Bollettino
Unione Matematica Italiana

S. V. Bolotin, P. H. Rabinowitz

A variational construction of chaotic trajectories for a Hamiltonian system on a torus

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **1-B** (1998), n.3, p. 541–570.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_1998_8_1B_3_541_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 1998.

Bollettino U. M. I.
(8) **1**-B (1998), 541-570

A Variational Construction of Chaotic Trajectories for a Hamiltonian System on a Torus.

S. V. Bolotin - P. H. Rabinowitz (*)

Abstract. – *A geometric criterion for the existence of chaotic trajectories of a Hamiltonian system with two degrees of freedom and the configuration space a torus is given. As an application, positive topological entropy is established for a double pendulum problem.*

1. – Introduction.

Consider a Hamiltonian system with compact two-dimensional configuration space M and Hamiltonian $H = H(p, q)$ on the phase space

$$T^* M = \{(p, q) \mid q \in M, \ p \in T_q^* M\}.$$

The symplectic structure $dp \wedge dq$ is standard. Suppose that H satisfies the following condition:

(H) $H \in C^3$ is strictly convex in the momentum p, i.e. the Hessian $H_{pp}(p, q)$ is positive definite for all (p, q) and $H(p, q) \to \infty$ as $p \to \infty$.

The last condition implies that all energy levels $\Sigma_h = \{H = h\}$ are compact. Define the function V on M by the formula $V(q) = H(0, q)$. If the system is reversible, i.e. $H(p, q) = H(-p, q)$, then V can be regarded as the potential energy. Without loss of generality it can be assumed that $\max\limits_M V = 0$. In fact, all the results of this paper will hold without requiring that $H \to \infty$ as $p \to \infty$, if the zero energy level is compact.

(*) The first author was supported by the RFFI under grant #96-01-00747 and by the NSF under grant #MCS-8110556. The second author was supported by the NSF under grant #MCS8110556 and by the U.S. Army under contract #DAAL03-87-12-0043. Any reproduction for the purposes of the U.S. Government is permitted.

The main example to keep in mind is a classical Hamiltonian system. Let

$$H(p,\, q) = \frac{1}{2}\,\langle A(q)p,\, p\rangle + \langle v(q),\, p\rangle + V(q)\,, \tag{1.1}$$

where $A(q)$ is positive definite for all q and v is a vector field on M. Then condition (H) is obviously satisfied. This system can be represented in Lagrangian form with

$$L(\dot q,\, q) = \frac{1}{2}\,\langle \dot q - v(q),\, A^{-1}(q)(\dot q - v(q))\rangle - V(q)\,.$$

If $v(q) = 0$ for all $q \in M$, the system is reversible. Then it is called a natural mechanical system.

The properties of the system strongly depend on the topology of the configuration space M and the energy value h. First suppose that the Euler characteristic $\chi(M) < 0$ and $h > 0$. Kozlov showed that an analytic natural Hamiltonian system is nonintegrable on the energy level Σ_h [27, 28]. For a natural system with $V = 0$ (geodesic flow), the topological entropy on Σ_h was estimated from below by Katok [25]. For any Hamiltonian system satisfying condition (H) the topological entropy is positive on Σ_h with $h > 0$ if $\chi(M) < 0$. The proof is essentially the same as for a geodesic flow and will not be given here. It is based on the classical Maupertuis principle, which reduces the problem to Finsler geometry.

Define the Jacobi metric $\|\cdot\|_h$ on the set $M_h = \{V < h\} \subset M$ by the formula

$$\|\dot q\|_h = \max_p \{\langle p,\, \dot q\rangle \,|\, H(q,\, p) = h\}\,, \qquad q \in M_h\,. \tag{1.2}$$

This function on TM_h is convex and homogeneous of degree one in $\dot q$, and so it is a Finsler metric on M_h which degenerates on the boundary ∂M_h. For a nonreversible system, the Jacobi metric is nonreversible: $\|-\dot q\|_h \neq \|\dot q\|_h$. For example, for the classical system with Hamiltonian (1.1),

$$\|\dot q\|_h = \sqrt{(2(h - V(q)) + \langle v(q),\, A^{-1}(q)\, v(q)\rangle)\langle \dot q,\, A^{-1}(q)\, \dot q\rangle} + \langle \dot q,\, A^{-1}(q)\, v(q)\rangle.$$

Since any trajectory $\sigma\colon [a,\, b] \to \Sigma_h$, $\sigma(t) = (p(t),\, q(t))$, is uniquely determined by its projection $q\colon [a,\, b] \to M$, such a curve q will be also referred to as a trajectory of the Hamiltonian system. The Maupertuis principle states that if $q\colon [a,\, b] \to M_h$ is a trajectory of the Hamiltonian system with energy h, then $q\colon [a,\, b] \to M_h$ is a geodesic of the Finsler metric $\|\cdot\|_h$, i.e. an extremal of the length functional (Maupertuis action)

$$J_h(\gamma) = \int_a^b \|\dot{\gamma}(t)\|_h \, dt \tag{1.3}$$

on the set of absolutely continuous curves $\gamma : [a, b] \to M$ with fixed boundary points. Conversely, any extremal of the functional J_h in M_h after a reparameterization becomes a trajectory of the Hamiltonian system with energy h.

For $h < 0$ or $\chi(M) \geqslant 0$ there are no purely topological obstructions to integrability. Indeed, there are many classical integrable systems on a sphere or a torus. Kozlov and Ten [30] have shown that for any 2-dimensional manifold N with boundary there exists an integrable natural Hamiltonian system with the domain of possible motion $\overline{M}_h \cong N$. So for $h < 0$ or $\chi(M) \geqslant 0$ additional assumptions are necessary for proving chaotic behavior.

In this paper, the case $\chi(M) = 0$ and h small will be treated. For $\chi(M) > 0$ (M is a sphere or a projective plane) there are many integrable systems, and so the conditions for chaotic behavior are necessarily more complicated [10, 11]. Without loss of generality, let M be orientable. Then M is topologically a torus $\boldsymbol{T}^2$. Suppose that V satisfies

(V) V has a strict nondegenerate maximum $q_0 \in M$ and the point $z_0 = (0, q_0)$ is an equilibrium of the system, i.e. $H_p(0, q_0) = 0$.

Note that for a reversible system a maximum of V is always an equilibrium, but in general this is an additional assumption. For example, for the classical system (1.1), this holds if $v(q_0) = 0$. Of course, performing a canonical transformation $p \to p - \nabla f(q)$, one can always assume that $v(q_0) = 0$, but then also V will be changed.

An equilibrium satisfying condition (V) is always unstable and the characteristic exponents $\pm\lambda_1$, $\pm\lambda_2$ have nonzero real parts. There are two possibilities:

(SF) The equilibrium is a saddle-focus, i.e. the characteristic exponents are complex: $\lambda_1 = \overline{\lambda}_2 \notin \boldsymbol{R}$.

(S) The equilibrium is a saddle, i.e. the characteristic exponents are real: $0 < \lambda_1 \leqslant \lambda_2$. We will assume for simplicity that $\lambda_1 \neq \lambda_2$.

Let W^s and W^u be the stable and unstable manifolds of the equilibrium. Under conditions above, there always exists a homoclinic orbit $\gamma \subset W^s \cap W^u$. This was proved in [6] by variational methods. Generically γ is transversal, i.e. the intersection of W^s and W^u along γ is transversal in Σ_0. If (SF) holds, the system has chaotic trajectories on the energy level Σ_0. In particular, the topological entropy is positive. This was proved by Buffoni and Séré [13] for a classical analytic Hamiltonian system (1.1) by using variational methods for homoclinic orbits. They have also shown that this result holds for C^3 Hamiltonians (private communication). Essentially the same proof works for any system satisfying (H), (V) and (SF). This result is a variational version of the theorem of Devaney [19], where chaotic trajectories were constructed under the assumption that there exists a transversal homoclinic trajectory γ to the equilibrium z_0. See also the recent papers of Kalies and VanderVorst [23] and Kalies, Kwapisz, and VanderVorst [24] who also treat interesting saddle-focus settings.

In this paper we assume that (S) holds. Recall that for a reversible system a maximum point of V is always a saddle (Lagrange Theorem). This case is more subtle than (SF). Indeed, Devaney [20] gave an example (Neumann problem), where there exist 4 transversal homoclinics to a saddle equilibrium, but the system is integrable. In his example, the configuration space M is a projective plane $\boldsymbol{R}P^2$, but it is easy to give an example with $M = \boldsymbol{T}^2$:

EXAMPLE. – Consider two disconnected mathematical pendulums:

$$H = \frac{1}{2}\,(p_1^2 + p_2^2) + \lambda_1^2(\cos q_1 - 1) + \lambda_2^2(\cos q_2 - 1)\,.$$

Then $(0, 0) \in \boldsymbol{T}^2$ is a saddle equilibrium which possesses 4 transversal homoclinic trajectories. But certainly the mathematical pendulum is an integrable system.

Thus additional assumptions are needed for chaotic behavior. Sufficient conditions were discovered by Turayev and Shilnikov [39]. They proved that if there exist 3 transversal homoclinics to a saddle equilibrium and they don't belong to the strong stable $W^{ss} \subset W^s$ or strong unstable $W^{uu} \subset W^u$ manifolds, corresponding to larger eigenvalues $\pm\lambda_2$, then the Hamiltonian system has a subsystem which is the suspension over a topological Markov chain.

The main result of the present paper is a variational version of the theorem of Turayev and Shilnikov. Instead of assuming the existence of several transversal homoclinics an assumption of a geometrical nature is made.

Under the conditions above, the Jacobi metric (1.2) corresponding to energy zero is defined everywhere on M, positive definite on $M \backslash \{q_0\}$, and vanishes at the equilibrium q_0. Let $J = J_0$ be the corresponding Jacobi functional. Let Γ be a simple free homotopy class of closed curves $\gamma : [0, 1] \to M$, $\gamma(0) = \gamma(1)$, and $\Omega \subset \Gamma$ the set of loops passing through q_0:

$$\Omega = \{\gamma \in \Gamma \,|\, \gamma(0) = \gamma(1) = q_0\}\,.$$

Obviously, $\inf\limits_{\Omega} J \geqslant \inf\limits_{\Gamma} J$.

THEOREM 1.4. – *Suppose that M is a torus and conditions* (H), (V), (S) *are satisfied. If*

$$c_1 = \inf_{\Omega} J > c_0 = \inf_{\Gamma} J \,, \tag{1.5}$$

there exists $\delta > 0$ such that one or both of the following statements hold true.

- *The system possesses chaotic trajectories on any energy level Σ_h with $0 < h < \delta$;*

- *the system possesses chaotic trajectories on any energy level with $-\delta < h < 0$.*

In particular, the system has positive topological entropy on Σ_h. In most cases Theorem 1.4 holds both for $h \in (0, \delta)$ and $h \in (-\delta, 0)$ as will be seen in Proposition 1.8. For a reversible system, Theorem 1.1 always holds for $h \in (-\delta, 0)$. Condition (1.5) was introduced in [11] for a natural system on a torus. It was used in [34, 15, 12] for systems with strong force singularities of the potential and in [35] for natural systems on a torus.

A sketch of the proof of Theorem 1.4 for an analytic natural Hamiltonian system was given in [11]. It is based on constructing by variational methods topologically transversal heteroclinics from the equilibrium z_0 to a minimizing periodic orbit in Γ and then applying a version of the Turayev-Shilnikov theorem [39]. This proof doesn't work for smooth H, since it relies heavily on the analyticity of the stable and unstable manifolds of equilibrium and periodic orbits. The following corollary of Theorem 1.4 was proved in [12] for a natural system by a different method, which doesn't require the assumption $\lambda_1 \neq \lambda_2$.

COROLLARY 1.6. – *If H is analytic, the system has no analytic integrals independent of H in a neighborhood of Σ_0.*

Also without condition (1.5), the infimum c_0 of the functional J on Γ is attained at some curve $\gamma_0 \in \Gamma$. If γ_0 doesn't pass through q_0, it is a periodic orbit of energy 0. If γ_0 passes through q_0 once, it is a homoclinic orbit. If γ_0 passes through q_0 several times, it is a chain of homoclinics. Condition (1.4) implies that only the first variant is possible. It is satisfied if there exists a closed curve in the homotopy class Γ with action which is less than the action of any homoclinic to the equilibrium z_0.

