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Interior Regularity for Weak Solutions of Ultraparabolic Equations in Divergence Form with Discontinuous Coefficients.

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Sunto. – *Abbiamo considerato il problema della regolarità interna delle soluzioni deboli della seguente equazione differenziale*

$$\sum_{i,j=1}^{m_0} \partial_{x_i}(a_{i,j}(x,t)\,\partial_{x_j}u) + \sum_{i,j=1}^{N} b_{i,j}x_i\partial_{x_j}u - \partial_t u = \sum_{j=1}^{m_0}\partial_{x_j}F_j(x,t),$$

dove $(x,t)\in\mathbb{R}^{N+1}$, $0<m_0\le N$ *ed* $F_j\in L^p_{\mathrm{loc}}(\mathbb{R}^{N+1})$ *per* $j=1,\dots,m_0$. *I nostri principali risultati sono una stima a priori interna del tipo*

$$\sum_{j=1}^{m_0}\|\partial_{x_j}u\|_p \le c\left(\sum_{j=1}^{m_0}\|F_j\|_p + \|u\|_p\right),$$

e la regolarità hölderiana di u. La stima a priori delle derivate viene ottenuta utilizzando una tecnica analoga a quella introdotta da Chiarenza, Frasca e Longo in [3], *per gli operatori ellittici in forma di non divergenza, supponendo che i coefficienti* $a_{i,j}$ *verifichino una condizione di «debole» continuità. Il risultato di hölderianità è conseguenza delle suddette stime e di una formula di rappresentazione basata sulla espressione esplicita della soluzione fondamentale dell'operatore «congelato».*

1. – Introduction.

In this note we are concerned with the interior regularity of the weak solutions of the second order differential equation

$$Lu := \sum_{i,j=1}^{m_0} \partial_{x_i}(a_{i,j}(z)\,\partial_{x_j}u) + \sum_{i,j=1}^{N} b_{i,j}x_i\partial_{x_j}u - \partial_t u = \sum_{j=1}^{m_0}\partial_{x_j}F_j(z), \tag{1.1}$$

where $z=(x,t)\in\mathbb{R}^{N+1}$, $0<m_0\le N$ and $F_j\in L^p_{\mathrm{loc}}(\mathbb{R}^{N+1})$ for $j=1,\dots,m_0$. In our treatment we shall always assume the following hypothesis

HYPOTHESIS H. – *The matrix* $A_0(z)=(a_{i,j}(z))_{i,j=1,\dots,m_0}$ *is symmetric* *and such that, for a suitable* $\mu > 0$,

$$\mu^{-1} |\xi|^2 \leq \langle A_0(z)\, \xi, \xi \rangle \leq \mu |\xi|^2$$

for every $z \in \mathbb{R}^{N+1}$ *and for every* $\xi \in \mathbb{R}^{m_0}$.

The matrix $B = (b_{i,j})_{i,j=1,\dots,N}$ *can be written as*

$$B = \begin{pmatrix} 0 & B_1 & 0 & \cdots & 0 \\ 0 & 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_r \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where each B_j *is a* $m_{j-1} \times m_j$ *block matrix of rank* m_j, $j = 1, 2, \dots, r$, *with* $m_0 \geq m_1 \geq \dots \geq m_r \geq 1$ *and* $m_0 + m_1 + \dots + m_r = N$.

In the sequel we shall write equation (1.1) in the following way:

$$\operatorname{div}(A(z)\, Du) + Yu = \operatorname{div}(F),$$

where $F = (F_1, \dots, F_{m_0}, 0, \dots, 0)$, $Yu = \langle x, BDu \rangle - \partial_t u$ and the $N \times N$ matrix $A(z)$ is defined as

$$A(z) = \begin{pmatrix} A_0(z) & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.2}$$

We shall say that u is a *weak solution* of (1.1) in an open set $\Omega \subset \mathbb{R}^{N+1}$ if $u, \partial_{x_1} u, \dots, \partial_{x_{m_0}} u, Yu \in L^2_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega \langle A(z)\, Du(z), D\psi(z) \rangle dz - \int_\Omega Yu(z)\, \psi(z)\, dz = \int_\Omega \langle F(z), D\psi(z) \rangle\, dz$$

for every $\psi \in C_0^\infty(\Omega)$.

We next state the main results of this paper, while we refer to Section 2 for the precise meaning of the hypothesis

$$a_{i,j} \in \mathrm{VMO}_L, \qquad i, j = 1, \dots, m_0.$$

THEOREM 1.1. – *Suppose that* L *satisfies Hypothesis H and that* $a_{i,j} \in \mathrm{VMO}_L$ *for* $i, j = 1, \dots, m_0$. *Let* Ω *be an open subset of* $\mathbb{R}^{N+1}$ *and let* u *be a weak solution in* Ω *of*

$$\operatorname{div}(A(x,t)\, Du) + \langle x, BDu \rangle - \partial_t u = \operatorname{div}(F)$$

where $u, F_j \in L^p(\Omega)$ *for* $j = 1, \dots, m_0$, $1 < p < \infty$. *Then, for any compact set*

$K \subseteq \Omega$ the derivatives $\partial_{x_j} u$, $j = 1, \ldots, m_0$ belong to $L^p(K)$. Moreover

$$\|\partial_{x_j} u; L^p(K)\| \le c_1 \left(\sum_{k=1}^{m_0} \|F_k; L^p(\Omega)\| + \|u; L^p(\Omega)\| \right), \tag{1.3}$$

where c_1 is a positive constant depending only on p, K, Ω and on the operator L.

THEOREM 1.2. – *Suppose that L satisfies Hypothesis H and that $a_{i,j} \in$ VMO$_L$ for $i, j = 1, \ldots, m_0$. Let Ω be an open subset of $\mathbb{R}^{N+1}$ and let u be a weak solution in Ω of*

$$\operatorname{div}(A(x,t)\,Du) + \langle x, BDu \rangle - \partial_t u = \operatorname{div}(F)$$

where $u, F_j \in L^p(\Omega)$ for $j = 1, \ldots, m_0$, $p > Q+2$. Then

$$\frac{|u(z) - u(\zeta)|}{\|\zeta^{-1} \circ z\|^{1-(Q+2)/p}} \le c_2 \left(\sum_{k=1}^{m_0} \|F_k; L^p(\Omega)\| + \|u; L^p(\Omega)\| \right), \quad \forall z, \zeta \in K, z \ne \zeta, \tag{1.4}$$

where c_2 is a positive constant depending only on p, K, Ω and on the operator L.

When the coefficients of the matrix A belong to suitable classes of Hölder continuous functions, many existence and regularity results for equations like (1.1) have been recently proved.

More precisely: existence and estimates of the fundamental solution for the operator L, Harnack inequality for non-negative solutions, existence and uniqueness results for the Cauchy problem, have been proved in [11], [12] and [13]. In addition, Schauder estimates and existence results for boundary value problems have been established in [8].

In this work we provide a first Hölder continuity result of the weak solutions of equation (1.1), when the coefficients of the matrix A are not regular. Unlike to the classical elliptic and parabolic case, we do not use Nash or Moser techniques [10], [9]. That methods, in fact, seem to us not easily adaptable to our setting. We use an approach based on a representation formula requiring a weak regularity assumption on the $a_{i,j}$'s (the VMO$_L$ hypothesis).

This note is organized as follows. In Section 2 we give some notations and we recall some known results. Section 3 contains the proofs of some representation formulas for the weak solutions of (1.1) and for their derivatives.

In Section 4 we prove Theorem 1.1, by using a technique introduced by Chiarenza, Frasca and Longo in [3] for elliptic operators in non-divergence form. This technique has been subsequently extended to the classical parabolic operators in [1] and to the operators of type (1.1) in *non-divergence form* in [2]. The same technique was also used by Di Fazio for elliptic operators in divergence form in [4].

The main difficulty in adapting the previous method to our setting is the lack of a uniqueness result which does not allows, as in the elliptic case, to approximate any weak solution by a sequence of solutions of suitably regular equations.

Section 5 contains the proof of Theorem 1.2 which relies on the following remark: if we denote by I_{m_0} the identity matrix $m_0 \times m_0$, and define the $N \times N$ constant matrix

$$J = \begin{pmatrix} I_{m_0} & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.5}$$

then we can write equation (1.1) in the following form

$$L_0 u := \operatorname{div}(JDu) + \langle x, BDu \rangle - \partial_t u = \operatorname{div}(\tilde{F}), \tag{1.6}$$

where $\tilde{F}_j = F_j + \partial_{x_j} u - \sum_{k=1}^{m_0} a_{j,k} \partial_{x_k} u$ for $j = 1, \ldots, m_0$, and $\tilde{F}_j = 0$ for $j = m_0 + 1, \ldots, N$. As a consequence of Theorem 1.1, $\tilde{F}_j \in L^p_{\mathrm{loc}}(\mathbb{R}^{N+1})$, then Hölder continuity result follows by using the representation formula for the solutions of (1.6) and the inequality in Proposition 5.2 for the kernels appearing in such a formula.

