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Rita Giuliano Antonini

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Subgaussianity and Exponential Integrability of Real Random Variables: Comparison of the Norms (*).

RITA GIULIANO ANTONINI

Sunto. – *Nello spazio delle variabili aleatorie subgaussiane definite su* $(\Omega, A, \mathbb{P})$, *si studia l'equivalenza tra la norma subgaussiana e la norma di Fernique, dando valutazioni numeriche delle costanti di equivalenza. A tale scopo si fa uso di una nuova caratterizzazione della norma subgaussiana delle variabili aleatorie simmetriche.*

0. – Introduction.

The concept of subgaussian random variable (which was introduced in 1960 by Kahane [1]) has been studied in detail by several authors: see [2], [3], [4], [5], [6].

In particular, a norm τ was introduced on the space of all subgaussian random variables in the paper [2].

Another concept which has drawn the attention of mathematicians is that of exponentially integrable random variable. As to our knowledge, the term «exponentially integrable» is used for the first time in [5], but the space of such variables has been studied earlier, for instance by Fernique [7], who had introduced a suitable norm σ on it.

Moreover, it is well known that the space of subgaussian random variables coincides with the subspace of exponentially integrable centered random variables.

In view of these facts, a question that quite naturally arises in whether the two mentioned norms are equivalent (i.e. there exist positive constants a and b such that $a\sigma(X) \leqslant \tau(X) \leqslant b\sigma(X)$).

Indeed, this question has been answered affirmatively in [6], and, in the present paper we are concerned with the problem of giving numerical evaluations for a and b (sections 2 and 4 respectively). The result of section 4 is obtained by means of a new characterisation of the subgaussian norm τ, which we discuss in section 3.

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We point out that our results improve those of [6] (see remarks (2.6) and (4.18)).

We briefly sum up some preliminary results in section 1, in order to make the exposition as self-contained as possible.

1. – The spaces $SG(\Omega)$ and $EI(\Omega)$.

Let X be a random variable, defined on the probability space $(\Omega, A, \mathbb{P})$. The following definitions are found in the paper [2]:

(1.1). DEFINITION. – X is said to have *subgaussian distribution* (or simply to be *subgaussian*) if there exists a real number $a \geqslant 0$ such that, for every $t \in \mathbb{R}$,

$$E[e^{tX}] \leqslant exp\left(\frac{t^2}{2}a^2\right).$$

(1.2). REMARK. – It easily follows from the definition that any subgaussian random variable is centered.

(1.3). DEFINITION. – The *standard gaussian* of X is the number

$$\tau(X) = \inf\left\{a \geqslant 0 \colon E[e^{tX}] \leqslant \exp\left(\frac{t^2}{2}a^2\right),\ \forall t \in \mathbb{R}\right\}. \tag{1.4}$$

We shall denote by $SG(\Omega)$ the set of subgaussian random variables. The following result is proved in [2]:

(1.5). THEOREM. – *$SG(\Omega)$ is a vector space and, moreover, τ is a norm on $SG(\Omega)$.* ∎

The following definition is given in the paper [5]:

(1.6). DEFINITION. – X is said to be *exponentially integrable* if there exists a number $\varepsilon > 0$ such that

$$E[e^{\varepsilon X^2}] < +\infty .$$

We shall denote by $EI(\Omega)$ the space of exponentially integrable variables. As to the structure of $EI(\Omega)$, we have the following result, due to Fernique [7]:

(1.7). THEOREM. – (a) *$EI(\Omega)$ is a vector space.*

(b) *For every r.v. X in $EI(\Omega)$, define*

$$\sigma(X) = \inf\{a > 0 : E[e^{X^2/a^2}] \leq 2\}.$$

Then σ is a norm on $EI(\Omega)$.

The following theorem relates $EI(\Omega)$ and $SG(\Omega)$; it follows from the results of [2] and an application of Stirling formula:

(1.8). THEOREM. – *Let X be a centered random variable. The following conditions are equivalent*:

(a) *X is subgaussian*;

(b) *we have*

$$\sup_k \left(\frac{2^k k!\, EX^{2k}}{(2k)!} \right)^{1/2k} < +\infty\ ;$$

(c) *X is exponentially integrable.*

We immediately deduce the

(1.9). COROLLARY. – *$SG(\Omega)$ coincides with the subspace of $EI(\Omega)$ consisting of the random variables which are centered and have finite moments of any order.*

(1.10). REMARK. – (a) $\Rightarrow$ (c) of theorem (1.8) follows also from the proof of proposition (2.1) of section 2.

2. – Comparison of τ and σ. First part.

