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Unique sequential limits.

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Summary. - *The paper contains a discussion of separation axioms between T_1 and T_2 and an example of a space in which sequential limits are unique but compact subsets are not closed.*

Introduction. - Halfar, in [1], has established that first countable HAUSDORFF spaces are exactly those first countable spaces in which compact subsets are closed. It is the purpose of this note to generalize slightly HALFAR'S theorem and [to give an] to give an example of a space in which sequential limits are unique but compact subsets are not closed.

It is well known that the T_2-axiom implies that sequential limits are unique and that unique sequential limits imply the the T_1-axiom.

THEOREM 1. - If $\{X, \tau\}$ is a space with the property that compact subsets are closed then any sequence in $\{X, \tau\}$ has at most one limit.

PROOF - Let $(x_1, x_2, \ldots, x_n, \ldots)$ denote a sequence in $\{X, \tau\}$ which converges to α in $\{X, \tau\}$. Let τ denote any open covering of the set $\{x_1, x_2, \ldots, x_n, \ldots\} \cup \{\alpha\}$. Let G_α denote a set of τ which contains α. G_α contains all but a finite number of elements of the set $\{x_1, x_2, \ldots, x_n, \ldots\}$. Hence, $\{x_1, x_2, \ldots, x_n, \ldots\} \cup \{\alpha\}$ is compact and, by hypothesis, also closed. Now, let β be any point of $\{X, \tau\}$ which is different from α. Since $\{x\}$ for x in X is compact, $\{x\}$ is closed and $\{X, \tau\}$ is T_1. Hence, α has a neighborhood which does not contain β. Thus there exists a natural number N such that for $\geq N$, $x_n \neq \beta$. Now, $\{x_N, x_{N+1}, \ldots, x_{N+k}, \ldots\} \cup \{\alpha\}$ is compact and, hence, closed. Therefore, $(x_N, x_{N+1}, \ldots, x_{N+k} \ldots)$ can not converge to β and, so, $(x_1\ x_2, \ldots, x_n, \ldots)$ can not converge to β.

THEOREM 2. - If $\{X, \tau\}$ is first countable then $\{X, \tau\}$ is HAUSDORFF if and only if sequential limits are unique.

PROOF. - The necessity is well known. For the sufficiency, let α and β be two distinct points of $\{X, \tau\}$. Let $\{U_1, U_2, \ldots, U_n, \ldots\}$ and $\{V_1, V_2, \ldots, V_n, \ldots\}$ denote countable neighborhood bases at α and β, respectively. Assume that every neighborhood of α

intersects every neighborhood of β on a non-empty set. Define $W_1 = U_1$, $W_2 = U_1 \cap V_1$, $W_3 = U_1 \cap V_1 \cap U_2$, ...

$$.., W_{2n} = u_1 \cap v_1, ..., u_n \cap v_n,$$

$$W_{2n+1} = U_1 \cap V_1 \cap ... \cap U_n \cap V_n \cap U_{n+1}, ...$$

$$..., W_1 \supset W_2 \supset ... \supset W_n \supset ....$$

Choose x_1 in W_1, x_2 in W_2, ..., x_n in W_n, ... Consider the sequence $(x_1, x_2, ..., x_n, ...)$. Let G_1 be an open set containing α and G_2 be an open set containing β. There exists a natural number N such that if $n > N$, $W_n \subset G_1 \cap G_2$. Hence, $(x_1, x_2, ..., x_n, ...)$ converges to both α and β. This is a contradiction and so $\{X, \tau\}$ is HAUSDORFF.

COROLLARY 2.1. - A first countable space is HAUSDORFF if and only if compact subsets are closed.

PROOF. - The corollary follows from theorems 1 and 2.

The previous corollary is Halfar's theorem.

COROLLARY 2.2. - If $\{X, \tau\}$ is a first countable space then compact subsets are closed if and only if sequential limits are unique.

PROOF. - The corollary follows from theorem 2 and corollary 2.1.

EXAMPLE 1. - Let R^* denote the set $R \cup \{\omega\}$, where R denotes the set of real numbers and ω denotes some element which is not à real number. Let τ denote the topology for R^* defined by the usual open sets in R along with $\{G / \sim G \subset R$ and $\sim G$ is the union of the ranges of a finite number of converging sequences of real numbers and the limits to which these converge$\}$. ($\sim G$ is the complement of G). In $\{R^*, \tau\}$ sequential limits are unique and compact subsets are not closed: no sequence which converges to a real number can converge to any other real number nor to ω and the set $M = \{x : 0 \leq x \leq 1\}$ is compact but not closed since ω is a limit point of M.

Thus compact subsets being closed is a sligtly stronger property than sequences having unique limits.

REFERENCES

[1] HALFAR, EDWARD, *A Note on Hausdorff Separation*, «The American Mathematical Montly», Vol. 68, 1961, p. 164.

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