We conclude this introduction by observing that our results are new also in the parabolic case. Indeed, if $m_0 = N$, Hypothesis H simply means that L is uniformly parabolic and $B = 0$.

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2. – Some known results.

We first recall some known results about operators (1.1) when the matrix A is constant. Note that, as a consequence of Hypothesis H, the «frozen» operator

$$L_\zeta := \operatorname{div}(A(\zeta)\, Du) + Yu \tag{2.1}$$

is hypoelliptic for every $\zeta \in \mathbb{R}^{N+1}$ (see [6]) and it is invariant with respect to the *translations* and the *dilations* of the following groups.

DEFINITION 2.1. – *For every* (x, t), $(\xi, \tau) \in \mathbb{R}^{N+1}$ *and* $\lambda > 0$ *we define*

$$(x, t) \circ (\xi, \tau) = (\xi + E(\tau)\, x, t + \tau), \qquad E(t) = \exp(-tB^T),$$

$$D(\lambda) = \mathrm{diag}\,(\lambda I_{m_0}, \lambda^3 I_{m_1}, \ldots, \lambda^{2r+1} I_{m_r}),$$

where I_{m_j} *denotes the* $m_j \times m_j$ *identity matrix.*

We will say that $(\mathbb{R}^{N+1}, \circ)$ *and* $(D(\lambda), \lambda^2)_{\lambda > 0}$ *are, respectively, the «translation group» and the «dilation group» associated to* L.

We next introduce a norm which is homogeneous of degree 1 with respect to the dilations $(D(\lambda), \lambda^2)_{\lambda > 0}$ and a corresponding quasi-distance which is invariant with respect to translation group.

DEFINITION 2.2. – *Let* $\alpha_1, \ldots, \alpha_N$ *be the positive integers such that*

$$\mathrm{diag}\,(\lambda^{\alpha_1}, \ldots, \lambda^{\alpha_N}) = D(\lambda).$$

If $\|z\| = 0$ *we set* $z = 0$ *while, if* $z \in \mathbb{R}^{N+1} \backslash \{0\}$ *we define* $\|z\| = \varrho$ *where* ϱ *is the unique positive solution to the equation*

$$\frac{x_1^2}{\varrho^{2\alpha_1}} + \frac{x_2^2}{\varrho^{2\alpha_2}} + \ldots + \frac{x_N^2}{\varrho^{2\alpha_N}} + \frac{t^2}{\varrho^4} = 1 \ (^1).$$

We will define the quasi-distance d *by*

$$d(z, w) = \|z^{-1} \circ w\|, \qquad z, w \in \mathbb{R}^{N+1}$$

and we denote by $B_r(z)$ *the* d*-ball of center* z *and radius* r.

Next proposition contains some properties of $\|\cdot\|$ and d, proved in [2], Proposition 1.3.

PROPOSITION 2.3. – *The function* $z \mapsto \|z\|$ *has the following properties*:

(i) $\|D(\lambda)\, z\| = \lambda \|z\|$ *for every* $z \in \mathbb{R}^{N+1}$ *and for every* $\lambda > 0$;

(ii) *The set* $\{z : \|z\| = 1\}$ *is the euclidean sphere* $\Sigma_{N+1} = \{(x, t) : |x|^2 + t^2 = 1\}$;

(1) We choose this definition, given by Fabes and Rivière [5], since it is convenient for applying the Fourier expansions in spherical harmonics of some singular integral kernels.

(iii) *There exists a constant* $c_0 = c_0(B) > 0$ (*depending only on matrix* B) *such that*

$$\|z \circ \zeta\| \leqslant c_0(\|z\| + \|\zeta\|)$$

for every z, $\zeta \in \mathbb{R}^{N+1}$;

(iv) *There exists a constant* $c_1 = c_1(B) > 0$ *such that*

$$\frac{1}{c_1}\|z\| \leqslant \|z^{-1}\| \leqslant c_1\|z\| \quad \textit{for every } z \in \mathbb{R}^{N+1};$$

(v) *For every compact set* $K \subset \mathbb{R}^{N+1}$ *there exists* $c_2 = c_2(B, K) > 0$ *such that*

$$|z - \zeta| \leqslant c_2\|\zeta^{-1} \circ z\|, \qquad \textit{for every } z,\ \zeta \in K \textit{ with } \|\zeta^{-1} \circ z\| \leqslant 1,$$

where $|\cdot|$ *denotes the euclidean norm in* $\mathbb{R}^{N+1}$.

REMARK 2.4. – The Lebesgue measure is invariant with respect to the translation group associated to L, since $\det E(t) = e^{t \operatorname{trace} B} = 1$, where $E(t)$ is the exponential matrix of Definition 2.1. Moreover, since $\det D(\lambda) = \lambda^Q$, where

$$Q = m_0 + 3m_1 + \ldots + (2r+1)\, m_r,$$

we also have

$$\operatorname{meas}(B_r(0)) = r^{Q+2} \operatorname{meas}(B_1(0)).$$

We shall call «homogeneous dimension» of $\mathbb{R}^{N+1}$ the integer $Q+2$.

We next introduce the BMO_L and VMO_L spaces, naturally related to the groups introduced in Definition 2.1. In the following definition we shall denote by B any ball of $\mathbb{R}^{N+1}$, by B_r any ball of radius r and, for every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^{N+1})$,

$$u_B = \frac{1}{\operatorname{meas}(B)} \int_B u.$$

DEFINITION 2.5. – *For every function* $u \in L^1_{\mathrm{loc}}(\mathbb{R}^{N+1})$ *let*

$$\|u\|_* = \sup_B \frac{1}{\operatorname{meas}(B)} \int_B |u(z) - u_B|\, dz,$$

$$\eta_u(r) = \sup_{\varrho \leqslant r} \frac{1}{\operatorname{meas}(B_\varrho)} \int_{B_\varrho} |u(z) - u_{B_\varrho}|\, dz.$$

Then we define

$$\mathrm{BMO}_L(\mathbb{R}^{N+1}) = \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^{N+1}) \colon \|u\|_* < +\infty\},$$

$$\mathrm{VMO}_L(\mathbb{R}^{N+1}) = \left\{u \in \mathrm{BMO}_L(\mathbb{R}^{N+1}) \colon \lim_{r \to 0} \eta_u(r) = 0\right\}.$$

We recall some results of [6] and [10]:

PROPOSITION 2.6. – *For every $z_0 \in \mathbb{R}^{N+1}$ let*

$$C(t, z_0) = \int_0^t E(s)\, A(z_0)\, E^T(s)\, ds\,.$$

Then the matrix $C(t, z_0)$ is positive for every $t > 0$ and the fundamental solution of L_{z_0} defined in (2.1), *with pole at zero, is*

$$\Gamma(z_0; x, t) = \frac{1}{(4\pi)^{N/2} (\det C(t, z_0))^{1/2}} \exp\left(-\frac{1}{4}\langle C^{-1}(t, z_0)\, x, x\rangle\right) \tag{2.2}$$

if $t > 0$, $\Gamma(z_0; x, t) = 0$ if $t \leq 0$.

The fundamental solution of L_{z_0} with pole at (ξ, τ) is the «left translated» of $\Gamma(z_0; \cdot)$ with respect to the group $(\mathbb{R}^{N+1}, \circ)$:

$$\Gamma(z_0; (\xi, \tau)^{-1} \circ (x, t))\,. \tag{2.3}$$

(2.3), *as a function of the variables (ξ, τ), is the fundamental solution, with pole at (x, t), of the adjoint operator $L^*_{z_0} = \mathrm{div}\,(A(z_0)\, D) - Y^*$ (since* trace $B =$ 0 *it results $Y^* = -Y$).*

There exists an operator L^+ with the same structure as (1.1) *and with constant coefficients $a_{i,j}$, such that, if Γ^+ denotes the fundamental solution of L^+, then*

$$\begin{cases} \Gamma(z_0; \zeta^{-1} \circ z) \leq c^+ \Gamma^+ (\zeta^{-1} \circ z), \\ |\partial_{x_j} \Gamma(z_0; \zeta^{-1} \circ z)| \leq c^+ \dfrac{\Gamma^+ (\zeta^{-1} \circ z)}{\sqrt{t - \tau}}, \\ |\partial_{x_i x_j} \Gamma(z_0; \zeta^{-1} \circ z)| \leq c^+ \dfrac{\Gamma^+ (\zeta^{-1} \circ z)}{t - \tau}, \end{cases} \tag{2.4}$$

for some positive constant c^+ and for every $i, j = 1, \ldots, m_0$, $z_0 \in \mathbb{R}^{N+1}$, $z = (x, t)$, $\zeta = (\xi, \tau)$, with $t < \tau$, (see [10], *Proposition* 2.4 *and Corollary* 2.1).

