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On the nonnegativity of Green's functions.

Nota di RICHARD BELLMAN (California U. S. A.) (*)

Summary. - *In previous papers we discussed various methods of estalishing the nonnegativity of the Green's function associated with the ordinary differential equation*

$$u'' + q(x)u = f(x), \; u(0) = u(1) = 0.$$

In this paper we wish to present another method which has certain merits. It clarifies the role played by the characteristic values of the associated Sturm-Liouville problem and it indicates how useful it may be to study the behavior of the solution of $Lu = v$, *where* L *is a linear operator, by means of the limiting behavior of the solution of*

$$\frac{\partial u}{\partial t} = Lu - v,$$

as $t \to \infty$. *This method has been used by Arrow and Hearon to study the inverse of input-output matrices.*

1. - Introduction.

In previous papers [1], [2], [3], we discussed various methods of establishing the nonnegativity of the GREEN'S function associated with the ordinary differential equation

$$u'' + q(x)u = f(x), \; u(0) = u(1) = 0. \tag{1.1}$$

(*) Pervenuta alla Segreteria dell' U. M. I. il 29 aprile 1963.

In this paper we wish to present another method which has certain merits. It clarifies the role played by the characteristic values of the associated STURM–LIOUVILLE problem and it indicates how useful it may be to study the behavior of the solution of $Lu = v$, where L is a linear operator, by means of the limiting behavior of the solution of

$$\frac{\partial u}{\partial t} = Lu - v, \tag{1.2}$$

as $t \to \infty$. This method has been used by Arrow and Hearon to study the inverse of input-output matrices; see [4].

2. – Nonnegativity of solutions of partial differential equations.

Consider the partial differential equation of parabolic type,

$$u_t + q(x)u - f(x) = u_{xx} \tag{2.1}$$

with the initial condition $u(x, 0) = h(x)$, with $h(x) \geq 0$, $0 \leq x \leq 1$, and $u(0, t) = u(1, t) = 0$. If we suppose that $q(x)$ is uniformly bounded, $0 \leq x \leq 1$, it is easy to show, under the hypotheses that $h(x)$, $f(x) \geq 0$, that $u(x, t) \geq 0$ for $t \geq 0$. If $q(x) \geq 0$, we use the finite difference approximation

$$u(x, t + \Delta^2) = \frac{u(x + \Delta, t) + u(x - \Delta, t)}{2} \tag{2.2}$$

$$+ q(x)u(x, t)\Delta + f(x)\Delta,$$

$t = 0$, Δ^2, ..., which establishes inductively that $u(x, t) \geq 0$. As $\Delta \to 0$, the solution of the finite difference equation converges to that of the partial differential equation, thus establishing the required nonnegativity.

If $q(x)$ is not nonnegative, but bounded from below by a constant so that $M + q(x) \geq 0$, we write $u = e^{-Mt} w(x)$. Substituting, we obtain the equation

$$w_t = w_{xx} + (q(x) + M)w - f(x)e^{-Mt}, \tag{2.3}$$

which we can treat as in (2.2) to establish nonnegativity.

3. - Nonnegativity of solution of ordinary differential equation.

Returning (2.1), let us allow t to become infinite. If all characteristic values of the equation

$$u_{xx} + q(x)u = \lambda u, \quad u(0) = u(1) = 0, \tag{3.1}$$

are negative, the solution of (2.1) converges as $t \to \infty$ to the solution of (1.1), establishing thereby the nonnegativity of the solution of (1.1).

4. - Extension.

There is no difficulty in eytending the proof to cover ordinary differential equations with more general boundary conditions and multidimensional partial differential equations of parabolic type.

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[3] E. F. Beckenbach, and R. Bellman, *Inequalities, Ergebnisse der Math.*, « Springer, Berlin », 1961.

[4] R. Bellman, *An Introduction to Matrix Analysis*, « McGraw-Hill Book Company », Inc., New York, 1960.