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A Geometric Description of Hazama's Exceptional Classes

Federica Galluzzi

Sunto. – *Sia X una varietà abeliana complessa di tipo Mumford. In queste note daremo una descrizione esplicita delle classi eccezionali in $B^2(X \times X)$ trovate da Hazama in [Ha] e le descriveremo geometricamente usando la grassmaniana delle rette di $\mathbb{P}^7$.*

Introduction.

In this paper we will give an explicit description of the exceptional classes found by Hazama (see [Ha, 5.1]) in the product of two varieties of Mumford-type.

Varieties of Mumford-type occur as general fibers of the 1-dimensional families of 4-dimensional polarized abelian varieties introduced by Mumford in [Mu2].

The main interest in studying these varieties comes from the fact that the Mumford-Tate group is strictly contained in $Sp(8)$, while the Mumford-Tate group of the general abelian variety is the whole symplectic group. Moreover, varieties of Mumford-type gave the first example of abelian varieties not characterized by their endomorphism algebra and having a «small» Mumford-Tate group, in the sense just explained.

Let X be such a variety. In [Ha, 5.1.], Hazama found that $B^i(X) \cong \mathbb{Q}$, $i = 0, 1, 2$ and he also proved that $B^2(X \times X) \neq \bigwedge^2 B^1(X \times X)$.

We investigate the Hodge structure of X and, using some of the results contained in [Ga], we are able to describe explicitly the exceptional classes in $B^2(X \times X)$. Then, using the grassmanian $Gr(1, 7)$, we give a geometric description for these classes.

The paper is organized as follows.

In Section 1 we first recall some general definitions and properties of Mumford-Tate groups to introduce the techniques we use to find the exceptional classes of a variety of Mumford-type. If X is such a variety, we write $V = H^1(X, \mathbb{Q})$. The Mumford-Tate group $MT(X)$ of X is defined as a subgroup of $GL(V)$, so it acts in a natural way on V and we can also consider the exterior powers of this representation on $\bigwedge^n V$. A classical result of [DMOS] tells us that the link between such representations and Hodge classes of X is that the (p, p)-Hodge classes $B^p(X)$ are precisely the invariants under the action of $MT(X)$ in $\bigwedge^{2p} V$ (see Prop. 1.2).

In Section 2 we recall briefly some properties of varieties of Mumford-type and we focus on the fact that they have complex Mumford-Tate group isogeneous to $SL(2)^3$.

In Section 3 we introduce the techniques we use to study the Hodge structure of X. These techniques involve the study of the representations (over $\mathbb{C}$) of $SL(2)^3$ and $Sp(8)$. Using these methods one sees that $B^i(X) \cong \mathbb{Q}$, $i = 0, 1, 2$.

We find the exceptional classes in $B^2(X \times X) \subseteq \bigwedge^2 V \otimes \bigwedge^2 V$ looking at the Killing form of the Lie algebra $\mathfrak{sl}(2)^3$. In Section 5 we give also an explicit description of such classes and thus we are also able to explain their geometry in terms of lines geometry in $\mathbb{P}^7$ in Section 6.

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1. – Mumford-Tate groups of abelian varieties.

The Mumford-Tate group (or Hodge group) was introduced by Mumford in [Mu1] for abelian varieties but in general it is associated to rational Hodge structures. Since we are interested in polarized abelian varieties, we introduce the definition for this case only. Good references are [DMOS], [G] and [vG]. This group has proved to be a powerful tool for studying the Hodge structure of an abelian variety. Here we recall its construction for the convenience of the reader.

