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Jürgen Berndt, Lieven Vanhecke

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φ-Symmetric Spaces and Weak Symmetry.

JÜRGEN BERNDT - LIEVEN VANHECKE

Sunto. – *Proviamo che tutti gli spazi semplicemente connessi φ-simmetrici sono debolmente simmetrici e quindi commutativi.*

1. – Introduction.

A connected Riemannian homogeneous space (M, g) is said to be *G-commutative* if G is a subgroup of the isometry group $I(M, g)$ of (M, g) acting transitively on M and if the algebra of all G-invariant differential operators on M is commutative. It is known that if (G, K) is a Riemannian symmetric pair, then the associated Riemannian symmetric space G/K is G-commutative (see [5, p. 343] for details and references). A Riemannian symmetric space is both $I(M, g)$-commutative and $I^0(M, g)$-commutative, where $I^0(M, g)$ denotes the identity component of $I(M, g)$. A. Selberg [13] considered commutativity for the more general class of weakly symmetric spaces. A connected Riemannian manifold is said to be *weakly symmetric* if any two points can be interchanged by an isometry. This is not the original definition given by Selberg, but equivalent to it [3]. As proved in [13], any weakly symmetric space is $I(M, g)$-commutative, but the converse is not true [9], [10].

In this note we provide new examples of weakly symmetric spaces, and hence of $I(M, g)$-commutative spaces. But we do not know whether these spaces are $I^0(M, g)$-commutative or not (see Remark).

THEOREM. – *Any simply connected φ-symmetric space is weakly symmetric and therefore $I(M, g)$-commutative.*

These φ-symmetric spaces were introduced by Takahashi [14] and are certain circle or line bundles over Hermitian symmetric spaces. For example, the simply connected ones over compact irreducible Hermitian symmetric spaces are $SU(n+m)/SU(n) \times SU(m)$, $SO(n+2)/SO(n)$, $SO(2n)/SU(n)$, $Sp(n)/SU(n)$, $E_6/SO(10)$ and E_7/E_6. The complete classification of simply connected φ-symmetric spaces has been achieved by J. A. Jiménez and O. Kowalski [7]. The proof of the theorem is based on the existence of real forms (that is, fixed point sets of anti-holomorphic involutions) in Hermitian symmetric spaces with the same rank. Such a real form can be lifted horizontally into the bundle space, up to some isometric covering map. The reflections of the bundle space in these lifted real forms give the isometries interchanging two points.

2. – Proof of the Theorem.

We refer to [4] as a basic reference for Sasakian structures. A φ-symmetric space is a connected, complete Sasakian manifold for which the reflection in any flow line of its characteristic vector field is a global automorphism of the Sasakian structure (φ, ξ, η, g). Given a φ-symmetric space M, the space $N := M/\xi$ consisting of the flow lines of ξ can be equipped with the structure of a Riemannian manifold such that the canonical projection $\pi\colon M \to N$ becomes a Riemannian submersion. The Sasakian condition implies that N is a Kähler manifold, and the φ-symmetry condition implies that N is Hermitian symmetric. There exists, up to isomorphism, at most one simply connected φ-symmetric space fibering over a Hermitian symmetric space. The explicit classification of simply connected φ-symmetric spaces is given by J. A. Jiménez and O. Kowalski in [7]. It is worthwhile to mention that a φ-symmetric space is always irreducible as a Riemannian manifold, even when it fibers over the Riemannian product of Hermitian symmetric spaces.

Let M be a simply connected φ-symmetric space with Sasakian structure (φ, ξ, η, g) and $\pi\colon\ M \to N$ the Riemannian submersion onto the corresponding simply connected Hermitian symmetric space N. Let p and q be any two distinct points in M. We now construct explicitly an isometry interchanging these two points. To begin with, we connect p and q by a geodesic γ and denote by m the midpoint between p and q on γ, and by X the unit vector tangent to γ at m.

