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Voisin's conjecture on Todorov surfaces

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Voisin's conjecture on Todorov surfaces

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To my sons Milo and Elvio, so they'll never give up on a dream.
To my husband Federico for his great companionship
in my most ambitious challenges.

Contents

Introduction

Le panorama qui avait commencé à s'ouvrir devant moi
et que je m'efforçais de scruter et de capter,
dépassait de très loin en ampleur et en profondeur
les hypothétiques besoins d'une démonstration,
et même tout ce que ces fameuses conjectures
avaient pu d'abord faire entrevoir
(...) c'est un monde nouveau et insoupçonné
qui s'était ouvert soudain.
(Grothendieck, [Gro85, 2.17])

The influence of Chow groups on singular cohomology is motivated by classical results by Mumford and Roitman and has been investigated extensively. On the other hand, the converse influence is rather conjectural and it takes place in the framework of the "philosophy of mixed motives", which is due to some great mathematicians such as Grothendieck, Bloch and Beilinson. For example, Bloch's conjecture on surfaces is still open and the Bloch–Beilinson's filtration conjecture is still far from being solved. In the spirit of exploring this influence, Voisin formulated in 1996 [Voi96] the following conjecture on 0–cycles on the self–product of surfaces of geometric genus one, which is implied by the generalized Bloch conjecture adapted to motives.

Conjecture 3.4.1. *Let S be a smooth complex projective surface such that $h^{2,0}(S) = 1$ and $h^{1,0}(S) = 0$. Let a, a' be two 0–cycles homologically trivial. Then*

$$a \times a' = a' \times a \in CH^4(S \times S).$$

There are few examples in which Conjecture 3.4.1 has been verified (see Section 3.4 for an overview on the known cases), but it is still open for a general $K3$ surface.
Our aim is to present a new example in which Conjecture 3.4.1 is true, namely a family of Todorov surfaces. Todorov surfaces were introduced by Todorov to provide counterexamples to Local and Global Torelli ([Tod81]).

In [Mor88] Morrison proved that there are exactly 11 non–empty irreducible families of Todorov surfaces.

Conjecture 3.4.1 has been proven by Laterveer for two of these families ([Lat16c], [Lat18a]). For both of these families the core of the proof was that an explicit description as complete intersections of the family was available. Up to now, the main obstacle to prove Conjecture 3.4.1 for all Todorov surfaces is the lack of an explicit and nice description of the remaining nine families.

In this work we focus on the family of Todorov surfaces with fundamental invariants $(\alpha, k) = (2, 12)$. We present an explicit description for this family as quotient of the complete intersection of four quadrics in $\mathbb{P}^6$. Our main result is the following theorem.

Theorem 5.3.5. *Let S be a general Todorov surface with fundamental invariants $(\alpha, k) = (2, 12)$.*
Then Conjecture 3.4.1 is true for S.

In Chapter 1 we introduce the theory of algebraic cycles and we focus on Chow groups. We present the classical equivalence relations on cycles and the related conjectures, i.e. Grothendieck's standard conjectures and the Hodge conjecture.

In Chapter 2 we give a brief introduction to Groethendieck's theory of motives, we present Kimura's conjecture on finite–dimensionality. Next we introduce the Chow–Künneth decomposition for Chow motives and Murre's related conjecture $C(X)$.

In Chapter 3 we explore the relation between singular cohomology and Chow groups, presenting the known results and the conjectural ones. In particular, we focus on the problem of measuring the size of the Chow group of 0–cycles $CH_0(X)$. We present Mumford's theorem and its conjectural converse, i.e. Bloch's conjecture for surfaces. Next we present the famous and deep Bloch–Beilinson's conjecture which aims to describe the deep relation between cohomology and Chow-groups. In relation to this conjecture, we present Murre's filtration which is a suitable candidate to fulfill the conjecture. Finally, we study Voisin's conjecture 3.4.1 on 0–cycles on the self–product of a surface with geometric genus 1. We present Voisin's result ([Voi96, Theorem 3.4]) on a 10–dimensional family of K3 surfaces on which the conjecture holds. In particular, this family is obtained as a desingularization of a double cover of $\mathbb{P}^2$ branched along the union of two cubics.

In Chapter 4 we focus on useful techniques to deal with algebraic cycles. First we introduce Bloch's higher Chow groups and their relation with Borel–

Moore homology. Next, we present a crucial tool for our purpose, Voisin's spreading of cycles ([Voi13] and [Voi15]) for the fibered self–product of a family.

In Chapter 5 we focus on the family of Todorov surfaces of type $(2, 12)$. First of all, we give an explicit description of the family as quotient of complete intersection of four quadrics in $\mathbb{P}^6$ by a linearized action of $(\mathbb{Z}/2\mathbb{Z})^2$. Our main result is the following theorem.

Theorem 5.1.20. *Let S be a general Todorov surface of type $(2, 12)$.*
Then the canonical model of S is a quotient surface V/G, where $G \cong (\mathbb{Z}/2\mathbb{Z})^2$ and V is the smooth complete intersection of four independent quadrics $Q_0, Q_1, Q_2, Q_3 \in H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))^G$.
Conversely, any such surface V/G, is a Todorov surface of type $(2, 12)$.

Next we focus on 0–cycles. By Rito's result [Rit09] and Voisin's ([Voi96, Theorem 3.4], any Todorov surface has an associated K3 surface for which Conjecture 3.4.1 holds. To conclude that the conjecture holds for all Todorov surfaces we need to be able to relate 0–cycles on the Todorov surface to 0–cycles on the associated K3. In particular, we show that the irreducible family of Todorov surfaces considered $\mathcal{S} \to B$ has a nice enough description to prove

$$CH^2(\mathcal{S} \times_B \mathcal{S})_{\mathbb{Q}} = 0.$$

To show this, we express the fibered self–product of the total space of the family as an open set of a variety with trivial Chow groups.
Next, we apply Voisin's spreading of cycles in order to prove the following key relation on 0–cycles (Theorem 5.3.2) between the Todorov surface S and the associated K3 surface M

$$CH^2(S)_{\mathbb{Q}} \cong CH^2(M)_{\mathbb{Q}}.$$

Following Laterveer's approach [Lat18a], we explore some motivic consequences of the results on 0–cycles. In particular, we prove that a Todorov surface of type $(2, 12)$ has the transcendental part of the motive isomorphic to the associated K3 surface's one. Moreover, we prove that the motive is of abelian type if the Picard number is big enough, so it is finite–dimensional in the sense of Kimura and O'Sullivan.

Although the techniques we use are not original (they have been developed by C. Voisin, R. Laterveer and many others), these results add some knowledge in a world governed by conjectures with very few scattered results. Indeed at the moment Voisin's conjecture 3.4.1 is still open for a general K3 surface and we know very little about Chow motives.
With this work, we hope to contribute to a bit of enlightening in this direction.

"Les conjectures" y occupaient une place centrale, certes, un peu comme le ferait la capitale d'un vaste empire ou continent, aux provinces innombrables, mais dont la plupart n'ont que des rapports des plus lointains avec ce lieu brillant et prestigieux. Sans avoir eu à me le dire jamais, je me savais le serviteur désormais d'une grande tâche: explorer ce monde immense et inconnu, appréhender ses contours jusqu'aux frontières les plus lointaines; et aussi, parcourir en tous sens et inventorier avec un soin tenace et méthodique les provinces les plus proches et les plus accessibles, et en dresser des cartes d'une fidélité et d'une précision scrupuleuse, où le moindre hameau et la moindre chaumière auraient leur place...
(Grothendieck, [Gro85, 2.17])

Chapter 1

Algebraic cycles and conjectures

In this chapter we introduce the main object of our interest, the group of algebraic cycles of a variety. We give some basic properties and operations on it. Next we present some equivalence relations on algebraic cycles and we focus on Chow groups, i.e. algebraic cycles modulo rational equivalence. Finally we present some classical conjectures about algebraic cycles.

Notation and conventions. *We work on an algebraically closed field k, that usually denotes the field of the complex numbers $\mathbb{C}$. A variety is intended to be a reduced (not necessarily irreducible) scheme. We denote as X_n an irreducible variety of dimension n. We denote as* $\mathrm{SmProj}(k)$ *the category of smooth projective varieties over k.*

1.1 Algebraic Cycles and related conjectures

We consider X a scheme over a field k.

Definition 1.1.1. *An i–*algebraic cycle *Z on X is a finite formal sum*

$$Z = \sum_\alpha n_\alpha [V_\alpha],$$

where $n_\alpha \in \mathbb{Z}$ and V_α are i–dimensional subvarieties of X. The group of i–dimensional algebraic cycles *of X is the free abelian group generated by all the reduced and irreducible closed subvarieties of X of dimension i. We denote this group as $Z_i(X)$.*
Given a subscheme $Y \subset X$ of dimension $j \leq i$, we can consider its associated cycle

$$Z = \sum_\alpha n_\alpha W_\alpha \in Z_i(X),$$

where for any α

$$\begin{cases} n_\alpha := l(\mathcal{O}_{Y,W_\alpha}) \in \mathbb{Z} \text{ is the length of the localization of } \mathcal{O}_Y \text{ at } W_\alpha; \\ W_\alpha \text{ is a irreducible reduced component of } Y; \\ \dim W_\alpha = i. \end{cases}$$

We notice that the subvarieties generating the group are not necessarily smooth. In general, we consider X a smooth irreducible variety of dimension n, and we denote the group of algebraic cycles of X of codimension $n-i$ as

$$Z^{n-i}(X) := Z_i(X).$$

Example 1.1.2.

- $Z^1(X) = \mathrm{Div}(X)$, *when X is smooth, this is the group of Weil divisors on X.*
- $Z^n(X) = Z_0(X)$ *is the group of 0–cycles of X_n. These are formal sums of points on X_n, such as*

$$Z = \sum_\alpha n_\alpha p_\alpha \in Z_0(X), \qquad \text{where } p_\alpha \in X \text{ are points.}$$

  For 0–cycles it is defined the degree *as* $\deg Z = \sum_\alpha n_\alpha$.

We denote the *group of algebraic cycles* on X as

$$Z(X) = \bigoplus_{i=0}^{\dim X} Z^i(X).$$

We can consider also the group of algebraic cycles with coefficients in a field $\mathbb{F}$, usually we consider $\mathbb{F} = \mathbb{Q}$. We denote such a group as

$$Z^i(X)_\mathbb{F} = Z^i(X) \otimes_\mathbb{Z} \mathbb{F}, \qquad Z(X)_\mathbb{F} = \bigoplus_{i \geq 0} Z^i(X)_\mathbb{F}.$$

1.1.1 Operations on cycles

On the group of algebraic cycles we can consider several operations, although they are not always defined. Here we give a brief overview of these operations, and we refer to [Mur14, MNP13] for further details.
Let us consider two smooth irreducible varieties X and Y, then we have the following possible operations.

- **Cartesian Product:** is the natural extension of the cartesian product of subvarieties

$$\begin{aligned} Z_{q_1}(X) \times Z_{q_2}(Y) &\to Z_{q_1+q_2}(X \times Y) \\ (A, B) &\mapsto A \times B. \end{aligned}$$

- **Push–forward:** we consider a proper morphism of varieties $f\colon X \to Y$, and an irreducible subvariety $Z \subseteq X$. We recall that

$$\deg(Z/f(Z)) = \begin{cases} [k(Z)\colon k(f(Z))] \ \text{ if } \dim(f(Z)) = \dim Z; \\ 0 \ \text{ if } \dim(f(Z)) < \dim Z. \end{cases}$$

We define $f_*(Z) = \deg(Z/f(Z))f(Z)$. By linearity, we get a homomorphism of groups that preserves the dimension of the cycles

$$f_*\colon Z_q(X) \to Z_q(X).$$

- **Intersection:** we consider two irreducible subvarieties $V, W \subseteq X$ of codimension i, j respectively. We recall that the intersection number of V and W is defined only for those irreducible components $U_\alpha \subseteq X$ where the intersection is proper, i.e. the codimension is the maximal one $i + j$ (see [Ful98] for further details). When the intersection is proper at every irreducible component, we can define the intersection product as

$$V \cdot W = \sum_\alpha i(V \cdot W; U_\alpha) U_\alpha \in Z^{i+j}(X).$$

We can extend this definition by linearity to obtain the intersection product between algebraic cycles:

$$\begin{cases} Z_1 = \sum_\alpha n_\alpha V_\alpha \in Z^i(X) \\ Z_2 = \sum_\beta m_\beta W_\beta \in Z^j(X) \end{cases} \quad \Rightarrow \quad Z_1 \cdot Z_2 = \sum_{\alpha,\beta} n_\alpha m_\beta \left(V_\alpha \cdot W_\beta\right).$$

We stress out that the intersection product between algebraic cycles is only defined if the intersection as subvarieties is proper on every irreducible components.

- **Pull–back:** we consider a morphism of varieties $f\colon X \to Y$, and its graph $\Gamma_f := \{(x, y) \in X \times Y \colon f(x) = y\}$. Let $Z \subseteq Y$ be a subvariety. We define the pullback of Z as

$$f^* Z = (\pi_X)_* \left(\Gamma_f \cdot (X \times Z)\right),$$

where $\pi_X\colon X \times Y \to X$ is the first projection of the cartesian product. We notice that this is only defined if the intersection $\Gamma_f \cap (X \times Z)$ is proper[1].
We can extend this definition by linearity, and we obtain a homorphism preserving the codimension of the cycles:

$$f^*\colon Z^i(Y) \to Z^i(X).$$

[1]This condition is always satisfied if the morphism f is flat.

- **Correspondence:** it is a cycle in the cartesian product $X \times Y$, i.e. an element of $Z(X \times Y)$. We denote the group of all the correspondences between X and Y as $\mathrm{Corr}(X, Y)$. The transpose of a correspondence $\Gamma \in \mathrm{Corr}(X, Y)$ is ${}^t\Gamma \in \mathrm{Corr}(Y, X)$. Intuitively, correspondences are the graph of multivalued maps. The peculiar aspect of correspondences is that they induce an action on cycles. Indeed, let us consider a correspondence $\Gamma \in Z^d(X \times Y)$, then the induced action is defined by:

$$\begin{aligned} \Gamma \colon Z^i(X_n) &\to Z^{i+d-n}(Y_m) \\ Z &\mapsto (\pi_Y)_* \left(\Gamma \cdot (Z \times Y)\right), \end{aligned}$$

  where $\pi_Y \colon X \times Y \to Y$ is the projection on the second factor. We notice that the action is defined whenever the intersection product $\Gamma \cdot (Z \times Y)$ is defined.

1.1.2 Equivalence relations on algebraic cycles

The operations considered in the last section are not always defined, so it seems natural to look for some equivalence relation on algebraic cycles that could avoid this problem. Meaning that in each equivalence class we can find a suitable element that makes the desired operation defined. There are several equivalence relations on algebraic cycles. In particular, in 1958 Pierre Samuel [Sam60] introduced the notion of *adequate equivalence relation* to denote an equivalence relation with the desired properties. We point out that, when referring to an equivalence relation on algebraic cycles, we are considering a family of equivalence relations, each one defined on $Z^i(X)$ for each $i \geq 0$.

Definition 1.1.3. *The following are the properties defining an* adequate equivalence relation *which we denote as* $\sim$*.*

RA1. Compatibility with grading and addition of cycles. *This condition implies in particular that we have subgroups of trivial cycles of* $Z^i(X)$ *for each* $i \geq 0$*:*

$$Z^i_{\sim}(X) = Z^i(X)/_{\sim} = \{Z \in Z^i(X) \colon Z \sim 0\}.$$

RA2. Compatibility with product of cycles. *If* $Z \in Z^i_{\sim}(X)$ *for some* $i \geq 0$*, then, for every variety* Y*, it holds* $Z \times Y \sim 0$ *on* $X \times Y$*.*

RA3 Compatibility with intersections of cycles. *If* $Z_1 \sim 0$ *and the intersection product* $Z_1 \cdot Z_2$ *is defined , then* $Z_1 \cdot Z_2 \sim 0$*.*

RA4. Compatibility with projections. *If* $Z \in Z^i(X \times Y)_{\sim}$ *for some* $i \geq 0$ *and* π_X *is the first projection, then* $(\pi_X)_* Z \sim 0$ *in* $Z(X)$*.*

RA5. Moving Lemma. *Given $Z, W_1, \ldots, W_l \in Z(X)$, there exists $Z' \sim Z$ such that $Z' \cdot W_i$ is defined for every $i = 1, \ldots, l$.*

If we consider the group of algebraic cycles with coefficients in a field $\mathbb{F}$, we use the following notation

$$Z^i_\sim(X)_\mathbb{F} = Z^i_\sim(X) \otimes_\mathbb{Z} \mathbb{F}.$$

After having defined an equivalence relation on a group, it is natural to consider the quotient with respect to this relation, i.e for any $i \geq 0$ we define

$$C^i_\sim(X) := Z^i(X)/_{Z^i_\sim(X)}, \quad \text{and} \quad C_\sim(X) := \bigoplus_{i\geq 0} C^i_\sim(X).$$

The following proposition characterizes the adequate equivalence relations as the ones that make all the operations on cycles defined.

Proposition 1.1.4 ([Mur14], Proposition 1.8 and 1.9)**.** *Let $\sim$ be an adequate equivalence relation defined on* SmProj(k)*, then*

i) *$C_\sim(X) = \bigoplus_{i\geq 0} C^i_\sim(X)$ is a commutative ring with the intersection product;*

ii) *given a proper morphism of varieties $f \colon X \to Y$, the push–forward $f_* \colon C^\sim_q \to C^\sim_q(Y)$ is an additive homomorphism for any $q \geq 0$, and* $\deg f_* = \dim Y - \dim X$*;*

iii) *for any morphism of varieties $f \colon X \to Y$, the pull–back $f^* \colon C_\sim(Y) \to C_\sim(X)$ is a homomorphism of graded ring, so in particular* $\deg f^* = 0$*;*

iv) *any correspondence $\Gamma \in \mathrm{Corr}(X, Y)$ of degree r, induces a homomorphism $C^i_\sim(X) \to C^{i+r}_\sim(Y)$ that depends only on the equivalence class of the correspondence in $Z(X \times Y)$.*

There are several adequate equivalence relations defined on algebraic cycles. We briefly recall the classical ones.

Rational equivalence

This is the equivalence relation we are mainly interested in. Rational equivalence is a generalization of the linear equivalence for divisors. It was introduced in the thirties by Severi (see for instance [Sev34]) and then formalized in modern terms in 1956 by Samuel and Chow independently ([Cho56], [Sam56]).

We recall that if X is an irreducible variety, and $\varphi \in k(X)^*$ is a rational function on X, then we can define a Weil divisor associated to φ as

$$\mathrm{div}(\varphi) = \sum_{Y \subset X} \mathrm{ord}_Y(\varphi) \cdot Y,$$

where the sum is taken over all the irreducible subvarieties of codimension one. Two divisors $D_1, D_2 \in \mathrm{Div}(X)$ are said to be linearly equivalent, $D_1 \sim_{\mathrm{lin}} D_2$, if there exists a rational function $\varphi \in k(X)^*$ such that $D_1 - D_2 = \mathrm{div}(\varphi)$. The quotient group $\mathrm{Div}(X)/_{\sim_{\mathrm{lin}}}$ is denoted as $CH^1(X)$. For further details on linear equivalence of divisors we refer the reader to [Har77, II.6]. It seems natural to look for a generalization of the idea of linear equivalence to higher codimensional cycles. We consider the subgroup $Z^i_{\mathrm{rat}}(X) \subset Z^i(X)$ generated by cycles of type $Z = \mathrm{div}(\varphi)$, where $\varphi \in k(Y)^*$ and $Y \subseteq X$ is an irreducible subvariety of codimension $i-1$.

Remark 1.1.5. First of all, we notice that Y need not to be smooth. Secondly, let's check that the codimension required are correct: $\mathrm{div}(\varphi) \in Z^1(Y)$, hence

$$\mathrm{codim}_X(\mathrm{div}(\varphi)) = \mathrm{codim}_Y(\mathrm{div}(\varphi)) + \mathrm{codim}_X(Y) = 1 + i - 1 = i.$$

So, $\mathrm{div}(\varphi) \in Z^i(X)$.

Equivalently, we can define an algebraic cycle $Z \in Z^i(X)$ to be *rationally equivalent* to 0 if and only if there exists a finite colloection $\{Y_\alpha, \varphi_\alpha\}_\alpha$, where $\{Y_\alpha\}_\alpha$ are irreducible subvarieties of codimension $i-1$, and $\varphi_\alpha \in k(Y_\alpha)^*$ such that $Z = \sum_\alpha \mathrm{div}(\varphi_\alpha)$.

Remark 1.1.6. In [Voi14b], the more general case of a scheme X over a field k is considered. In this setting, $Z_k^{\mathrm{rat}}(X)$ is defined as the subgroup generated by cycles of the form $\tau_* \mathrm{div}(\varphi)$, where $\varphi \in k(\widetilde{W})^*$, $W \subset X$ is an irreducible closed subvariety of dimension $k+1$, and $\tau \colon \widetilde{W} \to W$ is the normalization.

We recall here an alternative definition of rational equivalence which is due to Samuel and Chow (see [Ful98, 1.6]).

Definition 1.1.7. *Two algebraic cycles $W, W'' \in Z^i(X)$ are* rationally equivalent *if and only if there exists an algebraic cycle $Z \in Z^i(X \times \mathbb{P}^1)$ such that $Z|_{X\times\{0\}} = W$ and $Z|_{X\times\{1\}} = W'$.*

Given the definition of rational equivalence, we can consider the quotient group

$$CH^i(X) := Z^i(X)/_{Z^i_{\mathrm{rat}}(X)},$$

this group is called the i^{th}–*Chow group.*
Usually we are interested in Chow groups with rational coefficients that we denote as

$$CH^i(X)_{\mathbb{Q}} := Z^i(X)/_{Z^i_{\mathrm{rat}}(X)} \otimes \mathbb{Q}.$$

If X is purely dimensional and $\dim X = d$, we use the convention

$$CH_j(X) := CH^{d-j}(X).$$

Example 1.1.8. *If X is smooth, $CH^1(X) = \mathrm{Pic}(X)$ is the most famous Chow group, i.e. the Picard group of X, i.e. the group of isomorphism classes of line bundles on X.*

We recall some basic results on rational equivalence, we refer the reader to [Mur14, ch.1] for the proofs .

Theorem 1.1.9. *i) Rational equivalence is an adequate equivalence relation.*

ii) Homotopy property. *Let $\mathbb{A}^n$ be an affine space of dimension n and let us consider the first projection $\pi\colon X \times \mathbb{A}^n \to X$. Then the pullback*

$$\pi^*\colon CH^i(X) \to CH^i(X \times \mathbb{A}^n)$$

is an isomorphism for any $i \geq \dim X$.

iii) Localization sequence. *Let $Y \subset X$ be a closed subvariety of X and let $U = X - Y$. Let $i\colon Y \hookrightarrow X$, $j\colon U \hookrightarrow X$ be the inclusions. Then we have an exact sequence*

$$CH_i(Y) \xhookrightarrow{i_*} CH_i(X) \xrightarrow{j^*} CH_i(U) \to 0.$$

Algebraic equivalence

Algebraic equivalence was first introduced by Weil in 1952 ([Wei54]), and essentially it is obtained as a generalization of rational equivalence, substituting $\mathbb{P}^1$ with a smooth curve in Definition 1.1.7.

Definition 1.1.10. *We say that an algebraic cycle $Z \in Z(X)$ is* algebraically *trivial, $Z \sim_{\text{alg}} 0$, if there exist a smooth irreducible curve C, a correspondence $\Gamma \in Z^i(C \times X)$ and two points $a, b \in C$ such that $Z = \Gamma(a) - \Gamma(b)$.*

We get an equivalent definition by replacing Γ with $\Gamma - (C \times \Gamma(b))$ and asking that

$$(\Gamma - (C \times \Gamma(b)))\,(a) = Z, \qquad \text{and} \qquad (\Gamma - (C \times \Gamma(b)))\,(b) = 0.$$

Remark 1.1.11. It clearly holds that $Z^i_{\text{rat}}(X) \subset Z^i_{\text{alg}}(X)$, but in general the inclusion is strict. Indeed, if we consider the case of an elliptic curve X and we take two points on it $a, b \in X$, we can consider the cycle $Z = a - b$. Then Z is not the divisor of a function, hence $Z \nsim_{\text{rat}} 0$, but $Z \sim_{\text{alg}} 0$.

In particular, algebraic equivalence is an adequate equivalence relation (see [Sam60]), and Propostion 1.1.4 holds for $\sim_{\text{alg}}$.

Smash-nilpotent equivalence

Smash–nilpotent equivalence was introduce in 1995 by Voevodsky ([Voe95]) with the purpose to address a classical conjecture concerning equivalence relation on cycles (see conjecture $D(X)$ in Section 1.2).

Definition 1.1.12. *An algebraic cycle $Z \in Z(X)$ is* smash–nilpotent *trivial, $Z \sim_{\otimes} 0$, if there exists a natural number $N > 0$ such that the product of N copies of Z is rationally trivial, i.e. $Z^N = Z \times \cdots \times Z \sim_{\text{rat}} 0$. We denote the subgroup of smash–nilpotent trivial cycles as*

$$Z^i_{\otimes}(X) = \{Z \in Z(X) \colon Z \sim_{\otimes} 0\}.$$

This is indeed a subgroup of $Z^i(X)$ (see [MNP13, Proposition 1.2.10]).

In 1995–1996 Voisin and Voevodsky proved independently the following important result.

Theorem 1.1.13 ([Voe95], [Voi96])**.**

$$Z^i_{\text{alg}}(X)_{\mathbb{Q}} \subset Z^i_{\otimes}(X)_{\mathbb{Q}}$$

for any $i \geq \dim X$.

Remark 1.1.14. The inclusion $Z^i_{\text{alg}}(X)_{\mathbb{Q}} \subset Z^i_{\otimes}(X)_{\mathbb{Q}}$ is strict. Indeed, in 2009 Kahn and Sebastian proved that for any abelian variety A of dimension 3 it holds $Z^2_{\otimes}(A) = Z^2_{\text{hom}}(A)$ ([KS09]). Let us consider then the *Ceresa cycle*, i.e. we consider a very general curve of genus $g \geq 3$ and the cycle $Z = C - C^-$ in the Jacobian $J(C)$, where C^- is the image of C in $J(C)$ under the map $x \mapsto -x$. Then Z is is not algebraically equivalent to zero, but it is homologically equivalent to zero. So this gives an example of a cycle such that $Z \in Z_1^{\otimes}(J(C))$, but $Z \notin Z_1^{\text{alg}}(J(C))$ (see [Mur14, Theorem 4.14]).

Homological equivalence

In order to give the definition of homological equivalence, we need first to introduce the notion of Weil cohomology. In general we can work on any field $\mathbb{F}$ of characteristic 0, considering the category $\text{GrVect}_{\mathbb{F}}$ of finite dimensional graded vector spaces over $\mathbb{F}$. Let $\text{SmProj}(k)^{\text{opp}}$ denote the opposite category of $\text{SmProj}(k)$, i.e. the category with the same objects as $\text{SmProj}(k)$ but with reverse arrows.

A *Weil cohomology theory* is a functor

$$H \colon \text{SmProj(k)}^{\text{opp}} \to \text{GrVect}_{\mathbb{F}}$$

such that it satisfies all of the following properties.

1. There exists a graded, super–commutative *cup product*

$$\cup \colon H(X) \times H(X) \to H(X),$$

i.e. such that for any $a \in H^i(X)$ and any $b \in H^j(X)$ we get

$$b \cup a = (-1)^{ij} a \cup b \in H^{i+j}(X).$$

2. *Poincaré duality* holds: there exists a trace isomorphism

$$\mathrm{Tr}\colon H^{2d}(X_d) \xrightarrow{\sim} \mathbb{F}$$

such that we get a perfect pairing

$$H^i(X_d) \times H^{2d-i}(X_d) \xrightarrow{\cup} H^{2d}(X_d) \xrightarrow{\mathrm{Tr}} \mathbb{F}.$$

3. *Künneth formula* holds: there is a graded isomorphism given by

$$(\pi_X)^* \otimes (\pi_Y)^*\colon H(X) \otimes H(Y) \to H(X \times Y),$$

where π_X, π_Y are the projections of $X \times Y$ on the first and second factor respectively.

4. For any $i \geq \dim X$, there exists a *cycle class map*

$$\mathrm{cl}^i_X\colon CH^i(X) \to H^{2i}(X)$$

such that it satisfies the following conditions.

- Functioriality. In the sense that if we consider a morphism in $\mathrm{SmProj}(k)$
$$f\colon X \to Y,$$
then pull–backs and push–forwards commute with the cycle class maps, i.e.
$$\begin{cases} f^* \circ \mathrm{cl}_Y = \mathrm{cl}_X \circ f^*; \\ f_* \circ \mathrm{cl}_X = \mathrm{cl}_Y \circ f_*. \end{cases}$$
- Multiplicitivity. Cycle class maps are compatible with products, i.e.
$$\mathrm{cl}_X(Z \times W) = \mathrm{cl}_X(Z) \cup \mathrm{cl}_X(W)$$
for any $Z, W \in CH^i(X)$.
- Calibration. Cycle class maps are compatible with points, in the sense that the following diagram commutes for any point $p \in X$
$$\begin{tikzcd} CH^0(p) \arrow[r, "\mathrm{cl}_p"] \arrow[d, "\deg"] & H^0(p) \arrow[d, "\mathrm{Tr}"] \\ \mathbb{Z} \arrow[r, hook] & \mathbb{F}. \end{tikzcd}$$
- *Weak Lefschetz* thoerem holds, i.e. if we consider a smooth hyperplane section $Y_{d-1} \overset{\iota}{\hookrightarrow} X_d$, then the induced pullback $H^i(X) \overset{\iota^*}{\to} H^i(Y)$ is an isomorphism for any $i < d-1$ and it's injective for $i = d-1$. Refer to Theorem A.2.19 for further details.

- *Hard Lefschetz* theorem holds. If we consider a smooth hyperplane section $Y_{d-1} \overset{\iota}{\hookrightarrow} X_d$ we define the Lefschetz operator $L\colon H^i(X) \to H^{i+2}(X)$ by $L(\alpha) = \alpha \cup \mathrm{cl}_X(Y)$. Then there are induced isomorphisms

$$L^{d-i}H^{d-i}(X) \xrightarrow{\sim} H^{d+i}(X).$$

We refer the reader to A.2.17 for a more detailed discussion on this topic.

Examples of Weil cohomology theories are Betti cohomology, De Rham cohomology, étale cohomology and cristalline cohomology.

Remark 1.1.15. The properties listed above imply that the cycle class map cl_X is compatible with the intersection products. Indeed let us consider two properly intersecting algebraic cycles $Z, W \in Z(X)$ and the diagonal map $\delta\colon X \to X \times X$, defined by $x \mapsto (x, x)$, then

$$\mathrm{cl}_X(Z \cdot W) = \mathrm{cl}_X \delta^*(Z \times W) = \delta^*(\mathrm{cl}_X(Z) \otimes \mathrm{cl}_X(W)) = \mathrm{cl}_X(Z) \cdot \mathrm{cl}_X(W).$$

We denote the image of the cycle class map as

$$A^i(X) := \mathrm{Im}\left(\mathrm{cl}^i_X\colon CH^i(X) \to H^{2i}(X)\right),$$

and we call elements of $A^i(X) \subset H^{2i}(X)$ *algebraic classes.*

Definition 1.1.16. *We fix a Weil cohomology theory H. We say that a cycle $Z \in Z(X)$ is* homologically *trivial, $Z \sim_{\mathrm{hom}} 0$, if its cycle class is trivial, i.e. $\mathrm{cl}_X(Z) = 0$.*

Remark 1.1.17. We stress out that a priori, this relation depends on the choice of the Weil cohomology theory. We refer to Section 1.2 fur further details on this issue.

We denote the subgroup of homologically trivial cycles as

$$Z^i_{\mathrm{hom}}(X) := \{Z \in Z^i(X)\colon Z \sim_{\mathrm{hom}} 0\}.$$

When no confusion can arise, we denote the cycle class of an algebraic cycle Z as $[Z] := \mathrm{cl}(Z)$.

Let us have a look now at the relation between homological equivalence and the other adequate equivalence relations.

- *Algebraic equivalence and homological equivalence.*
First of all, we notice that there is an inclusion $Z^i_{\mathrm{alg}}(X) \subset Z^i_{\mathrm{hom}}(X)$. Indeed, let us consider two points on a curve C. They are algebraically equivalent and it holds that $\mathrm{cl}_C(a) = \mathrm{cl}_C(b)$, so we get a homologically trivial cycle $\mathrm{cl}_C(a) - \mathrm{cl}_C(b)$. Let us consider an algebraically trivial

cycle $Z \in Z(X)$, then there exists a correspondence $\Gamma \in Z(C \times X)$ such that $Z = \Gamma(a) - \Gamma(b) = (\pi_X)_*(\Gamma \cdot (\mathrm{cl}_C(a) - \mathrm{cl}_C(b))$ and so $Z \sim_{\mathrm{hom}} 0$.
For divisors, Matsusaka proved in [Mat57] that equality with rational coefficients holds, i.e. $Z^1_{\mathrm{alg}}(X)_{\mathbb{Q}} = Z^1_{\mathrm{hom}}(X)_{\mathbb{Q}}$.
However, for $1 < i < \dim X$, it was proved by Griffiths in 1969 that these two relation are different (see [Gri69]).

- *Smash–nilpotent equivalence and homological equivalence.*
By Künneth formula, we have that inclusion $Z^i_{\otimes}(X) \subset Z^i_{\mathrm{hom}}(X)$ holds. Indeed, if we consider a smash–nilpotent trivial cycle $Z \in Z(X)$, then $Z^N \sim_{\mathrm{rat}} 0$ for some integer $N > 0$. Then $0 = \mathrm{cl}_X(Z^N) = \mathrm{cl}_X(Z) \cup \cdots \cup \mathrm{cl}_X(Z)$, and we get that $\mathrm{cl}_X(Z) = 0$, so $Z \sim_{\mathrm{hom}} 0$.
Equality has been conjectured by Voevodsky, and this would imply that homological equivalence does not depend on the choice of the Weil cohomology theory.

Numerical equivalence

This is the relation that Grothendieck had in mind when developing the theory of motives (see Chapter 2).

Definition 1.1.18. *We say that an algebraic cycle $Z \in Z^i(X)$ is* numerically trivial*, $Z \sim_{\mathrm{num}} 0$, if for any cycle $W \in Z^{d-i}(X)$ for which the intersection product $Z \cdot W = \sum_\alpha n_\alpha p_\alpha \in Z_0(X)$ is defined the degree is zero, i.e.*

$$\deg(Z \cdot W) = \sum_\alpha n_\alpha = 0.$$

Here the sum is taken over the irreducible components $\{U_\alpha\}_\alpha$ of X (see the definition of intersection product in Section 1.1.1).

Remark 1.1.19. If the intersection product $Z \cdot W$ is not defined, by the Moving Lemma (RA5 for adequate equivalence relations in Definition 1.1.3), we can find an equivalent cycle Z' such that the intersection is proper, hence it makes sense to define $\deg(Z \cdot W) := \deg(Z' \cdot W)$.

The inclusion $Z^i_{\mathrm{hom}}(X) \subset Z^i_{\mathrm{num}}(X)$ follows by the compatibility properties of Weil cohomology theories. Indeed let us consider $Z \in Z_0(X)$, we have that $\deg(Z) = \mathrm{Tr} \circ \mathrm{cl}_X(Z)$, so that homological and numerical equivalence coincide for 0–cycles. If $Z \in Z^i_{\mathrm{hom}}(X)$ with $i < d = \dim X$ and we consider $W \in Z^{d-i}(X)$, then

$$\deg(Z \cdot W) = \mathrm{Tr} \circ \mathrm{cl}_X(Z \cdot W) = \mathrm{Tr}\,(\mathrm{cl}_X(Z) \cup \mathrm{cl}_X(W)) = 0.$$

For $i = 1$, Matsusaka proved in [Mat57] that equality holds in arbitrary characteristic for divisors. Lieberman's Theorem ([Lie68, Theorem 1]) gives some conditions in the case $\mathrm{char}(k) = 0$ for which equality actually holds for any $i \geq \dim X$.

In 1934 Severi was already familiar with the notion of equivalence relations between algebraic cycles. He refers to numerical equivalence as arithmetic equivalence, and he believed that it actually coincides with algebraical equivalence, which however is false. He was so sure about this, that he decided to denote these two relations with the same notation:

> *I don't change the sign because, as we are going to see further, it is extremely likely that algebraic equivalence and arithmetic equivalence are nothing more than different aspects of the same concept .* [2]
> (Severi, [Sev34, p. 143])

Remark 1.1.20. We summarize here the relation between numerical equivalence and other adequate equivalence relations (see [MNP13, Lemma 1.2.18]):

i) $Z^i_{\mathrm{alg}}(X) \subset Z^i_{\mathrm{hom}}(X)$;

ii) $Z^i_{\otimes}(X) \subset Z^i_{\mathrm{hom}}(X)$;

iii) $Z^i_{\mathrm{hom}}(X) \subset Z^i_{\mathrm{num}}(X)$.

Inclusions between adequate equivalence relations

Let us summarize the known result on the adequate equivalence relations we introduced. First of all, Samuel proved in [Sam60] that rational equivalence is the finest one among adequate equivalence relations, and it is known that numerical is the coarsest one (see Remark 1.1.20). We have the following inclusions

$$Z^i_{\mathrm{rat}}(X) \subset Z^i_{\mathrm{alg}}(X) \subset Z^i_{\mathrm{hom}}(X) \subseteq Z^i_{\mathrm{num}}(X) \subsetneq Z^i(X).$$

When we consider $\mathbb{Q}$ coefficients we get

$$Z^i_{\mathrm{alg}}(X)_{\mathbb{Q}} \subset Z^i_{\otimes}(X)_{\mathbb{Q}} \subseteq Z^i_{\mathrm{hom}}(X)_{\mathbb{Q}} \subseteq Z^i_{\mathrm{num}}(X)_{\mathbb{Q}}. \tag{1.1.21}$$

Hence we get inclusions of Chow groups:

$$CH^i_{\mathrm{alg}}(X)_{\mathbb{Q}} \subset CH^i_{\mathrm{hom}}(X)_{\mathbb{Q}} \subset CH^i_{\mathrm{num}}(X)_{\mathbb{Q}} \subsetneq CH^i(X)_{\mathbb{Q}}.$$

1.2 Standard conjectures

We give a brief presentation of Grothendieck's standard conjectures on algebraic cycles, for further details we refer the reader to the original paper of 1968 ([Gro69]) and Keliman's paper ([Kle68]).
Standard conjectures are motivated by Grothedieck's attempt to solve Weil's

[2]Free translation from the original italian sentence "*Non cangio il segno perché, come vedremo in seguito, é estremamente probabile che equivalenza algebrica ed equivalenza aritmetica non siano che aspetti diversi di un medesimo concetto*".

conjecture on the ζ–functions of algebraic varieties (which was then solved by Deligne with a completely different approach). However, their consequences are far more deep and they form the basis of the "yoga of motives" (see Chapter 2). In particular, standard conjectures imply that the category of numerical equivalent motives is semisimple abelian, and numerical equivalence is the only equivalence relation on cycles that provides this result. Grothendieck described this fact as a "miracle". In his words:

> "*Alongside the problem of resolution of singularities, the proof of the standard conjectures seems to me the most urgent task in algebraic geometry*".
> (Grothendieck, [Gro69]).

