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On a (third) functional equation, connected with the Weierstrassian function $\wp(z)$ (*).

Nota di HARI DAS BAGCHI e PHATIK CHAND CHATTERJI (a Calcutta).

Sunto. - *Si studia una equazione funzionale connessa alle funzioni ellittiche di* WEIERSTRASS.

The present paper aims at finding the complete solution of the functional equation:

$$\text{(I)} \quad f(x+y)f(x-y) = \frac{\{f(x)f(y)+a\}^2 + b\{f(x)+f(y)\}}{\{f(x)-f(y)\}^2}, \ (a, b \text{ are constants})$$

compatible with the limitation that $f(z)$ shall be devoid of any essential singularity in the *finite* part of the plane. A particular solution of (I) being known to be $f(z) = \wp(z)$, [WHITTAKER and WATSON, 1], we propose to take account of all other solutions, consistent with the afore-said restrictions. This paper is, in a sense, supplementary to two previous papers [2] of our , bearing on two other functional equations, satisfied by $\wp(z)$.

We are not aware whether the functional equation (I) has been scrutinised heretofore by any other writer.

1. A simple glance at (I) obviously suggests that the origin ($z = 0$) must be a singularity for $f(z)$. For if that were not so, the L. S. of (I) would be *finite* and the R. S. would be *infinite* on setting $y = x$. So $f(z)$ must have the origin for a singularity, which in the present set-up cannot but be a *pole*. Supposing the order of this pole to be n we may take the associated principal part as:

$$\frac{a_1}{z} + \frac{a_2}{z^2} + \dots + \frac{a_n}{z^n}, \qquad (a_n \neq 0).$$

(*) Vedi nota redazionale, pag. 277.

Consequently when ε is very small, we have the approximation:

$$(1) \qquad f(\varepsilon) = \frac{a_1}{\varepsilon} + \frac{a_2}{\varepsilon^2} + \dots + \frac{a_n}{\varepsilon^n}, \qquad \text{(nearly).}$$

Now putting $x = y + \varepsilon$ in (I), we get:

$$(2) \quad f(2y+\varepsilon)f(\varepsilon)\{f(y+\varepsilon) - f(y)\}^2 = \{f(y)f(y+\varepsilon) + a\}^2 + b\{f(y+\varepsilon) + f(y)\}.$$

If in this relation we insert the value of $f(\varepsilon)$, as given by (1), and substitute TAYLOR's expansions for $f(y+\varepsilon)$ and $f(2y+\varepsilon)$, (2) assumes the form:

$$\left\{f(2y) + \varepsilon f'(2y) + \frac{\varepsilon^2}{2!}f''(2y) + \dots\right\}\left(\frac{a_1}{\varepsilon} + \frac{a_2}{\varepsilon^2} + \dots + \frac{a_n}{\varepsilon^n}\right)$$
$$\left\{\varepsilon f'(y) + \frac{\varepsilon^2}{2!}f''(y) + \dots\right\}^2 = \left[f(y)\left\{f(y) + \varepsilon f'(y) + \frac{\varepsilon^2}{2!}f''(y) + \dots\right\} + a\right]^2 +$$
$$+ b\left[2f(y) + \varepsilon f'(y) + \frac{\varepsilon^2}{2!}f''(y) + \dots\right],$$

which, on being multiplied by ε^n, becomes:

$$(3) \quad \left\{f(2y) + \varepsilon f'(2y) + \frac{\varepsilon^2}{2!}f''(2y) + \dots\right\}(a_n + a_{n-1}\varepsilon + \dots + a_1\varepsilon^{n-1})$$
$$\left\{\varepsilon f'(y) + \frac{\varepsilon^2}{2!}f''(y) + \dots\right\}^2 = \varepsilon^n\left[f(y)\left\{f(y) + \varepsilon f'(y) + \frac{\varepsilon^2}{2!}f''(y) + \dots\right\} + a\right]^2 +$$
$$+ b\varepsilon^n\left[2f(y) + \varepsilon f'(y) + \frac{\varepsilon^2}{2!}f''(y) + \dots\right].$$

Inasmuch as the lowest orders of the (infinitesimal) terms on the L. S. and R. S. of (3) are 2 and n respectively, we infer immediately that $n = 2$. That is to say, if a function $f(z)$, analytic except for poles in the *finite* region of the plane, is to satisfy (I), it must have the origin for a *quadratic* pole. Other consequences of this result will be considered in 2.

