Bollettino Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **3-B** (2000), n.1, p. 79–94.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2000_8_3B_1_79_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2000.

Bollettino U. M. I.
(8) **1**-B (2000), 79-94

Some Results on Invariant Measures in Hydrodynamics.

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Sunto. – *In questa nota, si presentano risultati di esistenza e di unicitá di misure invarianti per l'equazione di Navier-Stokes che governa il moto di un fluido viscoso incomprimibile omogeneo in un dominio bidimensionale soggetto a una forzante che ha due componenti: una deterministica e una di tipo rumore bianco nella variabile temporale.*

The Navier-Stokes equations describe the motion of a viscous incompressible fluid. The treatment of the equations varies according to the dimension of the space. Let us consider a bounded domain of $\mathbb{R}^2$: in this case a unique solution is defined for arbitrary large time. Therefore, we can investigate the long time behaviour of the solution. This corresponds to the practical problem of determining which permanent regime will be observed after a short transient initial period. It is known that if the viscosity and the forcing are suitably related (so that the Reynolds number is small), the set of stationary solutions reduces to one point and the system converges to this equilibrium configuration. Otherwise, for too high forcing term or too low viscosity, there are many stationary solutions and asymptotically the system has a more complex behaviour. We say that the motion is turbulent. However, analysing statistically the results of numerical simulations or of experimental measurements, it has been observed that after an initial transient time the averages of some observables become constant. This means that an equilibrium in the statistical sense has been attained.

We keep in mind this summary on the classical 2D Navier-Stokes equations, to compare them with the matter of our study: the 2D stochastic Navier-Stokes equations (for the 1D case, i.e. the Burgers'equation, the statistics of the motion has been computed (cf. [3], [18]). The related mathematical results have been proven in [4], [6]). By this we mean the Navier-Stokes system, where the momentum conservation equation is perturbed by a white noise term. In § 2, we shall show that a well defined generalized solution exists for arbitrary large time. Even the long time behaviour of the system is well defined: there exists a unique invariant measure and the convergence takes

(*) Comunicazione presentata a Napoli in occasione del XVI Congresso U.M.I.

place, that is this measure represents the asymptotic equilibrium behaviour of the system. This is the subject of § 4. Roughly speaking, invariant measures play a role similar to that of the stationary solutions of the deterministic case. They are good candidates to represent the asymptotic behavior of the system. But, stationary solutions are in general not unique. Adding a suitable random perturbation, we get existence and uniqueness of an invariant measure (existence is hence obtained by means of the dissipativity property peculiar to the Navier-Stokes equations, uniqueness is strictly related to the white noise).

Let us point out that both the viscosity and the noise are necessary to obtain that there exists a unique invariant measure. But restriction neither on the viscosity nor on the forcing term will appear. On the other hand, when the viscosity vanishes and there are no forcing terms, the equation reduces to the Euler case without forcing term. This is a conservative system, for which the existence of an invariant measure has been proven in [1].

Let us summarize the contents of this note.

In § 1, there will be introduced the basic spaces and operators in order to give the abstract formulation (2) of our problem. § 2 will deal with existence and uniqueness of solutions of equation (2), while regularity results will be presented in § 3. Finally, invariant measures will be studied in § 4.

1. – Mathematical setting.

We consider a viscous homogeneous incompressible fluid in a bounded domain $\mathcal{D}$ of $\mathbb{R}^2$ with smooth boundary $\partial\mathcal{D}$. The motion of the fluid in the time interval $[t_0, T]$ is governed by the Navier-Stokes equations

$$\begin{cases} \dfrac{\partial u(t,\xi)}{\partial t} - \Delta u(t,\xi) + (u(t,\xi)\cdot\nabla)\, u(t,\xi) + \nabla p(t,\xi) = f(t,\xi) + n(t,\xi) \\ \operatorname{div} u(t,\xi) = 0 \end{cases} \tag{1}$$

where $u(t,\xi) = \{u_1(t,\xi), u_2(t,\xi)\}$ and $p(t,\xi)$ denote respectively the velocity and pressure fields of the fluid for $\xi \in \mathcal{D}$ and $t \in [t_0, T]$; $\nabla =$ grad and $\Delta =$ Laplacian w.r.t. the space variable ξ. The viscosity coefficient is considered to be equal 1, without loss of generality. Actually our (asymptotic) results do not depend on it whereas they do in the deterministic case. In the right-hand side there are two forcing terms: a deterministic component f and a stochastic one n, white noise in time. They can be understood as an average value and a fluctuation rapidly varying in time, respectively.

To equation (1) we associate the homogeneous boundary condition

$$u(t,\xi) = 0 \qquad \text{for } t \in [t_0, T],\ \xi \in \partial\mathcal{D}$$

and the initial condition

$$u(t_0, \xi) = u_{t_0}(\xi) \qquad \text{for } \xi \in \mathcal{D}.$$

We now introduce some tools to define the weak formulation corresponding to equation (1) with these conditions (cf. [24], [25]). Let $\mathbb{L}^2 = [L^2(\mathcal{D})]^2$, $\mathbb{H}^\alpha = [H^\alpha(\mathcal{D})]^2$. We define

$$\mathcal{V} = \{u \in [C_0^\infty(\mathcal{D})]^2 \colon \text{div } u = 0\}$$

and take the closure of this space in $\mathbb{L}^2$ and $\mathbb{H}^1$; we obtain respectively

$$H = \{u \in \mathbb{L}^2 \colon \text{div } u = 0 \text{ in } \mathcal{D},\, u \cdot n = 0 \text{ on } \partial\mathcal{D}\}$$

$$V = \{u \in \mathbb{H}_0^1 \colon \text{div } u = 0 \text{ in } \mathcal{D}D\}$$

where n is the outer normal to $\partial\mathcal{D}$. The space H is equipped with the scalar product $(\cdot, \cdot)$ induced by $\mathbb{L}^2$ and we denote by $|\cdot|$ the norm in H; the space V is a Hilbert space with the scalar product $((u, v)) = \sum_{i=1}^{2} (D_i u, D_i v)$, since $\mathcal{D}$ is bounded.