We fix a Weil cohomology theory H on a field $\mathbb{F}$ with $\mathrm{char}\mathbb{F} = 0$, usually $\mathbb{F}$ is $\mathbb{Q}$ or $\mathbb{Q}_l$.

Künneth conjecture $C(X)$

We consider the diagonal of X, $\Delta := \{(x, x) : x \in X\} \subset X \times X$ and its cycle class in cohomology $\mathrm{cl}_X(\Delta) \in H^{2d}(X \times X)$. By the Künneth decomposition ([Voi07a, Theorem 11.38]) we have a decomposition of the diagonal

$$\Delta = \bigoplus_{i=0}^{2d} \Delta_i,$$

where $\Delta_i \in H^{2d-i}(X) \otimes H^i(X)$ is called the ith Künneth component of Δ.

Conjecture 1.2.1 (C(X)). *The Künneth components of the diagonal are algebraic, i.e. for any $i \geq 2d$ there exists algebraic cycles $\pi_i \in CH^d(X \times X)_{\mathbb{Q}}$ such that $\mathrm{cl}_{X\times X}(\pi_i) = \Delta_i$.*

This conjecture is known for projective spaces, Grassmanians, flag varieties, curves, surfaces and abelian varieties.

Lefschetz type conjectures $B(X)$ and $A(X, L)$

We consider the Lefschetz operator

$$\begin{aligned} L \colon H^i(X) &\to H^{i+2}(X) \\ \alpha &\to \alpha \cup \mathrm{cl}_X(H), \end{aligned}$$

where $H \hookrightarrow X$ is a smooth hyperplane section of X. In particular, we notice that, by construction, L is algebraic. Since we are considering a Weil cohomology theory H, we can assume that Hard Lefschetz theorem holds, i.e. for any $i \geq d$ we have isomorphisms given by the iteration of L:

$$L^j \colon H^{d-i}(X) \xrightarrow{\sim} H^{d+i}(X).$$

By means of the Hard Lefschetz theorem, for any $2 \geq i \geq 2d$ we can define a unique linear map $\Lambda\colon H^i(X) \to H^{i-2}(X)$ such that the following diagrams commute

$$\text{for } 0 \geq j \geq d-2: \quad \begin{tikzcd} H^{d-j}(X) \arrow[r, "L^j", "\sim"'] \arrow[d, dashed, "\Lambda"'] & H^{d+j}(X) \arrow[d, "L"] \\ H^{d-j-2}(X) \arrow[r, "L^{j+2}", "\sim"'] & H^{d+j+2}(X) \end{tikzcd}$$

$$\text{for } 2 \geq j \geq d: \quad \begin{tikzcd} H^{d-j+2}(X) \arrow[r, "L^{j-2}", "\sim"'] & H^{d+j-2}(X) \\ H^{d-j}(X) \arrow[u, "L"] \arrow[r, "L^j", "\sim"'] & H^{d+j}(X) \arrow[u, dashed, "\Lambda"'] \end{tikzcd}$$

$$\text{and } H^{d-1}(X) \rightleftarrows H^{d+1}(X),$$

i.e. Λ is almost an inverse of the Lefschetz operator L.

Conjecture 1.2.2 (B(X)). *Λ is algebraic, i.e. there exists an algebraic cycle $Z \in CH^{d-1}(X \times X)_{\mathbb{Q}}$ such that $\Lambda = \mathrm{cl}_{X\times X}(Z)$.*

Here we are using the interpretation of Λ as a topological correspondence, i.e. an element of $H^*(X \times X) \supset H^*(X) \otimes H^*(X) \cong H^*(X)^\vee \otimes H^*(X)$.

We introduce another conjecture of Lefschetz type. We consider the commutative diagram

$$\begin{tikzcd} H^{2i}(X) \arrow[r, "L^{d-2i}", "\sim"'] & H^{2d-2i}(X) \\ A^i(X) \arrow[u, "\mathrm{cl}^i_X"] \arrow[r, hook, "l"] & A^{d-i}(X), \arrow[u, "\mathrm{cl}^{d-i}_X"'] \end{tikzcd} \tag{1.2.3}$$

where $A^i(X) = \mathrm{Im}(\mathrm{cl}^i_X) \subset H^{2i}(X)$ denotes the algebraic classes. By Hard Lefschetz theorem, we have that the lower map l is injective.

Conjecture 1.2.4 (A(X,L)). *The map l in the above diagram 1.2.3 is an isomorphism.*

There is an alternative formulation of this conjecture: the cup product pairing $A^i(X) \times A^{d-i}(X) \to \mathbb{Q}$ is non–degenerate (see [MNP13, Section 3.1.2]).

Remark 1.2.5. We briefly summarize some results on $B(X)$ and $A(X, L)$, for further details we refer the reader to [Kle68], [Gro69].

i) $B(X)$ implies $C(X)$;

ii) Λ^{d-1} is algebraic, indeed its given by the class of a divisor;

iii) $B(X)$ implies $A(X, L)$;

iv) $B(X)$ holds for projective spaces, Grasmmanians, curves, surfaces and abelian varieties.

Conjecture $D(X)$

We consider an algebraically closed field k.

Conjecture 1.2.6 (D(X)).

$$Z^i_{\text{hom}}(X)_{\mathbb{Q}} = Z^i_{num}(X)_{\mathbb{Q}}.$$

This conjecture was proven by Matsusaka for divisors in arbitrary characteristic ([Mat57]). For char$k = 0$ it was proven for $i = 2$, for dimension 1 and for abelian varieties ([Lie68]).

As an attempt to solve this conjecture, Voevodsky introduced the smash–nilpotent equivalence (see Definition 1.1.12) and formulated the following conjecture ([Voe95]).

Conjecture 1.2.7 (Voevodsky's conjecture).

$$Z^i_{\otimes}(X)_{\mathbb{Q}} = Z^i_{num}(X)_{\mathbb{Q}}.$$

This conjecture implies the standard conjecture $D(X)$ for every Weil cohomology theory, because its formulation is independent of the choice of the cohomology and the following inclusions hold (see (1.1.21))

$$Z^i_{\otimes}(X)_{\mathbb{Q}} \subset Z^i(X)_{\mathbb{Q}} \subset Z^i_{num}(X)_{\mathbb{Q}}.$$

Conjecture of Hodge type

Definition 1.2.8. *We define the* ith primitive cohomology *as the kernel of* L^{d-i+1}*, namely*

$$P^i(X) := \ker\left(L^{d-i+1} \colon H^i(X) \to H^{2d-i+2}(X)\right).$$

We define the primitive algebraic classes *as* $A^i_{\text{prim}} := A^i(X) \cap P^i(X)$.
For any $i \leq d/2$ *we have a pairing* $A^i_{\text{prim}} \times A^i_{\text{prim}} \xrightarrow{q} \mathbb{Q}$ *which is defined by* $(x, y) \mapsto (-1)^i \mathrm{Tr} \circ \left(L^{d-2i}(x) \cup y\right)$.

Conjecture 1.2.9 (Hdg(X))**.** *The pairing* q *of Definition A.2.16 is positive definite.*

This conjecture is true if char$k = 0$ and it holds for surfaces in arbitrary characteristic (see [MNP13, Section 3.1.3]).

1.3 The Hodge conjecture

The Hodge conjecture is one of the seven "Millennium Problems" in the list of the Clay Mathematics Institute. It was formulated by Hodge in 1941 as a question. As Beauville notices in [Bea08], the term "conjecture" does not seem so appropriate, since at the moment there is not enough evidence that a positive answer is plausible. Indeed, even if the conjecture does hold up to dimension three, from dimension four on it seems equally difficult to provide either a proof or a counterexample of it. Moreover, the original formulation by Hodge must be corrected in some way, since it is known to be false stated as it was.

The Hodge conjecture concerns only smooth complex projective varieties, so from now on X will be such a variety. In particular, we notice that every smooth projective variety inherits a Kähler metric from the projective space (see Corollary A.2.6), thus we are dealing with a Kähler manifold for which the Hodge decomposition (A.2.12) holds.

Definition 1.3.1. *An* Hodge class *of degree* $2k$*, or of type* (k,k)*, is an element of the set*

$$\mathrm{Hdg}^{2k}(X) = H^{k,k}(X,\mathbb{Q}) := H^{k,k}(X) \cap H^{2k}(X,\mathbb{Q}).$$

A Hodge class $\alpha \in \mathrm{Hdg}^{2k}(X)$ *is said to be* algebraic *if it is a rational linear combination of classes of algebraic cycles, i.e.*

$$\alpha = \sum_i q_i\,[Z_i] \qquad \textit{with } q_i \in \mathbb{Q},\ Z_i \in Z^k(X) \quad \forall i.$$

Let us now consider $Z \in Z^k(X)$ and the inclusion $i\colon Z \hookrightarrow X$. By applying the projection formula (A.1.24), we get that for any $\alpha \in H^*(X,\mathbb{Z})$ the following holds:

$$\begin{aligned} i_!\,(\beta \cup i^*(\alpha)) = i_!\,(\beta) \cup \alpha \quad \forall \beta \in H^*(Z,\mathbb{Z}) \\ \Rightarrow\ i_! i^*(\alpha) = i_!(1) \cup \alpha \\ \Leftrightarrow\ i_! i^*\,(\alpha) = [Z] \cup \alpha. \end{aligned} \tag{1.3.2}$$

Now we consider $\alpha \in H^{2n-2k}(X,\mathbb{Z})$ and $[Z] \in H^{2k}(X,\mathbb{Z})$, with $\dim_{\mathbb{R}} X = 2n$ and $\dim_{\mathbb{R}} Z = 2n-2k$. We consider the Gysin morphism (see (A.1.23)) $i_! = \mathrm{PD}^{-1} \circ i_* \circ \mathrm{PD}$:

$$\begin{array}{ccccccc} H^{2n-2k}(Z,\mathbb{Z}) & \overset{\mathrm{PD}}{\cong} & H_0(Z,\mathbb{Z}) & \xrightarrow{i_*} & H_0(X,\mathbb{Z}) & \overset{\mathrm{PD}^{-1}}{\cong} & H^{2n}(X,\mathbb{Z}) \\ i^*\,(\alpha) & \mapsto & \mathrm{PD}\,(i^*\,(\alpha)) & \mapsto & i_*\,(\mathrm{PD}\,(i^*\,(\alpha))) & \mapsto & i_!\,(i^*\,(\alpha)). \end{array}$$

Since the Poincaré duality map at top degree is given by integration, we have that

$$\begin{aligned} \int_X \alpha \cup [Z] \overset{(1.3.2)}{=} \int_X i_! i^*\,(\alpha) = \mathrm{PD}\,(i_! i^*\,(\alpha)) = i_*\,(\mathrm{PD}\,(\mathrm{i}^*(\alpha))) \\ = i_* \int_Z i^*(\alpha). \end{aligned} \tag{1.3.3}$$

The following proposition states that the fundamental class of any algebraic cycle is an integral Hodge class.

Proposition 1.3.4. *If $Z \in Z^k(X)$, then $[Z] \in H^{k,k}(X, \mathbb{Z})$.*

Proof. We prove the result for a smooth projective subvariety Z of codimension k, i.e. $\dim_{\mathbb{R}} Z = 2n - 2k$.
We recall that, by Poincaré duality (A.1.21) and the Hodge decomposition (A.2.12), we have a perfect pairing given by

$$\begin{array}{ccccc} H^{a,b}(X) \times H^{n-a,n-b}(X) & \xrightarrow{\cup} & H^{n,n}(X) & \xrightarrow{\sim} & \mathbb{Z} \\ (\alpha, \beta) & \mapsto & \alpha \cup \beta & \mapsto & \int_X \alpha \cup \beta. \end{array} \tag{1.3.5}$$

In particular, for any $\alpha \in H^{2n-2k}(X, \mathbb{C})$ and any $[Z] \in H^{2k}(X, \mathbb{Z})$ it holds

$$\int_X \alpha \cup [Z] = \sum_{a+b=k} \int_X \alpha_{n-a,n-b} \cup [Z]_{n-a,n-b}\,.$$

Hence we want to show that $\int_X \alpha_{n-a,n-b} \cup [Z]_{n-a,n-b} = 0$ if $(a, b) \neq (k, k)$. Let $a, b \in \mathbb{N}$, such that $a+b = 2k$ and $a, b \neq k$, then we have two possibilities: or $a > k$, or $b > k$. Let us suppose that $b > k$, so we consider $a < k$ and $n - a > n - b$. For dimensional reasons, it is clear that $H^{n-a,n-b}(Z, \mathbb{C}) = 0$ and so $i^* (\alpha_{n-a,n-b}) = 0$. Finally we have

$$\int_X \alpha_{n-a,n-b} \cup [Z]_{n-a,n-b} = \int_X i_! \left(i^* \left(\alpha_{n-a,n-b}\right)\right) = i_* \int_Z i^* \left(\alpha_{n-a,n-b}\right) = 0.$$

Since this is true for any $\alpha_{n-a,n-b} \in H^{n-a,n-b}(X, \mathbb{Z})$ and the pairing (1.3.5) is perfect, we get that $[Z]_{a,b} = 0$ for all $a, b \neq k$ and so we get that $[Z]$ is a Hodge class, namely $[Z] = [Z]_{k,k} \in H^{k,k}(X, \mathbb{Z})$.
The case $a > k$ is proved analogously. □

By the above proposition, we can restrict the target space of the cycle class map to Hodge classes

$$\begin{array}{rcl} Z^k(X) & \xrightarrow{\mathrm{cl}_X} & H^{k,k}(X, \mathbb{Z}) := H^{k,k}(X) \cap H^{2k}(X, \mathbb{Z}) \\ Z & \mapsto & [Z]\,. \end{array}$$

The original formulation of the Hodge conjecture asks if this map is surjective, i.e. $\forall\, [Z] \in H^{k,k}(X, \mathbb{Z})$ we can express it as $[Z] = \sum_i n_i [Z_i]$ where $n_i \in \mathbb{Z}$ and $Z_i \in Z^k(X)$ for all i.
In 1962 M. F. Atiyah and F. Hirzebruch in [AH62] found an example of an integral class with torsion which is not algebraic, so at that point the question was whether the conjecture does hold for torsion-free class. The counterexample of Kollár in [Kol92, Lemma p.134] made it clear that for integral classes the conjecture does not hold even in the torsion-free case. Hence we have to consider cohomology classes with rational coefficients and the modern formulation of the Hodge conjecture is the following.

Conjecture 1.3.6 (**HC**: Hodge conjecture)**.** *For any smooth complex projective variety X and any k, all classes in $H^{k,k}(X, \mathbb{Q})$ are algebraic.*

We present some reductions of the conjecture to help realize the difficulty of the problem.
For sake of simplicity we use the following notation:

- $\mathrm{HC}(X,k)_{(\mathrm{prim})}$: for any $\alpha \in \mathrm{Hdg}^{2k}(X)_{(\mathrm{prim})}$, α is algebraic;
- $\mathrm{HC}(X)$: for any k, $\mathrm{HC}(X,k)$;
- $\mathrm{HC}(d)$: for any variety X of dimension d, $\mathrm{HC}(X)$;
- HC: for any X, $\mathrm{HC}(X)$.

In the following proposition we see a first easy reduction of the Hodge conjecture to primitive cohomology.

Proposition 1.3.7. *$HC(X) \iff$ All the primitive Hodge classes of X are algebraic.*

Proof. One implication is easy: if the Hodge conjecture holds for a variety X, then all the Hodge classes are algebraic, also the primitive ones. So we just need to prove that the converse holds.
Let us suppose now that any $\alpha \in \mathrm{Hdg}^*(X)_{\mathrm{prim}} = \mathrm{Hdg}^*(X) \cap H^{2k}(X, \mathbb{Q})_{\mathrm{prim}}$ is algebraic. We notice that the cup product between algebraic classes is algebraic, and it is given by the intersection form. Indeed, if we consider two algebraic classes $\eta \in H^k(X, \mathbb{Q})$ and $\gamma \in H^l(X, \mathbb{Q})$, then

$$\eta = \sum_i q_i \, [Z_i] \,, \quad \gamma = \sum_j r_j \, [Q_j] \; \Rightarrow \; \eta \cup \gamma = \sum_{i,j} r_j q_i \, [Z_i \, .Q_j] \,,$$

with $r_j, q_i \in \mathbb{Q}$ for any i, j; $Z_i \in Z^k(X)$ for any i and $Q_j \in Z^l(X)$ for all j. Moreover, we notice that the Kähler form is algebraic since it is the fundamental class of a hyperplane section $\omega = [H]$. Hence the Lefschetz operator is algebraic, i.e. for any $\eta = \sum_i q_i \, [Z_i]$:

$$L(\eta) = \eta \cup \omega = \sum_i q_i \, [H.Z_i] \, .$$

Due to the Lefschetz decomposition (A.2.18) we have:

$$H^{2k}(X, \mathbb{Q}) = \bigoplus_{i \geq 0} L^i H^{2k-2i}(X, \mathbb{Q})_{\mathrm{prim}},$$

so $\mathrm{HC}(X,k)$ follows for any k, and we conclude. □

Next we present another application of Hard Lefschetz Theorem A.2.17 that actually halves the problem.

Proposition 1.3.8. HC(X,k) *with* $k \leq \frac{n}{2}$ $\Rightarrow$ HC$(X, n-k)$.

Proof. Let $k \leq \frac{n}{2}$ and let $\zeta \in H^{n-k,n-k}(X,\mathbb{Q}) = \mathrm{Hdg}^{2(n-k)}(X)$ be a Hodge class. By the Hard Lefschetz Theorem we have an isomorphism $L^{n-2k}\colon H^{k,k}(X,\mathbb{Q}) \cong H^{n-k,n-k}(X,\mathbb{Q})$ given by $\alpha \mapsto \alpha \cup \omega^{n-2k}$. In particular, it is surjective and there exists $\alpha \in H^{k,k}(X,\mathbb{Q})$ such that $\zeta = \alpha \cup \omega^{n-2k}$. In our hypothesis $\omega = [H]$ is algebraic, α is algebraic and the cup product sends algebraic classes to algebraic classes; hence ζ is algebraic. □

By the above results, the Hodge conjecture holds in the following cases.

- $k = 0$: in this case the Hodge decomposition is simply
$$H^0(X,\mathbb{Z}) = H^{0,0}(X)$$
and $[X]$ generates it;
- $k = 1$: by Lefschetz Theorem A.2.14 on $(1,1)-$classes;
- $k = n-1$: by Proposition 1.3.8 and the previous point;
- $k = n$: in this case the Hodge decomposition is trivial
$$H^{2n}(X,\mathbb{Z}) = H^{n,n}(X),$$
and $H^{2n}(X,\mathbb{Z})$ is generated by the fundamental class of any point.

Thus we can summarize our knowledge about the Hodge conjecture in the following table.

Variety	*class* $(0,0)$	*class* $(1,1)$	*class* $(2,2)$	*class* $(3,3)$	*class* $(4,4)$
curve	✓	✓	/	/	/
surface	✓	✓	✓	/	/
3−*fold*	✓	✓	✓	✓	/
4−*fold*	✓	✓	?	✓	✓

Hodge conjecture does hold for varieties up to dimension 3, and the first non trivial case is the class $(2,2)$ for a fourfold.

1.3.1 More on the Hodge conjecture

We introduce some useful notation:

- HC$^*(X,k)_{(\mathrm{prim})}$: for any $\alpha \in \mathrm{Hdg}^{2k}(X)_{(\mathrm{prim})}$ such that $\alpha \neq 0$, there exists $W \in Z^{n-k}(X)$ algebraic cycle such that $\alpha \cup W \neq 0$ in $H^{2n}(X)$;
- HC$^*(X)$: for any k, HC$^*(X,k)$;

- $\mathrm{HC}^*(d)$: for any X of dimension d, $\mathrm{HC}^*(X)$.

Lemma 1.3.9. *For any X such that $\dim X = n$, and for any k it holds:*

(i) $\mathrm{HC}(X, k) \Leftrightarrow \mathrm{HC}^*(X, n-k)$;

(ii) $\mathrm{HC}(X, k)_{\mathrm{prim}} \Leftrightarrow HC^*(X, n-k)_{\mathrm{prim}}$;

(iii) $\mathrm{HC}(X) \Leftrightarrow \mathrm{HC}^*(X)$;

(iv) $\mathrm{HC}(n) \Leftrightarrow \mathrm{HC}^*(n)$.

Proof. First we notice that if (i) holds for every k, then (iii) follows. Similarly, if for any variety X we have (iii), then (iv) holds too. Thus we only have to prove (i) and (ii). We begin by proving (i).

$\Rightarrow$ Let us suppose that every Hodge class of degree $2k$ is algebraic and let $0 \neq \alpha \in \mathrm{Hdg}^{2(n-k)}(X) = H^{n-k,n-k}(X, \mathbb{Q})$. We want to find a $W \in Z^k(X)$ such that $\alpha \cup [W] \neq 0$.
By Poincaré duality (A.1.21), we have a perfect pairing

$$\begin{aligned} H^{n-k,n-k}(X, \mathbb{Q}) \otimes H^{k,k}(X, \mathbb{Q}) &\to \mathbb{Q} \\ (\alpha, \beta) &\mapsto \int_X \alpha \cup \beta. \end{aligned}$$

Since $\alpha \neq 0$, there exists $\beta \in H^{k,k}(X, \mathbb{Q}) = \mathrm{Hdg}^{2k}(X)$ such that $\alpha \cup \beta \neq 0$. In our hypotheses β is algebraic, i.e. $\beta = \sum_i q_i [Z_i]$ with $q_i \in \mathbb{Q}$ and $Z_i \in Z^k(X)$ $\forall i$. We have

$$0 \neq \alpha \cup \beta = \sum_i q_i \alpha \cup [Z_i],$$

thus there exists i such that $\alpha \cup [Z_i] \neq 0$.

$\Leftarrow$ We recall that, by Hard Lefschetz Theorem A.2.17, we have an isomorphism $L^{n-2k}\colon H^{k,k}(X, \mathbb{Q}) \cong H^{n-k,n-k}(X, \mathbb{Q})$ given by $\alpha \mapsto \alpha \cup \omega^{n-2k}$. We denote by φ the isomorphism of the Hard Lefschetz Theorem, in particular:

$$\varphi = \begin{cases} L^{n-2k} \text{ if } 2k \leq n; \\ \left(L^{2k-n}\right)^{-1} \text{ if } 2k > n. \end{cases}$$

Moreover, we get a non-degenerate pairing on $H^{n-k,n-k}(X, \mathbb{Q})$

$$\begin{aligned} H^{n-k,n-k}(X, \mathbb{Q}) \otimes H^{n-k,n-k}(X, \mathbb{Q}) &\to \mathbb{Q} \\ (\varphi(\alpha), \beta) &\mapsto \int_X \alpha \cup \beta \end{aligned} \tag{1.3.10}$$

with $\alpha \in H^{k,k}(X, \mathbb{Q})$. Let us denote by $A^{k,k}(X) \subseteq \mathrm{Hdg}^{2k}(X)$ the subset of algebraic Hodge classes of degree $2k$. By hypotheses, for any

$\alpha \in A^{k,k}(X)$, there exists $[W] \in A^{k,k}(X)$ with $W \in Z^k(X)$, such that $\varphi(\alpha) \cup [W] \neq 0$ in $H^{2n}(X)$ and so $\int_X \varphi(\alpha) \cup \beta \neq 0$. We can conclude that the pairing (1.3.10) is non-degenerate on $\varphi\left(A^{k,k}(X)\right)$:

$$\varphi\left(A^{k,k}(X)\right) \otimes H^{n-k,n-k}(X,\mathbb{Q}) \to \mathbb{Q}$$
$$(\varphi(\alpha), \varphi([W])) \mapsto \int_X \varphi(\alpha) \cup [W] \,.$$

We get that $\varphi\left(A^{k,k}(X)\right)^{\perp} = 0$. Since all the cohomology groups are finite dimensional, it follows that

$$\begin{aligned} H^{n-k,n-k}(X,\mathbb{Q}) &= \varphi\left(A^{k,k}(X)\right)^{\perp} \oplus \varphi\left(A^{k,k}(X)\right) \\ &= \varphi\left(A^{k,k}(X)\right). \end{aligned}$$

By the following diagram

$$\begin{tikzcd} H^{k,k}(X,\mathbb{Q}) \arrow[r, "\varphi", "\cong"'] & H^{n-k,n-k}(X,\mathbb{Q}) \\ A^{k,k}(X) \arrow[u, hook] \arrow[r, "\varphi", "\cong"'] & \varphi\left(A^{k,k}(X)\right). \arrow[u, equal] \end{tikzcd}$$

we conclude that $A^{k,k}(X) = H^{k,k}(X,\mathbb{Q})$.

Finally we notice that, by the Hodge-Riemann bilinear relations (Lemma A.2.20), the pairing (1.3.10) is perfect also when restricted to the primitive classes and so the same procedure would lead to the proof of (*ii*). □

Remark 1.3.11. If $2k \leq n$ the above proof of point (*i*) also implies that $\mathrm{HC}(X, n-k)$. Indeed in this case we have that $\varphi = L^{n-2k}$ is algebraic since it is given by the cup product $\alpha \mapsto \alpha \cup \omega^{n-2k}$. Therefore:

$$\varphi\left(A^{k,k}(X)\right) = L^{n-2k}\left(A^{k,k}\right) \subseteq A^{n-k,n-k}(X) \subseteq H^{n-k,n-k}(X,\mathbb{Q}).$$

We have proved that $H^{n-k,n-k}(X,\mathbb{Q}) = \varphi\left(A^{k,k}(X)\right)$, then we have also that $A^{n-k,n-k}(X) = H^{n-k,n-k}(X,\mathbb{Q})$ and so $\mathrm{HC}(X, n-k)$ holds too.
On the other hand, if $2k > n$ we do not get a similar result, since $\varphi = \left(L^{2k-n}\right)^{-1}$ and we do not know if φ is an algebraic operator. In particular, this is the conjectures of Lefschetz type $B(X)$ (see Section 1.2 and [Kle68, §2]).

Induction argument

Since the Hodge conjecture holds for variety up to dimension three, we could try to use this as the base step of an induction proof on the dimension of

X. In this section we see that in the induction step we find some problems. In particular, we are not able to go from the odd to the even dimensional varieties, and the induction is incomplete.
We state the main result, and we refer to [Ans12, Proposition 2.6] for the proof of it.

Proposition 1.3.12. $\mathrm{HC}(n-1) \Rightarrow$ *for any* X *such that* $\dim X = n$*, all the Hodge classes of degree* $(k,k) \neq \left(\frac{n}{2}, \frac{n}{2}\right)$ *are algebraic.*

Thus the only obstacle to the inductive step is to prove that for a $n-$dimensional variety the class $\left(\frac{n}{2}, \frac{n}{2}\right)$ is algebraic. We distinguish the two cases of even dimensional and odd dimensional varieties.

Corollary 1.3.13. $\mathrm{HC}(2n) \Rightarrow \mathrm{HC}(2n+1)$.

Proof. Let $\dim X = 2n+1$, then $\mathrm{Hdg}^{2n+1}(X) = \varnothing$, since the degree of a Hodge class is always an even number. By Proposition 1.3.12 we have that any Hodge class of degree $2k \neq 2n+1$ is algebraic, hence we conclude. □

Corollary 1.3.14. $\mathrm{HC}(2n-1)$*, then for any* X *such that* $\dim X = 2n$ *it holds* $\mathrm{HC}(X) \Leftrightarrow \mathrm{HC}(X,n)_{\mathrm{prim}} \Leftrightarrow \mathrm{HC}^*(X,n)_{\mathrm{prim}}$.

Proof. Let us suppose that $\mathrm{HC}(2n-1)$ holds, then by Proposition 1.3.12 for any $k \neq n$ any $\alpha \in \mathrm{Hdg}^{2k}(X)$ is algebraic. By the Lefschetz decomposition (A.2.18):

$$H^{n,n}(X,\mathbb{Q}) = \bigoplus_{i \geq 0} L^i H^{n-i,n-i}(X,\mathbb{Q})_{\mathrm{prim}}.$$

Thus the only statement left to prove is that for any $\alpha \in H^{n,n}(X,\mathbb{Q})_{\mathrm{prim}}$, α is algebraic, i.e. $\mathrm{HC}(X,n)_{\mathrm{prim}}$, which, by Lemma 1.3.9, is equivalent to $\mathrm{HC}^*(X,n)_{\mathrm{prim}}$. □

Standard reductions of the Hodge conjecture

By combining all the previous results, we have that the Hodge conjecture can be reduced to apparently weaker statements.

Proposition 1.3.15. *The following statements are equivalent:*

(i) $\forall X$, $\mathrm{HC}(X)$;

(ii) $\forall n$: $\mathrm{HC}(2n-1) \Rightarrow \mathrm{HC}(X,n)$ *for every* X *s.t.* $\dim X = 2n$;

(iii) $\forall n$: $\mathrm{HC}(2n-1) \Rightarrow \mathrm{HC}(X,n)_{\mathrm{prim}}$ *for every* X *s.t.* $\dim X = 2n$;

(iv) $\forall n$: $\mathrm{HC}^*(2n-1) \Rightarrow \mathrm{HC}^*(X,n)$ *for every* X *s.t.* $\dim X = 2n$;

(v) $\forall n$: $\mathrm{HC}^*(2n-1) \Rightarrow \mathrm{HC}^*(X,n)_{\mathrm{prim}}$ *for every* X *s.t.* $\dim X = 2n$.

Proof. First of all, we notice that the following implications follow directly from Lemma 1.3.9: $(i) \Rightarrow (ii) \Rightarrow (iii)$, $(ii) \Leftrightarrow (iv)$, and $(iii) \Leftrightarrow (v)$. Thus, it is enough to prove that $(iii) \Rightarrow (i)$ by induction on $\dim X = n$. We consider as base steps $n = 1, 2, 3$, and we take as inductive hypothesis $\mathrm{HC}(2n-1)$. We distinguish two cases:

- if $n-1 = 2m$ is even, then by Corollary 1.3.13, we have $\mathrm{HC}(2m+1) = \mathrm{HC}(n)$;
- if $n-1 = 2m-1$ is odd, then by Corollary 1.3.14, we have $\mathrm{HC}(X)$ for every X such that $\dim X = 2m = n$ if and only if $\mathrm{HC}(X, m)_{\mathrm{prim}}$ and this holds by (iii).

□

Chapter 2

Motives

Parmi toutes les choses mathématiques que j'avais eu le privilège de découvrir et d'amener au jour, cette réalité des motifs m'apparaît encore comme la plus fascinante, la plus chargée de mystère - au coeur même de l'identité profonde entre "la géométrie" et "l'arithmétique".
(Grothendieck, [Gro85, 2.16])

In algebraic geometry we have different cohomology theories, such as Hodge cohomology, algebraic De Rham cohomology, crystalline cohomology, the étale ℓ–adic cohomology for every prime ℓ. The "*yoga of motives*" was introduced by Grothendieck in 1963–1969 aiming to associate to any variety $X \in \mathcal{V}\text{ar}$ a motive $h(X) \in \mathcal{M}$ such that $h(X)$ is responsible for all different cohomology theories that one can possibly associate to X.
In Grothendieck's words:

Contrary to what occurs in ordinary topology, one finds oneself confronting a disconcerting abundance of different cohomological theories. One has the distinct impression (but in a sense that remains vague) that each of these theories "amount to the same thing", that "they give the same results". In order to express this intuition, of the kinship of these different cohomological theories, I formulated the notion of "motive" associated to an algebraic variety. By this term, I want to suggest that it is the "common motive" (or "common reason") behind this multitude of cohomological invariants attached to an algebraic variety, or indeed, behind all cohomological invariants that are a priori possibile. [...] So, the motive associated to an algebraic variety would represent the "ultimate", "par excellence" cohomological invariant, from which it would be possibile to deduce all the other (associated to the different possible cohomological theories), as many musical "incarnations", or different "realizations".
(Grothendieck, [Gro85, 2.16])

He was particularly fond of his construction, that he defined the "heart in the heart" of the new geometry ([Gro85, 2.16]). The quite simple and elegant idea is to replace the category of varieties by another one which has the same objects, but whose morphisms are correspondences modulo an adequate equivalence relation on cycles. Grothendieck had in mind motives modulo numerical equivalence, because he sensed that it would be the only way to gain good properties. He was right, indeed Jannsen proved that this is the only equivalence relation that gives an abelian semi–simple category [Jan92]. We focus instead on the category of Chow motives, i.e. modulo rational equivalence, because it carries informations about cohomology but also about Chow groups. The disadvantage of this choice is that we do not work with an abelian category.

Remark 2.0.1. The construction of motives is unconditional and simple, but, in order to gain nice properties and have a concrete description of the category, we need the standard conjectures. In particular, Grothendieck constructed the category of pure motives, associated to smooth projective varieties, by means of algebraic cycles and equivalence relation. Any attempt though to generalize this construction to the category of mixed motives associated to arbitrary varieties over a field k up to now has been unsatisfactory.

2.1 Basics on Motives

We briefly recall the main steps of the construction of motives, defining the category of correspondences $\mathcal{CV}\mathrm{ar}_\sim$, the category of effective motives $\mathcal{M}_\sim^{\mathrm{eff}}$ and finally the more general category of motives $\mathcal{M}_\sim$:

$$\mathcal{V}\mathrm{ar} \to \mathcal{CV}\mathrm{ar}_\sim \to \mathcal{M}_\sim^{\mathrm{eff}} \to \mathcal{M}_\sim \,.$$

Fur further details we refer the reader to [MNP13] and [Jan94].

The Category of Correspondences

We denote by $\sim$ an adequate equivalence relation on cycles.
For the definition and basic properties of correspondences see Chapter 1. If we consider a correspondence Γ modulo an adequate equivalence relation $\sim$, we write $\Gamma \in \mathrm{Corr}_\sim(X \times Y)$.
We recall that correspondences act on cycles. Indeed, let us consider a correspondence $\Gamma \in Z^d(X \times Y)$, the induced action is defined by

$$\begin{aligned} \Gamma \colon Z^i(X) &\to Z^{i+d-\dim X}(Y) \\ Z &\mapsto (\pi_Y)_* \left(\Gamma \cdot (Z \times Y)\right), \end{aligned}$$

where $\pi_Y \colon X \times Y \to Y$ is the natural projection on the second factor of the cartesian product.

We notice that the action of the correspondence Γ is defined whenever the intersection product $\Gamma \cdot (Z \times Y)$ is defined. In particular, when we consider cycles modulo an adequate equivalence relation, the action of a correspondence is always defined.

If $\Gamma \in Z^d(X \times Y)$ and $d \neq \dim X$, the induced action does not preserve the codimension of cycles, and $r = d - \dim X$ is called the *degree* of the correspondence. We denote correspondences of degree r as

$$\mathrm{Corr}^r_\sim(X, Y) = Z^d(X \times Y)/_\sim.$$

Example 2.1.1. *Let us consider a morphism of varieties $f\colon X \to Y$. Then f gives two correspondences: its graph $\Gamma_f \in CH^{\dim X}(X \times Y)/_\sim$, and the transpose ${}^t\Gamma_f \in CH^{\dim X}(Y \times X)/_\sim$.*

Definition 2.1.2. $\mathcal{CV}\mathrm{ar}_\sim$ *is the category in which the objects are the same as in* $\mathcal{V}\mathrm{ar}$*, i.e. smooth projective varieties over k, and the arrows are degree–0 correspondences. So the arrows in this category are elements of*

$$\mathrm{Hom}_{\mathcal{CV}\mathrm{ar}_\sim}(X, Y) := \mathrm{Corr}^0_\sim(X, Y) := CH^{\dim X}(X \times Y)/_\sim.$$

Correspondences can be composed in the following way:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{CV}\mathrm{ar}_\sim}(X, Y) \otimes \mathrm{Hom}_{\mathcal{CV}\mathrm{ar}_\sim}(Y, Z) &\to \mathrm{Hom}_{\mathcal{CV}\mathrm{ar}_\sim}(X, Z) \\ (a, b) &\mapsto b \circ a, \end{aligned}$$

where $b \circ a := (p_{XZ})_* \left((p_{XY})^* (a) \cdot (p_{YZ})^* (b)\right)$, and we are considering the natural projections

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl, "p_{XY}"'] \arrow[d, "p_{XZ}"] \arrow[dr, "p_{YZ}"] & \\ X \times Y & X \times Z & Y \times Z. \end{tikzcd}$$

Definition 2.1.3. *An idempotent correspondence $p \in \mathrm{Corr}_\sim(X, X)$, i.e. such that $p \circ p = p$, is called a* projector*. In particular, a projector needs to be a correspondence of degree* 0*, and* $\mathrm{Corr}^0_\sim(X, X) \subseteq \mathrm{Corr}_\sim(X, X)$ *is a subring.*

Example 2.1.4. *Let us consider two morphisms of varieties $f\colon Y \to X$ and $g\colon Z \to Y$, then ${}^t\Gamma_g \circ^t \Gamma_f =^t \Gamma_{f\circ g}$.*
Indeed, by definition

$${}^t\Gamma_g \circ^t \Gamma_f = (p_{XZ})_* \left((p_{XY})^*({}^t\Gamma_f) \cdot (p_{YZ})^*({}^t\Gamma_g)\right) \in \mathrm{Hom}_{\mathcal{CV}\mathrm{ar}_\sim}(X, Z)$$

and ${}^t\Gamma_{f\circ g} = \{(x, z) \in X \times Z \mid f(g(z)) = x\} \in \mathrm{Hom}_{\mathcal{CV}\mathrm{ar}_\sim}(X, Z)$. In particular, let $(\bar{x}, \bar{z}) \in {}^t\Gamma_{f\circ g}$, then there exists $\bar{y} \in Y$ such that $g(\bar{z}) = \bar{y}$ and $f(\bar{y}) = \bar{x}$. So, $(\bar{x}, \bar{y}) \in {}^t\Gamma_f$, $(\bar{y}, \bar{z}) \in {}^t\Gamma_g$ and $(\bar{x}, \bar{z}) \in {}^t\Gamma_g \circ {}^t\Gamma_f$.