In the following we will denote $\Gamma_i(z_0; \cdot) = \partial_{x_i} \Gamma(z_0; \cdot)$ and $\Gamma_{ij}(z_0; \cdot) = \partial_{x_i x_j} \Gamma(z_0; \cdot)$, for $i, j = 1, \ldots, m_0$. We explicitly note that $\Gamma(z_0; \cdot)$ is homogeneous of degree $-Q$ with respect to the group $(D(\lambda), \lambda^2)_{\lambda > 0}$ and that $\Gamma_i(z_0; \cdot)$

and $\Gamma_{ij}(z_0;\cdot)$ are homogeneous of degree $-(Q+1)$ and $-(Q+2)$, respectively, for $i, j = 1, \dots, m_0$.

We recall that the restriction to Σ_{N+1} of any homogeneous polynomial which is harmonic is said *spherical harmonic*. We denote by $(K_m)_{m\in\mathbb{N}}$ an orthonormal complete system of spherical harmonics in $L^2(\Sigma_{N+1})$. We shall extend every function K_m to $\mathbb{R}^{N+1}\setminus\{0\}$ by setting

$$K_m(x,t) \equiv K_m(D(\|(x,t)\|^{-1})x, \|(x,t)\|^{-2}t)\,.$$

The following results concerning the Fourier expansion of homogeneous functions with respect to spherical harmonics have been proved in [2].

PROPOSITION 2.7. – *Let $F\in C^\infty(\mathbb{R}^{N+1}\setminus\{0\})$ be a homogeneous function of degree α. Then there exists a sequence $(b_m)_{m\in\mathbb{N}}$ such that*

$$F(z) = \sum_{m=1}^{\infty} b_m \|z\|^{\alpha} K_m(z)\,;$$

moreover, for every $r\in\mathbb{N}$ there exists $c(r)>0$, such that

$$|b_m| \sup_{z\in\Sigma_{N+1}} |K_m(z)| \leq c(r)\, m^{-r}\,.$$

Furthermore, if we consider the Fourier expansion of the function $\Gamma(z_0;\cdot)$,

$$\Gamma(z_0;z) = \sum_{m=1}^{\infty} b_m(z_0)\,\|z\|^{-Q} K_m(z)\,,$$

we also have that, for any $r\in\mathbb{N}$, there exists $c(r)>0$ such that

$$\sup_{z_0\in\mathbb{R}^{N+1}} |b_m(z_0)| \sup_{z\in\Sigma_{N+1}} |K_m(z)| \leq c(r) m^{-r}.$$

An analogous result holds for the derivatives $\Gamma_j(z_0;\cdot)$, $\Gamma_{i,j}(z_0;\cdot)$ and $Y\Gamma(z_0;\cdot)$.

We end this Section recalling two results about singular integrals. The first one is proved in [2], Theorem 3.1; for the second one see [13], Chap. I, Remark 8.21 (in that setting the volume function is $V(x,y) = \operatorname{meas}(B_1(0))\|y^{-1}\circ x\|^{Q+2}$).

THEOREM 2.8. – *For every $a\in \mathrm{BMO}_L(\mathbb{R}^{N+1})$, $g\in L^p(\mathbb{R}^{N+1})$, $1<p<\infty$ and for any $i,j=1,\dots,m_0$, we define*

$$T_{ij}g(z) = \lim_{\varepsilon\to 0} \int_{\|\zeta^{-1}\circ z\|\geq\varepsilon} \Gamma_{ij}(z;\zeta^{-1}\circ z)\, g(\zeta)\, d\zeta\,,$$

$$C_{ij}[a, g](z) = \lim_{\varepsilon \to 0} \int_{\|\zeta^{-1} \circ z\| \geq \varepsilon} \Gamma_{ij}(z; \zeta^{-1} \circ z)[a(z) - a(\zeta)]\, g(\zeta)\, d\zeta\, .$$

Then $T_{ij} g$, $C_{ij}[a, g] \in L^p(\mathbb{R}^{N+1})$ *and there exists a positive constant* $c = c(p)$ *such that*

$$\|T_{ij} g\|_p \leq c \|g\|_p\, ; \qquad \|C_{ij}[a, g]\|_p \leq c \|a\|_* \|g\|_p\, .$$

THEOREM 2.9. – *Let* $\alpha \in]0, Q+2[$ *and let* $K \in C(\mathbb{R}^{N+1} \setminus \{0\})$ *be a homogeneous function of degree* $-\alpha$ *with respect to the group* $(D(\lambda), \lambda^2)_{\lambda > 0}$. *If* $g \in L^q(\mathbb{R}^{N+1})$ *then the function*

$$Tg(z) = \int_{\mathbb{R}^{N+1}} K(\zeta^{-1} \circ z)\, g(\zeta)\, d\zeta\, ,$$

is defined almost everywhere, and there exists $c = c(q) > 0$ *such that*

$$\|Tg\|_p \leq c \max_{\|z\| = 1} |K(z)| \cdot \|g\|_q\, ,$$

where p *is given by the following relation*: $1/q + \alpha/(Q+2) = 1/p + 1$.

REMARK 2.10. – In what follows we shall also explicitly use the following result, which is contained in the proof of Theorem 3.1 of [2]. Consider the expansion of the function $\Gamma_{i,j}(z_0, \cdot)$:

$$\Gamma_{i,j}(z_0, \zeta) = \sum_{m=1}^{\infty} b_m(z_0) \frac{K_m(\zeta)}{\|\zeta\|^{Q+2}}\ ;$$

by letting $b_m^+ = \sup_{z_0} |b_m(z_0)|$ and

$$T_{i,j}^+ g(z) = \lim_{\varrho \to 0} \sum_{m=1}^{\infty} b_m^+ \left| \int_{\|\zeta^{-1} \circ z\| \geq \varrho} \frac{K_j(\zeta^{-1} \circ z)}{\|\zeta^{-1} \circ z\|^{Q+2}} g(\zeta)\, d\zeta \right|,$$

we have $T_{ij}^+ g \in L^p(\mathbb{R}^{N+1})$ and

$$\|T_{ij}^+ g\|_p \leq c^+ \|g\|_p$$

for some positive constant $c^+ = c^+(p)$.

3. – Representation formulas.

In this section we will prove some representation formulas for the weak solutions of (1.1) in terms of the fundamental solution $\Gamma(\cdot; \cdot)$ defined in (2.2). We first introduce some notations and prove some preliminary results. For fixed $r, s \in \mathbb{R}$, $0 < s < r$, let φ be a function belonging to $C^\infty(\mathbb{R})$ such that

$$\varphi(t) = 1 \quad \text{for every } 0 \leq t \leq s\,, \qquad \varphi(t) = 0 \quad \text{for every } t \geq r\,. \tag{3.1}$$

For every $\zeta_0 \in \Omega$ and $r > 0$ such that $B_r(\zeta_0) \subset \Omega$ we set

$$\eta(z) = \varphi(\|\zeta_0^{-1} \circ z\|)\,. \tag{3.2}$$

Then if u is solution of (1.1) we have

$$L(\eta u) = \operatorname{div}(G) + g\,,$$

where

$$G = \eta F + uAD\eta\,, \qquad g = \langle ADu, D\eta\rangle - \langle F, D\eta\rangle + uY^*\eta\,, \tag{3.3}$$

and Y^* denotes the adjoint of the operator Y.

We remark that, if $F_1, \ldots, F_{m_0}, u \in L^p_{\text{loc}}(\Omega)$ and $\partial_{x_1} u, \ldots, \partial_{x_{m_0}} u \in L^q_{\text{loc}}(\Omega)$, with $q < p$, then $G_1, \ldots, G_{m_0} \in L^p(B_r(\zeta_0))$, $g \in L^q(B_r(\zeta_0))$ and there exists a positive constant c, which depends only on r, s, p and q, such that

$$\begin{cases} \sum\limits_{j=1}^{m_0} \|G_j\|_p \leq c\left(\|u\|_p + \sum\limits_{j=1}^{m_0} \|F_j\|_p\right), \\ \|g\|_q \leq c\left(\|u\|_p + \sum\limits_{j=1}^{m_0} (\|F_j\|_p + \|\partial_{x_j} u\|_q)\right), \end{cases} \tag{3.4}$$

where $\|\cdot\|_p$ indicates the norm $\|\cdot\|_{L^p(B_r(\zeta_0))}$.

For the sake of brevity, in the sequel we shall write B_r instead of $B_r(\zeta_0)$.