Let X be a complex abelian variety, we write

$$V = H^1(X, \mathbb{Q}), \qquad V_{\mathbb{R}} = H^1(X, \mathbb{R}).$$

A complex structure on $V_{\mathbb{R}}$ is a $\mathbb{R}$-linear map

$$J \colon H^1(X, \mathbb{R}) \to H^1(X, \mathbb{R}) \qquad \text{such that } J^2 = -I .$$

A *polarization* for X is a cycle $E \in B^1(X) \subseteq H^2(X, \mathbb{Q})$, i.e. a map $E' \colon \Lambda^2 H_1(X, \mathbb{Q}) \to \mathbb{Q}$ that satisfies Riemann's Relations:

$$\dot{E}'(J'x, J'y) = E'(x, y), \qquad E'(x, J'x) \geq 0 ,$$

where J' is the dual of J. We introduce the representation of real algebraic groups

$$h\colon S^1 \to GL(V_{\mathbb{R}}),$$

$$a + ib \mapsto aI + bJ,$$

where $S^1 = \{z \in \mathbb{C}^*\colon |z| = 1\}$. For all n there are representations

$$\wedge^n h\colon S^1 \to GL(\wedge^n H^1(X, \mathbb{R})) = GL(H^n(X, \mathbb{R})).$$

DEFINITION 1.1. – *The special Mumford-Tate group* (*also called Hodge group*) $MT(X)$ *of the abelian variety* X *is the smallest algebraic subgroup* $G \subseteq GL(V)$ *which is defined over* $\mathbb{Q}$, *such that*

$$h(S^1) \subseteq G(\mathbb{R}).$$

Let $Sp(E)$ be the algebraic subgroup of $\mathrm{SL}(V)$ which fixes a polarization E of X. It can be easily proved that

$$MT(X)(\mathbb{R}) \subseteq Sp(E)(\mathbb{R}),$$

indeed $Sp(E)$ is defined over $\mathbb{Q}$ and $h(S^1) \subseteq Sp(E)(\mathbb{R})$. Thus $MT(X) \subseteq Sp(E)$. It can be proved that for the general abelian variety the isomorphism holds.

Mumford-Tate groups are useful tools to study the space

$$B^p(X) := H^{2p}(X, \mathbb{Q}) \cap H^{n,n}(X)\,(H^{2p}(X, \mathbb{C}))$$

of the Hodge classes of a complex abelian variety. Indeed we have the following results

THEOREM 1.2. – *Let* $\varrho_k\colon GL(V) \to GL(\bigwedge^k V)$ *be the* k^{th}*-exterior power of the standard representation* ϱ_1 *of* $GL(V)$. *For all* p *the space of Hodge classes of* X *is the subspace of* $MT(X)$*-invariants in* $H^{2p}(X, \mathbb{Q})$, *i.e.*

$$B^p(X) = H^{2p}(X, \mathbb{Q})^{MT(X)}.$$

PROOF. – See [DMOS] and also [G, 2.4].

PROPOSITION 1.3. – *There is a bijection* (*see* [DMOS] *and also* [G, 2.4.])

$$\left\{\begin{matrix} MT(X) \\ \mathbb{Q}\text{-subrepresentations} \\ \text{of } V^{\otimes n} \end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} \text{Hodge substructures} \\ \text{of } V^{\otimes n} \end{matrix}\right\}.$$

REMARK 1.4. – As $MT(X)(\mathbb{C})$ is a connected reductive group (see [G, 2.3, 2.5]), its representations (or the representations of its Lie algebra) are well known. So one can study the representations of $MT(X)(\mathbb{C})$ over $V^{\otimes n}$ and, if these representations are defined over $\mathbb{Q}$, one gets rational Hodge substructures in $V^{\otimes n}$. Moreover, one has

$$B^p(X) \otimes_{\mathbb{Q}} \mathbb{C} = (H^{2p}(X, \mathbb{C}))^{MT(X)(\mathbb{C})}.$$

So, to find the Hodge classes in $H^{2p}(X, \mathbb{Q}) = \bigwedge^{2p} V \subset V^{\otimes n}$ one first studies the invariants on $\bigwedge^{2p} V_{\mathbb{C}}$ and then one tries to find the invariants defined over $\mathbb{Q}$.

2. – Varieties of Mumford-type.

In [Mu2] Mumford defines a family of 4-dimensional polarized abelian varieties where the Mumford-Tate group of any fiber is not $Sp(8)$ but the general fiber does have no nontrivial endomorphisms. Over $\mathbb{C}$ the Mumford-Tate group of these fibers is isogenous to $SL(2, \mathbb{C})^3$, we call these fibers *variety of Mumford-type*.

