Tommaso Pacini

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TOMMASO PACINI

Sunto. – *Data una varietà Riemanniana orientata (M, g), il fibrato principale $SO_g(M)$ di basi ortonormali positive su (M, g) ha una parallelizzazione canonica dipendente dalla connessione di Levi-Civita. Questo fatto suggerisce la definizione di una classe molto naturale di strutture quasi-complesse su (M, g). Dopo le necessarie definizioni, discutiamo qui l'integrabilità di queste strutture, esprimendola in termini della struttura Riemanniana g.*

1. – Introduction.

Let M be a smoothly parallelizable m-dimensional differentiable manifold. A parallelization of M is, basically, the choice of an isomorphism between the tangent plane $T_x M$ and $\mathbb{R}^m$ that varies smoothly with respect to the parameter $x \in M$. Such a choice allows one to smoothly transfer a fixed structure, such as a complex structure, from $\mathbb{R}^m$ to the tangent bundle TM over M, thus giving M the additional structure of, for example, an almost complex manifold.

This is enough to prove that any even-dimensional parallelizable manifold admits an almost complex structure.

Let us now consider, for a fixed oriented m-dimensional Riemannian manifold (M, g), the $SO(m)$-principal fibre bundle of positively oriented orthonormal frames on (M, g): call it $SO_g(M)$, and let $\pi: SO_g(M) \to M$ be the usual projection.

It is well known that $SO_g(M)$ possesses a standard parallelization. It is defined as follows.

Given a principal fibre bundle $P(M, G)$, the action of the Lie group G on the total space P induces a homomorphism σ of the Lie algebra $\mathfrak{g}$ of G into the Lie algebra $\Lambda^0(TP)$ of vector fields on P.

For $A \in \mathfrak{g}$, we will denote $\sigma(A)$ by A^*.

For $u \in P$, let V_u be the tangent space to the fibre in u.

Since the action of G sends each fibre into itself, for each $u \in P$ σ induces a homomorphism $\sigma_u: \mathfrak{g} \to V_u$ defined by $A \mapsto A_u^*$ which is an isomorphism because G acts freely on P and $\dim(\mathfrak{g}) = \dim(V_u)$.

We have thus proved that for each $u \in SO_g(M)$, V_u is canonically isomorphic to the Lie algebra $\mathfrak{o}(m)$ of $SO(m)$.

Consider now a connection on $SO_g(M)$, i.e. a right-invariant distribution H on $SO_g(M)$ such that for all $u \in SO_g(M)$, $H_u \oplus V_u = T_u SO_g(M)$.

The differential of π at u, $\pi_*[u]$: $T_u SO_g(M) \to T_{\pi(u)} M$, restricts to an isomorphism between H_u and $T_{\pi(u)}M$, which we will continue to denote by $\pi_*[u]$. Remember that each $u \in SO_g(M)$ is a basis of $T_{\pi(u)}M$; the frame $u = \{u_i\}$ pulls back to a frame of H_u and thus defines the isomorphism

$$B_u: \mathbb{R}^m \to H_u,$$

$$e_i \mapsto \pi_*[u]^{-1}(u_i),$$

where $\{e_i\}$ is the standard basis of $\mathbb{R}^m$.

We have thus shown that any connection defines an isomorphism (which is smoothly dependent on u) between $T_u SO_g(M) = H_u \oplus V_u$ and $\mathbb{R}^m \oplus \mathfrak{o}(m)$, i.e. a parallelization of $SO_g(M)$.

In what follows we will sometimes not specify the subscripts of the above isomorphisms, so as to avoid a too cumbersome notation.

The particular structure of this parallelization suggests a refinement of the previous construction. Namely, we define an almost complex structure on $SO_g(M)$ by transfering a fixed structure on $\mathbb{R}^m$ to H_u and a fixed structure on $\mathfrak{o}(m)$ to V_u, via the above isomorphisms. This requires, as only additional hypotheses, that $\mathbb{R}^m$ and $\mathfrak{o}(m)$ admit complex structures, i.e. that they be even-dimensional. A quick calculation shows this to be true when $m = 4n$.

The goal of this article is to examine the integrability of such a class of almost complex structures. To do this, we fix the connection to be the Levi-Civita connection on $SO_g(M)$ induced by g and the structure on $\mathbb{R}^{4n}$ to be the standard complex structure J_0. The structure J on $\mathfrak{o}(4n)$ has, instead, no a priori restrictions.

It quickly becomes apparent that integrability requires additional hypotheses on J, i.e. that J be compatible both with J_0 and with g in the sense defined by theorem 1. Though clearly expressed, these conditions are of a fairly technical nature. We therefore proceed to show how a natural strengthening of our initial hypotheses suffices to express the above conditions in a much more elegant manner: theorem 2 basically states that, under the right hypotheses, the class of almost complex structures on $SO_g(M)$ is integrable if and only if

$$\begin{cases} n = 1: (M, g) \text{ is an autodual Einstein manifold}, \\ n > 1: (M, g) \text{ has constant sectional curvature}. \end{cases}$$

The author wishes to thank professor de Bartolomeis for suggesting the problem and for his help in reaching this solution.

2. – Preliminaries.

Let (M, g) be an oriented $4n$-dimensional Riemannian manifold.

Let $SO_g(M)$ be the associated $SO(4n)$-bundle of positive orthonormal frames.

We will adopt the following notation:

$P := SO_g(M)$,

$\mathfrak{o}(4n) :=$ Lie algebra of $SO(4n)$: antisymmetric $\mathbb{R}$-valued matrices .

$$R: SO(4n) \to \mathrm{Diff}\,(P) \qquad \text{action of } SO(4n) \text{ on } P\,,$$

$$g \mapsto R_g .$$

Let B and π_* be the isomorphisms defined in par. 1 and let $x := \pi[u]$. Then the following diagram is commutative:

$$\begin{array}{ccc} H_u & \xrightarrow{\pi_*[u]} & T_x M \\ {\scriptstyle B_u}\uparrow & & \downarrow{\scriptstyle u^{-1}} \\ \mathbb{R}^{4n} & \xrightarrow{\mathrm{id}} & \mathbb{R}^{4n} \end{array}$$

where u^{-1} simply associates to each vector in $T_x M$ its coordinates with respect to u.

