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Dermot J. Marron, T. Brian M. McMaster

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Splittability for Ordered Topological Spaces.

DERMOT J. MARRON - T. BRIAN M. MCMASTER

Sunto. – *In quest'articolo dimostriamo come il concetto «spezzabilità», formulato e sviluppato di Arhangel'skii, viene trasferito dallo studio di spazi topologici a quello di spazi topologici parzialmente ordinati. Otteniamo numerosi risultati in forma «sè X e spezzabile (facendo uso di funzioni appropriatamente scelte) su spazi che hanno una proprietà, allora anche X soddisfa la stessa proprietà».*

1. – Introduction.

In the context of topology, where splittability was first formulated by Arhangel'skii, and in which it has been extensively developed by him and by his co-workers (see, for example, [1], [2], [3] and the articles referenced therein), a space X is said to be *splittable* (or *cleavable*) over a class $\mathcal{P}$ of spaces if, for every subset A of X, there is a space Y in $\mathcal{P}$ and a continuous mapping f from X into Y such that $f^{-1}(f(A)) = A$. When the mappings may be chosen to lie within a particular class $\mathcal{M}$ of continuous mappings, one speaks of *$\mathcal{M}$-splittability* instead of splittability; and when the subsets A are constrained to be singletons, the term *pointwise splittability* is employed. It is often convenient to assume that all of the mappings involved are surjections; where this assumption is made in the present article, we shall use the notations *s-splittable, pointwise s-splittable, $\mathcal{M}$-s-splittable* and so on.

Recent publications have pointed out that the basic ideas just outlined will transfer readily to many other settings, in particular to those of semigroups [8] and to those of partially-ordered sets [9]. The purpose of the present note is to effect their extension to the realm of partially-ordered topological spaces and, in particular, to obtain analogues of some classical theorems which assert that a space that is splittable over a family of T_1 (T_2, T_3 *et cetera*) spaces must itself be T_1 (T_2, $T_3 \dots$ respectively). By a *partially-ordered topological space* (or *ordered space*) we here intend merely a set equipped with both a topology and a partial order, no *a priori* connection between the two structures being presupposed. The conditions for these spaces which are analogous to separation axioms in pure topology are the so-called *order-separation axioms* which were first systematically studied by L. Nachbin and subsequently developed by other writers. We refer the interested reader to Nachbin [10], McCartan

[5] and Choe [4] for fuller discussion, but we shall also now briefly describe some of the more important of these conditions in order to make the present account reasonably self-contained.

Let $X = \{X, \leqslant, \tau\}$ denote an ordered space. For each point x of X the notations $L(x)$ and $M(x)$ represent the sets of lower and upper bounds of x:

$$L(x) = \{y \in X\colon y \leqslant x\},$$

$$M(x) = \{z \in X\colon x \leqslant z\}$$

and X is called *lower T_1-ordered* if (for every x) $L(x)$ is closed, *upper T_1-ordered* if (for every x) $M(x)$ is closed, and *T_1-ordered* if it is both lower and upper T_1-ordered. A subset S of X is termed *increasing* if $s \in S$, $s \leqslant t$ together imply $t \in S$, and it is called *decreasing* if it satisfies the dual condition ($s \in S$, $t \leqslant s$ together imply $t \in S$). The space X is *T_2-ordered* if, whenever $x \nleqslant y$ in X, there exist disjoint neighbourhoods U of x and V of y such that U is increasing and V is decreasing. If, whenever F is a closed decreasing subset of X and x lies in $X \backslash F$, there are disjoint neighbourhoods U of x and V of F with U being increasing and V decreasing, then X is called *lower regularly ordered* and if, in addition, it is lower T_1-ordered, the term *lower T_3-ordered* is applied to it. The duals of these conditions are called *upper regularly ordered* and *upper T_3-ordered*, while a space that is both lower and upper regularly ordered (lower and upper T_3-ordered) is said to be *regularly ordered* (*T_3-ordered*)). By a *completely separated* space X, we understand one within which, whenever $x \nleqslant y$, there is a continuous order-preserving function f from X into the unit interval $[0, 1]$ such that $f(y) < f(x)$. If, in addition to this last condition, there exist for each x in X and each open neighbourhood V of x two continuous functions f and g from X into $[0, 1]$ such that f is order-preserving, g is order-reversing, $f(x) = g(x) = 1$ and $X \backslash V \subseteq f^{-1}(0) \cup g^{-1}(0)$, then X is designated as *completely regularly ordered*. Lastly, X is called *normally ordered* if, whenever A and B are disjoint closed subsets of X with A decreasing and B increasing, there exist disjoint neighbourhoods G of A and H of B where G is decreasing and H is increasing, and the conjunction of normally ordered and T_1-ordered is defined as *T_4-ordered*. Note that the term «neighbourhood» is not assumed to be *open* in this discussion. Note also that the parallelism between order-separation axioms and separation axioms, though compelling, is imperfect: for instance, T_4-ordered does not imply completely regularly ordered; again, it appears not to be known whether T_3-orderedness is always inherited by subspaces.