If X is such a variety, the Mumford-Tate group is defined using a quaternion algebra A that is, a central simple algebra of dimension four over its center.

Mumford chooses the algebra A so that its cener is a totally real cubic number field K. We can write an element $a \in A$ as

$$a = a_0 + a_1 \varepsilon_1 + a_2 \varepsilon_2 + a_3 \varepsilon_3, \qquad a_i \in K,$$

with $\varepsilon_1^2, \varepsilon_2^2 \in K$ and $\varepsilon_1 \varepsilon_2 = -\varepsilon_2 \varepsilon_1 = \varepsilon_3$. The Mumford-Tate group of X is defined as

$$MT(X) := \{x \in A^* : x\overline{x} = 1\},$$

where the «$-$» stands for the canonical involution in A: if $a = a_0 + a_1 \varepsilon_1 + a_2 \varepsilon_2 + a_3 \varepsilon_3 \in A$, then $\overline{a} = a_0 - a_1 \varepsilon_1 - a_2 \varepsilon_2 - a_3 \varepsilon_3$. Over $\mathbb{C}$, this group is isogeneous to $SL(2)^3$.

The complex structure for an abelian variety of Mumford-type is given by the real representation

$$h: S^1 \to SU(2) \times SU(2) \times SL(2, \mathbb{R}) \sim SO(4, \mathbb{R}) \times SL(2, \mathbb{R}) \hookrightarrow GL(8, \mathbb{R}),$$

$$e^{i\theta} \mapsto \left(I, I, \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right).$$

In this case, the multiplication by i on $\mathbb{R}^8$ is given by $J = h(i)$.

3. – Representations of $\mathfrak{sl}(2)^3$ and $\mathfrak{sp}(8)$.

It is possible to study the Hodge structure of a variety of Mumford-type following the strategy explained in Remark 1.4. In our case this means that we have to study representations of $SL(2)^3$ as a subgroup of $Sp(8)$. To do this, we look at the Lie algebras $\mathfrak{sl}(2)^3$ and $\mathfrak{sp}(8)$. We recall here some of the results of [Ga] that allow us to find the exceptional classes we are looking for.

Let W_n be the irreducible $(n+1)$-dimensional representation of $\mathfrak{sl}(2)$, the Lie algebra of $SL(2)$, so $W_n = S^n W_1$. If V_1, V_2, V_3 are irreducible representations of $\mathfrak{sl}(2)$, then we denote by $V_1 \boxtimes V_2 \boxtimes V_3$ the representation of $\mathfrak{sl}(2)^3$ on $V_1 \otimes V_2 \otimes V_3$ where $(g_1, g_2, g_3) \in \mathfrak{sl}(2)^3$ acts by g_i on the i-th tensor component. We write

$$W_{a,b,c} := W_a \boxtimes W_b \boxtimes W_c \quad \text{with} \quad a, b, c \in \mathbb{Z}_{\geq 0}$$

and any irreducible $\mathfrak{sl}(2)^3$-representation is of this type. We write $W := W_1$ with standard basis $\{e^1, e^{-1}\}$ and we denote the products $e^i \boxtimes e^j \boxtimes e^k$ with e^{ijk}. Let $\langle , \rangle$ be the alternating form on W which is invariant for $SL(2)$ and which is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

On the eight dimensional space $V = W \boxtimes W \boxtimes W$ we have an alternating form defined as follows

$$E(e^{ijk}, e^{i'j'k'}) = \langle e^i, e^{i'} \rangle \langle e^j, e^{j'} \rangle \langle e^k, e^{k'} \rangle \qquad i, \ldots, k' \in \{-1, 1\}.$$

One has

$$E(-,-) = \begin{cases} 0 & \text{if } i = i' \text{ or } j = j' \text{ or } k = k', \\ \pm 1 & \text{if } i \neq i', \quad j \neq j', \quad k \neq k'. \end{cases}$$