Denoting by H' and V' the dual spaces, if we identify H with H' we get the following continuous embeddings $V \subset H \subset V'$, where each space is dense in the following one.

Let now Π be the orthogonal projector from $\mathbb{L}^2$ onto H and define the Stokes operator A as

$$Au = -\Pi \Delta u\,, \qquad \forall u \in D(A) = \mathbb{H}^2 \cap V.$$

The operator A is a linear closed positive unbounded self-adjoint operator with discrete spectrum in H; $\{e_i\}_{i=1}^\infty$ is the sequence of its eigenvectors corresponding to the eigenvalues $\{\lambda_i\}_{i=1}^\infty$ $(0 < \lambda_1 \leq \lambda_2 \leq \ldots)$. For any $\alpha \in \mathbb{R}$, the Hilbert space $D(A^\alpha)$ can be characterized by

$$D(A^\alpha) = \left\{ u = \sum_{i=1}^\infty u_i e_i \colon \sum_{i=1}^\infty \lambda_i^{2\alpha} u_i^2 < +\infty \right\}.$$

The operator A is an isomorphism from $D(A^\alpha)$ onto $D(A^{\alpha-1})$. In particular $V = D(A^{1/2})$, $V' = D(A^{-1/2})$ and $H = D(A^0)$. $-A$ generates in H a bounded analytic semigroup e^{-tA} of class C_0.

Consider, now, the bilinear operator B defined as

$$\langle B(u, v), z\rangle = \int_{\mathcal{D}} [(u \cdot \nabla)\, v] \cdot z \, d\xi\,,$$

where $\langle \cdot, \cdot \rangle$ denotes duality paining between $\mathbb{H}^{-\alpha}$ and $\mathbb{H}^\alpha$.

Often, we shall write for short $B(x)$ instead of $B(x, x)$.

The main point in the analysis of our problem is to define the spaces on which B acts as a bilinear continuous operator. An easy case is $B : D(A^{1/4}) \times D(A^{1/4}) \to D(A^{-1/2})$. Indeed

$$|\langle B(u, v), z\rangle| \leqslant |u|_{\mathbb{L}^4} |\nabla v|_{\mathbb{L}^2} |z|_{\mathbb{L}^4} \leqslant C|A^{1/4}u| |A^{1/2}v| |A^{1/4}z|$$

since $\dim \mathcal{D} = 2$.

Other cases will be presented later on, when needed to estimate the non linear term.

Moreover,

$$\langle B(u, v), z\rangle = -\langle B(u, z), v\rangle$$

whenever both sides make sense.

We project equation (1) onto H, assuming for semplicity $\Pi f = f$. Since $\mathbb{L}^2 = H \oplus G$, where $G = \{u \in \mathbb{L}^2 : \exists \phi \in L^2(\mathcal{D}) \text{ s.t. } u = \nabla\phi\}$, we get the abstract formulation

$$\begin{cases} du(t) + [Au(t) + B(u(t))]\, dt = f(t)\, dt + G\, dw(t) \\ u(t_0) = u_{t_0} \end{cases} \tag{2}$$

to be understood as

$$u(t) + \int_{t_0}^{t} [Au(s) + B(u(s))]\, ds = u_{t_0} + \int_{t_0}^{t} f(s)\, ds + G[w(t) - w(t_0)].$$

We assume that $w(t)$ is a cylindrical Wiener process in H defined for all real t on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with filtration $\mathcal{F}_t = \sigma\{w(s) - w(\tau) : \tau \leqslant s \leqslant t\}$ (so that the Wiener process is adapted to the filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$ and, for any $t > s$, the increments $w(t) - w(s)$ are independent of $\mathcal{F}_s$). G is a linear continuous operator in H, satisfying further assumptions we shall specify later.

Because of the white noise term, we get a weak solution less regular than in the deterministic case, as we shall see in the next section. For this reason, the following definition of solution for (2) is given

DEFINITION 1.1. – *A stochastic process $u(t, \omega)$ is a generalized solution in $[t_0, T]$ of equation* (2) *if*

$$u(\cdot, \omega) \in C([t_0, T]; H) \cap L^2(t_0, T; D(A^{1/4}))$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$, u is progressively measurable in these topologies and equa-

tion (2) *is satisfied* $\mathbb{P}$-*a.s. in the integral sense*

$$\langle u(t), \phi\rangle + \int_{t_0}^{t} \langle u(s), A\phi\rangle\, ds - \int_{t_0}^{t} \langle B(u(s), \phi), u(s)\rangle\, ds =$$

$$\langle u_{t_0}, \phi\rangle + \int_{t_0}^{t} \langle f(s), \phi\rangle\, ds + \langle w(t) - w(t_0), G^* \phi\rangle$$

for all $t \in [t_0, T]$ *and all* $\phi \in D(A)$.