Notice that, as u is an orthonormal frame, u^{-1} is an isometry between $(T_x M, g_x)$ and $\mathbb{R}^{4n}$ with the standard euclidean metric.

Let H be the Levi-Civita connection on P and Ω be its curvature. We recall that $\Omega \in \Lambda^2(P) \otimes \mathfrak{o}(4n)$, i.e. is a $\mathfrak{o}(4n)$-valued 2-form on P.

In a standard way, each Ω_u can be alternatively viewed as an element of $\mathrm{End}\,(\mathfrak{o}(4n))$. Let us review the reasoning.

Ω has the property that $\Omega_u(X, Y) = 0$ if $Y \in V_u$. It follows that Ω_u can be viewed, with no loss of information, as $\Omega_u \in \Lambda^2(H_u^*) \otimes \mathfrak{o}(4n)$ or, through the isomorphism B, as $\Omega_u \in \Lambda^2(\mathbb{R}^{4n})^* \otimes \mathfrak{o}(4n)$.

If we now identify $\Lambda^2(\mathbb{R}^{4n})$ with $\mathfrak{o}(4n)$ via the canonical isomorphism

$$\Lambda^2(\mathbb{R}^{4n}) \to \mathfrak{o}(4n)$$

$$\xi \wedge \eta \mapsto \frac{1}{2}(\xi^t \eta - \eta^t \xi) \qquad \text{(matrix multiplication)},$$

we get $\Omega_u \in \mathfrak{o}(4n)^* \otimes \mathfrak{o}(4n)$, i.e. $\Omega_u \in \mathrm{End}(\mathfrak{o}(4n))$.

It may be useful to underline the fact that, according to the above conventions, $\Omega_u(B\xi, B\eta) = \Omega_u(\xi \wedge \eta)$, $\forall \xi, \eta \in \mathbb{R}^{4n}$.

The following lemma translates the usual properties of Ω into this new setting:

LEMMA 1. – 1. $\forall g \in SO(4n)$, $\Omega_u \circ ad(g) = ad(g) \circ \Omega_{ug}$.

2. *Ω_u is symmetric with respect to the standard metric on* $\mathfrak{o}(4n)$.

PROOF. – 1) Let us first prove that $(R_g)_*[u]\, B_u \xi = B_{ug}(g^{-1}\xi)$: the fact that the connection H is R-invariant shows that

$$(R_g)_*[u]\, B_u \xi = B_{ug}\eta \;\text{ for some }\; \eta \in \mathrm{R}^{4\mathrm{n}}$$

the fact that $\pi \circ R_g = \pi$ shows that

$$\pi_*[u]\, B_u \xi = \pi_*[ug](R_g)_*[u]\, B_u \xi = \pi_*[ug]\, B_{ug}\eta$$

finally, the commutativity of the above diagram implies that

$$\eta = (ug)^{-1}\pi_*[ug]\, B_{ug}\eta = (ug)^{-1}\pi_*[u]\, B_u \xi = g^{-1}u^{-1}\pi_*[u]\, B_u \xi = g^{-1}\xi\,.$$

The proof of the first claim is then based upon the fact (cfr. [KN]) that Ω has the property that

$$\forall g \in SO(4n), \qquad \forall X, Y \in T_u P\,,$$
$$\Omega_{ug}((R_g)_*[u]\, X, (R_g)_*[u]\, Y) = \mathrm{ad}\,(g^{-1})\; \Omega_u(X, Y)\,.$$

This leads to:

$$\mathrm{ad}\,(g) \circ \Omega_u(\xi \wedge \eta) = \mathrm{ad}\,(g)\; \Omega_u(B\xi, B\eta) =$$
$$\Omega_{ug^{-1}}((R_{g^{-1}})_*[u]\, B_u \xi, (R_{g^{-1}})_*[u]\, B_u \eta) = \Omega_{ug^{-1}}(B_{ug^{-1}}(g\xi), B_{ug^{-1}}(g\eta)) =$$
$$\Omega_{ug^{-1}}(g\xi \wedge g\eta) = \Omega_{ug^{-1}} \circ \mathrm{ad}\,(g)(\xi \wedge \eta)\,.$$

2) The standard metric on $\mathfrak{o}(4n)$ is $(M, N) := -\,\mathrm{tr}\, MN$. It is easy to check that

$$\forall M \in \mathfrak{o}(4n), \qquad \forall \alpha, \beta \in \mathbb{R}^{4n}, \qquad (M, \alpha \wedge \beta) = -\,(M\alpha, \beta)$$

where the product on the right-hand side is now the usual metric on $\mathbb{R}^{4n}$.

Let $\xi, \eta, \alpha, \beta \in \mathbb{R}^{4n}$ and let X, Y, A, B be the corresponding vectors in $T_{\pi(u)}M$.

Let R be the curvature tensor on (M, g) of type $(4, 0)$, so that $R(X, Y, A, B) = (\Omega_u(\xi \wedge \eta)\, \alpha, \beta)$.

The proof of the second claim is then based upon the well known fact that $R(X, Y, A, B) = R(A, B, X, Y)$:

$$(\Omega_u(\xi \wedge \eta), \alpha \wedge \beta) = -(\Omega_u(\xi \wedge \eta)\, \alpha, \beta) =$$

$$-R(X, Y, A, B) = -R(A, B, X, Y) = -(\Omega_u(\alpha \wedge \beta)\, \xi, \eta) =$$

$$(\Omega_u(\alpha \wedge \beta), \xi \wedge \eta) = (\xi \wedge \eta, \Omega_u(\alpha \wedge \beta)). \quad \blacksquare$$

It is well known that (M, g) has constant sectional curvature c if and only if

$$R(X, Y)\, Z = c(g(Z, Y)\, X - g(Z, X)\, Y)$$

where R is now the curvature tensor of type $(3, 1)$ on (M, g).