2. – Results on order-separation.

All the conclusions of this section take the same basic form: if X is splittable (or pointwise-splittable, or s-splittable ...) using suitable mappings over a family of ordered spaces each of which satisfies an axiom listed in the previous paragraph, then X satisfies it also. Our first concern is which family of mappings we should use and, as was pointed out in [6], the most obvious choice, that of the continuous and order-preserving maps, fails to give satisfactory results. Indeed, we presented in [7] a simple example of ordered spaces X and Y where X is splittable over Y using (continuous) order-preserving mappings and Y is T_4-ordered, but X is not even lower T_1-ordered. This compels us to identify more appropriate classes of transformations.

DEFINITION. – A map $f\colon X \to Y$ between ordered spaces is *L-preserving* (or L_p for short) if

$$f(L(x)) = L(f(x))\,, \qquad \forall x \in X$$

and it is *M-preserving* (or M_p briefly) if

$$f(M(x)) = M(f(x))\,, \qquad \forall x \in X\,.$$

It is easy to see that these conditions are stronger than order-preserving, and independent of one another. The classes of continuous L_p and continuous M_p maps, with domain and codomain specified by the context of the problem, will be denoted by $\mathcal{L}_p$ and $\mathcal{M}_p$ respectively.

THEOREM 1. – *Let $\mathcal{S}$ denote one of the three properties*

(*a*) *lower T_1-ordered,*

(*b*) *upper T_1-ordered,*

(*c*) *T_1-ordered,*

and let the ordered space X be either

(i) *pointwise $\mathcal{L}_p$-splittable or*

(ii) *pointwise $\mathcal{M}_p$-splittable or*

(iii) *$(\mathcal{L}_p \cup \mathcal{M}_p)$-splittable*

over a family of $\mathcal{S}$-spaces. Then X also is $\mathcal{S}$.

PROOF. – First, consider (*a*)(i). If X is not lower T_1-ordered we can find x and z such that $z \in \overline{L(x)} \setminus L(x)$, and then a continuous L_p map $f\colon X \to Y$ where $f^{-1}(f(z)) = \{z\}$ and Y is lower T_1-ordered. Now $f(z) \notin f(L(x)) = L(f(x))$ which is closed, so its preimage under f is a closed superset of $L(x)$ excluding z, a contradiction.

Next, we examine (*b*)(i). For any x in X choose a continuous L_p map f from X into an upper T_1-ordered space Y such that $f^{-1}(f(x)) = \{x\}$. Since f is readily checked to be order-preserving, we see that $f^{-1}(M(f(x)))$ is an increasing set and it includes x, so $M(x) \subseteq f^{-1}(M(f(x)))$. On the other hand, $z \in f^{-1}(M(f(x)))$ implies $f(x) \in L(f(z)) = f(L(z))$, so $f(x) = f(p)$ for some $p \in L(z)$. This forces $x = p$ and $x \leqslant z$ and $z \in M(x)$. We now have $M(x) = f^{-1}(M(f(x)))$ and it follows that $M(x)$ is closed, as required.