2. The point $z = 0$ being a pole of the second order, the corresponding principal part may be taken as:

$$\frac{a_1}{z} + \frac{a_2}{z^2}; \qquad (a_2 \neq 0)$$

so that

$$(1) \qquad f(\varepsilon) = \frac{a_1}{\varepsilon} + \frac{a_2}{\varepsilon^2} \qquad \text{(nearly), when } \varepsilon \text{ is very small.}$$

If we now put $x = 2\varepsilon$ and $y = \varepsilon$ in (I), it becomes:

$$(2) \qquad f(\varepsilon)f(3\varepsilon)\{f(2\varepsilon) - f(\varepsilon)\}^2 = \{f(2\varepsilon)f(\varepsilon) + a\}^2 + b\{f(2\varepsilon) + f(\varepsilon)\}.$$

When the values of $f(2\varepsilon)$ and $f(3\varepsilon)$, derived from (1), are substituted in (2), and the resulting relation is simplified, it reduces to:

$$(3) \qquad (a_2 + a_1\varepsilon)\left(\frac{a_2}{9} + \frac{a_1\varepsilon}{3}\right)\left(\frac{3a_2}{4} + \frac{a_1\varepsilon}{2}\right)^2 =$$

$$= \left\{ \left(\frac{a_2}{4} + \frac{a_1\varepsilon}{2}\right)(a_2 + a_1\varepsilon) + a\varepsilon^4 \right\}^2 + b\varepsilon^6 \left\{ (a_2 + a_1\varepsilon) + \left(\frac{a_2}{4} + \frac{a_1\varepsilon}{2}\right) \right\}.$$

Comparison of the coefficients of ε on both sides of (3) leads to

$$a_1 a_2^3 = 0,$$

which, by virtue of the inequality $a_2 \neq 0$, gives:

$$a_1 = 0.$$

Writing k for a_2, we may now represent the principal part of $f(z)$ at $z = 0$ in the form $\frac{k}{z^2}$ (4).

3. If we now fall back upon the original equation (I), and put $y = \varepsilon$ (very small) and allow unrestricted variation to x, we get:

$$(1) \quad f(x+\varepsilon)f(x-\varepsilon)\{f(x) - f(\varepsilon)\}^2 = \{f(x)f(\varepsilon) + a\}^2 + b\{f(x) + f(\varepsilon)\}.$$

Substituting TAYLOR's expansions for $f(x+\varepsilon)$ and $f(x-\varepsilon)$ and writing $f(\varepsilon) = \frac{k}{\varepsilon^2}$ on the strength of (4) of 2, we can exhibit (1) in the form:

$$(2) \quad \left[\left\{f(x) + \frac{\varepsilon^2}{2!}f''(x) + \dots\right\}^2 - \left\{\varepsilon f'(x) + \frac{\varepsilon^3}{3!}f'''(x) + \dots\right\}^2\right]\left[f(x) - \frac{k}{\varepsilon^2}\right]^2 =$$

$$= \left\{\frac{kf(x)}{\varepsilon^2} + a\right\}^2 + b\left\{f(x) + \frac{k}{\varepsilon^2}\right\}.$$

If we now multiply (2) by ε^4, and then equate the coefficients of ε on both sides, we derive:

$$k[f(x)f''(x) - \{f'(x)\}^2] = 2[f(x)]^3 + 2af(x) + b,$$

which can be thrown into the form:

$$(3) \qquad V\frac{d^2V}{dx^2} - \left(\frac{dV}{dx}\right)^2 = lV^3 + mV + n,$$

provided that

$$(4) \qquad V = f(x) \text{ and } l = \frac{2}{k}, \quad m = \frac{2a}{k} \text{ and } n = \frac{b}{k}.$$

