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The Hartogs-Type Extension Theorem for Meromorphic Mappings into q-Complete Complex Spaces.

SERGEI IVASHKOVICH - ALESSANDRO SILVA

Sunto. – *Si dimostra un risultato di prolungamento per applicazioni meromorfe a valori in uno spazio q-completo che generalizza direttamente il risultato classico di Hartogs e milgliora risultati di K. Stein..*

0. – Introduction.

The purpose of this paper is to prove a result in extending meromorphic mappings which can be considered as a direct generalization of the original Hartogs' extension theorem for holomorphic functions. The extension of an analytic object in the sense of Hartogs' has been shown to imply extensions across subvarieties (Riemann extension) or irreducible branches of fixed dimension (Thullen extension) (see [S_1]). Moreover, it is clear that to solve the problem of extending an analytic object in the sense of Hartogs', it is enough to show that it extends across the «hole» of a standard Hartogs' figure.

To be more precise, let Δ_r^n be the polydisc of radius r in $\mathbb{C}^n$, and set $\Delta^n := \Delta_1^n$; then the open subset of $\mathbb{C}^{n+q}$:

$$H_n^q(r) := \Delta^n \times (\Delta^q \setminus \overline{\Delta}_{1-r}^q) \cup \Delta_r^n \times \Delta^q \tag{1}$$

is called the q-concave Hartogs' figure. Note that Δ^{n+q} is the envelope of holomorphy of $H_n^q(r)$. Let Y be a reduced complex space. Meromorphic mappings $f: H_n^q(r) \to Y$ are said to satisfy a Hartogs-type extension Theorem if they extend to meromorphic mappings $\hat{f}: \Delta^{n+q} \to Y$. It will be necessary sometimes to express this property more precisely by saying that the space Y possess a meromorphic extension property in bidimension (n, q).

Hartogs-type extension Theorems for meromorphic mappings have been proved when Y is compact Kähler, and when Y is compact with some weaker metric properties by the first author in [I_1], [I_2] and [I_3]. In this paper we shall prove the following

THEOREM. – *Every meromorphic mapping* $f\colon H_n^q(r) \to Y$, *where* Y *is a* q*-complete complex space, extends to a meromorphic mapping from* Δ^{n+q} *to* Y.

We recall that a *strictly* q*-convex* function ϱ on the complex space Y with $\dim Y = N$ is a real valued C^2 function such that the hermitian matrix of the coefficients of the $(1, 1)$-form $dd^c \varrho$ has at least $N - q + 1$ positive eigenvalues at all points of Y. (Smooth objects on a complex space Y are by definition the pullbacks of smooth objects in domains of $\mathbb{C}^M$ under appropriate local embeddings. The number q is independent of such embeddings).

The complex space Y is called q*-complete* if there exists a strictly q-convex exhaustion function $\varrho\colon Y \to \mathbb{R}^+$.

We remark that in the case $q = 1$, that is when Y is Stein, and when f is holomorphic, our Theorem, via proper embedding of Y into $\mathbb{C}^M$, reduces to the extension of holomorphic functions and it gives back the classical theorem of Hartogs, [H].

More generally our Theorem provides Hartogs' type extension of meromorphic mappings into complex subspaces of $\mathbb{CP}^N \setminus \mathbb{CP}^{N-q}$, the Stein case being of course included as $\mathbb{CP}^N \setminus \mathbb{CP}^{N-1}$.

Another point, which we would like to mention in this Introduction is that our Theorem improves on the following result due to K. Stein, [St]:

Let D *be a domain in* $\mathbb{C}^{q+2}$, $q \geqslant 1$, *and* $K \subset\subset D$ *be a compact subset in* D *with connected complement. Let* Y *be a normal complex space of dimension* q. *Then every holomorphic mapping* $f\colon D \setminus K \to Y$ *extends to a holomorphic mapping from* D *to* Y.

Indeed, since every noncompact irreducible complex space of dimension q is q-complete by a theorem of Ohsawa, [O], we have the following immediate corollary of our Theorem:

COROLLARY. – *Let* Y *be an irreducible non compact complex analytic space of dimension* q. *Every meromorphic mapping* $f\colon H_n^q(r) \to Y$, $n \geqslant 1$, *extends to a meromorphic mapping from* Δ^{n+q} *to* Y.

