall that in [3] it is shown that if $G := D_4(3) \times (SL_3(7) \rtimes \langle \alpha \rangle)$, where α is an automorphism of $SL_3(7)$ with order 2 acting as the inversion map on the centre of $SL_3(7)$, then the 3rd term of the derived series of $\mathrm{Aut}_{sn}(G)$ is a non-trival abelian-by-(completely reducible) group.

Finally we consider the group $\Gamma := \mathrm{Aut}_\gamma(G)$, where G is a nilpotent group with class c. Note that by Lemma 2, $\mathrm{Aut}_\gamma(G)$ is just the group of all automorphisms of G which induce power automorphisms on G/G'. Clearly Γ' is nilpotent with class at most $c-1$. Moreover Γ is parasoluble with paraheight at most $\frac{c(c-1)}{2} + 1$, provided either G' has finite exponent or G/G' is not a periodic group with infinite exponent, by [5]. Moreover, if G is periodic then $\mathrm{Aut}_\gamma(G) = \mathrm{Cr}_p \mathrm{Aut}_\gamma(G_p)$, where G_p is the p-component of G.

PROPOSITION 5. – *Let G be a nilpotent group of class c and $\Gamma := \mathrm{Aut}_\gamma(G)$.*

If G/G' has finite exponent p^n, then $\Gamma = \Delta_0 \rtimes \langle \gamma \rangle$, where γ has order dividing $p-1$ and $\Delta_0 := C_\Gamma(G/G'G^p)$ has exponent dividing p^{cn-1} and is nilpotent of class at most $cn-1$.

If G/G' is non-periodic, then $\Delta := C_\Gamma(G/G')$ is a nilpotent subgroup of class at most $c-1$ and index at most 2. Moreover $\pi(\Delta) \subseteq \pi(G')$.

PROOF. – Assume G/G' has finite exponent p^n and set $\Delta_i := C_\Gamma(G/\gamma_i(G))$ for $i \geqslant 2$. Then the group Δ_i/Δ_{i+1} can be seen as a subgroup of $\mathrm{Aut}(G/\gamma_{i+1}(G))$ stabilizing the series $G \geqslant \gamma_i(G) \geqslant \gamma_{i+1}(G)$ and so by Lemma 2, its exponent divides p^n. Thus $\Delta_2 = C_\Gamma(G/G')$ has exponent dividing $p^{(c-1)n}$, since $\Delta_{c+1} = 1$. On the other hand, as a group of power automorphisms of G/G', the group Γ/Δ_2 is abelian with order dividing $p^{n-1}(p-1)$. If Γ_p/Δ_2 denotes its p-component, then Γ/Γ_p is cyclic with order dividing $p-1$ (where $\Gamma_2 = \Gamma$) and Γ_p stabilizes all refinements of the lower central series of G with elementary abelian factors. The shortest of them has length at most cn, hence Γ_p is nilpotent of class at most $cn-1$ and has exponent at most p^{cn-1}. Clearly $\Gamma_p = \Delta_0$.

Assume now that G/G' is non-periodic. Since Γ induces power automorphisms on G/G', we have that $|\Gamma/\Delta| \leqslant 2$. Moreover Δ stabilizes the lower central series of G, and so it is nilpotent of class at most $c-1$. The remaining part of the statement follows by Lemma 2. ■

REMARK. – We have seen that in the above cases the picture of $\mathrm{Aut}_\gamma(G)$ is not that different from that of the group $\mathrm{Aut}_n(G)$ of automorphisms fixing all normal subgroups setwise (see [7]). However it is easily seen that *if* $G = \mathrm{Dr}_{n \in \mathbb{N}} U(3, \mathbb{Z}_{p^n})$, *then the group* $\mathrm{Aut}_\gamma(G)$ *is not nilpotent*, while $\mathrm{Aut}_n(G)$ is. Here $U(3, \mathbb{Z}_{p^n})$ is the group of unitriangular 3×3 matrices with entries in the ring of integers modulo p^n.

REFERENCES

[1] R. A. BRYCE, *Subgroups like Wielandt's in finite soluble groups*, Math. Proc. Camb. Phil. Soc., **107** (1990), 239-259.
[2] C. D. H. COOPER, *Power automorphisms of a group*, Math. Z., **107** (1968), 335-356.
[3] J. COSSEY, *Automorphisms fixing every subnormal subgroup of a finite group*, Glasgow Math. J., **39** (1997), 111-114.
[4] M. DALLE MOLLE, *Sugli automorfismi che fissano i sottogruppi subnormali di un gruppo risolubile*, Boll. Un. Mat. Ital., B(7), **9-A** (1995), 483-491.
[5] U. DARDANO - C. FRANCHI, *On groups paralyzing a subgroup series*, to appear on Rend. Circ. Mat. Palermo.
[6] K. DOERK - T. HAWKES, *Finite soluble groups*, De Gruyter expositions in mathematics, **4**, Walter de Gruyter, Berlin, New York, 1992.
[7] S. FRANCIOSI - F. DE GIOVANNI, *On automorphisms fixing normal subgroups of nilpotent groups*, Boll. Un. Mat. Ital., B(7), **1** (1987), 1161-1170.
[8] S. FRANCIOSI - F. DE GIOVANNI, *On automorphisms fixing subnormal subgroups of soluble groups*, Atti Acc. Naz. Linc. Rend. Cl. Sci. Fis. Mat. Nat. (8), **82** (1988), 217-222.
[9] D. J. S. ROBINSON, *A Course in the Theory of Groups*, Springer V., Berlin, 1982.
[10] D. J. S. ROBINSON, *Automorphisms fixing every subnormal subgroup of a finite group*, Arch. Math., **64** (1995), 1-4.
[11] B. A. F. WEHRFRITZ, *On locally supersoluble groups*, J. Algebra, **43** (1976), 665-669.

Ulderico Dardano: Dipartimento di Matematica e Appl. «R. Caccioppoli»
Università di Napoli «Federico II»
Via Cintia, Monte S. Angelo, I-80126 Napoli, Italy
dardano@unina.it

Clara Franchi: Dipartimento di Matematica Pura e Applicata
Università degli Studi di Padova, Via Belzoni 7, I-35131 Padova, Italy
cfranchi@math.unipd.it

Pervenuta in Redazione
il 3 settembre 1999