Let now $\mathfrak{sl}(2)^3 := \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ where $\mathfrak{g}_i := \mathfrak{sl}(2)$. Let $\mathfrak{h}_i$ be the standard Cartan algebra of $\mathfrak{g}_i$ and $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3$ a Cartan algebra of $\mathfrak{sl}(2)^3$. For $i, j, k \in \mathbb{Z}$ we define a weight $(i, j, k) \in \mathfrak{h}^*$ by

$$(i, j, k) \colon \mathfrak{h} \to \mathbb{C},$$

$$(H_1, H_2, H_3) \mapsto i s_1 + j s_2 + k s_3,$$

where $H_i = \begin{pmatrix} s_i & 0 \\ 0 & -s_i \end{pmatrix} \in \mathfrak{h}_i$. We choose a symplectic basis $\{f_1, \ldots, f_8\}$ for (V, E) and consider $Sp(8) := Sp(V, E)$. By definition, $SL(2)^3 \subset \mathrm{Sp}(8)$. Let $\tilde{\mathfrak{h}}$ be the Cartan algebra of $\mathfrak{sp}(8)$. It has basis $\{H_1, \ldots, H_4\}$ with $H_i = E_{i,i} - E_{4+i,4+i}$ and $E_{4+i,4+i}$ is an elementary matrix. Let $\{L_1, \ldots, L_4\} \subseteq \tilde{\mathfrak{h}}^*$ be the dual basis. Given $H \in \tilde{\mathfrak{h}}$ one has

$$\begin{cases} Hf_i = L_i(H)\, f_i\,, \\ Hf_{i+4} = -L_i(H)\, f_{i+4}\,, \qquad i = 1, \ldots, 4\,. \end{cases}$$

In this way we can restrict the weigths to $\mathfrak{sl}(2)^3$ and we get

PROPOSITION 3.1 [Ga, 3.1.]. – *The $\mathbb{C}$-linear restriction map $\tilde{\mathfrak{h}}^* \to \mathfrak{h}^*$ acts on the roots in the following way*

$$L_1 \mapsto (1, 1, 1),$$

$$L_2 \mapsto (1, -1, -1),$$

$$L_3 \mapsto (-1, 1, -1),$$

$$L_4 \mapsto (-1, -1, 1).$$

PROOF. – If $X, Y \in \mathfrak{sl}(2)$, $v, w \in W$, one has

$$(X, Y)(v \boxtimes w) = (Xv) \boxtimes w + v \boxtimes (Yw)\,.$$

For example, $Hf_1 = L_1(H)\, f_1$ gives $H(e^{111}) = L_1(H) e^{111}$, and in general from the above formulas we get

$$(H_1, H_2, H_3)(e^{ijk}) = (as_1 + bs_2 + cs_3)(e^{ijk})\,.$$

Hence, the weights of the representation $W \boxtimes W \boxtimes W$ under the action of $\mathfrak{h}$ are $(\pm 1, \pm 1, \pm 1)$. ■

COROLLARY 3.2 [Ga, 3.2.]. – *The standard representation $V = W \boxtimes W \boxtimes W$ is irreducible under the action of $\mathfrak{sl}(2)^3$ with highest weight* $(1, 1, 1)$.

Using this result one can understand how tensor powers of the representation V decompose under the action of $SL(2)^3$.

4. – Exceptional classes.

We recall that the Hodge classes in $H^{2p}(X, \mathbb{Q}) = \bigwedge^2 V$ are precisely the invariants under the action of the Mumford-Tate group. Thus, in our case, we are looking for invariants for $SL(2)^3$ which are not invariants for $Sp(8)$.

We can prove the following (see [Ga, 3.4])

PROPOSITION 4.1 [Ga,3.4.-3.6.]. – *There is an isomorphism of* $\mathfrak{sp}(8)$*-representations*

$$\bigwedge^2 V \cong P_2 \oplus W_{0,0,0},$$

where P_2 *is the irreducible representation of* $\mathfrak{sp}(8)$ *containing the highest weight vector of* $\bigwedge^2 V$. *Moreover,* P_2 *decomposes as* $\mathfrak{sl}(2)^3$*-representation*:

$$P_2 \cong W_{2,2,0} \oplus W_{2,0,2} \oplus W_{0,2,2}.$$

Thus, from this Proposition follows that in $\bigwedge^2 V$ there is an invariant represented by $W_{0,0,0}$ and we know that it has to be the polarization $E \in \bigwedge^2 V$ which is invariant also for $Sp(8)$.