Remark 2.1.5. We saw that correspondences act on cycles. We notice that if $Z^i_\sim(X) \subset Z^i_{\text{hom}}(X)$, then we have also an induced action of correspondences on the Weil cohomology theory H^*. Indeed, let us consider a correspondence $\Gamma \in \mathrm{Corr}^r_\sim(X,Y)$, then

$$\begin{aligned}\Gamma_*\colon H^i(X) &\to H^{i+2r}(Y)\\ \alpha &\mapsto p_Y\{\mathrm{cl}_{X\times Y}(\gamma) \cup p_X^*(\alpha)\}.\end{aligned}$$

In particular, we notice that a correspondence in $\mathrm{Corr}_{\text{num}}(X,Y)$ acts on cohomology only if the standard conjecture $D(X)$ is true, namely if numerical and homological equivalence coincide (see Section 1.2).

Example 2.1.6. *Let $f\colon Y \to X$ be a morphism of varieties, then*

$$\left({}^t\Gamma_f\right)_* = f^*\colon Z^i(X)_\sim \to Z^i(Y)_\sim,$$

and it is an example of correspondence of degree 0*. The same happens in cohomology.*
Indeed by definition, for any $c \in Z^i(X)_\sim$

$$\begin{aligned}({}^t\Gamma_f)_*(c) &= (p_Y)_*\left((p_X)^*(c) \cdot^t \Gamma_f\right)\\ f^*(c) &= (p_Y)_*\left(\Gamma_f \cdot (Y \times c)\right),\end{aligned}$$

and we notice that

$${}^t\left[\Gamma_f \cdot (Y \times c)\right] = (p_X)^*(c) \cdot^t \Gamma_f.$$

Analogously, for the push–forward we get

$$\left(\Gamma_f\right)_* = f_*\colon Z_i(X)_\sim \to Z_i(Y)_\sim.$$

Since f_* preserves the cycles dimension, the degree of Γ_f is

$$\deg(f_*) = \dim Y - \dim X.$$

The Category of Effective Motives

Given $X \in \mathcal{V}\mathrm{ar}$, we can consider correspondences in the self–product $X \times X$.

Definition 2.1.7. *The* category of effective motives $\mathcal{M}^{\text{eff}}_\sim$ *is the category in which the objects are pairs (X,p) with $X \in \mathcal{V}\mathrm{ar}$ and p is a projector. Given two objects (X,p) and (Y,q), the arrows in the category are elements of*

$$\mathrm{Hom}_{\mathcal{M}^{\text{eff}}_\sim}\left((X,p),(Y,q)\right) := q \circ \mathrm{Corr}^0_\sim(X,Y) \circ p \subset \mathrm{Corr}^0_\sim(X,Y).$$

Then $\mathcal{M}^{\text{eff}}_\sim$ is a pseudo–abelian category. In particular it is the pseudo–completion of $\mathcal{CV}\mathrm{ar}_\sim$ ([MNP13, 2.2.1]). This means that the projectors have kernels and images, i.e. for any $M = (X,p) \in \mathcal{M}_\sim$ we have a decomposition $\mathrm{Im}(p) \oplus \mathrm{Im}(\mathrm{id}_X - p) \cong X$, where $\mathrm{Im}(\mathrm{id}_X - p) = \mathrm{Ker}(p)$.

The Category of Motives

Definition 2.1.8. *The* category of motives $\mathcal{M}_\sim$ *is the category in which objects are triplets* (X, p, m)*, where* $X \in \mathcal{V}\mathrm{ar}$*,* p *is a projector and* $m \in \mathbb{Z}$*. Given two objects* (X, p, m) *and* (Y, q, n)*, the arrows are elements of*

$$\mathrm{Hom}_{\mathcal{M}_\sim}((X, p, m), (Y, q, n)) := q \circ \mathrm{Corr}^{n-m}_\sim(X, Y) \circ p,$$

where $\mathrm{Corr}^{n-m}_\sim(X, Y) = Z^{n-m+\dim X}(X \times Y)/_\sim$.

Remark 2.1.9. We point out that there is a faithful full embedding of categories

$$\mathcal{M}^{\mathrm{eff}}_\sim \hookrightarrow \mathcal{M}_\sim\,.$$

We can construct motives using any adequate equivalence relation on cycles, so in particular we have

$$\begin{tikzcd} & \mathcal{V}\mathrm{ar} \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\ \mathcal{M}_{\mathrm{rat}} \arrow[r] & \mathcal{M}_{\mathrm{alg}} \arrow[r] & \mathcal{M}_{\mathrm{hom}} \arrow[r] & \mathcal{M}_{\mathrm{num}}. \end{tikzcd}$$

The most interesting categories are obtained by considering the finest and the coarsest adequate equivalence relation on algebraic cycles. Respectively: $\mathrm{CHM} := \mathcal{M}_{\mathrm{rat}}$, the category of *Chow motives*, and $\mathrm{NUM} := \mathcal{M}_{\mathrm{num}}$. We briefly recall two results on these two categories.

Theorem 2.1.10 ([Sch94], Theorem 1.6 and Corollary 3.5)**.** $\mathcal{M}_\sim$ *is an additive,* $\mathbb{Q}$*–linear and pseudo–abelian category. Moreover, the category of Chow motives is not abelian.*

Theorem 2.1.11 ([Jan92], Theorem 1)**.** $\mathcal{M}_\sim$ *is an abelian semi–simple category if and only if* $\mathcal{M}_\sim = \mathrm{NUM}$*.*

By construction, we have a contravariant functor

$$\begin{aligned} \mathcal{V}\mathrm{ar} &\longrightarrow \mathcal{M}_\sim \\ X &\mapsto \quad h_\sim(X) := (X, \Delta_X, 0) \\ f\colon Y \to X &\mapsto \quad {}^t\Gamma_f \in \mathrm{Hom}_{\mathcal{M}_\sim}(h_\sim(X), h_\sim(Y)), \end{aligned}$$

where $\Delta_X \subset X \times X$ is the diagonal of X.

Notation and conventions. *Since we deal mostly with Chow motives, we denote the* Chow motive *associated to a variety* X *simply as*

$$h(X) := h_{\mathrm{rat}}(X).$$

We briefly present some basic operations between motives.

- *Tensor product of motives.* Given two motives $(X,p,m),(Y,q,n) \in \mathcal{M}_\sim$ we define the tensor product as
$$(X,p,m) \otimes (Y,q,n) := (X \times Y, p \times q, m+n).$$
- *Direct sum of motives.* Let $(X,p,m),(Y,q,n) \in \mathcal{M}_\sim$ be two motives such that $m=n$, then we define the direct sum as
$$(X,p,m) \oplus (Y,q,m) := (X \coprod Y, p+q, m).$$
For the case $m \neq n$ we refer the reader to [Sch94, 1.14].

Example 2.1.12. *We present some natural examples just to give some practical ideas.*

a) We can consider the motive of a point $x \in X$
$$\mathbb{1} := (\mathrm{Spec}(k), \mathrm{id}, 0) = h_\sim(x).$$
This is called the unit motive*, because it is the identity with respect to the tensor product of motives.*

b) The Lefschetz motive *is defined as*
$$\mathbb{L}_\sim := (\mathrm{Spec}(k), \mathrm{id}, -1).$$

c) The Tate motive *is defined as*
$$\mathbb{T}_\sim := (\mathrm{Spec}(k), \mathrm{id}, 1).$$

d) Let X be irreducible such that $\dim X = d$ and there is a k–rational point $x \in X(k)$. Then we can define two orthogonal projectors
$$p_0 = \{x\} \times X, \quad p_{2d} = X \times \{x\}.$$
We have then two effective motives associated to X
$$h^0_\sim(X) = (X, p_0, 0), \quad h^{2d}_\sim(X) = (X, p_{2d}, 0).$$
In particular, it holds that $h^0_\sim(X) \cong \mathbb{1}$ and $h^{2d}_\sim(X) \cong \mathbb{L}^d$.

Symmetric and exterior powers of motives

We briefly recall some basics about group representations, for further details we refer the reader to [FH91], [Ser77] and [MNP13, Sections 4.2, 4.3].
We consider the symmetric group $\mathfrak{S}_n$ of permutations of n elements. One can construct ([FH91, Theorem 4.3]) a bijective correspondence between irreducible representations of $\mathfrak{S}_n$ and partitions of n, i.e. sets of positive integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that $\lambda_1 \leq \cdots \leq \lambda_r$ and $\sum_{i=1}^r \lambda_i = n$.

Example 2.1.13. *i) The partition $\lambda = (n)$ corresponds to the trivial representation $\sigma(v) = v$, with*

$$e_{sym} := e_{(n)} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma.$$

ii) The partition $\lambda = (1, \dots, 1)$ corresponds to the alternating representation $\sigma(v) = \operatorname{sgn}(\sigma)v$, with

$$e_{alt} := e_{(1,\dots,1)} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma.$$

By considering the action of the symmetric group $\mathfrak{S}_n$ on the product $X^n := X \times \cdots \times X$, i.e.

$$\begin{aligned} X \times \cdots \times X &\to X \times \cdots \times X \\ (x_1, \dots, x_n) &\mapsto (x_{\sigma_1}, \dots, x_{\sigma_n}), \end{aligned}$$

we get a correspondence $\Gamma_\sigma(X)$ which is the graph of this map. We can generalize this considering the correspondence Γ_r associated to an element $r = \sum_{\sigma \in \mathfrak{S}_n} r(\sigma)\sigma$ of the group ring $R = \mathbb{Q}[G]$. We get

$$\Gamma_r(X) = \sum_{\sigma \in \mathfrak{S}_n} r(\sigma)[\Gamma_\sigma(X)] \in \operatorname{Corr}^0_\sim(X^n),$$

where $r(\sigma) \in \mathbb{Z}$. Then we notice that the product of elements in the group ring R, translates into the composition of correspondences, i.e.

$$\Gamma_{rs}(X) = \Gamma_r(X) \circ \Gamma_s(X). \tag{2.1.14}$$

So for any λ partition of n, we obtain a projector defining

$$d_\lambda := \Gamma_{e_\lambda}(X) \in \operatorname{Corr}^0_\sim(X^n).$$

Analogously, we can consider a motive $M = (X, p, m) \in \mathcal{M}_\sim$ and its product $M^{\otimes n} = (X \times \cdots \times X, p \times \cdots \times p, nm) = (X^n, p^{\otimes n}, nm)$. Then we get (see [MNP13, Sections 4.3])

$$\Gamma_\sigma(M) := \Gamma_\sigma(X) \circ p^{\otimes n} = p^{\otimes n} \circ \Gamma_\sigma(X) \in \operatorname{Hom}_{\mathcal{M}_\sim}(M^{\otimes n}, M^{\otimes n}). \tag{2.1.15}$$

We can generalize this to any element $r = \sum_{\sigma \in \mathfrak{S}_n} r(\sigma)\sigma$ of the group ring:

$$\Gamma_r(M) := \sum_{\sigma \in \mathfrak{S}_n} r(\sigma)\Gamma_\sigma(M) \in \operatorname{Hom}_{\mathcal{M}_\sim}(M^{\otimes n}, M^{\otimes n}).$$

From (2.1.15), it follows that $d_\lambda \circ p^{\otimes n} = p^{\otimes n} \circ d_\lambda$ is a projector on X^n (see [MNP13, Lemma 4.3.1]).

Definition 2.1.16. *Let us consider a motive $M = (X, p, m) \in \mathcal{M}_\sim$ and a partion λ of n. We define*

$$\mathbb{T}_\lambda M := (X^n, d_\lambda \circ p^{\otimes n}, nm) \in \mathcal{M}_\sim.$$

In particular, we denote as

$$\begin{aligned} \mathrm{Sym}^n M &:= \mathbb{T}_{(n)} M = (X^n, d_{sym} \circ p^{\otimes n}, nm) \in \mathcal{M}_\sim; \\ \bigwedge^n M &:= \mathbb{T}_{(1,\ldots,1)} M = (X^n, d_{alt} \circ p^{\otimes n}, nm) \in \mathcal{M}_\sim. \end{aligned}$$

By definition, we have that

$$\begin{aligned} CH(\mathbb{T}_\lambda M) &= \mathrm{Im}(d_\lambda) \subset CH(M^{\otimes n}); \\ CH(\mathrm{Sym}^n M) &= \mathrm{Im}(d_{sym}) \subset CH(M^{\otimes n}); \\ CH\left(\bigwedge^n M\right) &= \mathrm{Im}(d_{alt}) \subset CH(M^{\otimes n}). \end{aligned}$$

2.1.1 Dimension of a motive

The idea of how to define the dimension of a motive was developed independently by Kimura [Kim05] and O'Sullivan [O'S05].

Definition 2.1.17. *We consider a motive $M = (X, p, m) \in \mathcal{M}_\sim$.*

i) *M is* evenly finite dimensional *if $\bigwedge^n M = 0$ for some $n > 0$. Then we define the* dimension *of M as*

$$\dim M := \max\left\{n > 0 : \bigwedge^n M \neq 0\right\}.$$

ii) *M is* oddly finite dimensional *if $\mathrm{Sym}^n M = 0$ for some $n > 0$. Then we define the* dimension *of M as*

$$\dim M := \max\left\{n > 0 : \mathrm{Sym}^n M \neq 0\right\}.$$

iii) *M is* finite dimensional *if there exists a decomposition*

$$M = M_+ \oplus M_-,$$

where M_+ is evenly finite dimensional, and M_- is oddly finite–dimensional. We define the dimension of M as the sum of the two dimensions

$$\dim M := \dim M_+ + \dim M_-.$$

The principle inspiring this definition is clear if we recall that a characterization for a vector space V having dimension d is that $\bigwedge^{d+1} V = 0$.

Remark 2.1.18. i) If $M \in \mathcal{M}_\sim$ is finite dimensional, then the decomposition $M = M_+ \oplus M_-$ is unique up to isomorphisms on the evenly and oddly parts. In particular, $\dim M$ is well defined (see [MNP13, Proposition 5.3.3]).

ii) If a motive $M \in \mathcal{M}_\sim$ is both evenly and oddly finite dimensional, then $M = 0$ (see [MNP13, Corollary 5.3.2]).

Example 2.1.19. *We present two practical examples.*

- *We can compute the dimension of the unit motive* $\mathbb{1}$*. Since*

$$\bigwedge^2 \mathbb{1} = (\mathrm{Spec}(k) \times \mathrm{Spec}(k), \mathrm{id} - \mathrm{id}, 0) = 0,$$

the dimension is 1 *and* $\mathbb{1}$ *is evenly finite dimensional. The same happens for the Lefschetz motive* $\mathbb{L} = (\mathrm{Spec}(k), \mathrm{id}, -1)$.

- *A curve has finite dimensional Chow motive ([Kim05], [K$\ddot{9}$3].)*

In particular, Kimura formulated the following conjecture in [Kim05].

Conjecture 2.1.20 (Kimura)**.** *Every Chow motive is finite dimensional.*

We briefly recall some basic properties of finite dimensional motive, for the proofs and further details we refer the reader to [MNP13, Chapter 5]. These properties are the basic tools to construct examples in which Kimura's conjecture is verified.

Proposition 2.1.21 (Direct sum)**.** *i) Let* $M, N \in \mathcal{M}_\sim$ *be two evenly (respectively oddly) finite dimensional motives. Then the direct sum* $M \oplus N$ *is an evenly (resp. oddly) finite dimensional motive.*
The converse implication holds: if $M \oplus N$ *is evenly (resp. oddly) finite dimensional, then* M *and* N *are evenly (resp. oddly) finite dimensional.*

ii) If $M, N \in \mathcal{M}_\sim$ *are finite dimensional, then also their direct sum* $M \oplus N$ *is finite dimensional and* $\dim(M \oplus N) \leq \dim M + \dim N$.

Proposition 2.1.22 (Tensor product)**.** *i) We consider two evenly (respectively oddly) finite dimensional motives* $M, N \in \mathcal{M}_\sim$*. Then the tensor product* $M \otimes N$ *is evenly (resp. oddly) finite dimensional and*

$$\dim(M \otimes N) \leq \dim M \cdot \dim N.$$

ii) If $M, N \in \mathcal{M}_\sim$ *are finite dimensional of different parity, then their tensor product* $M \otimes N$ *is oddly finite dimensional, and*

$$\dim(M \otimes N) \leq \dim M \cdot \dim N.$$

Corollary 2.1.23 (Cartesian product)**.** *If $h(X)$ and $h(Y)$ are finite dimensional, then also $h(X \times Y)$ is finite dimensional.*
In particular, if we consider a curve C, then $h(C^n)$ is finite dimensional for every $n \in \mathbb{N}$.

Theorem 2.1.24 ([Kim05] and [Via17])**.** *Let $X \in \mathcal{V}\mathrm{ar}$, then the following hold:*

i) if X is dominated by a product of curves $C_1 \times \cdots \times C_m$, then X has finite–dimensional motive;

ii) if $n := \dim X \leq 3$ and X is rationally dominated by a product of curves $C_1 \times \cdots \times C_m$, then X has finite–dimensional motive.

This last result is the most useful tool up to now to construct examples of varieties with finite dimensional motives. In particular, we have the following useful corollaries.

Corollary 2.1.25 (Proposition 5.6.13, [MNP13])**.** *Let S be a surface with $p_g(S) = 0$ and $h(S)$ finite dimensional. Then $CH^2_{\mathrm{AJ}}(S) = 0$, i.e. Bloch's conjecture is true for S. In particular, this is true if S is rationally dominated by $C_1 \times C_2$.*

Corollary 2.1.26 (Corollary 5.4.7, [MNP13])**.** *Abelian varieties have finite dimensional Chow motive.*

This follows from the fact that any abelian variety is dominated by the Jacobian of a curve, and such curve is dominated by the self–product of the curve times its genus.

Corollary 2.1.27 (Corollary 5.4.8, [MNP13])**.** *Motive that are direct summands of some tensor product of motives of curves are finite dimensional. Such kind of motives form a full tensor–subcategory inside the category of Chow motives, called the category of* motives of abelian type *(see [Via17]).*

Remark 2.1.28. Kimura's conjecture 2.1.20 is still open, but there are some examples in which it has been verified. In particular, all the known examples up to now are in the category of motives of abelian type. So there are no examples of finite–dimensional motives not generated by curves, even if we have examples of motives that are not in the category of motives of abelian type, e.g the motive of a very general quintic hypersurface in $\mathbb{P}^3$ (see [Del72, 7.6]).
We list some examples in which the finite–dimensionality conjecture 2.1.20 has been proved: Fermat hypersurfaces (that are dominated by product of curves) [Kim05]; K3 surfaces with Picard number 19 or 20 [Ped12]; surfaces not of general type with $p_g(S) = 0$ [GP02, Theorem 2.11]; some surfaces of general type with $p_0 = 0$ such as Catanese and Barlow surfaces [Voi14a],

[BF15],[PW16]; Godeaux surfaces [GP02]; Hilbert schemes of surfaces with finite–dimensional motive [dCM02]; generalized Kummer varieties [Xu18, Remark 2.9 (ii)]; varieties X with Abel–Jacobi trivial Chow groups [Via13, Theorem 4]; log–homogeneous varieties and 3–folds with nef tangent bundle [Iye09], (see also [Via17, Example 3.16]); 4–folds with nef tangent bundle [Iye11]; some 3–folds of general type [Via15][Section 8]; complex Fano 3–folds [GG12, Therem 5.1]; some other examples can be found in Laterveer's work [Lat19b], [Lat19c], [Lat18c], [Lat19e], [Lat19a], [Lat18e], [Lat19d], [Lat18b].

2.2 Chow–Künneth decomposition

We briefly introduce the definition of the Chow–Künneth decomposition.

Definition 2.2.1. *Let $X_d \in \mathrm{SmProj}(k)$, and let $h(X) \in \mathcal{M}_{rat}$ denote the Chow motive of S. The* Chow–Künneth decomposition *of $h(X)$ in $\mathcal{M}_{rat}$ is*

$$h(X) = \bigoplus_{i=0}^{2d} h_i(X),$$

where $h_i(X) = (X, \pi_i, 0)$, $\pi_i \in CH^d(X \times X)$ are orthogonal projectors, i.e. $\pi_i \circ \pi_i = \pi_i$ and $\pi_i \circ \pi_j = 0$ for $i \neq j$, and they are the Künneth components of the diagonal Δ_X, i.e.

$$[\Delta_X] = \sum_{i=0}^{2d} [\pi_i] \in H^{2d}(X \times X, \mathbb{Q});$$
$$\mathrm{cl}^d(\pi_i) \in H^{2d-i}(X, \mathbb{Q}) \otimes H^i(X, \mathbb{Q}) \subset H^{2d}(X \times X, \mathbb{Q}).$$

In particular, this decomposition is self–dual, in the sense that $\pi_i = \pi^t_{2d-i}$. We use the short–hand "CK decomposition".

In 1993, Murre formulated the following conjecture [Mur93, Part I].

Conjecture 2.2.2 ($CK(X)$)**.** *Every smooth projective variety admits a Chow-Künneth decomposition.*

Remark 2.2.3.
i) The CK conjecture is know for curves (see [MNP13, 6.1.3]), surfaces (see [Mur90], [KMP07, Proposition 2.1]), products of curves and surfaces (since it is stable under the cartesian product), abelian varieties (see [Š71], [DM91], [Kϋ93]), some threefolds (see [dAMS98]) and some Kuga–Satake varieties (see [GHM03]).

ii) The Chow–Künneth conjecture implies the weaker Künneth conjecture $C(X)$. In particular, the converse implication holds assuming Kimura's conjecture 2.1.20 (see [MNP13, Proposition 6.1.4]).

We recall a useful result on the Chow–Künneth decomposition for surfaces ([KMP07, Theorem 3.10]).

Theorem 2.2.4 (Kahn–Murre–Pedrini)**.** *Let S and S' be two smooth projective surfaces with a CK decomposition*

$$h(S) = \bigoplus_{i=0}^{4} h_i(S), \quad h(S') = \bigoplus_{i=0}^{4} h_i(S'),$$

as in Definition 2.2.1. Then

$$\mathcal{M}_{rat}(h_i(S), h_j(S')) = 0 \qquad \text{for all } j < i \text{ and } 0 \leq i \leq 4,$$

where $\mathcal{M}_{rat}(h_i(S), h_j(S')) = \pi_i(S) \circ CH^2(S \times S') \circ \pi_i(S')$ are the morphisms in the category $\mathcal{M}_{rat}$.

Remark 2.2.5. If there exists Chow–Künneth decomposition $\{\pi_i\}_i$ for X, then it provides a decreasing filtration on the Chow groups:

$$F^\nu CH^j(X)_{\mathbb{Q}} = \bigcap_{i > 2j-\nu} \ker \pi_i = \bigoplus_{i \leq 2j-\nu} \mathrm{Im}(\pi_i).$$

Chapter 3

Relation between singular cohomology and Chow groups

When dealing with a smooth projective variety X, we can associate to it two different kind of algebraic objects, namely singular cohomology groups and Chow groups. These two algebraic objects give informations about the differentiable structure of the variety (via Hodge theory), and about the algebraic cycles on X respectively. To investigate the relation between these two different approaches is then a natural problem, the answer, however, is not simple at all. While the influence of Chow groups on singular cohomology is motivated by some classical results by Mumford and Roitman, the converse influence is rather conjectural, and it depends on some deep and difficult conjectures that are far from being solved, such as Bloch's conjecture for surfaces and the more general Bloch–Beilinson's conjecture. The aim of this chapter is to introduce the reader to this conjectures, and give an idea of the context in which Voisin's conjecture on 0–cycles on surfaces arises.
For further details we refer the reader to [Voi07b, Chapter 10,11], [Voi96].

3.1 Preliminaries

3.1.1 The Abel–Jacobi map

We give a brief introduction to the Abel–Jacobi map, fur further details we refer the reader to [Voi07a, Chapter 12] and [Voi07b, Chapter 7.2].

Let X be a compact Kähler manifold of dimension n. Then we have a natural filtration $F^\bullet$ on the singular cohomology

$$H(X,\mathbb{C}) = \bigoplus_{0\leq p+q\leq 2n} H^{p,q}$$

given by the Hodge decomposition

$$0 \subseteq F^k H^k(X) \subseteq F^{k-1} H^k(X) \subseteq \cdots \subseteq F^0 H^k(X) = H^k(X, \mathbb{C}).$$

This is called the *Hodge filtration* (see Example A.3.3), and it is defined by

$$F^r H^k(X) = \bigoplus_{p \geq r} H^{p,k-p}(X), \qquad 0 \leq r \leq k.$$

When considering the singular cohomology groups for odd degrees, we get a decomposition as a direct sum

$$H^{2k-1}(X, \mathbb{C}) = F^k H^{2k-1}(X) \oplus \overline{F^k H^{2k-1}(X)}.$$

Then $F^k H^{2k-1}(X) \cap H^{2k-1}(X, \mathbb{R}) = \{0\}$ and we get an isomorphism of $\mathbb{R}$–vector spaces

$$H^{2k-1}(X, \mathbb{R}) \xrightarrow{\sim} H^{2k-1}(X, \mathbb{C}) / F^k H^{2k-1}(X).$$

Since the rank of $H^{2k-1}(X, \mathbb{R})$ is $\dim H^{2k-1}(X, \mathbb{R})$, we have a full lattice

$$\Lambda_k := \operatorname{Im}\left(H^{2k-1}(X, \mathbb{Z}) \to H^{2k-1}(X, \mathbb{C}) / F^k H^{2k-1}(X)\right)$$

in the complex vector space $V_k := H^{2k-1}(X, \mathbb{C}) / F^k H^{2k-1}(X)$.

Definition 3.1.1. *The* kth intermediate Jacobian $J^{2k-1}(X)$ *of* X *is the complex torus defined as*

$$\begin{aligned} J^{2k-1}(X) := V_k / \Lambda_k &= H^{2k-1}(X) / \left(F^k H^{2k-1}(X) \oplus H^{2k-1}(X, \mathbb{Z})\right) \\ &\cong \left(F^{n-k+1} H^{2n-2k+1}(X)\right)^{\vee} / H_{2n-2k+1}(X, \mathbb{Z}), \end{aligned}$$

where the last isomorphism follows from Poincarè duality.

Example 3.1.2. *For $k = 1$, the first intermediate Jacobian is an abelian variety. Namely $J^1(X) = \operatorname{Pic}^0(X)$, where*

$$\operatorname{Pic}^0(X) = \ker\left(H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z})\right)$$

parametrizes the isomorphism classes of invertible sheaves with null Chern class (see [Voi07a, Proposition 7.16]).
For $k = \dim X = n$, the intermediate Jacobian is called the Albanese variety $\operatorname{Alb}(X) := J^{2n-1}(X)$.

Definition 3.1.3. *The* Abel–Jacobi map *is the morphism defined by*

$$\begin{aligned} \mathrm{AJ} = \Phi_X^k \colon CH^k(X)_{\hom} &\to J^{2k-1}(X) \\ [Z] &\mapsto \left(\omega \mapsto \int_\gamma \omega\right), \end{aligned}$$

where γ is a piecewise smooth $(2n - 2k + 1)$–chain such that $\partial\gamma = Z$.

Example 3.1.4. *For $k = 1$, the Abel–Jacobi map is an isomorphism, $CH^1(X) \cong \mathrm{Pic}^0(X)$, but this is not true in general for $k \neq 1$.*
For $k = \dim X = n$, the Abel–Jacobi map is the Albanese map

$$\mathrm{alb}_X : CH_0(X)_{\mathrm{hom}} \to Alb(X).$$

3.1.2 How to measure the size of $CH_0(X)$

We consider a smooth complex projective variety X of dimension n.
Homologically trivial 0–cycles, are the degree–0 cycles, namely

$$Z = \sum_i n_i p_i \in CH(X) \text{ such that } \sum_i n_i = 0.$$

We can write any 0–cycle as the difference of two effective 0–cycles considering its positive and its negative part

$$Z = Z^+ - Z^-, \text{ where } Z^+ = \sum_{i, n_i > 0} n_i p_i \quad \text{and} \quad Z^- = \sum_{i, n_i < 0} n_i p_i.$$

We can parametrize effective cycles Z^+ of degree $d = \sum_{i, n_i > 0} n_i$ with the symmetric product $X^{(d)} = X^d / \mathrm{Sym}^d$ whose elements are unordered sets of d points of X. Given an effective 0–cycle, we can consider its class modulo rational equivalence via the map

$$\begin{aligned} \mathrm{c} \colon X^{(d)} &\to CH_0(X) \\ Z &\mapsto [Z]/\sim_{\mathrm{rat}} . \end{aligned}$$

Then we can consider the map

$$\begin{aligned} \sigma_d \colon X^{(d)} \times X^{(d)} &\to CH_0(X)_{\mathrm{hom}} \\ (Z^+, Z^-) &\mapsto c(Z^+) - c(Z^-). \end{aligned}$$

Definition 3.1.5. *The Chow group of 0–cycles $CH_0(X)$ is* representable *if there exits $d >> 0$ such that σ_d is surjective.*

One can prove that the fibers of σ_d are countable unions of closed algebraic subsets ([Voi07b, Lemma 10.7]). So it makes sense to define the dimension of a fiber of σ_d as the maximum dimension among the dimensions of its algebraic components. Moreover, there exists a countable union of proper algebraic subsets of $X^{(d)} \times X^{(d)}$ such that, by considering a point outside this locus, we get $\dim \sigma^{-1}(\sigma_d(x)) = r$ for some constant r, that is the dimension of the very general fiber. So we can define

$$\dim \mathrm{Im}\, \sigma_d := 2nd - r,$$

where $2nd = \dim \left(X^{(d)} \times X^{(d)} \right)$.

Definition 3.1.6. *We say that* $CH_0(X)_{\mathrm{hom}}$ *is* infinite–dimensional *if*

$$\lim_{d\to\infty} \dim \operatorname{Im} \sigma_d = +\infty.$$

Otherwise, we say that it is finite–dimensional.

Being representable gives an idea of the size of the group $CH_0(X)$, indeed it holds that $CH_0(X)$ is representable if and only of it is finite–dimensional (see [Voi07b, Proposition 10.10]). To enforce this result, we have a theorem by Roitman which proves that $CH_0(X)_{\mathrm{hom}}$ is an algebraic group when it is finite–dimensional,

Theorem 3.1.7 (Roitman, 1972)**.** *If* $CH_0(X)$ *is representable, then the Albanese map* $\mathrm{alb}_X\colon CH_0(X)_{\mathrm{hom}} \to \mathrm{Alb}(X)$ *is an isomorphism.*

3.2 Mumford's Theorem and Bloch's conjecture

In 1968 Mumford proved that, in general, we cannot expect the group of 0–cycles of a surface to be "small", in the sense that we cannot compare its size with the one of an algebraic variety. Then, considering the group of 0–cycles, we are dealing with a huge group in general. Moreover, this theorem testifies to the influence of Chow groups on singular cohomology.

Theorem 3.2.1 (Mumford, 1968)**.** *Let* S *be a smooth projective surface. If there exists a curve* $C \overset{j}{\hookrightarrow} S$ *such that* $j_*\colon CH_0(C) \to CH_0(S)$ *is surjective, then* $p_g(S) = \dim H^{2,0}(S) = 0$.

Mumford's theorem can be proven by Bloch–Srinivas decomposition of the diagonal. For a detailed overview of this argument we refer the reader to [Voi07b, Chapter10.2.2].

Mumford's theorem admits the following equivalent formulation.

Theorem 3.2.2 (Mumford)**.** *If* S *is a smooth projective surface such that* $p_g(S) \neq 0$*, then* $CH_0(S)$ *is infinite–dimensional.*

It is possible to generalize Mumford's and Roitman's theorems to higher dimensional varieties ([Voi07b, Theorem 10.17]).

Theorem 3.2.3. *Let* X *be a complex smooth projective variety. If there exists a subvariety* $Y \overset{j}{\hookrightarrow} X$ *such that* $j_*\colon CH_0(Y) \to CH_0(X)$ *is surjective, then* $H^0(X, \Omega^k_X) = 0$ *for all* $k > \dim Y$.

Bloch conjectured the converse of Mumford's theorem 3.2.2 in 1976 [Blo76, Lecture 1] (see also [Blo10]).

Conjecture 3.2.4 (Bloch, 1976)**.** *Let* S *be a smooth complex projective surface such that* $H^{2,0}(S) = 0$.
Then the Albanese map $\mathrm{alb}_S\colon CH_0(S)_{\mathrm{hom}} \to \mathrm{Alb}(S)$ *is an isomorphism.*

In 1942 the Italian mathematician Severi in [Sev42] made the following claim.

If S is a smooth projective surface, $CH_0(S)_{\mathrm{hom}} \otimes \mathbb{Q} = 0$ if and only if $p_g = q = 0$ for S.

Since Severi believed his claim was true, he didn't formulate it as conjectural, and so his contribution to the problem is usually neglected and this conjecture is known nowadays as "Bloch's conjecture". For an interesting review of Severi's work we refer the reader to [BCP04].

Remark 3.2.5. i) Bloch's conjecture was proved for surfaces not of general type by Bloch–Kas–Lieberman in 1976 [Voi07b, Chapter 11.1.3], for Godeaux surfaces [Voi07b, Chapter 11.1.4], uniruled surfaces [Voi07b, Chapter 11.1].

ii) If the surface is regular, i.e. $q(S) = 0$, then Alb(S) $= 0$, and the conjecture predicts that also $CH_0(S)_{\mathrm{hom}}$ is zero.

3.3 Bloch–Beilinson's conjecture

Bloch's conjecture is part of a big conjectural framework known as "philosophy of mixed motives" developed essentially by Bloch and Beilinson. The main idea is to gain informations about the kernel of the cycle class map, or better on the kernel of the Abel–Jacobi map

$$CH^i(X)_{\mathrm{AJ}} := \ker\left(\mathrm{AJ}\colon CH^i(X)_{\mathrm{hom}} \to J^i(X)\right).$$

So this conjecture deals with the injectivity of the cycle class maps, whether the standard conjectures (and in particular the Hodge conjecture) involve the surjectivity of these maps. The importance of Bloch–Beilinson's conjecture is huge, indeed, if it was true, it would provide a deep understanding of the representability of the Chow groups. Bloch–Beilinson's conjecture has different equivalent formulations, here we present one which does not involve mixed motives. This formulation conjectures the existence of a filtration on rational Chow groups, and it was presented by Beilinson in 1987 in [Bei87, 5.10]. For further details on this topic, and an introduction on the other formulations, we refer the reader to Jannsen's paper [Jan94].

Conjecture 3.3.1 (Bloch–Beilinson, 1987)**.** *Let k be a field and let X be any smooth projective variety over k. Then Künneth conjecture $C(X)$ holds, i.e. the Künneth components Δ_i of the diagonal of X are algebraic: $\mathrm{cl}_{X\times X}(\pi_i) = \Delta_i$ (see Section 1.2). Moreover, for all $i \geq 0$, there exists a descending filtration F on $CH^j(X)_{\mathbb{Q}}$ such that for a fixed Weil cohomology theory $H^*(X)$, the following properties hold:*

a)
$$\begin{cases} F^0CH^i(X)_{\mathbb{Q}} = CH^i(X)_{\mathbb{Q}}, \\ F^1CH^i(X)_{\mathbb{Q}} = CH^i_{\text{hom}}(X)_{\mathbb{Q}}, \\ F^2CH^i(X)_{\mathbb{Q}} = CH^i_{\text{AJ}}(X)_{\mathbb{Q}}; \end{cases}$$

b) $F^\bullet$ *is compatible with the intersection product, i.e.*
$$F^rCH^i(X)_{\mathbb{Q}} \cdot F^sCH^j(X)_{\mathbb{Q}} \subseteq F^{r+s}CH^{i+j}(X)_{\mathbb{Q}};$$

c) $F^\bullet$ *is compatible with push–forward* f_* *and pull–back* f^* *for any morphism* $f\colon X \to Y$ *in* $\mathrm{SmProj}(k)$*;*

d) $\pi_i|_{\mathrm{Gr}^\nu CH^j(X)_{\mathbb{Q}}} = \begin{cases} \mathrm{id} & \text{if } i = 2j - \nu, \\ 0 & \text{otherwise}; \end{cases}$.

e) $F^\nu CH^j(X)_{\mathbb{Q}} = 0$ *for* $\nu \gg 0$.

Conditions *b)* and *c)* imply that the filtration is compatible with the action of correspondences. Condition *a)* applied to $X \times X$ means that the action of correspondences on the graded pieces of the filtration
$$\mathrm{Gr}^\nu_F CH^i(X)_{\mathbb{Q}} = F^\nu CH^i(X)_{\mathbb{Q}}/F^{\nu+1}CH^i(X)_{\mathbb{Q}}$$
only depends on the class modulo $\sim_{\text{hom}}$, i.e. $CH^{\dim X}(X \times X)_{\text{hom}}$ acts as zero. Assuming that the Künneth conjecture holds for X, we can define the effective motive $h^i(X) = (X, \pi_i, 0)$. Then condition *d)* means that the graded pieces $Gr^\nu CH^j(X)_{\mathbb{Q}}$ depends only on the motive $h^{2j-\nu}(X)$ modulo homological equivalence.

Remark 3.3.2. We point out that Bloch's conjecture 3.2.4 for surfaces follows by Bloch–Beilinson's conjecture 3.3.1 (see [Jan94, Lemma 3.2]). Indeed, if we consider a surface S, such that $p_g(S) = \dim H^0(S, \Omega^2) = 0$, then we have that $H^2(S, \mathbb{Q})$ is algebraic. Then if we consider the Künneth component of the diagonal $\pi_2 \in H^2(S, \mathbb{Q}) \otimes H^2(S, \mathbb{Q})$, we have $\pi_2 = \sum_{i,j} n_{ij} C_i \times C_j$, where C_i and C_j are curves on S. By condition *d)*, then we have that $F^2CH^2(S)_{\mathbb{Q}} = CH^2_{\text{AJ}}(S)_{\mathbb{Q}}$ depends only on π_2. We get that π_2 acts trivially on $CH^2(S)_{\mathbb{Q}}$, because we can move a 0–cycle so it does not intersect a finite union of curves on S. Hence we conclude that $F^2CH^2(S)_{\mathbb{Q}} = 0$.

