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Deformations of CR-Structures on a Real Lie-Algebra.

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Sunto. – *Sia* $\boldsymbol{g}_0$ *un'algebra di Lie e* $(\boldsymbol{p}, J)$ *una sua struttura di Cauchy-Riemann, vale a dire* J *è una struttura complessa integrabile del sottospazio vettoriale* $\boldsymbol{p}$*. Come è stato fatto per il caso delle strutture complesse, cfr. [GT], introduciamo il concetto di deformazione di una struttura* CR*. Per mezzo dei gruppi di coomologia* $H^k(\boldsymbol{g}, \boldsymbol{q})$ *vengono provati risultati di rigidità. In particolare ogni struttura di Lie-*CR *che è semisemplice è rigida. Alcuni esempi chiariscono le soluzioni particolari esposte.*

1. – Introduction.

This paper is devoted to the study of *deformations* of CR-*structures*. A CR-structure on a real Lie-algebra $\boldsymbol{g}_0$ is a triple $(\boldsymbol{g}_0, \boldsymbol{p}, J)$ such that $\boldsymbol{p}$ is a linear subspace of $\boldsymbol{g}_0$ and J is an endomorphism of $\boldsymbol{p}$ whose square is $-id_{\boldsymbol{p}}$ and such that, $\forall x, y \in \boldsymbol{p}$, $[x, y] - [Jx, Jy] \in \boldsymbol{p}$ and $[Jx, Jy] = [x, y] + J[Jx, y] + J[x, Jy]$. If one considers the set $\boldsymbol{q} = \{x - iJx / x \in \boldsymbol{p}\}$, $\boldsymbol{q}$ is a complex subalgebra of the complexified $\boldsymbol{g} = \boldsymbol{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ such that $\boldsymbol{q} \cap \overline{\boldsymbol{q}} = \{0\}$, $\boldsymbol{g} = \boldsymbol{q} \oplus \overline{\boldsymbol{q}} \oplus v$. In the following, we say that $\boldsymbol{q}$ itself is a CR-structure, [SN], [GT], [GO3].

We recall the definition given by Nijenhuis and Richardson, [NR]. Let μ_0 be the Lie-product on V_0 (i.e. $\boldsymbol{g}_0 = (V_0, \mu_0)$) and $\varphi \colon V_0 \oplus V_0 \to V_0$ be an alternating bilinear map. When $\mu = \varphi + \mu_0$ is a Lie-product, φ is said to be a *deformation* of μ_0. Let us denote with (W, μ_0) the subalgebra $\boldsymbol{q}$, where W is the underlying linear subspace. In this context, we are interested in a deformation φ for which the deformated product is equivalent to μ_0 and admits (W, μ_0) as a subalgebra, namely it defines a new CR-structure. This formulation suggests to restrict our considerations to the subset $\{\mu$ Lie-product$/(W, \mu)$ is a subalgebra$\}$ of the space m_0 of all the Lie-products. Obviously, a geometric interest is taken by the rigid CR-structures, i.e. the CR-structures with respect to μ_0 which are CR-structures with respect to every deformed Lie-product $\mu = \varphi + \mu_0$ equivalent to μ_0. An exact definition of rigidity is given in terms of isolated points of a certain moduli space $\mathcal{M}(d)$. In particular, we shall consider the algebraic set of products equivalent to the given μ_0 and which admit W as a subalgebra. Finally, we quotient it by the group of linear (not Lie's) invertible morphisms which let W and $\overline{W}$ invariant. In such a quotient space, we study the isolated points.

Algebraic properties of this moduli space bring us to introduce some cohomology groups. The point $[\mu_0]$ (i.e. the CR-structure $\boldsymbol{q}$) is rigid if the second cohomology group vanishes (Section 4).

In Section 2, we recall the main definitions about deformations (following Nijenhuis and Richardson) and the main results on Lie's and Levi-flat CR-structures, proved in [GO2]. In Section 3, we define the rigidity of a CR-structure in the terms of the isolated points of the moduli space $\mathcal{M}(d)$. Hence is given a concept of Levi-rigidity (resp. Lie-rigidity) for the Levi-flat CR-structures (resp Lie-CR-structures). Finally, Section 4 introduces the cohomology groups $H^k(\boldsymbol{g}, \boldsymbol{q})$ and links the vanishing of $H^2(\boldsymbol{g}, \boldsymbol{q})$, with the CR-rigidity of $\boldsymbol{q}$. A trivial example of rigid CR-structure without vanishing second cohomology group is given. In the last part of the paper, some results on the rigidity of semisimple CR-structures are proved.

2. – Deformations of Lie-algebra structures.

Let V_0 be a real linear n-dimensional space. A *real Lie-algebra structure* over V_0 is given by a *Lie-product* μ_0, i.e by a bilinear alternating map $\mu_0 \colon V_0 \oplus V_0 \to V_0$ which satisfies the Jacobi identity

$$\sum_{x,\,y,\,z} \mu_0(x, \mu_0(y, z)) = 0\,,$$

where $\sum_{x,\,y,\,z}$ denotes that the sum is computed over the permutations of the set $\{x, y, z\}$. The corresponding Lie-algebra is the pair $\boldsymbol{g}_0 = (V_0, \mu_0)$. Its complexification is denoted by $\boldsymbol{g} = (V, \mu_0)$ and is endowed with the conjugation map τ with respect to $\boldsymbol{g}_0$. Notice that μ_0 will denote both the real and the complexified Lie-product.