REMARK. – The above relation corresponds to (2) and all the terms in it make sense. In fact

$$|\langle B(u(s), u(s)), \phi\rangle| = |\langle B(u(s), \phi), u(s)\rangle| \leq C|A^{1/2}\phi|\,|A^{1/4}u(s)|^2.$$

Moreover, given $G \in L(H)$, the last term $\langle w(t) - w(t_0), G^* \phi\rangle$ is a well defined family of random variables for $t \in [t_0, T]$ (we refer to [8], [7] for any results on stochastic processes).

2. – Existence and uniqueness of solutions.

We are interested in defining a stochastic dynamics such that the probability transition functions or, equivalently, the associated Markovian semigroup are well defined. In this way, the evolution of the process solution's law is represented by the adjoint semigroup. Later on, properties of the Markovian semigroup will be asked for, in order to face the problem of existence and uniqueness of the invariant measure.

Following [2], instead of considering the non linear stochastic equation (2), we divide our problem into two parts: a linear Itô equation

$$dz_a(t) + Az_a(t)\, dt + az_a(t)\, dt = Gdw(t) \qquad \forall a \geq 0 \text{ real} \tag{3}$$

and for $v_a := u - z_a$

$$\frac{dv_a(t)}{dt} + Av_a(t) + B(v_a(t) + z_a(t)) = az_a(t) + f(t) \tag{4}$$

which is a random, non linear equation but is studied pathwise by means of deterministic methods.

In the following Proposition, we collect the main properties of the Ornstein-Uhlenbeck process $z_a(t) = \int_{-\infty}^{t} e^{-(t-s)(A+a)} Gdw(s)$, which is a stationary ergodic solution of equation (3) (for the proof, cf. [14], [13]. The first part is a standard result on stochastic convolution)

PROPOSITION 2.1. – *Assume that G is a linear bounded operator in H such that*

$$\mathcal{R}(G) \subseteq D(A^{\beta+\varepsilon})$$

for some $\varepsilon > 0$ *and* β; *hence for every* $a \geqslant 0$, z_a *is a continuous stationary Gaussian process on* $D(A^\beta)$. *Moreover, there exist* $\mathbb{P}$*-a.s. finite random variables* C' *and* C'' *such that* $\mathbb{P}$*-a.s.*

$$|z_a(t)|_{D(A^\beta)} \leqslant C' + C''|t|$$

for all $t \leqslant 0$.

Finally

$$\lim_{a \to +\infty} \mathbb{E}|z_a(t)|^4_{D(A^\beta)} = 0\,.$$

For the second auxiliary equation, methods based on Galerkin procedure can be used, in a way similar to the study of the deterministic Navier-Stokes equation. We have

PROPOSITION 2.2. – *Let* $[t_0, T]$ *be a given finite time interval and fix any real* $a \geqslant 0$.

Assume that the continuous operator G in H has range $\mathcal{R}(G) \subseteq D(A^{1/4+\varepsilon})$ *for some* $\varepsilon > 0$. *Then, for arbitrary* $u_{t_0} \in H$, $f \in L^2(t_0, T; D(A^{-1/2}))$, *there exists a unique solution* v_a *of equation* (4) *with initial condition* $v_a(t_0) = u_{t_0} - z_a(t_0)$ *and such that*

$$v_a(\cdot, \omega) \in C([t_0, T]; H) \cap L^2(t_0, T; D(A^{1/2})) \qquad \mathbb{P}\text{-a.e. } \omega \in \Omega$$

PROOF. – From now on, $\omega \in \Omega$ is fixed; all the results for v_a are obtained pathwise. We multiply equation (4) by v (avoiding the subscript a) and integrate in $\mathcal{O}$

$$\frac{1}{2}\frac{d}{dt}|v|^2 + |A^{1/2}v|^2 = \left\langle \frac{dv}{dt}, v \right\rangle + \langle Av, v\rangle =$$

$$-\langle B(v+z, v+z), v\rangle + a\langle z, v\rangle + \langle A^{-1/2}f, A^{1/2}v\rangle =$$

$$+\langle B(v+z, v), z\rangle + a\langle z, v\rangle + \langle A^{-1/2}f, A^{1/2}v\rangle \leqslant$$

$$C|A^{1/4}(v+z)||A^{1/2}v||A^{1/4}z| + a|z||v| + |A^{-1/2}f||A^{1/2}v|$$

$$\leqslant C_1|v|^{1/2}|A^{1/2}v|^{3/2}|A^{1/4}z| + C_2|A^{1/4}z|^2|A^{1/2}v| + a|z||v| + |A^{-1/2}f||A^{1/2}v|$$

By means of Young's inequality and Gronwall's lemma, we obtain

$$\sup_{t\in[t_0,T]} |v(t)| < +\infty \qquad \int_{t_0}^{T} |A^{1/2}v(s)|^2 ds < +\infty .$$

We conclude first that if the solution v exists, then it belongs to $L^\infty(t_0, T; H) \cap L^2(t_0, T; D(A^{1/2}))$.