For an introduction to further consequences of the Bloch–Beilinson's conjecture, we refer the reader to [Jan94].

3.3.1 Murre's filtration

In 1993 Murre constructed a filtration on $CH^*(X)_{\mathbb{Q}}$ in [Mur93] using Chow motives, so the main idea is to focus on rational equivalence. The starting point is the Chow–Künneth decomposition for surfaces, which induces a decomposition in submotives. The Chow groups of these motives induce a filtration on the Chow ring of the surface ([Mur90, Theorem 3]) which actually coincides with Bloch's filtration ([Blo10, pp. 1–12]). Then the idea is to construct a filtration of Chow groups using Chow motives. To construct such a filtration we fix a Weil cohomology theory H^*, and we consider $X \in \mathrm{SmProj}(\mathrm{k})$ irreducible of dimension d such that $C(X)$ holds, i.e. the Künneth components of the diagonal π_i are algebraic. We recall (see 2.2) that if there exists Chow–Künneth decomposition $\{\pi_i\}_i$ for X, then it provides a filtration

$$F^{\nu}CH^j(X)_{\mathbb{Q}} = \bigcap_{i>2j-\nu} \ker \pi_i = \bigoplus_{i\leq 2j-\nu} \mathrm{Im}(\pi_i).$$

Then, Murre formulates the following conjecture.

Conjecture 3.3.3 (Murre). *(A) X admits a Chow–Künneth decomposition (this is also denoted as $CK(X)$, see Section2.2);*

(B) the correspondences $\pi_{2j+1}, \dots, \pi_{2d}$ act as zero on $CH^j(X)_{\mathbb{Q}}$;

(C) the induced filtration $F^{\bullet}$ is independent on the choice of the Cow–Künneth decomposition;

(D) $F^1CH^j(X)_{\mathbb{Q}} = CH^j_{\mathrm{hom}}(X)_{\mathbb{Q}}$.

Jannsen in [Jan94] proved the following beautiful result.

Theorem 3.3.4 (Jannses, 1994). *Murre's conjecture is equivalent to the Bloch–Beilinson's conjecture, and the filtrations coincide, i.e., if one filtration exists then the other does and they agree.*

As a corollary, we get that the Bloch–Beilinson's conjectural filtration is unique, and if it exists it coincides with the one constructed by Murre.

3.4 Voisin's conjecture on 0–cycles

Here we present the conjecture on which we focus in the next chapters. Inspired by the Bloch–Beilinson's conjecture, Voisin formulated in 1996 in [Voi96] the following conjecture on 0–cycles on the self–product of surfaces of geometric genus one.

Conjecture 3.4.1 (Voisin, 1996). *Let S be a smooth complex projective surface with $p_g(S) = 1$ and $q(S) = 0$. Let $a, a' \in CH^2_{\mathrm{hom}hom}(S)$ be two 0–cycles of degree 0 (i.e. homologically trivial 0–cycles). Let p_1, p_2 be the projections on the first and on the second factor of $S \times S$ respectively. Then*

$$(p_1^*a) \cdot (p_2^*a') = (p_1^*a') \cdot (p_2^*a) \text{ in } CH^4(S \times S). \tag{3.4.2}$$

Remark 3.4.3. To ease the notation, we use the following convention: $a \times a' := (p_1^* a) \cdot (p_2^* a')$. So (3.4.2) becomes

$$a \times a' = a' \times a \in CH^4(S \times S).$$

Voisin gave also a generalized formulation of this conjecture.

Conjecture 3.4.4. *Let S be a projective surface with $q(S) = 0$ and let $a_1, \dots, a_k$ be 0–cycle of degree 0 on S, then for $k \geq p_g(S) + 1$, one has $\sum_{\sigma \in \mathrm{Sym}_k} \mathrm{sgn}(\sigma) \sigma^*(a_1 \times \cdots \times a_k)$.*

We get Conjecture 3.4.1 when we consider for example K3 surfaces, so $p_g(S) = 1$ and the condition is $k \geq 2$.

Remark 3.4.5. One can see Voisin's conjecture as a consequence of a motivic version of Bloch's conjecture 3.2.4 (see [LV20, Remark 5.2]). Indeed, let us consider a surface S with $p_g(S) = 1$ and $q(S) = 0$. Since S is a surface, it admits a Chow–Künneth decomposition $\{\pi_S^0, \pi_S^2, \pi_S^4\}$, where $\pi_S^0 = \{x\} \times S$ and $\pi_S^4 = S \times \{x\}$, where $x \in S$ (see Defintion 2.2.1 and Remark 2.2.3). Then we can consider the motive M defined as

$$M := \bigwedge^2 h^2(S) := \left(S \times S, \frac{1}{2} \sum_{\sigma \in \mathfrak{S}_2} \mathrm{sgn}(\sigma) \Gamma_\sigma \circ \left(\pi_S^2 \times \pi_S^2 \right), 0 \right).$$

By the hypotheses on S we get that $h^{j,0}(M) = 0$ for all j, then a motivic version of Bloch's conjecture would imply that $CH_0(M) = 0$. Moreover, since π_S^0 and π_S^4 do not act on $CH^2_{\mathrm{hom}}(S)$, it holds that

$$CH^2_{\mathrm{hom}}(S) = CH^2_{AJ}(S) = (\pi_S^2)_* CH^2(S).$$

If we consider two 0–cycles $a, a' \in CH^2_{\mathrm{hom}}(S)$, then we have

$$a \times a' - a' \times a = (\pi_S^2 \times \pi_S^2)(a \times a') - \iota_*(\pi_S^2 \times \pi_S^2)(a \times a') \text{ in } CH_0(M),$$

where $\iota \in \mathfrak{S}_2$ is the non–trivial element, and conjecturally $CH_0(M) = 0$.

There are few examples in which Conjecture 3.4.1 has been verified (see [Voi96], [Lat16c], [Lat18a], [Lat16a]), but this conjecture is still open for a general $K3$ surface. There are examples in which the conjecture is true for surfaces with geometric genus greater than one (see [Lat18e]). There is also an analogous version of the conjecture for higher dimensional varieties, this version is studied in [Voi96],[Lat16b], [Lat17], [Lat18d], [BLP20], [LV20], [Via18], [Bur18].

Conjecture 3.4.1 has been proven by Laterveer for two families of Todorov surfaces ([Lat16c] and [Lat18a]). For both of these families the proof relies on the following useful case proved by Voisin ([Voi96, Theorem 3.4]).

3.4.1 A family of K3 surfaces on which the conjecture holds

Here we present the case of a 10 dimensional family of K3 surfaces, obtained as a desingularization of a double cover of $\mathbb{P}^2$ branched along the union of two cubics.

Remark 3.4.6. This case is particularly interesting in our setting, because when considering the family of Todorov surfaces of type $(2, 12)$, by Rito's theorem 5.0.2, we can associate to every Todorov surface a K3 surface that can be described as the blow–up of a double cover of $\mathbb{P}^2$ ramified along the union of two cubics. So by Theorem 3.4.10, Voisin's conjecture 3.4.1 holds on the K3 surface associated to a Todorov surface. Thus, in order to prove Conjecture 3.4.1 for the family of Todorov surfaces of type $(2, 12)$, we have to establish a relation between zero–cycles on the Todorov surface and zero–cycle on the associated K3 (see Section 5.3).

Let $W \subset \mathbb{P}^5$ be the sextic 4–fold defined by

$$F_1(x)F_2(x) - F_1(y)F_2(y) = 0,$$

where $(x : y) = (x_0 : x_1 : x_2 : y_0 : y_1 : y_2) \in \mathbb{P}^5$, F_1 and F_2 are the equations of two cubics E_1 and E_2 in $\mathbb{P}^2$ which are smooth and meet transversally. Let $S \to \mathbb{P}^2$ the blow–up of the double cover of $\mathbb{P}^2$ ramified along the union of the cubics $E_1 \cup E_2$. We denote as τ the involution on $S \times S$ exchanging factors, namely $\tau(s_1, s_2) = (s_2, s_1)$. We denote as π_1 and π_2 the natural projections on the first and the second factor of $S \times S$ respectively.

We notice that the singular locus of W is

$$\{(x : y) \in \mathbb{P}^5 \colon F_i(x) = F_i(y) = 0 \text{ for } i = 1, 2\},$$

where it has non–degenerate quadratic singularities. We consider its desingularization $\widetilde{W}$, and the involution ι of $\mathbb{P}^5$ defined by $\iota(x : y) = (y : x)$. We notice that W is invariant under the action of ι, and so $\widetilde{W}$ is invariant under the induced involution $\tilde{\iota}$, i.e. $CH_0(\widetilde{W}) = CH_0(\widetilde{W})^-$.

Lemma 3.4.7 (Lemma 3,4,1,[Voi96])**.** *There exists a correspondence*

$$\Gamma \in CH^4(\widetilde{S} \times \widetilde{S} \times \widetilde{W})$$

equivariant with respect to τ and $\tilde{\iota}$ which induces an inclusion

$$[\Gamma] \colon \left(\ker \left(CH_0(S \times S) \right) \xrightarrow{(\pi_1)_*} CH_0(S) \right) \hookrightarrow CH_0(\widetilde{W}).$$

Proof. The argument relies on the results on Shioda's paper [Shi79]. Let $\Sigma \subset \mathbb{P}^3$ defined by the equation $u^6 = F_1(x)F_2(x)$, where we use the notation

$(u : x) = (u : x_0 : x_1 : x_2)$ to denote a point in $\mathbb{P}^3$. Then there exists a rational map

$$\begin{aligned} \phi \colon \Sigma \times \Sigma &\dashrightarrow W \\ \big((u : x), (u', x')\big) &\mapsto (u'x : ux'). \end{aligned}$$

The image of the map is indeed inside W, since we have

$$\begin{aligned} F_1(u'x)F_2(u'x) - F_1(ux')F_2(ux') &= (u')^6 F_1(x)F_2(x) - u^6 F_1(x')F_2(x') \\ &= (u')^6 u^6 - u^6 (u')^6 = 0. \end{aligned} \tag{3.4.8}$$

By resolving the indeterminacies we get a morphism

$$\widetilde{\phi} \colon \widetilde{\Sigma \times \Sigma} \to \widetilde{W},$$

where $\widetilde{\Sigma \times \Sigma}$ is the blow–up of $\Sigma \times \Sigma$. Next we consider the natural morphism

$$\begin{aligned} \psi \colon \Sigma &\to S \\ (u : x) &\mapsto (u^3 : x), \end{aligned}$$

which is a degree 3 covering. Since

$$S \cong \{(u : x) \in \mathbb{P}(1,1,1,3) \colon u^2 = F_1(x)F_2(x)\},$$

the morphism ψ corresponds to the quotient map into the weighted projective plane $\mathbb{P}^3 \xrightarrow{3:1} \mathbb{P}(1,1,1,3)$, i.e. our situation is the following

$$\begin{tikzcd} \widetilde{\Sigma} \arrow[r, "res"] \arrow[d, "\psi"] & \Sigma \arrow[r, hook] & \mathbb{P}^3 \arrow[d, "3:1"] \\ \widetilde{S} \arrow[r, "res"] & S \arrow[r, hook] & \mathbb{P}(1,1,1,3). \end{tikzcd}$$

We have the following diagram, in which all the maps are morphisms

$$\begin{tikzcd} \widetilde{\Sigma \times \Sigma} \arrow[r, "\widetilde{\phi}"] \arrow[d, "\widetilde{\psi} \times \widetilde{\psi}"] & \widetilde{W} \subset \mathbb{P}^5 \\ \widetilde{S} \times \widetilde{S}. & \end{tikzcd}$$

Then we have a correspondence Γ whose induced action on cycles is

$$\Gamma_* = \widetilde{\phi}_* \left(\widetilde{\psi} \times \widetilde{\psi}\right)^* : CH_i\left(\widetilde{S} \times \widetilde{S}\right) \to CH_i(\widetilde{W}).$$

Since $\left(\widetilde{\psi}\times\widetilde{\psi}\right)^*$ preserves codimensions on cycles, we have

$$CH_i\left(\widetilde{S}\times\widetilde{S}\right) = CH^{4-i}\left(\widetilde{S}\times\widetilde{S}\right) \overset{\left(\widetilde{\psi}\times\widetilde{\psi}\right)^*}{\longrightarrow} CH^{4-i}(\widetilde{\Sigma\times\Sigma}) = CH_i(\widetilde{\Sigma\times\Sigma}),$$

so it preserves also dimensions. Moreover, since

$$\widetilde{\phi}_*\colon CH_i(\widetilde{\Sigma\times\Sigma}) \to CH_i(\widetilde{W}),$$

we get that Γ_* is a degree–0 correspondence, i.e. $\Gamma \in CH^4(\widetilde{S}\times\widetilde{S}\times\widetilde{W})$. We notice that $\widetilde{\phi}\colon \widetilde{\Sigma\times\Sigma}\to\widetilde{W}$ corresponds to the quotient map for the diagonal action of $\mathbb{Z}/6\mathbb{Z}$ on $\widetilde{\Sigma\times\Sigma}$, i.e.

$$\big(\lambda,(u:x),(u':x')\big) \overset{\mathbb{Z}/6\mathbb{Z}}{\mapsto} \big((\lambda u:x),(\lambda u',x')\big) \overset{\widetilde{\phi}}{\mapsto} (\lambda u'x:\lambda ux') = (u'x:ux').$$

So that over a point $(u'x:ux')\in\widetilde{W}$ there are 6 points of $\widetilde{\Sigma\times\Sigma}$. In particular,

$$\left(\ker\left(CH_0(S\times S)\right) \overset{(\pi_1)_*}{\longrightarrow} CH_0(S)\right) \hookrightarrow CH_0(\widetilde{W})_{\text{hom}}$$

is injective. Moreover, we claim that for any $a,a'\in CH_0(S)_{\text{hom}}$ it holds

$$\Gamma_*\left(a\times a' - a'\times a\right) \subset CH_0(\widetilde{W})^-,$$

where $CH_0(\widetilde{W})^- = \{\eta\in CH_0(\widetilde{W})\colon \widetilde{\iota}^*\eta = -\eta\}$ is the (-1)–eigenspace[1] for the action of $\widetilde{\iota}$ on $\widetilde{W}$, which is

$$\begin{aligned}\iota\colon \mathbb{P}^5 &\to \mathbb{P}^5\\ (x:y)&\mapsto(y:x).\end{aligned}$$

To prove our claim, first of all we notice that the action of $\mathbb{Z}/6\mathbb{Z}$ on

$$(\widetilde{\psi}\times\widetilde{\psi})^*CH_0(\widetilde{S}\times\widetilde{S})$$

is reduced to the action of $\mathbb{Z}/2\mathbb{Z}$, i.e. the action of the involution j of S over $\mathbb{P}^2$. The situation is indeed the following

$$\begin{tikzcd} \mathbb{Z}/6\mathbb{Z} \arrow[loop right] & \widetilde{\Sigma\times\Sigma} \arrow[r, "\widetilde{\phi}"] \arrow[d, "\widetilde{\psi}\times\widetilde{\psi}"', "3:1"] & \widetilde{W} \\ \mathbb{Z}/6\mathbb{Z} \arrow[loop right] & \widetilde{S}\times\widetilde{S}. \arrow[ur, "\Gamma"'] & \end{tikzcd}$$

[1] As an example, one can take two points $P,Q\in\widetilde{W}$ with $Q=\widetilde{\iota}(P)$ and consider the cycle $[P-Q]$. Then $\widetilde{\iota}^*[P-Q]=\widetilde{\iota}[P]-\widetilde{\iota}[Q]=[Q-P]=-[P-Q]$, so that $[P-Q]\in CH_0(\widetilde{W})^-$.

Since j acts as -1 on $CH_0(S)_{\text{hom}}$, then its action on $CH_0(S)_{\text{hom}} * CH_0(S)_{\text{hom}}$ is the identity, indeed for any $\sum_i n_i p_i, \sum_l m_l q_l \in CH_0(S)_{\text{hom}}$ we have

$$j\left(\left(\sum_i n_i p_i\right) \cdot \left(\sum_l m_l q_l\right)\right) = \left(-\sum_i n_i p_i\right) \cdot \left(-\sum_l m_l q_l\right).$$

Then, looking at the action $\Gamma_* = \widetilde{\phi}_* \left(\widetilde{\psi} \times \widetilde{\psi}\right)^*$ on 0–cycles of degree 0, we get

$$CH_0(\widetilde{S})_{\text{hom}} * CH_0(\widetilde{S})_{\text{hom}} \overset{(\widetilde{\psi}\times\widetilde{\psi})^*}{\hookrightarrow} CH_0(\widetilde{\Sigma \times \Sigma})^{\mathbb{Z}/6\mathbb{Z}}_{\text{hom}} \overset{\widetilde{\phi}_*}{\cong} CH_0(\widetilde{W})_{\text{hom}}.$$

Since $\widetilde{W}$ is invariant under the action of $\widetilde{\iota}$, then if $\Gamma_*\left(a \times a'\right) \in CH_0(\widetilde{W})_{\text{hom}}$ also $\Gamma_*(a' \times a) \in CH_0(\widetilde{W})_{\text{hom}}$.
Indeed we have $\widetilde{\iota}^* \colon CH_0(\widetilde{W}) \xrightarrow{\sim} CH_0(\widetilde{W})$, and

$$\begin{aligned} \widetilde{\iota}^*\left(\Gamma_*\left(a \times a' - a' \times a\right)\right) &= \widetilde{\iota}^*\left(\Gamma_*(a \times a')\right) - \widetilde{\iota}^*\left(\Gamma_*(a' \times a)\right) \\ &= \Gamma_*(a' \times a) - \Gamma_*(a \times a') \\ &= \Gamma_*(a' \times a - a \times a'). \end{aligned}$$

So $\Gamma_*(a \times a' - a' \times a) \in CH_0(\widetilde{W})^-$. □

We have to prove the following to conclude that $\Gamma_*(a \times a' - a' \times a) = 0$.

Proposition 3.4.9 (Prosposition 3.4.2, [Voi96]). $CH_0(\widetilde{W})^- = 0$.

Proof. We recall that $W = \{(x : y) \in \mathbb{P}^5 \colon F_1(x)F_2(x) - F_1(y)F_2(y) = 0\}$. For each $\alpha \in \mathbb{C}$ we define a Calabi–Yau 3–fold

$$W_\alpha := \{(x : y) \in \mathbb{P}^5 \colon F_1(x) = \alpha F_2(y), F_1(y) = \alpha F_2(x)\}.$$

If $(x : y) \in W_\alpha$, then we have

$$F_1(x)F_2(x) - F_1(y)F_2(y) = \alpha F_2(y)F_2(x) - \alpha F_2(x)F_2(y) = 0,$$

so W is covered by $\{W_\alpha\}_{\alpha \in \mathbb{C}}$. We notice that W_α is the complete intersection of two cubics in $\mathbb{P}^5$, the general element of the family is smooth, and each W_α is invariant under the action of ι. Then $CH_0(W_\alpha)^- = 0$ by [Voi96, Proposition 3.4.3] (see also [Voi92, Théorème 2.20] where this is proved for a quintic 3–fold in $\mathbb{P}^4$ invariant under an involution satisfying the same conditions). Since any 0–cycle on W can be supported on finitely many W_α, we conclude that $CH_0(\widetilde{W})^- = 0$. □

Then Conjecture 3.4.1 holds for S by Proposition 3.4.9 and Lemma 3.4.7.

Theorem 3.4.10 (Theorem 3.4, [Voi96]). *Conjecture 3.4.1 holds for S.*

Remark 3.4.11. Laterveer proved Voisin's conjecture 3.4.1 for the desingularization of a double cover of $\mathbb{P}^2$ ramified along the union of an irreducible quartic and an irreducible quadric ([Lat16c, Proposition 14]). Laterveer proved Voisin's conjecture 3.4.1 also for the desingularization of a double cover of $\mathbb{P}^2$ brached along the union of 6 lines in general position ([Lat16c, Proposition 16]).

To prove Voisin's conjecture for the desingularization of a double cover of $\mathbb{P}^2$ branched along the unione of a line and a quintic, even though it would be desirable, seems not possible adapting this approach. The problem in this case is to define a cover $\{W_\alpha\}$ such that any W_α is invariant under the action of the involution and it has trivial canonical bundle.

Chapter 4

Techniques to deal with algebraic cycles

4.1 Bloch's higher Chow groups and Borel–Moore homology

We briefly introduce some useful tools to deal with algebraic cycles, for further details we refer the reader to [Voi02].

4.1.1 Bloch's higher Chow groups

Let X be an irreducible variety over a field k. We denote as

$$\Delta_i = \left\{ (x_1, \dots x_{i+1}) \in \mathbb{A}_k^{i+1} \colon \sum_{j=1}^{i+1} x_1 = 1 \right\}$$

the affine simplex of dimension i. For each $j \leq i+1$ we have a face map $l_j \colon \Delta_{i-1} \hookrightarrow \Delta_i$ that corresponds to the inclusion into Δ_i of the hyperplane $(x_j = 0) \cap \Delta_i$. We denote as $Z^r(X \times \Delta_i)_{pr}$ the subgroup of codimension r cycles generated by subvarieties meeting properly all the $X \times \Delta_j \subseteq X \times \Delta_i$ (i.e. the intersection has the minimum possible dimension), where Δ_j is a face of Δ_i, i.e. $j \leq i$ (we are considering faces of any codimension). We can consider the pullback of the face map

$$\begin{array}{rcl} l_j^* \colon Z^r(X \times \Delta_i)_{pr} & \longrightarrow & Z^r(X \times \Delta_{i-1})_{pr} \\ W & \mapsto & l_j^* W = W \cap (X \times l_j(\Delta_{i-1})) \end{array}$$

where $W \cap (X \times l_j(\Delta_{i-1})) \subset X \times l_j(\Delta_{i-1}) \cong X \times \Delta_{i-1}$.

Then we have a differential map

$$\begin{aligned} d_i\colon Z^r(X\times\Delta_i)_{pr} &\longrightarrow Z^r(X\times\Delta_{i-1})_{pr}\\ W &\mapsto d_iW:=\sum_{j=1}^{i+1}(-1)^j l_j^* W. \end{aligned}$$

Definition 4.1.1. *We define the i–th* Bloch's higher Chow group *of codimension r cycles of X as the i–th homotopy group of the chain–complex $(Z^r(X\times\Delta_\bullet)_{pr}, d_i)$:*

$$\begin{aligned} \mathrm{CH}^r(X,i) &:= H_i\left(Z^r(X\times\Delta_\bullet)_{pr}, d_i\right)\\ &= \frac{\ker\left(d_i\colon Z^r(X\times\Delta_i)_{pr}\to Z^r(X\times\Delta_{i-1})_{pr}\right)}{\mathrm{Im}\left(d_{i+1}\colon Z^r(X\times\Delta_{i+1})_{pr}\to Z^r(X\times\Delta_i)_{pr}\right)}\\ &= \frac{\ker\left(\sum_{j=1}^{i+1}(-1)^j l_j^*\colon Z^r(X\times\Delta_i)_{pr}\to Z^r(X\times\Delta_{i-1})_{pr}\right)}{\mathrm{Im}\left(\sum_{j=1}^{i+2}(-1)^j l_j^*\colon Z^r(X\times\Delta_{i+1})_{pr}\to Z^r(X\times\Delta_i)_{pr}\right)}. \end{aligned}$$

Remark 4.1.2. Since $X\times\Delta_0\cong X$, we have that $\ker d_0 = Z^r(X)$. As for the boundaries one can easily see that they are the subgroup of codimension–r cycles rationally trivial, so that

$$CH^r(X,0)=CH^r(X).$$

We mention some useful properties of Bloch's higher Chow groups $CH^r(X,i)$ and we refer the reader to [Blo86] for more details on this construction.

i) *Functoriality.* Bloch's higher Chow groups are covariant for proper maps, contravariant for flat maps and contravariant for arbitrary maps when X is smooth.

ii) *Localization.* If $Y\subset X$ is closed of pure codimension d, then for any $r\le\dim X$ we have a long exact sequence for Bloch's higher Chow groups:

$$\begin{aligned} \ldots\to CH^r(X-Y,i+1)\to CH^{r-d}(Y,i)\to CH^r(X,i)\\ \to CH^r(X-Y,i)\to CH^{r-d}(Y,i-1)\to\ldots\\ \ldots\to CH^{r-d}(Y,0)\to CH^r(X,0)\to CH^r(X-Y,0)\to 0. \end{aligned}$$

4.1.2 Borel–Moore homology

We briefly recall some basic facts about singular homology and cohomology. For further details, we refer to the Appendix A, and [Ara12, Chapter 7], [PS08, Appendix B].

Let X be a topological space and R be a ring, in our case it would be $R = \mathbb{Z}$. A singular q–simplex is a continuous map $\Delta_q \to X$. We denote as $C_q(X;R)$ the free R–module generated by the singular q–simplices, so we get a chain–complex $C_\bullet(X;R)$. The homology of this complex is called *singular homology* and we denote the q–th singular homology group as $H_q(X;R)$. Considering homomorphisms, we get dually the cochain complex $C^q(X;R) := \mathrm{Hom}_R\,(S_q(X;R),R)$ and the q–th *singular cohomology* group $H^q(X;R)$. We recall that the behavior of the homology is covariant in the category of topological spaces and continuous maps, whereas the cohomology behaves contravariantly. In fact, given a continuous map $f\colon X \to Y$, we get a pull–back map in cohomology $f^*\colon H^\bullet(Y;R) \to H^\bullet(X;R)$, and a push–forward map in homology $f_*\colon H_\bullet(X;R) \to H_\bullet(Y;R)$.
Given $A \subset X$ closed, we get a chain–complex and a cochain–complex associated with the pair (X,R), i.e.

$$C_\bullet(X,A;R) := C_\bullet(X;R)/C_\bullet(A;R);$$
$$C^\bullet(X,A;R) := \ker(C^\bullet(X;R) \to C^\bullet(A;R)).$$

So we can consider the q–th singular homology group of the pair $H_q(X,A;R)$, given by the chain–complex $C_\bullet(X,A;R)$. Dually, we have the q–th singular cohomology group of the pair $H^q(X,A;R)$, given by the cochain complex $C^\bullet(X,A;R)$.

Definition 4.1.3. *We define the* R–module of the q–cochains

$$C^q(X,X-A;R) := \ker(C^q(X;R) \to C^q(X-A;R))$$

which are the cochains that vanish on the chains in $C_q(X-A;R)$, i.e. the cochains supported on A.
Varying $A \subset X$ compact, we get a direct system. Taking the direct limit, we define the R–module of the q–cochains with compact support

$$C^q_c(X;R) := \varinjlim C^q(X,X-A) \quad \textit{with } A \subset X \textit{ compact.}$$

Taking the cohomology of this complex, we get the q–th compact support cohomology *group $H^q_c(X;R)$.*
Dually, taking the inverse limit, we define the Borel–Moore *q–chain R–module*

$$C^{BM}_q(X;R) := \varprojlim C_q(X,X-A) \quad \textit{with } A \subset X \textit{ compact.}$$

We then can define the Borel–Moore homology *groups $H^{BM}_q(X;R)$ as the homology groups of this last chain–complex.*

Considering the category of locally compact spaces with proper maps $f\colon X \to Y$, we have that the Borel–Moore homology is a covariant functor and we have induced maps

$$f_*\colon H^{BM}_q(X;R) \to H^{BM}_q(Y;R).$$

Remark 4.1.4. We notice that, when X is compact, the Borel–Moore homology is the singular cohomology.

We recall some useful results on Borel–Moore homology. [PS08, Lemma 6.25]

Lemma 4.1.5. *Borel-Moore homology has a mixed Hodge structure given by the isomorphisms*

$$H_k^{BM}(X,\mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Q}}(H_c^k(X,\mathbb{Q});\mathbb{Q})$$

which has at most weights in the interval $[-k,0]$.

We consider the Deligne's weight filtration W on the Borel–Moore homology (see [PS08, Appendix B] and [Del75]).

We recall that, since W is an increasing filtration, the associated graded piece is

$$\mathrm{Gr}^W_{-2i}H_{2i+j}(V,\mathbb{Q}) := \frac{W_{-2i}H_{2i+j}(V,\mathbb{Q})}{W_{-2i-1}H_{2i+j}(V,\mathbb{Q})}.$$

The lemma implies that

$$\mathrm{Gr}^W_{-k}\, H_k(X) = W_{-k}H_k(X,\mathbb{Q}),$$

is the smallest weight subspace of the Borel–Moore homology of X [Tot14].

4.1.3 Relation between BM–homology and Bloch's higher Chow groups

There is a natural map (see [Jan90, Section 8], [Tot14, Section 4]) relating Bloch's higher Chow groups and Borel–Moore homology groups

$$CH_i(X,j)_{\mathbb{Q}} \to W_{-2i}H^{BM}_{2i+j}(X,\mathbb{Q}) \cap F^{-i}H^{BM}_{2i+j}(X).$$

In particular, there are cycle class maps

$$CH_i(X,j)_{\mathbb{Q}} \to \mathrm{Gr}^W_{-2i}\, H_{2i+j}(X)$$

which are compatible with long exact sequences.

We state a useful localization result for the Borel–Moore homology and the Bloch's higher Chow groups, fur further details see the original result in [Blo94, Corollary 0.2] and its extension in [Lev01, Theorem 1.7].

Theorem 4.1.6. *Let X be a quasi–projective variety, $Y \subset X$ a closed subvariety and let $U = X\backslash Y$ be its complement. Then we have a long exact sequence for Bloch's higher Chow groups and for the Borel–Moore homology, in particular we have a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
CH_i(U,j+1)_{\mathbb{Q}} & \to & CH_i(Y,j)_{\mathbb{Q}} & \to & CH_i(X,j)_{\mathbb{Q}} & \to & CH_i(U,j)_{\mathbb{Q}} & \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\mathrm{Gr}^W_{-2i}\, H_{2i+j+1}(U) & \to & \mathrm{Gr}^W_{-2i}\, H_{2i+j}(Y) & \to & \mathrm{Gr}^W_{-2i}\, H_{2i+j}(X) & \to & \mathrm{Gr}^W_{-2i}\, H_{2i+j}(U) & \to
\end{array}$$

The commutativity of the diagram is a consequence of the cycle class map being compatible with long exact sequences (see [Tot14, Section 4]).

Definition 4.1.7. *([Tot14]) We say that X has the* weak property *if there are isomorphisms induced by the cycle class maps*

$$CH_i(X)_{\mathbb{Q}} \xrightarrow{\sim} W_{-2i}H_{2i}(X,\mathbb{Q}) \quad \forall i.$$

We say that X has the strong property *if it has the weak property and there are surjections induced by the cycle class maps*

$$CH_i(X,1)_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gr}^W_{-2i}H_{2i+1}(X,\mathbb{Q}) \quad \forall i.$$

We have the following useful results.

Lemma 4.1.8. *[Lat18a, Lemmas 4.2;4.3;4.4] Let X be a quasi–projective variety.*

1. *Let $Y \subset X$ be a closed subvariety and $U = X \backslash Y$ be its complement. If Y and U have the strong property, then X has the strong property too.*

2. *Suppose that X admits a stratification by strata of the form $\mathbb{A}^k \backslash L$, where L is a finite union of linearly embedded affine subspaces. Then X has the strong property.*

3. *If X has the strong property and $P \to X$ is a projective bundle, then P has the strong property too.*

Proof. Proof of part (1).
Using Theorem 4.1.6, we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
CH_i(U,j+1)_{\mathbb{Q}} & \to & CH_i(Y,j)_{\mathbb{Q}} & \to & CH_i(X,j)_{\mathbb{Q}} & \to & CH_i(U,j)_{\mathbb{Q}} & \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\mathrm{Gr}^W_{-2i}H_{2i+j+1}(U) & \to & \mathrm{Gr}^W_{-2i}H_{2i+j}(Y) & \to & \mathrm{Gr}^W_{-2i}H_{2i+j}(X) & \to & \mathrm{Gr}^W_{-2i}H_{2i+j}(U) & \to
\end{array}$$

Considering the diagram for $j = 0$ we get

$$\begin{array}{ccccccccc}
CH_i(U,1)_{\mathbb{Q}} & \xrightarrow{r} & CH_i(Y)_{\mathbb{Q}} & \xrightarrow{s} & CH_i(X)_{\mathbb{Q}} & \xrightarrow{t} & CH_i(U)_{\mathbb{Q}} & \xrightarrow{u} & 0 \\
cl^1_U \downarrow & & cl_Y \downarrow\cong & & cl_X \downarrow & & cl_U \downarrow\cong & & \\
\mathrm{Gr}^W_{-2i}H_{2i+1}(U) & \xrightarrow{\overline{r}} & W_{-2i}H_{2i}(Y) & \xrightarrow{\overline{s}} & W_{-2i}H_{2i}(X) & \xrightarrow{\overline{t}} & W_{-2i}H_{2i}(U) & \xrightarrow{\overline{u}} & 0.
\end{array}$$

Since U has the strong property, the first arrow is surjective and U has the weak property so the last arrow is an isomorphism. Since Y has the weak property, the second arrow is an isomorphism too.
First, we prove that X has the weak property, i.e. $CH_i(X)_{\mathbb{Q}} \xrightarrow{cl_X} W_{-2i}H_{2i}(X)$ is an isomorphism. The strategy of the proof is to perform a diagram chase, and essentially it follows from the Five Lemma.

i) The map cl_X is surjective: let $\eta \in W_{-2i}H_{2i}(X,\mathbb{Q})$. Since cl_U is surjective, there exists $a \in CH_i(U)_{\mathbb{Q}}$ such that $cl_U(a) = \bar{t}(\eta)$. By commutativity, $\overline{u}(cl_U(a)) = \overline{u}(\bar{t}(\eta)) = u(a) = 0$, so that $a \in \ker u = \operatorname{Im} t$. Then, there exists $b \in CH_i(X)_{\mathbb{Q}}$ such that $t(b) = a$. Again, by commutativity, we have that $\bar{t}(cl_X(b)) = cl_U(t(b)) = cl_U(a) = \bar{t}(\eta)$.
Since $\bar{t}$ is a homomorphism, we get $\bar{t}(cl_X(b) - \eta) = 0$, hence by exactness $cl_X(b) - \eta \in \ker \bar{t} = \operatorname{Im} \overline{s}$. Then, there exists $\omega \in W_{-2i}H_{2i}(Y,\mathbb{Q})$ such that $\overline{s}(\omega) = cl_X(b) - \eta$. Since cl_Y is surjective, there exists $c \in CH_i(Y)_{\mathbb{Q}}$ such that $cl_Y(c) = \omega$. By commutativity, we get $\overline{s}(cl_Y(c)) = cl_X(s(c))$, so $cl_X(s(c)) = \overline{s}(\omega) = cl_X(b) - \eta$. Being cl_X a homomorphism, we get $cl_X(b - s(c)) = \eta$.

ii) The map cl_X is injective: let $a \in CH_i(X)_{\mathbb{Q}}$ be such that $a \in \ker cl_X$. Then $\bar{t}(cl_X(a)) = 0$ and by commutativity we have that $cl_U(t(a)) = 0$. Since cl_U is injective, this implies that $t(a) = 0$, so $a \in \ker t = \operatorname{Im} s$ by exactness. Then, there exists $b \in CH_i(Y)_{\mathbb{Q}}$ such that $s(b) = a$. By commutativity, we have $\overline{s}(cl_Y(b)) = cl_X(s(b)) = cl_X(a) = 0$, so $cl_Y(b) \in \ker \overline{s} = \operatorname{Im} \overline{r}$. Then, there exists $\omega \in \operatorname{Gr}^W_{-2i} H_{2i+1}(U,\mathbb{Q})$ such that $\overline{r}(\omega) = cl_Y(b)$. Since, cl^1_U is surjective, there exists $c \in CH_i(U,1)_{\mathbb{Q}}$ such that $cl^1_U(c) = \omega$ and by commutativity we get $\overline{r}(cl^1_U(c)) = cl_Y(r(c))$. So $cl_Y(r(c)) = \overline{r}(\omega) = cl_Y(b)$ and being cl_Y a homomorphism, we get $cl_Y(r(c) - b) = 0$. Since cl_Y is injective, in particular we have that $r(c) = b$, so $b \in \operatorname{Im} r = \ker s$, by exactness. Then we conclude that $0 = s(b) = a$.

In order to prove that X has the strong property, it is enough to prove that $CH_i(X,1)_{\mathbb{Q}} \twoheadrightarrow \operatorname{Gr}^W_{-2i}H_{2i+1}(X,\mathbb{Q})$ is surjective. We can continue the above diagram to the left and we get:

$$\begin{array}{ccccccccc}
CH_i(Y,1)_{\mathbb{Q}} & \xrightarrow{p} & CH_i(X,1)_{\mathbb{Q}} & \xrightarrow{q} & CH_i(U,1)_{\mathbb{Q}} & \xrightarrow{r} & CH_i(Y)_{\mathbb{Q}} & \longrightarrow \\
cl^1_Y \downarrow & & cl^1_X \downarrow & & cl^1_U \downarrow & & cl_Y \downarrow\cong & \\
\operatorname{Gr}^W_{-2i} H_{2i+1}(Y) & \xrightarrow{\bar{p}} & W_{-2i}H_{2i}(X) & \xrightarrow{\bar{q}} & W_{-2i}H_{2i}(U) & \xrightarrow{\bar{r}} & W_{-2i}H_{2i}(Y) & \longrightarrow
\end{array}$$

Since Y has the strong property, we have that cl_Y is an isomorphism and cl^1_Y is surjective. Analogously, since U has the strong property, we have that cl^1_U is surjective. Then, doing a diagram chase, we can see that cl^1_X is also surjective (as before, it is just an application of the Five Lemma).
Let $\omega \in \operatorname{Gr}^W_{2i} H_{2i+1}(X)$. Since cl^1_U is surjective, there exists $\eta \in CH_i(U,1)_{\mathbb{Q}}$ such that $cl^1_U(\eta) = \bar{q}(\omega)$.
By commutativity, we get that $\bar{r}(\bar{q}(\omega)) = \bar{r}(cl^1_U(\eta)) = cl_Y(r(\eta))$. By exactness, we get that $\operatorname{Im} \bar{q} = \ker \bar{r}$, so $0 = \bar{r}(\bar{q}(\omega)) = \bar{r}(cl^1_U(\eta)) = cl_Y(r(\eta))$. Since cl_Y is injective, we get $r(\eta) = 0$. Again, by exactness, we have $\eta \in \ker r = \operatorname{Im} q$, so there exists $\zeta \in CH_i(X,1)_{\mathbb{Q}}$ such that $q(\zeta) = \eta$. By commutativity, it holds $\bar{q}(cl^1_X(\zeta)) = cl^1_U(q(\zeta)) = cl^1_U(\eta) = \bar{q}(\omega)$. Since $\bar{q}$ is a homomorphism, $cl^1_X(\zeta) - \omega \in \ker \bar{q} = \operatorname{Im} \bar{p}$.