Given a base (e_i) in V_0, the map μ_0 is described by the *structure constants* c^i_{jk} such that $\mu_0(e_j, e_k) = c^i_{jk} e_i$. In the sequel, m_0 will be the set of all the bilinear alternating maps φ which satisfy the Jacobi identity. Such a set, via the structure constants c^i_{jk}, may be identified with an intersection of quadrics and hyperplanes in $\mathbb{R}^N$, $N = n\begin{bmatrix} n \\ 2 \end{bmatrix}$. Since $GL(n, \mathbb{R})$ acts on m_0 as a group of transformations, via the action $GL(n, \mathbb{R}) \times m_0 \to m_0 \colon (g, \varphi) \mapsto g_\varphi$ where $g_\varphi(x, y) = g^{-1}\varphi(gx, gy)$, the elements of the orbit $GL(n, \mathbb{R})\mu_0$ are said to be *equivalent* to μ_0: $\varphi \sim \mu_0$ if and only if $\exists g \in G$ such that $\varphi(x, y) = g^{-1}\mu_0(gx, gy)$.

In order to introduce the notion of *deformations* of a Lie-algebra structure $\boldsymbol{g} = (V, \mu_0)$, let us define the sets

$$\mathrm{Alt}^1_0(V) = \{F \colon V \oplus V \to V / F(x, x) = 0,\ F(V_0 \oplus V_0) \subset V_0\},$$

$$\mathrm{Alt}^n_0(V) = \{F \colon V \oplus \ldots \oplus V \to V / F \text{ is skew symmetric},\ F(V_0 \oplus \ldots \oplus V_0) \subset V_0\},$$

Given $\alpha \in \mathrm{Alt}_0^n(V)$ and $\beta \in \mathrm{Alt}_0^m(V)$, we set a product $\alpha \wedge \beta \in \mathrm{Alt}_0^{n+m}(V)$ as

$$\alpha \wedge \beta(x_0, \ldots x_{n+m}) = \sum_{\pi} \mathrm{sgn}(\pi)\, \alpha(\beta(x_{\pi(0)}, \ldots, x_{\pi(m)}), x_{\pi(m+1)}, \ldots, x_{\pi(m+n)}),$$

where π is the generic permutation of $\{o, \ldots n+m\}$. And we define the product $[,]$ in $\mathrm{Alt}_0(V)$ setting

$$[\alpha, \beta] = \alpha \wedge \beta - (-1)^{mn} \beta \wedge \alpha .$$

When $\alpha, \beta \in \mathrm{Alt}_0^1(V), [\alpha, \beta] = \alpha \wedge \beta + \beta \wedge \alpha = [\beta, \alpha]$. Furthermore, we define on $\mathrm{Alt}_0(V)$ the graded operator $\delta : \mathrm{Alt}_0(V) \to \mathrm{Alt}_0(V)$ by putting

$$\delta^{n+1} \alpha(x_0, \ldots x_n) = \sum_{1=0}^{n} (-1)^i \mu_0(x_i, \alpha(x_0, \ldots, \widehat{x}_i, \ldots, x_n) +$$

$$\sum_{i<j} (-1)^{i+j} \alpha(\mu_0(x_i, x_j), x_0, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_n),$$

where $\alpha \in \mathrm{Alt}_0^n(V)$ and the terms $\widehat{x}_h$ are omitted.

Since $\delta^{n+1} \circ \delta^n = 0$, δ is a *coboundary operator* and we define $B^1(\boldsymbol{g}) \doteq \mathrm{ad}(\boldsymbol{g})$, $B^{i+1}(\boldsymbol{g}) \doteq \mathrm{Im}\, \delta^i$, $Z^i(\boldsymbol{g}) \doteq \mathrm{Ker}\, \delta^i$; and $H^i(\boldsymbol{g}) \doteq Z^i(\boldsymbol{g})/B^i(\boldsymbol{g})$.

By an easy computation, one can see that $\delta\varphi = -[\mu_0, \varphi]$, $\forall \varphi \in \mathrm{Alt}_0^1(V)$, and that the Jacobi identity becomes $[\mu_0, \mu_0] = 0$. So m_0 is the nonempty set $\{\mu \in \mathrm{Alt}_0^1(V)/[\mu, \mu] = 0\}$ endowed with the topology of $\mathrm{Alt}_0^1(V)$.

The next Lemma caracterizes which alternating bilinear map μ satisfies the Jacobi identity, (i.e. which μ define a Lie-algebra structure).

LEMMA 2.1. – *Let μ_0 be a Lie-product and μ be an alternating bilinear map. Then μ is a Lie-product if and only if $\delta\varphi = (1/2)[\varphi, \varphi]$, where $\varphi = \mu - \mu_0$.*

In fact, $[\mu, \mu] = [\varphi + \mu_0, \varphi + \mu_0] = [\varphi, \varphi] + [\varphi, \mu_0] + [\mu_0, \varphi] = [\varphi, \varphi] + 2[\mu_0, \varphi] = [\varphi, \varphi] - 2\delta\varphi$. ∎

The solutions of $\delta\varphi = (1/2)[\varphi, \varphi]$, are called the *deformations* of μ_0. The corresponding linearized equation $\delta\varphi = 0$ defines the *infinitesimal deformations*, which, obviously, are the 2-cocycles of δ.