THEOREM 3.1. – *If u is a weak solution of* (1.1) *and $\eta, G_1, \ldots, G_{m_0}, g$ are the functions defined in* (3.3), *then*

$$(\eta u)(z) = \sum_{h,k=1}^{m_0} \int_{\mathbb{R}^{N+1}} \Gamma_h(z; \zeta^{-1} \circ z)[(a_{h,k}(z) - a_{h,k}(\zeta))\,\partial_{x_k}(\eta u)(\zeta) + G_h(\zeta)]\,d\zeta - \int_{\mathbb{R}^{N+1}} \Gamma(z; \zeta^{-1} \circ z)\, g(\zeta)\, d\zeta\,, \tag{3.5}$$

moreover, for every $j = 1, \ldots, m_0$,

$$\partial_{x_j}(\eta u)(z) = -\int_{\mathbb{R}^{N+1}} \Gamma_j(z; \zeta^{-1} \circ z)\, g(\zeta)\, d\zeta - \sum_{k=1}^{m_0} G_k(z) \int_{\Sigma_{N+1}} \Gamma_j(z; \zeta)\, \nu_k(\zeta)\, d\sigma_\zeta + \sum_{h,k=1}^{m_0} \lim_{\varepsilon \to 0} \int_{\|\zeta^{-1} \circ z\| \geq \varepsilon} \Gamma_{jh}(z; \zeta^{-1} \circ z)[(a_{h,k}(z) - a_{h,k}(\zeta))\,\partial_{x_k}(\eta u)(\zeta) + G_h(\zeta)]\,d\zeta\,, \tag{3.6}$$

for almost every $z \in \mathbb{R}^{N+1}$ $((\nu_1, \ldots, \nu_{N+1})$ *is the outer normal at the set* $\Sigma_{N+1})$.

REMARK 3.2. – We will prove representation formulas (3.5) and (3.6) under the weaker assumptions that: $u, \partial_{x_1} u, \ldots, \partial_{x_{m_0}} u \in L^q_{\mathrm{loc}}(\Omega)$ for some $q > 1$ and

$$\int_\Omega \langle A(z) Du(z), D\psi(z) \rangle \, dz + \int_\Omega u(z)\, Y^* \psi(z) \, dz = \int_\Omega \langle F(z), D\psi(z) \rangle \, dz$$

for every $\psi \in C_0^\infty(\Omega)$ (note that we don't assume the existence of Yu).

To prove Theorem we shall make use of the following result.

LEMMA 3.3. – *There exists a positive constant k, depending only on the matrix B, such that*

$$\left| \frac{\partial \| \zeta^{-1} \circ z \|}{\partial \xi_j} \right| \leq k \quad \text{for } j = 1, \ldots, m_0, \qquad | Y^* \| \zeta^{-1} \circ z \| | \leq \frac{1}{\| \zeta^{-1} \circ z \|}$$

for every $z, \zeta \in \mathbb{R}^{N+1}$, $z \neq \zeta$.

Here and in the following we agree to let the operator Y^* acting on the variable ζ: $Y^* f(\zeta) = -\langle \xi, BD_\xi f(\zeta) \rangle + \partial_\tau f(\zeta)$. Lemma 3.3 will be proved at the end of this Section.

PROOF OF THEOREM 3.1. – We split the proof of (3.5) in three steps. In step **A** we prove a representation formula for smooth functions with compact support. In step **B**, by using the classical tool of expansion in spherical harmonics, we get some uniform estimates that allows us to conclude the proof of Theorem in step **C**, by a density arguments.

Let v be a function belonging to $C_0^\infty(\Omega)$. For every $z_0 \in \Omega$ and for every function $\psi \in C_0^\infty(\Omega)$, the definition of weak solution and (3.3) immediately give

$$(3.7) \qquad -\int_\Omega L_{z_0} v(\zeta)\, \psi(\zeta) \, d\zeta = -\int_\Omega g(\zeta)\, \psi(\zeta) \, d\zeta + \int_\Omega \langle G(\zeta), D\psi(\zeta) \rangle \, d\zeta +$$

$$\int_\Omega \langle [A(z_0) - A(\zeta)]\, D(\eta u)(\zeta), D\psi(\zeta) \rangle \, d\zeta +$$

$$\int_\Omega \langle A(z_0)\, D(v - \eta u)(\zeta), D\psi(\zeta) \rangle \, d\zeta + \int_\Omega (v - \eta u)(\zeta)\, Y^* \psi(\zeta) \, d\zeta \, .$$

A. We shall prove that (3.7) also holds (almost everywhere) if we replace $\psi(\zeta)$ with the function $\Gamma(z_0; \zeta^{-1} \circ z)$. First of all we note that the functions in

(3.7) have compact support so that (3.7) also holds for every $\psi \in C^\infty(\mathbb{R}^{N+1})$.

If φ is the function defined in (3.1) with $r=1$, $s=1/2$, then we set for every z, $z_0 \in \Omega$ and for every $\delta > 0$

$$\psi_\delta(z_0; z; \zeta) = \left[1 - \varphi\left(\frac{\|\zeta^{-1} \circ z\|}{\delta}\right)\right] \Gamma(z_0; \zeta^{-1} \circ z), \tag{3.8}$$

and we remark that being v, $L_{z_0} v \in C_0^\infty(\mathbb{R}^{N+1})$, we have

$$\lim_{\delta \to 0} \int_\Omega (L_{z_0} v)(\zeta)\, \psi_\delta(z_0; z; \zeta)\, d\zeta = \int_\Omega (L_{z_0} v)(\zeta)\, \Gamma(z_0; \zeta^{-1} \circ z)\, d\zeta = -v(z).$$

We next consider the term $\int_\Omega g(\zeta)\, \psi(\zeta)\, d\zeta$ in (3.7). By using the first inequality of (2.4), we get

$$\left| \int_\Omega [\psi_\delta(z_0; z; \zeta) - \Gamma(z_0; \zeta^{-1} \circ z)]\, g(\zeta)\, d\zeta \right| \leqslant$$

$$c^+ \int_\Omega \varphi\left(\frac{\|\zeta^{-1} \circ z\|}{\delta}\right) \Gamma^+(\zeta^{-1} \circ z)\, |g(\zeta)|\, d\zeta \equiv R_\delta^+(g)(z).$$

Note that, for every fixed $z \in \mathbb{R}^{N+1}$, the function $\delta \mapsto R_\delta^+(g)(z)$ is non-increasing and

$$\|R_\delta^+(g)\|_p \leqslant \|g\|_p \left\| \Gamma^+(\eta)\, \varphi\left(\frac{\|\eta\|}{\delta}\right) \right\|_1 \underset{\delta \to 0}{\longrightarrow} 0,$$

then $R_\delta^+(g)(z) \to 0$ as $\delta \to 0$, for every $z \in \mathbb{R}^{N+1} \backslash H$ (here and in the sequel H denotes a zero measure set), and so

$$\lim_{\delta \to 0} \int_\Omega \psi_\delta(z_0; z; \zeta)\, g(\zeta)\, d\zeta = \int_\Omega \Gamma(z_0; \zeta^{-1} \circ z)\, g(\zeta)\, d\zeta, \tag{3.9}$$

for every $z \in \mathbb{R}^{N+1} \backslash H$ and for every $z_0 \in \mathbb{R}^{N+1}$. In the same manner, we shall estimate the next three integrals appearing in (3.7). In that order, setting

$$\varphi_1(t) \equiv \max\{|\varphi'(s)|,\, s \geqslant t\},$$

we obtain from Lemma 3.3 that

$$\left| \partial_{\xi_j} \varphi\left(\frac{\|\zeta^{-1} \circ z\|}{\delta} \right) \right| = \frac{1}{\delta} \left| \varphi'\left(\frac{\|\zeta^{-1} \circ z\|}{\delta} \right) \right| \cdot$$

$$\left| \frac{\partial \|\zeta^{-1} \circ z\|}{\partial \xi_j} \right| \leqslant \frac{k}{\delta} \left| \varphi_1\left(\frac{\|\zeta^{-1} \circ z\|}{\delta} \right) \right|,$$

for every $j = 1, \ldots, m_0$. Then, by using the second inequality in (2.4), we get

$$\text{(3.10)} \quad \begin{cases} \lim\limits_{\delta \to 0} \int\limits_{\Omega} \langle G(\zeta), D\psi_\delta(z_0; z; \zeta) \rangle \, d\zeta = \int\limits_{\Omega} \langle G(\zeta), D\Gamma(z_0; \zeta^{-1} \circ z) \rangle \, d\zeta \,, \\ \lim\limits_{\delta \to 0} \int\limits_{\Omega} \langle A(z_0) \, D(v - \eta u)(\zeta), D\psi_\delta(z_0; z; \zeta) \rangle \, d\zeta = \\ \qquad \int\limits_{\Omega} \langle A(z_0) \, D(v - \eta u), D\Gamma(z_0; \zeta^{-1} \circ z) \rangle \, d\zeta \,, \\ \lim\limits_{\delta \to 0} \int\limits_{\Omega} \langle [A(z_0) - A(\zeta)] \, D(\eta u)(\zeta), D\psi_\delta(z_0; z; \zeta) \rangle \, d\zeta = \\ \qquad \int\limits_{\Omega} \langle [A(z_0) - A(\zeta)] \, D(\eta u), D\Gamma(z_0; \zeta^{-1} \circ z) \rangle \, d\zeta \,, \end{cases}$$

for every $z \in \mathbb{R}^{N+1} \backslash H$ and for every $z_0 \in \mathbb{R}^{N+1}$.