If we now set:

$$U = \left(\frac{dV}{dx}\right)^2,$$

(3) can without much difficulty be presented in the form of a differential equation (having U for the dependent variable and V for the independent variable), viz.,

$$\frac{dU}{dV} + PU = Q, \tag{5}$$

where

$$P \equiv -\frac{2}{V} \quad \text{and} \quad Q = 2\left(lV^2 + m + \frac{n}{V}\right).$$

Manifestly (5) can be integrated in the form:

$U = 2lV^3 - 2mV - n + \lambda V^2$, (where λ is the constant of integration)

$$\text{i. e.,} \quad \left(\frac{dV}{dx}\right)^2 = 2lV^3 + \lambda V^2 - 2mV - n. \tag{6}$$

The two variables V and x being changed respectively into ξ and x', according to the trasforming scheme:

$$\left\{\begin{aligned} &\xi = V + \frac{\lambda}{6l} \qquad \text{and} \\ &x' = \sqrt{\frac{l}{2}} \cdot x, \end{aligned}\right. \tag{7}$$

the differential equation (6) can be carried over into:

$$\left(\frac{d\xi}{dx'}\right)^2 = 4\xi^3 - g_2(\lambda)\xi - g_3(\lambda), \tag{8}$$

where

$$g_2(\lambda) = \frac{\lambda^2}{3l^2} + \frac{4m}{l} \quad \text{and} \quad g_3(\lambda) = \frac{2n}{l} - \frac{2m\lambda}{3l^2} - \frac{\lambda^3}{27l^3}. \tag{9}$$

Evidently the relation (8) can be *inverted* into the form:

$$\xi = \wp(x'), \tag{10}$$

where $\wp$ denotes the Weierstrassian elliptic function, formed with the two invariants $g_2(\lambda)$ and $g_3(\lambda)$, as defined by (9).

Now restoring the actual values of ξ, x', as given by (7) and (4) we can re-write (10) in the form:

$$f(x) = \wp(\mu x) + \nu, \tag{11}$$

where μ, ν are respectively the two constants $\sqrt{\frac{1}{k}}$ and $-\frac{k\lambda}{12}$.

Before we can declare (11) to be the complete solution of (I), we have to ascertain what restrictions (if any) are to be placed upon the parametric constants μ, ν. This will be done in the next article.

4. The relation:

$$f(x) = \wp(\mu x) + \nu \tag{1}$$

being now taken as the starting point, and the corresponding values of $f(y)$, $f(x-y)$ and $f(x+y)$ being formed and then substituted in (I), *viz.*,

$$f(x+y)f(x-y)\{f(x)-f(y)\}^2 = \{f(x)f(y)+a\}^2 + b\{f(x)+f(y)\},$$

we get:

$$\begin{aligned}(2)\quad & [\{\wp(\mu x)+\nu\}\{\wp(\mu y)+\nu\}+a]^2 + b[\wp(\mu x)+\wp(\mu y)+2\nu] - \\ & - \{\wp(\mu x+\mu y)+\nu\}\{\wp(\mu x-\mu y)+\nu\}\{\wp(\mu x)-\wp(\mu y)\}^2 = 0.\end{aligned}$$

Certainly if (1) is to satisfy (I), the relation (2) must hold for *all* values of x, y, μ, ν. If we now as a matter of pleasure keep x, y, μ *fixed* and allow ν only to vary, (2) ought to hold for *all* values of ν. *i. e.*, (2) ought to be an *identity* in ν. But this is impossible, for the coefficient of the highest power $\nu^4 \neq 0$, being in fact *unity*; in fact for any *prescribed* set of values of x. y, μ, the relation (2), as it stands, can be solved as a biquadratic in ν, having, of course, only four roots. So the logical conclusion is that the parameter ν must be *absent* in (1). As for the other parameter μ, it can be easily verified that, whatever value be assigned to it in (1), the equation (I) will be satisfied.

In other words, *the most general solution of the functional equation (I), subject to the afore-said conditions, is*

$$f(z) = \wp(\mu z),$$

where μ is an arbitrary constant.

REFERENCES

[1] WHITTAKER and WATSON, *Modern Analysis*, 1915. Ch. XX. p. 449, Ex. 8.

[2] BAGCHI and CHATERJI, *Note on a functional equation connected with the Weierstrassian function* (« Bull. Cal. Math. Soc. ») March, 1950.