The following lemma translates this in terms of $\Omega_u \in \mathrm{End}(\mathfrak{o}(4n))$.

LEMMA 2. – *(M, g) has constant sectional curvature if and only if $\Omega = \lambda Id$.*

PROOF. – Recall that, according to the usual definitions, if $\xi, \eta, \zeta \in \mathbb{R}^{4n}$ are the coordinates of $X, Y, Z \in T_x M$ with respect to the basis u, then $\Omega_u(\xi \wedge \eta)$ is simply the matrix with respect to u of $R(X, Y) \in \mathrm{End}(T_x M)$.

It follows that $u^{-1} R(X, Y)\, Z = \Omega_u(\xi \wedge \eta)\, \zeta$, so that

$$\Omega(\xi \wedge \eta) = \lambda(\xi \wedge \eta) \Leftrightarrow \Omega(\xi \wedge \eta)\, \zeta = \lambda(\xi \wedge \eta)\, \zeta, \qquad \forall \zeta \in \mathbb{R}^{4n}$$

$$\Leftrightarrow u^{-1} R(X, Y)\, Z = \lambda/2(\xi^t \eta \zeta - \eta^t \xi \zeta) = \lambda/2(\xi g(Y, Z) - \eta g(X, Z)), \qquad \forall \zeta \in \mathbb{R}^{4n}$$

$$\Leftrightarrow R(X, Y)\, Z = \lambda/2(g(Z, Y)\, X - g(Z, X)\, Y). \quad \blacksquare$$

Let us end this section with the following

DEFINITION 1. – *(M, g) is an Einstein manifold if* $\mathrm{Ric} = \lambda g$, *where* Ric *is the Ricci tensor and λ is a constant.*

It is a well known fact that, if $\dim M \geq 4$, (M, g) is an Einstein manifold if and only if $\mathrm{Ric} = \lambda g$ where $\lambda \in C^\infty(M)$.

3. – Some almost complex structures on $SO_g(M)$ and their integrability.

Let J_0 denote both the $4n \times 4n$ (or $2n \times 2n$, as needed) matrix $\begin{bmatrix} O & -I \\ I & O \end{bmatrix}$ and the complex structure on $\mathbb{R}^{4n}$ defined by:

$$\begin{aligned} &\mathbb{R}^{4n} \to \mathbb{R}^{4n} \\ &\quad x \mapsto J_0 x \qquad \text{(matrix multiplication)}. \end{aligned}$$

Let J be any complex structure on $\mathfrak{o}(4n)$.

As seen in the introduction, we define an almost complex structure $\mathfrak{J}$ on P in the following way:

$$\begin{aligned} &\mathfrak{J}\colon T_u P \to T_u P\,, \\ &\mathfrak{J}_{|H_u} := B_u \circ J_0 \circ B_u^{-1}, \\ &\mathfrak{J}_{|V_u} := \sigma_u \circ J \circ \sigma_u^{-1}. \end{aligned}$$

We will call $\mathfrak{J}$ the «constant almost complex structure induced by a complex structure of type (J_0, J)».

We want to investigate the integrability of $\mathfrak{J}$. The main tool for this is provided by a classical theorem by Newlander and Nirenberg (cfr. [NN]), which states that an almost complex structure $\mathfrak{J}$ on a manifold is integrable if and only if $N_{\mathfrak{J}} \equiv 0$, where $N_{\mathfrak{J}}$ is the Nijenhuis tensor defined by

$$N_{\mathfrak{J}}(X, Y) := [\mathfrak{J}X, \mathfrak{J}Y] - [X, Y] - \mathfrak{J}[\mathfrak{J}X, Y] - \mathfrak{J}[X, \mathfrak{J}Y].$$

Performing this calculation in our case requires a closer look at the structure of $\mathfrak{o}(4n)$ and of the curvature tensor. For this purpose, we introduce the following notation.

$$\mathrm{Sym}(n) := \{n \times n \text{ real symmetric matrices}\},$$

$$\mathrm{Sym}_0(n) := \{A \in \mathrm{Sym}(n) : \mathrm{tr}\, A = 0\},$$

$$\mathfrak{u}(n) := \{A \in \mathfrak{o}(2n) : AJ_0 = J_0 A\} = \left\{\begin{bmatrix} S & -T \\ T & S \end{bmatrix} : S \in \mathfrak{o}(n),\ T \in \mathrm{Sym}(n)\right\},$$

$$\mathfrak{u}_0(n) := \left\{\begin{bmatrix} S & -T \\ T & S \end{bmatrix} : S \in \mathfrak{o}(n),\ T \in \mathrm{Sym}_0(n)\right\},$$

$$s(n) := \{A \in \mathfrak{o}(2n) \colon AJ_0 = -J_0 A\} = \left\{\begin{bmatrix} S & T \\ T & -S \end{bmatrix} : S, T \in \mathfrak{o}(n)\right\}.$$

It is well known that $\mathfrak{u}(n)$ is the Lie algebra of the group of unitary matrices $U(n)$ and that $\mathfrak{u}_0(n)$ is the Lie algebra of the group of special unitary matrices $SU(n)$.