Hence, if Y is compact and $f\colon H_n^q(r) \to Y$ is not surjective, if we delete from Y one point which is not in the image of f and we call the resulting space Y', let us apply Corollary 1 to $f\colon H_n^q(r) \to Y'$ and we obtain in particular (see sect. 4) the following improvement of the theorem of Stein:

COROLLARY. – *Let* Y *be an irreducible non compact complex analytic space of dimension* q. *For every domain* $D \subset \mathbb{C}^{q+1}$, $q \geqslant 1$, *and for every compact subset* $K \subset\subset D$ *with connected complement, every nonsurjective* *meromorphic mapping* $f\colon D\backslash K \to Y$ *extends to a meromorphic mapping from* D *to* Y.

In the last section we shall discuss among other open questions also someones arising from the attempts to remove the condition on Y to be noncompact in this last statement.

1. – Preliminaries.

Let X and Y be reduced complex spaces with X normal. A *meromorphic mapping* $f\colon X \to Y$ is defined as an irreducible, locally irreducible analytic subset $\Gamma_f \subset X \times Y$ (the graph of f), such that the restriction to Γ_f, $\pi|_{\Gamma_f}\colon \Gamma_f \to X$, of the natural projection $\pi\colon X \times Y \to X$ is proper, surjective and generically one to one, see [R].

The set $f[x] := \{y \in Y : (x, y) \in \Gamma_f\}$ is a compact subvariety in Y and the set of points $x \in X$ such that $\dim f[x] \geqslant 1$ is analytic by the Remmert proper mapping theorem and has codimension at least two, because of the condition of irreducibility of Γ_f. This set is called the *fundamental set* of f or the *set of points of indeterminancy* of f and will be denoted by F.

If X_1 is a normal subspace of X, $X_1 \not\subset F$, we denote by $f|_{X_1}$ the meromorphic mapping with a graph equal to the (unique!) irreducible component of $\Gamma_f \cap (X_1 \times Y)$, which projects *onto* X_1.

We shall list now some statements needed for the proof of our Theorem. First of all let us define the set

$$E_n^q(r) := (\Delta^{n-1} \times \Delta_r^1 \times \Delta^q) \cup (\Delta^{n-1} \times \Delta^1 \times A^q(1-r, 1)) = \Delta^{n-1} \times H_1^q(r), \tag{2}$$

where $A^q(1-r, 1) := \{z \in C^q : 1 - r < \|z\| < 1\}$, and $\|\cdot\|$ is a polydisc norm in $\mathbb{C}^q$.

The following lemma for $q = 1$ can be found in [I_4], Lemma 2.2.1. No changes in the proof are needed for $q \geqslant 1$:

LEMMA 1. – *If any meromorphic* map $f\colon E_n^q(r) \to Y$ *extends to a meromorphic* map $\hat{f}\colon \Delta^{n+q} \to Y$ *then the space* Y *possesses a meromorphic extension property in bidimension* (n, q).

We shall make use also of one result on *meromorphic families of analytic subsets* from [I_4].

Let S be a set, and $W \subset\subset \mathbb{C}^q$ an open subset equipped with the usual Euclidean metric. Y is again some complex space.

DEFINITION. – (i) *By a family of q-dimensional analytic subsets in the complex space* $X = W \times Y$ *we shall understand a subset* $\mathcal{F} \subset S \times W \times X$ *such* *that, for every $s \in S$ the set $\mathcal{F}_s = \mathcal{F} \cap \{s\} \times W \times X$ is the graph of a meromorphic mapping of W into X.*

(ii) *If S is a topological space and the complex space X is equipped with some Hermitian metric h, we say that the family $\mathcal{F}$ is continuous at the point $s_0 \in S$ if the $\mathcal{H} - \lim_{s \to s_0} \mathcal{F}_s = \mathcal{F}_{s_0}$.* (Here by $\mathcal{H} - \lim_{s \to s_0} \mathcal{F}_s$ we denote the limit of closed subsets of $\mathcal{F}_s$ in the Hausdorff metric on $W \times X$).