Under condition (1.5) the class Ω contains a minimizing homoclinic γ_1 of action c_1. Let Ω_k denote the k-fold iterated homotopy class. The set of curves in Ω_k which don't intersect γ_0 and γ_1 consists of two connected components Ω_k^+ and Ω_k^-. The proof of Theorem 1.4 is based on the following proposition.

PROPOSITION 1.7. – *Under condition* (1.5), *there exist* $n_\pm \in \boldsymbol{N}$ *such that for any* $k > n_\pm$ *there exists a minimizing homoclinic in* $\Omega_k^\pm$.

For analytic H, in [12] a stronger result was obtained: there exist an infinite number of transversal homoclinic orbits to z_0. The existence of an infinite number of homoclinics (in general, not transversal) was established by Caldiroli and JeanJean [15] by variational methods for the case of a point moving in $\boldsymbol{R}^2$ in a potential field with a strong force singularity. A result related to Proposition 1.7 was obtained in [35].

Recall that without condition (1.5), there are at least $\operatorname{rank} \pi_1(M) + 1$ homotopy classes containing minimizing homoclinics [6]. Using this result, one can establish the existence of chaotic trajectories for small $|h|$ if $\chi(M) < 0$. However, for a torus, the existence of four homotopically different minimizing homoclinics to a saddle equilibrium doesn't yield chaotic trajectories, as the example above shows. Thus the existence of a sufficient number of homoclinics in Proposition 1.7 is necessary to establish chaos. Proposition 1.8 shows that seven homotopically different minimizing homoclinics would have been enough. Five probably is sufficient.

Proposition 1.7 is proved in § 2. In § 3 a version of the λ-lemma is proved which provides a solution of the boundary value problem for the system near the equilibrium. In § 4 the homoclinic trajectories are glued together by using these local results to provide first periodic and then chaotic trajectories. This yields Theorem 1.4. The Poincaré map of the system is semiconjugate to a topological Markov chain [26] of arbitrary order on an invariant subset in Σ_h. The construction of the Markov chain is a variational version of the theorem of Turayev and Shilnikov [39].

Now a sketch of the description of the topological Markov chain will be given. The details are contained in § 4. Take a connected component in the set of minimizing homoclinics in every homotopy class $\Omega_k^\pm$ with $k > n_\pm$. Let E be the set of components not containing homoclinics in W^{ss} and W^{uu}. Let $W_\pm^s$ and $W_\pm^u$ be connected components of the sets $W^s \backslash W^{ss}$ and $W^u \backslash W^{uu}$. Put $\sigma(k) = \pm 1$ depending on whether the homoclinics from the class $k \in E$ belong to $W_\pm^s$. Similarly, put $\tau(k) = \pm 1$ depending on whether the homoclinics from the class $k \in E$ belong to $W_\pm^u$. The following improvement of Theorem 1.4 holds.

PROPOSITION 1.8. – *Under the hypotheses of Theorem* 1.4, *for any finite set* $K \subset E$ *there exists* $\delta > 0$ *such that for any* $h \in (-\delta, \delta) \backslash \{0\}$, *the Poincaré map on* Σ_h *has a subsystem semiconjugate to the topological Markov chain over* K

with the following matrix A:

$$A_{kj} = \begin{cases} \delta_{\sigma(k),\, \tau(j)}, & h < 0, \\ \delta_{\sigma(k),\, -\tau(j)}, & h > 0. \end{cases} \tag{1.9}$$

Here δ is the Kronecker symbol. Thus if all the numbers $\sigma(k)$, $\tau(k)$, $k \in E$, are the same, then the matrix A is nontrivial only for $h < 0$. If $\sigma(k) \neq \tau(j)$ for all k, j, it is nontrivial only for $h > 0$. For a reversible system, homotopy classes $\Omega_k^{\pm}$, with $k < 0$, contain minimizing homoclinics, and $\sigma(-k) = \tau(k)$. Hence A is always nontrivial for $h < 0$.

To conclude this section an application of Theorem 1.4 will be given.

EXAMPLE. – Consider the double mathematical pendulum with weightless rods of length l_1, l_2, and mass points m_1, m_2. Let q_1, q_2 be the angles between the rods and the vertical. This is a natural system on $\boldsymbol{T}^2 = \boldsymbol{R}^2/2\pi\boldsymbol{Z}^2$ and the Lagrangian has the form $L = T - V$ with

$$T(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{1}{2}(m_1 + m_2)\, l_1^2 \dot{q}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{q}_2^2 + m_2 l_1 l_2 \cos(q_1 - q_2)\, \dot{q}_1 \dot{q}_2,$$

$$V(q_1, q_2) = (m_1 + m_2)\, g l_1 (\cos q_1 - 1) + m_2 g l_2 (\cos q_2 - 1).$$

The point $0 = (0, 0) \in \boldsymbol{T}^2$ is the maximum point of the potential energy V and $V(0) = 0$. Let Γ be the homotopy class of the closed curve $\gamma_0 : [0, 2\pi] \to \boldsymbol{T}^2$, $\gamma_0(t) = (t, t + \pi)$. The Maupertuis action of γ_0 is an elliptic integral

$$J(\gamma_0) = \int_0^{2\pi} 2\sqrt{-V(t, t+\pi)\, T(t, t+\pi, 1, 1)}\, dt <$$

$$2\pi \sqrt{2g(m_1 l_1^2 + m_2 (l_1 - l_2)^2)((m_1 + m_2)\, l_1 + m_2 l_2)}\,.$$

It was shown in [12] that for the corresponding class $\Omega \subset \Gamma$ of curves passing through 0,

$$\inf_{\gamma \in \Omega} J(\gamma) > \frac{16}{3} m_2 \sqrt{g}\, (\max\{l_1, l_2\})^{3/2}.$$

Hence Theorem 1.4 yields

COROLLARY 1.10. – *The double pendulum has positive topological entropy on* Σ_h *for* $0 < |h| < \delta$ *if*

$$9\pi^2 (m_1 l_1^2 + m_2 (l_1 - l_2)^2)((m_1 + m_2)\, l_1 + m_2 l_2) < 32 m_2^2 (\max\{l_1, l_2\})^3. \tag{1.11}$$

For the double pendulum, there exist chaotic trajectories both for $h > 0$

and $h<0$ due to the symmetry $q \to -q$. Indeed, this involution maps W_+^s to W_-^s and W_+^u to W_-^u.

Of course, condition (1.11) is quite restrictive. Making estimates more carefully gives a better condition. However, our method certainly doesn't work if one of the rods is much shorter than the other.

Nonintegrability of the physical double pendulum was first proved by Burov [14] by using the Poincaré-Melnikov-Arnold method. However, his method works only for a very special physical pendulum which is close to a direct product of a mathematical pendulum and a rotator. Hence this result doesn't work for the classical double pendulum. In principle, it is possible to use the Poincaré-Melnikov-Arnold method in the limit $h \to \infty$. Indeed, this is equivalent to small gravity $g \to 0$. However, the corresponding integral turns out to be very complicated. Our method seems simpler.

Henceforth the Jacobi metric and the Maupertuis action corresponding to zero energy are simply denoted by $\|\cdot\|$ and J.

2. – Existence of homoclinic orbits.

In this section, it is sufficient to assume that only conditions (H), (V) and (1.5) hold. Moreover, it isn't necessary that the strict maximum point q_0 of V is nondegenerate. The results of this section use the methods of Morse [31] and Hedlund [22] on minimizing geodesics on compact surfaces. The difference is that now the Jacobi metric is a nonreversible Finsler metric and it degenerates at the point q_0. However, under condition (1.5), the proofs are practically the same as that of Morse and Hedlund.

Let d be the distance on M defined by the Jacobi metric:

$$d(a, b) = \inf\{J(\gamma) \mid \gamma \in C^1([0, 1], M), \gamma(0) = a, \gamma(1) = b\}. \tag{2.1}$$

This metric satisfies the axioms of a metric space except for symmetry when the system is nonreversible. A ball with center $a \in M$ can be defined in two ways:

$$B_\varepsilon^+(a) = \{q \in M \mid d(q, a) \leq \varepsilon\}, \qquad B_\varepsilon^-(a) = \{q \in M \mid d(a, q) \leq \varepsilon\}.$$

Both systems of balls define the same standard topology on M. Denote

$$U_\varepsilon^+ = B_\varepsilon^+(q_0), \qquad U_\varepsilon^- = B_\varepsilon^-(q_0), \qquad U_\varepsilon = U_\varepsilon^+ \cap U_\varepsilon^- . \tag{2.2}$$

The length $0 \leq J(\gamma) \leq +\infty$ can be defined for any oriented continuous curve $\gamma \in C^0([0, 1], M)$ by the formula

$$J(\gamma) = \sup\left\{\sum_{i=1}^n d(\gamma(t_{i-1}), \gamma(t_i)) \mid 0 = t_0 < t_1 < \ldots < t_n = 1\right\}.$$

Let $\Pi \subset C^0([0, 1], M)$ be the set of all rectifiable curves γ such that $J(\gamma) < \infty$. It is well known that the length J is lower semicontinuous on Π. Since the length of a curve is independent of the parameterization, a curve will mean an oriented nonparameterized curve. Thus curves differing by a parameterization are identified, and Π is the corresponding quotient space. Every element in Π will be represented by a curve $\gamma : [0, 1] \to M$ parameterized proportionally to the arc length. Then for any $c > 0$ the set $\{\gamma \in \Pi \mid J(\gamma) \leq c\}$ is compact in the C^0 topology.

Let $\pi : \widetilde{M} = \boldsymbol{R}^2 \to M$ be the universal covering of M. Lift the Jacobi metric to $\widetilde{M}$. The corresponding distance in $\widetilde{M}$ will be also denoted by d. Recall that a curve $\gamma : [0, 1] \to M$ is called *minimizing* if its lift $\tilde{\gamma} : [0, 1] \to \widetilde{M}$ to the covering plane $\widetilde{M}$ minimizes the distance d between any of its points. Thus $J(\gamma|_{[a, b]}) = d(\tilde{\gamma}(a), \tilde{\gamma}(b))$ for any $0 \leq a < b \leq 1$. Any minimizer is a geodesic of the Jacobi metric everywhere, except the point q_0, where the metric degenerates.

Just as in Riemannian geometry, any two minimizers $\alpha, \alpha' : [0, 1] \to \widetilde{M}$ have the following nonintersection property.

LEMMA 2.3. – *If there exist $0 < a < b < 1$ and $0 < a' < b' < 1$ such that $\alpha(a) = \alpha'(a)$, $\alpha(b) = \alpha'(b)$ and the sets $\alpha([a, b])$, $\alpha'([a', b'])$ are contained in $\widetilde{M} \setminus \pi^{-1}(q_0)$, then, up to a reparameterization, $\alpha|_{[a, b]} = \alpha'|_{[a', b']}$.* ■

The ordering is important because the metric is nonreversible.

The metric d is complete on $\widetilde{M}$, i.e. closed bounded sets are compact. Hence any two points in $\widetilde{M}$ can be connected by a minimizer. The standard proof of the Hopf-Rinov theorem doesn't work in the present setting, because the geodesic flow isn't complete: geodesics entering q_0 can't be continued further. They are asymptotic trajectories to q_0 [29]. This yields the following result which (for natural systems) is essentially due to Kozlov [29].

LEMMA 2.4. – *Any two points $a, b \in M$ can be connected by a minimizing curve γ from a given homotopy class. The curve γ is either a geodesic (i.e. trajectory of zero energy) or a broken geodesic (i.e. chain of trajectories of zero energy) $\alpha \cup \gamma_1 \cup \ldots \cup \gamma_n \cup \beta$, where α connects a with q_0, β connects q_0 with b, and γ_i are geodesic loops with origin q_0, i.e. trajectories homoclinic to q_0. Any free homotopy class of closed curves in M contains a minimizer which is either a closed geodesic (periodic trajectory of zero energy) or a chain of homoclinics.*

In particular, any point $a \in M$ can be connected with q_0 by minimizing asymptotic trajectories $\gamma_a^+ : [0, +\infty) \to M$ and $\gamma_a^- : (-\infty, 0] \to M$ of zero energy such that $\gamma_a^\pm(0) = a$ and $\gamma_a^\pm(\pm\infty) = q_0$. Any homotopy class of loops with origin q_0 contains a minimizer which is a homoclinic or a chain of homoclinics.