Let $\mathfrak{o}(4n)$ have the usual metric:

$$(A, B) := \mathrm{tr}\, A^t B = -\,\mathrm{tr}\, AB\,.$$

Then the equality

$$A = \frac{A - J_0 A}{2} + \frac{A + J_0 A}{2}, \qquad \forall A \in \mathfrak{o}(2n)$$

shows that

$$\mathfrak{o}(2n) = \mathfrak{u}(n) \oplus s(n), \qquad \text{orthogonal decomposition}.$$

Notice also that

$$\mathfrak{u}(n) = \mathfrak{u}_0(n) \oplus \mathbb{R}J_0, \qquad \text{orthogonal decomposition}.$$

The algebra $\mathfrak{u}_0(n)$ is simple.
The algebra $\mathfrak{o}(n)$ is simple if and only if $n \neq 4$.
The algebra $\mathfrak{o}(4)$ is semisimple with orthogonal decomposition

$$\mathfrak{o}(4) = \mathfrak{o}_+(4) \oplus \mathfrak{o}_-(4)$$

where $\mathfrak{o}_+(4)$ and $\mathfrak{o}_-(4)$ are simple ideals defined as the eigenspaces of the involution

$$\phi \colon \mathfrak{o}(4) \to \mathfrak{o}(4)$$

$$\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -c & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & f & -e & d \\ -f & 0 & c & -b \\ e & -c & 0 & a \\ -d & b & -a & 0 \end{bmatrix}.$$

It can easily be seen that $\mathfrak{o}_+(4) = \mathfrak{u}_0(4)$ and that $\mathfrak{o}_-(4) = \mathbb{R}J_0 \oplus s(2)$; this leads us quickly to a characterization of the corresponding normal subgroups of $SO(4)$.

The subgroup corresponding to $\mathfrak{o}_+(4)$ is obviously $SU(2)$.
Let $\overline{SU(2)}$ be the subgroup corresponding to $\mathfrak{o}_-(4)$.
Since $e^{(\pi/2)J_0} = J_0$, $J_0 \in \exp(\mathfrak{o}_-(4))$ so $J_0 \in \overline{SU(2)}$.
As $\overline{SU(2)}$ is normal in $SO(4)$, $\mathrm{ad}(g)\, J_0 \in \overline{SU(2)}$, $\forall g \in SO(4)$.
As $\overline{SU(2)}$ is simple, it can thus be described as the closure of the Lie subgroup generated by $\{\mathrm{ad}(g)\, J_0 \colon g \in SO(4)\}$.
Finally, it is interesting that neither the adjoint action of $SU(2)$ on $\mathfrak{o}_-(4)$ nor of $\overline{SU(2)}$ on $\mathfrak{o}_+(4)$ are irriducible.
Let us now go back to the curvature tensor Ω.
Let $\mathrm{Sym}(\mathfrak{o}(4n)) := \{\phi \in \mathrm{End}(\mathfrak{o}(4n))$ symmetric with respect to the standard metric on $\mathfrak{o}(4n)\}$.
Lemma 1 shows that $\Omega_u \in \mathrm{Sym}(\mathfrak{o}(4n))$.
Referring the reader to [Be] for further details, we recall that Ω_u admits

a canonical decomposition as sum of three elements in $\mathrm{Sym}(\mathfrak{o}(4n))$; we will write $\Omega_u = E_u + Z_u + W_u$.

The decomposition shows that $E_u = \lambda Id$ while Z_u and W_u are traceless. Furthermore, it shows that $Z_u = 0$ if and only if (M, g) is an Einstein manifold, and that $W_u = Z_u = 0$ if and only if (M, g) has constant sectional curvature. W_u is known as the Weyl tensor.

When $n = 1$ and one considers the splitting $\mathfrak{o}(4) = \mathfrak{o}_+(4) \oplus \mathfrak{o}_-(4)$, it can be shown that $W_u(\mathfrak{o}_+(4)) \subseteq \mathfrak{o}_+(4)$, $W_u(\mathfrak{o}_-(4)) \subseteq \mathfrak{o}_-(4)$, $Z_u(\mathfrak{o}_+(4)) \subseteq \mathfrak{o}_-(4)$, $Z_u(\mathfrak{o}_-(4)) \subseteq \mathfrak{o}_+(4)$. Furthermore, $Z_{u_{|\mathfrak{o}_+(4)}} = {}^t Z_{u_{|\mathfrak{o}_-(4)}}$.

If follows that, with respect to the above splitting of $\mathfrak{o}(4)$ and omitting the subscripts, Ω_u admits the block-matrix representation

$$\Omega \simeq \begin{bmatrix} W^+ + \lambda Id & Z \\ {}^t Z & W^- + \lambda Id \end{bmatrix},$$

where $W^+ := W_{|\mathfrak{o}_+(4)}$, $W^- := W|\mathfrak{o}_{(4)}$ and $Z := Z_{|\mathfrak{o}_-(4)}$.

It is also true that W^+ and W^- are traceless operators; they are the positive and negative Weyl tensors, respectively.

We can now go back to our initial problem of studying the integrability of $\mathfrak{J}$.

Definition 2. – *A complex structure J on a Lie algebra $\mathfrak{g}$ is integrable if the left-invariant almost complex structure induced by J on the corresponding Lie group G is integrable, or, equivalently, if*

$$N_J(X, Y) := [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] = 0\,, \quad \forall X, Y \in \mathfrak{g}\,.$$

We can now prove the following

Theorem 1. – *Let (M, g) be a $4n$-dimensional oriented Riemannian manifold.*

Let $\mathfrak{J}$ be the constant almost complex structure on $SO_g(M)$ induced by a structure of type (J_0, J).

Then $\mathfrak{J}$ is integrable if and only if the following two conditions are satisfied:

1. *J is integrable and satisfies the following compatibility condition with respect to J_0:*

$$\forall X \in \mathfrak{o}(4n), \qquad [J_0, X] = J(X) + J_0 J(X)\, J_0 .$$

2. $\Omega_u(J_0 X) = J \Omega_u(X)$ $\forall u \in P$, $\forall X \in s(2n)$.

Proof. – The proof is basically the calculation of the Nijenhuis tensor $N_{\mathfrak{J}}$ on P defined above.