(iii) *When S is a complex space itself, we shall call the family $\mathcal{F}$ meromorphic if the closure $\widehat{\mathcal{F}}$ of the set $\mathcal{F}$ is an analytic subset of $S \times W \times X$.*

We say that $\mathcal{F}$ is continuous if it is continuous at each point of S. If W_0 is open in W then the restriction $\mathcal{F}_{W_0}$ is naturally defined as $\mathcal{F} \cap (S \times W_0 \times X)$.

Then the statement about meromorphic families that we are going to need can be formulated as follows. (For standard notions and facts from pluripotential theory we refer to [Kl]).

Let us consider a meromorphic mapping $f \colon V \times W_0 \to X$ into a complex space X, where V is a domain in $\mathbb{C}^p$. Let S be some closed subset of V and $s_0 \in S$ some accumulation point of S. Suppose that for each $s \in S$ the restriction $f_s = f|_{\{s\} \times W_0}$ extends to a meromorphic mapping on $W \supset\supset W_0$. We suppose additionally that there is a compact subset $K \subset\subset X$ such that for all $s \in S$, $f_s(W) \subset K$.

Let ν_j denote the minima of volumes of j-dimensional compact analytic subsets contained in our compact $K \subset X$. We have $\nu_j > 0$ by Lemma 2.3.1 from [I_4]. Fix some $W_0 \subset\subset W_1 \subset\subset W$ and put

$$\nu = \min \{ \mathrm{vol}(A_{q-j}) \cdot \nu_j : j = 1, \dots, q \}, \tag{3}$$

where A_{q-j} are running over all $(q-j)$-dimensional analytic subsets of W, intersecting $\overline{W}_1$. Clearly $\nu > 0$. We are are going to express our statement in terms of the volumes of graphs over W. Indeed, let $w_e = dd^c \|z\|^2$ be an euclidean metric form on $W \subset \mathbb{C}^q$ and w_h be an hermitian metric form on X and let us consider Γ_{f_s}, for $s \in S$, as analytic subsets of $W \times X$.Their volumes are going to be

$$\mathrm{vol}(\Gamma_{f_s}) = \int_{\Gamma_{f_s}} (p_1^* w_e + p_2^* w_h)^q = \int_W (w_e + (p_1)_* p_2^* w_h)^q , \tag{4}$$

where $p_1 \colon W \times X \to W$ and $p_2 \colon W \times X \to X$ are the projections. Then:

LEMMA 2. – *Let us suppose that there exists a neighbourhood U of s_0 in V such that, for all $s_1, s_2 \in S \cap U$*

$$|\mathrm{vol}(\Gamma_{f_{s_1}}) - \mathrm{vol}(\Gamma_{f_{s_2}})| < \nu/2 . \tag{5}$$

If s_0 is a (locally) regular point of S then there exists a neighbourhood v' of s_0 in V such that f extends to a meromorphic mapping on $V' \times W_0$.

Here a locally regular point is ment in the sense of pluripotential theory, (see [Kl]). Further, slightly modifying arguments from [I_3] we shall derive now the following version of so called Continuity Principle.

Let $f: H_1^q(r) \to Y$ be a given meromorphic mapping and let us define $A_s^q(1-r, 1) := \{s\} \times A^q(1-r, 1)$ for $s \in \Delta^n$. We suppose that for s in some nonempty subset $S \subset \Delta^1$ the restriction $f_s := f|_{A_s^q(1-r,1)}$ is well defined and extends to a meromorphic mapping on the polydisc Δ^q.

LEMMA 3. – *Let us suppose that $f: H_1^q(r) \to Y$ is a meromorphic mapping and:*

(i) *there is a compact subset $K \subset\subset Y$ such that $f(\Delta^1 \times A^q(1-r, 1)) \subset K$ and $f(\{s\} \times \Delta^q) \subset K$ for all $s \in S$;*

(ii) *there is a constant $C_0 < \infty$ such that* $\mathrm{vol}(\Gamma_{f_s}) \leq C_0$ *for all $s \in S$.*

Then:

1. *Either there is a neighborhood U of* 0 *in Δ^1 and a meromorphic extension of f on $U \times \Delta^q$, or*
2. 0 *is an isolated point of S.*

The volumes here are measured with respect to the Euclidean metric on $\mathbb{C}^q$ and some Hermitian metric h on Y. The condition of boundedness in (ii) clearly does not depend on the particular choice of h. In the following, we shall refer to this statement as to C.P. The condition $n=1$ is important here, (cfr. Example 1 in [I_3]). We shall also discuss related questions in the last section.