Now suppose that condition (1.5) is satisfied. Take some $\delta \in (0, c_1 - c_0)$ and let $U = U_{\delta/4}$ be the set defined in (2.2).

LEMMA 2.5. – *Let $\Gamma^0 \subset \Gamma$ be the set of closed curves $\gamma \in \Gamma$ such that $J(\gamma) = c_0$. Then any $\gamma \in \Gamma^0$ is contained in $M \backslash U$, is a minimizing geodesic, has no self-intersections and $\gamma \cap \gamma' = \emptyset$ for different $\gamma, \gamma' \in \Gamma^0$. The set Γ^0 is compact in Π if every geodesic is parameterized by the arc length.*

PROOF. – The first statement follows from Lemma 2.4 since any curve $\gamma \in \Gamma$ with $J(\gamma) < c_0 + \delta/2$ doesn't intersect U. The fact that any $\gamma \in \Gamma^0$ is minimizing was proved by Morse [31] for geodesics on a closed surface of genus greater than one, and by Hedlund [22] for geodesics on a torus. The same proof carries over in the present setting. Lemma 2.3 yields the nonintersection property. ■

Morse also established the existence of heteroclinics connecting neighboring geodesics in Γ^0. However, this fact is not needed here.

COROLLARY 2.6. – *There exists a closed cylinder $N \subset M$ bounded by minimizing closed geodesics $\gamma_{\pm} \in \Gamma^0$ (possibly with $\gamma_+ = \gamma_-$) such that $q_0 \in N \backslash \partial N$, $J(\gamma_{\pm}) = c_0$ and $J(\gamma) > c_0$ for any curve $\gamma \in \Gamma$ such that $\gamma \subset N$ and $\gamma \neq \gamma_{\pm}$.*

Corollary 2.6 follows from Lemma 2.5. An orientation of M makes it possible to speak of the left and right sides of an oriented curve. Let γ_- be the right most geodesic from Γ^0 having q_0 on its right side, and define γ_+ similarly. Then the oriented geodesic γ_- is the left boundary of N, and γ_+ the right boundary. When the curves $\gamma_{\pm}$ coincide: $\gamma_- = \gamma_+$, then M is obtained from N by gluing the opposite faces of the cylinder. ■

Let $\Omega^0 \subset \Omega$ be the set of minimizing homoclinics:

$$\Omega^0 = \{\gamma \in \Omega \,|\, J(\gamma) = c_1\}.$$

The curves in Ω^0 are non self-intersecting and don't intersect each other (except at the common point q_0) and the boundary curves $\gamma_{\pm}$ of the cylinder N. The set Ω^0 is compact if every homoclinic is parameterized by the arc length. Let σ_+ and σ_- be the rightmost and the leftmost minimizing homoclinics from the set Ω^0 respectively. Let N_+ and N_- be compact cylinders in N bounded by the closed curves σ_+ and γ_+, and γ_- and σ_- respectively. When there is a unique minimizing homoclinic in Ω^0, the curves $\sigma_{\pm}$ coincide and $N = N_+ \cup N_-$. However, the possibility that $\sigma_+ \neq \sigma_-$ can't be ruled out.

Let $\pi\colon S\subset \widetilde{M}\to N$ be the universal covering of the cylinder N and $\pi\colon S_\pm\subset S\to N_\pm$ the coverings of the cylinders $N_\pm$. Then $S_\pm$ are infinite strips bounded by the curves $\beta_\pm=\pi^{-1}(\gamma_\pm)$ and $\alpha_\pm=\pi^{-1}(\sigma_\pm)$ respectively. Let T be the covering transformation of $\widetilde{M}$ corresponding to the homotopy class Γ. Then $\pi\circ T=\pi$ and $N_\pm=S_\pm/T$. If the curves $\alpha_\pm$ and $\beta_\pm$ are parameterized by the arc length, then $T\alpha_\pm(t)\equiv\alpha_\pm(t+c_1)$ and $T\beta_\pm(t)\equiv\beta_\pm(t+c_0)$. Fix some point $P_0\in S$ such that $\pi(P_0)=q_0$. Then $\pi^{-1}(q_0)=\{P_k\}_{k\in\boldsymbol{Z}}$ and $\pi^{-1}(U)=\cup_k V_k$, where $P_k=T^k(P_0)$ and $V_k=T^k(V_0)$ is a neighborhood of P_k. The curves $\alpha_\pm$ are broken geodesics: they are not smooth at the points P_k.

The distance $d_\pm$ on $S_\pm$, defined by a formula similar to (2.1), is complete. Hence any points $a,\,b\in S_\pm$ can be connected by a minimizing curve $\gamma\subset S_\pm$ of length $d_\pm(a,\,b)$. Although the boundary $\partial S_\pm$ consists of minimizing geodesics except at the points P_k, contrary to the reversible case, γ can have common points with the boundary. However, if, for example, $b=T^k a$ with $k\geqslant 0$, then by Lemma 2.3, γ will not touch $\beta_\pm$, and will not touch $\alpha_\pm$ except maybe at one of the break points P_k. Thus γ is a geodesic everywhere in $S_\pm\setminus\{P_k\}_{k\in\boldsymbol{Z}}$.

In order to prove Proposition 1.7, recall the notation introduced in § 1. Let $\Omega_k^\pm\in\pi_1(M,\,q_0)$ be the class of loops with origin q_0 which are contained in $N_\pm$ and homotopic to curves from Ω iterated k times. Denote

$$c_k^\pm=\inf_{\Omega_k^\pm} J\,,\qquad k\in\boldsymbol{Z}\setminus\{0\}\,.$$

By definition, $c_k^\pm=d_\pm(P_0,\,P_k)$, $c_{-k}^\pm=d_\pm(P_k,\,P_0)$ for $k\in\boldsymbol{N}$, and $c_1^\pm=c_1$. A minimizing curve from the class $\Omega_k^\pm$ corresponds to a curve of length $c_k^\pm$ connecting the points P_0 and P_k in $S_\pm$. If the system is reversible, then $c_k^\pm=c_{-k}^\pm$.

Using the intersection property, Morse proved [31] that if Γ_k is the free homotopy class of curves in Γ iterated k times, then

$$\inf_{\Gamma_k} J=kc_0\,,\qquad k\in\boldsymbol{N}\,. \tag{2.7}$$

Probably this fact goes back to the last century. Exactly the same proof yields:

LEMMA 2.8. – *For $k\in\boldsymbol{N}$, the function $k\to c_k^\pm$ is increasing and concave*:

$$c_k^\pm\geqslant c_{k-1}^\pm+c_0\,,\qquad c_k^\pm-c_{k-1}^\pm\geqslant c_{k+1}^\pm-c_k^\pm\,. \tag{2.9}$$

REMARK. – Condition (1.5) is not needed here. Moreover, such result holds for any metric satisfying triangle inequality and defining the standard topology on an orientable two-dimensional surface. Inequalities similar to (2.9) are also common in the Aubry-Mather theory of monotone twist maps [3]. Inequality (2.9) was proved by Caldiroli and JeanJean [15] for the case of a point moving in the plane in a force field with a singular potential. Hedlund's example [22] shows that Lemma 2.8 doesn't hold for $\dim M \geq 3$.

PROOF OF LEMMA 2.8. – The proof is the same as that of (2.7) [31]. For simplicity write $c_k = c_k^{\pm}$. It is sufficient to show that for any $k > 1$ there exists $\omega \in \Gamma$ such that

$$c_{k+1} - c_k \leq J(\omega) \leq c_k - c_{k-1} . \tag{2.10}$$

Let $\sigma:[0,1] \to S_{\pm}$ be a minimizer connecting P_0 with P_k in $S_{\pm}$. Then $J(\sigma) = c_k$. There exist $s < t$ such that $\sigma(t) = T\sigma(s)$. Indeed, this means the existence of a self-intersection point of the curve $\pi(\sigma) \in \Omega_k^{\pm}$ such that the corresponding loop goes about the cylinder $N_{\pm}$ once and thus belongs to Γ. Thus the curve $\sigma|_{[0,s]} \cup T^{-1}\sigma|_{[t,c_k]}$ connects P_0 with P_{k-1} and the curve $\sigma|_{[0,t]} \cup T\sigma|_{[s,t]} \cup T\sigma|_{[t,c_k]}$ connects P_0 with P_{k+1}. Inequality (2.11) follows immediately with $\omega = \pi\sigma|_{[s,t]}$. ■

Now under condition (1.5) the existence of a minimizing homoclinic orbit to the equilibrium q_0 in the class $\Omega_k^{\pm}$ will be proved for all large k. A modification of the approach of Caldiroli and JeanJean [15] will be used. Let $D_{\pm} = d_{\pm}(P_0, \beta_{\pm}) + d_{\pm}(\beta_{\pm}, P_0)$ Then

$$kc_0 + D_{\pm} \geq c_k^{\pm} \geq kc_0 + \delta . \tag{2.11}$$

Let

$$n_{\pm} = \sup\{k \in \boldsymbol{N} | c_k^{\pm} = kc_1\} .$$

By (2.11), $n_{\pm}$ is finite: $1 \leq n_{\pm} \leq D_{\pm}/\delta$. Let $\nu_{\pm} = (n_{\pm} + 1)\, c_1^{\pm} - c_{n+1}^{\pm} > 0$.

PROPOSITION 2.12. – *For $k > n_{\pm}$ and any $i, j \in \boldsymbol{Z} \backslash \{0\}$ such that $i + j = k$,*

$$c_i^{\pm} + c_j^{\pm} \geq c_k^{\pm} + \nu_{\pm} . \tag{2.13}$$

This is a general property of concave functions. However, for completeness the proof will be included. Again $\pm$ will be dropped from the notation.

PROOF. – For all $k \geq n$,

$$c_{k+1} \leq \frac{k+1}{k} c_k - \frac{\nu n}{k} . \tag{2.14}$$

Since $c_n = nc_1$ and $c_{n+1} < (n+1)\, c_1$, inequality (2.14) holds for $k = n$. Suppose that (2.14) is already proved for all integers less than k. Then by (2.9),

$$c_{k+1} < 2c_k - c_{k-1} < 2c_k - \left(c_k + \frac{\nu n}{k-1}\right)\frac{k-1}{k} = c_k\,\frac{k+1}{k} - \frac{\nu n}{k}\,.$$

Inequality (2.14) is proved. Induction yields the inequalities:

$$c_k \leqslant \frac{k}{i}\,c_i - \frac{k-i}{i}\,n\nu \tag{2.15}$$

for $k \geqslant i \geqslant n$, and

$$c_k \leqslant \frac{k}{i}\,c_i - (k-n)\,\nu \tag{2.16}$$

for $n \geqslant i > 0$.

Inequality (2.13) obviously holds if $i < 0$ or $j < 0$. First suppose that $k > i > n$ and $k > j > n$. Then by (2.15)

$$c_i \geqslant \frac{i}{k}\,c_k + \frac{j}{k}\,n\nu\,, \qquad c_j \geqslant \frac{j}{k}\,c_k + \frac{i}{k}\,n\nu\,.$$

Therefore,

$$c_i + c_j \geqslant c_k + n\nu\,.$$

Now suppose that $1 \leqslant i \leqslant n$ and $k > j > n$. Then by (2.15) and (2.16),

$$c_i + c_j \geqslant \frac{i}{k}\,(c_k + (k-n)\,\nu) + \frac{j}{k}\,c_k + \frac{i}{k}\,n\nu = c_k + i\nu\,.$$

Finally, suppose that $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$. Then

$$c_i + c_j \geqslant \frac{i}{k}\,(c_k + (k-n)\,\nu) + \frac{j}{k}\,(c_k + (k-n)\,\nu) = c_k + (k-n)\,\nu\,.$$

Proposition 2.12 is proved. ∎

Take $\varepsilon > 0$ so small that $U_\varepsilon^- \cup U_\varepsilon^+ \subset U$, where the notations from (2.2) are used. Let $0 < \varrho < \min(\varepsilon/2, \nu/4)$ and $U_\varrho = U_\varrho^- \cap U_\varrho^+$. Denote by W_0 the connected component of the set $\pi^{-1}(U_\varrho)$ containing P_0. Then $W_k = T^k W_0$ is a ϱ-neighborhood of the point P_k.

