Bollettino
Unione Matematica Italiana

Bill Watson

Superminimal fibres in an almost Hermitian submersion

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **3-B** (2000), n.1, p. 159–172.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2000_8_3B_1_159_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2000.

Bollettino U. M. I.
(8) **1**-B (2000), 159-172

Superminimal Fibres in an Almost Hermitian Submersion.

BILL WATSON (*)

Sunto. – *Se la varietà base, N, di una submersione quasi-Hermitiana, $f: M \to N$, è una G_1-varietà e le fibre sono subvarietà superminimali, allora lo spazio totale, M, è G_1. Se la varietà base, N, è Hermitiana e le fibre sono subvarietà bidimensionali e superminimali, allora lo spazio totale, M, è Hermitiano.*

It is natural question to ask what conditions on an almost Hermitian submersion, $f: (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$, are sufficient to induce a given almost Hermitian structure from the Gray-Hervella list [Gr-He] onto the total space, (M^{2m}, g, J). The first study of the induction of a given almost Hermitian structure onto the total space of an almost Hermitian submersion was that of L. Vanhecke and the author [Wa-Va2], in which we characterized almost semi-Kähler submersions. Here we significantly extend the list of induced almost Hermitian structures.

All manifolds considered herein are assumed to be smooth, complete, and connected. All mappings, vector fields, sections, etc., are assumed to be smooth.

1. – Almost Hermitian manifolds.

An *almost complex structure* on a smooth manifold M is a tensor field J of type $(1, 1)$ such that $J^2 = -Id$. If M is equipped with a chosen almost complex structure, J, then the *almost complex manifold*, (M, J), is necessarily orientable and of even dimension, $2m$. The *Nijenhuis tensor* of the almost complex structure J on (M, J) is the tensor field N of type $(1, 2)$ given by

$$N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]. \tag{1.1}$$

The well-known Newlander-Nirenberg Theorem [Ko-No] states that J is the almost complex structure associated to a complex manifold structure on M if and only if the Nijenhuis tensor of J vanishes. If N is zero, we say that J is *integrable.*

A Riemannian manifold (M^{2m}, g) equipped with an orthogonal almost com-

(*) E-mail: to watsonw@stjohns.edu

plex structure, J, is called an *almost Hermitian manifold.* The *Kähler 2-form* on the almost Hermitian manifold (M^{2m}, g, J) is given by $\omega(X, Y) = g(X, JY)$. An almost Hermitian manifold, (M^{2m}, g, J), on which $d\omega = 0$ is called *almost Kähler*. In the lattice of almost Hermitian structures defined by A. Gray and L. Hervella [Gr-He], the class of almost Kähler structures corresponds to the $U(n)$-invariant subspace $\mathfrak{W}_2$ of the representation space $\mathfrak{W}_1 \oplus \mathfrak{W}_2 \oplus \mathfrak{W}_3 \oplus \mathfrak{W}_4$. An almost Hermitian manifold, (M^{2m}, g, J) , with an integrable almost complex structure, J, is called a *Hermitian manifold.* The Gray-Hervella class of Hermitian structures is $\mathfrak{W}_3 \oplus \mathfrak{W}_4$. An almost Hermitian manifold with a coclosed Kähler 2-form is called an *almost semi-Kähler* manifold ($\delta\omega = 0$ and Gray-Hervella class $\mathfrak{W}_1 \oplus \mathfrak{W}_2 \oplus \mathfrak{W}_3$). A Hermitian almost semi-Kähler manifold is called *semi-Kähler* (Gray-Hervella class $\mathfrak{W}_3$). *Nearly Kähler* manifolds have $(\nabla_E J)\, E = 0$, for all local vector fields, E, and Gray-Hervella class $\mathfrak{W}_1$. Both almost Kähler manifolds and nearly Kähler manifolds are *quasi-Kähler* ($(\nabla_E J)\, F + (\nabla_{JE} J)\, JF = 0$ for all local vector fields, E and F, and Gray-Her-