To derive the proof of this statement from the reasonings in [I_3] we shall need some notions and results from the theory of cycle spaces (due to D.Barlet, see [B_2]) as they where adapted to our «noncompact» situation in [I_3].(See also [Fj]).

In what follows all the complex spaces considered are supposed to be reduced, normal and countable at infinity. Let us recall that an analytic cycle of dimension q in a complex space Y is a formal sum $Z = \sum_j n_j Z_j$, where $\{Z_j\}$ is a locally finite sequence of analytic subsets (always of pure dimension q) and n_j are positive integers called multiplicities of the Z_j's. The space $|Z| := \bigcup_j Z_j$ is called the support of Z.

We are going to associate the following space of cycles to a given meromorphic mapping $f: \Delta^n \times A^q(1-r, 1) \to X$, satisfying conditions of Lemma 3. Let us fix some $0 < c < 1$ and let us consider the set $\mathcal{C}'_{f,C}$ of all analytic cycles Z of pure dimension q in $Y := \Delta^{1+q} \times X$, such that:

(a) $Z \cap [\Delta^n \times A^q(1-r, 1) \cap X] = \Gamma_{f_z} \cap \{z\} \times A^q(1-r, 1) \times X$ for some $z \in \Delta(c)$.

(b) $\mathrm{vol}(Z) < C$, where C is some constant, $C > C_0$, C_0 as in Lemma 3.

Condition (a) means in particular that for such z the mapping f_z extends meromorphically from $A^q_z(1-r, 1)$ to $\{z\} \times \Delta^q$.

Let us define $\overline{\mathcal{C}}_{f,C}$ to be the closure of $\mathcal{C}'_{f,C}$ in the usual topology of currents. In [I_3] it was shown that $\mathcal{C}_{f,C} := \{Z \in \overline{\mathcal{C}}_{f,C} : \mathrm{vol}(Z) < C\}$ is an analytic space of finite dimension in the neighborhood of each of its points.

Let $f: \Delta^n \times A^q(1-r, 1) \to X$ be a meromorphic mapping and let us denote by $\mathcal{C}_0$ the subset of $\overline{\mathcal{C}}_{f,C}$ consisting of cycles which are limits of $\{\Gamma_{f_{s_n}}\}$ for $s_n \to 0$, $s_n \in S$. It is a compact subset (by Bishop's theorem) of the topological space $\mathcal{C}_{f,2C}$. For every cycle $Z \in \mathcal{C}_0$ let us choose a neighborhood W_Z such that $\mathcal{C}_{0|W_Z}$ is analytic. Let $W_{Z_1}, \ldots, W_{Z_N}$ be a finite covering of $\mathcal{C}_0$ and let us remark that there is an $\varepsilon_0 > 0$ such that for any $s \in S \cap \Delta^n(\varepsilon_0)$ we have $\Gamma_{f_s} \subset \bigcup_{j=1}^{N} W_{Z_j}$.

Now we are prepared to give a proof of Lemma 3. Let us consider a universal family $\mathcal{Z} := \{Z_a : a \in \mathcal{C}_{f,2C_0}\}$. It is complex space of finite dimension. We have an evaluation map