About last integral in (3.7), it directly follows from (3.9) and (3.10) that there exists a function $T(v - \eta u)(z_0, z)$ belonging to $L^p(B_r)$ such that

$$\int\limits_{\Omega} (v - \eta u)(\zeta) \, Y^* \psi_\delta(z_0; \zeta^{-1} \circ z) \, d\zeta \underset{\delta \to 0}{\longrightarrow} T(v - \eta u)(z_0, z) \,,$$

for every $z \in \mathbb{R}^{N+1} \backslash H$ and for every $z_0 \in \mathbb{R}^{N+1}$. Then, setting

$$\text{(3.11)} \quad w(z_0, z) = -\int\limits_{\Omega} \Gamma(z_0; \zeta^{-1} \circ z) \, g(\zeta) \, d\zeta + \int\limits_{\Omega} \langle G(\zeta), D\Gamma(z_0; \zeta^{-1} \circ z) \rangle \, d\zeta \, + \\ + \int\limits_{\Omega} \langle [A(z_0) - A(\zeta)] \, D(\eta u)(\zeta), D\Gamma(z_0; \zeta^{-1} \circ z) \rangle \, d\zeta$$

and

$$S(v - \eta u)(z_0, z) = \int\limits_{\Omega} \langle A(z_0) \, D(v - \eta u)(\zeta), D\Gamma(z_0; \zeta^{-1} \circ z) \rangle \, d\zeta \,,$$

we have

$$(3.12)\quad (\eta u)(z) - w(z_0, z) = (\eta u)(z) - v(z) - S(v - \eta u)(z_0, z) - T(v - \eta u)(z_0, z)$$

for every $z \in \mathbb{R}^{N+1} \backslash H$ and for every $z_0 \in \mathbb{R}^{N+1}$.

B. In order to let $v \to \eta u$, we shall find two operators S^+ and T^+ such that

$$(3.13)\quad \begin{cases} |S(v - \eta u)(z_0, z)| \leq S^+(v - \eta u)(z), & \text{for every } z,\ z_0 \in \mathbb{R}^{N+1}, \\ \|S^+(v - \eta u)\|_p \leq c(B_r) \sum_{j=1}^{m_0} \|\partial_{x_j}(v - \eta u)\|_p . \end{cases}$$

and

$$(3.14)\quad \begin{cases} |T(v - \eta u)(z_0, z)| \leq T^+(v - \eta u)(z), & \text{for any } z \in \mathbb{R}^{N+1} \backslash H,\ z_0 \in \mathbb{R}^{N+1}, \\ \|T^+(v - \eta u)\|_p \leq k \|v - \eta u\|_p , \end{cases}$$

for some positive constants $c(B_r)$ and k.

In view of the second inequality of (2.4) and Theorem 2.9 it is clear that the operator

$$S^+(v - \eta u)(z) = \mu c^+ \int_\Omega \frac{\Gamma^+(\zeta^{-1} \circ z)}{\sqrt{t - \tau}} \sum_{j=1}^{m_0} |\partial_{x_j}(v - \eta u)(\zeta)|\, d\zeta$$

satisfies condition (3.13).

In order to prove (3.14) we first show that the function $T(v - \eta u)$ satisfies the following statement, (which allows us to use some results of singular integral theory in homogeneous space):

$$(3.15)\ T(v - \eta u)(z_0, z) = \lim_{j \to \infty} \int_{\|\zeta^{-1} \circ z\| \geq \delta_j} (v - \eta u)(\zeta)\, Y^* \Gamma(z_0;\, \zeta^{-1} \circ z)\, d\zeta ,$$

for some sequence $(\delta_j)_{j \in \mathbb{N}}$ such that $\delta_j \to 0$ as $j \to \infty$, for every $z \in \mathbb{R}^{N+1} \backslash H$ and for every $z_0 \in \mathbb{R}^{N+1}$. To this end we note that

$$\int_{\mathbb{R}^{N+1}} w(\zeta)\, Y^* \psi_\delta(z_0;\, \zeta^{-1} \circ z)\, d\zeta = \int_{\|\zeta^{-1} \circ z\| \geq \delta/2} w(\zeta)\, Y^* \Gamma(z_0;\, \zeta^{-1} \circ z)\, d\zeta -$$

$$\int_{\delta/2 \leq \|\zeta^{-1} \circ z\| \leq \delta} w(\zeta)\, Y^* \left(\varphi \left(\frac{\|\zeta^{-1} \circ z\|}{\delta} \right) \Gamma(z_0;\, \zeta^{-1} \circ z) \right) d\zeta$$

for every $w \in L^p(\mathbb{R}^{N+1})$. By using the third estimate in (2.4) and Lemma 3.3

we find that there exists a bounded function $\tilde{\varphi}$ such that $\tilde{\varphi}(t) = 0$ for $t > 1$ and

$$\left| Y^* \left(\varphi \left(\frac{\|\zeta^{-1} \circ z\|}{\delta} \right) \Gamma(z_0;\, \zeta^{-1} \circ z) \right) \right| \leqslant \frac{1}{\delta^{Q+2}}\, \tilde{\varphi} \left(\frac{\|\zeta^{-1} \circ z\|}{\delta} \right)$$

for every $z_0, z, \zeta \in \mathbb{R}^{N+1}$ with $\delta/2 \leqslant \|\zeta^{-1} \circ z\| \leqslant \delta$. Moreover

$$\int_{\delta/2 \leqslant \|\zeta^{-1} \circ z\| \leqslant \delta} Y^* \left(\varphi \left(\frac{\|\zeta^{-1} \circ z\|}{\delta} \right) \Gamma(z_0;\, \zeta^{-1} \circ z) \right) d\zeta = 0\,,$$

then, setting

$$T_\delta w(z, z_0) = \int_{\delta/2 \leqslant \|\zeta^{-1} \circ z\| \leqslant \delta} w(\zeta)\, Y^* \left(\varphi \left(\frac{\|\zeta^{-1} \circ z\|}{\delta} \right) \Gamma(z_0;\, \zeta^{-1} \circ z) \right) d\zeta$$

we get

$$|T_\delta w(z, z_0)| =$$

$$\left| \int_{\delta/2 \leqslant \|\zeta^{-1} \circ z\| \leqslant \delta} Y^* \left(\varphi \left(\frac{\|\zeta^{-1} \circ z\|}{\delta} \right) \Gamma(z_0;\, \zeta^{-1} \circ z) \right) [w(\zeta) - w(z)]\, d\zeta \right| \leqslant$$

$$\frac{1}{\delta^{Q+2}} \int_{\delta/2 \leqslant \|\zeta^{-1} \circ z\| \leqslant \delta} \tilde{\varphi} \left(\left\| D\left(\frac{1}{\delta} \right) \zeta^{-1} \circ z) \right\| \right) |w(\zeta) - w(z)|\, d\zeta \equiv T_\delta^+ w(z)\,.$$

Hence, from Minkowskii's inequality, we obtain

$$\|T_\delta^+ w\|_p \leqslant \int_{1/2 \leqslant \|\eta\| \leqslant 1} \tilde{\varphi}(\eta) \left(\int_{\mathbb{R}^{N+1}} |w(z \circ D(\delta)\, \eta^{-1}) - w(z)|^p\, dz \right)^{1/p} d\eta \underset{\delta \to 0}{\longrightarrow} 0\,,$$

from which it follows the statement (3.15).