The pairings (a)(ii) and (b)(ii) are clearly dual to what has just been done, while (c)(i) and (c)(ii) follow by combining appropriate «lower» and «upper» halves, but the arguments involving (iii) are somewhat different. First, we may readily verify the following:

(I) If $f\colon X \to Y$ is an M_p-map and $f(X) \subseteq A \subseteq Y$ then the co-restriction $f^A\colon X \to A$ is M_p also.

(II) If $f\colon X \to Y$ is an M_p surjection, $x \in X$ and $f^{-1}(f(L(x))) = L(x)$ then $f(L(x)) = L(f(x))$.

Now suppose that X is $(\mathcal{L}_p \cup \mathcal{M}_p)$-splittable over a family of lower T_1-ordered spaces. Given $x \in X$ we can thus choose $f\colon X \to Y$ where Y is lower T_1-ordered, $f^{-1}(f(L(x))) = L(x)$ and f is either L_p or M_p. If f is L_p, we immediately get $L(x) = f^{-1}(f(L(x))) = f^{-1}(L(f(x)))$ to be closed. If f is M_p, then (I) and the observation that lower T_1-order is inherited by subspaces together show that there is no loss of generality in assuming f to be a surjection. Now (II) gives us $L(x)$ closed, as before. This establishes (a)(iii), the dual argument provides (b)(iii) and their conjunction yields (c)(iii) and concludes the demonstration.

THEOREM 2. – *Let X be either pointwise $\mathcal{L}_p$-splittable, pointwise $\mathcal{M}_p$-splittable or $(\mathcal{L}_p \cup \mathcal{M}_p)$-splittable over a family of T_2-ordered spaces. Then X is T_2-ordered also.*

PROOF. – Let $x \nleqslant y$ in X. If we can find a continuous L_p map $f\colon X \to Y$ where $f^{-1}(f(x)) = \{x\}$ and Y is T_2-ordered, then $f(x) \notin f(L(y)) = L(f(y))$, that is, $f(x) \nleqslant f(y)$ so there are disjoint neighbourhoods U of $f(x)$ and V of $f(y)$ such that U is increasing and V is decreasing. Bearing in mind that f is order-preserving, it follows that the disjoint neighbourhoods $f^{-1}(U)$ of x and $f^{-1}(V)$ of y are increasing and decreasing respectively, so X is T_2-ordered. The dual argument runs in case f is M_p instead of L_p.

Turning now to the case where X is $(\mathcal{L}_p \cup \mathcal{M}_p)$-splittable, we choose $f\colon X \to Y$ where $f^{-1}(f(M(x))) = M(x)$, Y is T_2-ordered and f is either L_p or M_p. Now $x \nleqslant y$ so $M(x) \cap L(y) = \phi$, from which we see that $f(M(x)) \cap f(L(y))$ is empty also. Since $x \in M(x)$, $f(x) \notin f(L(y))$ so, if f is L_p, $f(x) \nleqslant f(y)$. Equally, if f is M_p we get $f(y) \notin f(M(x)) = M(f(x))$ yielding $f(x) \nleqslant f(y)$ again. The proof that X is T_2-ordered now completes as before.

THEOREM 3. – *If X is pointwise (closed, $\mathcal{L}_p$)-splittable over a class of lower regularly ordered spaces, then X itself is lower regularly ordered.*

PROOF. – Given $x \in X \backslash F$ where F is both closed and decreasing, choose a closed, continuous L_p map $f\colon X \to Y$ where Y is lower regularly ordered and $f^{-1}(f(x)) = \{x\}$. Then $f(F)$ is closed in Y, decreasing because

$$f(F) = f(\cup \{L(x)\colon x \in F\}) = \cup \{f(L(x))\colon x \in F\} = \cup \{L(f(x))\colon x \in F\}$$

and $f(x)$ lies in its complement, so we can choose disjoint neighbourhoods U of $f(x)$ and V of $f(F)$ where U is increasing and V is decreasing in Y. It follows that $f^{-1}(U)$, $f^{-1}(V)$ separate x and F in the desired fashion.