Let us consider, in $GL(V_0)$, a smooth curve $\sigma = \sigma(t)$, $|t| < \varepsilon$, such that $\sigma(0) = I$. Suppose that $\mu_t(x, y) = \sigma(t)^{-1} \mu_0(\sigma(t)\, x, \sigma(t)\, y)$ is a deformation of μ_0. Then we have $\sigma = I + t\varphi + O(t^2)$, $\mu_t = \mu_0 + tF + O(t^2)$ and $F = -\delta^1 \varphi$. Vice versa, a generic 2-cocycle does not come from a family $\{\mu_t\}$ as its first-order term. The deformation equation shows that the «tangent space» to m_0 at μ_0 is a subspace of $Z^2(\boldsymbol{g})$.

With these computations, we may also prove a classical result on Lie-algebras

PROPOSITION 2.1. – *The space of derivations* $\mathrm{Der}(\boldsymbol{g}_0)$ *is the tangent space at* I *of the group of automorphisms* $\mathrm{Aut}(\boldsymbol{g}_0)$.

PROOF. – Let $\sigma = \sigma(t)$, $|t| < \varepsilon$, be a smooth curve in $\mathrm{Aut}(\boldsymbol{g}_0)$, such that $\sigma(0) = I$. Let us write $\sigma_t = I + t\varphi + O(t^2)$ and $\mu_t = \mu_0 + tF + O(t^2)$. Since $\sigma(t) \in \mathrm{Aut}(\boldsymbol{g}_0)$, we have $\mu_t = \mu_0$ and hence $\delta^1 \varphi = -F = 0$. So, $\varphi \in Z^1(\boldsymbol{g}) = \{l \in \mathrm{End}\, V_0 / -l\mu_0(x, y) + \mu_0(lx, y) + \mu_0(x, ly) = 0\} = \mathrm{Der}(\boldsymbol{g}_0)$.

Vice versa, let $\varphi \in \mathrm{Der}(\boldsymbol{g}_0)$. Then the curve $\sigma_t = I + t\varphi + O(t^2)$ takes value in $\mathrm{Aut}(\boldsymbol{g}_0)$, $\forall |t| < \varepsilon$. ■

DEFINITION 2.1. – *The structure* $\boldsymbol{g} = (V, \mu_0)$ *is said to be rigid when the orbit* $GL(V_0)\mu_0$ *is an open set in* m_0, *or, equivalently, when* $[\mu_0]$ *is isolated in* $m_0/GL(V_0)$.

Nijenhuis and Richardson show that a sufficient condition for rigidity is the vanishing of $H^2(\boldsymbol{g})$. Such a condition is not necessary.

Finally, we remark that the elements of $H^3(\boldsymbol{g})$ may be interpreted as obstructions to expanding an infinitesimal deformation of μ_0 into a one-parameter family of deformations of μ_0. For this, we consider $\mu_t = \mu_0 + t\varphi_1 + t^2\varphi_2 + \ldots$. The condition of Jacobi on μ_t give rise to an infinite sequence of conditions on μ_0 and on the $\varphi_j s$:

1. $[\mu_0, \mu_0] = 0$,
2. $[\mu_0, \varphi_1] = 0$,
3. $[\mu_0, \varphi_2] + (1/2)[\varphi_1, \varphi_1] = 0$,
4. $\ldots$

The first one says that μ_0 is a Lie-product; the second one that φ_1 is an infinitesimal deformation of μ_0; the third one that $[\varphi_1, \varphi_1]$ is a coboundary, and that the class $[\varphi_1, \varphi_1] \in H^3(\boldsymbol{g})$ vanishes. So we may conclude with the

PROPOSITION 2.4. – *Let* $H^3(\boldsymbol{g})$ *vanish. Then every infinitesimal deformation is the first-order term of a one-parameter family.*

3. – CR-structures on a real Lie-algebra and their rigidity.

A CR-structure on a real Lie-algebra $\boldsymbol{g}_0$ is a d-dimensional subalgebra $\boldsymbol{q}$ of the complexified $\boldsymbol{g}$ whose intersection with its conjugated $\overline{\boldsymbol{q}}$ vanishes. Such a datum corresponds to a real subset $\boldsymbol{p} \subseteq \boldsymbol{g}_0$ endowed with an endomorphism J, such that $J^2 = -id$, $\forall x, y \in \boldsymbol{p}$, $[x, y] - [Jx, Jy] \in \boldsymbol{p}$ and $[Jx, Jy] = [x, y] + J[Jx, y] + J[x, Jy]$. We are also interested in two particular subcases. Precisely we shall consider *Lie*-CR-*structures* and *Levi-flat* ones. In particular, in the first case, $\boldsymbol{p}$ is an ideal and J is ad_X-invariant (i.e. $\boldsymbol{q}$ is a complex ideal). In the second one, $\boldsymbol{p}$ is a real subalgebra (i.e. $\boldsymbol{q} \oplus \overline{\boldsymbol{q}}$ is a complex one). Obviously a Lie-CR-structure is Levi-flat. These particular cases are described in [GO1], [GO2].