Hence, there exists $\vartheta \in \mathrm{Gr}^W_{-2i} H_{2i+1}(Y)$ such that $\bar{p}(\vartheta) = cl^1_X(\zeta) - \omega$. By surjectivity of cl^1_Y, we get that there exists $\lambda \in CH_i(Y,1)_{\mathbb{Q}}$ such that $cl^1_Y(\lambda) = \vartheta$. By commutativity, we get that $cl^1_X(\zeta) - \omega = \bar{p}(\vartheta) = \bar{p}(cl^1_Y(\lambda)) = cl^1_X(p(\lambda))$. Since cl^1_X is a homomorphism, we conclude $\omega = cl^1_X(p(\lambda) - \zeta)$, hence cl^1_X is surjective.

Proof of Part (2):
First of all, we notice that affine spaces have the strong property (see [Tot14, Lemma 5]), so both $\mathbb{A}^k$ and L have the strong property. Then we want to prove that $\mathbb{A}^k \backslash L$ has the strong property (see [Tot14, Lemma 6]).
We start by showing that $\mathbb{A}^k \backslash L$ has the weak property. We use the localization sequence in Theorem 4.1.6 to get the following diagram:

$$\begin{array}{ccccccc} CH_i(L)_{\mathbb{Q}} & \xrightarrow{p} & CH_i(\mathbb{A}^k)_{\mathbb{Q}} & \xrightarrow{q} & CH_i(\mathbb{A}^k \backslash L)_{\mathbb{Q}} & \xrightarrow{r} & 0 \\ cl_L \downarrow\cong & & cl_k \downarrow\cong & & cl \downarrow & & i \downarrow \\ W_{-2i}H_{2i}(L) & \xrightarrow{\bar{p}} & W_{-2i}H_{2i}(\mathbb{A}^k) & \xrightarrow{\bar{q}} & W_{-2i}H_{2i}(\mathbb{A}^k \backslash L) & \xrightarrow{\bar{r}} & 0 \end{array}$$

Since $\mathbb{A}^k$ and L have the strong property, then cl_k and cl_L are isomorphisms. To prove that cl is an isomorphism too, we do a diagram chase as before.

i) The map cl is surjective: let $\omega \in W_{-2i}H_{2i}(\mathbb{A}^k \backslash L, \mathbb{Q})$, then $\bar{r}(\omega) = 0$. By exactness, there exists $\eta \in W_{-2i}H_{2i}(\mathbb{A}^k, \mathbb{Q})$ such that $\bar{q}(\eta) = \omega$. Since cl_k is surjective, there exists $\zeta \in CH_i(\mathbb{A}^k)_{\mathbb{Q}}$ such that $cl_k(\zeta) = \eta$, so $cl(q(\zeta)) = \bar{q}(cl_k(\zeta)) = \bar{q}(\eta) = \omega$.

ii) The map cl is injective: let $\alpha \in CH_i(\mathbb{A}^k \backslash L)_{\mathbb{Q}}$ be such that $cl(\alpha) = 0$. We notice that $\alpha \in \ker r = \operatorname{Im} q$, so there exists $\beta \in CH_i(\mathbb{A}^k)_{\mathbb{Q}}$ such that $q(b) = \alpha$.
By commutativity, we get $\bar{q}(cl_k(\beta)) = cl(q(\beta)) = cl(\alpha) = 0$. So $cl_k(\beta) \in \ker \bar{q} = \operatorname{Im} \bar{p}$, i.e. there exists $\delta \in W_{-2i}H_{2i}(L, \mathbb{Q})$ such that $\bar{p}(\delta) = cl_k(\beta)$. Since cl_L is surjective, there exists $\gamma \in CH_i(L)_{\mathbb{Q}}$ such that $cl_L(\gamma) = \delta$.
By commutativity, we have $cl_k(\beta) = \bar{p}(\delta) = \bar{p}(cl_L(\gamma)) = cl_k(p(\gamma))$. Since cl_k is an injective homomorphism, we have that $p(\gamma) = \beta$ and so $\alpha = q(\beta) = q(p(\gamma)) = 0$ by exactness.

Part (1) of the Lemma, assures us that the union of two manifolds satisfying the strong property has still the strong property, so we conclude that X has the strong property.

Proof of Part (3):
The result follows from the projective bundle formula for higher Chow groups (see [Blo86, Theorem 7.1]). Indeed, we get a commutative diagram

$$\begin{tikzcd} CH_i(P)_{\mathbb{Q}} \arrow[r, "\cong"] \arrow[d, "\mathrm{cl}_i"] & \bigoplus_{j=0}^{\operatorname{rank} P} CH_{i-j}(X)_{\mathbb{Q}} \arrow[d, "\oplus\, \mathrm{cl}_{i-j}"] \\ W_{-2i}H_{2i}(P)_{\mathbb{Q}} \arrow[r, "\cong"] & \bigoplus_{j=0}^{\operatorname{rank} P} W_{-2i}H_{2i-2j}(X, \mathbb{Q}), \end{tikzcd}$$

which proves that P has the weak property.
To prove that P has the strong property too, we use again the projective bundle formula and we get another commutative diagram

$$\begin{tikzcd} CH_i(P,1)_{\mathbb{Q}} \arrow[r, "\cong"] \arrow[d, "\mathrm{cl}^1_i"] & \bigoplus_{j=0}^{\operatorname{rank} P} CH_{i-j}(X,1)_{\mathbb{Q}} \arrow[d, two heads, "\oplus\, \mathrm{cl}^1_{i-j}"] \\ \mathrm{Gr}^W_{-2i} H_{2i+1}(P)_{\mathbb{Q}} \arrow[r, "\cong"] & \bigoplus_{j=0}^{\operatorname{rank} P} \mathrm{Gr}^W_{-2i} H_{2i+1-2j}(X, \mathbb{Q}). \end{tikzcd}$$

□

4.2 Voisin's spreading of cycles

The spreading of cycles appeared for the first time in Nori's work [Nor93], and since then it has been used profusely to deal with algebraic cycles. We give a brief presentation of this subject, for further details we refer the reader to the work of Voisin (see [Voi13], [Voi14b, 4.3.3] and [Voi15]) and Green–Griffiths (see [GG03]).

Notation and conventions. *We consider a family $\mathcal{X}$ of varieties, which is parametrized by a smooth projective morphism $\pi\colon \mathcal{X} \to B$, with B a smooth, irreducible, and quasi–projective variety. We denote the fiber over $b \in B$ as $\mathcal{X}_b := \pi^{-1}\{b\}$. If we consider an algebraic cycle of codimension r, $\mathcal{Z} \in Z^r(\mathcal{X})$, we denote its restriction to the fiber over $b \in B$ as $\mathcal{Z}_b := \mathcal{Z}|_{\mathcal{X}_b}$. If we consider a subset $U \subset B$, we use the following notation $\mathcal{X}_U := \pi^{-1}(U)$.*

When dealing with a family of varieties $\mathcal{X} \to B$, we can consider an algebraic cycle supported on the general fiber, $Z_b \in Z^*(\mathcal{X}_b)$. The main idea of the spreading is that we can construct a relative cycle $\mathcal{Z}$ on the whole family, such that, when we restrict this cycle to the general fiber, we get that it coincides with Z_b, modulo a restriction of the basis. Moreover, the cohomology class of the relative spread cycle predicts the behavior of the cycle restricted to the general fiber.
The spreading of cycles is particularly interesting for rational equivalence, as we can see from the following result [Voi14b, Theorem 1.2, Corollary 1.3, Theorem 3.1, Corollary 3.4].

Theorem 4.2.1 (General spreading principle for rational equivalence)**.** *Let $\mathcal{X} \to B$ be a smooth projective family, with B smooth, irreducible and quasi–projective. Let $\mathcal{Z} \in Z^r(\mathcal{X})$ be a codimension r algebraic cycle of $\mathcal{X}$, and let $b \in B$ be a very general point. If the restriction to the fiber $\mathcal{Z}_b$ is rationally trivial, then there exists a Zariski open set $U \subset B$ and $N \in \mathbb{N}\backslash\{0\}$ such that $N\mathcal{Z}|_{\mathcal{X}_U}$ is rationally trivial too.*
In particular, the cohomology class $[\mathcal{Z}] \in H^{2r}(\mathcal{X}, \mathbb{Q})$ vanishes on $\mathcal{X}_U$.

Remark 4.2.2. This spreading result does not hold for other equivalence relations on algebraic cycles, it works only on the finest adequate one, rational equivalence (see [Voi14b, 3.1] for a counter example in the algebraic equivalence case).

Theorem 4.2.1 relies on the following lemma.

Lemma 4.2.3 (Lemma 3.2, [Voi14b])**.** *Let $f: X \to Y$ be a projective fibration, where X and Y are smooth, and let $Z \in CH^k(X)$. Then the set $\{y \in Y$ such that $Z_y := Z|_{f^{-1}(y)} \sim_{rat} 0\}$ is a countable union of closed algebraic subsets of Y, where $Z|_{X_y} := j_y{*}Z$ and $j_y: f^{-1}(y) \hookrightarrow X$ is the inclusion of the fiber.*

Here we focus on another spreading result that works on the fibered self–product of a family of $\mathcal{X}$ over B, i.e. on $(\pi, \pi): \mathcal{X} \times_B \mathcal{X} \to B$. This result is explained extensively by Voisin in [Voi13] and [Voi14b, Section 4.3, Proposition 4.25].
We consider a codimension–r algebraic cycle in the fibered self–product

$$\mathcal{Z} \subset \mathcal{X} \times_B \mathcal{X},$$

and we denote its restriction to the fiber over $b \in B$ as $\mathcal{Z}_b := \mathcal{Z}|_{\mathcal{X}_b \times \mathcal{X}_b}$.

Proposition 4.2.4 (Spreading over the fibered self–product)**.** *Let $\bar{b} \in B$ be a very general point. We assume there exists a closed algebraic subset $Y_{\bar{b}} \subset \mathcal{X}_{\bar{b}}$ of codimension c, and there exists a codimension r algebraic cycle $W \in Z^r(Y_{\bar{b}} \times Y_{\bar{b}})_{\mathbb{Q}}$ whose cohomology class coincides with the relative cycle's one restricted to the very general fiber, i.e.*

$$[W] = [\mathcal{Z}_{\bar{b}}] \in H^{2r}(\mathcal{X}_{\bar{b}}, \mathbb{Q}).$$

Then, the cycle W exists relatively. More precisely, there exists a closed algebraic subset $\mathcal{Y} \in \mathcal{X}$ of codimension c and there exists a relative cycle $\mathcal{W} \in Z^r(\mathcal{X} \times_B \mathcal{X})_{\mathbb{Q}}$ such that $\mathcal{W}_{\bar{b}} = W$, $\mathcal{W} \subset \mathcal{Y} \times_B \mathcal{Y}$ and for any $b \in B$ it holds that

$$[\mathcal{W}_b] = [\mathcal{Z}_b] \in H^{2r}(\mathcal{X}_b, \mathbb{Q}).$$

Proof. The proof relies on the countability of the Hilbert schemes for smooth projective varieties (see [Voi07b, Section 3.3]). Indeed, there is a countable set of algebraic varieties M_i parametrizing the tuples $(\bar{b}, Y_{\bar{b}}, W_{(\bar{b})})$ such that $\bar{b} \in B$ is very general, $Y_{\bar{b}} \in Z^c(\mathcal{X}_{\bar{b}})$ and $W_{(\bar{b})} \in Z^r(Y_{\bar{b}} \times Y_{\bar{b}})_{\mathbb{Q}}$ satisfies the hypotheses. There exist countable many varieties

$$\mathcal{Y}_i \to M_i \stackrel{a_i}{\to} B$$

such that $(\mathcal{Y}_i)_{a_i^{-1}(\bar{b})} \cong Y_{\bar{b}}$ and $\mathcal{Y}_i \subset \mathcal{X}_{M_i} = \mathcal{X} \times_B M_i$. We can consider also the universal objects $\mathcal{W}_i \subset \mathcal{Y}_i \times_{M_i} \mathcal{Y}_i$, such that

$$[\mathcal{W}_{i,\bar{b}}] = [\mathcal{Z}_{\bar{b}}] \in H^{2r}(\mathcal{X}_{\bar{b}} \times_{M_i} \mathcal{X}_{\bar{b}}, \mathbb{Q}).$$

We consider a parametrization M_i such that there is an element $m \in M_i$ whose image is $\bar{b}$, i.e. $a_i(m) = \bar{b}$. We have that

$$\bar{b} \in \bigcup_{i \in \mathbb{N}} a_i(M_i) := X.$$

Since X is written as a countable union of closed sets, we can apply a corollary of Baire's category theorem, and we conclude that at least one of the closed sets has nonempty interior. So there exists $j \in \mathbb{N}$ such that $a_j(M_j)$ has a nonempty interior, in particular it contains a Zariski open set, hence $a_j(M_j) = B$, and a_j is a dominating morphism. Modulo a restriction to a subvariety of M_j, we can assume that $a_j \colon M_j \to B$ is also generically finite and proper. Then we can consider the proper generically finite morphism $r_j \colon \mathcal{X}_{M_j} \to \mathcal{X}$ induced by a_j, and we have the following situation

$$\begin{array}{ccc} \mathcal{X}_{M_j} & \xrightarrow{r_j} & \mathcal{X} \\ \downarrow & & \downarrow \\ M_j & \xrightarrow{a_j} & B. \end{array}$$

We consider $\mathcal{Y}_j \subset \mathcal{X}_{M_j}$ and the restriction of r_j to $\mathcal{Y}_j$ which we denote as $r'_j \colon \mathcal{Y}_j \to \mathcal{Y} := r_j(\mathcal{Y}_j)$. Since r_j is generically finite, it follows that $\operatorname{codim} \mathcal{Y} \geq c$. Next, we define a codimension r algebraic cycle on $\mathcal{X} \times_B \mathcal{X}$ as $\mathcal{W} := (r'_j, r'_j)_*(\mathcal{W}_j)$, where $\mathcal{W}_j \subset \mathcal{Y}_j \times_{M_j} \mathcal{Y}_j$ and

$$\mathcal{Y}_j \times_{M_j} \mathcal{Y}_j \stackrel{(r'_j, r'_j)}{\longrightarrow} \mathcal{Y} \times_B \mathcal{Y}.$$

Then $\mathcal{W}$ is supported on the image $(r'_j, r'_j)(\mathcal{Y}_j \times_{M_j} \mathcal{Y}_j) \subset \mathcal{Y} \times_B \mathcal{Y}$. Let us denote $N = \deg r_j$, then for any $b \in B$ it holds

$$[\mathcal{W}_b] = N[\mathcal{Z}_b] \in H^{2r}(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q}).$$

We can conclude by replacing $\mathcal{W}$ with $\frac{1}{N}\mathcal{W}$, which is possible since we consider cycles with $\mathbb{Q}$–coefficients. □

4.2.1 Leray–spectral sequence argument

The spreading result in Proposition 4.2.4, is particularly useful to spread fiberwise homologically trivial cycles, i.e. when we deal with cycles whose cohomology class on the fiber is zero. In this case we would like to spread the cycle and consider the relative one.

First we present a result due to Voisin (see [Voi13, Lemma 2.11]), which gives a decomposition of a homologically trivial cycle which vanishes on the fibers.

Notation and conventions. *Let X be a smooth projective variety. Let $\{L_i\}_{i\leq r}$ be a set of very ample line bundles on X. We consider a family of smooth complete intersections of hypersurfaces in $|L_i|$, parametrized by the open set $B \subset \prod_i^r \left(\mathbb{P}\left(H^0(X, L_i)\right)\right)$, and we consider the universal family $\pi\colon \mathcal{X} \to B$ with $\mathcal{X} \subset B \times X$. We denote the fiber over a point $b \in B$ as $\mathcal{X}_b := \pi^{-1}(b)$.*

Lemma 4.2.5. *Let $\alpha \in H^{2n-2r}(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q})$ be a cohomology class homologically trivial when restricted to a fiber, i.e. $\alpha|_{\mathcal{X}_b \times \mathcal{X}_b} = 0$ for any $b \in B$. Then, there exist two classes $\beta_1 \in H^{2n-2r}(X \times \mathcal{X}, \mathbb{Q})$ and $\beta_2 \in H^{2n-2r}(\mathcal{X} \times X, \mathbb{Q})$ such that*

$$\alpha = \beta_1|_{\mathcal{X}\times_B \mathcal{X}} + \beta_2|_{\mathcal{X}\times_B\mathcal{X}}.$$

Proof. Let us consider the smooth proper morphism on the fibered self–product $(\pi, \pi)\colon \mathcal{X} \times_B \mathcal{X} \to B$. We consider the sheaf $R^k(\pi, \pi)_*\mathbb{Q}$ on B, which is the higher direct image sheaf of (π, π), i.e. the sheafification of the presheaf $U \mapsto H^i((\pi, \pi)^{-1}(U), \mathbb{Q})$ (see SectionA.3.1). So on the stalks we have $R^k(\pi, \pi)_*\mathbb{Q}(b) = H^k(X_b \times X_b, \mathbb{Q})$ and $R^k\pi_*\mathbb{Q}(b) = H^i(X_b, \mathbb{Q})$.

By the relative Künneth decomposition we have

$$R^k(\pi, \pi)_*\mathbb{Q} = \bigoplus_{i+j=k} R^i\pi_*\mathbb{Q} \otimes R^j\pi_*\mathbb{Q}. \tag{4.2.6}$$

Equation (4.2.6) holds since a presheaf and its sheafification have canonically isomorphic stalks and the stalks of the above sheaves are isomorphic (see Lemma A.3.5) by the Künneth decomposition ([Voi07a, Theorem 11.38]). Now we consider the Leray spectral sequence of (π, π) which gives the Leray filtration L on $H^{2n-2r}(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q})$ (see Theorem A.3.6). Let us consider the graded pieces of the Leray filtration and apply equation (4.2.6), we get

$$\begin{aligned} Gr^l_L H^{2n-2r}(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q}) &= H^l(B, R^{2n-2r-l}(\pi, \pi)_*\mathbb{Q}) \\ &= \bigoplus_{i+j=2n-2r-l} H^l(B, R^i\pi_*\mathbb{Q} \otimes R^j\pi_*\mathbb{Q}). \end{aligned}$$

The first equality holds since the Leray filtration degenerates at E_2 (see Theorem A.3.7), i.e. by Theorem A.3.6 it holds

$$\begin{aligned} H^l(B, R^{2n-2r-l}(\pi,\pi)_*\mathbb{Q}) = E_2^{l,2n-2r-l} \\ \cong E_\infty^{l,2n-2r-l} = Gr_L^l H^{2n-2r}(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q}). \end{aligned}$$

By hypothesis, α vanishes on the quotient

$$Gr_L^0 H^{2n-2r}(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q}) = H^0(B, R^{2n-2r}(\pi,\pi)_*\mathbb{Q}),$$

so $\alpha \in L^1 H^{2n-2r}(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q})$. Let us focus then on the case $l > 0$ and the remaining graded pieces $H^l(B, R^i\pi_*\mathbb{Q} \otimes R^j\pi_*\mathbb{Q})$ for $i + j = 2n - 2r - l$. We have two possible cases.

- If $i < n - r$, by Lefschetz hyperplane section theorem A.2.19, we have that $R^i\pi_*\mathbb{Q}$ is the constant sheaf with stalk $H(X, \mathbb{Q})$. Then we get

$$\begin{aligned} H^l(B, R^i\pi_*\mathbb{Q} \otimes R^j\pi_*\mathbb{Q}) &= H^i(X, \mathbb{Q}) \otimes H^l(B, R^j\pi_*\mathbb{Q}) \\ &= Gr_L^l \left(H^i(X, \mathbb{Q}) \otimes H^{l+j}(\mathcal{X}, \mathbb{Q}) \right). \end{aligned}$$

- If $j < n - r$, similarly, we get that

$$\begin{aligned} H^l(B, R^i\pi_*\mathbb{Q} \otimes R^j\pi_*\mathbb{Q}) &= H^l(B, R^i\pi_*\mathbb{Q}) \otimes H^j(X, \mathbb{Q}) \\ &= Gr_L^l \left(H^{l+i}(\mathcal{X}, \mathbb{Q}) \otimes H^j(X, \mathbb{Q}) \right). \end{aligned}$$

So we conclude that the map

$$\begin{aligned} \bigoplus_{i<n-r} H^i(X,\mathbb{Q}) \otimes L^1 H^{2n-2r-i}(\mathcal{X},\mathbb{Q}) \oplus \bigoplus_{j<n-r} L^1 H^{2n-2r-j}(\mathcal{X},\mathbb{Q}) \otimes H^j(X,\mathbb{Q}) \\ \to L^1 H^{2n-2r}(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q}) \end{aligned}$$

is surjective. Since $\alpha \in L^1 H^{2n-2r}(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q})$, we can find

$$\begin{aligned} \beta_1 &\in \bigoplus_{i<n-r} H^i(X, \mathbb{Q}) \otimes L^1 H^{2n-2r-i}(\mathcal{X}, \mathbb{Q}), \\ \beta_2 &\in \bigoplus_{j<n-r} L^1 H^{2n-2r-j}(\mathcal{X}, \mathbb{Q}) \otimes H^j(X, \mathbb{Q}) \end{aligned}$$

which decompose α as $\alpha = \beta_1|_{\mathcal{X}\times_B\mathcal{X}} + \beta_2|_{\mathcal{X}\times_B\mathcal{X}}$. □

We can gain some extra informations if we ask that the ambient variety has trivial Chow groups.

Definition 4.2.7. *[Voi14b, 4.3] Let us consider a smooth complex algebraic variety X. We say that X has* trivial Chow groups *if the cycle class map is injective, i.e.*

$$\mathrm{cl}\colon CH^*(X)_{\mathbb{Q}} \hookrightarrow H^{2*}(X, \mathbb{Q}).$$

Example of varieties with trivial Chow groups are all the varieties admitting a stratification by affine spaces, as Grassmanians, and smooth toric varieties, as projective spaces.

Remark 4.2.8. If X is projective and has trivial Chow groups, then (see [Lat98], [Lew95]) we have that

i) the cycle class map are isomorphisms for any i:

$$\mathrm{cl}\colon CH^i(X)_{\mathbb{Q}} \xrightarrow{\sim} H^{2i}(X, \mathbb{Q}).$$

ii) $H^{2i+1}(X, \mathbb{Q}) = 0$ for any i.

We briefly recall the main basic results on trivial Chow groups. We refer the reader to [Voi13] and [Voi14b, Section 4.3.1] for the complete proofs.

Proposition 4.2.9. *Let X be a smooth complex variety with trivial Chow groups.*

i) Let E be a locally free sheaf on X and let $p\colon \mathbb{P}(E) \to X$ be a projective bundle on X. Then $\mathbb{P}(E)$ has trivial Chow groups too.

ii) Let Y be a smooth projective variety with trivial Chow groups, then $X \times Y$ has trivial Chow groups.

We present a result due to Voisin (see [Voi13, Lemma 2.12]). In the hypotheses of Lemma 4.2.5 we add that the ambient variety has trivial Chow groups and the cohomology class is algebraic. We use the same notation 4.2.1 as in Lemma 4.2.5.

Lemma 4.2.10. *Assume that X has trivial Chow groups and consider a cohomology class $\alpha \in H^{2n-2r}(\mathcal{X} \times_B \mathcal{X}, \mathbb{Q})$ homologically trivial when restricted to a fiber such that it is algebraic. Then we can choose the β_i's to be the restriction of classes of algebraic cycles on $B \times X \times X$.*

Proof. First of all we claim that it is enough to prove that we can choose the β_i's to be the restriction of classes of algebraic cycles on $X \times \mathcal{X}$, where we recall that $\mathcal{X} \subset B \times X$. Indeed, we define

$$\mathbb{P} := \{(\sigma_1, \ldots, \sigma_r, x)\colon \sigma_i(x) = 0 \quad \forall i \text{ s.t. } 1 \le i \le r\} \subset \prod_{i=1}^{r} \mathbb{P}(H^0(X, L_i)) \times X.$$

Then $\mathcal{X}$ is a Zariski open set in the natural fibration $f\colon \mathbb{P} \to X$, indeed $\mathcal{X} \subset B \times X \subset \mathbb{P}$. Since we took the L_i's to be very ample, we have that this is a fibration into products of projective spaces. Hence, also $X \times \mathcal{X}$ is a Zariski open set in the corresponding fibration into products of projective spaces $X \times \mathbb{P} \to X \times X$. By [Voi07b, Section 9.3.2], we have the surjectivity of the restriction map

$$CH\left(X \times X \times \prod_i^r \mathbb{P}(H^0(X, L_i))\right) \to CH(X \times \mathbb{P}).$$

Then, we get the surjectivity of the composition with the restriction map to $X \times \mathcal{X}$

$$CH(X \times \mathbb{P}) \to CH(X \times \mathcal{X}),$$

and the same holds for the map

$$CH(X \times X \times X) \to CH(X \times \mathcal{X}).$$

Let us prove now that we can choose the β_i's to be the restriction of classes of algebraic cycles on $X \times \mathcal{X}$. By Lemma 4.2.5 we get that

$$\alpha = \beta_1|_{\mathcal{X}\times_B\mathcal{X}} + \beta_2|_{\mathcal{X}\times_B\mathcal{X}}, \tag{4.2.11}$$

with

$$\beta_1 \in \bigoplus_{i<n-r} H^i(X,\mathbb{Q}) \otimes L^1H^{2n-2r-i}(\mathcal{X},\mathbb{Q}) = H^{*<n-r}(X,\mathbb{Q}) \otimes L^1H^*(\mathcal{X},\mathbb{Q}),$$
$$\beta_2 \in \bigoplus_{j<n-r} L^1H^{2n-2r-j}(\mathcal{X},\mathbb{Q}) \otimes H^j(X,\mathbb{Q}) = L^1H^*(\mathcal{X},\mathbb{Q}) \otimes H^{*<n-r}(X,\mathbb{Q}).$$

Since X has trivial Chow groups, by Remark 4.2.8, its cohomology is generated by algebraic classes, i.e. classes $[z_{i,j}] \in H^{2i}(X,\mathbb{Q})$, where $z_{i,j}$ are algebraic cycles in $CH^i(X,\mathbb{Q})$. So we can choose a basis of $H^{*<n-r}(X,\mathbb{Q})$

$$\{[z_{i,j}]\}_{i,j}$$

with $2i < n-r$. By Künneth decomposition ([Voi07a, Theorem 11.38]), there exists $\gamma_{i,j}, \gamma'_{i,j} \in L^1H^*(\mathcal{X},\mathbb{Q})$ such that we can write

$$\begin{aligned} \beta_1 &= \sum_{i,j} [z_{i,j}] \cup \gamma_{i,j}; \\ \beta_2 &= \sum_{i,j} \gamma'_{i,j} \cup [z_{i,j}]. \end{aligned} \tag{4.2.12}$$

We denote the projections on the factors as follows:

$$
\begin{tikzcd}
 & & \mathcal{X} \times_B \mathcal{X} \arrow[dll, "p_B"'] \arrow[ddl, "\pi'_1"'] \arrow[d, "p_{1,X}"] \arrow[ddrr, "\pi'_2"] \arrow[drrr, "p_{2,X}"] & \subset B \times X \times X & & \\
B & & X & & & X \\
 & \mathcal{X} & & & \mathcal{X} &
\end{tikzcd}
$$

The second projection π'_2 is a smooth projective morphism whose fiber over any point of $\mathcal{X}_b \subset \mathcal{X}$ is $\mathcal{X}_b$. Let us choose now classes of algebraic cycles $[z^*_{i,j}] \in H^{2n-2r-2i}(X, \mathbb{Q})$ such that their restriction to the fibers $[z_{i,j}]^*|_{\mathcal{X}_b}$ form the dual basis of $H^{*>n-r}(\mathcal{X}_b, \mathbb{Q})$. Then, by (4.2.12) we have

$$
\begin{aligned}
\pi'_{2*}\left(p^*_{1,X}[z_{i,j}]^* \cup \beta_1\right) &= \pi'_{2*}\left(p^*_{1,X}[z_{i,j}]^* \cup \left(\sum_{i,j}[z_{i,j}] \cup \gamma_{i,j}\right)\right) = \gamma_{i,j}; \\
\pi'_{2*}\left(p^*_{1,X}[z_{i,j}]^* \cup \beta_2\right) &= \pi'_{2*}\left(p^*_{1,X}[z_{i,j}]^* \cup \left(\sum_{i,j}\gamma'_{i,j} \cup [z_{i,j}]\right)\right) = 0;
\end{aligned}
\tag{4.2.13}
$$

where we recall that $\gamma_{i,j}, \gamma'_{i,j} \in L^1H^*(\mathcal{X}, \mathbb{Q})$ and $\beta_1 \in H^{2n-2r}(X \times \mathcal{X}, \mathbb{Q})$, $\beta_2 \in H^{2n-2r}(\mathcal{X} \times X, \mathbb{Q})$. Since α is algebraic by hypothesis, then also the class $\pi'_{2*}\left(p^*_{1,X}[z_{i,j}]^* \cup \alpha\right) = \gamma_{i,j}$ is algebraic, where the equality follows from (4.2.11) and (4.2.13). Similarly we get that

$$
\begin{aligned}
\pi'_{1*}\left(p^*_{2,X}[z_{i,j}]^* \cup \beta_1\right) &= \pi'_{1*}\left(p^*_{2,X}[z_{i,j}]^* \cup \left(\sum_{i,j}[z_{i,j}] \cup \gamma_{i,j}\right)\right) = 0; \\
\pi'_{1*}\left(p^*_{2,X}[z_{i,j}]^* \cup \beta_2\right) &= \pi'_{1*}\left(p^*_{2,X}[z_{i,j}]^* \cup \left(\sum_{i,j}\gamma'_{i,j} \cup [z_{i,j}]\right)\right) = \gamma'_{i,j}.
\end{aligned}
\tag{4.2.14}
$$

Hence $\pi'_{1*}\left(p^*_{2,X}[z_{i,j}]^* \cup \alpha\right) = \gamma'_{i,j}$ is algebraic too. Then we conclude by (4.2.12) that also β_1, β_2 are algebraic. $\square$

Chapter 5

A result on Todorov surfaces of type (2,12)

Our aim is to present a new example in which Voisin's conjecture 3.4.1 on 0–cycles is true, namely a family of Todorov surfaces.

Definition 5.0.1. *A* Todorov surface *is a smooth projective surface S of general type, with $p_g(S) = 1$, $q(S) = 0$ and such that the bicanonical map $\phi_{|2K_S|}$ factors as*

$$\phi_{|2K_S|} \colon S \xrightarrow{\sigma} S \dashrightarrow \mathbb{P}^r,$$

where $\sigma \colon S \to S$ is an involution such that S/σ is birational to a $K3$ surface with rational double points. We call S/σ the singular $K3$ surface associated to S.
We call the minimal resolution of S/σ the K3 surface associated to S.

Todorov surfaces were introduced by Todorov to provide counterexamples to Local and Global Torelli ([Tod81]). They were classified by Morrison ([Mor88]) up to *fundamental invariants* (α, k), where the 2–torsion group of $\mathrm{Pic}(S)$ has order 2^α and $k = 8 + K_S^2$. With this classification Morrison proves that there are exactly 11 non–empty irreducible families of Todorov surfaces corresponding to

$$(\alpha, k) \in \{(0,9), (0,10), (0,11), (1,10), (1,11), (1,12),$$
$$(2,12), (2,13), (3,14), (4,15), (5,16)\}.$$

Todorov surfaces of type (0,9) are also known as *Kunev surfaces.*

Conjecture 3.4.1 has been proven by Laterveer for the family of Todorov surfaces of type (0,9)([Lat16c]). Laterveer also proved the conjecture for the family of Todorov surfaces of type (1,10) ([Lat18a]). For both of these families the core of the proof was that an explicit description of the family as complete intersections was available. The following result is an essential tool for both cases, indeed it allows the reduction to the case of a double cover of $\mathbb{P}^2$ ramified along the union of two cubics, for which Conjecture 3.4.1 has been proven by Voisin (see Theorem 3.4.10, [Voi96, Theorem 3.4]).

Theorem 5.0.2 (Rito [Rit09])**.** *Let S be a Todorov surface and let M be the $K3$ surface associated to S, i.e. the the smooth minimal model of S/σ. Then there exists a generically finite degree–2 cover $M \to \mathbb{P}^2$ ramified along the union of two cubics.*

So any Todorov surface has an associated K3 surface for which Conjecture 3.4.1 holds. To conclude that the conjecture holds for all Todorov surfaces one needs to be able to establish a relation between 0–cycles on S and 0–cycles on M. The technique used to prove the conjecture in the known cases is based on Voisin's principle of "spreading of cycles" [Voi13], [Voi14b, Ch. 4] (see also Section 4.2). This approach works as long as the irreducible family of Todorov surfaces considered $\mathcal{S} \to B$ has a nice enough description to prove the following property

$$CH^2_{\text{hom}}(\mathcal{S} \times_B \mathcal{S})_{\mathbb{Q}} = 0.$$

The lack of such an explicit description for the other families of Todorov surfaces prevents to apply the same method to them.

We focus on the family of Todorov surfaces with fundamental invariants $(\alpha, k) = (2, 12)$. We present an explicit description for this family as quotients of the complete intersection of four quadrics in $\mathbb{P}^6$. Our main result is the following theorem.

Theorem 5.3.5. *Let S be a general Todorov surface with fundamental invariants $(\alpha, k) = (2, 12)$.*
Then Conjecture 3.4.1 is true for S.

In Section 5.1 we give an explicit description of the family studying the universal cover of the surfaces. To do so, we use a similar approach to [BFNP14] and [NP14]. In Section 5.2 we focus on 0–cycles by exploiting the idea of realizing the fibered self–product of the family of surfaces as a Zariski open set of a variety with trivial Chow groups. In Section 5.3 we prove Theorem 5.3.5 applying the "spreading" of algebraic cycles on a family following the approach of Laterveer ([Lat18a]) and Voisin ([Voi13], [Voi15]). In Section 5.4 we give a motivic version of the main result and some applications, following the approach in [Lat18a].

Notation and conventions. *We work on the field of complex numbers $\mathbb{C}$. A* variety *is a quasi–projective separated scheme of finite type over $\mathbb{C}$ with the Zariski topology. A* subvariety *is a reduced equidimensional subscheme. A* curve *is a variety of dimension one, a* surface *is a variety of dimension*

two.
We denote the Euler-Poincaré characteristic *of a projective surface as*

$$\chi(S) := \sum_{i=0}^{2}(-1)^i h^i(S, \mathcal{O}_S) = 1 - q(S) + p_g(S).$$

For $d \geq 0$, we denote the dth plurigenerus *of S as*

$$P_d(S) := h^0(dK_S).$$

We denote by $CH^j(X)_{AJ}$ the kernel of the Abel–Jacobi map:

$$AJ\colon CH^j(X) \longrightarrow J^{2k-1}(X),$$

where $J^{2k-1}(X)$ is the k–th Intermediate Jacobian of X.
We denote a projective point in $\mathbb{P}^6$ with homogeneous coordinates as

$$x := (x_0\colon \ldots : x_6).$$

5.1 Explicit description of the family

We want to give an explicit description of the family of Todorov surfaces of type $(2, 12)$.

Let us consider S to be a Todorov surface of type $(2, 12)$, then we have that $K_S^2 = 12-8 = 4$ and 2–Tor(Pic(S)) $\cong (\mathbb{Z}/_{2\mathbb{Z}})^2$. This implies that there is a Galois cover $V \xrightarrow{q} S$ with Galois group $(\mathbb{Z}/_{2\mathbb{Z}})^2$, which is also étale, so it has no branch locus. Then we have the following numerical situation:

$$\begin{cases} \chi(V) = 4\chi(S) = 8, \\ q(V) = q(S) = 0, \\ p_g(V) = \chi(V) - 1 + q(V) = 7, \\ K_V^2 = 4K_S^2 = 16. \end{cases}$$

Our aim is to describe the family of Todorov surface of type $(2, 12)$ as complete intersection of four general quadrics in $\mathbb{P}^6$ modulo the action of a group of automorphisms of $\mathbb{P}^6$.

Definition 5.1.1. *We define the following involutions on $\mathbb{P}^6$:*

$$\begin{aligned} \sigma_1\colon (x_0\colon \ldots : x_6) &\mapsto (x_0\colon -x_1\colon -x_2\colon -x_3\colon -x_4\colon x_5\colon x_6), \\ \sigma_2\colon (x_0\colon \ldots : x_6) &\mapsto (x_0\colon x_1\colon x_2\colon -x_3\colon -x_4\colon -x_5\colon -x_6), \\ \sigma_1 \circ \sigma_2\colon (x_0\colon \ldots : x_6) &\mapsto (x_0\colon -x_1\colon -x_2\colon x_3\colon x_4\colon -x_5\colon -x_6), \\ \sigma\colon (x_0\colon \ldots : x_6) &\mapsto (x_0\colon -x_1\colon -x_2\colon -x_3\colon -x_4\colon -x_5\colon -x_6). \end{aligned}$$

We define the group $G =< \sigma_1, \sigma_2 >$ of automorphisms of $\mathbb{P}^6$.

Remark 5.1.2. Formulas in Definition 5.1.1 describe an action of $G \cong (\mathbb{Z}/_{2\mathbb{Z}})^2$ on $H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1)))$, and therefore on $H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(d)))$ for any $d \in \mathbb{N}$, which is compatible with the action of G on $\mathbb{P}^6$. For this action we have:

$$\begin{aligned} W &:= H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))^G \\ &= \langle x_0^2, x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2, x_1x_2, x_3x_4, x_5x_6\rangle_{\mathbb{C}} \cong \mathbb{C}^{10}. \end{aligned} \tag{5.1.3}$$

Definition 5.1.4. *We define*

$$\widetilde{U} \subset \mathrm{Gr}(4, W)/_{\mathrm{GL}(7,\mathbb{C})^G}$$

to be the open set that parametrizes the complete intersection of four quadrics $V = \bigcap_{i=0}^3 Q_i$ *with* $Q_0, \ldots, Q_3 \in H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))^G$.