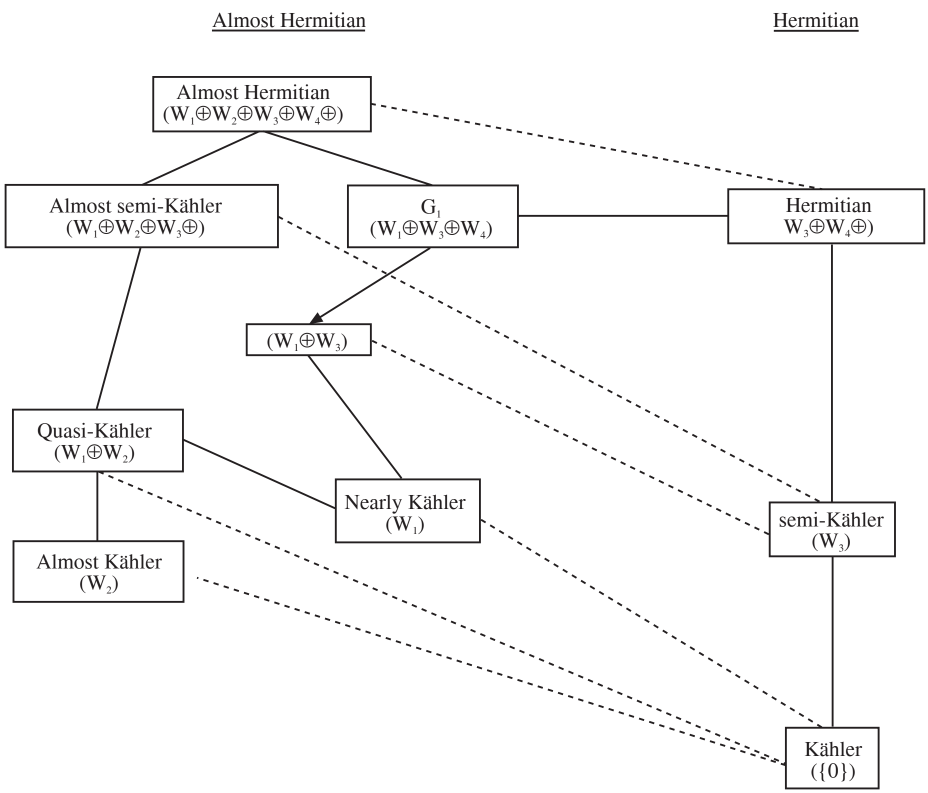

Figure 1. – Strict Inclusion Lattice of Almost Hermitian Structures.

vella class $\mathcal{W}_1 \oplus \mathcal{W}_2$) [Gr-He]. Recall that all holomorphically immersed submanifolds of a quasi-Kähler manifold are minimal [Gr1]. If $\nabla J = 0$, then (M^{2m}, g, J) is *Kähler* and $N = 0$. The Gray-Hervella class of Kähler structures is $\{0\}$. These classes of almost Hermitian structures are related by the strict inclusion lattice in fig. 1 in which $W_i = \mathcal{W}_i$.

A manifold mapping $f: (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ from one almost Hermitian manifold to another is called a *holomorphic mapping* if its differential commutes with the two almost complex structures; i.e., $df \circ J = \check{J} \circ df$. Holomorphic mappings are sometimes called «$(J, \check{J})$-pseudoholomorphic,» or, simply, «pseudoholomorphic.»

2. – Almost Hermitian submersions.

We recall the definition of a Riemannian submersion [Gr2], [O'N], which is a natural generalization of a Riemannian product projection mapping.

DEFINITION 2.1. – *A surjective mapping $f: (M, g) \to (N, \check{g})$ between Riemannian manifolds is called a Riemannian submersion if*:

i) *f has maximal rank, and*

ii) *df, restricted to* $(\mathrm{Ker}(df))^{\perp}$, *is a linear isometry.*

We shall denote attributes of the base space of a Riemannian submersion by an upside down caret; e.g., $\check{g}$. The *fibre submanifold*, F_x, of the Riemannian submersion $f: (M, g) \to (N, \check{g})$, over the point $x \varepsilon N$ is $f^{-1}(x)$. Since M is assumed to be complete, the fibre submanifolds are closed and regularly embedded. We let $\hat{g}$ denote the induced metric on the fibre, F, and will denote attributes of the fibres by a caret, $\hat{}$. Vector fields on M which are in the kernel of df are tangent to the fibres and are called *vertical* vector fields. The vertical distribution $V(M)$ on the tangent bundle of M induced by df is completely integrable. We shall denote vertical vector fields by the letters U, V, W, etc. Vector fields on M which are g-orthogonal to the vertical distribution are said to be *horizontal*. The horizontal distribution, which is not necessarily completely integrable, is denoted $H(M)$. We shall denote horizontal vector fields by the letters X, Y, Z, etc. A horizontal vector field X on the total space M is said to be *basic* if it is f-related to a vector field X_* on the base space, N. We shall often use basic vector fields to establish tensorial relationships involving horizontal vector fields. The projection mappings from the orthogonal decomposition of the tangent bundle $T(M) = V(M) \oplus H(M)$ are denoted by $\mathcal{V}$: $T(M) \to V(M)$ and $\mathcal{H}$: $T(M) \to H(M)$, respectively.

The *O'Neill configuration tensors,* T and A, of the Riemannian submersion $f: (M, g) \to (N, \check{g})$ have been well-studied [Gr2], [O'N]. Let E and F be arbitrary vector fields on M. (The use of the symbol F for a general vector field and for the fibre submanifolds should cause no confusion). Define

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F \tag{2.1}$$

$$A_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F \tag{2.2}$$

The properties of T and A have been developed extensively in the fundamental papers of A. Gray [Gr2] and B. O'Neill [O'N]. Essentially, T is the second fundamental form of the fibre submanifolds, and A is the complete integrability tensor of the horizontal distribution.