$$F: \mathcal{Z} \to \Delta^{1+q} \times X$$

defined by $Z_a \in \mathcal{Z} \to Z_a \subset \Delta^{1+q} \times X$. Let us consider the union $\widehat{\mathcal{C}}_0$ of those components of $\mathcal{C}_{f,2C_0}$ which intersect $\mathcal{C}_0$ and recall, that $\mathcal{C}_0$ stands here for the set of all limits of $\{\Gamma_{f_{s_n}}, s_n \in S\}$. At least one of those components, say $\mathcal{K}$, contains two points s_1 and s_2 s.t. Z_{s_1} projects onto $\{0\} \times \Delta^k$ and Z_{s_2} projects onto $\{s\} \times \Delta^k$ with $s \neq 0$. This is just because S contains more then one point. Let us consider the restriction $\mathcal{Z}|_{\mathcal{K}}$ of the universal space to $\mathcal{K}$. It is an irreducible complex space of finite dimension. Choose points $z_1 \in Z_{s_1}$ and $z_2 \in Z_{s_2}$ and join them by an analytic disc $\phi: \Delta \to \mathcal{Z}|_{\mathcal{K}}$, $\phi(0) = z_1$, $\phi(1/2) = z_2$. Then the composition $\psi = \pi \circ F \circ \phi: \Delta \to \Delta$ is non degenerate because $\psi(0) = 0 \neq s = \psi(1/2)$. Thus ψ is proper and obviously so is the map $F: \mathcal{Z}|_{\phi(\Delta)} \to F(\mathcal{Z}|_{\phi(\Delta)}) \subset \Delta^{1+q} \times X$. Thus $F(\mathcal{Z}|_{\phi(\Delta)})$ is an analytic set in $U \times \Delta^k \times X$ for small enough U extending Γ_f by reason of dimension.

We shall make use also of the following result due to D. Barlet ([B_1] Proposition 3):

LEMMA 4. – *Let X be a reduced complex space (of finite dimension) and let $\varrho: X \to \mathbb{R}^+$ be a strictly q-convex function. Let h be some C^2-smooth*

Hermitian metric on X. Then there exists an Hermitian metric h_1 *and a function* $c\colon \mathbb{R}^+ \to \mathbb{R}^+$ *(both of class* C^2*) such that*:

(i) $h_1 \geq h$;

(ii) *the* (q, q)*-form* $\Omega = dd^c[(c \circ \varrho)\, w_{h_1}^{q-1}]$ *is strictly positive on* X.

Here w_h is the $(1, 1)$-form canonically associated with h. In our case we need $X = \Delta^{n+q} \times Y$ and we shall use only the fact that on X there exists a strictly positive (q, q)-form which is dd^c-exact: in fact d-exactness is going to be sufficient for us. We recall that a (q, q)-form Ω is called strictly positive if for any $x \in X$ and linearly independent vectors $v_1, \ldots, v_q \in T_x X$ one has $\Omega_x(iv_1 \wedge \bar{v}_1, \ldots, iv_q \wedge \bar{v}_q) > 0$.

2. – Proof of the Theorem.

We are going to give in this section the proof of the main result of this note:

THEOREM. – *Every meromorphic mapping* $f\colon H_n^q(r) \to Y$, *where* Y *is a* q*-complete complex space, extends to a meromorphic mapping from* Δ^{n+q} *to* Y.

Step 1. *Case* $n = 1$.

Let us denote by W the biggest open subset of Δ^1 such that f extends meromorphically to $H_W(r) := (\Delta^1 \times A^q(1-r, 1)) \cup (W \times \Delta^q)$, and let us remark explicitly that the complex space $X = \Delta^{1+q} \times Y$ is (obviously) q-complete.

We apply Barlet's Theorem, Lemma 4, sect. 1, by taking as ϱ a strictly q-convex exhaustion of X in order to have a strictly positive dd^c-exact (q, q)-form Ω on X. Let w be a fixed $(q-1, q-1)$-form of class C^2 such that $dd^c w = \Omega$. Let us denote by F the set of points of indeterminancy of f.

By shrinking the polydisc Δ^{1+q}, we may suppose, without loss of generality, that f_z is defined in a neighborhood of $\overline{\Delta}^q$ for all $z \in W$. In the same way we may suppose that $w \in C^2(\overline{\Delta}^{1+q} \times Y)$, i.e. it is smooth up to the boundary.

We need to prove that $W = \Delta^1$. Suppose not, and fix a point $z_0 \in \partial W \cap \Delta^1$. Let us denote by V some disc centered at z_0 which is contained in Δ^1. For $z \in V \cap W$ one has

$$\operatorname{vol}(\Gamma_{f_z}) = \int_{\Gamma_{f_z}} \Omega = \int_{\Gamma_{f| \partial \Delta_z^q}} d^c w \leq C \tag{6}$$

where the constant C does not depend on $z \in V \cap W$, while $d^c w$ is of f class C^1 on $\overline{\Delta}^{1+q} \times Y$.

To obtain the estimate (6) we have used the fact that we can measure the volumes of analytic sets of pure dimension q contained in some compact part of X by means of $\int \Omega$ with Ω a strictly positive (q, q)-form on X.