As $N_{\mathfrak{J}}$ is a tensor, $N_{\mathfrak{J}} \equiv 0$ if and only if the following three cases are true:

1. $N_{\mathfrak{J}}(X^*, Y^*) = 0, \ \forall X, Y \in \mathfrak{o}(4n)$.
2. $N_{\mathfrak{J}}(X^*, B\xi) = 0, \ \forall \xi \in \mathbb{R}^{4n}, \ \forall X \in \mathfrak{o}(4n)$.
3. $N_{\mathfrak{J}}(B\xi, B\eta) = 0, \ \forall \xi, \eta \in \mathbb{R}^{4n}$.

We will consider the three cases separately.

1) $$N_{\mathfrak{J}}(X^*, Y^*) = [\mathfrak{J}X^*, \mathfrak{J}Y^*] - [X^*, Y^*] - \mathfrak{J}[\mathfrak{J}X^*, Y^*] - \mathfrak{J}[X^*, \mathfrak{J}Y^*] =$$
$$[(JX)^*, (JY)^*] - [X^*, Y^*] - \mathfrak{J}[(JX)^*, Y^*] - \mathfrak{J}[X^*, (JY)^*] =$$
$$[JX, JY]^* - [X, Y]^* - (J[JX, Y])^* - (J[X, JY])^*,$$

where the final identity follows from the fact that the above mentioned $\sigma \colon \mathfrak{o}(4n) \to \Lambda^0(TP)$ is a Lie algebra homomorphism.

Therefore

$$N_{\mathfrak{J}}(X^*, Y^*) = 0 \Leftrightarrow [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y] = 0$$

so that

$$N_{\mathfrak{J}}(X^*, Y^*) = 0\,, \qquad \forall X, Y \in \mathfrak{o}(4n) \Leftrightarrow J \text{ is integrable}\,.$$

2) We first show that $[X^*, B\xi] = B(X\xi)$.

Let $\alpha_t := \exp(tX)$.

Notice that X^* is, by definition, the vector field induced by the 1-parameter group of diffeomorphisms R_{α_t}.

Remember (cfr. proof of lemma 1) that $dR_g[u](B_u \xi) = B_{ug}(g^{-1}\xi)$. Then:

$$[X^*, B\xi] = \lim_{t \to 0} \frac{B\xi - dR_{\alpha_t}[\alpha(-t)](B\xi)}{t} = \lim_{t \to 0} \frac{B\xi - B(\alpha(t)^{-1}\xi)}{t} =$$
$$B\left(\lim_{t \to 0} \frac{\xi - \exp(-tX)\,\xi}{t}\right) = B\frac{d}{dt}(-\exp(-tX)\,\xi)|_{t=0} = B(X\xi)\,.$$

Consequently:

$$N_{\mathfrak{J}}(X^*, B\xi) = [\mathfrak{J}X^*, \mathfrak{J}B\xi] - [X^*, B\xi] - \mathfrak{J}[\mathfrak{J}X^*, B\xi] - \mathfrak{J}[X^*, \mathfrak{J}B\xi] =$$
$$[(JX)^*, B(J_0\xi)] - B(X\xi) - \mathfrak{J}[(JX)^*, B\xi] - \mathfrak{J}[X^*, B(J_0\xi)] =$$
$$B(J(X)\,J_0\xi) - B(X\xi) - B(J_0 J(X)\,\xi) - B(J_0 X J_0 \xi)\,.$$

Therefore

$$N_{\mathfrak{J}}(X^*, B\xi) = 0 \Leftrightarrow J(X)\, J_0\, \xi - X\xi - J_0 J(X)\, \xi - J_0 X J_0\, \xi = 0$$

so that

$$N_{\mathfrak{J}}(X^*, B\xi) = 0\,, \qquad \forall\xi \Leftrightarrow J(X)\, J_0 - X - J_0 J(X) - J_0 X J_0 = 0\,.$$

Left multiplication by J_0 proves that

$$N_{\mathfrak{J}}(X^*, B\xi) = 0, \quad \forall\xi, \forall X \Leftrightarrow [J_0, X] = J(X) + J_0 J(X)\, J_0, \qquad \forall X\,.$$

3) We first prove that $[B\xi, B\eta]_u \in V_u$.

Let θ be the unique $\mathbb{R}^{4n}$-valued 1-form on P such that

$$\theta(X) = 0\,, \quad \forall X \in V_u \qquad \text{and} \qquad \theta(B\xi) = \xi\,.$$

θ defines a $\mathbb{R}^{4n}$-valued 2-form, called the torsion of the connection, in the following way:

$$\Theta(X, Y) := d\theta(X^h, Y^h)$$

or, equivalently,

$$\Theta(X, Y) := \frac{1}{2}\{X^h\, \theta(Y^h) - Y^h\, \theta(X^h) - \theta[X^h, Y^h]\}\,,$$

where X^h, Y^h denote the horizontal components of X, Y.

Recall that, by definition, the Levi-Civita connection has $\Theta \equiv 0$.

Since $\theta(B\xi)$ and $\theta(B\eta)$ are constant, it then follows that

$$\theta[B\xi, B\eta] = -2\,\Theta(B\xi, B\eta) = 0$$

that is,

$$[B\xi, B\eta] \in V_u.$$

Let ω be the $\mathfrak{o}(4n)$-valued 1-form defined on P by the connection. We recall that

$$\omega(X) = 0\,, \qquad \forall X \in H_u$$

and that

$$\Omega(X, Y) = d\omega(X^h, Y^h) = \frac{1}{2}\{X^h\, \omega(Y^h) - Y^h\, \omega(X^h) - \omega[X^h, Y^h]\}\,.$$

From the preceding result it follows that $N_{\mathfrak{J}}(B\xi, B\eta) \in V_u$, so that

$$N_{\mathfrak{J}}(B\xi, B\eta) = 0 \Leftrightarrow \omega N_{\mathfrak{J}}(B\xi, B\eta) = 0\,.$$

Noticing that $\omega[B\xi, B\eta] = -2\Omega(B\xi, B\eta)$ and $\omega\mathfrak{J} = J\omega$ proves that

$$N_{\mathfrak{J}}(B\xi, B\eta) = 0 \Leftrightarrow \Omega(BJ_0\xi, BJ_0\eta) - \Omega(B\xi, B\eta) - J\Omega(BJ_0\xi, B\eta) - J\Omega(B\xi, BJ_0\eta) = 0\,.$$

Let us now use the identification described in par. 1, viewing Ω_u as $\Omega_u\colon \Lambda^2(\mathbb{R}^{4n}) \to \mathfrak{o}(4n)$.