A holomorphic Riemannian submersion $f: (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ between almost Hermitian manifolds is called an *almost Hermitian submersion* [Wa1]. The vertical and horizontal distributions of an almost Hermitian submersion are J-invariant. Therefore, the fibre submanifolds, $(F^{2(m-n)}, \hat{g}, \hat{J})$, are closed, regularly embedded, almost Hermitian submanifolds of M^{2m} of dimension $2m - 2n$. If the total space (M^{2m}, g, J) belongs to one of the classes: quasi-Kähler, almost Kähler, Kähler, or Hermitian, then both the fibre submanifolds,$(F^{2(m-n)}, \hat{g}, \hat{J})$, and the base space,$(N^{2n}, \check{g}, \check{J})$, inherit the same almost Hermitian structure. For this reason, if $f: (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ is an almost Hermitian submersion with M belonging to the Gray-Hervella class $\mathfrak{P}$, then we shall call f, a $\mathfrak{P}$-*submersion.*

3. – Almost semi-Kähler submersions.

We recall from [Wa-Va2] the main results on almost semi-Kähler submersions which will be needed here. Unlike many other almost Hermitian structures, the inheritance of the almost semi-Kähler property by the fibres and base space of an almost Hermitian submersion whose total space is almost semi-Kähler is not automatic.

THEOREM 3.1. – [Wa1] *Let $f: (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ be an almost Hermitian submersion whose total space, (M^{2m}, g, J), is almost semi-Kähler. Then the base space, $(N^{2n}, \check{g}, \check{J})$, is almost semi-Kähler if and only if the fibres, $(F^{2m-2n}, \hat{g}, \hat{J})$, are minimally embedded.*

In order to examine the inheritance of the almost semi-Kähler property onto the fibre submanifolds, $(F, \hat{g}, \hat{J})$, we defined [Wa-Va2] the B tensor of an almost Hermitian submersion to be:

$$B(E, F) = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}JF - \mathcal{V}\nabla_{\mathcal{H}JE}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}JF - \mathcal{H}\nabla_{\mathcal{H}JE}\mathcal{V}F\,.$$

Then $B(X, Y) = A_X JY - A_{JX} Y$ on horizontal vector fields, X and Y, on M. On quasi-Kähler submersions, B vanishes. However, the trace of B

does give important information about almost semi-Kähler submersions.

THEOREM 3.2 [Wa-Va2]. – *Let $f:(M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ be an almost Hermitian submersion whose total space, (M^{2m}, g, J), is almost semi-Kähler. Then the fibres, $(F^{2m-2n}, \hat{g}, \hat{J})$, are almost semi-Kähler if and only if $trB = 0$.*

In [Wa-Va2], we were concerned with what additional hypotheses allow the induction of the Gray-Hervella structure $\mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \mathfrak{M}_3$ onto the total space of an almost Hermitian submersion from the same structure on the fibres and on the base space.

THEOREM 3.3. – *Let $f:(M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ be an almost Hermitian submersion with both the fibres, $(F^{2m-2n}, \hat{g}, \hat{J})$, and the base space, $(N^{2n}, \check{g}, \check{J})$, almost semi-Kähler. Then the total space, (M^{2m}, g, J), is almost semi-Kähler if and only if the fibres are minimally embedded and $trB = 0$.*

4. – Superminimal fibres.

Superminimal submanifolds of an almost Hermitian manifold are a specialization of minimal submanifolds. They are variously called *negatively oriented-isoclinic* by T. Friedrich [Fr1], *isotropic harmonic* by M. J. Micallef and J. D. Moore [Mi-Mo], and *superminimal* by R. Bryant [Br]. They were known to E. Calabi [Ca], but were not explicitly so named in his 1967 article. Friedrich [Fr2] traces the concept to St. Kwietniewski [Kw] in 1902 and O. Borüvka [Bo] in 1928. Superminimal submanifolds have enjoyed a recent resurgence in the work of S. Gudmundsson and J. C. Wood (see, e.g., [Gu-Wo]).

DEFINITION 4.1 [Gu-Wo]. – *An almost Hermitian submanifold $(F, \hat{g}, \hat{J})$ of an almost Hermitian manifold (M^{2m}, g, J) is superminimally immersed (or F is superminimal) if $\nabla_V J = 0$ for all vector fields, V, tangent to F.*

Let us now consider superminimal fibres in an almost Hermitian submersion, $f:(M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$. There are four components of $g((\nabla_V J)E, F)$ on an almost Hermitian submersion. We easily find:

SM-1) $g((\nabla_V J)\, U, W) = \hat{g}(\widehat{\nabla}_V JU - J\widehat{\nabla}_V U, W)$,

SM-2) $g((\nabla_V J)\, U, X) = g(JT_V U - T_V JU, X)$,

SM-3) $g((\nabla_V J)\, X, U) = -g((\nabla_V J)\, U, X)$,

SM-4)) $g((\nabla_U J)\, X, Y) = -g(A_{JX} Y + A_X JY, U)$, for X, basic.

PROPOSITION 4.2. – *Superminimal fibres of an almost Hermitian submersion are minimal.*

PROOF. – The vanishing of the calculation SM-2 is equivalent to $JT_V U = T_V JU$. Therefore, $T_U U + T_{JU} JU = 0$, and the mean curvature vector field, H, of the fibre submanifolds is zero. *q.e.d.*

In fact, superminimal submanifolds are always minimal [Gu-Wo]. The vanishing of calculation SM-1 ensures that superminimal fibres are Kähler.