PROPOSITION 2.17. – *For $k > n_\pm$, any curve σ connecting P_0 with P_k in $S_\pm$ such that $J(\sigma) \leqslant c_k^\pm + \varrho$ can't pass through the neighborhoods W_i with $i \neq 0, k$.*

PROOF. – If σ passes through some point $q \in W_i$ with $i \neq 0, k$, then q can be connected with P_i and P_i with q by curves in $N_\pm$ of length $\leqslant \varrho$. This gives a curve α connecting the points P_0 and P_i, and the curve β connecting the points P_i and P_k. Moreover

$$J(\alpha) + J(\beta) = J(\sigma) + 2\varrho \leqslant c_k^\pm + 3\varrho .$$

But by Proposition 2.12,

$$c_k + 3\varrho \geqslant J(\alpha) + J(\beta) \geqslant c_i^\pm + c_{k-i}^\pm \geqslant c_k^\pm + \nu_\pm ,$$

which is a contradiction. ■

In particular, any minimizer in the homotopy class $\Omega_k^\pm$ is homoclinic orbit. The existence of a homoclinic in a homotopy class $\Omega_k^\pm$ under the condition of Proposition 2.12 is a particular case of a result of [6].

COROLLARY 2.18. – *Any homotopy class $\Omega_k^\pm$ with $k > n_\pm$ contains a minimizing homoclinic $\gamma_k^\pm \subset N_\pm$ such that $J(\gamma_k^\pm) = c_k^\pm$.* ■

Thus Proposition 1.7 is proved (see Figure 1). If the system is analytic, the homoclinics $\gamma_k^\pm$ are isolated and topologically transversal. For a smooth system there can be a continuum of minimizing homoclinics in $\Omega_k^\pm$.

Denote $U^\pm = U_\varepsilon^\pm$ and let $\Sigma^- = \partial U^-$, $\Sigma^+ = \partial U^+$. Take any minimizing homoclinic $\gamma \in \Omega_k^\pm$ and parameterize $\gamma : [0, 1] \to N_\pm$ proportionally to the arc length. Then $\gamma(\varepsilon / c_k^\pm) \in \Sigma^-$ and $\gamma(1 - \varepsilon / c_k^\pm) \in \Sigma^+$ are respectively the first intersection point of γ with the curve Σ^- and the last intersection point with Σ^+. By Proposition 2, $\gamma(t) \in M_\varrho = M \backslash U_\varrho$ for $t \in [\varepsilon / c_k^\pm, 1 - \varepsilon / c_k^\pm]$. Let $A_k^\pm \subset \Sigma^-$ and $B_k^\pm \subset \Sigma^+$ be the sets of first and last intersection points with Σ^- and Σ^+ for all minimizing homoclinics $\gamma \in \Omega_k^\pm$. The sets $A_k^\pm$ and $B_k^\pm$ are closed and

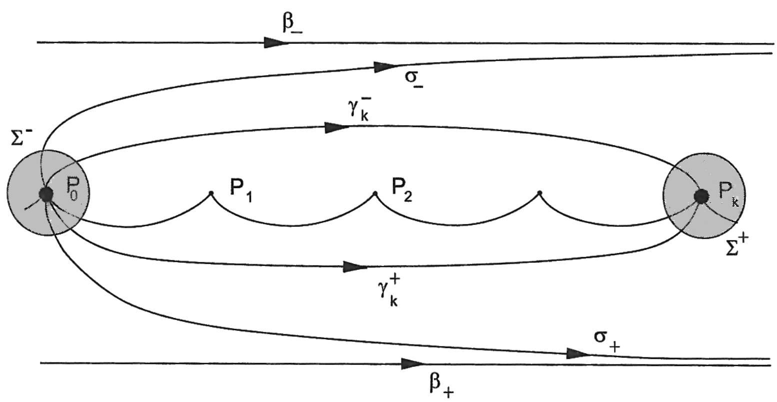

Fig. 1. – Minimizing homoclinics and heteroclinics.

$A_k^{\pm} \cap A_i^{\pm} = \emptyset$ and $B_k^{\pm} \cap B_i^{\pm} = \emptyset$ for $k \neq i$. For a reversible system, $A_k^{\pm} \cap B_i^{\pm} = \emptyset$ except for $k = -i$, when $A_k^{\pm} = B_{-k}^{\pm}$. Similarly, let A_1 and B_1 be the sets of first and last points in $\Sigma^{\pm}$ respectively of minimizing homoclinics from the class Ω. Introduce the clockwise cyclic order on the circles $\Sigma^{\pm}$. Lemma 2.3 yields

PROPOSITION 2.19. – *The following sequences of closed sets in* $\Sigma^{\pm}$ *are well ordered*:

$$\dots < A_{k+1}^- < A_k^- < \dots < A_{n_-}^- < A_1 < A_{n_+}^+ < \dots < A_k^+ < A_{k+1}^+ < \dots$$

$$\dots < B_{k+1}^+ < B_k^+ < \dots < B_{n_+}^+ < B_1 < B_{n_-}^- < \dots < B_k^- < B_{k+1}^- < \dots .$$

The existence of minimizing heteroclinics from q_0 to $\gamma_{\pm}$ under condition (1.4) was established in [11, 12, 35] for a natural system on a torus. A similar result was proved by Morse [31] for the geodesic problem. The same proof works in the present setting and yields:

PROPOSITION 2.20. – *There exist minimizing heteroclinic orbits* $\sigma_{\pm}$ *from the equilibrium* q_0 *to the periodic orbits* γ_+, γ_-, *and minimizing heteroclinic orbits* $\tau_{\pm}$ *from these periodic orbits to the equilibrium.*

See Figure 1. If these heteroclinic orbits are parameterized by the arc length so that $\sigma_{\pm} : [0, \infty) \to N_{\pm}$ and $\tau_{\pm} : (-\infty, 0] \to N_{\pm}$, then $\sigma_{\pm}(t) \in M_{\varrho} = M \backslash U_{\varrho}$ for $t \geqslant \varepsilon$, $\tau_{\pm}(t) \in M_{\varrho}$ for $t \leqslant -\varepsilon$, and $\sigma_{\pm}(\varepsilon) \in \Sigma^-$, $\tau_{\pm}(-\varepsilon) \in \Sigma^+$. Let $A_{\infty}^{\pm} \subset \Sigma^-$ be the set of first intersection points $\sigma_{\pm}(\varepsilon)$ with Σ^- for minimizing heteroclinics from the equilibrium q_0 to the orbit $\gamma_{\pm}$. Similarly, $B_{\infty}^{\pm}$ is the set of last intersection points with Σ^+ for minimizing heteroclinics from the orbit $\gamma_{\pm}$ to the equilibrium.

PROPOSITION 2.21. – *The sets* $A_{\infty}^{\pm}$ *and* $B_{\infty}^{\pm}$ *are closed. For all* k,

$$A_{\infty}^- < A_k^- < A_k^+ < A_{\infty}^+ , \qquad B_{\infty}^+ < B_k^+ < B_k^- < B_{\infty}^- .$$

As $k \to \infty$, *the set* A_k^+ *tends to* $\inf A_{\infty}^+$, *the set* A_k^- *to* $\sup A_{\infty}^-$, *and similarly for* B_k^{∞}. *In the reversible case* $B_{\infty}^- < A_{\infty}^-$ *and* $A_{\infty}^+ < B_{\infty}^+$.

The last statement follows from the fact that when $k \to \infty$, minimizing homoclinics, if represented by minimizers connecting P_0 with P_k in $S_{\pm}$, tend to minimizing heteroclinics from the equilibrium q_0 to periodic orbits $\gamma_{\pm}$. Similarly, if represented by minimizers connecting P_{-k} with P_0 in $S_{\pm}$, they tend to minimizing heteroclinics from the periodic orbits $\gamma_{\pm}$ to the equilibrium q_0. ■

These sets will be used in § 4 to get an improvement of Theorem 1.4.

3. – Boundary value problem.

Suppose that conditions (V) and (S) hold: q_0 is a point of nondegenerate maximum of the function V and the eigenvalues $0 < \lambda_1 < \lambda_2$ are real and different. In this section the existence and uniqueness of orbits with small energy h connecting two points a, b in a neighborhood of q_0 will be proved. The sign of h depends on the position of the points a, b. Let the curves $\Delta^{\pm} \subset U^{\pm}$ passing through q_0 be the unions of the trajectories of asymptotic orbits with the characteristic exponent $\mp\lambda_2$ respectively. Then $\Delta^{\pm}$ divides $U^{\pm}$ into two components $U_+^{\pm}$ and $U_-^{\pm}$. The energy h will be negative if a and b both lie either to the right or to the left of Δ^+ and Δ^-, respectively, and positive if they lie on different sides. The case when one of the points lies on $\Delta^{\pm}$ is more complicated, and won't be consided.

This result certainly isn't new: a similar proposition must have been used by Turayev and Shilnikov [39]. However, [39] contains no formulation of such a result and, as far as we know, the proof was never given.

In the next section this local proposition will be used to glue together the homoclinics obtained in § 2.

Under condition (V), the equilibrium z_0 is hyperbolic. Hence the existence of the local stable and unstable manifolds $W^{s,u}_{\mathrm{loc}}$ of the equilibrium follows from the Hadamard-Perron theorem [1]. The projections of $W^{s,u}_{\mathrm{loc}}$ to M are diffeomorphisms of small neighborhoods of z_0 in W^s_{loc} and W^u_{loc} to a neighborhood U of q_0 in M. It is well known that $W^{s,u}_{\mathrm{loc}}$ are Lagrangian manifolds, i.e., the restrictions of the 1-form $\langle p, dq\rangle$ to $W^{s,u}_{\mathrm{loc}}$ are closed. Hence these manifolds are defined by generating function $s^{\pm}$ on U such that $s^{\pm}(q_0) = 0$:

$$W^u_{\mathrm{loc}} = \{(p, q)\,|\,p = \nabla s^-(q),\ q \in U\}, \qquad W^s_{\mathrm{loc}} = \{(p, q)\,|\,p = -\nabla s^+(q),\ q \in U\}.$$

For a reversible system, $s^+ = s^-$.

The classical calculus of variations implies that $s^+(q) = d(q, q_0)$ and $s^-(q) = d(q_0, q)$, where the distance d is defined by the Jacobi metric. Indeed, s^- satisfies the Hamilton-Jacobi equation $H(\nabla s^-(q), q) = 0$. By the definition (1.2) of the Jacobi metric,

$$\|\dot q\| = \max\{\langle p, \dot q\rangle\,|\,H(p, q) = 0\} \geqslant \langle \nabla s^-(q), \dot q\rangle = \dot s^-(q),$$

and equality holds iff $\dot q = \lambda H_p(\nabla s^-(q), q)$ for some $\lambda > 0$

The functions $s^{\pm}$ each have a nondegenerate minimum at the point q_0. For small $\varepsilon > 0$, the ε-balls $U^{\pm} = U^{\pm}_{\varepsilon}$ are disks in M with smooth boundaries $\Sigma^{\pm}$. The calculation above yields the following analogue of the Gauss lemma. For a generalization see [5].

LEMMA 3.1. – *For any point $a \in U^{\pm}$, there exist unique trajectories*

$\gamma_a^+ : [0, \infty) \to U^+$ and $\gamma_a^- : (-\infty, 0] \to U^-$ with zero energy such that

$$\lim_{t \to \pm\infty} \gamma_a^{\pm}(t) = q_0$$

and $\gamma_a^{\pm}(0) = a$. The curves $\gamma_a^{\pm}$ are minimizing: $J(\gamma_a^{\pm}) = s^{\pm}(a)$. ■

For the trajectories $\gamma_a^{\pm}(t)$, let $z_a^{\pm}(t) = (p_a^{\pm}(t), \gamma_a^{\pm}(t))$ be the corresponding orbits in the phase space. Then $z_a^+(t) \in W^s_{\mathrm{loc}}$ and $z_a^-(t) \in W^u_{\mathrm{loc}}$. For $a \in \Sigma^{\pm}$, the curves $\gamma_a^{\pm}$ intersect the circles $\Sigma^{\pm}$ orthogonally: $p_a^{\pm}(0) \perp T_a \Sigma^{\pm}$.