Remark 5.1.5. We are considering all the four–dimensional subspaces in $W \cong \mathbb{C}^{10}$, so we are taking four quadrics in $W = H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))^G$ that are linearly independent. Then we are quotienting by

$$GL(7, \mathbb{C})^G = \{f \in GL(7) \text{ such that } \forall g \in G\, f \circ g = g \circ f\}.$$

Since $GL(7, \mathbb{C})$ acts naturally on $H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1))$ with basis $(x_0, \ldots, x_6)$, we have an induced action of $GL(7, \mathbb{C})$ on $H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(d))$ for any $d \in \mathbb{N}$. In particular, we can consider its induced action on $H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))$. Since we can see G as a subgroup of $GL(7, \mathbb{C})$, we can also consider the subgroup of the invariants, i.e. $GL(7, \mathbb{C})^G$ and its action on $H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))$. In particular, we have an action of $GL(7, \mathbb{C})^G$ on W, which induces an action of $GL(7, \mathbb{C})^G$ on $\mathrm{Gr}(4, W)$.

Definition 5.1.6. *We define* $U \subset \mathrm{Gr}(4, 10)/_{\mathrm{GL}(7)^G}$ *to be the open set that parametrizes only the smooth complete intersections*

$$V = \bigcap_{i=0}^{3} Q_i \ \textit{with}\ Q_0, \ldots, Q_3 \in H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))^G$$

such that $V \cap \mathrm{Fix}_G = \varnothing$, *so that the action of* G *on* V *is free and we do not have any fixed point. So we consider the following situation:* $\mathcal{U} \xrightarrow{p} U \subset \mathrm{Gr}(4, 10)/_{\mathrm{GL}(7)^G}$, *where*

$$\mathcal{U} := \left\{ \Big([Q_0, Q_1, Q_2, Q_3], x\Big) \in U \times \mathbb{P}^6 \colon x \in \bigcap_{i=0}^{3} Q_i \right\} \subset U \times \mathbb{P}^6,$$

and $V = V_u \cong p^{-1}(u) = \{u\} \times V_u$ *for some* $u \in U$.
We define $S := V/G$ *to be the quotient under the action of* G.

Remark 5.1.7. We get the following numerical situation:

$$K_S^2 = 4, \quad q(S) = 0, \quad \chi(\mathcal{O}_S) = 2, \quad p_g(S) = 1.$$

Remark 5.1.8. Since $K_V = \mathcal{O}_V(1)$ by adjunction, the canonical divisor is ample and V is a canonical model of a surface of general type and it is minimal.

Lemma 5.1.9. *The quotient surface $S/\sigma = (V/G)/\sigma$ is a K3 surface with at most nodes as singularities, and the bicanonical map of S factors through it.*

Proof. We consider the bicanonical maps of V and V/G. Since V is complete intersection, we have that $h^1(2K_V) = 0$. Since K_V is ample, by Serre's duality we get also $h^2(2K_V) = 0$. In particular, by Riemann–Roch Theorem, we have

$$P_2(V) := h^0(2K_V) = \chi(\mathcal{O}_V(2K_V)) = K_V^2 + \chi(\mathcal{O}_V) = 24.$$

So the bicanonical map is $\phi_{2K_V} : V \to \mathbb{P}^{23}$.
Since $K_V = \mathcal{O}_V(1)$ is ample, then K_S is ample and so $h^1(2K_S) = 0$ by Mumford's vanishing theorem. Then $P_2(S) = 6$ by Riemann-Roch theorem.

We have the following commutative diagram:

$$\begin{tikzcd}
V \arrow[r, "\phi_{2K_V}"] \arrow[d, "q"] \arrow[drr, dashed] & \mathbb{P}^{23} \arrow[d, dashed] \arrow[dr, dashed] & \\
S = V/G \arrow[r, "\psi", "\phi_{2K_S}"'] & \mathbb{P}^5 \arrow[r, hook] & \mathbb{P}^9.
\end{tikzcd}$$

Then it holds that

$$\begin{aligned} H^0(2K_{V/G}) &= H^0(2K_V)^G \\ &= \langle x_0^2, \dots, x_6^2, x_1x_2, x_3x_4, x_5x_6 \rangle_{\mathbb{C}} \mod H^0(I_V(2)). \end{aligned}$$

It is convenient to look at the bicanonical image in $\mathbb{P}^9$, so we study the map

$$\begin{aligned} \psi \colon \mathbb{P}^6 &\longrightarrow \mathbb{P}^9 \\ (x_0\colon \dots : x_6) &\mapsto (x_0^2\colon \dots : x_6^2\colon x_1x_2\colon x_3x_4\colon x_5x_6). \end{aligned}$$

The map ψ is given by the chosen monomial quadrics. Since V is the complete intersection of four quadrics in this system, the restrictions of the quadrics of W to V are elements of $H^0(2K_V)^G \cong \mathbb{C}^6$ which is isomorphic to $H^0(2K_{V/G})$. So, we get that $\psi(V) \subset \mathbb{P}^5$, and this $\mathbb{P}^5$ is defined by the four linear equations in $\mathbb{P}^9$ given by those quadrics defining V.
We notice that all quadrics in W are $\sigma-$invariant. Since σ commutes with σ_1 and σ_2, then $H := \langle \sigma, \sigma_1, \sigma_2 \rangle \cong (\mathbb{Z}/_{2\mathbb{Z}})^3$ is a group of automorphisms of V of order 8 such that ψ factors through its action. A standard direct computation shows that ψ is finite of degree 8. It follows that ψ is the quotient

by H. Then we have a commutative diagram

$$\begin{tikzcd} V \arrow[r, "\psi"] \arrow[d, "q"] & V/H \\ V/G \arrow[ur, "r"] & \end{tikzcd}$$

where r is the quotient by the involution induced by σ.
We want to show that $\psi(V)$ is a K3 surface. In particular, we claim that it is a complete intersection of three quadrics in $\mathbb{P}^5$. Indeed, in such a case we have that by Adjunction formula

$$K_{\psi(V)} = \mathcal{O}_{\psi(V)}(-6+2+2+2) = \mathcal{O}_{\psi(V)}. \tag{5.1.10}$$

Being $\psi(V) = V/H \subset \mathbb{P}^5$ complete intersection, then $q(\psi(V)) = 0$ and so $\psi(V)$ is a K3 surface.

Now let us prove that $\psi(V)$ is a complete intersection of three quadrics in $\mathbb{P}^5$. To ease the notation, we name the coordinates in $H^0(2K_{V/G})$ as

$$z_0 = x_0^2, \dots, z_6 = x_6^2, z_{12} = x_1x_2, z_{34} = x_3x_4, z_{56} = x_5x_6.$$

Then the image of $\psi\colon \mathbb{P}^6 \to \mathbb{P}^9$ has dimension 6 and

$$\psi(\mathbb{P}^6) = \{(z_0, \dots, z_6, z_{12}, z_{34}, z_{56}) \in \mathbb{P}^9 \colon z_{12}^2 = z_1z_2,\ z_{34}^2 = z_3z_4, z_{56}^2 = z_5z_6\}.$$

Indeed, $\psi(\mathbb{P}^6)$ is contained in this locus. Since the intersection of these three quadrics defines an irreducible 6–dimensional variety which is complete intersection, this is indeed $\psi(\mathbb{P}^6)$. When we restrict to V, we get that $\psi(V)$ is a complete intersection of three quadrics and four linear forms in $\mathbb{P}^9$ given by the four quadrics defining V. $\square$

Proposition 5.1.11. *Let $G =< \sigma_1, \sigma_2 >\cong (\mathbb{Z}/_{2\mathbb{Z}})^2$ be the group of automorphisms of $\mathbb{P}^6$ acting as in Definition 5.1.1 and let V be a smooth complete intersection of four quadrics $Q_0, Q_1, Q_2, Q_3 \in W = H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))^G$ parametrized by $\mathcal{U}$ as in Defintion 5.1.6. Then the quotient surface V/G is a Todorov surface of type* $(2, 12)$.

Proof. We need to prove that the involution σ on V/G is such that the quotient is a (singular) K3 surface so that the biconical map of $(V/G)/\sigma$ factors through it.
Since σ acts as identity on W and, in particular, on the equation defining V, we have that $\sigma \in \mathrm{Aut}(V)$. Moreover, σ commutes with G, so we can consider its action on the quotient, i.e. $\sigma[p] = [\sigma(p)]$ is well defined for any $[p] \in V/G = S$. Then, the result follows directly from Lemma 5.1.9. $\square$

Proposition 5.1.12. *The family of Definition 5.1.6 of Todorov surfaces of type (2,12) is 12–dimensional.*

In order to prove this result we need a preliminary lemma on the dimension of the stabilizer.

Lemma 5.1.13. *A generic point in the Grassmanian* $\mathrm{Gr}(4, W)$ *has a 1–dimensional stabilizer.*

Proof. First of all, we notice that a multiple of the identity matrix λI with $\lambda \in \mathbb{C}^*$ acts trivially. So the generic orbit has dimension greater or equal to one. So it is enough to find a generic point in the Grassmanian which has 1–dimensional stabilizer to prove the claim.
Let us consider the point in $\mathrm{Gr}(4, 10)$ given by the following four quadrics in $H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))^G$:

$$\begin{aligned} Q_0 &= \{(x_0, \dots, x_6) \in \mathbb{P}^6 \colon x_0^2 + x_1^2 + x_3^2 + x_5^2 = 0\}; \\ Q_1 &= \{(x_0, \dots, x_6) \in \mathbb{P}^6 \colon x_2^2 + x_4^2 + x_6^2 = 0\}; \\ Q_2 &= \{(x_0, \dots, x_6) \in \mathbb{P}^6 \colon x_1x_2 + x_3x_4 + x_5^2 = 0\}; \\ Q_3 &= \{(x_0, \dots, x_6) \in \mathbb{P}^6 \colon x_3x_4 + x_5x_6 = 0\}. \end{aligned}$$

Since any element of a finite subgroup of $\mathrm{GL}(7)$ is diagonalizable, asking to commute with the group G for an element $f \in \mathrm{GL}(7)$ is equivalent to ask for f to preserve the eigenspaces, Indeed, let us consider an eigenvector, i.e an element $v \in \mathbb{C}^7$ such that for any $g \in G$ it holds $gv = \lambda v$ for some $\lambda \in \mathbb{C}^*$. Then we have

$$g(f(v)) = f(g(v)) = f(\lambda v) = \lambda f(v).$$

Let us denote $G \cong (\mathbb{Z}/_{2\mathbb{Z}})^2 = \mathbb{Z}/_{2\mathbb{Z}}e_1 \oplus \mathbb{Z}/_{2\mathbb{Z}}e_2$, $G^* \cong \mathbb{Z}/_{2\mathbb{Z}}\epsilon_1 \oplus \mathbb{Z}/_{2\mathbb{Z}}\epsilon_2$. When we look at the action of G on $H^0(\mathcal{O}_{\mathbb{P}^6}(1))$, we have a decomposition into irreducible components on the characters: $W_{\chi_0} \oplus 2W_{\chi_{\epsilon_1}} \oplus 2W_{\chi_{\epsilon_2}} \oplus 2W_{\chi_{\epsilon_1+\epsilon_2}}$. Then, $\{x_0\}$ generates W_{χ_0}, $\{x_1, x_2\}$ is a basis of $2W_{\chi_{\epsilon_1}}$, $\{x_3, x_4\}$ is a basis of $2W_{\chi_{\epsilon_2}}$ and $\{x_5, x_6\}$ is a basis of $2W_{\chi_{\epsilon_1+\epsilon_2}}$.
So a general $M \in \mathrm{GL}(7, \mathbb{C})^G$ would be a matrix of the type

$$M = \begin{pmatrix} a & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & b & c & 0 & \dots & \dots & 0 \\ 0 & d & e & 0 & \dots & \dots & 0 \\ 0 & \dots & 0 & f & g & 0 & 0 \\ 0 & \dots & 0 & h & i & 0 & 0 \\ 0 & \dots & \dots & \dots & 0 & l & m \\ 0 & \dots & \dots & \dots & 0 & n & r \end{pmatrix}, \tag{5.1.14}$$

for some coefficients $a, b, c, d, e, f, g, h, i, l, m, n, r \in \mathbb{C}$. Now we analyze the action of such a matrix M on the quadrics.

- Q_1 is sent to $(cx_1+ex_2)^2+(gx_3+ix_4)^2+(mx_5+rx_6)^2$. So it is sent to a linear combination of the four quadrics, which does not contain Q_0, since Q_0 is the only quadric depending on x_0^2. So Q_1 goes to $\alpha Q_1+\beta Q_2+\gamma Q_3$, for some $\alpha,\beta,\gamma\in\mathbb{C}$. Since Q_0 is the only quadric depending on x_1^2 and x_3^2, we conclude that $c^2=g^2=0$, so $c=g=0$. This implies that the monomial x_1x_2 does not appear in the image of Q_1. Since Q_2 is the only quadric in which x_1x_2 appears, we get that $\beta=0$. From $g=0$, we get that also the monomial x_3x_4 cannot appear in the image of Q_1. Since x_3x_4 appears only in the equations of Q_2 and Q_3 and since $\beta=0$, we conclude that $\gamma=0$. So Q_1 is sent by M into its multiple αQ_1. Since Q_2 is the only quadric containing x_5^2, this implies that $m=0$ and we conclude that
$$\alpha=e^2=i^2=r^2\neq 0, \tag{5.1.15}$$
since M is invertible.

- Q_3 is sent by M to
$$\begin{aligned}(fx_3+hx_4)\cdot(gx_3+ix_4)+(lx_5+nx_6)\cdot(mx_5+rx_6)\\ =(fx_3+hx_4)\cdot ix_4+(lx_5+nx_6)\cdot rx_6.\end{aligned}$$
As before, since Q_0 is the only quadric containing x_0^2, Q_1 is the only one which depends on x_2^2 and Q_2 is the only one that has x_5^2, we have that Q_3 is sent by M to a multiple of itslef. So $M(Q_3)=\lambda Q_3$ for some $\lambda\in\mathbb{C}$. In particular, it holds $hi=nr=0$. By (5.1.15), we get that $h=n=0$. Since M is invertible, we have that
$$\lambda=fi=lr\neq 0. \tag{5.1.16}$$

- Q_2 is sent by M to
$$\begin{aligned}(bx_1+dx_2)\cdot(cx_1+ex_2)+(fx_3+hx_4)\cdot(gx_3+ix_4)+(lx_5+nx_6)^2\\ =(bx_1+dx_2)\cdot ex_2+fix_3x_4+l^2x_5^2.\end{aligned}$$
Since Q_1 is the only quadric containing x_6^2 and x_4^2, we have that x_2^2 cannot appear in the equation of $M(Q_2)$, so it has to be $de=0$. By (5.1.15) we have $e\neq 0$, so $d=0$ and the matrix M is diagonal. So Q_2 is sent by M to μQ_2 for some $\mu\in\mathbb{C}^*$ and
$$\mu=be=fi=l^2\neq 0. \tag{5.1.17}$$
By (5.1.16) we get $lr=\lambda=fi=\mu=l^2$, so $l=r$. By (5.1.15) we have $be=e^2=l^2=fi=i^2$, so $b=e=\pm l$ and $f=i=\pm l$.

- Q_0 is sent by M to

$$\begin{aligned} a^2x_0^2 + (bx_1 + dx_2)^2 + (fx_3 + hx_4)^2 + (lx_5 + nx_6)^2 \\ = a^2x_0^2 + b^2x_1^2 + f^2x_3^2 + l^2x_5^2. \end{aligned}$$

Since the matrix is diagonal, $M(Q_0) = \omega Q_0$, for some $\omega \in \mathbb{C}^*$, so

$$0 \neq \omega = a^2 = b^2 = f^2 = l^2 \Rightarrow a = \pm l.$$

We conclude that M is of the form

$$M = \begin{pmatrix} \pm l & 0 & \dots\dots\dots\dots\dots & & & & 0 \\ 0 & \pm l & 0 & \dots\dots\dots & & & 0 \\ 0 & 0 & \pm l & 0 & \dots\dots & & 0 \\ 0 & \dots & 0 & \pm l & \dots\dots & & 0 \\ 0 & \dots\dots & & 0 & \pm l & 0 & 0 \\ 0 & \dots\dots\dots & & & 0 & l & 0 \\ 0 & \dots\dots\dots\dots & & & & 0 & l \end{pmatrix}, \tag{5.1.18}$$

for some $l \in \mathbb{C}^*$. $\square$

Proof. of Proposition 5.1.12 Now let S, S' be two Todorov surfaces of type $(2, 12)$. If S and S' give different points in $U \subset \mathrm{Gr}(4, 10)/_{\mathrm{GL}(7)^G}$, then S and S' cannot be isomorphic. Indeed any isomorphism $S \xrightarrow{\sim} S'$ lifts to the universal cover $V \xrightarrow{\sim} V'$. Since we embed the universal cover in $\mathbb{P}^6$ via the canonical map, such isomorphism extends to an automorphism of $\mathbb{P}^6$.
In order to compute the dimension of the family we are describing, we compute the dimension of the base

$$U \subset \mathrm{Gr}(4, 10)/_{\mathrm{GL}(7,\mathbb{C})^G}.$$

We have that the dimension of the Grassmanian variety is:

$$\dim \mathrm{Gr}(4, 10) = 4(10 - 4) = 24.$$

Let us compute now $\dim \mathrm{GL}(7, \mathbb{C})^G$. We have that

$$\dim \mathrm{GL}(7)^{\mathrm{G}} = 1 + 3 \cdot 2^2 = 13.$$

However, we notice that the action on $\mathrm{GL}(7)^G$ is not faithful. By Lemma 5.1.13, we conclude that

$$\dim(U) = \dim \mathrm{Gr}(4, 10)/_{\mathrm{GL}(7)^G} = 24 - 13 + 1 = 12. \tag{5.1.19}$$

$\square$

So we have found a 12-dimensional family of Todorov surfaces of type $(2, 12)$, whose general element is $S = V/G$, where V is a smooth complete intersection of four linearly independent quadrics in $\mathbb{P}^6$ which are G–invariant.

We are finally able to prove our main result to describe the family of Todorov surfaces of type $(2, 12)$.

Theorem 5.1.20. *Let S be a general Todorov surface with fundamental invariants $(\alpha, k) = (2, 12)$. Then the canonical model of S is a quotient surface V/G where V is the smooth complete intersection of four independent quadrics $Q_0, Q_1, Q_2, Q_3 \in H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))^G$ parametrized by $\mathcal{U}$ (as in Definition 5.1.6), and $G \cong (\mathbb{Z}/_{2\mathbb{Z}})^2$ is the group of automorphisms of $\mathbb{P}^6$ acting as in Definition 5.1.1. Conversely, any such surface V/G is a Todorov surface of type $(2, 12)$.*

Proof. By Proposition 5.1.11 it follows that V/G is a Todorov surface of type (2,12). In order to prove the first part of the theorem, we use a dimensional argument. Since the number of moduli of the family of Todorov surfaces of type (2,12) is 12, and the family is irreducible (see [Tod81], [Mor88, Theorem 7.5] for details on the moduli spaces of Todorov surfaces ,and [Usu91, Remark 5.3.5], [LP15, Section 4.2]), by (5.1.19) we conclude that we are describing the general element of the family. □

5.1.1 The complete family

By means of Theorem 5.1.20, we can give an explicit description of the general member of the family of Todorov surfaces of type $(2, 12)$. Now we want to introduce the complete family of Todorov surfaces of this type, which is more useful when dealing with cycles.

Definition 5.1.21. *We define*

$$\overline{B} := \prod_{i=0}^{3} \mathbb{P}\left(H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))^G\right) \cong \mathbb{P}^9 \times \mathbb{P}^9 \times \mathbb{P}^9 \times \mathbb{P}^9 \,. \tag{5.1.22}$$

Let $\mathcal{V} \xrightarrow{p} B$ denote the total space of the family of the complete intersections $\bigcap_{i=0}^{3} Q_i \subset \mathbb{P}^6$, where $b \in B$ and $B \subseteq \overline{B}$ is a Zariski open set which parametrizes only the smooth intersections, i.e. $\overline{B}$ is the projective closure of B.
We are in the following situation:

$$\mathcal{V} := \left\{ \Big([Q_0], [Q_1], [Q_2], [Q_3], x\Big) \in B \times \mathbb{P}^6 \colon x \in \bigcap_{i=0}^{3} Q_i \right\} \subset B \times \mathbb{P}^6.$$

For any $b = ([Q_0], [Q_1], [Q_2], [Q_3]) \in B$, *we define*

$$V_b := \bigcap_{i=0}^{3} Q_i \cong p^{-1}(b) = \{b\} \times V_b.$$

In particular, the morphism p *corresponds to the first projection of* $B \times \mathbb{P}^6$ *restricted to* $\mathcal{V}$. *Since the action of* $G \cong (\mathbb{Z}/_{2\mathbb{Z}})^2$ *on* $\overline{B} \times \mathbb{P}^6$ *is non trivial only on the second component, we can consider its action on* $\mathcal{V}$ *and we get* $\mathcal{S} := \mathcal{V}/G \to B$. *By Theorem 5.1.20,* $\mathcal{S}$ *is the complete family of smooth Todorov surfaces with fundamental invariants* $(2, 12)$.

So we have the following situation

$$\begin{array}{ccccccc} \mathcal{V} & & \subset B \times \mathbb{P}^6 & \qquad & \mathcal{S} & & \subset B \times \mathbb{P}^6/G \\ {\scriptstyle \pi}\big\downarrow & & & & {\scriptstyle \overline{\pi}}\big\downarrow & & \\ B & & & & B. & & \end{array}$$

Proposition 5.1.23. $\mathcal{V}$ *is a smooth quasi–projective variety.*

Proof. Let us consider the second projection of $B \times \mathbb{P}^6$ restricted to $\mathcal{V}$, i.e. $\psi \colon \mathcal{V} \to \mathbb{P}^6$ is the morphism such that

$$\psi^{-1}(p) = \{(b, p) \in B \times \mathbb{P}^6 \colon p \in V_b\}.$$

For each $p \in \mathbb{P}^6$, there exists a quadric $Q \in H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))^G$ such that $Q(p) \neq 0$. Indeed, let $p = (\overline{x}_0 \colon \ldots \colon \overline{x}_6)$ be a point in $\mathbb{P}^6$, then there exists $i \in \{0, \ldots 6\}$ such that $\overline{x}_i \neq 0$, so it is enough to choose $Q(x) = x_i^2$, so that $p \notin \ker(Q)$.
Let us consider $(b_1, p), (b_2, p) \in \psi^{-1}(p)$, where $b_1 = ([Q_0], [Q_1], [Q_2], [Q_3])$, $b_2 = ([R_0], [R_1], [R_2], [R_3])$ and $[Q_i], [R_i] \in \mathbb{P}\left(H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))^G\right)$.
Then, if we consider a linear combination $\lambda b_1 + \mu b_2$ with $\lambda, \mu \in \mathbb{C}^*$, we have that $p \in V_{\lambda b_1 + \mu b_2}$. So the fiber over p is a linear system, hence $\mathcal{V}$ is smooth. $\square$

By Definition 5.0.1, to each Todorov surface $S_b = V_b/G$ we have two associated $K3$ surfaces, one is the singular $K3$ surface $\overline{M_b} = S_b/\sigma$ obtained as the quotient by the involution, and the other is its resolution of singularities $M_b = (\overline{M_b})^{res}$. We are in the following situation:

$$\begin{array}{ccc} & & \mathcal{V} \subset B \times \mathbb{P}^6 \\ & & {\scriptstyle q}\big\downarrow{\scriptstyle 4:1} \\ & & \mathcal{S} = \mathcal{V}/G \\ & & {\scriptstyle f}\big\downarrow{\scriptstyle 2:1} \\ \mathcal{M} & \longrightarrow & \overline{\mathcal{M}}, \end{array}$$

where $\overline{\mathcal{M}}$ parametrizes the singular $K3$ surfaces associated to the Todorov surfaces and $\mathcal{M}$ parametrizes the smooth ones obtained by resolving the quotient singularities. Fiberwise we have

$$\begin{tikzcd} & G \curvearrowright V_b \arrow[d, "q_b"', "4:1"] \\ & \sigma \curvearrowright S_b = V_b/G \arrow[d, "f_b"', "2:1"] \\ M_b \arrow[r, "ros"] & \overline{M_b}. \end{tikzcd}$$

5.2 Results on 0–cycles

In order to prove that the family of Todorov surfaces of type $(2,12)$ verifies Voisin's conjecture 3.4.1, first we prove some preliminary results on 0–cycles for this family.
The core of the proof of Theorem 5.3.5, is the following result

$$CH^2_{\text{hom}}(\mathcal{S} \times_B \mathcal{S})_{\mathbb{Q}} = CH^2_{\text{hom}}(\mathcal{V} \times_B \mathcal{V})_{\mathbb{Q}} = 0, \tag{5.2.1}$$

where $\mathcal{S}$ and $\mathcal{V}$ are the family defined in 5.1.21.

The proof of (5.2.1) is based on the results in [Lat18a, Proposition 4.5], [Voi15], [Voi13]. The main idea is to see the fiber product $\mathcal{V} \times_B \mathcal{V}$ as a Zariski–open set of a variety X whose Chow groups are trivial. We recall that having trivial Chow groups means to have injective cycle class maps $CH^i(X)_{\mathbb{Q}} \hookrightarrow H^{2i}(X, \mathbb{Q})$ for any i (see Definition 4.2.7).

Since $\mathcal{V} \subset B \times \mathbb{P}^6$, we have a morphism $\pi \colon \mathcal{V} \times_B \mathcal{V} \to \mathbb{P}^6 \times \mathbb{P}^6$ such that

$$\begin{aligned} \pi^{-1}(p,q) = \Big\{ \big(([Q_0] : [Q_1] : [Q_2] : [Q_3]), (p,q)\big) \in \mathcal{V} \times_B \mathcal{V} \colon \\ Q_i(p) = Q_i(q) = 0 \, \forall i \Big\} \\ \cong \Big\{ b = \big([Q_0] : [Q_1] : [Q_2] : [Q_3]\big) \in B : Q_i(p) = Q_i(q) = 0 \, \forall i \Big\}. \end{aligned}$$

Definition 5.2.2. *We recall that* $\overline{B} := \prod_{i=0}^{3} \mathbb{P}\left(H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))^G\right)$ *is the projective closure of* B*, so that* B *is a Zariski open set which parametrizes the smooth complete intersections.*
We define the variety $X \subset \overline{B} \times \mathbb{P}^6 \times \mathbb{P}^6$ *as*

$$\begin{aligned} X := \Big\{ \Big(([Q_0] : [Q_1] : [Q_2] : [Q_3]), (p,q)\Big) \in \overline{B} \times \mathbb{P}^6 \times \mathbb{P}^6 : \\ Q_i(p) = Q_i(q) = 0 \, \forall i \Big\}. \end{aligned}$$

Then X *contains the fiber product* $\mathcal{V} \times_B \mathcal{V}$ *as a Zariski open set.*

Identity (5.2.1) follows by proving that X has trivial Chow groups (see Proposition 5.2.7).

Proposition 5.2.3. *Suppose that $B \subset \overline{B}$ is small enough to have a smooth morphism $\mathcal{V} \to B$.*
Then $CH^2_{\mathrm{hom}}(\mathcal{V} \times_B \mathcal{V})_{\mathbb{Q}} = CH^2_{\mathrm{hom}}(\mathcal{S} \times_B \mathcal{S})_{\mathbb{Q}} = 0$.

Proof. As the map $g\colon \mathcal{V} \xrightarrow{4:1} \mathcal{S} = \mathcal{V}/G$ is a finite surjective morphism, $CH^2_{\mathrm{hom}}(\mathcal{V} \times_B \mathcal{V})_{\mathbb{Q}} = 0$ implies $CH^2_{\mathrm{hom}}(\mathcal{S} \times_B \mathcal{S})_{\mathbb{Q}} = 0$. So it is enough to prove the statement for the fibered product of the complete intersections family $\mathcal{V} \times_B \mathcal{V}$.

Let $D := X \backslash (\mathcal{V} \times_B \mathcal{V})$ be the boundary divisor and $m := \dim X$. Let $a \in CH^2_{\mathrm{hom}}(\mathcal{V} \times_B \mathcal{V})_{\mathbb{Q}}$ be a homologically trivial cycle. So a is the restriction of a cycle in X, i.e. there exists $\overline{a} \in CH_{m-2}(X)_{\mathbb{Q}}$ such that $\overline{a}|_{\mathcal{V}\times_B\mathcal{V}} = a$ and $[\overline{a}]|_{\mathcal{V}\times_B\mathcal{V}} = 0 \in H^4(\mathcal{V} \times_B \mathcal{V}, \mathbb{Q})$.
Performing a resolution of singularities on X

$$\begin{tikzcd} \widetilde{X} \arrow[r, "ros"] & X \\ \widetilde{D} \arrow[u, hook] \arrow[r, "ros", "r"'] & D, \arrow[u, hook, "i"'] \end{tikzcd}$$

we find out that the class $[\overline{a}]$ comes from a Hodge class $\beta \in H^2(\widetilde{D}, \mathbb{Q})$ since $\overline{a} \in CH_{m-2}(X)_{\mathbb{Q}}$ and since it is homologically trivial on $\mathcal{V} \times_B \mathcal{V}$. By Lefschetz Theorem A.2.14 on $(1,1)$–classes, we have that β is algebraic, so there exists a cycle $b \in CH^1(\widetilde{D})_{\mathbb{Q}}$ such that $[b] = \beta$. Let us define $\overline{\overline{a}} := \overline{a} - i_*(r_*b) \in CH^{\mathrm{hom}}_{m-2}(X)_{\mathbb{Q}}$, which is a trivial group since X has trivial Chow groups. Thus $\overline{\overline{a}} = 0$ in $CH^{\mathrm{hom}}_{m-2}(X)_{\mathbb{Q}}$ which yields $0 = \overline{\overline{a}}|_{\mathcal{V}\times_B\mathcal{V}} = a$, and we conclude that $CH^2_{\mathrm{hom}}(\mathcal{V} \times_B \mathcal{V}) = 0$. □

5.2.1 Stratification of $\mathbb{P}^6 \times \mathbb{P}^6$

To prove the key Proposition 5.2.3, it is left to prove that X, defined as in Definition 5.2.2, has trivial Chow groups, following the argument in [Lat18a].

We consider the projection

$$X \xrightarrow{\pi} \mathbb{P}^6 \times \mathbb{P}^6. \tag{5.2.4}$$

Then the fiber over a point is a product of projective spaces

$$\pi^{-1}(p,q) \cong \{b \in \overline{B} \colon Q_i(p) = Q_i(q) = 0\} \cong \mathbb{P}^r \times \mathbb{P}^r \times \mathbb{P}^r \times \mathbb{P}^r \subset \overline{B},$$

for some $r \leq 9$, where $\overline{B}$ is defined as in (5.1.22). However, the fiber does not have constant dimension on the whole space.
To prove that X has trivial Chow groups, the idea is to find a stratification of $\mathbb{P}^6 \times \mathbb{P}^6$ such that the fiber of π has constant dimension on each stratum.

Lemma 5.2.5. *There exists a Zariski-open set of points $U \subset \mathbb{P}^6 \times \mathbb{P}^6$ such that, for any $(p,q) \in U$, the fiber of the projection $\pi\colon X \to \mathbb{P}^6 \times \mathbb{P}^6$ is $\pi^{-1}(p,q) \cong \mathbb{P}^7 \times \mathbb{P}^7 \times \mathbb{P}^7 \times \mathbb{P}^7$. Moreover, for all the points (p',q') in the complement $Z = (\mathbb{P}^6 \times \mathbb{P}^6)\backslash U$, the fiber is $\pi^{-1}(p',q') \cong \mathbb{P}^8 \times \mathbb{P}^8 \times \mathbb{P}^8 \times \mathbb{P}^8$.*

Proof. By Proposition 5.1.23, we have that each point of $\mathbb{P}^6$ imposes one condition on each component $\mathbb{P}(H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))^G)$ of

$$\overline{B} = \Pi_{i=0}^3 \left(\mathbb{P}(H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))^G)\right) \cong \mathbb{P}^9 \times \mathbb{P}^9 \times \mathbb{P}^9 \times \mathbb{P}^9.$$

Indeed, let us consider $Q \in \mathbb{P}(H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))^G) \cong \mathbb{P}^9$. By (5.1.3), we can make the condition $Q(p) = 0$ explicit as

$$Q(p) = \alpha p_0^2 + \beta p_1^2 + \gamma p_2^2 + \delta p_3^2 + \epsilon p_4^2 + \zeta p_5^2 + \eta p_6^2 + \theta p_1 p_2 + \kappa p_3 p_4 + \lambda p_5 p_6 = 0,$$

with $p = (p_0 : \cdots : p_6) \in \mathbb{P}^6$ and $\alpha, \ldots, \lambda \in \mathbb{C}$. Given a point $(p,q) \in \mathbb{P}^6 \times \mathbb{P}^6$, we have two such conditions, which we can represent by a matrix

$$A(p,q) := \begin{pmatrix} p_0^2 & p_1^2 & p_2^2 & p_3^2 & p_4^2 & p_5^2 & p_6^2 & p_1p_2 & p_3p_4 & p_5p_6 \\ q_0^2 & q_1^2 & q_2^2 & q_3^2 & q_4^2 & q_5^2 & q_6^2 & q_1q_2 & q_3q_4 & q_5q_6 \end{pmatrix}.$$

In general, A has maximum rank, so that inside $\mathbb{P}^6 \times \mathbb{P}^6$ there is a Zariski–open set of pair of points (p,q), each one of them imposing one condition on each component of $\overline{B}$. So, for a general point in $\mathbb{P}^6 \times \mathbb{P}^6$, we have that the fiber is

$$\pi^{-1}(p,q) \cong \left\{b \in \overline{B} \colon Q_i(p) = Q_i(q) = 0\right\} \cong \mathbb{P}^7 \times \mathbb{P}^7 \times \mathbb{P}^7 \times \mathbb{P}^7 \subset \overline{B},$$

However, the rank of A is not always maximum. Indeed, inside $\mathbb{P}^6 \times \mathbb{P}^6$ there is locus Z, such that for every $(p,q) \in Z$ the dimension of the fiber increases by one on each component and the rank of A drops by one.

For any $j,k \in \{0,\ldots,6\}$ we define partial diagonals as follows

$$\begin{aligned} \Delta_{\pm}^{j,k} := \Big\{ &(p,q) \in \mathbb{P}^6 \times \mathbb{P}^6 : \exists \lambda \in \mathbb{C}^* \text{ s.t. } q_0 = \pm\lambda p_0, \\ &q_j = -\lambda p_j, q_k = -\lambda p_k, q_i = \lambda p_i \,\forall i \neq j,k \Big\}; \\ \Delta_{\pm}^{j,k,l,m} := \Big\{ &(p,q) \in \mathbb{P}^6 \times \mathbb{P}^6 : \exists \lambda \in \mathbb{C}^* \text{ s.t. } q_0 = \pm\lambda p_0, \\ &q_i = -\lambda p_i \,\forall i \in \{j,k,l,m\} \text{ and } q_i = \lambda p_i \,\forall i \neq j,k,l,m \Big\}; \\ \Delta^0 := \Big\{ &(p,q) \in \mathbb{P}^6 \times \mathbb{P}^6 : \exists \lambda \in \mathbb{C}^* \text{ s.t. } q_0 = \lambda p_0, q_i = -\lambda p_i \forall i \neq 0 \Big\}. \end{aligned}$$

Then we consider the union

$$\begin{aligned} Z := \Delta_{\mathbb{P}^6 \times \mathbb{P}^6} \cup \Delta_+^{1,2} \cup \Delta_-^{1,2} \cup \Delta_+^{3,4} \cup \Delta_-^{3,4} \cup \Delta_+^{5,6} \cup \Delta_-^{5,6} \cup \\ \Delta_+^{1,2,3,4} \cup \Delta_-^{1,2,3,4} \cup \Delta_+^{1,2,5,6} \cup \Delta_-^{1,2,5,6} \cup \Delta_+^{3,4,5,6} \cup \Delta_-^{3,4,5,6} \cup \Delta^0. \end{aligned}$$

Then, for any point $(p', q') \in Z$ we have that the rank of A is not maximum, so the fiber of such a point is

$$\pi^{-1}(p', q') \cong \left\{b \in \overline{B} \colon Q_i(p') = Q_i(q') = 0\right\} \cong \mathbb{P}^8 \times \mathbb{P}^8 \times \mathbb{P}^8 \times \mathbb{P}^8 \subset \overline{B}.$$

We define $U = \left(\mathbb{P}^6 \times \mathbb{P}^6\right) \setminus Z$. Then we claim that U is Zariski–open set in which the fiber has lower dimension, i.e. for any $(p, q) \in U$ we have that $\pi^{-1}(p, q) \cong \mathbb{P}^7 \times \mathbb{P}^7 \times \mathbb{P}^7 \times \mathbb{P}^7$. Indeed, for a point $(p, q) \in U$ there exist $i, j \in \{0, \ldots, 6\}$ and $\lambda, \mu \in \mathbb{C}^*$ with $\lambda \neq \pm\mu$ such that $q_i = \lambda p_i$ and $q_j = \mu p_j$. If we suppose that $\operatorname{rank} A(p, q) = 1$, then all the 2×2 minors vanish. In particular, we have $0 = p_i^2 q_j^2 - p_j^2 q_i^2 = (p_i q_j - p_j q_i)(p_i q_j + p_j q_i)$, and this holds if and only if $q_j = \pm\frac{q_i}{p_i} p_j$ so $\mu = \frac{q_i}{p_i}$. Substituting $q_i = \lambda p_i$, we get $q_j = \pm\frac{q_i}{p_i} p_j = \pm\lambda p_j$, so $\lambda = \pm\mu$ which is a contradiction. ☐

5.2.2 The variety X has trivial Chow group

In order to prove that X, defined as in Definition 5.2.2, has trivial Chow groups, we prove that X has the strong property in the sense of Definition 4.1.7.