We say that a generic CR-structure $\boldsymbol{q}$ corresponds to the pair (W, μ_0) if

1. $W \cap \overline{W} = \{0\}$,
2. $\mu_0(W \oplus W) \subset W$.

Moreover, in correspondence of the Lie-CR-structures and of the Levi-flat ones, the subspace W must satisfy, respectively,

3_{Lie}. $\mu_0(W \oplus V) \subset W$,

3_{Levi}. $\mu_0(W \oplus \overline{W}) \subset W \oplus \overline{W}$,

where the pair (V, μ_0) defines the complex Lie-algebra $\boldsymbol{g}$.

In the following, we define the rigidity of a CR-structure and we construct some cohomology groups which concern the $\boldsymbol{q}$ structure. To do that, we have to restrict ourselves to the Lie-products which admit $\boldsymbol{q}$ as the desired structure.

Remind that the rigid Lie-algebras correspond to the isolated points of $m_0/GL(V_0)$. In this Section, we consider the suitable subsets of m_0 and of $GL(V_0)$ which let us give an analogous construction for CR-structures.

Take the real algebraic set $n(d) \doteq GL(V_0)\mu_0 \cap \{\varphi \in m_0/\varphi(W \oplus W) \subset W\}$, composed by the Lie-products equivalent to μ_0 which admit W as a subalgebra. And consider the quotient by the group of the linear (even not Lie's) automorphisms which let W invariant: $G(V, W) = \{\sigma \in GL(V_0)/\sigma W = W\}$. A CR-structure $\boldsymbol{q}$ is said to be CR-*rigid* if $[\mu_0]$ is an isolated point in $n(d)/G(V, W)$.

The set $n(d)/G(V, W)$ can be described in the terms of a subset of the Grassmannian manifold $\mathrm{Gr}(n, d)$. To do that, define the set $S(d) = \{W \in \mathrm{Gr}(n, d)/W \cap \overline{W} = \{0\}, \mu_0(W \oplus W) \subset W\}$. The group $\mathrm{Aut}\,\boldsymbol{g}_0$ acts on $S(d)$ and $W, W' \in S(d)$ give equivalent CR-structures on $\boldsymbol{g}_0$ if and only if $W' = \sigma W$, $\sigma \in \mathrm{Aut}\,\boldsymbol{g}_0$. Thus, the moduli space for CR-structures on $\boldsymbol{g}_0$ is given by $S(d)/\mathrm{Aut}\,\boldsymbol{g}_0$. Then, the desired description is given by the

PROPOSITION 3.1. – *The spaces $S(d)/\mathrm{Alt}\,\boldsymbol{g}_0$ and $n(d)/G(V, W)$ are isomorphic. We shall pose $\mathfrak{M}(d) := n(d)/G(V, W) \simeq S(d)/\mathrm{Aut}\,\boldsymbol{g}_0$.*

PROOF. – Taken $W' \in S(d)$, there exists $g' \in GL(V_0)$ such that $W' = g'W$, and hence $g'\mu_0$ is in $n(d)$. Vice versa, if $g'\mu_0 \in n(d)$, the subspace $g'W$ is in $S(d)$.

Let us consider the quotient spaces, and take $[W'] \in S(d)/\mathrm{Aut}\,\boldsymbol{g}_0$. Then $W'' \in [W']$ if and only if $\exists \sigma \in \mathrm{Aut}\,\boldsymbol{g}_0$ such that $W'' = \sigma W'$, and then

$W = g''^{-1} \sigma g' W$. Moreover $g''^{-1} \sigma g' V_0 = V_0$, and this implies $[g' \mu_0] \in n(d)/G(V, W)$. The converse is analogous. ■

We derive four conditions equivalent to the rigidity of the CR-structure $\boldsymbol{q}$:

COROLLARY 3.2. – *The following facts are equivalent*:

1. $[W]$ *is isolated in* $S(d)/\mathrm{Aut}\,\boldsymbol{g}_0$;
2. $[\mu_0]$ *is isolated in* $n(d)/G(V, W)$;
3. $\mathrm{Aut}\,\boldsymbol{g}_0 W$ contains a neighborhood of W in $S(d)$;
4. $G(V, W)\,\mu_0$ contains a neighborhood of μ_0 in $n(d)$. ■

Analogous definitions are given for Levi-flat CR-structures and Lie-CR-structures. Precisely we shall consider the sets

$$n_{\mathrm{Levi}}(d) = GL(V_0)\,\mu_0 \cap \{\varphi \in m_0 / \varphi(W \oplus W) \subset W,\ \varphi(W \oplus \overline{W}) \subset W \oplus \overline{W}\},$$

$$S_{\mathrm{Levi}}(d) = \{W \in \mathrm{Gr}\,(n, d)/W \cap \overline{W} = \{0\},\ \mu_0(W \oplus W) \subset W,\ \mu_0(W \oplus \overline{W}) \subset W \oplus \overline{W}\},$$

$$n_{\mathrm{Lie}}(d) = GL(V_0)\mu_0 \cap \{\varphi \in m_0 / \varphi(W \oplus V) \subset W\},$$

$$S_{\mathrm{Lie}}(d) = \{W \in \mathrm{Gr}\,(n, d)/W \cap \overline{W} = \{0\},\ \mu_0(W \oplus V) \subset W\}.$$

The quotients $\mathcal{M}_{\mathrm{Levi}}(d)$ and $\mathcal{M}_{\mathrm{Lie}}(d)$ are the analogous of $\mathcal{M}(d)$ in the respective case as well as the notations of Levi-rigidity and Lie-rigidity. Trivially, $\mathcal{M}(d) \supset \mathcal{M}_{\mathrm{Levi}}(d) \supset \mathcal{M}_{\mathrm{Lie}}(d)$, so we may immediately remark that a Levi-flat CR-rigid CR-structure is Levi-CR-rigid and a Levi-CR-rigid Lie-CR-structure is Lie-CR-rigid.

