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SEZIONE SCIENTIFICA

BREVI NOTE

Generating function for a generalized function

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Sunto. - *Viene data la funzione generatrice di una funzione generalizzata introdotta dall' Autore.*

1. - In a recent paper [1] we have introduced the generalized function $F_n^{(r)}(x;\ a,\ k,\ p)$ defined by

$$F_n^{(r)}(x;\ a,\ k,\ p) = x^{-a}e^{px^r}D^n(x^{kn+a}e^{-px^r}) \tag{1.1}$$

and have studied some operational formulas connected with this generalized function. It is of great interest to introduce such a generalized function, because this function includes the Hermite, Laguerre, Bessel polynomials and the generalized function of Gould and Hopper [2], as special cases. In fact we note that

$$H_n(x) = (-1)^n F_n^{(2)}(x;\ 0,\ 0,\ 1) \tag{1.2}$$

$$L_n^a(x) = \frac{1}{n!} F_n^{(1)}(x;\ a,\ 1,\ 1) \tag{1.3}$$

$$y_n(x,\ a+2,\ b) = b^{-n} F_n^{(-1)}(x;\ a,\ 2,\ b) \tag{1.4}$$

$$H_n^r(x,\ a,\ p) = (-1)^n F_n^{(r)}(x;\ a,\ 0,\ p) \tag{1.5}$$

where $H_n(x)$, $L_n^a(x)$, $y_n(x,\ a,\ b)$ denote the polynomials of Hermite, Laguerre and Bessel, and $H_n^r(x,\ a,\ p)$ the generalized function of Gould and Hopper.

The object of this paper is to give a generating function for the generalized function $F_n^{(r)}(x;\ a,\ k,\ p)$, whereby the well known generating functions for the Hermite, Laguerre, Bessel polynomials and also for the generalized function of Gould-Hopper, are rendered intutive. A generating function for the special function $F_n^{(r)}(x;\ a-(k-1)n,\ k,\ p)$ is also obtained.

2. – If $\Phi(z)$ is derivable at $z = x$ and $\Phi(x) \neq 0$, and if

$$z = x + \omega\Phi(z),$$

then a function $f(z)$ which is derivable at $z = x$ can be expanded by Lagrange's formula

$$f(z) = f(x) + \sum_{n=1}^{\infty} \frac{\omega^n}{n!} D^{n-1}[\{\Phi(x)\}^n f'(x)].$$

Now differentiating with respect to ω and afterwards writing $F(z)$ instead of $\Phi(z)f'(z)$, we obtain the expansion [3]

$$\frac{F(z)}{1 - \omega\Phi'(z)} = \sum_{n=0}^{\infty} \frac{\omega^n}{n!} D^n[\{\Phi(x)\}^n F(x)]. \tag{2.1}$$

Let $\Phi(x) = x^k$, $F(x) = x^a e^{-px^r}$. Thus we have

$$z = x + \omega z^k.$$

Now

$$\begin{aligned}
&\sum_{n=0}^{\infty} F_n^{(r)}(x;\ a,\ k,\ p)\frac{\omega^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\omega^n}{n!} x^{-a} e^{px^r} D^n[(x^k)^n \cdot x^a e^{-px^r}] \\
&= x^{-a} e^{px^r} \sum_{n=0}^{\infty} \frac{\omega^n}{n!} D^n[(x^k)^n \cdot x^a e^{-px^r}] \\
&= x^{-a} e^{px^r} \frac{z^a e^{-pz^r}}{1 - \omega k z^{k-1}} \\
&= \left(\frac{z}{x}\right)^a (1 - \omega k z^{k-1})^{-1} e^{p(x^r - z^r)},
\end{aligned}$$

where $z = x + \omega z^k$.

Thus we get the desired generating function for $F_n^{(r)}(x;\ a, k, p)$:

$$\sum_{n=0}^{\infty} F_n^{(r)}(x;\ a,\ k,\ p)\frac{\omega^n}{n!} = \left(\frac{z}{x}\right)^a (1 - \omega k z^{k-1})^{-1} e^{p(x^r - z^r)} \tag{2.2}$$

where

$$z = x + \omega z^k.$$

It is interesting to note the following special cases:

(a) HERMITE Polynomials.

Using $r = 2$, $a = k = 0$, $p = 1$, we have from (2.2)

$$\sum_{n=0}^{\infty} F_n^{(2)}(x;\ 0,\ 0,\ 1)\frac{\omega^n}{n!} = e^{x^2-z^2}.$$

Now since $H_n(x) = (-1)^n F_n^{(2)}(x;\ 0,\ 0,\ 1)$, and $z = x + \omega$, we finally obtain

$$\sum_{n=0}^{\infty} \frac{H_n(x) \cdot (-\omega)^n}{n!} = e^{-\omega^2-2x\omega}; \tag{2.3}$$

which is the familiar generating function of the HERMITE polynomials.

b) LAGUERRE Polynomials.

