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Sunto. – *Uno dei metodi migliori per scoprire le proprietà di un cappio chiuso è studiarne il gruppo di moltiplicazione* [3], [4]. *In questo breve saggio descriviamo i gruppi di moltiplicazione di una classe importante di cappi, e cioè di quella dei cappi flessibili che posseggono la proprietà inversa.*

A *loop* is a set with a single binary operation, denoted by juxtaposition, such that in $xy = z$, knowledge of any two of x, y and z specifies the third uniquely, and with a unique two-sided indentity element, denoted by 1. A *flexible loop* is a loop that satisfies the indentity $(xy)\,x = x(yx)$. An *inverse property loop* is a loop in which, given x, y, there exists a unique x^{-1} such that $(yx)\,x^{-1} = x^{-1}(xy) = y$. A *right loop* is a set with a single binary operation that is right cancellative, and with a unique left identity element.

The multiplication group, *MltL*, of a loop L is the subgroup of the group of all bijections on L generated by right and left translations, that is $MltL :=$ $\langle R(x), L(x) \colon x \in L \rangle$, where $R(x)$ (respectively, $L(x)$) is right (respectively, left) translation by x.

It is easy to show that if L is an inverse property loop, then there is an involutory automorphism J on *MltL*, defined on generators by:

$$R(x)^J = L(x^{-1}), \qquad L(x)^J = R(x^{-1})$$

Recall that a *right transversal* T to a subgroup H of a group G is a complete set of right coset representatives. That is,

$$G = \bigcup_{t \in T} Ht .$$

There is a natural G action on T, denoted by $*$, given by:

$$t * g := u , \qquad \text{where } u \text{ is the representative in } T \text{ of the coset } Htg .$$

This action restricted to T itself, endows T with a binary operation. Thus, T is naturally a groupoid, and can be thought of as the *right quotient* of G by H. We use $\underline{T}$ to denote this groupoid. If H is normal, then $\underline{T}$ is identical with the standard quotient group structure on G/H. In the general case (H not necessarily normal), it is easy to show that $\underline{T}$ is right cancellative. It is not necessary, though, that $\underline{T}$ be left cancellative, i.e., that $\underline{T}$ be a loop. For example, let G be the symmetric group on three symbols. That is, let $G = \langle a, b \mid a^2 = b^3 = (ab)^2 = 1 \rangle$. then $T = \{1, b, ba\}$ is a right transversal to $\langle a \rangle$ in G. Of course, $\underline{T}$ is right cancellative. But the reader can easily check, $\underline{T}$ is not a loop.

The following fact [3] characterizes multiplication groups of loops (recall that the core of a subgroup H in a group G is the largest normal subgroup of G contained in H):

[NK] *A group G is the multiplication group of some loop L if and only if it has a subgroup H and two right transversals A and B to H in G, such that $\underline{A}$ is isomorphic to L, the core of H in G is trivial, $[A, B] \leqslant H$ and $\langle A, B \rangle = G$.*

Let G be a group with an involutory automorphism J, and a transversal T to $I = stab_G(J)$. The triple (G, T, J) is called a *group with partial biality* if both $[T, T^J] \leqslant I$, and $\langle T, T^J \rangle = G$.

In [4] we characterized multiplication groups of Moufang loops as being special images of groups with biality, where groups with biality were defined as in the above definition of groups with partial biality, but with two extra conditions; hence the definition above. Clearly, a group with partial biality is the multiplication group of a loop if the core of I in G is trivial.

LEMMA 1. – *If (G, T, J) is a group with partial biality and if G has unique square roots, then for each $t \in T, [t, t^{-J}] = 1$.*

PROOF. – Since, $t^{-1} t^J t t^{-J} \in I$, we have $t^{-1} t^J t t^{-J} = t^{-J} t t^J t^{-1}$. Rearranging gives, $(t^{-1} t^J)^2 = (t^J t^{-1})^2$. Since G has unique square roots, the result follows.