LEMMA 4.3. – *Let $f: (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ be an almost Hermitian submersion such that $A_X JX = 0$ for all horizontal local vector fields X. If the fibres, $(F^{2(m-n)}, \widehat{g}, \widehat{J})$, of f are superminimally embedded, then the horizontal distribution is completely integrable* $(A = 0)$.

PROOF. – $A_X JY - A_{JX} Y = 0$, by the standard polarization trick. Combining this with the vanishing of SM-4 yields the desired result. *q.e.d.*

5. – G_1-submersions.

DEFINITION 5.1. – *An almost Hermitian manifold (M^{2m}, g, J) is a G_1 manifold if $g(N(E, F), E) = 0$ for all vector fields, E, F, on M.*

G_1 manifolds were studied by L. Hervella and E. Vidal [He-Vid]. The Gray-Hervella class of G_1 structures is $\mathfrak{M}_1 \oplus \mathfrak{M}_3 \oplus \mathfrak{M}_4$. If the total space of an almost Hermitian submersion $f: (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ is G_1, then it is easy to verify that both the base space and the fibres, $(F^{2(m-n)}, \widehat{g}, \widehat{J})$, are G_1.

THEOREM 5.2. – *Let $f: (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ be an almost Hermitian submersion whose base space is a G_1 manifold. If the fibres are superminimally embedded, then the total space, (M^{2m}, g, J), is G_1.*

PROOF. – Let X be basic and U, V be vertical vector fields. In order to verify the G_1 property on M, we need only consider the following four calculations:

G1-1) $g(N(U, V), U) = \widehat{g}(\widehat{N}(U, V), U)$,

G1-2) $g(N(X, U), X) =$,

G1-3) $g(N(U, X), U) =$,

G1-4) $g(N(X, Y), X) = \check{g}(\check{N}(X_*, Y_*), X_*)$.

The fibres are Kähler, so G1-1 is zero. The second calculation, G1-2, vanishes on any almost Hermitian submersion because $[X, U]$ is vertical and the J-invariant vertical distribution of f is completely integrable. Only the third calculation, G1-3, requires analysis. Consider

$$g(N(U, X), U) = -g((\nabla_U J)\, JX, U) + g((\nabla_{JX} J)\, U, U) -$$

$$-g((\nabla_{JU} J)\, X, U) + g((\nabla_X J)\, JU, U) = g((\nabla_{JX} J)\, U, U) + g((\nabla_X J)\, JU, U)\,,$$

because $\nabla_U J = 0$. But $g((\nabla_{JX} J)\, U, U) + g((\nabla_X J)\, JU, U) = 0$ from eqn. (4.10) of [Gr1]. *q.e.d.*

6. – Hermitian submersions.

There are six calculations which must vanish in order to establish the integrability ($N = 0$) of the almost complex structure, J, on the total space of an almost Hermitian submersion (see [Gr1]):

N-1) $g(N(U, V), W) = \hat{g}(\widehat{N}(U, V), W)$,

N-2) $g(N(U, V), X) = 0$,

N-3) $g(N(X, U), V) = g((\nabla_{JU} J)X, V) + g((\nabla_U J)(JX), V) - g((\nabla_X J)(JU), V) - g((\nabla_{JX} J)\, U, V)$,

N-4) $g(N(X, U), Y) = 0$,

N-5) $g(N(X, Y), U) = (1/2)\, g(A_X Y + JA_{JX} Y + JA_X JY - A_{JX} JY, U)$,

N-6) $g(N(X, Y), Z) = \check{g}\,(\check{N}(X_*, Y_*), Z_*)$.

For N-2 $= 0$, note that the vertical distribution of an almost Hermitian submersion is J-invariant and completely integrable. For N-4 $= 0$, we may assume that X is basic. Then the Lie bracket $[X, U]$ is vertical.

THEOREM 6.1. – *Let* $f\colon (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ *be an almost Hermitian submersion whose base space is a Hermitian manifold. If the fibres are superminimally embedded and the vertical projection,* $\mathcal{V}(\nabla_X J)\, U$, *of* $(\nabla_X J)\, U$ *is* 0 *for all horizontal X and for all vertical U, then the total space,* (M^{2m}, g, J), *is Hermitian.*

PROOF. – The definition of superminimal fibres ($\nabla_U J = 0$) and $\mathcal{V}(\nabla_X J)\, U = 0$ imply that N-3 vanishes. Calculation SM-4 implies that $A_{JX} Y + A_X JY = 0$, for X, basic, which yields $A_{JX} JY = A_X Y$. Thus, N-5 is zero. *q.e.d.*

7. – $\mathfrak{M}_1 \oplus \mathfrak{M}_3$-submersions.