We are going to check if the conditions of the Continuity Principle, Lemma 3, sect. 1 are satisfied. The inequality (6) says that the second assumption of C.P. is satisfied.

To check if the first one is satisfied, let us suppose that there exists a sequence $\{z_n\} \subset V \cap W$, converging to $z_\infty \in \Delta^1$, such that $\{\Gamma_\nu := \Gamma_{f_{z_\nu}}\}$ is not contained in any relatively compact subset of $\overline{\Delta}^{1+q} \times Y$. If ν is big enough, the restriction $\varrho|_{\Gamma_\nu}$ will have then a strict maximum in the interior of Γ_ν. This is impossible because the Levi form of $\varrho|_{\Gamma_\nu}$ has at least one positive eigenvalue at each point of Γ_ν. Let us remark also that the q-complete space Y cannot contain any compact q-dimensional subspace.

C.P. says now that f meromorphically extends to $V_1 \times \Delta^q$ for some neighborhood V_1 of z_0 in Δ. This proves that $W = \Delta^1$.

Step 2. *Case* $n \geqslant 2$.

This will be done by induction on n. By Lemma 1, sect. 1 all we need is to extend meromorphic mappings from $E_n^q(r)$ to Δ^{n+q}. For $n=1$, we have $E_1^q(r) = H_1^q(r)$ and thus this is already done by Step 1.

Notice that $E_{n+1}^q(r) = \Delta^1 \times E_n^q(r)$, and let us denote by $E_{n,z}^q(r) := \{z\} \times E_n^q(r)$ for $z \in \Delta^1$. We remark that by the induction hypothesis the restriction $f|_{E_{n,z}^q(r)}$ meromorphically extends to $\Delta_z^{n+q} := \{z\} \times \Delta^{n+q}$ for all $z \in \Delta^1$. We denote by W the maximal open subset in Δ^1 such that our map f extends meromorphically to $W \times \Delta^{n+q}$.

Set $S = \Delta^1 \backslash W$ and let us consider the family $\{\Gamma_{f_s}\colon s \in S\}$ of analytic subsets in $X := \Delta^{n+q} \times Y$. Here, we denote the graph of the restriction $f_s := f|_{\Delta_s^{n+q}}$ by Γ_{f_s}, as usual.

Let us define $S_k := \{s \in S\colon \mathrm{vol}(\Gamma_{f_s}) \leqslant k \cdot (\nu/2)\}$, where ν is as in Lemma 2, sect. 1 with $W = \Delta^{n+q}$, and $W_0 = \Delta_{1-r/2}^{n+q}$. By the maximality of S and by Lemma 2, sect. 1 we see that all points of each S_k are (locally) regular, thus each S_k is polar. So S is a polar subset of Δ^1, in other words it is a set of harmonic measure zero in Δ^1.

By some linear coordinate transformation in $\mathbb{C}^{1+n+q}$ we are going to change a little bit the band of the Δ^{n+q}-direction, in order to prove in the same manner that f meromorphically extends to the whole of Δ^{1+n+q}. In fact, let us consider linear changes L of coordinate systems in $\mathbb{C}^{1+n+q}$ whose associated matrices are of the form (L_1, L_2), where L_1 is (a number) close to zero and L_2 is close to the identity map of $\mathbb{C}^{n+q}$ into itself. For each L of this form we can extend f onto $\Delta^{1+n+q} \backslash \Sigma)$, where $\Sigma := L^{-1}(S^L \times \Delta^{n+q})$ and S^L is a set of harmonic measure zero in Δ^1.

In an appropriate coordinate system the si,gularity set Σ is a the product of $n+q+1$ closed sets of harmonic measure zero in the plane. Thus Σ is pluripolar and of Hausdorff dimension zero.

Hence, by using the fact that $\Delta^{1+n+q} \times Y$ is (obviously) $(n+q)$-complete and Lemma 3, sect. 1, we can remove the singularity Σ and the proof is complete. q.e.d.

3. – Consequences and open questions.

Let us start with some direct consequences of the Theorem.