Using similar results, in [Ga, 3.7] we showed that $B^2(X) \cong \mathbb{Q}$ for $i = 0, 1, 2$ so, there are not exceptional classes in $B^i(X)$ and this was also proved by Hazama in [Ha, 5.1.]. Now, if we look at $\bigwedge^2 V \otimes \bigwedge^2 V$, we note that there are some special elements in it . More precisely, consider

$$P_2 \otimes P_2 \subseteq \bigwedge^2 V \otimes \bigwedge^2 V.$$

We have

$$W_{2,2,0} \otimes W_{2,2,0} \cong (W_2 \otimes W_2) \boxtimes (W_2 \otimes W_2) \boxtimes W_0. \tag{1}$$

In $W_2 \otimes W_2 \cong S^2 W \otimes S^2 W$ there is the invariant coming from the Killing form of $\mathfrak{sl}(2)$:

$$v = (e^1 \odot e^{-1}) \otimes (e^1 \odot e^{-1}) - 1/2[(e^1 \odot e^1) \otimes (e^{-1} \odot e^{-1}) + (e^{-1} \odot e^{-1}) \otimes (e^1 \odot e^1)]$$

(the $\odot$ denotes the symmetric product in $W_2 = S^2 W_1$). Thus the vector

$$\phi_1 = v \boxtimes v \boxtimes 1 \quad \text{with} \quad 1 \in \mathbb{C} = W_0$$

is an element of $(P_2 \otimes P_2)^{SL(2)^3}$. In the same way we obtain the invariants

$$\phi_2 = v \boxtimes 1 \boxtimes v \in W_{2,0,2} \otimes W_{2,0,2},$$

$$\phi_3 = 1 \boxtimes v \boxtimes v \in W_{0,0,2} \otimes W_{0,0,2}.$$

We have the following

PROPOSITION 4.2 [Ga, 3.8.]

i) *ϕ_1, ϕ_2, ϕ_3 are a basis of* $(P_2 \otimes P_2)^{SL(2)^3}$,

ii) $(P_2 \otimes P_2)^{Sp(8)} \cong \mathbb{C}$,

iii) *if $\boldsymbol{G}$ is a connected Lie group with $SL(2)^3 \subset \boldsymbol{G} \subset Sp(8)$ such that ϕ_1, ϕ_2, ϕ_3 are invariant for $\boldsymbol{G}$, then $\boldsymbol{G} = SL(2)^3$.*

Thus, the ϕ_i's represent exceptional classes in $B^2(X \times X)$. The fact that $B^2(X \times X) \neq \bigwedge^2 B^1(X \times X)$ for X a variety of Mumford-type was also proved by Hazama in [Ha, 5.1.].

5. – Explicit description of classes ϕ_1, ϕ_2, ϕ_3.

We give now an explicit formula for the invariants ϕ_1, ϕ_2, ϕ_3 in $\bigwedge^2 V \otimes \bigwedge^2 V$. In this section we write $W = \langle x, y \rangle$ and we use polynomial notations for symmetric products in W_2. We want to give an explicit isomorphism

$$\bigwedge^2 V \xrightarrow[\alpha]{\cong} W_{2,2,0} \oplus W_{2,0,2} \oplus W_{0,2,2} \oplus W_{0,0,0}$$

using bases. By symmetry we work with $\phi_1 = v \boxtimes v \boxtimes 1$ only. In this basis

$$v = xy \otimes xy - 1/2[x^2 \otimes y^2 + y^2 \otimes x^2].$$

The standard basis for $\bigwedge^2 V$ is $\{f_{ij} := f_i \wedge f_j,\ i < j\}$. First we choose a basis for P_2

$$B = \{f_{ij} \colon j \neq i+4\} \cup \{f_{15} - f_{26}, f_{15} - f_{37}, f_{15} - f_{48}\}.$$

