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Note on the Density Constant in the Distribution of Self-Numbers - II.

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Sunto. – *Dimostriamo che la costante che regola la distribuzione dei cosiddetti* self numbers *è un numero trascendente. Ciò precisa un risultato dimostrato in un precedente articolo dal medesimo titolo, ossia che tale costante sia irrazionale. Il metodo fa uso di una curiosa formula per l'espansione 2-adica di tale numero (già utilizzata nell'altro lavoro) e del profondo Teorema del Sottospazio.*

The present note is an addendum to [TZ]. In that paper we were concerned with the set $\mathcal{S}$ of integers which may be represented as sums of distinct terms of type $2^k + 1$, $k \in \boldsymbol{N}$. We agree that $0 \in \mathcal{S}$. These numbers appear in connection with *digitaddition sequences* and the complement of $\mathcal{S}$ in $\boldsymbol{N}$ is the set of the so-called *Self-Numbers* in the scale of 2 (see e.g. [S], [Z]).

It was proved in [Z] that $\mathcal{S}(x) = lx + O(\log^2 x)$ for large x, where $\mathcal{S}(x)$ denotes the number of elements of $\mathcal{S}$ up to x and $0 < l < 1$. In [TZ] the following rather curious formula was obtained (see equation (5) in [TZ])

$$l = 1 - \frac{1}{8}\alpha^2 \qquad \text{where } \alpha := \sum_{a \in \mathcal{S}} \frac{1}{2^a}\,. \tag{1}$$

The formula was then applied to show that l is irrational, by means of classical results in Pell's Equation theory and Diophantine Approximation.

In [TZ] we expressed the opinion that a possible proof of the transcendence of l (or α) could have been difficult, especially in view of the fact that it seems not easy to produce particularly good rational approximations to these numbers. (For instance we pointed out that α is not a Liouville number.) However we overlooked that the special form of the approximations found in [TZ] allows an application of the celebrated Schmidt Subspace Theorem. It is the purpose of this short note to outline such argument and prove the following

Theorem. – *The numbers l and α are transcendental.*

PROOF. – By (1) it is sufficient to prove that α is transcendental. In [TZ] it was shown that there exists an infinite sequence $\mathcal{M}$ of positive integers such that, for $m \in \mathcal{M}$,

$$|\alpha(2^m - 1) - B_m| \ll 2^{-m} \tag{2}$$

where the B_m are suitable positive integers. (See the calculations at p. 146 of [TZ], where α was denoted by $\sqrt{r/s}$. If E is the sequence in the statement of Lemma 2 of [TZ], we may take $\mathcal{M} = \{2^k + 1 : k \in E\}$.)

Now an application of Lemma 2 of [CZ] would be sufficient to conclude at once. For the sake of completeness we give however the short argument in the present case.

For the reader's convenience we state a version of Schmidt's Subspace Theorem due to H. P. Schlickewei; we have borrowed it from [Sch2, Thm. 1E, p. 178] (a complete proof requires also [Sch1]).

SUBSPACE THEOREM. – *Let S be a finite set of absolute values of $\boldsymbol{Q}$, including the infinite one and normalized in the usual way (i.e. $|p|_v = p^{-1}$ if $v|p$). Extend each $v \in S$ to $\overline{\boldsymbol{Q}}$ in some way. For $v \in S$ let $L_{1,v}, \ldots, L_{N,v}$ be N linearly independent linear forms in N variables with algebraic coefficients and let $\varepsilon > 0$. Then the solutions $\boldsymbol{x} := (x_1, \ldots, x_N) \in \boldsymbol{Z}^N$ to the inequality*

$$\prod_{v \in S} \prod_{i=1}^{N} |L_{i,v}(\boldsymbol{x})|_v < \|\boldsymbol{x}\|^{-\varepsilon}$$

where $\|\boldsymbol{x}\| := \max\{|x_i|\}$, are contained in finitely many proper subspaces of $\boldsymbol{Q}^N$.

Now suppose by contradiction that α is algebraic. We apply the Subspace Theorem with the following data. We let S consist of ∞ and the 2-adic valuation, denoted w. We put $N = 3$ and

$$L_{i,\infty} = L_{i,w} := x_i \quad \text{for } i = 1, 2, \quad L_{3,\infty} := x_1 + \alpha x_2 - \alpha x_3, \quad L_{3,w} := x_3.$$

Put now, for $m \in \mathcal{M}$, $\boldsymbol{x}_m := (B_m, 1, 2^m)$. By (2) we have $B_m \ll 2^m$, whence $\|\boldsymbol{x}_m\| \ll 2^m$. Again by (2) we have $|L_{3,\infty}(\boldsymbol{x}_m)| \ll 2^{-m}$, whence $\prod_{i=1}^{3} |L_{i,\infty}(\boldsymbol{x}_m)| \ll 1$.

Also, we have plainly $|L_{1,w}(\boldsymbol{x}_m)|_w \leq 1$, $|L_{2,w}(\boldsymbol{x}_m)|_w = 1$ while $|L_{3,w}(\boldsymbol{x}_m)|_w = 2^{-m}$. Hence $\prod_{i=1}^{3} |L_{i,w}(\boldsymbol{x}_m)|_w \leq 2^{-m}$.

We conclude that for $m \in \mathcal{M}$, $\prod_{v \in S} \prod_{i=1}^{3} |L_{i,v}(\boldsymbol{x}_m)|_v \ll 2^{-m} \ll \|\boldsymbol{x}_m\|^{-1}$.

By the Subspace Theorem there exist an infinite subsequence $\mathcal{M}'$ of $\mathcal{M}$ and rational numbers a, b, c, not all zero, such that for $m \in \mathcal{M}'$ we have

$$aB_m + b + c2^m = 0.$$

Necessarily $a \neq 0$ for otherwise also b and c would have to vanish. Hence we may substitute for B_m in (2), obtaining $|2^m(\alpha + c/a) + b/a - \alpha| \ll 2^{-m}$. For large m this inequality implies $\alpha = -c/a \in \boldsymbol{Q}$. However we have proved in [TZ] that α is irrational (see the beginning of p.147).

REFERENCES

[CZ1] P. Corvaja - U. Zannier, *Diophantine equations with power sums and Universal Hilbert Sets*, preprint (1997).

[Sch1] W. M. Schmidt, *Diophantine Approximation*, Springer-Verlag, Lecture Notes in Mathematics, **785** (1980).

[Sch2] W. M. Schmidt, *Diophantine Approximations and Diophantine Equations*, Springer-Verlag, Lecture Notes in Mathematics, **1467** (1991).

[S] K. B. Stolarsky, *The sum of a digitaddition sequence*, Proc. Amer. Math. Soc., **59** (1976), 1-5.

[TZ] G. Troi - U. Zannier, *Note on the Density Constant in the Distribution of Self-Numbers*, Boll. Un. Mat. Ital. (7), **9-A** (1995), 143-148.

[Z] U. Zannier, *On the distribution of self numbers*, Proc. Amer. Math. Soc., **85**, 1 (1982), 10-14.

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