In this section we give a numerical evaluation for the constant a (see introduction). We have indeed the following result:

(2.1). PROPOSITION. – *Let X be a subgaussian variable (we recall that, by* (1.9), *this amounts to saying that X is exponentially integrable, centered and has finite moments of any order). Then*

$$\sigma(X) \leq \sqrt{2 + 2\sqrt{2}}\ \tau(X).$$

PROOF. – The following inequality is proved in [2]: for every $k \in \mathbb{N}$, we have

$$E|X|^k \leq 2\left(\frac{k}{e}\right)^{k/2} \tau^k(X). \tag{2.2}$$

Fix now $a > 0$. By (2.2) we have

$$E[e^{X^2/a^2}] = 1 + \sum_{k \geqslant 1} \frac{1}{k!} EX^{2k} \frac{1}{a^{2k}} \leqslant 1 + 2 \sum_{k \geqslant 1} \frac{1}{k!} \frac{(2k)^k}{e^k} \tau^{2k}(X) \frac{1}{a^{2k}}. \tag{2.3}$$

It is easily seen, by induction on n, that, for every $n \geqslant 1$,

$$\frac{n^n}{e^n n!} \leqslant \frac{1}{e} < \frac{1}{2}. \tag{2.4}$$

For $n = 2k$, $k \geqslant 1$, (2.4) becomes

$$\frac{(2k)^{2k}}{e^{2k}(2k)!} < \frac{1}{2},$$

or equivalently,

$$2 \frac{(2k)^k}{e^k} < \frac{e^k (2k)!}{(2k)^k}.$$

Hence the last quantity in (2.3) is majorized with

$$1 + \sum_{k \geqslant 1} \frac{(2k)!}{k!(2k)^k} e^k \tau^{2k}(X) \frac{1}{a^{2k}}. \tag{2.5}$$

By Stirling formula, it is easy to see that

$$\frac{(2k)!}{k!(2k)^k} \leqslant \sqrt{2}\ 2^k e^{-k}.$$

Hence (2.5) is not greater than

$$1 + \sqrt{2} \sum_{k \geqslant 1} \left(\frac{2\tau^2(X)}{a^2} \right)^k \leqslant 1 + \sqrt{2} \sum_{k \geqslant 1} \left(\frac{2\tau^2(X)}{a^2} \right)^k =$$

$$= 1 + \sqrt{2} \frac{2\tau^2(X)}{a^2 - 2\tau^2(X)} = \frac{a^2 + 2(\sqrt{2} - 1)\tau^2(X)}{a^2 - 2\tau^2(X)},$$

for $a > \sqrt{2}\tau(X)$.

The last fraction above is not greater than 2 for

$$a \geqslant \sqrt{2 + 2\sqrt{2}}\ \tau(X);$$

hence we have the inclusion

$$[\sqrt{2 + 2\sqrt{2}}\ \tau(X), +\infty) \subset \{a > 0 : E[e^{X^2/a^2}] \leqslant 2\}.$$

We get the desired conclusion by taking the infima of the above sets. ■

(2.6). REMARK. – In the proof of theorem 4 in section 2 of [6] it is shown that $\sigma(X) \leqslant 3\sqrt{2}\tau(X)$, and we have

$$\sqrt{2+2\sqrt{2}} \cong 2,19\ldots < 3\sqrt{2} \cong 4,24\ldots .$$

3. – A characterisation of the gaussian standard for symmetric random variables.

In this section we shall consider only symmetric random variables, which we shall call, for the sake of brevity, simply random variables in the following.

For every real number $t \neq 0$, let M_t be the convex function defined by

$$M_t(x) = \frac{\cosh tx - 1}{e^{t^2/2} - 1}, \qquad x \in \mathbb{R}.$$

For $t = 0$, put

$$M_0(x) = x^2, \qquad x \in \mathbb{R}.$$

Clearly, for every random variable X, $t \to E[M_t(X)]$ is a symmetric function. Moreover,

$$E[M_t(X)] = \frac{E[e^{tX}] - 1}{e^{t^2/2} - 1}.$$

Now let X be a fixed random variable. Put, for each t,

$$A_t = \left\{ a > 0 \colon E\left[M_t\left(\frac{X}{a} \right) \right] \leqslant 1 \right\}.$$

We shall assume that A_t is nonemtpy.

We have

(3.1). PROPOSITION. $A_t = A_{-t}$. *Moreover A_t is a closed, left bounded half-line.*

The proof of (3.1) is an easy consequence of two lemmas:

(3.2). LEMMA. – *The (symmetric) function $t \to E[e^{tX}]$ is increasing for $t > 0$ (hence it is decreasing for $t < 0$).*

Now, for every t, put

$$\tau_t(X) = \inf A_t.$$

(3.3). LEMMA. –

$$E\left[M_t\left(\frac{X}{\tau_t(X)}\right)\right] \leqslant 1\,.$$

The proofs of (3.2) and (3.3) are straightforward.