The next lemma gives a solution of the boundary value problem similar to the problem studied by Shilnikov [38].

LEMMA 3.2. – *Let $T > 0$ be sufficiently large. Then for any points $a \in U^+$, $b \in U^-$ and $\tau \geqslant T$ there exists a unique trajectory*

$$z(t) = (p(t), q(t)) = f(a, b, \tau, t), \qquad (\tau, t) \in D_T = \{(\tau, t) \mid \tau \geqslant T,\ 0 \leqslant t \leqslant \tau\},$$

such that $q(0) = a$ and $q(\tau) = b$. The map f is C^2 on $U^+ \times U^- \times D_T$. Moreover, $z(0) = f(a, b, \tau, 0) \to z_a^+(0)$ and $z(\tau) = f(a, b, \tau, \tau) \to z_b^-(0)$ as $\tau \to \infty$ uniformly in $(a, b) \in U^+ \times U^-$.

Lemma 3.2 follows from the results of [38]. It follows also from the λ-lemma [33] and the fact that for $a, b \in U$ the planes $X = T_a^* M$ and $Y = T_b^* M$ intersect the invariant manifolds W^u_{loc} and W^s_{loc} transversally.

Later an estimate for $z(t)$ for all $t \in [0, \tau]$ will be needed:

$$f(a, b, \tau, t) = z_a^+(t) + z_b^-(t - \tau) + e^{-\lambda_1 \tau} \phi(a, b, \tau, t). \tag{3.3}$$

where $\|\phi\|_{C^1}$ is uniformly bounded on $U^+ \times U^- \times D_T$. Of course, when writing this formula, it is assumed that some local coordinates such that $z_0 = 0$ are chosen.

In [38] a weaker estimate was proved. Representation (3.3) doesn't follow from the usual λ-lemma, but it can be deduced from the strong λ-lemma proved by Deng [18]. Consider the differential equation $\dot{z} = v(z)$, where v is a C^2 vector field in a neighborhood of $0 \in \boldsymbol{R}^n$. Let g_t, $t \in \boldsymbol{R}$, be the phase flow. Suppose that $v(0) = 0$ and the matrix $D_z v(0)$ has no eigenvalues on the imaginary axis. Assume that there are eigenvalues both with positive and negative real parts. Let $\lambda = \min |\operatorname{Re} \operatorname{Spec} D_z v(0)|$. The system has nonempty stable W^s and unstable W^u local invariant manifolds at the equilibrium 0.

LEMMA 3.4. – *Let X, Y be manifolds in $\boldsymbol{R}^n$ intersecting the manifolds W^s, W^u, respectively, transversally at some points x_0 and y_0. Then for sufficiently large $T > 0$ and any $\tau \geqslant T$ there exists a solution $z(t) = g_t(z(0))$, $0 \leqslant t \leqslant \tau$, such that:*

- *$z(0) \in X$ and $z(\tau) \in Y$;*

• *there is a representation*

$$z(t) = g_t(x_0) + g_{t-\tau}(y_0) + e^{-\lambda\tau}\phi(\tau, t), \tag{3.5}$$

where ϕ is C^1 uniformly bounded on D_T, i.e. there exists a constant $C > 0$ such that

$$\|\phi\|_{C^1(D_T, \boldsymbol{R}^n)} \leqslant C, \qquad D_T = \{(\tau, t) \mid \tau \geqslant T, \qquad 0 \leqslant t \leqslant \tau\};$$

• *If the manifolds X, Y smoothly depend on a parameter c taking values in a compact manifold Z, then $\phi(\tau, t, c)$ is a C^1 function of $(t, \tau, c) \in D_T \times Z$, and $\|\phi\|_{C^1(D_T \times Z, \boldsymbol{R}^n)} \leqslant C$.*

A little weaker statement is proved in [38]. Lemma 3.4 can be deduced also from the exponential expansion proved by Deng [18].

PROOF OF LEMMA 3.4. – In a neighborhood $N = W^u \times W^s$ of the equilibrium 0 in $\boldsymbol{R}^n$, the coordinates $x \in W^u$ and $y \in W^s$ will be used. Set $z = (x, y)$. Suppose that the manifold X intersecting the manifold W^s at the point y_0 is given by the equation $y = f(x)$, where $x \in W^u$ and $y \in W^s$, and the manifold Y by the equation $x = h(y)$. Then $f(0) = y_0 \in X \cap W^s$ and $h(0) = x_0 \in Y \cap W^u$. Let the manifold $g_t(X) \cap N$ [33] be given by the equation $y = f_t(x)$. The strong λ-lemma [18] says that $\|f_t\|_{C^1} \to 0$ exponentially: $\|f_t\|_{C^1} \leqslant Ce^{-\lambda t}$ for $t \geqslant 0$ and such an estimate holds for the derivative $D_t f_t$. An analogous estimate $\|h_t\|_{C^1} \leqslant Ce^{\lambda t}$ also holds for the equation $x = h_t(y)$ of the manifold $g_t(Y) \cap N$ for $t \leqslant 0$.

According to the boundary conditions, $z(t) \in g^t(X) \cap g^{t-\tau}(Y)$. Thus $z(t) = (x, y)$ with x, y satisfying the equations

$$y = f_t(x), \qquad x = h_{t-\tau}(y).$$

By the above estimates and the implicit function theorem, equation $y = f_t(h_{t-\tau}(y))$ has a unique solution $y = g_t(y_0) + \eta(\tau, t)$ for any $t \in [0, \tau]$. Indeed,

$$y = f_t(h_{t-\tau}(y)) = g_t(y_0) + [f_t(h_{t-\tau}(y)) - f_t(0)] = g_t(y_0) + \Phi_{\tau, t}(y).$$

The term $\Phi_{\tau, t}(y)$ is quadratically small if f_t and $h_{t-\tau}$ are small: $\|\Phi_{\tau, t}\|_{C^1} \leqslant \|f_t\|_{C^1}\|h_{t-\tau}\|_{C^1}$. Hence $\|\Phi_{\tau, t}\|_{C^1} \leqslant C^2 e^{-\lambda\tau}$ and

$$|\eta(\tau, t)| \leqslant C^2 e^{-\lambda\tau}.$$

A similar inequality holds for the derivatives of η in t, τ. The representation (3.5) is proved for the coordinate $y \in W^s$. For the coordinate x, the proof is similar. When X, Y depend on a parameter $c \in Z$, the estimates above will be uniform in c. ■

Lemma 3.4 implies Lemma 3.2 and representation (3.3). Indeed, put $X =$

$T_a^* M$ and $Y = T_b^* M$. Then $g_t(z) = z_a^+(t)$ for $z \in W_{\rm loc}^s \cap X$ and $g_t(z) = z_a^-(t)$ for $z \in W_{\rm loc}^u \cap Y$. Equation (3.3) implies the following:

COROLLARY 3.6. – *Let $S(a, b, \tau)$ be the action*

$$S(a, b, \tau) = \int_0^\tau \langle p, dq \rangle = \int_0^\tau \langle p(t), H_p(z(t)) \rangle \, dt$$

of the trajectory $z(t) = (p(t), q(t))$ in Lemma 3.2. *Then $S \in C^2(U^+ \times U^- \times [T, \infty))$ and $S(a, b, \tau) \to s^+(a) + s^-(b)$ as $\tau \to \infty$ uniformly in $(a, b) \in U^+ \times U^-$.*

Now we seek to estimate the energy $h(a, b, \tau)$ of the trajectory $z(t) = (p(t), q(t))$. The energy h is a C^2 function on $U^+ \times U^- \times [T, \infty)$. Obviously, $h \to 0$ as $\tau \to \infty$, but for the sequel it is necessary to know the sign of h.

Let $W_{\rm loc}^{uu} \subset W_{\rm loc}^u$ be the strong unstable curve of the equilibrium q_0, i.e., the invariant curve tangent to the eigenvector with the larger eigenvalue $\lambda_2 > \lambda_1$. The definition of the strong stable curve $W_{\rm loc}^{ss} \subset W_{\rm loc}^s$ is similar. They are projected to the curves $\Delta^\pm \subset M$ which divide the neighborhoods $U^\pm$ of the point q_0 into open sets $U_+^\pm$ and $U_-^\pm$. Thus $U^\pm \setminus \Delta^\pm = U_+^\pm \cup U_-^\pm$ and $W_\pm^u = \pi^{-1}(U_\pm^-) \cap W_{\rm loc}^u$ and $W_\pm^s = \pi^{-1}(U_\pm^+) \cap W_{\rm loc}^s$.

LEMMA 3.7. – *There exists a C^1 function h_0 on $U^+ \times U^-$ such that $h_0 > 0$ on $(U_+^+ \times U_-^-) \cup (U_-^+ \times U_+^-)$ and $h_0 < 0$ on $(U_+^+ \times U_+^-) \cup (U_-^+ \times U_-^-)$ and the function h on $U^+ \times U^- \times [T, \infty)$ has the form*

$$h(a, b, \tau) = e^{-\lambda_1 \tau}(h_0(a, b) + h_1(a, b, \tau)),$$

where the function $h_1 \colon U^+ \times U^- \times [T, \infty) \to \boldsymbol{R}$ is small for large τ:

$$\|h_1\|_{C^1(U^+ \times U^- \times [\tau, \infty))} \to 0 \quad \text{as } \tau \to \infty .$$

PROOF. – This follows from a calculation in normal coordinates in a neighborhood of the equilibrium. Note first that there exist local symplectic coordinates x_1, x_2, y_1, y_2 such that $W^u = \{y = 0\}$, $W^s = \{x = 0\}$, and the Hamiltonian takes the form

$$H(x, y) = \lambda_1 x_1 y_1 (1 + O(x, y)) + \lambda_2 x_2 y_2 (1 + O(x, y)) . \tag{3.8}$$

Thus x is the coordinate on W^u, and y on W^s. The coordinates can be chosen in such a way that $W^{uu} = \{x_1 = 0, y = 0\}$ and $W^{ss} = \{y_1 = 0, x = 0\}$. Moreover, we can assume that W_+^u is given by the inequality $x_1 > 0$. Then it is easy to see that W_+^s is given by the inequality $y_1 > 0$.

The Hamiltonian system on the unstable manifold takes the form

$$\dot{x}_1 = (\lambda_1 + O(x))\, x_1\,, \qquad \dot{x}_2 = O(x)\, x_1 + (\lambda_2 + O(x))\, x_2\,.$$

The first equation can be transformed to a linear form $\dot{\xi}_1 = \lambda_1 \xi_1$ by a C^1 change of variables $\xi_1 = x_1 f(x)$, with $f(0) > 0$ [1]. The phase flow g_{-t} on W^u takes the form

$$g_{-t}(x) = (e^{-\lambda_1 t} x_1 f(x),\, 0) + e^{-\lambda_1 t} G(x, t)\,, \tag{3.9}$$

where $\|G\|_{C^1} \to 0$ uniformly as $t \to \infty$. In [18] such a representation is called an exponential expansion.

A similar representation holds for the flow on the stable manifold:

$$g_t(y) = (e^{-\lambda_1 t} y_1 g(y),\, 0) + e^{-\lambda_1 t} E(y, t)\,, \qquad g(0) > 0\,, \tag{3.10}$$

where $\|E\|_{C^1} \to 0$ uniformly as $t \to \infty$.