To prove existence of such a solution, the Galerkin method is used (cf., e.g., [24]). We consider the n-finite dimensional equation associated to (4), obtained projecting equation (2) onto span$\{e_1, \dots, e_n\}$; the solution v_n exists globally in time and the same a priori estimates as above are valid. Hence, the sequence $\{v_n\}$ remains in a bounded set of $L^\infty(t_0, T; H)$ and $L^2(t_0, T; D(A^{1/2}))$ and we pass to the limit as $n\to\infty$ considering the weak convergence (of a subsequence) in $L^\infty(t_0, T; H)$ and $L^2(t_0, T; D(A^{1/2}))$; moreover a strong convergence is necessary to consider the limit of the non linear term. But $v_n' = -Av_n - \Pi_n B(v_n + z_n) - \Pi_n f$. Thus, if $v_n \in L^\infty(t_0, T; H) \cap L^2(t_0, T; D(A^{1/2}))$, then $v_n' \in L^2(t_0, T; D(A^{-1/2}))$. Hence, there is strong convergence in $L^2(t_0, T; H)$, because $H^1(t_0, T; D(A^{-1/2})) \cap L^2(t_0, T; D(A^{1/2}))$ is compactly embedded in $L^2(t_0, T; H)$.

Concerning the continuity, a general theorem of interpolation says that if $v \in L^2(t_0, T; D(A^{1/2}))$ and $v' \in L^2(t_0, T; D(A^{-1/2}))$, then v is almost everywhere equal to a function continuous from $[t_0, T]$ into H and $(d/dt)|v(t)|_H^2 = 2\langle v'(t), v(t)\rangle$ in the distributional sense on $[t_0, T]$. Therefore the initial condition makes sense.

Concerning the uniqueness, we proceed in a classical way; if $v^{(1)}$ and $v^{(2)}$ are two solutions, consider the difference $V = v^{(1)} - v^{(2)}$. It satisfies

$$\frac{d}{dt}V + AV + B(v^{(1)} + z) - B(v^{(2)} + z) = 0$$

with $V(t_0) = 0$.

If we multiply this equation by V and integrate

$$\frac{1}{2}\frac{d}{dt}|V|^2 + |A^{1/2}V|^2 = -\langle B(v^{(1)} + z), V\rangle + \langle B(v^{(2)} + z), V\rangle \leqslant$$

$$\frac{1}{2}|A^{1/2}V|^2 + C|V|^2\,|A^{1/4}(v^{(1)} + z)|^4.$$

Then, by the usual method of Gronwall's inequality

$$|V(t)| = 0 \qquad \forall t \in [t_0, T].$$

We conclude that the solution v_a of system (4) is unique for any given $a \geqslant 0$.

The proof of this Proposition is achieved. ■

Therefore, merging the regularity of v_a and z_a, we have proved

THEOREM 2.3. – *Assume that the continuous operator G in H has range $\mathcal{R}(G) \subseteq D(A^{1/4+\varepsilon})$ for some $\varepsilon > 0$. Then, for each time interval $[t_0, T]$ and for arbitrary $u_{t_0} \in H$, $f \in L^2(t_0, T; D(A^{-1/2}))$, there exists a generalized solution u over $[t_0, T]$ of equation* (2) *such that*

$$u(\cdot, \omega) \in C([t_0, T]; H) \cap L^2(t_0, T; D(A^{\min\{1/4+\varepsilon', 1/2\}})) \qquad \mathbb{P}\text{-a.e. } \omega \in \Omega$$

for any $0 < \varepsilon' < \varepsilon$; only one of such solutions u satisfies the further property

$$u(t) - \int_{-\infty}^{t} e^{-(t-s)A} G dw(s) \in L^2(t_0, T; D(A^{1/2})) \qquad \mathbb{P}\text{-a.s.}$$

From now on, we shall refer to this solution u as to the canonical solution. Moreover, the process u is a Markovian process satisfying the Feller property in H.

Now, it is clear why we asked the *generalized* solution u to be in $L^2(t_0, T; D(A^{1/4}))$ instead of the *classical* $L^2(t_0, T; D(A^{1/2}))$. $\mathcal{R}(G) \subseteq D(A^{1/4+\varepsilon})$ $(\exists \varepsilon > 0)$ is the minimal assumption on the noise providing the existence of such a generalized solution, that is of a dynamics for the stochastic Navier-Stokes equation in the space H. This assumption can be even weakened to be $\mathcal{R}(G) \subseteq D(A^{\varepsilon})$ $(\exists \varepsilon > 0)$ in the case of periodic boundary condition (cf. [13]).

But, this solution is not regular enough to be unique (the technique used to prove uniqueness of v fails for u). This is why a (unique) canonical solution has to be introduced.

We refer to [13] for the remaining properties of the solution u, which are based on classical techniques for stochastic processes.

3. – Regular solutions.

As in the deterministic case, we can study the regularity of the solution u depending on the regularity of the data u_{t_0}, f and G.

As far as the Ornstein-Uhlenbeck equation is concerned, regularity results are provided by well-known methods (cf. statement of Proposition 2.1).

For the non linear equation (4), as a first attemp, a similar procedure to that of Proof 2.2 might be used. This means to multiply equation (4) by $A^{2\alpha} v$.