Observe that the classification of Levi-flat CR-structures is based on the bilinear form $\omega : \boldsymbol{p} \oplus \boldsymbol{p} \to \boldsymbol{p} : (X, Y) \mapsto [X, Y] - [JX, JY]$ which is shown to be a Lie-product; moreover, with respect of this product, the CR-structure is a Lie's one, [GO2]. Furthermore, if ω denotes even the complexified one, then it coincides with $2\mu_0$ on W. Now we may consider the following different structures on W: the CR-structure with respect to μ_0 and the Lie-algebra's one, with respect to ω. The next result links the rigidity of the algebra (W, ω) and the rigidity of the CR-structure of $\boldsymbol{g}_0$ (W, μ_0).

THEOREM 3.3. – *The Lie-algebra* (W, ω) *is rigid, whenever the* CR-*structure* $\boldsymbol{q}$ *is.*

PROOF. – Let us denote by $\boldsymbol{p}$ the subspace $\mathrm{Re}\,\boldsymbol{q}$ (the space of the real parts of elements of $\boldsymbol{q}$), and with $m_{\boldsymbol{p}}$ the set $\{\varphi : \boldsymbol{p} \oplus \boldsymbol{p} \to \boldsymbol{p} : \varphi$ is a Lie-product$\}$. Then ω is rigid if and only if there exists a neighborood U of ω in $m_{\boldsymbol{p}}$ such that $U \subset GL(\boldsymbol{p})\,\omega$.

Now take the map $j\colon m_{\boldsymbol{p}} \to n(d)\colon \varphi \mapsto \varphi^*$, where

$$\varphi^*(X, Y) = \begin{cases} \varphi(X, Y) & \text{if } X, Y \in \boldsymbol{q} \\ 0 & \text{otherwise}. \end{cases}$$

So j is a continuous and injective map and satisfies $j(2\omega) = \mu_0|_W = \widetilde{\mu}_0$, where

$$\widetilde{\varphi}(X, Y) = \begin{cases} \varphi(X, Y) & \text{if } X, Y \in \boldsymbol{q}, \\ 0 & \text{otherwise}. \end{cases}$$

Finally, consider the map $\pi\colon n(d) \to n(d)\colon \varphi \mapsto \widetilde{\varphi}$. Since π is an open map, then, if $\mu_0 \in n(d)$ is an isolated point, $\widetilde{\mu}_0$ is isolated, too. It follows that $2\omega = j^{-1}(\widetilde{\mu}_0)$ is isolated, which means that (W, ω) is rigid provided $\boldsymbol{q}$ is. ■

THEOREM 3.4. – *Let $\boldsymbol{q}$ be a Levi-flat CR-structure. Then (W, ω) is rigid if and only if $\boldsymbol{q}$ is Levi-rigid.*

PROOF. – Suppose $\boldsymbol{q}$ is Levi-CR-rigid. Then, in view of Theorem 3.3, we have $j(m_{\boldsymbol{p}}) \subset n_{\mathrm{Levi}}(d)$. Now, define

$$\widetilde{\widetilde{\varphi}}(X, Y) = \begin{cases} \varphi(X, Y) & \text{if } X, Y \in \boldsymbol{q} \oplus \overline{\boldsymbol{q}} \\ 0 & \text{otherwise}, \end{cases}$$

and π' by $n_{\mathrm{Levi}}(d) \to n_{\mathrm{Levi}}(d)\colon \varphi \mapsto \widetilde{\widetilde{\varphi}}$; then, π' is an open map, and consequentely, (W, ω) is rigid.

Vice versa, the map ϱ defined by $n_{\mathrm{Levi}}(d) \to m_{\boldsymbol{p}}\colon \varphi \mapsto \varphi|_W$ is continuous and $\mu_0 \in \varrho^{-1}(2\omega)$. Hence, when ω is an isolated point, μ_0 is isolated, too. ■

EXAMPLE 3.5. – In [GT] the authors proved that a semisimple complex structure in a semisimple Lie-algebra is rigid.

Thus, the complex structure considered on the Lie-algebra (W, ω) in Theorem 3.4 is rigid, whenever (W, ω) is semisimple.

Furthermore, in [GO2], we prove that (W, ω) is semisimple if and only if $\boldsymbol{q}$ is semisimple.

In conclusion, all the Levi-flat CR-structures, which are semisimple as Lie-algebras, provide examples of Levi-rigid CR-structures. Moreover, they are CR-rigid.

4. – The cohomology groups $H^k(\boldsymbol{g}, \boldsymbol{q})$.