Now put $t = \tau/2$ in (3.3) and estimate the energy H at the point $z(\tau/2)$. Denote $z_a^+(0) = (0, y) \in W^s$ and $z_b^-(0) = (x, 0) \in W^u$. By (3.3), (3.9) and (3.10),

$$z(\tau/2) = (e^{-\lambda_1 \tau/2} x_1 f(x),\, 0,\, e^{-\lambda_1 \tau/2} y_1 g(y),\, 0) + e^{-\lambda_1 \tau/2} F(x, y, \tau)\,, \tag{3.11}$$

where $\|F\|_{C^1} \to 0$ as $T \to \infty$. Substituting (3.11) into (3.8) gives

$$h(a, b, \tau) = H(z(\tau/2)) = e^{-\lambda_1 \tau} (\lambda_1 x_1 y_1 f(x) g(y) + h_1(x, y, \tau))\,,$$

where $\|h_1\|_{C^1} \to 0$ as $T \to \infty$. This completes the proof of Lemma 3.7. ∎

A similar argument was recently used by Buffoni and Séré [13] for the case of a saddle-focus equilibrium, following an earlier paper of Devaney [19]. In our setting, a result like Lemma 3.7 was probably the basis for the theorem of Turayev and Shilnikov [39], although it wasn't formulated in [39]. (This paper contains no proofs).

Let $K \subset (U^+ \backslash \Delta^+) \times (U^- \backslash \Delta^-)$ be a compact set. Then for $(a, b) \in K$ the function $h_0(a, b)$ is bounded away from zero. Thus $h(a, b, \tau)$ is monotone in τ for sufficiently large τ. Solving the equation $h = h(a, b, \tau)$ for τ yields a C^2 function
$\tau = \tau_h(a, b)$ provided h is chosen with the right sign. This gives:

PROPOSITION 3.12. – *Let $K \subset (U^+ \backslash \Delta^+) \times (U^- \backslash \Delta^-)$ be a compact connected set. There exists $\delta > 0$ such that for all $h \in [-\delta, 0)$ if $K \subset (U_+^+ \times U_+^-) \cup (U_-^+ \times U_-^-)$ and for all $h \in (0, \delta]$ if $K \subset (U_+^+ \times U_-^-) \cup (U_-^+ \times U_+^-)$ the following holds.*

- *For any point $(a, b) \in K$, there exists a unique trajectory*

$z^h_{a,b}:[0,\tau]\to U$ of energy h connecting the points a and b. As $h\to 0$, $z^h_{a,b}(0)\to z_a^+(0)$ and $z^h_{a,b}(\tau)\to z_b^-(0)$.

- *The time $\tau=\tau_h(a,b)$ is a C^2 function on K and $\tau_h\to\infty$ as $h\to 0$. Moreover*

$$\tau_h(a,b) = -\frac{\log|h|}{\lambda_1}+\mu(a,b,h), \tag{3.13}$$

where the function μ is bounded as $h\to 0$.

- *The action $f_h(a,b)=S(a,b,\tau_h(a,b))$ of this trajectory is a C^2 function on K and $f_h(a,b)\to s^+(a)+s^-(b)$ uniformly as $h\to 0$.*

Denote the projection to M of the trajectory $z^h_{a,b}$ of energy h connecting the points a, b by $\omega^h_{a,b}:[0,\tau_h(a,b)]\to M$. Then

$$f_h(a,b)=J_h(\omega^h_{a,b})\to s^+(a)+s^-(b) \quad \text{as } h\to 0\,. \tag{3.14}$$

4. – Gluing of homoclinics.

In this section the results of § 2-3 will be combined to obtain the existence of an infinite number of periodic and chaotic orbits. The variational problem for finding homoclinics in § 2 will be reformulated in a different way. Let $\Sigma^\pm=\partial U^\pm$ and $M_\varrho = M\backslash U_\varrho$. Consider the set Π of rectifiable curves $\gamma:[0,1]\to\overline{M_\varrho}$ such that $\gamma(0)\in\Sigma^-$ and $\gamma(1)\in\Sigma^+$. In fact, as in § 2, Π will denote the quotent space: curves obtained by a reparameterization are identified. The topology on Π is the C^0 topology if curves in Π are parameterized proportionally to the arc length in the Jacobi metric. The length functional J on Π is lower semicontinuous in the C^0 topology on $\Pi\subset C^0([0,1],M)$ and the set $\Pi^C=\{\gamma\in\Pi\,|\,J(\gamma)\le C\}$ is compact for any $C>0$.

Any curve $\gamma\in\Pi$ defines a curve $\tilde\gamma\in\Omega$ by connecting the points q_0 with $a=\gamma(0)$ and $b=\gamma(1)$ with q_0 by the minimizers γ_a^- and γ_b^+ given in Lemma 3.1. Thus $\tilde\gamma=\gamma_a^-\cup\gamma\cup\gamma_b^+$ and $J(\tilde\gamma)=J(\gamma)+2\varepsilon$. Let $\Pi^\pm\subset\Pi$ be the set of curves contained in $D_\pm=\overline{M_\varrho}\cap N_\pm$. Let $\Pi_k^\pm=\{\gamma\in\Pi^\pm\,|\,\tilde\gamma\in\Omega_k\}$. Proposition 2.17 and Corollary 2.18 imply the following

LEMMA 4.1. – *For $k>n_\pm$ the class $\Pi_k^\pm$ contains a minimizer $\gamma_k^\pm$ of the functional J such that*

$$J(\gamma_k^\pm)=c_k^\pm-2\varepsilon\,.$$

The curves $\tilde\gamma_k^\pm\in\Omega_k^\pm$ corresponding to minimizers $\gamma_k^\pm$ are trajectories of minimizing homoclinics from the class $\Omega_k^\pm$. Any curve $\gamma\in\Pi_k^\pm$ such that $J(\gamma)\le c_k^\pm-2\varepsilon+\varrho$ is contained in M_ϱ. For any $k\in\boldsymbol{N}$ there exists $c\in(0,\varrho]$ such that *any curve* $\gamma \in \Pi_k^\pm$ *with* $J(\gamma) \leqslant c_k^\pm - 2\varepsilon + c$ *is contained in* $D_\pm \setminus \partial D_\pm$. ■

For any $k \in \boldsymbol{N}$ such that $k > n_\pm$ let

$$\Lambda_k^\pm = \{\gamma \in \Pi_k^\pm \,|\, J(\gamma) = c_k^\pm - 2\varepsilon\}.$$

Then the sets $A_k^\pm$ and $B_k^\pm$ in Proposition 2.19 can be defined as

$$A_k^\pm = \{\gamma(0) \,|\, \gamma \in \Lambda_k^\pm\} \subset \Sigma^-, \qquad B_k^\pm = \{\gamma(1) \,|\, \gamma \in \Lambda_k^\pm\} \subset \Sigma^+. \tag{4.2}$$

Lemma 4.1 implies that the sets $A_k^\pm$ and $B_k^\pm$ are nonempty and closed for $k > n_\pm$.

To simplify the notation, set

$$\boldsymbol{Z} \times \boldsymbol{Z}_2 = \{k = (j, s) \,|\, j \in \boldsymbol{Z},\ s = \pm\}$$

and for any $k = (j, s)$ in this set write A_k and B_k instead of A_j^s and B_j^s, c_k instead of c_j^s, Π_k instead of Π_j^s and Λ_k instead of Λ_j^s.

Recall that the curves $\Delta^\pm \subset U^\pm$ dividing $U^\pm$ into two components $U_+^\pm$ and $U_-^\pm$ were defined in § 3. The intersection $\Sigma^\pm \cap \Delta^\pm$ consists of two points dividing $\Sigma^\pm$ into arcs $\Sigma_+^\pm = \Sigma^\pm \cap U_+^\pm$ and $\Sigma_-^\pm = \Sigma^\pm \cap U_-^\pm$.

Let $G \subset \boldsymbol{Z}$ be the set of indices $k \in \boldsymbol{Z} \times \boldsymbol{Z}_2$ such that A_k contains a point from the set $\Sigma^- \cap \Delta^-$ or B_k contains a point from the set $\Sigma^+ \cap \Delta^+$. There are at most four such indices k, and so G contains at most four elements. Denote

$$E = \{k = (j, s) \in \boldsymbol{Z} \times \boldsymbol{Z}_2 \,|\, j > n_s,\ k \notin G\}. \tag{4.3}$$

For any $k \in E$, denote by A_k' an arbitrary connected component of the set $A_k \subset \Sigma$ and let

$$B_k' = \{\gamma(1) \,|\, \gamma \in \Lambda_k,\ \gamma(0) \in A_k'\} \subset B_k.$$

Then $A_k' \subset \Sigma_+^-$ or $A_k' \subset \Sigma_-^-$, and $B_k' \subset \Sigma_+^+$ or $B_k' \subset \Sigma_-^+$.

REMARK. – The indices $k \in G$ are deleted because we would like to glue together minimizing curves in Π_k by connecting orbits with given energy $h \neq 0$ constructed in the previous section. For points near $\Sigma^\pm \cap \Delta^\pm$, the connecting orbits with given nonzero energy depend discontinuously on the boundary points.

Fix an arbitrary finite set $K \subset E$. Let $X_k \subset \Sigma^-$ be a small closed neighborhood of A_k' such that $X_k \cap (A_k \setminus A_k') = \emptyset$, and let $Y_k \subset \Sigma^+$ be an analogous neighborhood of B_k'. If these neighborhoods are sufficiently small, then $X_k \cap X_l = Y_k \cap Y_l = \emptyset$ for $k \neq l$, and $X_k \cap \Delta^- = Y_k \cap \Delta^+ = \emptyset$.

For the chosen finite set $K \subset E$, there exists $c \in (0, \varrho]$ such that

$$J(\gamma) > c_k - 2\varepsilon + c$$

for any $k \in K$ and any $\gamma \in \Pi_k$ such that $\gamma(0) \in \partial X_k$ or $\gamma(1) \in \partial Y_k$. Denote

$$Z_k = \{\gamma \in \Pi_k \,|\, J(\gamma) \leqslant c_k - 2\varepsilon + c,\ \gamma(0) \in X_k,\ \gamma(1) \in Y_k\}. \tag{4.4}$$

Then Z_k is compact and $\Lambda_k \subset Z_k$. The boundary ∂Z_k of the set Z_k in Π_k consists of the curves $\gamma \in \Pi$ such that $\gamma(0) \in \partial X_k$ or $\gamma(1) \in \partial Y_k$ or $J(\gamma) = c_k - 2\varepsilon + c$. If c is sufficiently small, the definition of the sets X_k and Y_k implies the following:

LEMMA 4.5. – *For any* $k \in K$,

$$\inf_{\partial Z_k} J \geqslant c_k - 2\varepsilon + c\,. \tag{4.6}$$

and $\gamma \subset D_\pm \backslash \partial D_\pm$ *for any curve* $\gamma \in Z_k$.

Now we start gluing the homoclinics. First the existence of multibump periodic orbits will be proved. Take an arbitrary $N \in \boldsymbol{Z}^+$ and an arbitrary sequence $k_i \in K$, $i = 1, \dots, N$, and extend it to a periodic sequence setting $k_{i+N} = k_i$. A sequence $\{k_i\}$ will be called *admissible* for negative and positive energy respectively if it satisfies one of the following conditions.

Connection condition for negative energy. If $X_{k_i} \subset \Sigma_+^-$ for some i, then $Y_{k_{i-1}} \subset \Sigma_+^+$, and if $X_{k_i} \subset \Sigma_-^-$, then $Y_{k_{i-1}} \subset \Sigma_-^+$.

Connection condition for positive energy. If $X_{k_i} \subset \Sigma_+^-$, then $Y_{k_{i-1}} \subset \Sigma_-^+$, and if $X_{k_i} \subset \Sigma_-^-$, then $X_{k_{i-1}} \subset \Sigma_+^+$.

Now a periodic orbit close to a sequence of minimizing homoclinics will be constructed.

THEOREM 4.7. – *For any finite set* $K \subset E$, *there exists* $\delta > 0$ *such that for any* $h \in [-\delta, \delta] \backslash \{0\}$ *and any admissible sequence* $\{k_i \in K\}_{i=1}^N$, *there exists a periodic trajectory* $z(t) = (p(t), q(t))$ *of energy* h *and a monotone periodic sequence*

$$\dots < a_i < b_i < a_{i+1} < b_{i+1} < \dots, \qquad a_{i+N} = a_i + T \quad \text{and} \quad b_{i+N} = b_i + T\,,$$

where T *is the period of the trajectory, such that for all* $i \in \boldsymbol{Z}$:

- $q(a_i) \in X_{k_i}$, $q(b_i) \in Y_{k_i}$, $q|_{[a_i, b_i]} \in Z_{k_i}$;
- $q(t) = \omega^h_{q(b_i), q(a_{i+1})}(t - b_i)$ *for all* $t \in [b_i, a_{i+1}]$;
- *there exists a constant* $C > 0$, *depending only on* K, *such that*

$$b_i - a_i \leqslant C\,, \qquad b_{i+1} - a_i + \frac{\log|h|}{\lambda_1} \leqslant C\,. \tag{4.8}$$

REMARKS. – The proof of Theorem 4.7 shows that the constant $\delta > 0$ depends only on the set K and is independent of N and of the sequence $k_i \in K$. The period T depends on h and goes to infinity like $N \log |h|$ when $h \to 0$.