The delicate point is in dealing with the non linear part, which would read

$$\langle A^{-1/2+\alpha} B(v+z), A^{1/2+\alpha} v \rangle . \tag{5}$$

From [17] we know that, for $0 < \alpha < 1/2$

$$|\langle A^{-1/2+\alpha} B(v+z), A^{1/2+\alpha} v \rangle| \leqslant C |A^{1/4+\alpha/2}(v+z)|^2 |A^{1/2+\alpha} v|$$

and therefore it can be proved, as in the previous section, the following Theorem (cf. [13] for the proof)

THEOREM 3.1. – *Given $\alpha \in (0, (1/2))$, assume that $\mathcal{R}(G) \subseteq D(A^{1/4+\alpha/2+\varepsilon})$ for some $\varepsilon > 0$. Then, for each time interval $[t_0, T]$ and $u_{t_0} \in D(A^{\alpha})$, $f \in L^2(t_0, T; D(A^{-1/2+\alpha}))$, there exists a unique generalized solution u over $[t_0, T]$ of equation (2) such that $\mathbb{P}$-a.s.*

$$u \in C([t_0, T]; D(A^{\alpha})) \cap L^{4/(1-2\alpha)}(t_0, T; D(A^{1/4+\alpha/2}))$$

and

$$u(t) - \int_{-\infty}^{t} e^{-(t-s)A} G dw(s) \in L^2(t_0, T; D(A^{1/2+\alpha})) .$$

It is a Markovian process satisfying the Feller property in $D(A^{\alpha})$.

For higher values of α, care has to be taken because of the boundary condition. Indeed, (5) makes sense only if $\alpha < 3/4$. In fact, recall that $B(u, v) = \Pi[(u \cdot \nabla) v]$; then $B(u, v)$ doesn't satisfy the Dirichlet boundary condition. And we can apply $A^{-1/2+\alpha}$ in front of B as long as $-1/2 + \alpha < 1/4$, i.e. whenever the space $D(A^{-1/2+\alpha})$ includes condition only on the normal component on the boundary.

For such values of the index α, but not included in the above Theorem, the bilinear part can be estimated by Sobolev embedding techniques.

On the other hand, for $\alpha \geqslant 3/4$, the space regularity of the process v (and thus of u) can be obtained seeking time-space regularity of the paths of v. In [23] Temam proceeds in this way for the deterministic equation. In our case, the presence of randomness leads to state new necessary conditions (cf. for the linear case [15]).

Summing up, we have stated only the assumptions providing the (pathwise) dynamics of equation (2) to be well defined in the space $D(A^{\alpha})$ for any $\alpha \in [0, (1/2))$. This will be needed in the next section to prove uniqueness of invariant measure. Indeed, the basic case $\alpha = 0$ will not be enough for this.

But similar results can be obtained for $\alpha \geqslant 1/2$. More precisely, a limitation arises in the case of Dirichlet boundary condition (cf. [12]). But dealing with the periodic case, the same procedure as in the previous section is successful as showed in [11] (let us remark that, the higher α is the easier and plainer is the estimate of the non linear part. Indeed, $B : \mathbb{H}^m \times \mathbb{H}^{m+1} \to \mathbb{H}^m$ for m integer ≥ 2, because $H^m(\mathcal{D})$ is a multiplicative algebra for $m \geq 2$ if dim $\mathcal{D} = 2$).

4. – Invariant measures.

We have proved that the stochastic Navier-Stokes equation is a well posed problem: for each initial data u_{t_0} there exists a unique canonical solution $u(t; u_{t_0})$ for $t \in [t_0, T]$. This allow us to introduce the probability transition functions $P(t - t_0, x, \Gamma) = \mathbb{P}\{u(t; u_{t_0} = x) \in \Gamma\}$ or equivalently the Markov semigroup $(P_s \phi)(x) = \int \phi(y)\, P(s, x, dy)$. The dual semigroup $\{P_s^\star\}_{s>0}$ governes the evolution in time of the law of $u(t)$. Hence, a probability measure μ is said to be invariant if $P_s^\star \mu = \mu$ for all $s > 0$.

From now on, f is assumed to be independent of time; therefore, both the forcing terms are stationary in time, since $n(t) = \partial w(t)/\partial t$ is a generalized stationary process.

We shall show that, under quite general assumptions, there exists an invariant measure and moreover this is unique, under more restrictive assumptions.

For the *existence* result we exploit the dissipativity property in order to get the existence of an invariant measure for the Navier-Stokes equation with a random perturbation. Our existence result is based on Prohorov's Theorem and Krylov-Bogoliubov's method. Actually, we have to check the tightness property in order to obtain that there exists a (sub-)sequence $\{\mu_n\}$ of measures converging to a limit measure μ; then this μ is invariant.

The *uniqueness* of the invariant measure is obtained by providing that the Markovian semigroup P_t is irreducible and strongly Feller. In fact by Doob's Theorem (cf. [9]), there exists at most one invariant measure if all the measures $P(t, x, \cdot)$ and $P(s, y, \cdot)$ are equivalent, that is mutually absolutely continuous, for arbitrary time $s, t > 0$ and initial data x, y. And a sufficient condition for this assumption is that the Markovian semigroup P_t is irreducible and strongly Feller (cf. [20], [22]). So, we shall first investigate separately these two properties, and later on choose a proper setting in which both hold.

Finally, putting together the results independently obtained, we get that there exists a unique invariant measure, which is the limit of the distribution law as $T \to \infty$.

4.1. *Existence.*

The Navier-Stokes equation is *dissipative* in the sense that starting from a point in H, the system will live not in the whole space H but in a smaller subspace (depending on the regularity of the forcing terms).