The Gray-Hervella class $\mathfrak{M}_1 \oplus \mathfrak{M}_3$ is the intersection of the classes G_1 ($\mathfrak{M}_1 \oplus \mathfrak{M}_3 \oplus \mathfrak{M}_4$) and the almost semi-Kähler class ($\mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \mathfrak{M}_3$). The G_1 property is inherited by the base space of a G_1-submersion from the total space. Therefore, by Thm. 3.1, the base space, $(N^{2n}, \check{g}, \check{J})$, of an almost Hermitian submersion $f\colon (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ whose total space is $\mathfrak{M}_1 \oplus \mathfrak{M}_3$ is a $\mathfrak{M}_1 \oplus \mathfrak{M}_3$ manifold if and only if the fibres, $(F^{2(m-n)}, \hat{g}, \hat{J})$, are minimally embedded. By Thm. 3.3, the fibres of a $\mathfrak{M}_1 \oplus \mathfrak{M}_3$-submersion are $\mathfrak{M}_1 \oplus \mathfrak{M}_3$ if and only if $trB = 0$. The induction of the $\mathfrak{M}_1 \oplus \mathfrak{M}_3$ property onto the total space of an almost Hermitian submersion follows from Thms. 3.3 and 5.2:

THEOREM 7.1. – *Let $f\colon (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ be an almost Hermitian submersion whose base space is a $\mathfrak{M}_1 \oplus \mathfrak{M}_3$ manifold. If the fibres are superminimally embedded and $trB = 0$, then the total space, (M^{2m}, g, J), is $\mathfrak{M}_1 \oplus \mathfrak{M}_3$.*

8. – Quasi-Kähler submersions.

We begin by observing that superminimal fibres in a quasi-Kähler submersion produce strong geometric restrictions on the horizontal distribution.

PROPOSITION 8.1. – *The horizontal distribution of a quasi-Kähler submersion with superminimal fibres is completely integrable $(A = 0)$.*

PROOF. – $A_X JX = 0$ on a quasi-Kähler submersion [Wa1]. Lemma 4.3 yields $A = 0$. *q.e.d.*

There are six calculations which must vanish in order to conclude that the total space of an almost Hermitian submersion is quasi-Kähler:

QK-1) $g((\nabla_U J)\, V + (\nabla_{JU} J)\, JV, W) = \hat{g}((\hat{\nabla}_U \hat{J})\, V + (\hat{\nabla}_{JU} \hat{J})\, \hat{J} V, W)$,

QK-2) $g((\nabla_U J)\, V + (\nabla_{JU} J)\, JV, X) = g(T_U JV - JT_U V - JT_U V - JT_{JU} JV, X)$,

QK-3) $g((\nabla_X J)\, U + (\nabla_{JX} J)\, JU, V) =$,

QK-4) $g((\nabla_X J)\, U + (\nabla_{JX} J)\, JU, Y) = -g(A_X JY - JA_X Y - JA_X U - JA_{JX} JU, Y)$,

QK-5) $g((\nabla_X J)\, Y + (\nabla_{JX} J)\, JY, U) = -g((\nabla_X J)\, U, Y) - g((\nabla_{JX} J)\, JU, Y)$,

QK-6) $g((\nabla_X J)\, Y + (\nabla_{JX} J)\, JY, Z) = \check{g}((\check{\nabla}_{X_*} \check{J})\, Y_* + (\check{\nabla}_{\check{J} X_*} \check{J})\, \check{J} Y_*, Z_*)$.

THEOREM 8.2. – *Let $f\colon (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ be an almost Hermitian submersion whose base space is quasi-Kähler. Assume that $A_X JX = 0$, for all horizontal local vector fields X. If the fibres are superminimal and $\mathcal{V}(\nabla_X J)\, U = 0$, for all horizontal X and for all vertical U, then the total space, (M^{2m}, g, J), is quasi-Kähler.*

PROOF. – $\mathcal{V}(\nabla_X J)\, U = 0$ implies that calculation QK-3 vanishes. The J-complex bilinearity of T from the assumed superminimality of the fibres gives QK-

$2 = 0$. From Lemma 4.3, $A_X JX = 0$ and SM-4 $= 0$ imply that $A = 0$, which then implies that the calculations QK-4 and QK-5 are zero. *q.e.d.*

9. – Almost Kähler submersions.

We know that the fibres of an almost Kähler submersion are minimal because they are holomorphically embedded submanifolds of the almost Kähler total space. It is well known [Wa1] that the horizontal distribution of a Kähler submersion is completely integrable. This is also true for the weaker class of almost Kähler submersions.

THEOREM 9.1. – *The horizontal distribution of an almost Kähler submersion, $f: (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$, is completely integrable* ($A \equiv 0$).

PROOF. – Let U be vertical, and let X and Y be horizontal vector fields on M with X, basic. Because M is almost Kähler,

$$0 = d\omega(U, X, Y) = g((\nabla_U J) X, Y) + g((\nabla_Y J) U, X) + g((\nabla_X J) Y, U) =$$

$$= g(\nabla_U JX - J\nabla_U X, Y) + g(\nabla_Y JU - J\nabla_Y U, X) + g(\nabla_X JY - J\nabla_X Y, U) =$$

$$= g(A_{JX} U - JA_X U, Y) + g(A_Y JU - JA_Y U, X) + g(A_X JY - JA_X Y, U).$$

On a quasi-Kähler submersion [Wa1], we have $A_X JY = A_{JX} Y = -A_X JY$ and $A_X JU = A_{JX} U = -JA_X U$. Therefore, $0 = -3g(A_X JY, U) - g(A_X Y, JU) = g(4A_X U, JY)$. Hence, $A \equiv 0$. *q.e.d.*

There are four types of components of the exterior differential, $d\omega$, of the fundamental Kähler 2-form, ω, on M to be considered when studying almost Kähler submersions:

AK-1) $d\omega(U, V, W) = \hat{d}\hat{\omega}(U, V, W)$,

AK-2) $d\omega(U, V, X) =$,

AK-3) $d\omega(U, X, Y) =$,

AK-4) $d\omega(X, Y, Z) = \check{d}\,\check{\omega}(X_*, Y_*, Z_*)$.