We next consider the expansion in spherical harmonics of the function $Y^* \Gamma(z_0; \cdot)$:

$$Y^* \Gamma(z_0;\, \zeta^{-1} \circ z) = \sum_{m=1}^{\infty} b_m(z_0)\, \frac{K_m(\zeta^{-1} \circ z)}{\|\zeta^{-1} \circ z\|^{Q+2}}\,.$$

Set $b_m^+ = \sup_{z_0} |b_m(z_0)|$ and

$$T^+(v - \eta u)(z) = \lim_{j \to \infty} \sum_{m=1}^{\infty} b_m^+ \left| \int_{\|\zeta^{-1} \circ z\| \geqslant \delta_j} \frac{K_m(\zeta^{-1} \circ z)}{\|\zeta^{-1} \circ z\|^{Q+2}}\, (v - \eta u)(\zeta)\, d\zeta \right|;$$

since

$$Y^* \Gamma(z_0;\, \zeta^{-1} \circ z) = \sum_{i,\,j=1}^{m_0} a_{i,\,j}(z_0)\, \partial^2_{\xi_i,\,\xi_j} \Gamma(z_0;\, \zeta^{-1} \circ z)$$

for all $z_0, z, \zeta \in \mathbb{R}^{N+1}$, with $z \neq \zeta$, we obtain from Remark 2.10 that T^+ is well defined and satisfies (3.14). Hence, if we set

$$\widetilde{T}(v - \eta u) = |(\eta u)(z) - v(z)| + S^+(v - \eta u)(z) + T^+(v - \eta u)(z), \tag{3.16}$$

from (3.13) and (3.14), it follows that there exists a positive constant $\tilde{c}$ such that

$$\begin{cases} |w(z_0;\, z) - (\eta u)(z)| \leq \widetilde{T}(v - \eta u)(z), & \text{for every } z \in \mathbb{R}^{N+1} \backslash H,\ z_0 \in \mathbb{R}^{N+1}, \\ \|\widetilde{T}(v - \eta u)\|_p \leq \tilde{c} \left(\|v - \eta u\|_p + \sum_{j=1}^{m_0} \|\partial_{x_j}(v - \eta u)\|_p \right). \end{cases} \tag{3.17}$$

C. Let $(v_k)_{k \in N}$ be a sequence of functions belonging to $C_0^\infty(\mathbb{R}^{N+1})$ such that

$$\|\eta u - v_k\|_p \underset{k \to \infty}{\longrightarrow} 0\,,$$

$$\|\partial_{x_j}(\eta u) - \partial_{x_j} v_k\|_p \underset{k \to \infty}{\longrightarrow} 0 \qquad \text{for all } j = 1, \ldots, m_0\,.$$

Setting

$$\widetilde{w}(z) = \inf \left\{ \widetilde{T}(v_k - \eta u)(z) \colon k \in \mathbb{N} \right\}$$

and using the results proved in step **B**, we find a zero measure set $H \subset \mathbb{R}^{N+1}$ such that

$$|w(z_0,\, z) - (\eta u)(z)| \leq \widetilde{w}(z)\,, \qquad \|\widetilde{w}\|_p \leq \inf\{ \|\widetilde{T}(v_k - \eta u)\|_p \colon k \in N\}\,,$$

for every $z \in \mathbb{R}^{N+1} \backslash H$ and for every $z_0 \in \mathbb{R}^{N+1}$.

Thus, by using inequalities (3.17) for every function v_k, we conclude that

$$(\eta u)(z) = w(z_0,\, z)$$

for every $z_0 \in \mathbb{R}^{N+1}$ and for every $z \in \mathbb{R}^{N+1} \backslash H$. By choosing $z_0 = z$, we finally find identity (3.5).

We are left with the proof of (3.6). We first remark that if $g \in C_0^\infty(\mathbb{R}^{N+1})$, it is quite standard (see for example Theorem 2.4 in [2]) to prove that

$$\partial_{x_j} \int_{\mathbb{R}^{N+1}} \Gamma(z_0; \zeta^{-1} \circ z)\, g(\zeta)\, d\zeta = \int_{\mathbb{R}^{N+1}} \Gamma_j(z_0; \zeta^{-1} \circ z)\, g(\zeta)\, d\zeta\,,$$

for all $j = 1, \ldots, m_0$ and for every $z, z_0 \in \mathbb{R}^{N+1}$. By a density argument and Theorem 2.9 we extend the last identity to every function $g \in L^p(\mathbb{R}^{N+1})$. Proceeding in much the same way for the other integrals which define the function $w(z_0, z)$ and by using Theorem 2.8 we get

(3.18) $\quad \partial_{x_j} w(z_0, z) =$

$$- \int_{\mathbb{R}^{N+1}} \Gamma_j(z_0; \zeta^{-1} \circ z)\, g(\zeta)\, d\zeta + \sum_{k=1}^{m_0} G_k(z) \int_{\|\zeta\| = 1} \Gamma_j(z_0; \zeta)\, \nu_k(\zeta)\, d\sigma +$$

$$\sum_{h,k=1}^{m_0} \lim_{\varepsilon \to 0} \int_{\|\zeta^{-1} \circ z\| \geq \varepsilon} \Gamma_{jh}(z_0; \zeta^{-1} \circ z)[(a_{h,k}(z_0) - a_{h,k}(\zeta))\, \partial_{x_k}(\eta u)(\zeta) + G_h(\zeta)]\, d\zeta\,,$$

for every $z_0 \in \mathbb{R}^{N+1}$ and for almost every $z \in \mathbb{R}^{N+1}$.

The proof of Theorem 3.1 will be concluded if identity (3.18) holds with $z_0 = z$.

For every fixed z_0 we consider the expansion in spherical harmonics of the terms in (3.18). In order to simplify the notations we write

$$\partial_{x_j}(\eta u)(z) = \partial_{x_j} w(z_0, z) = \sum_{m=1}^{\infty} b_m(z_0)\, T_m(g_m)(z)\,,$$

where g_m denotes one of the following functions: g, G_j, $\partial_{x_j}(\eta u)$ or $a_{j,h}\, \partial_{x_h}(\eta u)$, $(j, h = 1, \ldots, m_0)$, and T_m indicates the convolution with a suitable homogeneous function. By Proposition 2.7, the convergence is uniform with respect to z_0 then

$$\partial_{x_j}(\eta u)(z) = \sum_{m=1}^{\infty} b_m(z)\, T_m(g_m)(z)\,,$$

for almost every $z \in \mathbb{R}^{N+1}$. This concludes the proof of Theorem 3.1, since the last expression is the expansion in spherical harmonics of the second member of (3.6).

We are left with the

PROOF OF LEMMA 3.3. – Accordingly with Definition 2.2 the norm of $w = (y, s) \in \mathbb{R}^{N+1} \backslash \{0\}$ is the unique positive solution ϱ of the equation

$$M(w, \varrho) = \frac{y_1^2}{\varrho^{2\alpha_1}} + \ldots + \frac{y_N^2}{\varrho^{2\alpha_N}} + \frac{s^2}{\varrho^4} = 1\,;$$

then

$$\frac{\partial \|w\|}{\partial y_i} = \frac{-1}{(\partial M/\partial \varrho)(w, \|w\|)} \cdot \frac{\partial M(w, \|w\|)}{\partial y_i} ,$$

$$\frac{\partial \|w\|}{\partial s} = \frac{-1}{(\partial M/\partial \varrho)(w, \|w\|)} \cdot \frac{\partial M(w, \|w\|)}{\partial s} ,$$

and since $|y_i| \leqslant \|w\|^{\alpha_i}$, $|s| \leqslant \|w\|^2$, we have

$$\left| \frac{\partial \|w\|}{\partial y_i} \right| \leqslant \frac{|y_i|}{\|w\|^{2\alpha_i - 1}} \leqslant \|w\|^{1-\alpha_i}, \qquad \left| \frac{\partial \|w\|}{\partial s} \right| \leqslant \frac{|s|}{\|w\|^3} \leqslant \|w\|^{-1} . \tag{3.19}$$

That being stated, by writing $\zeta^{-1} \circ z = (x - E(t-\tau)\,\xi, t-\tau)$, we get

$$\frac{\partial \|\zeta^{-1} \circ z\|}{\partial \xi_j} = -\sum_{k=1}^{N} \frac{\partial \|\zeta^{-1} \circ z\|}{\partial y_k} e_{kj}(t-\tau),$$

where $E(s) = \exp(-sB^T)$ and B is the matrix in Hypothesis H. Then, since

$$\frac{\partial \|\zeta^{-1} \circ z\|}{\partial \xi_j} = \frac{\partial \|\zeta^{-1} \circ z\|}{\partial y_j} - (t-\tau) \sum_{k=m_0+1}^{m_0+m_1} e_{kj}(1) \frac{\partial \|\zeta^{-1} \circ z\|}{\partial y_k} + \ldots +$$
$$\frac{(-(t-\tau))^r}{r!} \sum_{k=m_0+m_1+\ldots+m_{r-1}+1}^{N} e_{kj}(1) \frac{\partial \|\zeta^{-1} \circ z\|}{\partial y_k} .$$

and $\alpha_1 = \ldots = \alpha_{m_0} = 1$, $\alpha_{m_0+1} = \ldots = \alpha_{m_0+m_1} = 3$, $\ldots$, $\alpha_{m_0+m_1+\ldots+m_{r-1}+1} = \ldots = \alpha_N = 2r+1$, the first assertion of Lemma 3.3 follows straightforwardly from (3.19).

The proof of the second assertion is again a consequence of (3.19), since a direct calculation gives

$$Y^* \|\zeta^{-1} \circ z\| = -\frac{\partial \|\cdot\|}{\partial s}(\zeta^{-1} \circ z).$$

4. – L^p estimates.

To simplify the notations, in this Section, we shall denote

$$D_0 u = (\partial_{x_1} u, \ldots, \partial_{x_{m_0}} u)$$

and we shall write $\|F\|_p$, $\|D_0 u\|_p$ instead of $\sum_{k=1}^{m_0} \|F_k\|_p$, $\sum_{k=1}^{m_0} \|\partial_{x_k} u\|_p$, respectively.