We consider the vector

$$(x^2 \boxtimes x^2 \boxtimes 1) \otimes (y^2 \boxtimes y^2 \boxtimes 1)$$

which is a summand of ϕ_1. We have

$$x^2 \boxtimes x^2 \boxtimes 1 \in W_{2,2,0} \text{ of weight } (2, 2, 0),$$

$$y^2 \boxtimes y^2 \boxtimes 1 \in W_{2,2,0} \text{ of weight } (-2, -2, 0).$$

There are f_{18}, f_{45} only which have the same weigths in the basis of P_2, thus

$$\alpha^{-1}(x^2 \boxtimes x^2 \boxtimes 1) \in \langle f_{18} \rangle, \qquad \alpha^{-1}(y^2 \boxtimes y^2 \boxtimes 1) \in \langle f_{45} \rangle.$$

Analogously

$$\alpha^{-1}(x^2 \boxtimes y^2 \boxtimes 1) \in \langle f_{27} \rangle, \qquad \alpha^{-1}(y^2 \boxtimes x^2 \boxtimes 1) \in \langle f_{36} \rangle.$$

We have to proceed in a different way for the other summands of ϕ_1 since we have many vectors of the same weight in B. We denote x_1, x_2 the vectors in the basis of $\mathfrak{sl}_2$ which are represented by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We consider the vector $x^2 \boxtimes xy \boxtimes 1$ of weight $(2, 0, 0)$. There are f_{78} e f_{12} which have the same weight in P_2. Now let the Lie algebra $\mathfrak{sl}_2^3$ acts. We see

$$(0, 0, x_1)(f_{78} + f_{12}) = 0\,, \qquad (0, 0, x_2)(f_{78} + f_{12}) = 0\,.$$

Thus $\alpha(\langle f_{78} + f_{12} \rangle) \subset W_{2,2,0}$. We proceed in the same way with the other vectors and we construct the isomorphism up to constants. We choose this constants in such a way that ϕ_1 is invariant in the new basis too. Put

$$\begin{aligned}
&A = f_{15} - f_{37} + f_{48} - f_{26}\,, && \\
&B = (f_{78} + f_{12}) \otimes (f_{34} + f_{56})\,, && \overline{B} = (f_{34} + f_{56}) \otimes (f_{78} + f_{12})\,, \\
&C = (f_{13} + f_{68}) \otimes (f_{57} + f_{24})\,, && \overline{C} = (f_{57} + f_{24}) \otimes (f_{13} + f_{68})\,, \\
&D = f_{18} \otimes f_{45}\,, && \overline{D} = f_{45} \otimes f_{18}\,, \\
&E = f_{27} \otimes f_{36}\,, && \overline{E} = f_{36} \otimes f_{27}\,.
\end{aligned}$$

Now we have the invariant as a vector of $\bigwedge^2 V \otimes \bigwedge^2 V$

$$\phi_1 = A \otimes A + 2[B + \overline{B} + C + \overline{C}] + 4[D + \overline{D} + E + \overline{E}].$$

6. – Some geometry.

We want to give an interpretation in terms of projective geometry of the invariants ϕ_1, ϕ_2, ϕ_3. Let $Gr(\mathbb{C}^2, V)$ be the grassmanian of 2-planes in V, i.e. straight lines in $\mathbb{P}(V)$, with the Plücker map

$$Gr(1, \mathbb{P}(V)) \hookrightarrow P\Big(\bigwedge^2 V\Big), \qquad \langle v_1, v_2 \rangle \mapsto v_1 \wedge v_2\,.$$

We take now the symplectic vector space (V, E) with the symplectic basis of Section 3 and the Segre-like map

$$X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\psi} \mathbb{P}^7 = \mathbb{P}(V),$$

$$((x_1 : x_2), (y_1 : y_2), (z_1 : z_2)) \mapsto (\ldots : x_i y_j z_k : \ldots).$$

PROPOSITION 6.1. – *For any* $\{\underline{x}\}$, $\{\underline{y}\} \in \mathbb{P}^1$ *the rational curve given by* $\psi(\{\underline{x}\} \times \{y\} \times \mathbb{P}^1)$ *is an isotropic straight line in* $\mathbb{P}^7$ *w.r.t. the symplectic form E and this is also true for the curves* $\psi(\{\underline{x}\} \times \mathbb{P}^1 \times \{\underline{z}\})$ *and* $\psi(\mathbb{P}^1 \times \{\underline{y}\} \times \{z\})$.