The above translates as

$$N_{\mathfrak{J}}(B\xi, B\eta) = 0 \Leftrightarrow \Omega(J_0\xi\wedge J_0\eta - \xi\wedge\eta) = J\Omega(J_0\xi\wedge\eta + \xi\wedge J_0\eta)\,.$$

We now, as before, identify $\Lambda^2(\mathbb{R}^{4n})$ with $\mathfrak{o}(4n)$. Then $J_0\xi\wedge J_0\eta - \xi\wedge\eta$ corresponds to an element $X\in s(2n)$, as can easily be seen by proving that it anticommutes with J_0, and $J_0\xi\wedge\eta + \xi\wedge J_0\eta = -J_0X$, so that

$$N_{\mathfrak{J}}(B\xi, B\eta) = 0 \Leftrightarrow \Omega(X) = J\Omega(-J_0X)$$

We can then conclude that

$$N_{\mathfrak{J}}(B\xi, B\eta) = 0 \ \ \forall\xi, \eta \Leftrightarrow \Omega(J_0X) = J\Omega(X) \qquad \forall X\in s(2n)\,. \qquad \blacksquare$$

The two conditions appearing in theorem 1 are of different nature. The first is algebraic, in the sense that, being J_0 fixed, it concerns only the complex structure J on the Lie algebra $\mathfrak{o}(4n)$. The second is twistor-like, in the sense that it implies a compatibility between the metric g and the complex structure $\mathfrak{J}$.

The canonical splitting $\mathfrak{o}(4n) = \mathfrak{u}(2n)\oplus s(2n)$ suggests restricting our attention to those J's such that $J(\mathfrak{u}(2n))\subseteq\mathfrak{u}(2n)$, $J(s(2n))\subseteq s(2n)$, i.e. defined as the sum of a complex structure J_1 on $\mathfrak{u}(2n)$ and a complex structure J_2 on $s(2n)$: we will say that J is of type (J_1, J_2).

The following lemma shows that, when J is of type (J_1, J_2), condition (1) of theorem 1 can be reformulated in a much simpler manner:

LEMMA 3. – *Let J be a complex structure on $\mathfrak{o}(4n)$ of type (J_1, J_2). The following conditions are equivalent*:

1. J_1 is integrable;

$$\forall A\in s(2n), \qquad J_2(A) = J_0A \qquad (\textit{matrix multiplication})\,.$$

2. $\forall A\in\mathfrak{o}(4n)$, $[J_0, A] = J(A) + J_0J(A)J_0$; *$J$ is integrable.*

PROOF. – $1\Rightarrow 2$:

$$\forall A\in s(2n), \quad [J_0, A] = J_0A - AJ_0 = J(A) + J_0^2AJ_0 = J(A) + J_0J(A)\,J_0,$$

$$\forall A\in\mathfrak{u}(2n), \quad [J_0, A] = 0 = J(A) + J_0J(A)J_0,$$

$$\forall A, B \in s(2n), \quad N_J(A, B) = [J_0 A, J_0 B] - [A, B] - J[J_0 A, B] - J[A, J_0 B] =$$
$$J_0 A J_0 B - J_0 B J_0 A - AB + BA - J(J_0 AB - BJ_0 A + AJ_0 B - J_0 BA) = 0\,,$$

$\forall A, B \in u(2n), \quad N_J(A, B) = 0$ by hypothesis,

$\forall A \in u(2n), \forall B \in s(2n),$

$$N_J(A, B) = [J_1(A), J_0 B] - [A, B] - J_0[J_1(A), B] - J_0[A, J_0 B] =$$
$$J_1(A) J_0 B - J_0 B J_1(A) - AB + BA - J_0 J_1(A)\, B + J_0 B J_1(A) -$$
$$J_0 A J_0 B - BA = 0\,.$$

$2 \Rightarrow 1$: as $N_{J_1} = N_{J|u(2n)}$, J_1 is obviously integrable;

$$\forall A \in s(2n), \qquad 2J_0 A = [J_0, A] = J(A) + J_0 J(A) J_0 = 2J(A) = 2J_2(A)\,. \qquad \blacksquare$$

DEFINITION 3. – *A complex structure on* $\mathbb{R}^{4n} \oplus \mathfrak{o}(4n)$ *is of type* (J_0, J_1, J_2) *if it is given by the sum of the standard complex structure on* $\mathbb{R}^{4n}$, *of any complex structure* J_1 *on* $\mathfrak{u}(2n)$ *and of any complex structure* J_2 *on* $s(2n)$.

A complex structure (J_0, J_1, J_2) *is of integrable type if* J_1 *is integrable and* J_2 *is the standard structure on* $s(2n)$ *defined by* $J_2(X) = J_0 X$ *(matrix multiplication).*

It is important to mention that integrable structures on $\mathfrak{u}(2n)$ exist (cfr. [Mo]) and have been extensively studied (cfr. [Sn]).

We will now examine the integrability of constant almost complex structures on $SO_g(M)$ induced by structures of type (J_0, J_1, J_2).

THEOREM 2. – *Let* (M, g) *be a* $4n$*-dimensional oriented Riemannian manifold.*

Let $\mathfrak{J}$ *be the constant almost complex structure on* $SO_g(M)$ *induced by a structure of type* (J_0, J_1, J_2).