We shall use the vanishing of the vertical projection of $(\nabla_X J) U$ to help establish the existence of many Gray-Hervella structures. This is a natural condition to consider in view of (see also Prop. 10.2):

THEOREM 9.2. – *Let $f: (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ be an almost Kähler submersion. Then,*

$$\mathcal{V}(\nabla_X J) U = 0 .$$

PROOF. – $d\omega(U, V, X) = g((\nabla_U J) V, X) + g((\nabla_V J) X, U) + g((\nabla_X J) U, V)$. The first two terms on the right sum to $g(T_{JV} U - JT_V U + JT_V U - T_V JU, X)$. But $T_{JV} U = T_V JU$ on a quasi-Kähler submersion [Wa1]. Therefore, $0 = d\omega(U, V, X) = g((\nabla_X J) U, V)$. *q.e.d.*

LEMMA 9.3. – *Let $f: (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ be an almost Hermitian submersion. If $A = 0$, then*

$$d\omega(U, X, Y) = 0 .$$

PROOF. – Consider $d\omega(U, X, Y) = g((\nabla_U J) X, Y) + g((\nabla_X J) Y, U) + g((\nabla_Y J) U, X)$. Now $g((\nabla_X J) Y, U) = g(A_X JY, U) + g(A_X Y, JU) = 0$. Similarly, $g((\nabla_Y J) U, X) = 0$. Assume now that X is basic. Then, $g((\nabla_U J) X, Y) = g(A_{JX} U, Y) + g(A_X U, JY) = 0$. *q.e.d.*

THEOREM 9.4. – *Let $f: (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ be an almost Hermitian submersion whose base space is almost Kähler. Assume that $A_X JX = 0$, for all horizontal local vector fields X. If the fibres are superminimal and $\mathcal{V}(\nabla_X J) U = 0$, for all horizontal X and for all vertical U, then the total space, (M^{2m}, g, J), is almost Kähler.*

PROOF. – Superminimal fibres are Kähler, so calculation AK-1 is 0. For calculation AK-2, consider $d\omega(U, V, X) = g((\nabla_U J) V, X) + g((\nabla_V J) X, U) + g((\nabla_X J) U, V)$. Then, $d\omega(U, V, X) = 0$ because $\nabla_U J = 0$ and $\mathcal{V}(\nabla_X J) U = 0$. Lemma 4.3 yields $A = 0$. Thus, for (AK-3), $d\omega(U, X, Y) = 0$, by Lemma 9.3. By hypothesis, AK-4 is 0. Therefore, M is almost Kähler. *q.e.d.*

THEOREM 9.5. – *Let $f: (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ be a quasi-Kähler submersion whose fibres and base space are almost Kähler. Assume that the horizontal distribution is completely integrable. If $\mathcal{V}(\nabla_X J) U = 0$, for all horizontal X and for all vertical U, then the total space, (M^{2m}, g, J), is almost Kähler.*

PROOF. – By Lemma 9.3, we need only consider calculation AK-2. In $d\omega(U, V, X) = g((\nabla_U J) V, X) + g((\nabla_V J) X, U) + g((\nabla_X J) U, V)$, $g((\nabla_V J) X, U) = g(JT_V U - T_V JU, X)$. Similarly, $g((\nabla_U J) V, X) = g(T_U JV - JT_U U, X)$. Hence, $g((\nabla_V J) X, U) + g((\nabla_U J) V, X) = g(T_U JV - T_V JU, X)$ which vanishes on the quasi-Kähler manifold, M . Thus, AK-2 is 0. *q.e.d.*

10. – Nearly Kähler submersions.

The O'Neill configuration tensor T is J-complex bilinear on a nearly Kähler submersion [Wa1]. In [Wa-Va1], L. Vanhecke and the author showed that the horizontal distribution of a nearly Kähler submersion is completely integrable ($A = 0$).

PROPOSITION 10.1. – *Let* $f\colon (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ *be a nearly Kähler submersion with Kähler fibres,* $(F^{2m-2n}, \hat{g}, \hat{J})$. *Then the fibres are superminimally embedded.*

PROOF. – Only calculation SM-1 must be checked because T is J-complex bilinear and $A = 0$. But the assumption of Kähler fibres yields SM-1$=0$. *q.e.d.*

PROPOSITION 10.2. – *Let* $f\colon (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ *be a nearly Kähler submersion. Then,*

$$(\nabla_X J)\, U = 0 \quad \textit{and}$$

$$(\nabla_U J)\, X = 0$$

for all horizontal X *and for all vertical* U.