Now put

$$E_t = \{X\colon \tau_t(X) < +\infty\}\,.$$

We are interested in analyzing the structure of E_t and the properties of τ_t on E_t. Since $A_t = A_{-t}$, we have

$$\tau_t(X) = \tau_{-t}(X)\,; \qquad E_t = E_{-t}.$$

Hence, there is no loss of generality in confining ourselves to the case $t \geqslant 0$.

We shall prove the following result:

(3.4). THEOREM. *E_t is a vector space and τ_t is a norm on E_t.*

PROOF. – It is easy to see that E_t is a vector space and τ_t is a seminorm on it (recall that M_t is convex). It remains to see that $\tau_t(X) = 0$ implies $X = 0$. The relation $\tau_t(X) = 0$ amounts to saying that, for every $a > 0$, we have

$$E\left[M_t\left(\frac{X}{a}\right)\right] \leqslant 1\,,$$

or, equivalently,

$$E[e^{tX/a}] \leqslant e^{t^2/2}.$$

By the exponential Chebicev inequality, we deduce that, for every $u > 0$

$$P(X > u) = P(e^{tX/a} > e^{tu/a}) \leqslant E[e^{tX/a}]\, e^{-tu/a} \leqslant e^{t^2/2} e^{-tu/a}.$$

By letting a tend to zero, we get $P(X > u) = 0$ for every $u > 0$, hence $P(X > 0) = 0$, and also $P(X \neq 0) = 0$ because of the symmetry of X. ∎

Now, for every random variable X, put

$$\hat{\tau}(X) = \sup_t \tau_t(X)\,,$$

and consider the set

$$G(\Omega) = \left\{ x \in \bigcap_t E_t : \hat{\tau}(X) < +\infty \right\}.$$

We are interested in the structure of the pair $(G(\Omega), \hat{\tau})$.

First of all, $G(\Omega)$ is obviously non-empty, since all gaussian variables belong to it. Moreover, it is clear, by its very construction, that

(3.5). PROPOSITION. – *$G(\Omega)$ is a vector space and $\hat{\tau}$ is a norm.*

We now turn to examine the relation between $(G(\Omega), \hat{\tau})$ and the normed space $(SG(\Omega), \tau)$ defined in section 1. We state the following:

(3.6). THEOREM. – *$G(\Omega)$ coincides with the subspace of $SG(\Omega)$ consisting of the symmetric random variables. Moreover, $\hat{\tau} = \tau$ on $G(\Omega)$.*

We need a simple lemma, whose proof is straightforward:

(3.7). LEMMA. – *Let X be a random variable. Put*

$$A = \left\{ a > 0 : E\left[M_t\left(\frac{X}{a} \right) \right] \leqslant 1 \ \forall t \right\} = \{ a > 0 : E[e^{tX/a}] \leqslant e^{t^2/2} \ \forall t \};$$

$$B = \{ b > 0 : E[e^{tX}] \leqslant e^{b^2 t^2/2} \forall t \}.$$

Then we have $A = B$.

PROOF of (3.6). – We have $A = \bigcap_t A_t$; since A_t is a left bounded half-line for each t, the same is true for A. Moreover, by the preceding lemma, for every random variable X, we have

$$\hat{\tau}(X) = \sup_t \tau_t(X) = \inf A = \inf B = \tau(X). \qquad \blacksquare$$

4. – Comparison of τ and σ. Second part.

In the first part of this section we shall consider symmetric random variables and we shall use the characterization of τ given in section 3, namely the formula

$$\tau(X) = \sup_t \tau_t(X). \tag{4.1}$$

We begin with the following

(4.2). PROPOSITION. – *For every symmetric random variable X, we have*

$$\sup_{|t| \leqslant \sqrt{2}} \tau_t(X) \leqslant \sigma(X).$$

For the proof, we need some preliminary facts.

Consider the function M defined by

$$M(x) = e^{x^2} - 1, \qquad x \in \mathbb{R}.$$

It is obvious that

$$\sigma(x) = \inf\left\{a > 0 \colon E\left[M\left(\frac{X}{a}\right)\right] \leqslant 1\right\}. \tag{4.3}$$

We shall use also the functions M_t defined in section 3. Then we have the easy

(4.4). LEMMA. – *For every t, with* $|t| \leqslant \sqrt{2}$, *we have*

$$M_t(x) \leqslant M(x),$$

for every x.