COROLLARY 1 (Thullen type extension Theorem). – *Let $\Omega \subset \mathbb{C}^n$ be an open subset, $V \subset \Omega$ be an analytic subvariety of dimension q and G be an open subset of Ω which intersects every q-dimensional branch of V. Every meromorphic mapping $f: (\Omega \setminus V) \cup G \to Y$, where Y is a q-complete complex space, extends to a meromorphic mapping from Ω to Y.*

In particular one has

COROLLARY 2 (Riemann type extension Theorem). – *Let $\Omega \subset \mathbb{C}^n$ be an open subset, $V \subset \Omega$ be an analytic subvariety of dimension $q-1$. Every meromorphic mapping f: $\Omega \setminus V \to Y$, where Y is a q-complete complex space, extends to a meromorphic mapping from Ω to Y.*

The proofs of Corollaries 1 and 2 are immediate after [S_1], p. 5.

A general Thullen type extension Theorem for meromorphic mappings has been proved by Siu when Y is compact Kähler in [S_2]. We have also:

COROLLARY 3. – *Let Y be a complex analytic space of dimension q and let us suppose that every irreducible component of Y of dimension q is non compact. Every meromorphic mapping f: $H_n^q(r) \to Y$, extends to a meromorphic mapping from Δ^{n+q} to Y.*

In fact, every complex space of dimension N with no compact irreducible component of dimension N is N-complete, by a Theorem of Ohsawa [O], Th. 1.

As it has been mentioned in the Introduction, Corollary 4 immediately gives an improvement on a Theorem of K. Stein [St]:

COROLLARY 4. – *Let Y be a compact complex space of pure dimension q. For every domain $D \subset \mathbb{C}^{q+1}$, $q \geqslant 1$, and for every compact subset $K \subset\subset D$ with connected complement, every nonsurjective meromorphic mapping f: $D \setminus K \to Y$ extends to a meromorphic mapping from D to Y.*

We shall end with discussing some open questions, which naturally arise from the results and attempts of this paper.

QUESTION 1. – *Let Y be a compact complex three-fold. Prove that every meromorphic* (*or holomorphic*) map $f\colon H_1^2(r)\to Y$ *extends to* $\Delta^3\setminus\{$*discrete set of points*$\}$.

In particular, if K is compact, $K\subset\Delta^3$ with connected complement then every meromorphic map $f\colon \Delta^3\setminus K\to Y$ *extends to* $\Delta^3\setminus\{$*finite set of points*$\}$.

For the proof of such type of statements one can try to use special metrics on Y. Namely, a compact complex threefold possesses an Hermitian metric h, such that its associated $(1,1)$-form ω_h satisfies $dd^c\omega_h^2=0$. This can help to bound the volumes of the images of two-discs in Y.

The next question arises when one tries to prove Corollary 3 without assuming Y to be noncompact

QUESTION 2. – *Let Y be a compact complex manifold* (*space*) *of dimension* $q\geqslant 2$. *Suppose that there exists a meromorphic* map $f\colon \boldsymbol{B}_*^{q+1}\to Y$ *from the punctured ball in* $\mathbb{C}^{q+1}$ *onto Y such that for any $\varepsilon>0$ the restriction* $f_\varepsilon := f|_{\boldsymbol{B}_*^{q+1}(\varepsilon)}$ *of f to the punctured ε-ball is still surjective. Prove that Y is Moishezon.*

In the case of positive answer to Question 2, one can extend f meromorphically across zero. A somewhat stronger statement may be needed:

QUESTION 2′. – *Let M be a strongly pseudoconvex hypersurface in the punctured ball* $\boldsymbol{B}_*^{q+1}$ *such that* $0\in M$ *and M divides* $\boldsymbol{B}_*^{q+1}$ *into two parts, say* B^+ *and* B^-. *Let* $f\colon B^+\to Y$ *be given as in Question* 2. *Let us suppose that M is concave from the side of B^+. Prove that if for any $\varepsilon>0$ the restriction* $f|_{\boldsymbol{B}_*^{q+1}(\varepsilon)\cap B^+}$ *of f is still surjective, f extends meromorphically across zero.*

QUESTION 3. – *Can one remove the condition $n=1$ from Lemma* 3?

Such attempt is likely to lead to the «non analytic» version of Remmert proper mapping theorem and to the questions of local flattenings. The major problem here is that $\mathcal{C}_{f,C}$ is not going to be an analytic space and F will not be proper in general.

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