PROOF. – $g((\nabla_X J)\, U, Y) = g(A_X JU - JA_X U, Y) = 0$. Use the standard polarization trick on $0 = (\nabla_{(X+U)} J)(X + U)$ to obtain $g((\nabla_X J)\, U, V) = g((\nabla_U J)\, X, V) = g(T_U JX - JT_U X, V) = 0$ on a nearly Kähler submersion [Wa1]. $g((\nabla_X J)\, U, Y) = g(A_X JU - JA_X U, Y) = 0$. Let X by basic. Then, $g((\nabla_U J)\, X, Y) = g(A_X JU - JA_X U, Y) = 0$. *q.e.d.*

There are four relevant calculations for confirming the existence of the nearly Kähler property on the total space of an almost Hermitian submersion:

NK-1) $g((\nabla_U J)\, U, V) = \hat{g}((\hat{\nabla}_U \hat{J})\, U, V)$,

NK-2) $g((\nabla_U J)\, U, X) = g(T_U JU - JT_U U, X)$,

NK-3) $g((\nabla_X J)\, X, U) = g(A_X JX, U)$,

NK-4) $g((\nabla_X J)\, X, Y) = \check{g}((\check{\nabla}_{X_*} \check{J})\, X_*, Y_*)$.

THEOREM 10.3. – *Let* $f\colon (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ *be an almost Hermitian submersion with the fibres,* $(F^{2m-2n}, \hat{g}, \hat{J})$, *and the base space both nearly Kähler, and with* $A_X JX = 0$. *Suppose that the O'Neill configuration tensor, T, is J-complex bilinear. Then the total space,* (M^{2m}, g, J), *is nearly Kähler.*

PROOF. – Each of the calculations, NK-1 through NK-4, vanishes. *q.e.d.*

THEOREM 10.4. – *Let* $f\colon (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ *be an almost Hermitian submersion with the base space nearly Kähler, and with* $A_X JX = 0$. *If the fibres,* $(F^{2m-2n}, \hat{g}, \hat{J})$, *are superminimal, then the total space,* (M^{2m}, g, J), *is nearly Kähler.*

PROOF. – Superminimal fibres are Kähler and the O'Neill configuration tensor, T, is then J-complex bilinear. By Thm. 10.3, (M^{2m}, g, J), is nearly Kähler. *q.e.d.*

COROLLARY 10.5. – *Let* $f\colon (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ *be a quasi-Kähler submersion with the base space nearly Kähler. If the fibres,* $(F^{2m-2n}, \hat{g}, \hat{J})$, *are superminimal, then the total space,* (M^{2m}, g, J), *is nearly Kähler.*

PROOF. – From Prop. 8.1, $A = 0$ on a quasi-Kähler submersion with superminimal fibres. Thm. 10.4 then yields the stated result. *q.e.d.*

While this article was in preparation, we became aware of a manuscript, «A note on almost Kähler and nearly Kähler submersions,» by M. Falcitelli and A. M. Pastore [Fa-Pa]. Some of these conclusions regarding nearly Kähler submersions and related results on almost Kähler submersions are proved therein.

11. – Kähler submersions.

Kähler submersions were originally defined in [Wa1], and were extensively studied by D. L. Johnson [Jo]. Obviously, T is J-complex bilinear and $A = 0$ on a Kähler submersion. Moreover, the fibres of a Kähler submersion are superminimal. In order to prove that the total space of an almost Hermitian submersion is Kähler, we must establish the vanishing of the following six calculations:

K-1) $g((\nabla_U J)\, V, W) = \hat{g}((\hat{\nabla}_U \hat{J})\, V, W)$,

K-2) $g((\nabla_U J)\, V, X) = g(T_U JV - JT_U V, X)$,

K-3) $g((\nabla_X J)\, U, V) =$,

K-4) $g((\nabla_X J)\, U, Y) = -g(A_X JY - JA_X Y, U)$,

K-5) $g((\nabla_X J)\, Y, U) = g(A_X JY - JA_X Y, U) = -g((\nabla_X J)\, U, Y)$,

K-6) $g((\nabla_X J)\, Y, Z) = \check{g}((\check{\nabla}_{X_*} \check{J})\, Y_*, Z_*)$.

THEOREM 11.1. – *Let* $f\colon (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ *be an almost Hermitian submersion with Kähler fibres and Kähler base space,* $(N^{2n}, \check{g}, \check{J})$. *If the*

O'Neill configuration tensors, T and A, are J-complex bilinear, and $\mathcal{V}(\nabla_X J)U = 0$, *then* (M^{2m}, g, J) *is Kähler.*

PROOF. – The vanishing of calculation K-2 is equivalent to the J -complex bilinearity of the O'Neill configuration tensor, T. The J -complex bilinearity of A implies that K-4 and K-5 are zero. K-1 and K-6 are zero because the fibres and base space are assumed to be Kähler. $\mathcal{V}(\nabla_X J)\, U = 0$ gives calculation K-3 = 0. Therefore, (M^{2m}, g, J) is Kähler. *q.e.d.*