Let $N = N_{-r} \times \ldots \times N_{-1} \times N_0 \times N_1 \times \ldots \times N_s$ be the de Rham decomposition of N, where $N_0 = \mathbb{C}^k$ with some $k \geqslant 0$ and where N_ν is an irreducible, simply connected, non-compact (if $\nu < 0$) or compact (if $\nu > 0$) Hermitian symmetric space. According to [11] or [15], in each N_ν there exists a connected, complete, totally real, totally geodesic submanifold P_ν with $\dim_{\mathbb{R}} P_\nu = \dim_{\mathbb{C}} N_\nu$ and $\operatorname{rank} P_\nu = \operatorname{rank} N_\nu$ $(\nu \neq 0)$. For $\nu = 0$ we take $P_0 = \mathbb{R}^k$. Then $P = P_{-r} \times \ldots \times P_{-1} \times P_0 \times P_1 \times \ldots \times P_s$ is a connected, complete, totally real, totally geodesic submanifold of N with $\dim_{\mathbb{R}} P = \dim_{\mathbb{C}} N$. By the homogeneity of N we may assume that $\pi(m) \in P$. Recall that a flat in a Riemannian symmetric space is a connected, complete, totally geodesic, flat submanifold whose dimension is the rank of the symmetric space. From standard theory of symmetric spaces it is known that the isotropy group of the isometry group of N_ν at $(\pi(m))_\nu$ acts transitively on the flats of N_ν containing $(\pi(m))_\nu$. Any tangent vector of a symmetric space is tangent to a suitable flat in that space. Since P_ν has the same rank as N_ν $(\nu \neq 0)$, we may therefore assume that $(\pi_* X)_\nu$ is tangent to P_ν. Clearly, for $\nu = 0$ we may assume this also.

According to [11], each P_ν is a reflective submanifold of N_ν, that is, the reflection of N_ν in P_ν is a well-defined global isometry of N_ν. Since P_ν is reflective, there exists a connected, complete, totally geodesic submanifold Q_ν of N_ν with $(\pi(m))_\nu \in Q_\nu$ and $T_{(\pi(m))_\nu} Q_\nu$ is the normal space of P_ν at $(\pi(m))_\nu$. Then $Q := Q_{-r} \times \ldots \times Q_{-1} \times Q_0 \times Q_1 \times \ldots \times Q_s$ is a connected, complete, totally real, totally geodesic submanifold of N with $\pi(m) \in Q$ and $\pi_* X$ perpendicular to $T_{\pi(m)} Q$.

The integrability condition for a submanifold in N to admit locally a horizontal lift is to be totally real [12]. So, let $\tilde{Q}$ be a local horizontal lift of Q with $m \in \tilde{Q}$. Then, clearly, X is perpendicular to $T_m \tilde{Q}$. Moreover, $\tilde{Q}$ is totally geodesic in M [12]. Now, Q is also reflective, which implies that at each point of Q there exists a totally geodesic, totally real submanifold of N containing this point and being tangent to the normal space of Q at this point. All these perpendicular totally real, totally geodesic submanifolds have local horizontal lifts, each of which is totally geodesic in M. So, if R^M denotes the Riemannian curvature tensor of M, the Gauss equation yields that $R^M(X, Y)Z$ is perpendicular to $\tilde{Q}$ whenever X, Y, Z are horizontal vectors perpendicular to $\tilde{Q}$. We also have

$$R^M(X, Y)\,\xi = g(Y, \xi)\,X - g(X, \xi)\,Y$$

for all $X, Y \in TM$ [4]. Using this we may conclude that $R^M(X, Y)Z$ is perpendicular to $\tilde{Q}$ for all X, Y, Z perpendicular to $\tilde{Q}$. Thus, the normal bundle of $\tilde{Q}$ is curvature-invariant. It follows that the local reflections of M in $\tilde{Q}$ are isometries [1]. As M is complete, connected, simply connected and real analytic, these local isometric reflections extend to a well-defined global isometric reflection f of M in some totally geodesic submanifold containing $\tilde{Q}$. Since X is perpendicular to $\tilde{Q}$, the reflection f reverses the direction of γ and hence interchanges p and q. By this the theorem is proved.

Remark. It is not known to the authors whether these φ-symmetric spaces are $I^0(M, g)$-commutative or not. The Stiefel manifold $V_2(\mathbb{R}^{n+2}) = SO(n+2)/SO(n)$ of orthonormal 2-frames in $\mathbb{R}^{n+2}$ is a φ-symmetric space fibering over the Hermitian symmetric space $G_2^+(\mathbb{R}^{n+2}) = SO(n+2)/SO(n) \times SO(2)$, the real Grassmann manifold of oriented 2-planes in $\mathbb{R}^{n+2}$. In [6] it has been stated that for $n > 29$ these Stiefel manifolds are not $I^0(M, g)$-commutative. This result has been mentioned in [1], [2] and [8]. Unfortunately, the proof in [6] is not correct due to a wrong use of the action of the isotropy group. So it is still open whether these Stiefel manifolds are $I^0(M, g)$-commutative or not.

Added in proof.

Recently, D. Akhiezer gave an affirmative answer to the question stated in the Remark. See his preprint *A remark on Stiefel manifolds* $SO(n+2)/SO(n)$.

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Jürgen Berndt: University of Hull, Department of Mathematics
Hull HU6 7RX, United Kingdom

L. Vanhecke: Katholieke Universiteit Leuven, Department of Mathematics
Celestijnenlaan 200 B, 3001 Leuven, Belgium

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