Then $\mathfrak{J}$ *is integrable if and only if*

1. (J_0, J_1, J_2) *is of integrable type.*
2. (M, g) *has the following property*:

$n = 1 : (M, g)$ *is an autodual Einstein manifold* (*i.e.* $Z \equiv W^- \equiv 0$).

$n > 1 : (M, g)$ *has constant sectional curvature*.

PROOF. – Given the additional hypotheses on $\mathfrak{J}$, the preceding lemma shows that condition (1) is equivalent to the first condition of theorem 1. We therefore only need to prove that condition (2) is equivalent to the second condition of theorem 1.

As usual, let $J := J_1 \oplus J_2$ denote the complex structure on $\mathfrak{o}(4n)$.

Notice that, as $J(s(2n)) \subseteq s(2n)$, the second condition of theorem 1 may be simply expressed by $[\Omega, J]_{|s(2n)} = 0$.

On the other hand, lemma 2 shows that (M, g) has constant sectional curvature if and only if $\Omega = \lambda Id$, while previous considerations prove that, in the case $n = 1$, (M, g) is an Einstein manifold with $W^- \equiv 0$ if and only if $\Omega_{|\mathfrak{o}_-(4)} = \lambda Id$.

To prove the theorem, it is thus sufficient to prove that $[\Omega, J]|_{s(2n)} = 0$ if and only if

$$n = 1 : \Omega_{|\mathfrak{o}_-(4)} = \lambda Id ,$$

$$n > 1 : \Omega = \lambda Id .$$

One of the two implications is obvious: that $\Omega_{|\mathfrak{o}_-(4)} = \lambda Id$ and $\Omega = \lambda Id$ imply $[\Omega, J]_{|s(2n)} = 0$.

We will prove the viceversa in two steps, by showing

1. $[\Omega, J]_{|s(2n)} = 0 \Rightarrow \Omega(\mathrm{ad}(g)\, s(2n)) \subseteq \mathrm{ad}(g)\, s(2n), \quad \forall g \in SO(4n).$

2. $\Omega(\mathrm{ad}(g)\, s(2n)) \subseteq \mathrm{ad}(g)\, s(2n) \Leftrightarrow \begin{cases} n = 1 : \Omega_{|\mathfrak{o}_-(4)} = \lambda Id , \\ n > 1 : \Omega = \lambda Id . \end{cases}$

1) Let $[\Omega_u, J]_{|s(2n)} = 0$, $\forall u \in P$.

In particular, $[\Omega_{ug}, J]_{|s(2n)} = 0$, $\forall g \in SO(4n)$.

We saw that $\Omega_{ug} = ad(g^{-1}) \circ \Omega_u \circ ad(g)$, $\forall g \in SO(4n)$.

Let $X \in s(2n)$ and $g \in U(2n)$. Then

$$\mathrm{ad}(g)\, X \in s(2n) \qquad \text{and} \qquad \mathrm{ad}(g)\, J(X) = \mathrm{ad}(g)\, J_0 X = J_0\, \mathrm{ad}(g)\, X = J\, \mathrm{ad}(g)\, X$$

so that, combining the above expressions,

$$0 = [\Omega_{ug}, J]_{|s(2n)} = [\mathrm{ad}(g^{-1})\, \Omega_u\, \mathrm{ad}(g), J]_{|s(2n)} = [\mathrm{ad}(g^{-1}), J]_{|\Omega_u(s(2n))},$$

$$\forall g \in U(2n).$$

This is enough to prove that $\Omega(s(2n)) \subseteq s(2n)$: by denoting with Δ the projection of $\Omega(s(2n))$ onto $\mathfrak{u}(2n)$ with respect to the decomposition $\mathfrak{o}(4n) = \mathfrak{u}(2n) \oplus s(2n)$, all we must do is to show that $\Delta = 0$.

As $[\mathrm{ad}(g), J]_{|s(2n)} = 0$, the above expression implies that

$$[\mathrm{ad}(g), J]_{|\Delta} = 0 , \qquad \forall g \in U(2n).$$

Let $\tilde{\Delta} := \{X \in \mathfrak{u}(2n) : [\mathrm{ad}(g), J]\, X = 0\}$, $\forall g \in U(2n)$.

It is easy to show that $\tilde{\Delta}$ is an ideal of $\mathfrak{u}(2n)$ and that $J(\tilde{\Delta}) \subseteq \tilde{\Delta}$. In particular, $\tilde{\Delta}$ has even dimension. As $\mathfrak{u}(2n)$ is reductive with decomposition

$\mathfrak{u}_0(2n)\oplus\mathbb{R}J_0$ and $\mathfrak{u}_0(2n)$ is a simple odd-dimensional ideal, $\tilde{\Delta}=\mathfrak{o}(2n)$ or $\tilde{\Delta}=0$.

Suppose $\tilde{\Delta}=\mathfrak{o}(2n)$, so that the Lie group associated to $\tilde{\Delta}$ would be $U(2n)$. J would define on $U(2n)$ a (left invariant) complex structure which, because $[\mathrm{ad}\,(g), J]\equiv 0$, would make $U(2n)$ a complex Lie group. This is impossible, as $U(2n)$ is compact and any compact complex Lie group is abelian.

If follows that $\tilde{\Delta}=0$, so, in particular, $\Delta=0$.

This proves that $\Omega_u(s(2n))\subseteq s(2n)$, $\forall u\in P$.