THEOREM 11.2. – *Let* $f\colon (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ *be an almost Kähler submersion with superminimal fibres onto a Kähler base manifold,* $(N^{2n}, \check{g}, \check{J})$. *Then* (M^{2m}, g, J) *is Kähler.*

PROOF. – The Kähler base space is G_1, so (M^{2m}, g, J) is G_1 by Thm. 5.2. An almost Kähler G_1 manifold is Kähler. *q.e.d.*

Thm. 11.2 leads to the affirmative resolution of the four-dimensional Goldberg Conjecture on the total space of an almost Kähler submersion [Wa2] by establishing that the fibres of an almost Kähler submersion $f\colon (M^4, g, J) \to (N^2, \check{g}, \check{J})$ with M, Einstein, are superminimal. This result also follows from the superminimal fibre conclusion using Thm. 6.1 along with the fact that $\mathcal{V}(\nabla_X J)\, U = 0$ on an almost Kähler submersion (Thm. 9.2).

THEOREM 11.3. – *Let* $f\colon (M^{2m}, g, J) \to (N^{2n}, \check{g}, \check{J})$ *be a nearly Kähler submersion with Kähler fibres and Kähler base space,* $(N^{2n}, \check{g}, \check{J})$. *Then,* (M^{2m}, g, J) *is Kähler.*

PROOF. – Prop. 10.2 gives $\mathcal{V}(\nabla_X J)\, U = 0$, and $A = 0$ from [Wa-Va2]. Thm. 9.5 now implies that M is almost Kähler, and, therefore, Kähler ($\mathfrak{M}_1 \cap \mathfrak{M}_2 = \{0\}$). *q.e.d.*

REFERENCES

[Bo] BORŮVKA O., *Sur une classe de surfaces minima plone'es dans un space á quatre dimensions a courbure constante*, C. R. Acad. Sci., **187** (1928), 334-336.

[Br] BRYANT R., *Conformal and minimal immersions of compact surfaces into the 4-sphere*, J. Differential Geom., **17** (1982), 455-473.

[Ca] CALABI E., *Minimal immersions of surfaces into Euclidean spheres*, J. Differential Geom., **1** (1967), 111-125.

[Fa-Pa] FALCITELLI M. - A. M. PASTORE, *A note on almost Kähler and nearly Kähler submersions*, (preprint), 1998.

[Fr1] FRIEDRICH T., *On surfaces in four-spaces*, Ann. Global Anal. & Geom., **2** (1984), 257-287.

[Fr2] FRIEDRICH T., *On superminimal surfaces*, Arch. Math. (Brno), **33** (1997), 41-56.

[Gr1] GRAY A., *Minimal varieties and almost Hermitian submanifolds*, Michigan Math. J., **12** (1965), 273-287.

[Gr2] GRAY A., *Pseudo-Riemannian almost product manifolds and submersions*, J. Math. Mech., **16** (1967), 715-738.

[Gr-He] GRAY A. - L. HERVELLA, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. **CXXIII** (1980), 35-58.

[Gu-Wo] GUDMUNDSSON S. - J. C. WOOD, *Harmonic morphisms between almost Hermitian manifolds*, Boll. Un. Mat. Ital. **11-B**, Supp. 2 (1997), 185-197.

[He-Vid] HERVELLA L. - E. VIDAL, *Nouvelles géométries pseudo-Kähleriennes G_1 et G_2*, C. R. Acad. Sci. Paris **283** (1976), 115-118.

[Jo] JOHNSON D. L., *Kähler submersions and holomorphic connections*, J. Diff. Geometry **15** (1980), 71-79.

[Ko-No] KOBAYASHI S. - K. NOMIZU, *Foundations of Differential Geometry, Vol. II*, Interscience, New York 1969.

[Kw] KWIETNIEWSKI ST., *Über Flachen des vierdimensionalen Raumes, deren sämtliche Tangentialbenen untereinander gleichwinklig sind, und ihre Beziehung zu ebenen Kurven*, Dissertation, Zürich, 1902.

[Mi-Mo] MICALLEF M. J. - J. D. MOORE, *Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes*, Ann. Math., **127** (1988), 199-227.

[O'N] O'NEILL, B., *The fundamental equations of a submersion*, Michigan Math. J., **13** (1966), 459-469.

[Wa1] WATSON B., *Almost Hermitian submersions*, J. Diff. Geometry, **11** (1976), 147-165.

[Wa2] WATSON B., *The four-dimensional Goldberg Conjecture is true for almost Kähler submersions* (to appear) J. of Geometry, 1999.

[Wa-Va1] WATSON B. - L. VANHECKE, *J-symmetries and J-linearities of the configuration tensors of an almost Hermitian submersion*, Simon Stevin, **51** (1977), 139-156.

[Wa-Va2] WATSON B. - L. VANHECKE, *The structure equation of an almost semi-Kähler submersion*, Houston Math. J., **5** (1979), 295-305.

Bill Watson: Dept. of Mathematics & Computer Science
St. John's University, Jamaica, NY 11439 USA

Pervenuta in Redazione
il 7 aprile 1999