LEMMA 2. – *If (G, T, J) is a group with partial biality and if G has unique square roots, then for each $t, v \in T$, $t^J v^{-1} v^J t^{-1} = v^{-1} t^{-1} t^J v^J$.*

PROOF. – Since $t^{-1} v^J t v^{-J} \in I$, we have $t^{-J} v t^J v^{-1} = t^{-1} v^J t v^{-J}$. Rearranging gives $t^J v^{-1} v^J t^{-1} = v^{-1} t^J t^{-1} v^J = v^{-1} t^{-1} t^J v^J$, where the last equality holds by lemma 1.

THEOREM 3. – *A group G with unique square roots is the multiplication group of some inverse property, flexible loop if (G, T, J) is group with partial biality, the core of I in G is trivial, and $T^{-1} = T$.*

PROOF. – [NK] implies both that $\underline{T}$ is a loop and that $G = Mlt\underline{T}$ [3]. $T^{-1} = T$ implies that $\underline{T}$ has the inverse property [3]. We must show that $\underline{T}$ satisfies the flexible law: $(a * b) * a = a * (b * a)$. Let $c = a * b$. Then $Iab = Ic$. Thus, $Iaba =$

Ica. Now let $b * a = d$. Then $Iba = Id$. Thus, let

$$(*) \qquad k = da^{-1} b^{-1} \in I\,.$$

So $Iad = Ia(da^{-1}b^{-1})\, ba = Iakba$. We have $d^{-1} d^J = a^{-1} b^{-1} k^{-1} k b^J a^J$ (by (*)) $= a^{-1} b^{-1} b^J a^J$. So

$$b^{-1} b^J = ad^{-1} d^J a^{-J} = aa^{-J} a^J d^{-1} d^J a^{-1} aa^{-J} = a^{-J} aa^J d^{-1} d^J a^{-1} a^{-J} a$$

(by lemma 1). Thus, $a^J b^{-1} b^J a^{-1} = a(a^J d^{-1} d^J a^{-1})\, a^{-J}$. And so $b^{-1} a^{-1} a^J b^J = a(d^{-1} a^{-1} a^J d^J)\, a^{-J}$ (by lemma 2). Thus, $ada^{-1} b^{-1} a^{-1} = a^J d^J a^{-J} b^{-J} a^{-J}$. Which means $aka^{-1} = ada^{-1} b^{-1} a^{-1} \in I$, and thus $Iaba = Iakba$.

Examples of multiplication groups—with unique square roots—of inverse property, flexible loops abound. The reader is directed especially to [1] and [4].

Let G be a finite group acting transitively on a set T. Let I be the stabilizer in G of some point in T. Let $C(T)$ be the vector space of all functions from T into the set of complex numbers. The action of G on T induces a representation r of G in $C(T)$:

$$r\colon G \to Aut(C(T))\,; \qquad g \mid\to (f(t) \mid\to f(g^{-1} t)$$

r decomposes into a direct sum of irreducible representations. If this decomposition is multiplicity free then (G, I) is called a *Gelfand pair*. Since it is sometimes quite difficult to determine the decomposition of $C(T)$, the following lemma is often useful:

LEMMA 4. – [2] (Gelfand's Lemma) *Lemma $J\colon G \to G$ be a monomorphism such that $\forall g \in G$, $g^{-1} \in Ig^J I$. Then (G, I) is a Gelfand pair.*

THEOREM 5. – *If (G, T, J) is a group with partial biality, and if G has odd order, then (G, I) is a Gelfand pair.*

PROOF It is easy to show that I, the subgroup of elements of G fixed by J, is also the stabilizer of 1 under the natural action of G on T. Now let $g = ht$, where $h \in I$ and $t \in T$. Notice that $gg^{-J} h = ht(ht)^{-J} h = htt^{-J} h^{-1} h = htt^{-J} = ht^{-J} t = h(t^{-J} h^{-1})(ht) = hg^{-J} g$. Thus, $g^{-J} hg^{-1} = g^{-1} hg^{-J}$. This means that $g^{-J} hg^{-1} = k$, for some $k \in I$. And this means that $g^{-1} = h^{-1} g^J k \in Ig^J I$, as desired.

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