For every $\zeta_0 \in \Omega$ and for $r, s > 0$ such that $B_r(\zeta_0) \subseteq \Omega$ and $s < r$ consider the function η defined in (3.2).

For the sake of brevity, in the sequel we shall write B_r instead of $B_r(z_0)$, and

$$v(z) = \eta(z)\, u(z);$$

note that, using the notations introduced in (3.3), we have

$$Lv = \operatorname{div}(G) + g .$$

In order to prove Theorem 1.1 we give some preliminary results.

LEMMA 4.1. – *Let u, $F \in L^p(B_r)$ and $D_0 u \in L^q(B_r)$, with $1/q = 1/p + 1/(Q+2)$. There exists a positive constant r_0, depending only on the operator L, such that, if $r \leq r_0$, then $D_0 u \in L^p_{\mathrm{loc}}(B_r)$ and*

$$\|D_0 u;\, L^p(B_s)\| \leq k(\|F;\, L^p(B_r)\| + \|u;\, L^p(B_r)\| + \|D_0 u;\, L^q(B_r)\|),$$

for every $s \in]0, r[$, where $k = k(r, s) > 0$.

PROOF OF LEMMA 4.1. – We first prove the claim assuming that $D_0 u \in L^p(B_r)$. Using the same notations of Theorems 2.8 and 2.9 and letting

$$c_{j,k}(z) = \int_{\Sigma_{N+1}} \Gamma_j(z;\, \zeta)\, \nu_k(\zeta)\, d\sigma,$$

accordingly to Theorem 3.1, we may write the derivatives of v as

$$v_{x_j}(z) = \sum_{h,k=1}^{m_0} C_{j,k}[a_{h,k}, v_{x_k}](z) - \sum_{h=1}^{m_0} T_{j,h}(G_h)(z) - Tg(z) - \sum_{h=1}^{m_0} c_{j,h}(z)\, G_h(z).$$

Then, Theorems 2.8 and 2.9 (since $c_{j,h}$ are bounded functions) give

$$\|\partial_{x_j} v\|_p \leq c\left(\sum_{h,k=1}^{m_0} \|a_{h,k}\|_* \,\|\partial_{x_k} v\|_p + \|G\|_p + \|g\|_q\right),$$

where the norm $\|\cdot\|_*$ is in fact taken in the set B_r. As a consequence of the VMO_L hypothesis on the coefficients $a_{h,k}$, there exists $r_0 > 0$ such that

$$\|D_0 v\|_p \leq c'(\|G\|_p + \|g\|_q)$$

for every $r \in]0, r_0]$. Since $p > q$ and η is a bounded function with support contained in the ball $B_r(z_0)$ the proof of the Lemma immediately follows from this inequality and from (3.4).

We next remove the assumption that $D_0 u \in L^p(B_r)$. First of all, we note that the map

$$\tilde{T}\colon (L^q(B_r))^{m_0} \to (L^q(B_r))^{m_0}, \tag{4.1}$$

defined for every $U \in (L^q(B_r))^{m_0}$ as

$$(\tilde{T}U)_j(z) = \sum_{h,\,k=1}^{m_0} C_{j,\,k}[a_{h,\,k},\, U_k](z) + \sum_{h=1}^{m_0} T_{j,\,h}(G_h)(z) - Tg(z) - \sum_{h=1}^{m_0} c_{j,\,h}(z)\, G_h(z),$$

$(j = 1, \dots, m_0)$ is a contraction and so $D_0 v$ is its unique fixed point. On the other hand, (if r_0 is small enough), $\tilde{T}$ is also a contraction in $L^p(B_r)$ then has a unique fixed point $U_p \in L^p(B_r)$. Since $L^p(B_r) \supset L^q(B_r)$ the function $D_0 v$ must coincide with $U_p \in L^p(B_r)$.

REMARK 4.2. – The idea of using the Banach fixed point theorem to show that $D_0 u \in L^p_{\mathrm{loc}}(B_r)$ is contained in [3]. By iterating the method we find that, if u is a weak solution of (1.1) and $u, F \in L^p_{\mathrm{loc}}(\Omega)$, then also $D_0 u \in L^p_{\mathrm{loc}}(\Omega)$. In the sequel we shall implicitly use this result.

LEMMA 4.3. – *Let* $\varrho, \sigma \in \mathbb{R}$, *with* $0 < \sigma < \varrho \leq r_0$, *where* r_0 *is as in the Lemma* 4.1, *and let* $p > 2$. *If* u, $F \in L^p(B_\varrho)$ *then there exists a constant* $c_0 = c_0(\varrho, \sigma, p) > 0$ *such that*

$$\|D_0 u;\, L^p(B_\sigma)\| \leq c_0 (\|F;\, L^p(B_\varrho)\| + \|u;\, L^p(B_\varrho)\| + \|D_0 u;\, L^2(B_\varrho)\|).$$

PROOF OF LEMMA 4.3. – If $1/p + 1/(Q+2) \geq 1/2$ the result immediately follows from Lemma 4.1, applied with $\varrho = r$ and $\sigma = s$, being $\eta(z) = 1$ for every $z \in B_s(z_0)$.

Otherwise we must iterate the method: set

$$m = \min\left\{k \in \mathbb{N}\colon\ \frac{k}{Q+2} \geq \frac{1}{2} - \frac{1}{p}\right\}, \qquad \delta = \left(\frac{\varrho}{\sigma}\right)^{1/m}$$

and, for $h = 1, \dots, m+1$,

$$s_h = \delta^{h-1}\sigma, \qquad r_h = \delta^h \sigma, \qquad q_h = \frac{p(Q+2)}{(h-1)p + (Q+2)}.$$

By Lemma 4.1 we obtain

$$\|D_0 u;\, L^{q_h}(B_{s_h})\| \leq c_h (\|F;\, L^{q_h}(B_{r_h})\| + \|u;\, L^{q_h}(B_{r_h})\| + \|D_0 u;\, L^{q_{h+1}}(B_{r_h})\|)$$

for every $h = 1, \dots, m$. Since $s_1 = \sigma$, $r_m = \varrho$ and $q_{m+1} \leq 2 < q_m < \dots < q_1 = p$, from these inequalities we get the proof of the Lemma.

LEMMA 4.4. – *If $u, F \in L^2(B_r)$ then for every s, with $0 < s < r$, there exists a constant $c_1 = c_1(r, s) > 0$ such that*

$$\|D_0 u;\, L^2(B_s)\| \leqslant c_1(\|F;\, L^2(B_r)\| + \|u;\, L^2(B_r)\|)\,.$$

PROOF OF LEMMA 4.4. – Let η be the function defined in (3.2), and let $(\psi_k)_{k\in\mathbb{N}}$ be a sequence of $C_0^\infty(\Omega)$ functions converging to $u\eta^2$ in the $L^2(\Omega)$ norm. Using ψ_k as a test function in the definition of weak solution and letting $k \to \infty$ we easily find

$$\int\limits_{\Omega}(\eta^2\langle ADu,\, Du\rangle + 2u\eta\langle ADu,\, D\eta\rangle) + \frac{1}{2}\int\limits_{\Omega}Y(\eta^2u^2) - \int\limits_{\Omega}\eta u^2 Y\eta =$$

$$\int\limits_{\Omega}(\eta^2\langle F,\, Du\rangle + 2u\eta\langle F,\, D\eta\rangle)\,,$$

and then

$$\mu^{-1}\|\eta D_0 u\|_2^2 \leqslant c(\|u\|_2^2 + \|u\|_2\|\eta D_0 u\|_2 + \|F\|_2\|\eta D_0 u\|_2 + \|u\|_2\|F\|_2) \leqslant c'(\varepsilon)(\|u\|_2^2 + \|F\|_2^2) + \varepsilon\|\eta D_0 u\|_2^2\,, \tag{4.2}$$

where the norms are taken in B_r, μ is the constant of Hypothesis H and ε is arbitrary. Since $\eta(z) = 1$ for every $z \in B_s$, the proof of the Lemma is a direct consequence of (4.2).

PROOF OF THEOREM 1.1. – First of all we note that it is sufficient to prove the Theorem when the open set Ω is a B_r and the compact set K is $\overline{B_s}$, with $s \in]0, r[$. Recall that in Remark 4.2 we showed that $D_0 u \in L^p_{\mathrm{loc}}(\Omega)$. Moreover, it is enough to consider the case $p > 2$, since the case $p = 2$ is contained in Lemma 4.4 and the case $1 < p < 2$ can be recovered by an elementary duality argument. After that, the proof of Theorem 1.1 is a simple consequence of Lemmas 4.3 and 4.4.

5. – Hölder continuity.

In this Section we shall prove the Hölder continuity result stated in Theorem 1.2. We first prove a mean value result that is essentially contained in [14], Proposition 1.15.