For the family of random variables $\{u_{(t_0)}(0, \omega): \omega \in \Omega\}_{t_0 \leq 0}$, an estimate uniform in t_0 is required. This is stronger than the result of Theorem 2.3. We have (cf. the very technical proof in [14], [13])

LEMMA 4.1. – *Denote by* $u_{(t_0)}(\cdot)$ *the solution of the equation* (2) *over* $[t_0, \infty)$ *with the initial condition* $u_{t_0} = 0$. *Assume that* $\mathcal{R}(G) \subseteq D(A^{1/4 + 2\varepsilon})$ *for some* $\varepsilon > 0$ *and* $f \in D(A^{-1/2 + \delta})$ *for some* $\delta > 0$. *Then there exists a real random variable* $r(\omega)$ ($\mathbb{P}$*-a.s. finite*) *such that*

$$\sup_{-\infty < t_0 \leq 0} |A^{\gamma} u_{(t_0)}(0, \omega)| \leq r(\omega)$$

for some $\gamma > 0$, *for* $\mathbb{P}-$ *a.e.* $\omega \in \Omega$.

Therefore, we get the *tightness* property in H for the family of laws of the random variables $\{u_{(t_0)}(0, \cdot)\}_{t_0 \leq 0}$. Indeed, if we put $\Omega_N := \{\omega : r(\omega) \leq N\}$ and notice that

$$\Omega_N \subset \left\{\omega : \sup_{-\infty < t_0 \leq 0} |A^{\gamma} u_{(t_0)}(0, \omega)| \leq N\right\}$$

then, choosing any $\varepsilon > 0$ there exists N_ε such that

$$1 - \varepsilon < \mathbb{P}(\Omega_{N_\varepsilon}) < \mathbb{P}\left\{\sup_{-\infty < t_0 \leq 0} |A^{\gamma} u_{(t_0)}(0, \omega)| \leq N_\varepsilon\right\}.$$

Since the embedding $D(A^{\gamma}) \subset H$ is compact for $\gamma > 0$, it follows that the family of laws of the random variables $u_{(t_0)}(0, \cdot)$ is tight, i.e.

$$\forall \varepsilon > 0 \;\; \exists K_\varepsilon \text{ compact in } H \text{ such that}$$

$$\mathbb{P}\{u_{(t_0)}(0, \omega) \in K_\varepsilon\} > 1 - \varepsilon \qquad \text{for all } t_0 \leq 0 .$$

Noticing that $u_{(t_0)}(0, \cdot)$ and $u_{(0)}(-t_0, \cdot)$ have the same law, we get tightness also for $\left\{\frac{1}{\tau}\int_0^\tau u_{(0)}(s, \cdot)\, ds\right\}_{\tau > 0}$. Since P_t is Feller, we have proved the following

THEOREM 4.2. – *The stochastic problem* (2) *has an invariant measure.*

4.2. *Uniqueness.*

As already explained, we shall look for irreducibility and strong Feller property. We refer to [13], [16] for any details of the proof.

We begin with studying *irreducibility.*

We say that P_t is irreducible in E if $P(t, x, \Gamma) > 0$ for any $t > 0$, $x \in E$, Γ open non empty subset of E.

Let us define the mapping

$$\Phi : X_z \to X_u, \qquad z \mapsto u = v + z$$

where z is the Ornstein-Uhlenbeck process solution of

$$\begin{cases} dz(t) + Az(t)\,dt = Gdw(t) \\ z(0) = 0 \end{cases} \tag{6}$$

Here we consider the system on a time interval $[0, T]$ for any $T > 0$; then z is a centered Gaussian process.

We give now a general outline of the proof of the irreducibility in order to point out what is important to know about the map Φ. Keeping in mind that by definition $P(t, x, \Gamma) = \mathbb{P}\{u(t; x) \in \Gamma\}$, it is enough to show that

$$\mathbb{P}\{u(\cdot, \omega) \in \mathcal{U}_\varrho\} \geqslant \mathbb{P}\{z(\cdot, \omega) \in \mathcal{Z}_{\delta_\varrho}\} > 0$$

where $\mathcal{U}_\varrho$, $\mathcal{Z}_{\delta_\varrho}$ are open sets in suitable spaces that we shall specify later on (δ_ϱ means that, given ϱ, $\mathcal{Z}_{\delta_\varrho}$ has to be chosen as a function of $\mathcal{U}_\varrho$).

Therefore, the law $\mathcal{L}(z)$ has to be a full measure on a suitable space in order to get the latter inequality. Sufficient conditions for this (linear) case are presented, e.g., in [19]. The first one is true if (proceeding pathwise), given $\overline{u}$, there exists a suitable $\overline{z}$ such that if z is in a ball $\mathcal{Z}_{\delta_\varrho}(\overline{z})$ of center $\overline{z}$ and radius δ_ϱ, then u belongs to a ball $\mathcal{U}_\varrho(\overline{u})$ of center $\overline{u}$ and radius ϱ in suitable topologies.

Therefore the spaces X_z and X_u have to be properly defined. In order to prove the validity of the first inequality the following three steps are checked (for any given $\alpha \in [0, (1/2))$)

1) $\Phi : C_0([0, T]; D(A^\alpha)) \cap L^{4/(1-2\alpha)}(0, T; \quad D(A^{1/4+\alpha/2})) \to C([0,T]; \quad D(A^\alpha))$ is well defined

2) given $\overline{u} \in C([0, T]; D(A^\alpha))$, there exists $\overline{z} \in C_0([0, T]; D(A^\alpha)) \cap L^{4/(1-2\alpha)}(0, T; D(A^{1/4+\alpha/2}))$ such that $\overline{u} = \Phi(\overline{z})$

3) Φ is continuous in the assigned topologies.