COROLLARY. – (i) If X is pointwise (closed, $\mathcal{M}_p$)-splittable over spaces, each of which is upper regularly ordered [or upper T_3-ordered], then so is X.

(ii) If X is pointwise (closed, $\mathcal{L}_p \cap \mathcal{M}_p$)-splittable over T_3-ordered spaces, then X is T_3-ordered.

The requirement in (ii) here that the relevant maps be both L-preserving and M-preserving, as well as closed and continuous, may well be seen as undesirably restrictive. One way to lighten this restriction becomes available if they may be chosen to be surjective:

THEOREM 4. – *If X is (closed, $\mathcal{L}_p \cup \mathcal{M}_p$)-s-splittable over T_3-ordered spaces, then X is T_3-ordered.*

PROOF. – Let $x \in X \backslash F$ where F is closed and decreasing in X. Choose a closed continuous surjection $f\colon X \to Y$ where Y is T_3-ordered, $f^{-1}(f(F)) = F$ and $f \in \mathcal{L}_p \cup \mathcal{M}_p$. In the case where f is L_p, we know that $f(F)$ is a decreasing set. On the other hand, when f is M_p, consider $y \leqslant f(v)$ where $v \in F$. Then $y = f(z)$ for some z in X because of surjectivity, and $f(v) \in M(f(z)) = f(M(z))$ yielding $f(v) = f(w)$ for some $w \geqslant z$. Next, $w \in F$ by choice of f, and $z \in F$ also since F is decreasing. Thus $y = f(z) \in f(F)$ and we again find that $f(F)$ is decreasing. The proof that X is lower regularly ordered completes as in Theorem 3, while the dual argument deals with upper regularly ordered. Lastly, we invoke part (c)(iii) of Theorem 1.

We have obtained variants of Theorems 3 and 4 for mappings that are open rather than closed. The proofs are similar to those already presented.

THEOREM 5. – (i) *If X is (open, $\mathcal{L}_p$)-splittable over upper T_3-ordered spaces, then X is upper T_3-ordered.*

(ii) *If X is (open, $\mathcal{L}_p$)-s-splittable over lower T_3-ordered spaces, then X is lower T_3-ordered.*

THEOREM 6. – *If X is $(\mathcal{L}_p \cup \mathcal{M}_p)$-splittable over a class of completely separated spaces, then X is completely separated.*

PROOF. – Given $x \nleq y$ in X, choose continuous $f\colon X \to Y$ such that $f^{-1}(f(M(x))) = M(x)$ and $f \in \mathcal{L}_p \cup \mathcal{M}_p$. If f is M_p then $y \notin M(x)$ gives $f(y) \notin f(M(x)) = M(f(x))$, that is, $f(x) \nleq f(y)$. If f is L_p, we saw in the proof of Theorem 2 how to reach the same conclusion. So there must exist continuous, order-preserving $g\colon Y \to [0, 1]$ that makes $g(f(x)) > g(f(y))$. The composite map $g \circ f$ shows us that X is completely separated also.

THEOREM 7. – *If X is (open, $\mathcal{L}_p \cup \mathcal{M}_p$)-splittable over a class of completely regularly ordered spaces, then X is completely regularly ordered.*

PROOF. – It is completely separated by Theorem 6. Now given $x \in X$ and an open neighbourhood V of x, choose a continuous, open and order-preserving map $f\colon X \to Y$ where Y is completely regularly ordered and $f^{-1}(f(V)) = V$. Since $f(V)$ is an open neighbourhood of $f(x)$, there exist continuous g, h: $Y \to [0, 1]$ such that $g(f(x)) = h(f(x)) = 1$, g is order-preserving, h is order-reversing and $Y \setminus f(V) \subseteq g^{-1}(0) \cup h^{-1}(0)$. Routine inspection of the functions $g \circ f$ and $h \circ f$ will confirm that X is completely regularly ordered.