In particular,

$$\Omega_{ug}(s(2n))\subseteq s(2n),\qquad \forall g\in SO(4n),$$

$$i.e.\qquad \Omega_u(\mathrm{ad}\,(g)\;s(2n))\subseteq \mathrm{ad}\,(g)\;s(2n),\qquad \forall g\in SO(4n).$$

2) Remembering that Ω is symmetric, it is essentially the content of the final lemma. ■

LEMMA 4. – *Let $\Omega\in\mathrm{End}\,(\mathfrak{o}(4n))$ be symmetric with respect to the standard metric on $\mathfrak{o}(4n)$. Then the following conditions are equivalent*:

1. $\Omega(\mathrm{ad}\,(g)\;s(2n))\subseteq \mathrm{ad}\,(g)\;s(2n),\qquad \forall g\in SO(4n).$

2. $\begin{cases} n=1:\ \Omega|_{\mathfrak{o}_-(4)}=\lambda Id\,,\\ n>1:\ \Omega=\lambda Id\,.\end{cases}$

PROOF. – 1 $\Rightarrow$ 2: Let us define

$$P:\ \mathfrak{o}(4n)\rightarrow\mathfrak{u}(2n)\qquad \text{orthogonal projection}\,.$$

The definition of $\mathfrak{u}(2n)$ shows that $P=\dfrac{1}{2}[I+\mathrm{ad}\,(J_0)]$.

Since $\mathrm{ad}\,(g)$ is an isometry of $\mathfrak{o}(4n)$, Ω is symmetric and $\mathfrak{u}(2n)\perp s(2n)$,

$$\Omega(\mathrm{ad}\,(g)\;s(2n))\subseteq \mathrm{ad}\,(g)\;s(2n)\Rightarrow\Omega(\mathrm{ad}\,(g)\,\mathfrak{u}(2n))\subseteq\mathrm{ad}\,(g)\;\mathfrak{u}(2n)\,.$$

It follows that $s(2n)$ and $\mathfrak{u}(2n)$ are invariant for the family $\mathrm{ad}\,(g^{-1})\circ\Omega\circ\mathrm{ad}\,(g)$, i.e.

$$[\mathrm{ad}\,(g^{-1})\circ\Omega\circ\mathrm{ad}\,(g), P]=0,\qquad i.e.$$

$$[\Omega,\ \mathrm{ad}\,(g)\circ P\circ\mathrm{ad}\,(g^{-1})]=0,\qquad i.e.$$

$$[\Omega,\ \mathrm{ad}\,(gJ_0g^{-1})]=0\,,\qquad \forall g\in SO(4n)$$

Let $H:=\langle\{gJ_0g^{-1}:\ g\in SO(4n)\}\rangle$.

H is, algebraically, a normal subgroup of $SO(4n)$ so $\overline{H}$ is a normal Lie subgroup of $SO(4n)$.

We must now distinguish between the cases $n=1$, $n>1$.

If $n>1$, $SO(4n)$ is a simple Lie group so $\overline{H}=SO(4n)$. It is easy to see that

$$[\Omega, \mathrm{ad}(h)] = 0, \qquad \forall h \in \overline{H}, \qquad i.e.\ [\Omega, \mathrm{ad}(g)] = 0, \quad \forall g \in SO(4n).$$

By Shur's lemma, $\Omega = \lambda I + \mu J$ for some $J: J^2 = -Id$.

Since Ω is symmetric, Ω is diagonalizable; as J isn't diagonalizable, it must be $\mu = 0$, i.e. $\Omega = \lambda I$.

If instead $n=1$, as seen above, $\overline{H}$ is the normal proper subgroup of $SO(4)$ corresponding to $\mathfrak{o}_-(4)$.

As before, this implies that

$$[\Omega, \mathrm{ad}(h)] = 0, \qquad \forall h \in \overline{H}.$$

Notice now that

$$\mathrm{span}\{\mathrm{ad}(g)\, s(2n): g \in SO(4)\} = \mathfrak{o}_-(4),$$

as $s(2n) \subseteq \mathfrak{o}_-(4)$ and $\mathfrak{o}_-(4)$ is a simple ideal of $\mathfrak{o}(4)$. It follows that $\Omega(\mathfrak{o}_-(4)) \subseteq \mathfrak{o}_-(4)$, so that

$$[\Omega_{|\mathfrak{o}_-(4)}, \mathrm{ad}(h)_{|\mathfrak{o}_-(4)}] = 0, \qquad \forall h \in \overline{H}.$$

Applying Shur's lemma to $\Omega_{|\mathfrak{o}_-(4)}$, we find $\Omega_{|\mathfrak{o}_-(4)} = \lambda I$.

$2 \Rightarrow 1$: Obvious, because $\mathrm{ad}(g)\, s(2) \subseteq \mathfrak{o}_-(4)$, $\forall g \in SO(4)$. ■

The second condition of theorem 2 requires a final consideration.

Up to Riemannian covering space equivalence and connectedness, complete Riemannian manifolds with constant sectional curvature k have been classified: depending on the sign of k (and disregarding an eventual normalization of the metric), they are either S^n, $\mathbb{R}^n$, or the hyperbolic space with their standard metrics.

When (M, g) is one of these three models, it is well known that $SO_g(M)$ is a Lie group, as it is diffeomorphic to the group of isometries of (M, g).

In general, when (M, g) is a generic Riemannian manifold with constant sectional curvature, $SO_g(M)$ is modelled on a Lie group, in the sense of having an atlas in which the transition functions are Lie group isomorphisms.

Regarding autodual Einstein manifolds, note that the scalar curvature s is constant. In the compact case (again disregarding metric normalization), Hitchin provides a classification when $s \geqslant 0$:

$$\begin{cases} s > 0 : (M, g) \text{ is isometric to } S^4 \text{or } CP^2 \text{ with their standard metrics,} \\ s = 0 : (M, g) \text{ is either flat or its universal covering space is a } K3 \\ \text{surface with the Calabi-Yau metric} \end{cases}$$

For further details, cfr. [Be].

No such classification is known for the case $s < 0$; the only known examples of such manifolds are the compact quotients of the real and complex hyperbolic spaces.

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Dept. of Mathematics, Pisa University, Italy
e-mail: pacini@dm.unipi.it.

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