PROOF. – We write a generic point of the first family in a suitable way

$$((x_1 : x_2), (y_1, y_2), (s : t)) \mapsto s(\ldots : x_2 y_2 : 0 \ldots : 0) + t(\ldots : 0 : x_1 y_1 : \ldots)$$

and we make computations using the bilinearity of E. The second assertion follows by symmetry. ■

Each of the above families can be considered as a $\mathbb{P}^1 \times \mathbb{P}^1$ in $Gr(1, \mathbb{P}^7)$, we denote them with Q_1, Q_2, Q_3.

Now we use the Plücker map to see the relations between the Q_i's and the irreducible representations $W_{a,b,c}$ in P_2. As usual, we work with Q_1. From the results above follows

PROPOSITION 6.2. – *The image of* $Q_1 \subseteq Gr(1, \mathbb{P}^7)$ *under the Plücker map is contained in* $\mathbb{P}(W_{2,2,0}) \subseteq P_2 \subseteq \mathbb{P}(\bigwedge^2 V)$.

We use now the Veronese and the Segre map to clarify the geometric situation

$$Q_1 = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8 = \mathbb{P}(W2, 2, 0) \subseteq \mathbb{P}(\bigwedge^2 V),$$

$$((x_1 : x_2), (y_1 : y_2)) \mapsto ((x_1^2 : ..), (y_1^2 : ..)) \mapsto (.. : x_i x_j y_k y_l : ..),$$

$$((.. : v_i : ..)(.. : w_j : ..)) \mapsto (.. : v_i w_j : ..).$$

Consider now the induced restrictions

$$H^0(\mathbb{P}^8, \mathcal{O}(1)) \to H^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(2, 2)) \to H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(4, 4))$$

and the subvariety of $\mathbb{P}^2 \times \mathbb{P}^2$

$$(C_1 \times \mathbb{P}^2) \cup (\mathbb{P}^2 \times C_2) := (v_1 v_3 - v_2^2)(w_1 w_3 - w_2^2) = 0\,.$$

Note that $v_1 v_3 - v_2^2 \in S^2(W_{2,2,0})$ corresponds to the Killing form. Then

$$\phi_1 := (v_1 v_3 - v_2^2)(w_1 w_3 - w_2^2) = 0 \in S^2(W_{2,2,0})$$

is the equation for a variety in $\mathbb{P}^8$ which cuts $\mathbb{P}^2 \times \mathbb{P}^2$ in $(C_1 \times \mathbb{P}^2) \cup (\mathbb{P}^2 \times C_2)$.

REFERENCES

[DMOS] P. DELIGNE - J. S. MILNE - A. OGUS - K. SHIH, *Hodge Cycles, Motives and Shimura Varieties,* LNM 900, Springer-Verlag (1982).

[F-H] W. FULTON-J. HARRIS, *Representation Theory*, GTM 129, Springer-Verlag (1991).

[Haz] F. HAZAMA, *Algebraic cycles on certain abelian varieties and powers of special surfaces*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **31** (1984), 487-520.

[Ga] F. GALLUZZI, *Hodge structure of an abelian fourfold of Mumford-type*, preprint.

[G] B. B. GORDON, *A survey of the Hodge conjecture for Abelian Varieties*, Duke preprint alg-geom 9709030, to appear in the second edition of «A survey of the Hodge conjecture» by James D. Lewis.

[Mu1] D. MUMFORD, *Families of abelian varieties*, in *Algebraic Groups and Discontinuous Subgroup*, Proc. Sympos.Pure Math., **9**, Amer. Math. Soc., Providence, R.I. (1966), 347-351.

[Mu2] D. MUMFORD, *A note of Shimura's Paper «Discontinuous Groups and Abelian Varieties»*, Math. Ann., **181**, (1969), 345-351.

[vG] B. VAN GEEMEN, *An introduction to the Hodge Conjecture for abelian varieties*, Algebraic cycles and Hodge Theory, Torino 1993 Lect. Notes in Math. 1594, Springer, Berlin, etc., (1994), 233-252.

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