LEMMA 5.1. – *Let $K \in C^1(\mathbb{R}^{N+1}\setminus\{0\})$ be a homogeneous function of degree α, with respect to the group $(D(\lambda), \lambda^2)_{\lambda>0}$. Then there exist two constants $c >$*

0 *and* $M > 1$ *such that*

$$|K(\zeta) - K(z)| \leqslant c\|z^{-1} \circ \zeta\| \cdot \|z\|^{\alpha - 1}$$

for every z, ζ *such that* $\|z\| \geqslant M\|z^{-1} \circ \zeta\|$.

An immediate consequence of Lemma 5.1 is the following result:

PROPOSITION 5.2. – *There exist two constants* $c > 0$ *and* $M > 1$ *such that*

$$|\Gamma(z_0; w^{-1} \circ \zeta) - \Gamma(z_0; w^{-1} \circ z)| \leqslant c \frac{\|\zeta^{-1} \circ z\|}{\|z^{-1} \circ w\|^{Q+1}},$$

$$|\Gamma_j(z_0; w^{-1} \circ \zeta) - \Gamma_j(z_0; w^{-1} \circ z)| \leqslant c \frac{\|\zeta^{-1} \circ z\|}{\|z^{-1} \circ w\|^{Q+2}},$$

$$|\Gamma_{ij}(z_0; w^{-1} \circ \zeta) - \Gamma_{ij}(z_0; w^{-1} \circ z)| \leqslant c \frac{\|\zeta^{-1} \circ z\|}{\|z^{-1} \circ w\|^{Q+3}},$$

for every $z, \zeta, w, z_0 \in \mathbb{R}^{N+1}$ *such that* $\|z^{-1} \circ w\| \geqslant M\|z^{-1} \circ \zeta\|$ *and for any* i, $j = 1, \ldots, m_0$.

PROOF OF LEMMA 5.1. – We first prove the result assuming that $\|z\| = 1$. Denote $w = z^{-1} \circ \zeta$, and note that, since $M > 1$, we obtain from Proposition 2.3

$$\|\zeta\| = \|z \circ w\| \leqslant c_0(1 + 1/M) \leqslant 2c_0,$$

and then, again by Proposition 2.3,

$$|\zeta - z| \leqslant c_2\|\zeta^{-1} \circ z\| = c_2\|w\|.$$

Moreover

$$1 = \|z\| = \|\zeta \circ (w^{-1})\| \leqslant c_0(\|\zeta\| + \|w^{-1}\|) \leqslant c_0(\|\zeta\| + c_1\|w\|) \leqslant c_0(\|\zeta\| + c_1/M),$$

from which we get, by choosing $M > c_0 c_1$,

$$\|\zeta\| \geqslant m \equiv \frac{1}{c_0} - \frac{c_1}{M} > 0.$$

Then, being $K \in C^1(\mathbb{R}^{N+1} \backslash \{0\})$, we obtain

$$|K(z) - K(\zeta)| \leqslant \sup_{m \leqslant \|\eta\| \leqslant 2c_0} |DK(\eta)|\,|\zeta - z| \leqslant c\|\eta\|,$$

where $c = c_2 \sup_{m \leqslant \|\eta\| \leqslant 2c_0} |DK(\eta)|$.

If now z is any point of $\mathbb{R}^{N+1}\setminus\{0\}$, then

$$|K(z)-K(\zeta)| = \|z\|^{\alpha} \, |K(D(\|z\|^{-1})\, z) - K(D(\|z\|^{-1})\, \zeta)| \leqslant$$

$$c\|z\|^{\alpha}\, \|(D(\|z\|^{-1})\, z)^{-1} \circ (D(\|z\|^{-1})\, \zeta)\| = c\|z\|^{\alpha-1} \|\zeta^{-1}\circ z\|\,,$$

since, for every $\lambda > 0$ we have

$$(D(\lambda)\, z)^{-1} \circ (D(\lambda)\, \zeta) = (D(\lambda)(z^{-1})) \circ (D(\lambda)\, \zeta) = D(\lambda)(z^{-1}\circ \zeta)\,.$$

This completes the proof of Lemma 5.1.

PROOF OF THEOREM 1.2. – As in Section 4, we shall prove the Theorem for the compact set $K=\overline{B_s(z_0)} \subseteq \Omega$. It is convenient to write equation (1.1) in the following form

$$L_0 u \equiv \operatorname{div}(JDu) + \langle x,\, BDu\rangle - \partial_t u = \operatorname{div}(\widetilde{F})\,,$$

where $\widetilde{F}_j = F_j + \partial_{x_j} u - \sum_{k=1}^{m_0} a_{j,k}\, \partial_{x_k} u$ for $j=1, \ldots, m_0$, $\widetilde{F}_j = 0$ for $j = m_0 + 1, \ldots, N$ and J is defined in (1.5).

Let $r>0$ be such that $B_r(z_0) \subset \Omega$ and let η be the function defined in (3.2). If we set

$$v(z) = \eta(z)\, u(z)\,,$$

we find

$$L_0 v = \operatorname{div}(\eta \widetilde{F}) - \langle \widetilde{F},\, D\eta\rangle + uL\eta + 2\langle JDu,\, D\eta\rangle\,.$$

Note that, by Theorem 1.1, we have

$$\|\widetilde{F};\, L^p(\overline{B_r})\| \leqslant c(\|F;\, L^p(\Omega)\| + \|u;\, L^p(\Omega)\|) \tag{5.1}$$

and, since the functions u and v coincide in the set $\overline{B_s}$, it is sufficient to prove (1.4) for the function v.

Denote by $\Gamma^0(z;w) \equiv \Gamma^0(w^{-1}\circ z)$ the fundamental solution of the operator L_0, with pole at w; then the following representation formula for the function v holds:

$$v(z) = \int \Gamma^0(z;\, w)(\langle \widetilde{F}(w),\, D\eta(w)\rangle - u(w)\, L\eta(w) - 2\langle JDu(w),\, D\eta(w)\rangle)\, dw - \tag{5.2}$$

$$\sum_{j=1}^{m_0} \int \Gamma^0_j(z;\, w)\, \eta(w)\, \widetilde{F}_j(w)\, dw \equiv v_0(z) - \sum_{j=1}^{m_0} v_j(z)\,.$$

Consider one of the integrals v_j in the last sum. It results

$$|v_j(z) - v_j(\zeta)| = \left| \int (\Gamma_j^0(z; w) - \Gamma_j^0(\zeta; w))\, \eta(w)\, \tilde{F}_j(w)\, dw \right| \leqslant$$

$$\int_{\|z^{-1} \circ w\| \leqslant M \|z^{-1} \circ \zeta\|} (|\Gamma_j^0(z; w)| + |\Gamma_j^0(\zeta; w)|)\, |\eta(w)\, \tilde{F}_j(w)|\, dw +$$

$$\int_{\|z^{-1} \circ w\| \geqslant M \|z^{-1} \circ \zeta\|} |\Gamma_j^0(z; w) - \Gamma_j^0(\zeta; w)|\, |\eta(w)\, \tilde{F}_j(w)|\, dw \leqslant$$

(from Propositions 2.3 and 5.2)

$$c \int_{\|z^{-1} \circ w\| \leqslant M \|z^{-1} \circ \zeta\|} \|z^{-1} \circ w\|^{-(Q+1)}\, |\eta(w)\, \tilde{F}_j(w)|\, dw +$$

$$c \int_{\|\zeta^{-1} \circ w\| \leqslant c_0 (M + c_1) \|z^{-1} \circ \zeta\|} \|\zeta^{-1} \circ w\|^{-(Q+1)}\, |\eta(w)\, \tilde{F}_j(w)|\, dw +$$

$$c \int_{\|z^{-1} \circ w\| \geqslant M \|z^{-1} \circ \zeta\|} \frac{\|\zeta^{-1} \circ z\|}{\|z^{-1} \circ w\|^{Q+2}}\, |\eta(w)\, \tilde{F}_j(w)|\, dw\,.$$

Applying the Hölder inequality it then follows that

$$|v_j(z) - v_j(\zeta)| \leqslant \tilde{c};\, \|\tilde{F};\, L^p(\overline{B_r})\| \cdot \|\zeta^{-1} \circ z\|^{1 - (Q+2)/p} \tag{5.3}$$

for $j = 1, \ldots, m_0$, where $\tilde{c}$ is a positive constant depending only on the operator L.

With a similar argument, using the Hölder inequality with q instead of p, where $1/q = 1/p + 1/(Q+2)$, we obtain

$$|v_0(z) - v_0(\zeta)| \leqslant \tilde{c}\, \|\tilde{F};\, L^q(\overline{B_r})\| \cdot \|\zeta^{-1} \circ z\|^{1 - (Q+2)/p}\,.$$

Being $q < p$, the proof of Theorem 1.2 follows from this inequality, from (5.3) and (5.1).

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