Concerning the other property: P_t is *strong Feller* in E if $P_t : B_b(E) \to C_b(E)$, $\forall t > 0$.

We have to consider intermediate auxiliary equations. We are seeking the property in the space $D(A^\alpha)$, i.e.

if $\|x - y\|_{D(A^\alpha)} \to 0$,

then $|(P_t\psi)(x) - (P_t\psi)(y)| \to 0$ for arbitrary $t > 0$, $\psi \in B_b(D(A^\alpha))$.

By the mean value Theorem, we get that the Markovian semigroup $\{P_t\}$ is

Lipschitz Feller if we are able to estimate the derivative of $P_t \psi$. Regularizing effect of P_t is due to the fact that $P_t \psi$ is the solution of a parabolic equation, i.e. the Kolmogorov equation associated to the SDE (2). The proof is pretty technical and starts from a representation for this quantity given by the Elworthy's formula (cf. [10], [5]), which holds for the n-finite dimensional Galerkin system associated to (2). The necessary estimate is obtainable only by modifying this equation by a cut-off function Θ_R in front of the non linear term B, i.e.

$$(7) \qquad \begin{cases} du^{(R)}(t) + Au^{(R)}(t)\, dt + \Theta_R(\|u^{(R)}(t)\|^2_{D(A^\alpha)})\, B(u^{(R)}(t))\, dt = f\, dt + G\, dw(t) \\ u^{(R)}(0) = x \end{cases}$$

where for $R \in (0, \infty)$ the cut-off function Θ_R is a C^∞ function equal to 1 in $[-R, R]$ and 0 outside $[-R-1, R+1]$; if $R = \infty$, then $\Theta \equiv 1$ and (7) reduces to (2).

Passing to the limit as $n \to \infty$ and $R \to \infty$, we shall obtain the strong Feller property for the principal system (2) in the space $E = D(A^\alpha)$ for arbitrary $\alpha \in [(1/4), (1/2))$, exploiting Theorem 3.1 (of course, easy generalizations hold for $\alpha \geqslant 1/2$ in the periodic boundary condition case). For technical reasons, the restriction $\alpha \geqslant 1/4$ has to be introduced.

Keeping in mind the procedure sketched at the beginning of this section, we get uniqueness of the invariant measure. For this, strong assumptions on the noise are required. Indeed, for any fixed $\alpha \in [(1/4), (1/2))$, it has to be assumed

[**H**] $G: H \to H$ is a linear bounded operator, injective, with range $\mathcal{R}(G)$ dense in $D(A^{1/4+\alpha/2})$ and such that $D(A^{2\alpha}) \subseteq \mathcal{R}(G) \subseteq D(A^{1/4+\alpha/2+\varepsilon})$ for some $\varepsilon > 0$.

Let us point out that these two crucial properties could be proved only in the space $D(A^{1/4})$; indeed, if the Markov process lives in the space $D(A^\alpha) \subset D(A^{1/4})$, it inherites these two properties from $D(A^{1/4})$. This is true in our case, according to Theorem 3.1. But a stronger upper bound for the range of G is assumed in Theorem 3.1. In this way we would make smaller the domain in which $\mathcal{R}(G)$ has to be bounded. On the other hand, by means of our procedure we check directly irreducibility and strong Feller property in $D(A^\alpha)$ for every $\alpha \in [(1/4), (1/2))$. In this way we make less restrictive the lower bound for $\mathcal{R}(G)$.

4.3. *Final theorem.*

We conclude relating the existence result of the invariant measure of § 4.1 to the uniqueness of § 4.2. We have proved

THEOREM 4.3. – *Fix* $\alpha \in [(1/4), (1/2))$. *Assume* [**H**] *and that the data have the regularity specified in Theorem* 3.1. *Then, there exists a unique invariant measure* μ *for the system* (2), *concentrated on* $D(A^{\alpha})$, *which is equivalent to each transition probability* $P(t, x, \cdot)$ *for arbitrary* $x \in D(A^{\alpha})$ *and* $t > 0$. *Moreover* μ *is strongly mixing, that is*

$$\lim_{t \to +\infty} P(t, x, \Gamma) = \mu(\Gamma)$$

for arbitrary $x \in D(A^{\alpha})$, $\Gamma \in \mathcal{B}(D(A^{\alpha}))$.

Such a unique invariant measure is ergodic, in the sense that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \Psi(u(t; x))\, dt = \int_{D(A^{\alpha})} \Psi\, d\mu \qquad \mathbb{P}\text{-a.s.}$$

for all $x \in D(A^{\alpha})$ and Borel measurable functions $\Psi : D(A^{\alpha}) \to \mathbb{R}$ such that $\int_{D(A^{\alpha})} |\Psi|\, d\mu < \infty$.

This has a physical relevance (e.g., cf. [21]). In fact, the solutions of the Navier-Stokes equations under realistic conditions are so highly oscillatory that usually one computes mean values of the solutions with some kind of time average. And the unique equilibrium measure (the ergodic measure) provides the way to make statistical averages. This is nothing but the ergodic principle, which lies at the basis of the statistical fluid dynamics; here a rigorous proof of its validity for the stochastic equation has been given.

5. – Conclusion.

The presence of a (suitable, not too degenerate) white noise provides the existence of a unique invariant measure. This result depends heavily on the noise. This procedure can not be applied to the deterministic equation. For instance, the irreducibility property is not fulfilled by the deterministic equation; in fact, if $G \equiv 0$ then $P(t, x, \cdot) = \delta_{u(t;\, x)}$.