Using $r = 1$, $k = p = 1$, we have from (2.2) and (1.3)

$$\sum_{n=0}^{\infty} n!\ L_n^a(x)\frac{\omega^n}{n!} = \left(\frac{z}{x}\right)^a (1-\omega)^{-1}e^{x-z}.$$

Here we have

$$z = x + \omega z \quad \text{i.e.,} \quad z = \frac{x}{1-\omega}.$$

Hence we get

$$\sum_{n=0}^{\infty} L_n^a(x)\omega^n = (1-\omega)^{-a-1}e^{\frac{-x\omega}{1-\omega}}; \tag{2.4}$$

which is the familiar generating function of the LAGUERRE polynomials.

c) BESSEL Polynomials.

Using $r = -1$, $k = 2$, $p = b$, we derive from (2.2) and (1.4)

$$\sum_{n=0}^{\infty} \frac{b^n y_n(x,\ a+2,\ b) \cdot \omega^n}{n!} = \left(\frac{z}{x}\right)^a (1-2\omega z)^{-1}e^{b(x^{-1}-z^{-1})}.$$

Here we notice that

$$z = x + \omega z^2,$$

so that we have

$$z = \frac{1}{2\omega}(1-\sqrt{1-4x\omega}),$$

because this root tends to x as ω tends to zero.

Hence we get

$$(2.5)\qquad \sum_{n=0}^{\infty} \frac{(b\omega)^n}{n!} y_n(x, a+2, b) = \\ = \left(\frac{1}{2} + \frac{1}{2}\sqrt{1-4x\omega}\right)^{-a} (1-4x\omega)^{-\frac{1}{2}} e^{\frac{b}{2x}(1-\sqrt{1-4x\omega})}$$

which is the familiar generating function for the BESSEL polynomials.

d) Generalized Function of GOULD-HOPPER.

Using $k=0$, we derive from (2.2) and (1.5)

$$\sum_{n=0}^{\infty} \frac{(-\omega)^n}{n!} H_n^r(x, a, p) = \left(\frac{z}{x}\right)^a e^{p(x^r - z^r)}$$

where

$$z = x + \omega.$$

Thus we have

$$(2.6)\qquad \sum_{n=0}^{\infty} \frac{(-\omega)^n}{n!} H_n^r(x, a, p) = x^{-a}(x+\omega)^a e^{p\{x^r - (x+\omega)^r\}};$$

which is the generating function for Gould-Hopper's function [2, p. 54].

3. - In this section we shall find the generating function for the special function $F_n^{(r)}(x; a-(k-1)n, k, p)$. From (1.1) we notice that

$$(3.1)\qquad F_n^{(r)}(x; a-(k-1)n, k, p) = x^{-a+(k-1)n} e^{px^r} D^n(x^{a+n} e^{-px^r}).$$

Now

$$(3.2)\qquad \sum_{n=0}^{\infty} F_n^{(r)}(x; a-(k-1)n, k, p) \frac{\omega^n}{n!} = \\ = x^{-a} e^{px^r} \sum_{n=0}^{\infty} \frac{(x^{k-1}\omega)^n}{n!} D^n(x^{a+n} e^{-px^r}).$$

Using the transformation $xz=1$, we obtain from (3.2)

$$\sum_{n=0}^{\infty} F_n^{(r)}(z^{-1}; a-(k-1)n, k, p) \frac{\omega^n}{n!} \\ = z^{a+1} e^{pz^{-r}} \sum_{n=0}^{\infty} \frac{(-\omega z^{2-k})^n}{n!} \left(\frac{d}{dz}\right)^n (z^{-a-1} e^{-pz^{-r}}).$$

Now by Taylor's theorem, we know that

$$\sum_{n=0}^{\infty} \frac{(-x^k t)^n}{n!} \left(\frac{d}{dx}\right)^n f(x) = f(x - x^k t).$$

Thus we have

$$\sum_{n=0}^{\infty} F_n^{(r)}(z^{-1};\ a - (k-1)n,\ k,\ p) \frac{\omega^n}{n!}$$

$$= z^{a+1} e^{pz^{-r}} (z - \omega z^{2-k})^{-a-1} e^{-p(z - \omega z^{2-k})^{-r}}.$$

In other words,

$$\sum_{n=0}^{\infty} F_n^{(r)}(x;\ a - (k-1)n,\ k,\ p) \frac{\omega^n}{n!} \tag{3.3}$$

$$= (1 - \omega x^{k-1})^{-a-1} e^{px^r \{1 - (1 - \omega x^{k-1})^{-r}\}}.$$

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[1] S. K. Chatterjea, *Some operational formulas connected with a function defined by a generalized Rodrigues' formula*, to appear in Acta Mathematica (Budapest) in Vol. XVII.

[2] H. W. Gould, and A. T. Hopper, *Operational formulas connected with two generalization of Hermite polynomials*, «Duke Math. Jour.» Vol. 29 (1962), pp. 51-64.

[3] E. G. C. Poole, *Introduction to the theory of linear differential equations*, Dover publications (1960).

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