The constructed periodic orbit belongs to the homotopy class Γ_r with $r = \sum_{i=1}^{N} j_i$, where j_i is determined from $k_i = (j_i, s_i)$.

The proof of Theorem 4.7 requires introducing of an appropriate function space. Consider the Cartesian product

$$Z = \Pi_{k_1} \times \ldots \times \Pi_{k_N}$$

with the product topology. The topology on Π_k was defined above. Thus Z is the set of sequences $z = (\gamma_1, \ldots, \gamma_N)$, where $\gamma_i \in \Pi_{k_i}$. Set $\gamma_{N+i} = \gamma_i$. Then any point $z \in Z$ defines a periodic sequence $\{\gamma_i\}_{i \in \mathbf{Z}}$.

Fix energy $h \in [-\delta, \delta] \setminus \{0\}$ as in Proposition 3.12. Later $\delta > 0$ will have to be decreased several times. Choose $\delta > 0$ so small that $V < -\delta$ on $\overline{M_\varrho}$. Then for $|h| < \delta$ the Jacobi metric $\|\cdot\|_h$ is positive definite on $\overline{M_\varrho}$. Hence the Maupertuis functional J_h corresponding to the energy h is well defined and lower semicontinuous on Π.

Fix some constant

$$C > \max_{k \in K} c_k - 2\varepsilon + c \,. \tag{4.9}$$

Then for any $\gamma \in \Pi$ such that $J(\gamma) \leqslant C$,

$$|J_h(\gamma) - J(\gamma)| \leqslant \lambda(\delta) \tag{4.10}$$

for some $\lambda(\delta) > 0$ independent of $\gamma \in \Pi$ and such that $\lambda(\delta) \to 0$ as $\delta \to 0$.

Indeed, for any $q \in \overline{M_\varrho}$, the set $\Sigma_h(q) = \{p \in T_q^* M \,|\, H(p, q) = h\}$ is compact and $\Sigma_h(q) \to \Sigma_0(q)$ in the Hausdorff metric as $h \to 0$. Then

$$\|\dot{q}\|_h = \max_{p \in \Sigma_h(q)} \langle p, \dot{q} \rangle \to \max_{p \in \Sigma_0(q)} \langle p, \dot{q} \rangle = \|\dot{q}\|$$

uniformly in $q \in \overline{M_\varrho}$. Hence

$$(1 - \alpha(\delta))\|\dot{q}\| \leqslant \|\dot{q}\|_h \leqslant (1 + \alpha(\delta))\|\dot{q}\| \,,$$

where $\alpha(\delta) \to 0$ as $\delta \to 0$. Hence (4.10) holds with $\lambda(\delta) = C\alpha(\delta)$.

Define the following functional Φ_h on Z:

$$\Phi_h(z) = \sum_{i=1}^{N} \left(J_h(\gamma_i) + f_h(\gamma_i(1), \gamma_{i+1}(0))\right) , \tag{4.11}$$

where the function f_h was defined in Proposition 3.12. The functional (4.11) is similar to the functional used in the Aubry-Mather theory [3, 32]. For $h = 0$,

$$\Phi_0(z) = \sum_{i=1}^{N} (J(\gamma_i) + s^+(\gamma_i(1)) + s^-(\gamma_{i+1}(0))) = \sum_{i=1}^{N} J(\gamma_i) + 2N\varepsilon \geqslant \sum_{i=1}^{N} c_{k_i}.$$

The inequality turns out to be an equality iff γ_i is a minimizer in its homotopy class Π_{k_i}. Thus the functional Φ_0 on Z has a minimum equal to

$$\min_Z \Phi_0 = \sum_{i=1}^{N} c_{k_i}.$$

By Corollary 2, the minimum is attained on the set of sequences $z = (\gamma_1, \ldots, \gamma_N)$ such that $J(\gamma_i) = c_{k_i} - 2\varepsilon$, i.e. $\gamma_i \in \Lambda_{k_i}$. Hence the set of minimum points of Φ_0 equals $\Lambda_{k_1} \times \ldots \times \Lambda_{k_N}$. Any minimum point of Φ_0 represents a sequence of homoclinics $\tilde{\gamma}_i \in \Omega_{k_i}$.

LEMMA 4.12. – *Suppose $z = (\gamma_1, \ldots, \gamma_N) \in Z$ gives a point of local minimum for the functional Φ_h, $h \in [-\delta, \delta] \setminus \{0\}$, such that $\gamma_i \subset D_\pm \setminus \partial D_\pm$ for all i. For each $i \in \boldsymbol{Z}$, connect the point $y_i = \gamma_i(1)$ with $x_{i+1} = \gamma_{i+1}(0)$ by the trajectory $\omega_i = \omega^h_{y_i, x_{i+1}}$ constructed in Proposition* 3.12. *Then the closed curve*

$$\gamma = \gamma_1 \cup \omega_1 \cup \gamma_2 \cup \omega_2 \cup \ldots \cup \gamma_N \cup \omega_N$$

is a trajectory of a periodic orbit with energy h provided that $\delta > 0$ is sufficiently small.

PROOF. – Since γ_i is a local minimum of J_h on Π_{k_i} and γ_i has no common points with the boundary $\partial D_\pm$, by the Maupertuis principle it is a trajectory of energy h. Thus it is a projection of a solution $z_i : [0, \tau_i] \to P$, $z_i(t) = (p_i(t), q_i(t))$ such that $q_i(0) = \gamma_i(0) = x_i$, $q_i(\tau_i) = \gamma_i(1) = y_i$. It is sufficient to show that for all i

$$z_i(\tau_i) = z^h_{y_i, x_{i+1}}(\tau_i), \qquad z_i(0) = z^h_{y_{i-1}, x_i}(\tau_h(y_{i-1}, x_i)).$$

Then γ is a smooth curve and hence it is a trajectory of Hamilton's equations.

Thus it is sufficient to show that the momentum has no jump at the points y_i and x_i (see Figure 2). It will be shown, for example, that the momentum has no jump at the point y_i. Take a variation of the point $y_i = \gamma_i(1) \in \Sigma^+$ and a

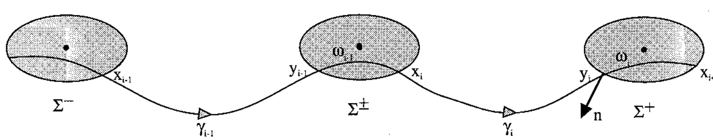

Fig. 2. – Gluing.

smooth variation of the curve γ_i, keeping the point x_i and all curves γ_j with $j \neq i$ fixed. By the first variation formula,

$$\delta J_h(\gamma_i) = \langle p_i(t), \delta q_i(t) \rangle |_0^{\tau_i} = \langle p_-, \delta y_i \rangle,$$

where $p_- = p_i(\tau_i) \in T^*_{y_i} M$. Similarly,

$$\delta J_h(\omega_i) = -\langle p_+, \delta y_i \rangle,$$

where $p_+ \in T^*_{y_i} M$ is defined by $z^h_{y_i, x_{i+1}}(0) = (p_+, y_i)$. Since $\delta y_i \in T_{y_i} \Sigma^+$ is arbitrary and $\delta J_h(\gamma_i) + \delta J_h(\omega_i) = 0$, the jump $p_+ - p_-$ of the momentum satisfies the condition

$$p_+ - p_- \perp T_{y_i} \Sigma^+, \qquad H(p_-, y_i) = H(p_+, y_i) = h. \tag{4.13}$$

Since H is convex in the momentum, for small $|h|$ and given p_+ equations (4.13) have two solutions $p_- = p_+$ and $p_- = p_+ + \lambda n$, where $n \in T^*_{y_i} M$ is the exterior normal to Σ^+ and $\lambda > 0$. Suppose that $p_+ \neq p_-$. By Proposition 3.12, for small $|h|$ the direction of the momentum p_+ is close to the interior normal $-n$ to Σ^+. Hence the direction of $p_- = p_+ + \lambda n$ is close to the exterior normal n. Since γ_i is a trajectory of energy h, $\dot\gamma_i(1)$ is directed outside Σ^+ for small $|h|$. This implies that γ_i enters the small neighborhood U_ϱ of the point q_0 and contradicts the assumption that $\gamma_i \subset M_\varrho$. ■

In view of Lemma 4.12, to establish the existence of a multibump periodic orbit it is sufficient to show that for small $|h|$ the functional Φ_h has a local minimizer $z = (\gamma_1, \ldots, \gamma_N)$ near $\Lambda_{k_1} \times \ldots \times \Lambda_{k_N}$ such that $\gamma_i \subset D_\pm \setminus \partial D_\pm$ for all i.

Let $Z_k \subset \Pi_k$ be the set defined in (4.4) and let

$$X = \{(\gamma_1, \ldots, \gamma_N) \in Z \,|\, \gamma_i \in Z_{k_i}\} = Z_{k_1} \times \ldots \times Z_{k_N}.$$

The set X is compact as a product of compact sets. By Lemma 4.5, $\gamma \subset D_\pm \setminus \partial D_\pm$ for any $\gamma \in Z_k$ and $k \in K$.

LEMMA 4.14. – *There exists $\delta > 0$ such that for $h \in [-\delta, \delta] \setminus \{0\}$ the functional Φ_h is well defined and lower semicontinuous on X.*

Indeed, the functional Φ_h is a sum of continuous and semicontinuous functions.

PROOF OF THEOREM 4.7. – By Lemma 4.14, for $h \in [-\delta, \delta] \setminus \{0\}$ the functional Φ_h has a minimum point $z \in X$. In view of Lemma 4.12, to complete the proof it remains to show that any minimum point $z = (\gamma_1, \ldots, \gamma_N)$ of Φ_h in X lies in the interior of X in Z. Since $\gamma_i \subset D_\pm \setminus \partial D_\pm$, Lemma 4.12 applies and gives the theorem. The boundary of X in Z consists of the points $z = (\gamma_1, \ldots, \gamma_N)$

with $\gamma_i \in \partial Z_{k_i}$ for some $i = 1, \dots, N$. Suppose that the minimum point z lies on the boundary, so that $\gamma_i \in \partial Z_{k_i}$ for some i. By (4.6) and (4.10),

$$J_h(\gamma_i) \geqslant c_{k_i} + c - 2\varepsilon - \lambda(\delta). \tag{4.15}$$

Perturb the curve γ_i, replacing it by an arbitrary curve $\widehat{\gamma}_i \in \Lambda_{k_i}$. The fact that the interaction between the components γ_i and $\gamma_{i\pm1}$ in (4.11) is very small for small $|h|$, implies that the new point $\widehat{z} = (\gamma_1, \dots, \widehat{\gamma}_i, \dots, \gamma_N) \in X$ satisfies $\Phi_h(\widehat{z}) < \Phi_h(z)$. Indeed, by (4.10),

$$J_h(\widehat{\gamma}_i) \leqslant c_{k_i} - 2\varepsilon + \lambda(\delta). \tag{4.16}$$

By (3.14), for $|h| \leqslant \delta$, the interaction term in (4.11) can be estimated as follows

$$|f_h(\gamma_{i-1}(1), \widehat{\gamma}_i(0)) + f_h(\widehat{\gamma}_i(1), \gamma_{i+1}(0)) - 4\varepsilon| \leqslant \mu(\delta), \tag{4.17}$$

where $\mu(\delta) \to 0$ as $\delta \to 0$. By (4.15), (4.16) and (4.17), if $\delta > 0$ is so small that $\mu(\delta) + 2\lambda(\delta) < c$, then for $h \in [-\delta, \delta] \setminus \{0\}$,

$$\Phi_h(\widehat{z}) = \Phi_h(\gamma_1, \dots, \widehat{\gamma}_i, \dots, \gamma_N) \leqslant \Phi_h(z) - c + \mu(\delta) + 2\lambda(\delta) < \Phi_h(z),$$

which is a contradiction. Theorem 4.7 is proved. ■

A similar result for infinite sequences $\{k_i \in K\}_{i \in \mathbf{Z}}$ will be proved next.