1. In this Section we link the rigidity of a CR-structure $\boldsymbol{q} = (W, \mu_0)$ with the vanishing of a new cohomology group $H^2(\boldsymbol{g}, \boldsymbol{q})$, subgroup of $H^2(\boldsymbol{g})$, de-

pending on $\boldsymbol{q}$; take the following sets of (multi)linear maps

$$L_0(V,\,W) = \{l \in \operatorname{End} V_0/lW \subset W\} \subset L_0(V) = \operatorname{End} V_0$$

$$\operatorname{Alt}_0^k(V,\,W) = \{F\colon V^{\oplus(k+1)} \to V/F\colon V_0^{\oplus(k+1)} \subset V_0,\, F\colon W^{\oplus(k+1)} \subset W\} \subset \operatorname{Alt}_0^k(V).$$

The coboundary operators δ^{k+1} defined in Section 2 may be immediately restricted to the new ones $\delta_W^{k+1}\colon \operatorname{Alt}_0^k(V,\,W) \to \operatorname{Alt}_0^{k+1}(V,\,W)$. The expression of δ_W^1 and of δ_W^2 are

$$\delta_W^1 l(x,\,y) = -l\mu_0(x,\,y) + \mu_0(lx,\,y) + \mu_0(x,\,ly)\,,$$

$$\delta_W^2 F(x,\,y,\,z) = \sum_{x,\,y,\,z} F(\mu_0(x,\,y),\,z) + \sum_{x,\,y,\,z} \mu_0(F(x,\,y),\,z).$$

Then we set $Z^j(\boldsymbol{g},\,\boldsymbol{q}) = \operatorname{Ker}\delta_W^j$, $B^1(\boldsymbol{g},\,\boldsymbol{q}) = \{x \in \boldsymbol{g}/\mathrm{ad}_x\boldsymbol{q} \subset \boldsymbol{q}\} = \boldsymbol{n}(\boldsymbol{q})$, $B^{j+1}(\boldsymbol{g},\,\boldsymbol{q}) = \operatorname{Im}\delta_W^j$, and define the *cohomology groups* $H^j(\boldsymbol{g},\,\boldsymbol{q}) = Z^j(\boldsymbol{g},\,\boldsymbol{q})/B^j(\boldsymbol{g},\,\boldsymbol{q})$. A beginning geometrical result is given by the

PROPOSITION 4.2. – *Let $\boldsymbol{q}$ be a Lie-CR-structure, then $H^1(\boldsymbol{g},\,\boldsymbol{q})$ is the set of the Lie-derivations of $\boldsymbol{g}$ which have W as invariant space and which are not inner derivations.*

In fact, we have

$$Z^1(\boldsymbol{g},\,\boldsymbol{q}) = \{l \in L_0(V,\,W)/l\mu_0(x,\,y) = \mu_0(lx,\,y) + \mu_0(x,\,ly)\} = \operatorname{Der}(\boldsymbol{g},\,\boldsymbol{q})\,.$$

Since $\boldsymbol{q}$ is an ideal, we have that $B^1(\boldsymbol{g},\,\boldsymbol{q}) = \mathrm{ad}\,\boldsymbol{g}$. It follows $H^1(\boldsymbol{g},\,\boldsymbol{q}) = \operatorname{Der}(\boldsymbol{g},\,\boldsymbol{q})/\mathrm{ad}\,\boldsymbol{g}$.

2. Now, we are in position to stay the main result of this Section and hence of the paper.

THEOREM 4.2. – *Let $\boldsymbol{q}$ be a CR-structure on $\boldsymbol{g}_0$. If $H^2(\boldsymbol{g},\,\boldsymbol{q}) = 0$, then $\boldsymbol{q}$ is* CR-*rigid.*

PROOF. – we have to show that $G(V,\,W)\mu_0$ contains a neighborhood of μ_0 in $n(d)$. This is equivalent to the fact that $G(V,\,W)\mu_0$ and $n(d)$ have the same Zariski tangent space at μ_0.

Let us denote by T_1, T_2, T_3 the Zariski tangent spaces at μ_0 of $G(V,\,W)\mu_0, n(d)$ and $m_W = \{\varphi \in m_0/\varphi(W \oplus W) \subset W\}$, respectively. And we prove the following inclusions, $Z^2(\boldsymbol{g},\,\boldsymbol{q}) \supset T_3 \supset T_2 \supset T_1 \supset B^2(\boldsymbol{g},\,\boldsymbol{q})$. (So we conclude using the vanishing of H^2).

Step 1: $Z^2(\boldsymbol{g},\,\boldsymbol{q}) \supset T_3$. If we write $\varphi = \mu_0 + F$, it is easy to remark that $m_W = \{F \in \operatorname{Alt}_0^1(V,\,W)/\delta^2 F = (1/2)[F,\,F]\}$. The map $\phi(x,\,y,\,z)\colon \operatorname{Alt}_0^1(V,\,W) \to V\colon F \mapsto \delta^2 F(x,\,y,\,z) - (1/2)[F,\,F](x,\,y,\,z)$ is an element of the ideal $\mathfrak{J}_0(m_W)$, and its linear part is $\delta^2 F(x,\,y,\,z)$. Hence $T_3 \subset \bigcap_{x,\,y,\,z \in V} \{F \in \operatorname{Alt}_0^1(V,\,W)/\delta^2\cdot$

$F(x, y, z) = 0\} = Z^2(\boldsymbol{g}, \boldsymbol{q})$.