PROOF of (4.2). – Let $a > 0$ be such that

$$E\left[M\left(\frac{X}{a}\right)\right] \leqslant 1.$$

From the preceding lemma we get

$$E\left[M_t\left(\frac{X}{a}\right)\right] \leqslant 1,$$

for each t, with $|t| \leqslant \sqrt{2}$.

This amounts to saying that the following inclusion holds:

$$\left\{a > 0 \colon E\left[M\left(\frac{X}{a}\right)\right] \leqslant 1\right\} \subset A_t,$$

so that, by (4.3), we obtain

$$\sigma(X) \geqslant \tau_t(X).$$

By taking the supremum with respect to t in the preceding relation, we conclude the proof. ■

We now pass to evaluate $\sup_{|t| > \sqrt{2}} \tau_t(X)$.

(4.5). LEMMA. – *For every pair of real numbers t,x, we have*

$$e^{tx} + e^{-tx} \leqslant e^{(1/2)(t^2+x^2)} + 1 \leqslant e^{(1/2)(t^2+x^2)} + 2\,.$$

(4.6). LEMMA. – *For every pair of real numbers t, x, we have*

$$M_t\left(\frac{x}{2}\right) \leqslant \frac{1}{2} M_t(x)\,.$$

The proofs of (4.5) and (4.6) are simple exercises.

(4.7). PROPOSITION. – *For every symmetric random variable X, we have*

$$\sup_{|t| > \sqrt{2}} \tau_t(X) \leqslant \sqrt{2}\;\sigma(X)\,.$$

PROOF. – From lemma (4.5) we easily get the relation

$$M_t(x) \leqslant \frac{1}{2}\left(1 + \frac{1}{e^{t^2/2} - 1}\right) e^{x^2/2}. \tag{4.8}$$

Since $|t| > \sqrt{2}$, we have

$$\frac{1}{2}\left(1 + \frac{1}{e^{t^2/2} - 1}\right) \leqslant \frac{1}{2}\left(1 + \frac{1}{e-1}\right) = \frac{e}{2(e-1)} = \beta\,. \tag{4.9}$$

We observe that $\beta < 1$, so that

$$\frac{1}{2\beta} > \frac{1}{2}\,. \tag{4.10}$$

Now let $a > 0$ be such that

$$E\left[M\left(\frac{X}{a}\right)\right] \leqslant 1\,;$$

from (4.8) and (4.9) we get

$$E\left[M_t\left(\frac{X\sqrt{2}}{a}\right)\right] \leqslant \beta E[e^{X^2/a^2}] = \beta E\left[M\left(\frac{X}{a}\right)\right] + \beta \leqslant 2\beta\,.$$

Hence, we deduce from lemma (4.6) and relation (4.10) that

$$E\left[M_t\left(\frac{X}{a\sqrt{2}}\right)\right] = E\left[M_t\left(\frac{1}{2}\,\frac{X\sqrt{2}}{a}\right)\right] \leqslant \frac{1}{2} E\left[M_t\left(\frac{X\sqrt{2}}{a}\right)\right] \leqslant \beta < 1\,.$$

The preceding relation says that $a\sqrt{2} \in A_t$, that is

$$a\sqrt{2} \geqslant \tau_t(X). \tag{4.11}$$

On taking the infimum with respect to a in (4.11), and recalling (4.3) again, we get

$$\sqrt{2}\ \sigma(X) \geqslant \tau_t(X),$$

and now, by taking the supremum with respect to t, we obtain the required relation. ■

Propositions (4.2) and (4.7), together with (4.1) yield

(4.12). PROPOSITION. – *For every symmetric subgaussian random variable X, we have*

$$\tau(X) \leqslant \sqrt{2}\,\sigma(X). \qquad ■$$

We now drop the assumption of symmetry, and use an argument of symmetrization: let X be any subgaussian variable and Y an independent copy of X. By Jensen inequality and (4.12) we have

$$E[e^{tX}] \leqslant E[e^{t(X-Y)}] \leqslant \exp\left[\frac{t^2}{2}2\sigma^2(X-Y)\right] \leqslant exp\left[\frac{t^2}{2}8\sigma^2(X)\right],$$

since σ is a norm and $\sigma(X) = \sigma(Y)$.

Hence we deduce the

(4.13). PROPOSITION. – *For every subgaussian random variable X, we have*

$$\tau(X) \leqslant 2\sqrt{2}\,\sigma(X).$$

(4.14). REMARK. – It is hardly possible to evaluate the constant b by the methods of [6] (see introduction).

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Dipartimento di Matematica, Università di Pisa, Via F. Buonarroti 2, 56100 Pisa, Italy

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