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A Note on Toeplitz Matrices and Unitary Equivalence.

Nota di C. R. Putnam (a Lafayette U. S.A.)

Summary. - *There is obtained a generalization of a condition assuring the unitary equivalence of a Toeplitz matrix* (c_{j-k}) *to a certain function of the matrix belonging to the qua ratic form* $2\sum_1^\infty x_n x_{n+1}$.

1. **Introduction.** Let $\{c_n\}$, where $n = 0, \pm 1, \pm 2, \ldots$, be a sequence of complex numbers satisfying

$$(1) \qquad c_{-n} = \bar{c}_n \quad \text{and} \quad \sum_1^\infty |c_n|^2 < \infty$$

and let $f(\theta)$ denote the real function belonging to $L^2[-\pi, \pi]$ defined by

$$(2) \qquad f(\theta) \sim \sum_{-\infty}^{\infty} c_n e^{in\theta}.$$

Let $d_{\rho jk}(\theta) = 2\pi^{-1} \sin j\theta \sin k\theta d\theta$, the differential of the spectral matrix belonging to the quadratic form $2\sum_1^\infty x_n x_{n+1}$ (cf. [7] and the

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references to HILBERT and HELLINGER cited there). It was shown in [7] that if, in addition to (1), it is assumed that the relations

$$(3) \qquad c_n \text{ are real}$$

and

$$(4) \qquad |c_n| \leq \text{const. } \alpha^n (n = 0, 1, 2, \ldots), \text{ where } 0 < \alpha = \text{Const.} < 1,$$

hold, then

$$(5) \qquad T = UFU^*,$$

where T and F are defined by

$$(6) \qquad T = (c_{j-k}) \text{ and } F = \left(\int_0^\pi f(\theta) d\rho_{jk}(\theta)\right),$$

and U is unitary.

In fact, it was shown *loc. cit.* that the unitary equivalence relation (5) is true if (3) holds and if (4) is replaced by the weaker hypothesis that

$$(7') \qquad \text{meas } \{\theta, f(\theta) \text{ in } Z\} = 0 \text{ whenever meas } Z = 0$$

and

$$(7'') \qquad \Sigma\, n\, |c_n| < \infty \text{ (or even } \sum_n (\sum_m c_{n+m}^2)^{1/2} < \infty).$$

(The condition (7′) amounts to the assumption that F be absolutely continuous. For the definition of absolute continuity used here, see [7], p. 840, also [3], p. 240, [9].) The proof of this theorem depended upon certain facts on commutators obtained in [5] and [6] and upon a result of ROSENBLUM [8] concerning the unitary equivalence of two absolutely continuous operators differing by a trace class operator. A generalization of the theorem has recently been obtained by ROSENBLUM [9] using results of KATO [3], [4].

That T and F need not be unitarily equivalent if (4) is assumed but the reality assumption (3) is dropped is easily seen. For let $c_1 = -i$, $c_{-1} = i$ and $c_n = 0$ otherwise; then the spectrum of T is the interval $-2 \leq \lambda \leq 2$ (see [1], also below). But $F = \left(2\int_0^\pi \sin\theta\, d\rho_{jk}(\theta)\right)$, from which it follows that the spectrum of F is the interval $0 \leq \leq \lambda \leq 2$.

2. **The Theorem.** In this paper there will be established, under a relaxation of the restriction (3), an equivalence relation similar to (5) but now existing between T and a matrix G closely related to F. Instead of (3), it will be supposed that

$$(8) \qquad c_n = a_n e^{in\varphi}, \; a_n \text{ and } \varphi \text{ real}, \; a_{-n} = a_n \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

It is to be noted that a_n may be negative and hence need not be $|c_n|$. (For the matrix T considered in the preceding paragraph, $a_1 = a_{-1} = 1$ and $\varphi = -\pi/2$). There will be proved the following.

THEOREM. – *Let the sequence* $\{c_n\}$ *satisfy* (4) *and* (8). *Then there exists a unitary matrix* U *for which* $T = (c_{j-k})$ *satisfies*

$$(9 \qquad T = UGU^*,$$

where

$$(10) \qquad G = \left(\int_0^\pi g(\theta) d\rho_{jk}(\theta) \right), \; g(\theta) \sim \sum_{-\infty}^{\infty} a_n e^{in\theta} = a_0 + 2 \sum_1^\infty a_n \cos n\theta.$$

Moreover, the assertion remains true if the assumption (4) *is replaced by the weaker hypothesis of* (7′) *and* (7″).

3. **Proof of the Theorem**. If the diagonal unitary matrix V is defined by $V = \operatorname{diag}(e^{i\varphi}, e^{2i\varphi}, e^{3i\varphi}, \ldots)$, a direct calculation and the use of (8) show that $VTV^* = (a_{j-k})$. In view of (8) and the implied relation $f(\theta) = g(\theta + \varphi)$, it is clear that conditions (4), (7′), and (7″) imply, respectively, the corresponding relations in which the c_n and $f(\theta)$ are replaced by a_n and $g(\theta)$. It now follows from the theorem of [7] mentioned above that there exists a unitary matrix W for which $(a_{j-k}) = WGW^*$, where G is defined by (10). Relation (9) with $U = V^*W$ now follows.

4. **Some Special Cases.** If $\varphi = 0$, so that $c_n = a_n$ (real) and $f(\theta) = a_0 + 2 \sum^\infty a_n \cos n\theta$ is even, the theorem mentioned earlier results.

In case $\varphi = \pi/2$ and $c_{2n} = 0$, one has $c_{2n-1} = -i(-1)^n a_{2n-1}$ and, hence, the (restricted type of) odd function $f(\theta) = 2 \sum_1^\infty (-1)^n a_{2n-1} \sin (2n-1)\theta$.

Another special case results if all $c_n = 0$ except for $n = N \neq 0$. Since c_N can be expressed in its polar form $c_N = |c_N| e^{i\psi} = |c_N| e^{iN(\psi/N)}$, it is clear that (8) is a consequence of (1). In this case (9) holds with

$$G = \left(\int_0^\pi 2\,|c_N| \cos N\theta \; d\rho_{jk}(\theta) \right).$$

Finally, it is seen that the Theorem is applicable when the c_n for $n \geq 0$ are the terms of a power series in z with real coefficients. For, if z is inside the circle of convergence, it is seen that the c_n, where $c_n = b_n z^n = b_n |z|^n e^{in\varphi}$ with c_{-n} defined to be $\bar{c}_n$ $(n = 0, 1, 2, \ldots)$, satisfy (4) and (8) with $a_n = b_n |z|^n (= a_{-n})$.

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