Remark 5.2.6. We notice that we have the following implications: strong property $\Rightarrow$ weak property $\Rightarrow$ trivial Chow groups. Indeed we have

$$CH_i(X)_{\mathbb{Q}} \xrightarrow{\sim} W_{-2i} H_{2i}(X, \mathbb{Q}) \hookrightarrow H_{2i}(X, \mathbb{Q}).$$

Proposition 5.2.7. *The variety X has trivial Chow groups, i.e.*

$$CH_*^{\mathrm{hom}}(X)_{\mathbb{Q}} = 0.$$

Proof. For any point $(p, q) \in \mathbb{P}^6 \times \mathbb{P}^6$ we can consider the fiber of the projection $X \xrightarrow{\pi} \mathbb{P}^6 \times \mathbb{P}^6$ as in (5.2.4). We have that

$$\pi^{-1}(p, q) \cong \{b \in \overline{B} \colon Q_i(p) = Q_i(q) = 0\} \cong \mathbb{P}^r \times \mathbb{P}^r \times \mathbb{P}^r \times \mathbb{P}^r \subset \overline{B},$$

for some $r \leq 9$, where $\overline{B}$ is defined as in (5.1.22). The dimension of the fiber is not constant, nonetheless, by Lemma 5.2.5, we have that it is constant on a Zariski open set $U \subset \mathbb{P}^6 \times \mathbb{P}^6$ and on its complement Z.
Our situation is the following

$$\begin{tikzcd} X_U \arrow[r, hook] & X \arrow[d, "\pi"] & X_Z \arrow[l, hook'] \\ U \arrow[r, hook] & \mathbb{P}^6 \times \mathbb{P}^6 & Z, \arrow[l, hook'] \end{tikzcd}$$

where $X_Z = \pi^{-1}(Z)$, $X_U = \pi^{-1}(U)$, U and Z are defined as in Lemma 5.2.5. We want to prove that X_Z and X_U have the strong property and then conclude that X has trivial Chow groups by means of Lemma 4.1.8(1).

The idea is to find a stratification of $\mathbb{P}^6 \times \mathbb{P}^6$ such that the fiber of π has constant dimension on each stratum.

For every $i = 0, \dots, 6$ we define

$$\begin{aligned} A_i &:= \{(p,q) \in \mathbb{P}^6 \times \mathbb{P}^6 \colon p_i \neq 0 \text{ and } q_i \neq 0\}; \\ B_i &:= \{(p,q) \in \mathbb{P}^6 \times \mathbb{P}^6 \colon p_i = 0 \text{ and } q_i = 0\}; \\ C_i &:= \{(p,q) \in \mathbb{P}^6 \times \mathbb{P}^6 \colon p_i \neq 0 \text{ and } q_i = 0\} \\ &\quad \cup \{(p,q) \in \mathbb{P}^6 \times \mathbb{P}^6 \colon p_i = 0 \text{ and } q_i \neq 0\}. \end{aligned}$$

First of all, we consider the locus Z. The intersection $Z \cap C_0$ is empty, whereas $\overline{A}_0 := Z \cap A_0$ is isomorphic to 14 copies of $\mathbb{A}^6$ via the map

$$\begin{gathered} \left(\left[1 : \tfrac{p_1}{p_0} : \tfrac{p_2}{p_0} : \tfrac{p_3}{p_0} : \tfrac{p_4}{p_0} : \tfrac{p_5}{p_0} : \tfrac{p_6}{p_0}\right], \lambda\left[\pm 1 : \pm\tfrac{p_1}{p_0} : \pm\tfrac{p_2}{p_0} : \pm\tfrac{p_3}{p_0} : \pm\tfrac{p_4}{p_0} : \pm\tfrac{p_5}{p_0} : \pm\tfrac{p_6}{p_0}\right]\right) \\ \downarrow \\ \left(\tfrac{p_1}{p_0}, \tfrac{p_2}{p_0}, \tfrac{p_3}{p_0}, \tfrac{p_4}{p_0}, \tfrac{p_5}{p_0}, \tfrac{p_6}{p_0}\right), \end{gathered}$$

with $\lambda \in \mathbb{C}^*$.

For the intersection $\overline{B}_0 := Z \cap B_0$, we can consider $\overline{A}_1 := \overline{B}_0 \cap A_1$ which is isomorphic to 14 copies of $\mathbb{A}^5$ via the map

$$\begin{gathered} \left(\left[0 : 1 : \tfrac{p_2}{p_1} : \tfrac{p_3}{p_1} : \tfrac{p_4}{p_1} : \tfrac{p_5}{p_1} : \tfrac{p_6}{p_1}\right], \left[0 : \pm\lambda : \pm\lambda\tfrac{p_2}{p_1} : \pm\lambda\tfrac{p_3}{p_1} : \pm\lambda\tfrac{p_4}{p_1} : \pm\tfrac{p_5}{p_1} : \pm\lambda\tfrac{p_6}{p_1}\right]\right) \\ \downarrow \\ \left(\tfrac{p_2}{p_1}, \tfrac{p_3}{p_1}, \tfrac{p_4}{p_1}, \tfrac{p_5}{p_1}, \tfrac{p_6}{p_1}\right), \end{gathered}$$

with $\lambda \in \mathbb{C}^*$. The intersection $\overline{B_0} \cap C_1$ is empty and next we can consider $\overline{B}_1 := \overline{B}_0 \cap B_1$. Iterating this process we get

$$\begin{cases} \overline{Z}_{-1} := Z, \\ \overline{Z}_i := Z \cap \left(\bigcap_{j=0}^{i} B_j\right) & \text{for } i \geq 0; \end{cases}$$

$$\overline{A}_i := \overline{Z}_{i-1} \cap A_i \cong \coprod_{j=1}^{14} \mathbb{A}^{6-i} \qquad \text{for } i \in \{0, \dots, 6\}.$$

So we have a stratification of Z by affine spaces as $Z = \coprod_{i=0}^{5} \mathbb{A}^5$.

We consider now $U = \left(\mathbb{P}^6 \times \mathbb{P}^6\right) \backslash Z$. Then $\overline{U}_0 := U \cap A_0$ is isomorphic to $\mathbb{A}^{12}$ minus 14 copies of $\mathbb{A}^6$ via the map

$$\begin{gathered} \left(\left[1 : \tfrac{p_1}{p_0} : \tfrac{p_2}{p_0} : \tfrac{p_3}{p_0} : \tfrac{p_4}{p_0} : \tfrac{p_5}{p_0} : \tfrac{p_6}{p_0}\right], \left[1 : \tfrac{q_1}{q_0} : \tfrac{q_2}{q_0} : \tfrac{q_3}{q_0} : \tfrac{q_4}{q_0} : \tfrac{q_5}{q_0} : \tfrac{q_6}{q_0}\right]\right) \\ \downarrow \\ \left(\tfrac{p_1}{p_0}, \tfrac{p_2}{p_0}, \tfrac{p_3}{p_0}, \tfrac{p_4}{p_0}, \tfrac{p_5}{p_0}, \tfrac{p_6}{p_0}, \tfrac{q_1}{q_0}, \tfrac{q_2}{q_0}, \tfrac{q_3}{q_0}, \tfrac{q_4}{q_0}, \tfrac{q_5}{q_0}, \tfrac{q_6}{q_0}\right). \end{gathered}$$

Iterating the process as above, we get

$$\begin{cases} \overline{T}_{-1} := U, \\ \overline{T}_i := U \cap \left(\bigcap_{j=0}^{i} B_j \right) \end{cases} \qquad \text{for } i \geq 0;$$

$$\begin{cases} \overline{V}_i := U \cap \left(\bigcap_{j=0}^{i} C_j \right) \cong \mathbb{A}^{12-i} \backslash \left(\coprod_{j=1}^{14} \mathbb{A}^{6-i} \right) \\ \overline{U}_i := \overline{T}_{i-1} \cap A_i \cong \mathbb{A}^{12-2i} \backslash \left(\coprod_{j=1}^{14} \mathbb{A}^{6-i} \right) \end{cases} \qquad \text{for } i \in \{0, \dots, 6\}.$$

So we can see $U = \coprod_{i=0}^{6} \left(\overline{V}_i \coprod \overline{U}_i \right)$ as a disjoint union of varieties of type $\mathbb{A}^k \backslash L$, where L is a finite union of linearly embedded affine spaces. By Lemma 4.1.8(2), U has the strong property and so does Z.
Since $X_Z = \pi^{-1}(Z)$ is a fibration over Z, whose fibers are product of projective spaces $\mathbb{P}^8 \times \mathbb{P}^8 \times \mathbb{P}^8 \times \mathbb{P}^8$, then, by means of Lemma 4.1.8(3), X_Z has the strong property too. With the same argument, $X_U = \pi^{-1}(U)$ has the strong property, since it is a fibration over U with fiber $\mathbb{P}^7 \times \mathbb{P}^7 \times \mathbb{P}^7 \times \mathbb{P}^7$. Then, by Lemma 4.1.8(1) we conclude that X has the strong property, and in particular it has trivial Chow groups. $\square$

5.3 Proof of Theorem 5.3.5

Our aim is to prove Voisin's conjecture 3.4.1 for Todorov surfaces of type $(2, 12)$ (see Definition 5.0.1). In order to do so, we show that, when dealing with homologically trivial 0–cycles on a Todorov surface, we can actually move the problem onto the associated $K3$ surface to gain some more informations.

Remark 5.3.1. We notice that $\overline{M} = S/\sigma$ is a singular variety with quotient singularities. In general, Chow groups of singular varieties do not admit intersection product or a ring structure. But, in our case, we have that $CH_*(\overline{M})_{\mathbb{Q}}$ inherits the intersection product and ring structure from $CH_*(S)_{\mathbb{Q}}$ since it is a subring of it. Indeed we have the following isomorphism (see [Ful98, Example 8.3.12])

$$CH_*(\overline{M})_{\mathbb{Q}} \cong (CH_*(S)_{\mathbb{Q}})^{\sigma}.$$

Theorem 5.3.2. *Let S be a Todorov surface with fundamental invariants $(\alpha, k) = (2, 12)$. Let $\overline{M}$ be the associated singular $K3$ surface to S.*
Then there is an isomorphism

$$CH^2_{\text{hom}}(S)_{\mathbb{Q}} \cong CH^2_{\text{hom}}(\overline{M})_{\mathbb{Q}}.$$

Proof. We want to find a correspondence in $CH^2(\mathcal{S} \times_B \mathcal{S})$ that is homologically trivial when restricted to each fiber (where $\mathcal{S}$ is defined as in Definition 5.1.21).

Let $\Delta_{\mathcal{S}} \in CH^2(\mathcal{S} \times_B \mathcal{S})$ denote the relative diagonal. We consider the following relative correspondence

$$\Gamma = 2\Delta_{\mathcal{S}} -^t \Gamma_f \circ \Gamma_f \in CH^2(\mathcal{S} \times_B \mathcal{S}),$$

where $\Gamma_f \subset \mathcal{S} \times \overline{\mathcal{M}}$ is the correspondence given by the graph of

$$f \colon \mathcal{S} = \mathcal{V}/G \to \overline{\mathcal{M}},$$

and ${}^t\Gamma_f$ is the transpose correspondence. We denote the restriction to the fiber as $\Gamma_b := \Gamma|_{S_b \times S_b}$.
Looking at the action induced on cohomology by Γ_b, we get

$$(\Gamma_b)_* = 2\,\mathrm{id}_{H^*(S_b)} - (f_b)^*(f_b)_* \colon H^*(S_b, \mathbb{Q}) \to H^*(S_b, \mathbb{Q}).$$

We claim that the action of Γ_b is zero on $H^{2,0}(S_b)$. This is true if and only if $(f_b)^*(f_b)_* = 2\,\mathrm{id}$ on $H^{2,0}(S_b)$.
By [IM79, Lemma 1] $(f_b)^*(f_b)_* = (\Delta_{S_b})_* + \sigma_*$ and the involution σ_* acts as the identity on $H^{2,0}(S_b)$ since $h^{2,0}(\overline{M}_b) = 1 = h^{2,0}(S_b)$. So we get $(f_b)^*(f_b)_* = (\Delta_{S_b})_* + \sigma_* = 2\,\mathrm{id}_{H^{2,0}(S_b)}$ and the claim is proved.
Now we consider the Künneth decomposition of the diagonal of S_b

$$[\Delta(S_b)] = \sum_{i=0}^{4} \left[\pi_i^b\right] = \left[\pi_0^b\right] + \left[\pi_2^b\right] + \left[\pi_4^b\right] \in H^4(S_b \times S_b, \mathbb{Q}),$$

where $\left[\pi_i^b\right] \in H^{4-i}(S_b, \mathbb{Q}) \otimes H^i(S_b, \mathbb{Q}) \subset H^4(S_b \times S_b, \mathbb{Q})$ is the i–th Künneth component. The first and third component are zero, due to the fact that $q(S_b) = h^{1,0}(S_b) = 0$. Since the Künneth conjecture $C(X)$ is known to be true for surfaces ([MNP13, ch. 3.1.1]), we know that the Künneth components are algebraic, i.e. they come from algebraic cycles $\pi_i^b \in CH^2(S_b \times S_b)_{\mathbb{Q}}$.

We recall that the action of π_i^b in cohomology is the identity on $H^i(S_b, \mathbb{Q})$ and it is zero elsewhere ([MNP13, Ch. 6.1]). We are mainly interested in the second component $\pi_2^b = \Delta(S_b) - \pi_0^b - \pi_b^4$, where $\pi_0^b = \{x\} \times S_b$, $\pi_4^b = S_b \times \{x\}$, and x is a point in S_b.

We consider now the composition of correspondences

$$\Psi_b := \Gamma_b \circ \pi_2^b = \left(2\Delta(S_b) -^t \Gamma_{f_b} \circ \Gamma_{f_b}\right) \circ \pi_2^b \in CH^2(S_b \times S_b)_{\mathbb{Q}}. \tag{5.3.3}$$

By definition of π_2^b, when we look at the action in cohomology we have that Ψ_b acts only on $H^2(S_b, \mathbb{Q})$. Moreover, since we proved that the action of Γ_b is zero on $H^{2,0}(S_b)$, we see that $[\Psi_b] \in H^4(S_b \times S_b, \mathbb{Q}) \cap \left(H^{1,1}(S_b) \otimes H^{1,1}(S_b)\right)$.

Now we would like to consider a relative version of the correspondence Ψ_b defined in (5.3.3).
In order to do so, we claim that π_2^b exists also relatively, i.e. there exists $\pi_2^{\mathcal{S}} = \Delta_{\mathcal{S}} - \pi_0 - \pi_4 \in CH^2(\mathcal{S} \times_B \mathcal{S})$ such that, for any $b \in B$, $\pi_2^{\mathcal{S}}|_b = \pi_2^b$ and $\pi_i|_b = \pi_i^b$ for any i. Indeed, let us consider the class of an ample divisor

$h \in CH^1(\mathbb{P}^6)$, and its self–intersection $h^2 = h \cdot h \in CH^2(\mathbb{P}^6)$. Next we consider $h^2 \times B \in CH^2(\mathbb{P}^6 \times B)$, and its restriction to $\mathcal{V} \subset B \times \mathbb{P}^6$, i.e. $\overline{h} := (h^2 \times B)|_{\mathcal{V}} \in CH^2(\mathcal{V})$.
Looking at the fiber, we have that for any point $b \in B$

$$\bar{h}|_{V_b} = x_1 + x_2 + \cdots + x_d \text{ in } CH^2(V_b),$$

where $d = \deg V_b = 16$. So we have that $\bar{h}|_{V_b \times V_b} = dx_1$ in $H^4(V_b)$ (since $H^4(V_b) \cong \mathbb{Q}$, so all points have the same class in $H^4(V_b)$). Then we define

$$\pi_0^{\mathcal{V}} := \frac{1}{d} pr_1^* \left(\overline{h}|_{\mathcal{V}}\right) \in CH^2(\mathcal{V} \times_B \mathcal{V});$$
$$\pi_4^{\mathcal{V}} := \frac{1}{d} pr_2^* \left(\overline{h}|_{\mathcal{V}}\right) \in CH^2(\mathcal{V} \times_B \mathcal{V});$$

where pr_1, pr_2 are the projections in the fiber product

$$\begin{array}{ccc} \mathcal{V} \times_B \mathcal{V} & \xrightarrow{pr_2} & \mathcal{V} \\ {\scriptstyle pr_1}\downarrow & & \downarrow \\ \mathcal{V} & \longrightarrow & B. \end{array}$$

When we restrict to each fiber and we pass to cohomology, by the Künneth decomposition, we have

$$[\pi_0^{\mathcal{V}}]|_{V_b} = [p] \times [V_b] \in H^4(V_b, \mathbb{Q}) \otimes H^0(V_b, \mathbb{Q}),$$
$$[\pi_4^{\mathcal{V}}]|_{V_b} = [V_b] \times [p] \in H^0(V_b, \mathbb{Q}) \otimes H^4(V_b, \mathbb{Q});$$

where $p \in V_b$ is a point. So we can define the relative Künneth component of the diagonal $\pi_2^{\mathcal{V}} = \Delta_{\mathcal{V}} - \pi_0^{\mathcal{V}} - \pi_4^{\mathcal{V}} \in CH^2(\mathcal{V} \times_B \mathcal{V})$. The we can use the push–forward of $\mathcal{V} \xrightarrow{q} \mathcal{S} = \mathcal{V}/G$ to get the relative Künneth component of $\mathcal{S}$, namely $\pi_2^{\mathcal{S}} = \Delta_{\mathcal{S}} - q_*\pi_0^{\mathcal{V}} - q_*\pi_4^{\mathcal{V}}$.

So we can consider also the relative correspondence

$$\Psi := \Gamma \circ \pi_2^{\mathcal{S}} = (2\Delta_{\mathcal{S}} -{}^t\Gamma_f \circ \Gamma_f) \circ \pi_2^{\mathcal{S}} \in CH^2(\mathcal{S} \times_B \mathcal{S})_{\mathbb{Q}},$$

where $\Psi|_b = \Psi_b$ defined in (5.3.3).
By applying Lefschetz Theorem A.2.14 on $(1,1)$ classes ([Huy05, Proposition 3.3.2]) on S_b for a general $b \in B$, we get that there exists a divisor $Y_b \subset S_b$ and a cycle $\gamma_b \in CH^2(S_b \times S_b)_{\mathbb{Q}}$ such that $\mathrm{Supp}(\gamma_b) \subseteq Y_b \times Y_b$, and

$$[\Psi_b] = [\gamma_b] \in H^4(S_b \times S_b, \mathbb{Q}).$$

By means of Voisin's "spreading of cycles" (Proposition4.2.4,[Voi13, Proposition 2.7]), we can see that γ_b exists relatively. More precisely, there exists a divisor $\mathcal{Y} \subset \mathcal{S}$ and a cycle $\gamma \in CH^2(\mathcal{S} \times_B \mathcal{S})_{\mathbb{Q}}$ supported on $\mathcal{Y} \times_B \mathcal{Y}$ such that

$$[\Psi_b] = [\gamma|_b] \in H^4(S_b \times S_b, \mathbb{Q}).$$

Finally we can define the correspondence

$$\Psi' := \Psi - \gamma = (2\Delta_{\mathcal{S}} - {}^t\Gamma_f \circ \Gamma_f) \circ \pi_2^{\mathcal{S}} - \gamma \in CH^2(\mathcal{S} \times_B \mathcal{S})_{\mathbb{Q}}.$$

Then Ψ' has the desired property of being homologically trivial when restricted to any fiber, i.e. for any $b \in B$

$$\left[\Psi'|_b\right] = [\Psi_b] - [\gamma|_b] = 0 \in H^4(S_b \times S_b, \mathbb{Q}).$$

Now we want to apply the Leray spectral sequence argument as in [Lat18a, proof of Theorem 3.1]. In order to do this, we apply Voisin's lemmas 4.2.5, 4.2.10.

So, we have

$$\left[\Psi'\right] = \beta_1|_{\mathcal{S}\times_B\mathcal{S}} + \beta_2|_{\mathcal{S}\times_B\mathcal{S}} = [\alpha_1]\,|_{\mathcal{S}\times_B\mathcal{S}} + [\alpha_2]\,|_{\mathcal{S}\times_B\mathcal{S}}$$

with $\beta_i = [\alpha_i]\,|_{\mathcal{S}\times_B\mathcal{S}}$ and $\alpha_i \in CH^2(B \times (\mathbb{P}^6/G) \times (\mathbb{P}^6/G))$. We can define

$$\left[\Psi''\right] = \left[\Psi'\right] - ([\alpha_1] + [\alpha_2])\,|_{\mathcal{S}\times_B\mathcal{S}} = 0 \in H^4(\mathcal{S} \times_B \mathcal{S}, \mathbb{Q}).$$

We notice that $[\Psi'']$ is algebraic because it's the difference between algebraic cycles, so $\Psi'' \in CH^2_{\text{hom}}(\mathcal{S} \times_B \mathcal{S})_{\mathbb{Q}} = 0$, where the last equality holds by Proposition 5.2.3.

Then we have that

$$\Psi'' = 0 \text{ in } CH^2_{\text{hom}}(\mathcal{S} \times_B \mathcal{S})_{\mathbb{Q}},$$
$$\Psi' = (2\Delta_{\mathcal{S}} - {}^t\Gamma_f \circ \Gamma_f) \circ \pi_2^{\mathcal{S}} - \gamma = (\alpha_1 + \alpha_2)|_{\mathcal{S}\times_B\mathcal{S}} \text{ in } CH^2_{\text{hom}}(\mathcal{S} \times_B \mathcal{S})_{\mathbb{Q}}.$$

When we restrict to each fiber, and we look at the action on cycles, we get $\forall b \in B$:

$$\begin{aligned} 2\,\mathrm{id}_* =& (2\Delta_{S_b} \circ \pi_2^b)_* \colon CH^2_{\text{hom}}(S_b)_{\mathbb{Q}} \to CH^2_{\text{hom}}(S_b)_{\mathbb{Q}} \\ =& (f_b)^*(f_b)_*(\pi_2^b)_* + (\gamma_b)_* + (\alpha_1 + \alpha_2)|_{S_b\times S_b} \\ =& (f_b)^*(f_b)_* + (\gamma_b)_* + (\alpha_1 + \alpha_2)|_{S_b\times S_b}, \end{aligned}$$

where last equality holds since π_2^b acts as the identity on $CH^2_{\text{hom}}(S_b)_{\mathbb{Q}}$. We recall that γ_b is supported on a divisor, hence it does not act on 0–cycles and $\alpha_1 + \alpha_2 \in CH^2(B \times (\mathbb{P}^6/G) \times (\mathbb{P}^6/G))$. So on the right the only term that acts on 0–cycles is $(f_b)^*(f_b)_*$. We get

$$(f_b)^*(f_b)_* = 2\,\mathrm{id}_* \colon CH^2_{\text{hom}}(S_b)_{\mathbb{Q}} \to CH^2_{\text{hom}}(S_b)_{\mathbb{Q}} \quad \text{where} \quad \begin{array}{c} \sigma \curvearrowright S_b = V_b/G \\ {\scriptstyle f_b}\Big\downarrow{\scriptstyle 2:1} \\ \overline{M}_b. \end{array}$$

Then we conclude that $CH^2_{\text{hom}}(S)_{\mathbb{Q}} \cong CH^2_{\text{hom}}(\overline{M})_{\mathbb{Q}}$. □

Proposition 5.3.4. $CH^2_{\text{hom}}(S)_{\mathbb{Q}} \cong CH^2_{\text{hom}}(\overline{M})_{\mathbb{Q}} \cong CH^2_{\text{hom}}(M)_{\mathbb{Q}}$

Proof. We have the following situation

$$\begin{array}{ccc} E & \hookrightarrow & M \\ \downarrow & & \downarrow{\scriptstyle ros} \\ \overline{E} & \hookrightarrow & \overline{M}, \end{array}$$

where E is the exceptional locus in M whose image is the singular locus $\overline{E}$ in $\overline{M}$. Then, by [Kim92], we get the following exact sequence

$$0 \to CH^2(\overline{M})_{\mathbb{Q}} \to CH^2(M)_{\mathbb{Q}} \oplus CH^2(\overline{E})_{\mathbb{Q}} \to CH^2(E)_{\mathbb{Q}} \to 0.$$

We have that $CH^2(\overline{E})_{\mathbb{Q}} = 0$, since we have only quotient singularities. So $\overline{E}$ is just a bunch of points, and $CH^2(E)_{\mathbb{Q}} = 0$ too, since E is a bunch of curves. So our claim is proved. □

As a corollary, we get then Theorem 5.3.5, i.e. that Conjecture 3.4.1 is true for the family of Todorov surfaces of type $(2, 12)$ we consider. The proof follows the one given in [Lat18a, Corollary 3.2].

Theorem 5.3.5. *Let S be a Todorov surface with fundamental invariants $(\alpha, k) = (2, 12)$.*
Then Conjecture 3.4.1 is true for S.

Proof. First of all, we notice that it is enough to prove the theorem with rational coefficients. Indeed, by Rojtman's Theorem ([Roj80]) there is no torsion in $CH^4_{\text{hom}}(S \times S)$.
Let M be the associated $K3$ surface to S, i.e. the minimal resolution of S/σ. We have a commutative diagram:

$$\begin{array}{ccc} CH^2_{\text{hom}}(S)_{\mathbb{Q}} \otimes CH^2_{\text{hom}}(S)_{\mathbb{Q}} & \longrightarrow & CH^4_{\text{hom}}(S \times S)_{\mathbb{Q}} \\ \uparrow & & \uparrow \\ CH^2_{\text{hom}}(M)_{\mathbb{Q}} \otimes CH^2_{\text{hom}}(M)_{\mathbb{Q}} & \longrightarrow & CH^4_{\text{hom}}(M \times M)_{\mathbb{Q}}. \end{array}$$

By Theorem 5.3.2, the left vertical arrow is an isomorphism. We recall that by Rito's result (Theorem 5.0.2) the K3 surface can be described as the blow-up of a double cover of $\mathbb{P}^2$ ramified along the union of two cubics. By Theorem 3.4.10 ([Voi96, Theorem 3.4]), Conjecture 3.4.1 is then true for M, i.e.

$$a \times a' = a' \times a \in CH^4(M \times M) \quad \forall a, a' \in CH^2_{\text{hom}}(M).$$

Hence the conjecture holds for S too. □

Remark 5.3.6. We notice that Theorem 5.3.5 holds for all Todorov surfaces of type (2,12). Indeed, by Lemma 4.2.3 ([Voi14b, Lemma 3.2]), it is enough to prove Voisin's conjecture 3.4.1 for the general (or even the very general by [Voi14b, Remark 3.3]) member of the family of Todorov surfaces of type (2,12). The upshot of this approach is that for the general member of this family we have an explicit description in terms of complete intersections and quotients by Theorem 5.1.20.

5.4 Motivic consequences

Here we present the motivic version of Theorem 5.3.5 with some interesting corollaries. The central result is that a Todorov surface of type $(2,12)$ has the transcendental part of the motive isomorphic to the associated K3 surface's one, in the sense of [KMP07] (see Section 2.1.1). The proof is directly inspired by Laterveer's work [Lat18a].

In order to study the groups of 0–cycles $CH_0(S)$, Bloch's conjecture suggests that the interesting part of the Chow–Künneth decomposition is $h_2(S) = (S, \pi_2, 0)$ where $\pi_2 = \Delta_S - \pi_0 - \pi_1 - \pi_3 - \pi_4$. To study this summand we use a further decomposition due to Kahn–Murre–Pedrini [KMP07, Proposition 2.3].

Proposition 5.4.1 (Kahn–Murre–Pedrini)**.** *Let S be a smooth projective surface with a Chow–Künneth decomposition as in Definition 2.2.1. There there is a unique splitting in orthogonal projectors*

$$\pi_2 = \pi_2^{\mathrm{alg}} + \pi_2^{\mathrm{tr}} \text{ in } CH^2(S \times S)_{\mathbb{Q}}.$$

This gives an induced decomposition on the motive

$$h_2(S) \cong h_2^{\mathrm{alg}} \oplus t_2(S) \text{ in } \mathcal{M}_{rat},$$

where $h_2^{\mathrm{alg}}(S) = (S, \pi_2^{\mathrm{alg}}, 0)$, $t_2(S, \pi_2^{\mathrm{tr}}, 0)$ and in cohomology we get

$$H^*(t_2(S), \mathbb{Q}) = H^2_{\mathrm{tr}}(S), \quad H^*(h_2^{\mathrm{alg}}(S), \mathbb{Q}) = NS(S)_{\mathbb{Q}},$$

where the transcendental cohomology $H^2_{\mathrm{tr}}(S)$ is defined as the orthogonal complement of the Néron–Severi group $NS(S)_{\mathbb{Q}}$ in $H^2(S, \mathbb{Q})$. The component $t^2(S)$ is called the transcendental part of the motive *of S.*
Moreover, we have that $CH^*(t_2(S))_{\mathbb{Q}} = CH^2_{AJ}(S)_{\mathbb{Q}}$.

Theorem 5.4.2. *Let S be a Todorov surface of type $(2,12)$, and let M be the K3 surface associated to S, i.e. the minimal resolution of S/σ (see Definition 5.0.1). Then there is an isomorphism of Chow motives*

$$t_2(S) \cong t_2(M) \text{ in } \mathcal{M}_{rat}.$$

Proof. In Section 5.1.1 we give an explicit description of the general member of the family if Todorov surfaces of type $(2, 12)$. We recall that in this description we have the following situation fiberwise

$$\begin{tikzcd} & G \curvearrowright V_b \arrow[d, "q_b"', "4:1"] \\ & \sigma \curvearrowright S_b = V_b/G \arrow[d, "f_b"', "2:1"] \\ M_b \arrow[r, "ros"] & \overline{M_b}. \end{tikzcd}$$

The idea is to prove the theorem for the general member S of the family of Todorov surfaces of type (2,12) using its explicit description by Theorem 5.1.20. By Remark 5.3.6 and Lemma 4.2.3 ([Voi14b, Lemma 3.2]) this is enough to conclude.

So by Theorem 5.1.20 we have that $S \cong S_b$ for some $b \in B$ and $M \cong M_b$. Let us consider now the Chow–Künneth decomposition $\{\pi_0^{S_b}, \pi_2^{S_b}, \pi_4^{S_b}\}$for S_b and $\{\pi_0^{M_b}, \pi_2^{M_b}, \pi_4^{M_b}\}$ for M_b, as in Definition 2.2.1. Then Proposition 5.4.1 gives a further decomposition in the algebraic and the transcendental part of the second component:

$$\pi_2^{S_b} = \pi_2^{S_b,\mathrm{alg}} + \pi_2^{S_b,\mathrm{tr}} \text{ and } \pi_2^{M_b} = \pi_2^{M_b,\mathrm{alg}} + \pi_2^{M_b,\mathrm{tr}}.$$

Let us consider now the following correspondence constructed in the proof of Theorem 5.3.2:

$$2\Delta_{S_b} \circ \pi_2^{S_b} =^t \Gamma_b \circ \Gamma_b \circ \pi_2^{S_b} + \gamma_b + (\alpha_1 + \alpha_2)|_{S_b \times S_b} \in CH^2_{\mathrm{hom}}(S_b \times S_b)_{\mathbb{Q}},$$

where Γ_b is the graph of f_b and ${}^t\Gamma_b$ is its transpose. We apply to this the composition with the correspondence $\pi_2^{S_b,\mathrm{tr}}$ on both sides:

$$\begin{aligned} 2\pi_2^{S_b,\mathrm{tr}} &= \pi_2^{S_b,\mathrm{tr}} \circ 2\Delta_{S_b} \circ \pi_2^{S_b} \circ \pi_2^{S_b,\mathrm{tr}} \\ &= \pi_2^{S_b,\mathrm{tr}} \circ \left({}^t\Gamma_b \circ \Gamma_b \circ \pi_2^{S_b} + \gamma_b + (\alpha_1 + \alpha_2)|_{S_b \times S_b}\right) \circ \pi_2^{S_b,\mathrm{tr}} \\ &= \left(\pi_2^{S_b,\mathrm{tr}} \circ^t \Gamma_b \circ \Gamma_b \circ \pi_2^{S_b} \circ \pi_2^{S_b,\mathrm{tr}}\right) + \left(\pi_2^{S_b,\mathrm{tr}} \circ \gamma_b \circ \pi_2^{S_b,\mathrm{tr}}\right) \\ &\qquad + \left(\pi_2^{S_b,\mathrm{tr}} \circ (\alpha_1 + \alpha_2)|_{S_b \times S_b} \circ \pi_2^{S_b,\mathrm{tr}}\right). \end{aligned} \tag{5.4.3}$$

We recall that γ_b is supported on $Y_b \times Y_b$ where $Y_b \subset S_b$ is a divisor, so γ_b is in the "irrelevant ideal" $\mathcal{J}(S_b \times S_b)$ which is generated by the classes of correspondences in $CH^2(S_b \times S_b)$ that are not dominant over S_b by the projections on the first or on the second factor (see [KMP07, Definition 4.2]). By [KMP07, Theorem 4.3] we can define a homomorphism

$$\begin{aligned} \phi\colon CH^2(S_b \times S_b) &\to \mathcal{M}_{rat}(t_2(S_b), t_2(S_b)) \\ Z &\mapsto \quad \pi_2^{S_b,\mathrm{tr}} \circ Z \circ \pi_2^{S_b,\mathrm{tr}} \end{aligned}$$

whose kernel is precisely $\mathcal{J}(S_b \times S_b)$, i.e there is an induced isomorphism

$$\bar{\phi}\colon \frac{CH^2(S_b \times S_b)}{\mathcal{J}(S_b \times S_b)} \simeq \mathcal{M}_{rat}(t_2(S_b), t_2(S_b)).$$

In particular, this shows that $\pi_2^{S_b,\mathrm{tr}} \circ \gamma_b \circ \pi_2^{S_b,\mathrm{tr}} = 0$ in $\mathcal{M}_{rat}(t_2(S_b), t_2(S_b))$.

Next we recall that $\alpha_i \in CH^2(B \times \mathbb{P}^6 \times \mathbb{P}^6)$. So we can write

$$(\alpha_1 + \alpha_2)|_{S_b \times S_b} = \sum_{i,j} D_i \times D_j = \left(\sum_{i,j} D_i \times D_j \right) \circ \pi_2^{S_b,\mathrm{alg}},$$

where $D_i, D_j \subset S_b$ are divisors and the last equality holds since $\pi_2^{S_b,\mathrm{alg}}$ is a projector on the Neron–Severi group $NS(S_b)_{\mathbb{Q}}$. Being $\pi_2^{S_b,\mathrm{alg}}$ and $\pi_2^{S_b,\mathrm{tr}}$ orthogonal we conclude that

$$\left(\pi_2^{S_b,\mathrm{tr}} \circ (\alpha_1 + \alpha_2)|_{S_b \times S_b} \circ \pi_2^{S_b,\mathrm{tr}}\right) = \left(\pi_2^{S_b,\mathrm{tr}} \circ \left(\sum_{i,j} D_i \times D_j \right) \circ \pi_2^{S_b,\mathrm{alg}} \circ \pi_2^{S_b,\mathrm{tr}} \right) = 0.$$

So in (5.4.3) the only summand that survives on the left is the first one, and we get

$$2\pi_2^{S_b,\mathrm{tr}} = \pi_2^{S_b,\mathrm{tr}} \circ^t \Gamma_b \circ \Gamma_b \circ \pi_2^{S_b} \circ \pi_2^{S_b,\mathrm{tr}} = \pi_2^{S_b,\mathrm{tr}} \circ^t \Gamma_b \circ \Gamma_b \circ \pi_2^{S_b,\mathrm{tr}}, \tag{5.4.4}$$

where last equality holds since $\pi_2^{S_b} = \pi_2^{S_b,\mathrm{alg}} + \pi_2^{S_b,\mathrm{tr}}$ and $\pi_2^{S_b,\mathrm{alg}}, \pi_2^{S_b,\mathrm{tr}}$ are orthogonal. Next we claim that

$$2\pi_2^{S_b,\mathrm{tr}} = \pi_2^{S_b,\mathrm{tr}} \circ^t \Gamma_b \circ \pi_2^{M_b,\mathrm{tr}} \circ \Gamma_b \circ \pi_2^{S_b,\mathrm{tr}} \text{ in } \mathcal{M}_{rat}(t_2(S_b), t_2(S_b)). \tag{5.4.5}$$

To prove the claim we recall that $\pi_2^{M_b,\mathrm{alg}}$ and $\pi_2^{M_b,\mathrm{tr}}$ are orthogonal and $\pi_2^{M_b} = \pi_2^{M_b,\mathrm{alg}} + \pi_2^{M_b,\mathrm{tr}}$, thus we get

$$\begin{aligned}
\pi_2^{S_b,\mathrm{tr}} \circ^t \Gamma_b \circ \pi_2^{M_b,\mathrm{tr}} \circ \Gamma_b \circ \pi_2^{S_b,\mathrm{tr}} &= \pi_2^{S_b,\mathrm{tr}} \circ^t \Gamma_b \circ \pi_2^{M_b} \circ \Gamma_b \circ \pi_2^{S_b,\mathrm{tr}} \\
&= \pi_2^{S_b,\mathrm{tr}} \circ^t \Gamma_b \circ \left(\Delta_{M_b} - \pi_0^{M_b} - \pi_4^{M_b}\right) \circ \Gamma_b \circ \pi_2^{S_b,\mathrm{tr}} \\
&= \pi_2^{S_b,\mathrm{tr}} \circ^t \Gamma_b \circ \Delta_{M_b} \circ \Gamma_b \circ \pi_2^{S_b,\mathrm{tr}} = \pi_2^{S_b,\mathrm{tr}} \circ^t \Gamma_b \circ \Gamma_b \circ \pi_2^{S_b,\mathrm{tr}},
\end{aligned}$$

where the last equalities follow from Theorem 2.2.4. Then we conclude the proof of the claim by means of (5.4.4).
Now we want to prove that, analogously, there is a rational equivalence of cycles

$$2\pi_2^{M_b,\mathrm{tr}} = \pi_2^{M_b,\mathrm{tr}} \circ \Gamma_b \circ \pi_2^{S_b,\mathrm{tr}} \circ^t \Gamma_b \circ \pi_2^{M_b,\mathrm{tr}} \text{ in } CH^2(M_b \times M_b)_{\mathbb{Q}}. \tag{5.4.6}$$

This follows easily since

$$2\Delta_{M_b} = \Gamma_b \circ^t \Gamma_b \text{ in } CH^2(M_b \times M_b)_{\mathbb{Q}}.$$

So applying twice on both sides $\pi_2^{M_b,\mathrm{tr}}$ we get:

$$\begin{aligned}
2\pi_2^{M_b,\mathrm{tr}} &= \pi_2^{M_b,\mathrm{tr}} \circ \Gamma_b \circ^t \Gamma_b \circ \pi_2^{M_b,\mathrm{tr}} \\
&= \pi_2^{M_b,\mathrm{tr}} \circ \Gamma_b \circ^t \circ \Delta_{S_b} \circ \Gamma_b \circ \pi_2^{M_b,\mathrm{tr}} \\
&= \pi_2^{M_b,\mathrm{tr}} \circ \Gamma_b \circ^t \circ \left(\Delta_{S_b} - \pi_0^{S_b} - \pi_4^{S_b}\right) \circ \Gamma_b \circ \pi_2^{M_b,\mathrm{tr}} \\
&= \pi_2^{M_b,\mathrm{tr}} \circ \Gamma_b \circ^t \circ \left(\pi_2^{S_b,\mathrm{alg}} + \pi_2^{S_b,\mathrm{tr}}\right) \circ \Gamma_b \circ \pi_2^{M_b,\mathrm{tr}} \\
&= \pi_2^{M_b,\mathrm{tr}} \circ \Gamma_b \circ \pi_2^{S_b,\mathrm{tr}} \circ^t \Gamma_b \circ \pi_2^{M_b,\mathrm{tr}}.
\end{aligned}$$

By (5.4.4) and (5.4.6), we conclude that $\Gamma_b\colon t_2(S_b) \to t_2(M_b)$ in $\mathcal{M}_{rat}$ is an isomorphism of motives, and its inverse is its transpose ${}^t\Gamma_b$. Since S_b is birational to S and M_b is birational to M, and since the transcendental part of the motive is a birational invariant, we conclude that there is also an isomorphism of motives $t_2(S) \cong t_2(M)$ in $\mathcal{M}_{rat}$. □

We present some corollaries of Theorem 5.4.2. First, we recall the definition of isogeny and a useful results of Huybrechts [Huy19].