NOTE. – If the map f is closed rather than open, then $f(X \setminus V) = f(X) \setminus f(V)$ is closed and so $V' = Y \setminus f(X \setminus V)$ is also an open neighbourhood of x. By re-working the previous proof with $f(V)$ replaced by V', we can show that (closed, $\mathcal{L}_p \cup \mathcal{M}_p$)-splittability over completely regularly ordered spaces implies completely regular order also.

THEOREM 8. – *If X is either (closed, $\mathcal{L}_p$)-s-splittable or (closed, $\mathcal{M}_p$)-s-splittable over a class of spaces, each of which is*

(i) *normally ordered, or*

(ii) *T_4-ordered*
then X enjoys the same property.

[The proof follows the lines of argument established earlier in this article.]

3. – Observations on other order-topological properties.

Although the «preservation under splittability» of order-separation axioms was our motivation for introducing the L_p and M_p mappings, it turns out that they are important for analysing the splittability behaviour of many other order-topological invariants. This section presents a brief account of aspects of this analysis, beginning with another group of definitions.

For each subset A of an ordered space X let $i(A)$ and $d(A)$ denote the increasing hull and the decreasing hull of A, that is,

$$i(A) = \cup\{M(x): x \in A\},$$

$$d(A) = \cup\{L(x): x \in A\}.$$

Then X is termed an I_i *space* (respectively, an I_d *space*) if, for every open subset G of X, $i(G)$ is open (respectively, $d(G)$ is open). When X is both I_i and I_d, it is called an I *space*. Analogously, X is a C_i *space* (respectively, a C_d *space*) if, for every closed subset K of X, $i(K)$ is closed (respectively, $d(K)$ is closed), and when both of these requirements are satisfied it is referred to as a C *space*. Once again, easy examples show that splittability by order-preserving maps with «good» topological behaviour can fail to preserve these conditions, but:

THEOREM 9. – (i) *If X is (open, $\mathcal{M}_p$)-splittable over I_i spaces, then X is an I_i space.*

(ii) *If X is (open, $\mathcal{M}_p$)-splittable over I_d spaces, then X is an I_d space.*

(iii) *The duals of* (i) *and* (ii) *are valid.*

(iv) *If X is (open, $\mathcal{L}_p \cup \mathcal{M}_p$)-splittable over a class of I spaces, then X is an I space.*

PROOF. – The key steps are that $f(i(A)) = i(f(A))$ whenever f is M_p, that $f(d(A)) = d(f(A))$ whenever f is L_p, and that if $f\colon X \to Y$ is M_p and Y is I_d then $f(X)$ is also I_d (and dually).

A parallel theorem for C_i, C_d and C spaces may be derived by substituting «closed» for «open».

The ordered space X is said to be *order separated* if X can be expressed as the disjoint union of two non-empty open subsets one of which is increasing (and therefore the other of which is decreasing). In the opposite eventuality it is called *order connected*. The following is easily verified.

THEOREM 10. – (i) *If X is order-preserving-s-splittable over a class of order separated spaces, then X is order separated.*

(ii) *If X is (open, $\mathcal{L}_p \cup \mathcal{M}_p$)-s-splittable over a class of order connected spaces, then X is order connected.*

Nachbin called a space X *locally convex* if, given $x \in X$ and a neighbourhood U of x, there is always a convex open set V such that $x \in V \subseteq U$. It is readily checked that:

THEOREM 11. – *If X is (open, order preserving)-splittable over locally convex spaces, then X is locally convex.*

To conclude on a related and as-yet-unresolved point: X is said to *have small intervals* if, given $x \in X$ and a neighbourhood U of x, there exist y, z in U with $[y, z] \subseteq U$ and $[y, z]$ is a neighbourhood of x. We have not identified a class $\mathcal{M}$ of mappings (other than homeomorphic order-isomorphisms!) for which $\mathcal{M}$-splittability over spaces that have small intervals implies the same property for the split space.

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Department of Pure Mathematics, The Queen University of Belfast
Belfast, BT7 1NN, Northern Ireland

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