THEOREM 4.18. – *For any finite set $K \subset E$, there exists $\delta > 0$ such that for all $h \in [-\delta, \delta] \setminus \{0\}$ and any admissible sequence $\{k_i \in K\}_{i \in \mathbf{Z}}$, there exists a trajectory of energy h and an infinite monotone sequence*

$$\dots < a_{i-1} < b_{i-1} < a_i < b_i < a_{i+1} < b_{i+1} < \dots$$

satisfying the properties in Theorem 4.7.

Theorem 4.18 implies Theorem 1.4 and Proposition 1.8.

PROOF OF THEOREM 4.18. – The proof is obtained by a limit procedure which is standard in the Aubry-Mather theory [3]. Fix a constant $C > 0$ as in (4.9) and consider the infinite product

$$Q = (\Pi^C)^{\mathbf{Z}}$$

with the product topology. Let $Y \subset Q$ be the set of sequences $z = \{\gamma_i\}_{i \in \boldsymbol{Z}}$ with $\gamma_i \in Z_{k_i}$ for all $i \in \boldsymbol{Z}$. A sequence $z \in Y$ will be called minimizing if for any $\hat{z} = \{\hat{\gamma}_i\}_{i \in \boldsymbol{Z}} \in Y$ such that $\hat{\gamma}_i = \gamma_i$ for all i except belonging to a finite set $I \subset \boldsymbol{Z}$,

$$\sum_{i \in I} (J_h(\hat{\gamma}_i) + f_h(\hat{\gamma}_i(1), \hat{\gamma}_{i+1}(0)) - J_h(\gamma_i) - f_h(\gamma_i(1), \gamma_{i+1}(0))) \geqslant 0 .$$

This is a standard definition in the Aubry-Mather theory [3, 32].

Exactly as in the proof of Theorem 4.7, for any minimizer $z \in Y$, it follows that $\gamma_i \in Z_{k_i} \backslash \partial Z_{k_i}$ for all i. Thus by Lemma 4.12 any minimizing sequence z gives a trajectory of energy h satisfying the condition of Theorem 4.18. Therefore, it is sufficient to prove the existence of a minimizing sequence.

Take a sequence $\{n_i \in K\}_{i \in \boldsymbol{Z}}$ and for any $N \in \boldsymbol{Z}^+$ replace it by a periodic sequence $\{k_i\}_{i \in \boldsymbol{Z}}$ such that $k_i = n_i$ for $i = -N, \ldots, N$ and $k_{i+2N+1} = k_i$ for all $i \in \boldsymbol{Z}$. Since the sequence $\{k_i\}_{i \in \boldsymbol{Z}}$ is periodic, Theorem 4.7 applies. This gives a minimum point of the functional Φ_h on $\prod_{i=-N}^{N} Z_{k_i}$ which defines a $2N+1$-periodic sequence $z^{(N)} \in Q$. Now let $N \to \infty$. Since Q is compact in the product topology, $z^{(N)}$ has a subsequence $z^{(N_j)}$ converging to a sequence $z^{(\infty)} \in Q$. Obviously, $z^{(\infty)} \in Y$. Since for any finite set $I \subset \boldsymbol{Z}$, $\gamma_i^{(N_j)} \to \gamma_i^{(\infty)}$ for all $i \in I$, the limit sequence is minimizing. The proof of Theorem 4.18 is complete. ■

REMARK. – One can extend the above construction to glue together not only homoclinics to q_0 but also heteroclinics from q_0 to γ_+, γ_- and from γ_+, γ_- to q_0. Namely fix the energy $h \in [-\delta, \delta] \backslash \{0\}$ and consider a finite sequence $k_-, k_1, \ldots, k_N, k_+$, where $k_i \in E$ and $k_-, k_+ \in \boldsymbol{Z}_2 = \{\pm\}$. Such a sequence is called *admissible* if the sequence $\{k_i\}$ is admissible, the sets $A_\infty^{k_+}$ and $B_\infty^{k_-}$ in Proposition 2.21 have connected components that don't intersect the curves $\Delta^{\mp}$ respectively and conditions similar to the connection conditions above are satisfied. For any admissible sequence and sufficiently small $\delta > 0$, there exists a heteroclinic trajectory σ of energy h with properties as in Theorem 4.7 which is asymptotic to a periodic orbit $\alpha \subset N_{k_-}$ as $t \to -\infty$ and to a periodic orbit $\beta \subset N_{k_+}$ as $t \to +\infty$. For small h, the orbit α is close to γ_{k_-} and β is close to γ_{k_+}.

Consider, for example the sequence k_-, k_+ containing only two terms. Then the corresponding orbit σ will be heteroclinic from the periodic orbit α to the periodic orbit β and passing near the equilibrium q_0 once. In general, it will be a multibump heteroclinic passing near the equilibrium $N+1$ times.

REFERENCES

[1] V. I. ARNOLD - Y. S. IL'YASHENKO - D. V. ANOSOV, *Ordinary differential equations,* Encyclopedia of Mathematical Sciences, Vol. **1**, Springer-Verlag, 1989.

[2] V. I. ARNOLD - V. V. KOZLOV - A. I. NEISHTADT, *Mathematical aspects of classical and celestial mechanics*, Encyclopedia of Mathematical Sciences, Vol. **3**, Springer-Verlag (1989).

[3] S. AUBRY - P. Y. LEDAERON, *The discrete Frenkel-Kontorova model and its extensions, I. Exact results for the ground-states*, Physica D, 8 (1983), 381-422.

[4] G. D. BIRKHOFF, *Dynamical Systems*, Amer. Math. Soc. Colloq. Publ., IX, New York, 1927.

[5] S. V. BOLOTIN, *Libration motions of natural dynamical systems*, Vestnik Moskov. Univ. Ser. I Matem. Mekhan., **6** (1978), 72-77.

[6] S. V. BOLOTIN, *Libration Motions of Reversible Hamiltonian Systems,* Moscow State University, 1981.

[7] S. V. BOLOTIN, *The existence of homoclinic motions*, Vestnik Moskov. Univ. Matem. Mekh., **6** (1983), 98-103.

[8] S. V. BOLOTIN, *Homoclinic orbits to invariant tori of Hamiltonian systems*, Amer. Math. Soc. Transl., Ser. 2, **168** (1995), 21-90.

[9] S. V. BOLOTIN - V. V. KOZLOV, *Libration in systems with many degrees of freedom*, J. Appl. Math. Mech. (PMM), **42** (1978), 245-250.

[10] S. V. BOLOTIN, *Variational methods for constructing chaotic motions in the rigid body dynamics*, Prikl. Matem. i Mekhan., **56** (1992), 230-239.

[11] S. V. BOLOTIN, *Variational criteria for nonintegrability and chaos in Hamiltonian systems*, in: *Hamiltonian Systems: Integrability and Chaotic Behavior*, NATO ASI Series, **331**, Plenum Press (1994), 173-179.

[12] S. V. BOLOTIN - P. NEGRINI, *Variational criteria for nonintegrability,* Russian J. Math. Phys., No 1, 1998.

[13] B. BUFFONI - E. SÉRÉ, *A global condition for quasi-random behavior in a class of conservative systems,* Comm. Pure Appl. Math. (1996).

[14] A. A. BUROV, *Nonexistence of an additional integral of the problem of a planar heavy double pendulum,* Prikl. Matem. i Mekhan., **50** (1986), 168-171.

[15] P. CALDIROLI - L. JEANJEAN, *Homoclinics and heteroclinics for a class of conservative singular dynamical systems*, Preprint, 1996.

[16] K. CIELEBAK - E. SÉRÉ, *Pseudo-holomorphic curves and multiplicity of homoclinic orbits,* Duke Math. J. (1996).

[17] V. COTI ZELATI - P. H. RABINOWITZ, *Homoclinic orbits for second order Hamiltonian systems posessing superquadratic potentials,* J. Amer. Math. Soc., **4** (1991), 693-727.

[18] B. DENG, *The Shilnikov problem, exponential expansion, strong λ-lemma, C^1-linearization and homoclinic bifurcation*, J. Differ. Equat., **79** (1989), 189-231.

[19] R. L. DEVANEY, *Homoclinic orbits in Hamiltonian systems*, J. Differ. Equat. **21** (1976), 431-438.

[20] R. L. DEVANEY, *Transversal homoclinic orbits in an integrable system*, Amer. J. Math., **100** (1978), 631-642.

[21] F. GIANNONI - P. H. RABINOWITZ, *On the multiplicity of homoclinic orbits on Riemannian manifolds for a class of second order Hamiltonian systems*, NoDEA, **1** (1993), 1-49.

[22] G. A. HEDLUND, *Geodesics on a two-dimensional Riemannian manifold with periodic coefficients*, Ann. Math., **33** (1932), 719-739.
[23] W. D. KALIES - R. C. A. M. VANDERVORST, *Multitransition homoclinic and heteroclinic solutions of the extended Fisher-Kolmogorov equation,* Preprint, 1995.
[24] W. D. KALIES - J. KWAPISZ - R. C. A. M. VANDERVORST, *Homotopy classes for stable connections between Hamiltonian saddle-focus equilibria,* Preprint, 1996.
[25] A. KATOK, *Entropy and closed geodesics,* Ergod. Theor. Dynam. Syst., **2** (1982), 339-367.
[26] A. KATOK - B. HASSELBLATT, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, Cambridge University Press (1995).
[27] V. V. KOZLOV, *Topological obstructions to the integrability of natural mechanical systems*, Dokl. Akad. Nauk. SSSR, **249** (1979), 1299-1302.
[28] V. V. KOZLOV, *Integrability and nonintegrability in classical mechanics*, Uspekhi Mat. Nauk, **38** (1983), 3-67.
[29] V. V. KOZLOV, *Calculus of variations in large and classical mechanics*, Uspekhi Matem. Nauk, **40** (1985), 33-60.
[30] V. V. KOZLOV - V. V. TEN, *Topology of domains of possible motion for integrable systems*, Matem. Sbornik, **187** (1996), 59-64.
[31] M. MORSE, *A fundamental class of geodesics in any closed surface of genus greater than one*, Trans. Amer. Math. Soc., **26** (1924), 25-61.
[32] J. MATHER, *Variational construction of connecting orbits*, Ann. Inst. Fourier, **43** (1993), 1349-1386.
[33] J. PALIS - W. DE MELO, *Geometric Theory of Dynamical Systems,* Springer-Verlag, 1982.
[34] P. H. RABINOWITZ, *Homoclinics for a singular Hamiltonian system*, to appear in *Geometric Analysis and the Calculus of Variations* (J. Jost, ed.), International Press (1996), 267-296.
[35] P. H. RABINOWITZ, *Heteroclinics for a Hamiltonian system of double pendulum type*, to appear in Top. Methods in Nonlin. Analysis, Vol. 9 (1997), 41-76.
[36] E. SÉRÉ, *Existence of infinitely many homoclinics in Hamiltonian systems*, Math. Z., **209** (1992), 27-42.
[37] E. SÉRÉ, *Looking for the Bernoulli shift*, Ann. Inst. H. Poincaré, Anal. Nonlin., **10** (1993), 561-590.
[38] L. P. SHILNIKOV, *On a Poincaré-Birkhoff problem*, Math. USSR Sbornik, **3** (1967), 353-371.
[39] D. V. TURAYEV - L. P. SHILNIKOV, *On Hamiltonian systems with homoclinic curves of a saddle*, Dokl. Akad. Nauk SSSR, **304** (1989), 811-814.

S. V. Bolotin: Department of Mathematics and Mechanics,
Moscow State University

P. H. Rabinowitz: Department of Mathematics, University of Wisconsin
Madison, USA

Pervenuta in Redazione
il 2 aprile 1997 e, in forma rivista, il 16 aprile 1997