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On the linear differential equation whose solutions are the products of solutions of two given linear differential equations

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Summary. - *The purpose of this paper is to illustrate the application of a result in matrix theory to the problem of determining the linear differential equation whose solutions are the products of the solutions of two given linear differential equations.*

§ 1. Introduction

It was observed by NEWTON that a simple way to obtain the power series expansion, for the function

$$u = (\text{arc sin } x)^2 \tag{1}$$

was to form the second order linear differential equation

$$(1 - x^2)\frac{d^2u}{dx^2} - x\frac{du}{dx} = 2, \tag{2}$$

and solve it by means of a power series expansion.

In another direction, it was shown by Appell, [1], that if u_1, u_2 represent two linearly independent solutions of

$$u'' + p(t)u' + q(t)u = 0, \tag{3}$$

then u_1^2, u_1u_2, u_2^2 represent three linearly independent solutions of the third order linear differential equation

$$u''' + 3p(t)u'' + (2p^2(t) + p'(t) + 4q(t))u' + (4p(t)q(t) + 2q'(t))u = 0 \tag{4}$$

cf., also, WHITTAKER and WATSON, [2].

This result is useful in connection with the determination of the power series expansion for the square of the hypergeometric function $f(a, b, c; x)$ [1], and plays a role in the study of the MATHIEU function, [2].

There are several ways of determining the equation in (4), since the problem is analogous to that of finding the polynomial whose roots are various symmetric combinations of the roots of a given polynomial.

In this paper, we shall present a new method based upon an interesting result concerning matrix differential equations.

§ 2. Preliminary Lemma

The crux of the method is the well-known

Lemma. Let Y *and* Z *be respectively the solutions of*

$$(1) \qquad \frac{dY}{dt} = A(t)Y, \qquad Y(0) = I,$$

$$\frac{dZ}{dt} = ZB(t), \qquad Z(0) = I.$$

Then the solution of

$$(2) \qquad \frac{dX}{dt} = A(t)X + XB(t), \qquad X(0) = C,$$

is given by

$$(3) \qquad X = YCZ.$$

Verification provides an immediate proof. Let us assume that $A(t)$ and $B(t)$ satisfy the condition of being integrable over any finite interval.

§ 3. Application

Let us now apply this result to the problem of finding the 3rd order linear differential equation whose solutions are u_1^2, $u_1 u_2$, u_2^2, vhere u_1 and u_2 are two linearly independent solutions of (1.3).

(1) While this paper was in the process of being typed, a paper on this theme appeared: H. F. SANDHAM, *A Square and a Product of Hypergeometric Functions*, Quart. Jour. of Math., Vol. 7(1956). pp. 153-154.

Without loss of generality, let u_1 and u_2 be determined by the boundary conditions.

$$u_1(0) = 1, \qquad u_1'(0) = 0, \tag{1}$$
$$u_2(0) = 0, \qquad u_2'(0) = 1.$$

Setting $u' = v$, we see that (1.3) is equivalent to the system

$$\begin{aligned} u' &= v \\ v' &= -p(t)v - q(t)u. \end{aligned} \tag{2}$$

Let

$$A(t) = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix}. \tag{3}$$

Then the matrix solution of

$$U' = A(t)U, \qquad U(0) = I, \tag{4}$$

is given by

$$U = \begin{pmatrix} u_1(t) & u_2(t) \\ u_1'(t) & u_2'(t) \end{pmatrix}, \tag{5}$$

and the solution of

$$V' = VA(t)^T, \qquad V(0) = I, \tag{6}$$

by $V = U^T$, the transpose of U.

From the lemma in § 2, we deduce that the solution of

$$X' = A(t)X + XA(t)^T, \quad X(0) = C, \tag{7}$$

is given by

$$X = UCU^T. \tag{8}$$

Taking C to be a symmetric matrix,

$$\begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix}, \tag{9}$$

we see that X is given by

$$X = \begin{pmatrix} c_1 u_1^2 + 2c_2 u_1 u_2 + c_3 u_2^2 & c_1 u_1 u_1' + c_2(u_2' u_1 + u_2 u_1') + c_3 u_2 u_2' \\ c_1 u_1 u_1' + c_2(u_2 u_1 + u_2 u_1') + c_3 u_3 u_3' & c_1 u_1'^2 + 2c_2 u_1' u_2' + c_3 u_2'^2 \end{pmatrix}. \tag{10}$$

Writing

$$X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}, \tag{11}$$

the equation in (7) is equivalent to the system

$$\begin{aligned} x_1' &= 2(a_{11} x_1 + a_{12} x_2) \\ x_2' &= a_{21} x_1 + (a_{11} + a_{22}) x_2 + a_{12} x_3 \\ x_3' &= 2(a_{21} x_2 + a_{22} x_3). \end{aligned} \tag{12}$$

Eliminating x_2 and x_3, we obtain a third order linear differential equation for x_1 whose general solution is $c_1 u_1^2 + 2c_2 u_1 u_2 + c_3 u_2^2$ where c_1, c_2, and c_3 are arbitrary constants.

Similarly, eliminating x_1 and x_2, we obtain the equation whose general solution is $c_1 u_1'^2 + 2c_2 u_1' u_2' + c_3 u_2'^2$; eliminating x_1 and x_3, we obtain an equation whose solutions are the derivatives of the equation obtained by the elimination of x_2 and x_3.

§ 4. The General Case

In stating the lemma in § 2, we ignored any discussion of the dimension of Y and Z. It is clear that the result is valid if $A(t)$ and Y are $m x m$ matrices, $B(t)$ and Z nxn matrices, and C and X $m x n$ matrices.

Using the technique sketched in § 3, we can obtain the linear differential equation of order $m n$ whose solutions are the products of the solutions of a linear differential equation of order m and one of order n.

BIBLIOGRAPHY

[1] C. APPELL, *Comptes Rendus*, XCI (1880). pp. 211-214.

[2] E. T. WHITTAKER and G. N. WATSON, *Modern Analysis*, Cambridge, 1935, p. 418.