Step 2: $T_3 \supset T_2 \supset T_1$. Indeed, $m_W \supset n(d) \supset G(V, W)\mu_0$.

Step 3: $T_1 \supset B^2(\boldsymbol{g}, \boldsymbol{q})$. Consider $l \in L_0(V, W)$, and $\sigma(t) = I + tl \in L_0(V, W)$. Then $\sigma(t)$ is in $G(V, W)$, when $|t| < \varepsilon$. Hence $\varphi(t) = \sigma(t)\mu_0 \in G(V, W)\mu_0$ and $\varphi(t) = \mu_0 + t(-\delta^1 l) + O(t^2)$. So you conclude that $\delta^1 l \in T_1$. ■

REMARK 4.3. – Let us consider the sets $\mathrm{Alt}^k_{\mathrm{Levi}}(V, W) = \{F\colon V^{\oplus(k+1)} \to V/F\colon V_0^{\oplus(k+1)} \subset V_0,\ F\colon W^{\oplus(k+1)} \subset W,\ F\colon W^{\oplus k} \oplus \overline{W} \subset W \oplus \overline{W}\}$ and $\mathrm{Alt}^k_{\mathrm{Lie}}(V, W) = \{F\colon V^{\oplus(k+1)} \to V/F\colon V_0^{\oplus(k+1)} \subset V_0,\ F\colon W^{\oplus k} \oplus V \subset W\}$. Then the coboundary operator δ^{k+1} sends $\mathrm{Alt}^k_{\mathrm{Levi}}(V, W)$ (resp. $\mathrm{Alt}^k_{\mathrm{Lie}}(V, W)$) in $\mathrm{Alt}^{k+1}_{\mathrm{Levi}}(V, W)$ (resp. $\mathrm{Alt}^{k+1}_{\mathrm{Lie}}(V, W)$). Thus we may define, in an obvious way, $H^k_{\mathrm{Levi}}(\boldsymbol{g}, \boldsymbol{q})$ and $H^k_{\mathrm{Lie}}(\boldsymbol{g}, \boldsymbol{q})$. With the same proof as in the above theorem, we see that a Levi-flat (resp. Lie's) CR-structure is Levi-rigid (resp Lie-rigid) when $H^2_{\mathrm{Levi}}(\boldsymbol{g}, \boldsymbol{q}) = \{0\}$ (resp. $H^2_{\mathrm{Lie}}(\boldsymbol{g}, \boldsymbol{q}) = \{0\}$).

The condition of the Theorem 4.2 is not necessary. In fact, we have the

PROPOSITION 4.4. – *Every* CR-*structure* $\boldsymbol{q}$ *on a real abelian Lie-algebra* $\boldsymbol{g}_0$ *is* CR-*rigid. While its cohomology groups are nonvanishing.*

Since $\boldsymbol{g}_0$ is abelian, the set $S(d)$ is given by $\{W \in \mathrm{Gr}(n, d)/W \cap \overline{W} = \{0\}\}$, while $\mathrm{Aut}\,\boldsymbol{g}_0$ coincides with $GL(V_0)$. Hence all the elements of $S(d)$ are mutually equivalent and $S(d)/GL(V_0) = \{0\}$. So, every CR-structure is rigid. Now, let us compute the cohomology groups. It is a trivial fact that the coboundary operator vanishes. This fact is equivalent to $\mathrm{Ker}\,\delta^1_W = L_0(V, W)$, $\mathrm{Ker}\,\delta^{j+1}_W = \mathrm{Alt}^j_0(V, W)$, $\mathrm{Im}\,\delta^j_W = \{0\}$, and hence $H^1(\boldsymbol{g}, \boldsymbol{q}) = L_0(V, W)$, $H^{j+1}(\boldsymbol{g}, \boldsymbol{q}) = \mathrm{Alt}^j_0(V, W)$.

3. In the last part of this Section we give some rigidity results. For this, we consider the decomposition $\boldsymbol{g} = \boldsymbol{q} \oplus \overline{\boldsymbol{q}} \oplus v$. Remind that the space v admits a real basis $\{X_i\}$.

THEOREM 4.5. – *Let* $\boldsymbol{g}_0$ *be a real Lie-algebra and* $\boldsymbol{q}$ *a semisimple* CR-*structure, such that* $[\boldsymbol{q}, V] \subset \boldsymbol{q}$. *Then* $\boldsymbol{q}$ *is* CR-*rigid.*

PROOF. – Because of the semisemplicity of $\boldsymbol{q}$ we have that $H^1(\boldsymbol{q}) = H^2(\boldsymbol{q}) = 0$, [VA]. Let now consider $\varphi \in Z^2(\boldsymbol{g}; \boldsymbol{q}) \cap B^2(\boldsymbol{g})$; by definition there is a $\beta \in L_0(V)$ such that $\varphi = \delta^1\beta$. Furthermore, φ induces a 2-cocycle of $Z^2(\boldsymbol{q})$. So, there is a linear map $\psi\colon \boldsymbol{q} \to \boldsymbol{q}$ such that $\delta^1_W\psi = \varphi$. In particular, $\psi - \beta$ is a coboundary of $\boldsymbol{q}$ with respect to the adjoint representation: i.e., exists $Z \in \boldsymbol{g}$ such that $(\psi - \beta)Q = \mathrm{ad}_Z Q$, $\forall Q \in \boldsymbol{q}$.