But, in general, without constraints on the Reynolds number, the deterministic Navier-Stokes equation has many stationary solutions. When a sufficiently distributed random perturbation is added, only one invariant measure exists. The effect of the noise is to mix up the dynamics of the system, allowing a unique asymptotic behaviour. No one of the deterministic stationary solutions is an invariant measure for the stochastic equation whose noise satisfies [**H**].

An easy example of noise satisfying our assumptions is $G = A^{-\gamma} L$ for any $3/8 < \gamma \leq 1$ given a linear bounded injective operator L.

Or even, let us represent the cylindrical Wiener process as a series (not converging in H but in bigger space) with respect to the Stokes eigenvectors $\{e_j\}_{j=1}^{\infty}$, i.e. $w(t) = \sum_{j=1}^{\infty} \beta_j(t)\, e_j$ (given a family $\{\beta_j\}_{j=1}^{\infty}$ of independent real valued standard Wiener processes), so that $Gw(t) = \sum_{j=1}^{\infty} \sigma_j \beta_j(t)\, e_j$; the condition is $\sigma_j \neq 0\ \forall j$ (no degenerate noise) and $c/j^{2\alpha} \leq \sigma_j < C/j^{1/4+\alpha/2}$, for j's large enough, since $\lambda_j \sim j$ as $j \to +\infty$.

Eventually, given any noise satisfying the assumption of Theorem 4.3, sufficient conditions on the data u_{t_0} and f are given so that there exists a unique invariant measure.

REFERENCES

[1] S. Albeverio - M. Ribeiro de faria - R. Høegh-krohn, *Stationary measures for the periodic Euler flow in two dimensions*, J. Stat. Phys., Vol. **20**, No. 6 (1979), 585-595.

[2] A. Bensoussan - R. Temam, *Equations stochastiques du type Navier-Stokes*, J. Funct. Anal., **13** (1973), 195-222.

[3] D. H. Chambers - R. J. Adrian - P. Moin - D. S. Steward - H. J. Sung, *Karhunen-Loéve expansion of Burgers' model of turbulence*, Phys. Fluids 31/9 (1988), 2573-2582.

[4] G. da Prato - A. Debussche - R. Temam, *Stochastic Burgers equation*, NoDEA, **1** (1994), 389-402.

[5] G. da Prato - K. D. Elworthy - J. Zabczyk, *Strong Feller property for Stochastic Semilinear Equations*, Stoch. Anal. and Appl. 1, Vol. **13** (1995), 35-45.

[6] G. da Prato - D. Gatarek, *Stochastic Burgers equation with correlated noise*, Stochastics & Stoc. Rep., **52** (1995), 29-41.

[7] G. da Prato - J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, Cambridge, 1996.

[8] G. da Prato - J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge, 1992.

[9] J. L. Doob, *Asymptotic properties of Markov transition probability*, Trans. Am. Math. Soc., **64** (1948), 393-421.

[10] K. D. Elworthy - X. M. Li, *Formulae for the Derivatives of Heat Semigroups*, J. Funct. Anal., **125** (1994), 252-286.

[11] B. Ferrario, *Stochastic Navier-Stokes equations: analysis of the noise to have a unique invariant measure*, to appear in Ann. Mat. Pura e Appl.

[12] B. Ferrario, *Pathwise regularity of non linear Ito equations. Application to a Stochastic Navier-Stokes equation.*, to appear in Stoc. Anal. Appl.

[13] B. Ferrario, *Invariant measures for the stochastic Navier-Stokes equations*, Ph.D., Scuola Normale Superiore (1998), Pisa.

[14] F. Flandoli, *Dissipativity and invariant measures for stochastic Navier-Stokes equations*, NoDEA, **1** (1994), 403-423.

[15] F. FLANDOLI, *Regularity theory and stochastic flows for parabolic SPDE's*, Stochastics Monographs 9 London: Gordon and Breach, 1995.
[16] F. FLANDOLI - B. MASLOWSKI, *Ergodicity of the 2-D Navier-Stokes equation under random perturbations*, Commun. Math. Phys., **171** (1995), 119-141.
[17] Y. GIGA - T. MIYAKAWA, *Solutions in L_r of the Navier-Stokes initial value problem*, Arch. Rational Mech. Anal., **89** (1985), 267-281.
[18] D. T. JENG, *Forced model equation for turbulence*, Phys. Fluids 12/10 (1969), 2006-2010.
[19] B. MASLOWSKI, *On probability distributions of solutions of semilinear stochastic evolution equations*, Stochastics & Stoc. Rep., **45** (1993), 17-44.
[20] B. MASLOWSKI - J. SEIDLER, *Ergodic properties of recurrent solutions of stochastic evolution equations*, Osaka J. Math., **31** (1994), 969-1003.
[21] A. S. MONIN - A. M. YAGLOM, *Statistical Fluid Mechanics*, Cambridge, Mass., MIT Press, 1971.
[22] J. SEIDLER, *Ergodic behaviour of stochastic parabolic equations*, to appear in Czechoslovak Math. J.
[23] R. TEMAM, *Behaviour at time $t = 0$ of the solutions of semi-linear evolution equations*, J. Diff. Eq., **43** (1982), 73-92.
[24] R. TEMAM, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New York, 1988.
[25] R. TEMAM, *Navier-Stokes Equations and Nonlinear Functional Analysis*, SIAM, Philadelphia, 1983.

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