Let $Z = Q_1 + \overline{Q_2} + v$, $Q_1, Q_2 \in \boldsymbol{q}$, and $B = Q_2 + \overline{Q_2}$. Consider the map $\sigma(X) = \beta(X) + [X, B]$. Then $\sigma(\overline{Q}) = \overline{\sigma(Q)}$ and $\delta^1_W\sigma = \delta^1_W\beta = \varphi$. Moreover, $\sigma(Q) =$

$\psi(Q) + [Q, Q_2 - Q_1 - v] \in \boldsymbol{q}$. This proves that $Z^2(\boldsymbol{g}; \boldsymbol{q}) \cap B^2(\boldsymbol{g}) = B^2(\boldsymbol{g}; \boldsymbol{q})$ and hence $H^2(\boldsymbol{g}; \boldsymbol{q}) = 0$, since the Zarisky tangent space T_3 is contained in $B^2(\boldsymbol{g})$. ■

The proof is based on the fact that the element Z may be written as a sum of a real vector B and of an element of the normalizer $\boldsymbol{n}(\boldsymbol{q}) = B^1(\boldsymbol{g}, \boldsymbol{q})$. Hence, we have the more general

THEOREM 4.6. – *Let $\boldsymbol{g}_0$ be a real Lie-algebra and $\boldsymbol{q}$ a semisimple* CR-*structure such that $\boldsymbol{g} = \boldsymbol{g}_0 + \boldsymbol{n}(\boldsymbol{q})$. Then $\boldsymbol{q}$ is rigid.* ■

Remind that when $\boldsymbol{q}$ is semisimple, the normalizer $\boldsymbol{n}(\boldsymbol{q})$ is the direct sum $\boldsymbol{q} \odot \boldsymbol{c}(\boldsymbol{q})$, where $\boldsymbol{c}(\boldsymbol{q})$ is the centralizer. Indeed, $\forall X \in \boldsymbol{n}(\boldsymbol{q})$ $\mathrm{ad}_X|_{\boldsymbol{q}}$ is a derivation of $\boldsymbol{q}$. Hence there exists an element $X' \in \boldsymbol{q}$, such that $\mathrm{ad}_X|_{\boldsymbol{q}} = \mathrm{ad}_{X'}|_{\boldsymbol{q}}$ (and such an element is unique). The map $\varphi \colon \boldsymbol{n}(\boldsymbol{q}) \to \boldsymbol{n}(\boldsymbol{q})\colon X \mapsto X'$ is an endomorphism such that $Im\varphi = \boldsymbol{q}$ and $Ker\varphi = \boldsymbol{c}(\boldsymbol{q})$, the thesis follows.

As a trivial consequence, we get the

COROLLARY 4.8. – *Let $\boldsymbol{q}$ be a semisimple Lie-CR-structure of a real Lie-algebra $\boldsymbol{g}_0$. Then $\boldsymbol{q}$ is rigid.*

Such result may be extended to the

EXAMPLE 4.8. – Take a reductive Lie-algebra $\boldsymbol{g}$. In particular, we have that $\boldsymbol{g}$ splits as $\boldsymbol{g} = \theta(\boldsymbol{g}) \oplus \mathcal{D}\boldsymbol{g}$: i.e. its radical is abelian since it is the center $\theta(\boldsymbol{g})$. Consequently, any LCR-structure splits analogously: $\boldsymbol{q} = \theta(\boldsymbol{q}) \oplus \mathcal{D}\boldsymbol{q}$, where $\theta(\boldsymbol{q}) = \boldsymbol{q} \cap \theta(\boldsymbol{g})$ and $\mathcal{D}\boldsymbol{q} = \boldsymbol{q} \cap \mathcal{D}\boldsymbol{g}$. Both $\theta(\boldsymbol{q})$ and $\mathcal{D}\boldsymbol{q}$ may be seen as LCR-structures of $\theta(\boldsymbol{g})$ and $\mathcal{D}\boldsymbol{g}$, respectively. Thus, they are CR-rigid. Since the elements of Aut$\boldsymbol{g}_0$ preserve the splitting $\boldsymbol{g} = \theta(\boldsymbol{g}) \oplus \mathcal{D}\boldsymbol{g}$, $\boldsymbol{q}$ is CR-rigid, too.

This Example is trivial, whenever $\boldsymbol{g}$ is compact. In fact, in this case, $\boldsymbol{q}$ is abelian, [GO1].

In the noncompact case, it is more interesting since this is the case of the CR-semisimple LCR-algebras introduced in [GO3], which are forced to be rigid.

EXAMPLE 4.9. – In conclusion, consider the «dual» case of the previous Example. Let $\boldsymbol{q}$ be a reductive Levi-flat CR-structure: $\boldsymbol{q} = \theta(\boldsymbol{q}) \oplus \mathcal{D}\boldsymbol{q}$. Then, $\mathcal{D}\boldsymbol{q}$ is CR-rigid. Otherwise, Proposition 4.4 assures that $\theta(\boldsymbol{g})$ is rigid with respect to ω. Since $Aut\boldsymbol{g}_0$ preserves the splitting, the CR-structure $\boldsymbol{q}$ is globally Levi-rigid.

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