Definition 5.4.7. *Let S and S' be two surfaces, we say that S and S' are* isogenous *if there exists a Hodge isometry $\varphi\colon H^2(S,\mathbb{Q}) \xrightarrow{\sim} H^2(S',\mathbb{Q})$, i.e. φ is a isomorphism of $\mathbb{Q}$–vector spaces which is compatible with the Hodge structure and the cup product on both sides*[1]*.*
In particular, this means that there is a Hodge isometry on the transcendental cohomology $H^2_{\mathrm{tr}}(S) \cong H^2_{\mathrm{tr}}(S')$ and on the algebraic one $H^2_{\mathrm{alg}}(S) \cong H^2_{\mathrm{alg}}(S')$.

Theorem 5.4.8 (Motivic Šafarevič conjecture, Theorem 0.2 [Huy19])**.** *Any Hodge isometry $H^2(S,\mathbb{Q}) \cong H^2(S',\mathbb{Q})$ between two complex projective K3 surfaces can be lifted to an isomorphism of Chow motives $h(S) \cong h(S')$. In particular, two isogenous K3 surfaces have isomorphic Chow motives.*

For the proof of this result we refer the reader to [Huy19].

Corollary 5.4.9. *Let S, S' be two isogenous Todorov surfaces of type* $(2,12)$*, then they have isomorphic Chow motives, i.e.*

$$h(S) \cong h(S') \textit{ in } \mathcal{M}_{rat}.$$

Proof. Let us denote by $\overline{M}, \overline{M}'$ the singular K3 surfaces associated to S and S' respectively, and by M, M' their resolutions of singularities. Then we have an isogeny given by the pullback $H^2_{\mathrm{tr}}(S) \cong H^2_{\mathrm{tr}}(\overline{M})$, since S is a double

[1] For a discussion on the meaning and different uses of the term "isogenous" see [Mor87].

cover of $\overline{M}$. Another isogeny is given by the pullback $H^2_{\mathrm{tr}}(M) \cong H^2_{\mathrm{tr}}(\overline{M})$, since transcendental cohomology is invariant when resolving quotient singularities (see [Lat18a, proof of Lemma 3.3]). By Theorem 5.4.2, since $H^*(t_2(S), \mathbb{Q}) = H^2_{\mathrm{tr}}(S)$ and $H^*(t_2(M), \mathbb{Q}) = H^2_{\mathrm{tr}}(M)$, we have also an isomorphism $H^2_{\mathrm{tr}}(S) \cong H^2_{\mathrm{tr}}(M)$. In particular, this isomorphism is compatible with the Hodge structure, since it comes from a correspondence, and it is compatible with the cup product. Thus we get also a Hodge isometry $H^2_{\mathrm{tr}}(M) \cong H^2_{\mathrm{tr}}(M')$. By Huybrechts theorem 5.4.8, we have that this Hodge isometry can be lifted to an isomorphism of Chow motives, i.e. $h(M) \cong h(M')$. In particular, we get an isomorphism on the transcendental part of the motives $t_2(M) \cong t_2(M')$. Then, by Theorem 5.4.2, we get an isomorphism of motives $t_2(S) \cong t_2(S')$ and we conclude that $h(S) \cong h(S')$ in $\mathcal{M}_{rat}$. □

Corollary 5.4.10. *Let S be a Todorov surface of type $(2, 12)$. Assume that P is a K3 surface such that there is a Hodge isometry $H^2_{\mathrm{tr}}(S) \cong H^2_{\mathrm{tr}}(P)$. Then, there is an isomorphism of Chow motives*

$$t_2(S) \cong t_2(P) \text{ in } \mathcal{M}_{rat}.$$

Proof. Let M be the K3 surface associated to S, then by Theorem 5.4.2 we have an isomorphism $H^2_{\mathrm{tr}}(S) \cong H^2_{\mathrm{tr}}(M)$. As we noticed in the proof of Corollary 5.4.9, this isomorphism is compatible with Hodge structure and cup product and so there is also a Hodge isometry $H^2_{\mathrm{tr}}(M) \cong H^2_{\mathrm{tr}}(M')$. Applying Huybrechts theorem 5.4.8, we can lift this isometry to an isomorphism of motives $t_2(M) \cong t_2(P)$ in $\mathcal{M}_{rat}$. By Theorem 5.4.2 we conclude that $t_2(S) \cong t_2(M) \cong t_2(P)$ in $\mathcal{M}_{rat}$. □

Corollary 5.4.11. *Let S be a Todorov surface of type $(2, 12)$ with very high Picard number, i.e. $\rho(S) \geq h^{1,1}(S) - 1$. Then S has finite dimensional motive.*

Proof. By [KMP07, Lemma 7.6.6], the motives $h_0(S), h_4(S), h_2^{\mathrm{alg}}(S)$ are finite–dimensional, hence all the summands of the Chow motive $h(S)$ are finite–dimensional except perhaps $t_2(S)$.
Since a direct sum of finite–dimensional motives is finite–dimensional (see Proposition 2.1.21), it is enough to prove that $t_2(S)$ is finite–dimensional.
Let M be the K3 surface associated to S. By Theorem 5.4.2 we have $t_2(S) \cong t_2(M)$, and so it suffices to show that $t_2(M)$ is finite–dimensional.
We recall that the Picard number of S, $\rho(S)$, is the rank of the Neron–Severi group $NS(S)_{\mathbb{Q}}$, and $\dim H^2_{\mathrm{tr}}(S) = b_2(S) - \rho(S) = 2 - \rho(S) \leq 3 - h^{1,1}(S) \leq 3$, since by hypothesis $\rho(S) \geq h^{1,1}(S) - 1$.
By the isomorphism $H^2_{\mathrm{tr}}(S) \cong H^2_{\mathrm{tr}}(M)$, we get that $\rho(M) \geq H^2_{\mathrm{tr}}(M) - 3 = b_2(M) - 3 = 19$. Since M has a large Picard number, it has finite dimensional motive [Ped12, Theorem 2]. □

Appendix A

Preliminary notions

The aim of this Appendix is to introduce the main objects and tools we need to address our problem. For a deeper and more detailed discussion on these subjects we refer the reader to the references given in each section.

A.1 Basics

A.1.1 Singular homology

Let X be an oriented compact differentiable variety of dimension n. We can associate to X two different kind of groups, homology and cohomology groups. These two algebraic objects read informations about topology and the differentiable structure of X respectively.
Let us briefly introduce the homology groups of X. For a deeper presentation of this topic, we refer to [Ara12, §7].

The *standard* $n-$*simplex* is the set

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for any } i \geq 0 \right\}.$$

The ith face of the standard $n-$simplex is denoted by Δ^n_i, and it is defined as the intersection of Δ^n with the hyperplane $t_i = 0$. Moreover, each face of an $n-$simplex is homeomorphic to a $(n-1)-$ simplex via the affine map $\delta_i \colon \Delta^{n-1} \to \Delta^n_i$. If $\{t_0, \dots, t_n\}$ are the vertices that define a $n-$simplex, then the ith face is given by

$$\delta_i(\{t_0, \dots, t_n\}) = \{t_0, \dots, \hat{t_i}, \dots, t_n\}.$$

By gluing simplices we can define different kinds of topological spaces. In particular, we recall that every manifold and algebraic variety can be triangulated, so it can be constructed in this fancy way.

Let X be a topological space. A *singular $n-$simplex* is defined to be a continuous map $\sigma\colon \Delta^n \to X$.
We denote as $C_n(X,\mathbb{Z})$ the free abelian group generated by the singular $n-$simplices of X, and for $n < 0$ we set $C_n(X,\mathbb{Z}) = \{\underline{0}\}$. Elements of $C_n(X,\mathbb{Z})$ are called *singular $n-$chains* and are finite formal sums of type $\sum_i g_i\sigma_i$ with $g_i \in \mathbb{Z}$ and $\sigma_i\colon \Delta^n \to X$ singular $n-$simplex for every i. Given a singular $n-$simplex σ, we can identify it with its set of vertices $\{v_0, \dots, v_n\}$, where the v_i are the $0-$simplices of σ, i.e. $v_i\colon \Delta^0 \to X$.
The *boundary map* $\partial_n\colon C_n(X,\mathbb{Z}) \to C_{n-1}(X,\mathbb{Z})$ is defined as in the simplicial case:

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0,\dots,\hat{v_i},\dots,v_n]}.$$

It is easy to verify that $\partial_{n-1}\circ\partial_n = 0$. Thus we can define the chain complex:

$$\dots C_n(X,\mathbb{Z}) \xrightarrow{\partial_n} C_{n-1}(X,\mathbb{Z}) \to \dots C_1(X,\mathbb{Z}) \xrightarrow{\partial_1} C_0(X,\mathbb{Z}) \xrightarrow{\partial_0} 0. \quad \text{(A.1.1)}$$

We use the following notation:

$$B_n(X,\mathbb{Z}) = \operatorname{Im}\partial_{n+1} \qquad \text{and} \qquad Z_n(X,\mathbb{Z}) = \ker\partial_n.$$

Elements in $Z_n(X,\mathbb{Z})$ are called *$n-$cycles*, elements in $B_n(X,\mathbb{Z})$ are called *$n-$boundaries*. By the property of the boundary operator, we have that $B_n(X,\mathbb{Z}) \subseteq Z_n(X,\mathbb{Z})$, hence the following definition is natural.

Definition A.1.2. *The n^{th}* singular homology group *of X is the quotient group*

$$H_n(X,\mathbb{Z}) = \frac{Z_n(X,\mathbb{Z})}{B_n(X,\mathbb{Z})}.$$

By applying the tensor operation, we can extend this definition to any commutative ring R, that usually will be $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, and we use the notation

$$C_n(X,R) = R \otimes_{\mathbb{Z}} C_n(X,\mathbb{Z}).$$

By extending the boundary operators in the natural way, we can extend the previous definitions in the general case. Indeed, by applying the functor $R \otimes_{\mathbb{Z}} \cdot$, we get the following complex:

$$\dots C_n(X,R) \xrightarrow{\partial_n} C_{n-1}(X,R) \to \dots C_1(X,R) \xrightarrow{\partial_1} C_0(X,R) \xrightarrow{\partial_0} 0. \quad \text{(A.1.3)}$$

We can extend these definitions, by defining the *singular $n-cycles$* and the *singular $n-$boundaries* in X with coefficients in R, they are respectively elements of $Z_n(X,R) = \ker\partial_n$ and $B_n(X,R) = \operatorname{Im}\partial_{n+1}$. These are both subgroups of $C_n(X,R)$.

We define the *singular homology* of X with coefficients in R as the homology of the complex (A.1.3), i.e. the nth singular homology group of X with coefficients in R is

$$H_n(X,R) = \frac{Z_n(X,R)}{B_n(X,R)}.$$

Next, we briefly recall some basic results about singular homology. For further details we refer the reader to [Kos80, §29].

Lemma A.1.4. *If X is a non-empty path–connected topological space, then*

$$H_0(X,\mathbb{Z}) \cong \mathbb{Z}.$$

Functorial properties

If we consider a continuous map between topological spaces $f\colon X \to Y$, then we can define the induced morphism between singular complexes

$$C_n(f)\colon C_n(X,R) \to C_n(Y,R)$$

by setting

$$C_n(f)\left(\sum_{i=0}^{n} g_i\sigma_i\right) = \sum_{i=o}^{n} g_i\left(f\circ\sigma_i\right).$$

It is easy to see that such a map gives a morphism between complexes of chains, i.e. the following diagram commutes $\forall n \in \mathbb{Z}$

$$\begin{array}{ccc}
C_n(X,R) & \xrightarrow{C_n(f)} & C_n(Y,R) \\
{\scriptstyle\partial_n}\big\downarrow & & {\scriptstyle\partial_n}\big\downarrow \\
C_{n-1}(X,R) & \xrightarrow{C_{n-1}(f)} & C_{n-1}(Y,R).
\end{array}$$

The following lemma summarizes the basic properties of this morphism.

Lemma A.1.5. *Let $f\colon X \to Y$, $g\colon Y \to Z$ be continuous maps between topological spaces, and let R be a commutative ring, then*

(i) $\partial_n \circ C_n(f) = C_{n-1}(f)\circ\partial_n$*;*

(ii) $C_n(\mathrm{id}_X) = \mathrm{id}_{C_n(X,R)}$*;*

(iii) $C_n(g\circ f) = C_n(g)\circ C_n(f)$*.*

Definition A.1.6. *Given $f\colon X \to Y$, continuous map between topological spaces, we define the* induced morphisms in homology

$$H_n(f)\colon H_n(X,R) \to H_n(Y,R)$$

by setting

$$H_n(f)\left(\left[\sum_{i=0}^{n} g_i\sigma_i\right]\right) = \left[C_n(f)\left(\sum_{i=0}^{n} g_i\sigma_i\right)\right] = \left[\sum_{i=o}^{n} g_i\left(f\circ\sigma_i\right)\right].$$

From Lemma A.1.5, one easily checks that $H_n(\mathrm{id}_X) = \mathrm{id}_{H_n(X,R)}$. and $H_n(g \circ f) = H_n(g) \circ H_n(f)$, so the following result follows.

Theorem A.1.7. *H_n is a covariant functor from the category of the topological spaces to the category of $R-$modules.*

Usually we will denote the induced morphism in homology as

$$f_*\colon H_\bullet(X, R) \to H_\bullet(Y, R).$$

We state now a fundamental result of homology theory and we refer the reader to [Hat02, Theorem 2.10] for its proof.

Theorem A.1.8. *If two continuous maps $f, g\colon X \to Y$ are homotopic, then they induced the same homomorphism in the homology groups, i.e. $f_* = g_*$.*

By means of this theorem and the functorial properties, we get the homotopical invariance of the homology groups.

Corollary A.1.9. *If $f\colon X \to Y$ is an homotopy equivalence, then the induced maps $f_*\colon H_n(X, R) \to H_n(Y, R)$ are isomorphisms for all n.*

Finally we state a useful result for manifold (see [Hat02, Theorem 3.26]).

Theorem A.1.10. *Let X a closed connected manifold of dimension n. Then $\forall k > n$, $H_k(X, R) = 0$.*

For further details on this topic we refer the reader to [Hat02, GH81].

Relative singular homology

Given a subspace $A \subseteq X$, we call (X, A) a *pair* of topological spaces. By taking the restriction of the boundary maps, we can consider two singular complexes of chains, $C(X, R) = \{C_n(X, R), \partial_n\}_n$ and $C(A, R) = \{C_n(A, R), \partial_n\}_n$. By means of the inclusion map $\iota\colon A \hookrightarrow X$ we obtain injective morphisms $C_n(\iota)\colon C_n(A, R) \to C_n(X, R)$ for each $n \in \mathbb{Z}$, given by

$$C_n(\iota)\left(\sum_{j=1}^{n} g_j \sigma_j\right) = \sum_{j=1}^{n} g_j \left(\iota \circ \sigma_j\right).$$

Hence, we can consider the *group of relative singular $n-$chains* defined as

$$C_n(X/A, R) = \frac{C_n(X, R)}{C_n(A, R)}.$$

We can construct the *complex of relative singular n-chains* by considering the induced morphisms $\overline{\partial}_n\colon C_{n+1}(X/A, R) \to C_n(X/A, R)$ obtained by

$$\overline{\partial}_n([\sigma]) = [\partial_n(\sigma)].$$

It is easy to see that these morphisms are well-defined, thus we get a complex of chains $C(X/A,R) = \{C_n(X/A,R), \overline{\partial}_n\}_n$. We can consider its homology groups, called the *relative singular homology* of the pair (X,A) with coefficients in R

$$H_n(X/A,R) = \frac{\ker \overline{\partial}_n}{\operatorname{Im} \overline{\partial}_{n+1}}.$$

In particular, we have a short exact sequence of complexes of chains

$$0 \to C(A,R) \to C(X,R) \to C(X/A,R) \to 0$$

which induces a long exact sequence in homology called the *long exact sequence of the pair*

$$\begin{aligned} &\dots H_{n+1}(X/A,R) \to H_n(A,R) \to H_n(X,R) \to \\ &\to H_n(X/A,R) \to H_{n-1}(A,R) \to H_{n-1}(X,R) \to \dots \end{aligned} \tag{A.1.11}$$

A.1.2 Singular cohomology

Cohomology is the dual construction of homology, hence it inherits analogous properties, the main difference being its functorial controvariance. This induces a product defined on cohomology groups that makes them into rings, gaining all the extremely useful properties that this extra structure carries. We begin this brief review on cohomology by introducing the singular cohomology groups for an arbitrary topological space X of dimension n.

Given an abelian group G, we define the group of *singular n–cochains with coefficients in G* as

$$C^n(X,G) = \operatorname{Hom}(C_n(X,\mathbb{Z}),G)$$

i.e. $C^n(X,G)$ is the dual group of the nth singular chain group $C_n(X,\mathbb{Z})$. Hence a singular $n-$cochain ϕ is a map which assigns to any $n-$simplex $\sigma\colon \Delta^n \to X$ an element $\phi(\sigma) \in G$.

We can define the *coboundary map* $\delta^n\colon C^n(X,G) \to C^{n+1}(X,G)$ as the dual of the boundary operator. Given a cochain $\phi \in C^n(X,G)$, the coboundary map is defined by

$$\delta\phi(\sigma) = \sum_i (-1)^i \phi\left(\sigma|_{[v_o,\dots,\hat{v}_i,\dots,v_{n+1}]}\right)$$

for any $\sigma\colon \Delta^{n+1} \to X$, $(n+1)-$singular simplex.

Moreover, since $\partial_n \circ \partial_{n+1} = 0$, it follows that $\delta^n \circ \delta^{n-1} = 0$.

We consider now the chain complex introduced in (A.1.1)

$$\dots \xrightarrow{\partial n+1} C_n(X,\mathbb{Z}) \xrightarrow{\partial n} C_{n-1}(X,\mathbb{Z}) \to \dots C_1(X,\mathbb{Z}) \xrightarrow{\partial_1} C_0(X,\mathbb{Z}) \xrightarrow{\partial_0} 0.$$

Applying the functor $\mathrm{Hom}_{\mathbb{Z}}(\cdot\,,G)$ we get the dual complex

$$0 \to C^0(X,G) \xrightarrow{\delta^0} C^1(X,G) \to \dots C^k(X,G) \xrightarrow{\delta^k} C^{k+1}(X,G) \to \dots \tag{A.1.12}$$

Elements of $Z^n(X,G) := \ker \delta^n$ are called $n-$*cocycles*, and elements in $B^n(X,G) := \operatorname{Im} \delta^{n-1}$ are called $n-$*coboundaries*. The properties of the coboundary operator imply $B^n(X,G) \subseteq Z^n(X,G)$ for any $n \in \mathbb{N}$.

Definition A.1.13. *The* nth singular cohomology group *of* X *is the quotient group*

$$H^n(X,G) = \frac{Z^n(X,G)}{B^n(X,G)}.$$

As before, given a continuous map $f\colon X \to Y$ between topological spaces, we have an induced cochain map $C^n(f)\colon C^n(Y,G) \to C^n(X,G)$ defined by

$$C^n\left(f\right)\left(\phi\right)\left(\sigma\right) = \phi\left(f \circ \sigma\right),$$

for all $\phi \in C^n(Y,G)$ and for all $\sigma\colon \Delta^n \to X$ singular $n-$simplex of X. By Lemma A.1.5 *(i)* it dually follows the property

$$\delta^n \circ C^n(f) = C^{n+1} \circ \delta^n.$$

Hence $C^n(f)$ defines a homomorphism in cohomology

$$H^n(f)\colon H^n(Y,G) \to H^n(X,G),$$

and H^n is a contravariant functor between the category of the topological spaces and the category of graded $\mathbb{Z}-$modules. Again, we get that this functor is homotopically invariant.

Proposition A.1.14. *Homotopic maps induce the same morphism in cohomology. In particular, homotopically equivalent spaces have the same cohomology.*

Usually we will denote the homomorphism $C^n(f)$ as

$$f^*\colon H^\bullet(Y,G) \to H^\bullet(X,G),$$

the so–called *pull-back map*. Given any field k of characteristic 0, it holds (see [Hat02, Theorem 3.2]):

$$H^j(X,k) \cong \mathrm{Hom}_{\mathbb{Z}}(H_j(X,\mathbb{Z}),k).$$

Cup product

We consider a topological space X, and its cohomology with coefficients in a commutative ring R that usually would be $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$.

Definition A.1.15. *Given two cochains $\phi \in C^k(X,R)$, $\psi \in C^l(X,R)$, we define the* cup product *between ϕ and ψ as the $(k+l)-$ cochain $\phi \cup \psi \in C^{k+l}(X,R)$ given by*

$$(\phi \cup \psi)(\sigma) = \phi\left(\sigma|_{[v_0,\dots,v_k]}\right) \cdot \psi\left(\sigma|_{[v_{k+1},\dots,v_{k+l}]}\right),$$

for any $\sigma\colon \Delta^{k+l} \to X$ singular simplex of X.

From this definition the following result can be proven (see [Hat02, Lemma 3.6]).

Lemma A.1.16. *For any $\phi \in C^k(X,R)$ any $\psi \in C^l(X,R)$*

$$\delta(\phi \cup \psi) = \delta\phi \cup \psi + (-1)^k \phi \cup \delta\psi.$$

Thus the cup product of cochain induces a cup product in cohomology:

$$H^k(X,R) \times H^l(X,R) \xrightarrow{\cup} H^{k+l}(X,R).$$

This cup product is not always symmetric, more precisely the following holds (see [Hat02, Theorem 3.11]):

$$\phi \cup \psi = (-1)^{kl} \psi \cup \phi \qquad \forall \phi \in C^k(X,R), \forall \psi \in C^l(X,R).$$

We define

$$H^*(X,R) = \bigoplus_k H^k(X,R),$$

which is a graded ring with the cup product in cohomology.

It is easy to see (see e.g. [Hat02, Proposition 3.10]) that the cup product commutes with the pull-back in cohomology, i.e. given a continuous map between topological spaces $f\colon X \to Y$, it holds:

$$f^*(\phi \cup \psi) = f^*(\phi) \cup f^*(\psi) \qquad \forall \phi, \psi \in C^\bullet(X,R).$$

De Rham theorem

There is an interpretation of the cohomology algebra $H^*(X,\mathbb{C})$ in terms of differentiable forms on X. Before stating this fundamental result, we briefly recall the main ideas of the De Rham cohomology. For a complete presentation of this topic we refer the reader to [GH94, §3].

We denote by $\Omega_p(X)$ the space of the $p-$forms on X and we set $\Omega^* = \bigoplus_p \Omega^p(X)$. We recall that on this space we have a differential operator $d\colon \Omega^p(X) \to \Omega^{p+1}(X)$ which commutes with the pull-back map and satisfies

$d^2 = 0$. A form $\omega \in \ker d$, i.e. such that $d\omega = 0$, is called a *closed* form and a form $\mu \in \operatorname{Im} d$, i.e. $\mu = d\eta$, is called *exact*. The qth De Rham cohomology group of X is the vector space

$$H^q_{DR}(X) = \frac{\ker d\colon \Omega^q(X) \to \Omega^{q+1}(X)}{\operatorname{Im} d\colon \Omega^{q-1}(X) \to \Omega^q(X)}.$$

The De Rham cohomology of X is defined as

$$H^*_{DR}(X) = \bigoplus_q H^q_{DR}(X).$$

We state now a key results to relate singular and De Rham cohomologies in the manifold case (see [Ara12, Corollary 7.1.11]).

Theorem A.1.17 (De Rham)**.** *Let X be a manifold.*
Then $H^k_{DR}(X) \cong H^k(X, \mathbb{R}) \cong H^k(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ as graded algebras.

A.1.3 Poincaré duality

We present the Poincaré duality theorem, which explicitly describes the dual relation between homology groups and cohomology groups.
First of all, we need to introduce a new kind of product between singular chains and cochains.

Definition A.1.18. *For all $k \geq l$, we define the* cap product $\cap\colon C_k(X, R) \times C^l(X, R) \to C_{k-l}(X, R)$ *by*

$$\sigma \cap \phi = \phi\left(\sigma|_{[v_0,\ldots,v_l]}\right) \sigma|_{[v_l,\ldots,v_k]} \colon \Delta^{k-l} \to X,$$

for any $\sigma \in C_k(X, R)$ and any $\phi \in C^l(X, R)$.

The cap product induces a $R-$bilinear map between classes, which is also called cap product, as follows:

$$\begin{aligned} H_k(X, R) \times H^l(X, R) &\xrightarrow{\cap} H^{k-l}(X, R) \\ ([\sigma], [\phi]) &\mapsto \quad [\sigma \cap \phi] \end{aligned}$$

Let us consider now a continuous map $f\colon X \to Y$ between topological spaces. Then for any $\sigma \in H_k(X, R)$ and any $\phi \in H^l(X, R)$ one can prove the so–called *projection formula*

$$f_*(\sigma) \cap \phi = f_*\left(\sigma \cap f^*(\phi)\right). \tag{A.1.19}$$

We are now ready to state the main theorem of this section, for the proof of it we refer the reader to [Hat02, Theorem 3.30].

Theorem A.1.20 (Poincaré duality)**.** *Let X be an $R-$orientable closed variety with fundamental class $[X] \in H_n(X, R)$. For every $k \leq n$, the map*

$$\begin{aligned} H^k(X, R) &\xrightarrow{\text{PD}} H_{n-k}(X, R) \\ \phi &\mapsto \quad [X] \cap \phi \end{aligned} \tag{A.1.21}$$

is an isomorphism.

We notice that, if we have an orientable compact variety of dimension n, Poincaré duality implies that the bilinear pairing

$$\begin{aligned} H^k(X, \mathbb{Z}) \otimes H^{n-k}(X, \mathbb{Z}) &\longrightarrow H^n(X, \mathbb{Z}) \xrightarrow{\int_X} \mathbb{Z} \\ (\alpha, \beta) \qquad &\mapsto \quad \alpha \cup \beta \quad \mapsto \int_X \alpha \cup \beta \end{aligned} \tag{A.1.22}$$

is non-degenerate. Thus the induced homomorphism

$$H^{n-k}(X, \mathbb{Z}) \to \operatorname{Hom}\left(H_k(X, \mathbb{Z}), \mathbb{Z}\right)$$

is an isomorphism up to torsions.

The Gysin morphism

Let us consider two smooth projective varieties X, Y of dimension n, m respectively, and a morphism $f : Y \to X$. The Gysin morphism $f_!$ is defined as $\text{PD}^{-1} \circ f_* \circ \text{PD}$, where PD is the isomorphism given by the Poincaré Duality (A.1.21), i.e.

$$f_! : H^k(Y, \mathbb{Z}) \overset{\text{PD}}{\cong} H_{2m-k}(Y, \mathbb{Z}) \xrightarrow{f_*} H_{2m-k}(X, \mathbb{Z}) \overset{\text{PD}^{-1}}{\cong} H^{k+2n-2n}(X, \mathbb{Z}). \tag{A.1.23}$$

Given a proper morphism $f : Y \to X$, then *projection formula* holds (see [SDTI95, §I.7.6]):

$$f_!(\alpha \cup f^*(\beta)) = f_!(\alpha) \cup \beta \qquad \forall \alpha \in H^*(X, \mathbb{Z}), \forall \beta \in H^*(Y, \mathbb{Z}). \tag{A.1.24}$$

A.2 Kähler manifolds

Kähler manifolds are a special case of complex manifolds. They have a complex structure and a Riemannian metric, so they represent a fundamental bridge between complex and Riemannian geometry. In particular, all smooth projective varieties are Kähler, and for Kähler manifolds we have a suitable decomposition of the cohomology groups which is due to Hodge.
We will give a brief presentation of Kähler manifolds and the Hodge decomposition, fur further details we refer the reader to [Ara12, §8], [Voi07a, §3] and [Wel08, §3].

First of all, let us briefly recall the definition of Riemannian manifolds.

Definition A.2.1. *Let X be a differentiable manifold, a* Riemannian metric *on X is a family $\{g_p\}_{p\in X}$ of positive definite inner products on the tangent spaces*

$$g_p\colon T_pX \times T_pX \longrightarrow \mathbb{R} \quad \forall p \in X$$

defining, for every smooth vector fields V, W on X, a smooth map given by

$$p \mapsto g_p\left(V\left(p\right), W\left(p\right)\right).$$

We can express the Riemannian metric by the metric tensor

$$g = \sum_{i,j} g_{ij}\, dx_i \otimes dy_j \quad \forall p \in X.$$

A real manifold with a Riemannian metric is called a Riemann manifold*.*

A.2.1 Kähler metrics

Let us consider now a complex manifold X.
For complex manifolds, the analogous of the Riemannian metric is the Hermitian metric, i.e. a family of positive definite inner products $\{h_p\}_{p\in X}$ on the complex tangent spaces varying in a smooth way such that

$$h_p\left(V\left(p\right), \overline{W\left(p\right)}\right) = \overline{h_p\left(W\left(p\right), \overline{V\left(p\right)}\right)}$$

for every V, W vector fields on X. As before, we can express a Hermitian metric by the metric tensor

$$h = \sum_{i,j} h_{ij}\, dz_i \otimes d\bar{z}_j,$$

where we are considering local coordinates $z_i = x_i + \mathrm{i}y_i$.
A Hermitian metric defines a Riemannian metric g on the smooth underlying real manifolds of X, which is defined to be the real part of h, namely

$$g = \frac{1}{2}\left(h + \bar{h}\right).$$

Moreover, to each Hermitian metric we can associate a complex differential form $\omega \in \Omega^{1,1}_X \cap \Omega^2_{X,\mathbb{R}}$ called the *Kähler form* that is the imaginary part of h, i.e. $\omega = \Im h$. In particular, the following hold

$$h = g - \mathrm{i}\omega \qquad \text{and} \qquad \omega = \frac{\mathrm{i}}{2}\sum_{i,j} h_{ij}\, dz_i \wedge d\bar{z}_j.$$

We state the following useful result (see [Voi07a, §3.1.3]).

Lemma A.2.2. *Let X be a complex manifold such that $\dim_{\mathbb{C}} X = n$. Then the volume form associated to a Hermitian metric h on X is $\frac{\omega^n}{n!}$.*

Proposition A.2.3 ([Ara12]. Proposition 10.1.2)**.** *Given a Hermitian metric h with Kähler form ω, the following are equivalent*

(i) the Kähler form is closed, i.e. $d\omega = 0$;

(ii) it is possible to express locally the Kähler form as $\omega = \partial\bar{\partial} f$, where f is a function.

If the Kähler form satisfies one of the previous conditions, we say that h is a Kähler metric and X is a *Kähler manifold*.
From Lemma A.2.2, we have the following result (see [Voi07a, Corollary 3.9]).

Corollary A.2.4. *If X is a compact Kähler manifold, then for every k s.t. $1 \leq k \leq n$ the closed form ω^k is not exact.*

Hence, the De Rham class $\left[\omega^k\right] \in H^{2k}_{DR}(M)$ is non-zero. We call $[\omega]$ the *Kähler class* of the Kähler metric of X.

The complex projective space $\mathbb{P}^n_{\mathbb{C}}$ has a natural Kähler metric, called the *Fubini–Study metric*, which is given by the Kähler form

$$\omega = \frac{\mathrm{i}}{2\pi} \partial\bar{\partial} \log\left(|z_0|^2 + \cdots + |z_n|^2\right).$$

The main examples of Käheler manifolds are constructed using the following results (see [Ara12, Lemma 10.1.6, Corollary 10.1.7]).

Lemma A.2.5. *Given a Kähler manifold X with Kähler metric h, any complex submanifold inherits a Kähler metric induced by h such that its Kähler class is the restriction of the Kähler class of X.*

Corollary A.2.6. *Every smooth projective variety is a Kähler manifold with the metric induced by the Fubini-Study metric.*

Finally we mention a result (see [Voi07a, Proposition 3.14]), that characterizes Kähler metrics as Euclidean metrics, up to second order approximation.

Proposition A.2.7. *Let X be a complex manifold of dimension n with a Kähler metric h. Then, for each point $x \in X$, there exist coordinates $z_1, \ldots, z_n$ centred at x, such that the matrix of h in these coordinates*

$$(h_{ij})_{ij} = \left(h\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right)\right)$$

is equal to $I_n + O\left(\sum_i |z_i|^2\right)$, where I_n denotes the Euclidean metric.

A.2.2 The Hodge decomposition

We recall that, in the case of a Riemannian manifold X, we have a differential operator $d\colon \Omega^k(X) \to \Omega^{k+1}(X)$ and its formal adjoint operator $d^*\colon \Omega^{k+1}(X) \to \Omega^k(X)$ (see [Ara12, §8]) for further details). Then we can define the *Hodge Laplacian* $\Delta = dd^* + d^*d$, and we say that a form $\alpha \in \Omega^k(X)$ is *harmonic* if it satisfies $\Delta\alpha = 0$. We denote the space of the harmonic $k-$forms as $\mathcal{H}^k = \ker\big(\Delta\colon \Omega^k(X) \to \Omega^k(X)\big)$. The following fundamental result (see [Voi07a, Theorem 5.23]) states that every cohomolgy class has an harmonic representative.

Theorem A.2.8. *There is a natural isomorphism for any $k \in \mathbb{N}$*

$$\begin{aligned} \mathcal{H}^k &\longrightarrow H^k(X,\mathbb{C}) \\ \alpha &\mapsto \quad [\alpha]\,. \end{aligned}$$

If X is a complex manifold, we have a decomposition (see [Voi07a, §2.3])

$$\Omega_{X,\mathbb{C}} = \Omega_X^{1,0} \oplus \Omega_X^{0,1}, \tag{A.2.9}$$

where a $(1,0)-$form α is locally expressible as $\alpha = \sum_i \alpha_i dz_i$ with $\alpha_i \in \mathcal{C}^k$ if α is $\mathcal{C}^k$ differentiable, and a $(0,1)-$form β is $\beta = \sum_i \beta_i d\bar{z}_i$ with $\beta_i \in \mathcal{C}^k$ if β is $\mathcal{C}^k$. We denote as

$$\Omega^{p,q}(X) = \bigwedge^p \Omega_X^{1,0} \otimes \bigwedge^q \Omega_X^{0,1}.$$

Then previous decomposition A.2.9 induces a decomposition on the $k-$forms:

$$\Omega^k_{X,\mathbb{C}} = \Omega^k_{X,\mathbb{R}} \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega_X^{p,q}.$$

In particular, the differential forms

$$dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}.$$

are local generators of the bundle $\Omega_X^{p,q}$ in holomorphic coordinates $z_1, \dots, z_n$.

Hence, we can write a $(p,q)-$form α as

$$\alpha = \sum_{I,J} \alpha_{I,J} dz_I \wedge d\bar{z}_J,$$

where the sum is taken over all the multi-indexes I, J such that $|I| = p$ and $|J| = q$. We have an expression for the differential of a $(p,q)-$form α:

$$d\alpha = \sum_{I,J} d\alpha_{IJ} dz_I \wedge d\bar{z}_J.$$

Given a $\mathcal{C}^1$ form α of type (p,q), we define $\partial\alpha$ as the component of type $(p+1,q)$ of $d\alpha$. Analogously, we define $\bar{\partial}\alpha$ as the $(p,q+1)$ component of

$d\alpha$. Moreover, since a $k-$form decomposes uniquely into $(p,q)-$components $\alpha^{p,q}$ with $p+q=k$, we can set

$$\partial\alpha = \sum_{p+q=k} \partial\alpha^{p,q} \quad \text{and} \quad \bar{\partial}\alpha = \sum_{p+q=k} \bar{\partial}\alpha^{p,q},$$

hence $d = \partial + \bar{\partial}$. In particular ∂ and $\bar{\partial}$ satisfy the Leibniz rule

$$\begin{aligned} \partial(\alpha \wedge \beta) &= \partial\alpha \wedge \beta \pm \alpha \wedge \partial\beta; \\ \bar{\partial}(\alpha \wedge \beta) &= \bar{\partial}\alpha \wedge \beta \pm \alpha \wedge \bar{\partial}\beta. \end{aligned}$$

Moreover, they are idempotent, i.e. $\partial^2 = 0$ and $\bar{\partial}^2 = 0$. For further details on these operators we refer the reader to [Voi07b, §2.3].

We can consider the corresponding adjoint operators ∂^* and $\bar{\partial}^*$. Thus we have two different Laplacian operators $\Delta_\partial = \partial\partial^* + \partial^*\partial$, and $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. In the case of a Kähler variety, we have the following fundamental result, that would be the key to the Hodge decomposition (see [Voi07a, Theorem 6.1]).

Theorem A.2.10. *If X is a Kähler manifold, then $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$.*

Corollary A.2.11. *If $\alpha \in \Omega^k(X)$ is an harmonic form, then its $(p,q)-$components are also harmonic. In particular we have $\mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q}$.*

By Theorem A.2.8, we have that $\mathcal{H}^{p,q} \cong H^q(X, \Omega_X^p) \cong H^{p,q}(X)$, where $H^{p,q}(X)$ contains the classes of closed form of type (p,q). Thus, when the metric is Kähler, we have an induced decomposition, called the *Hodge decomposition*

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X). \tag{A.2.12}$$

In particular, the Hodge decomposition does not depend on the choice of the Kähler metric (see [Voi07a, Proposition 6.11]).
We conclude this session with an useful corollary of the last result.

Proposition A.2.13. *The cup product $H^k(X,\mathbb{C}) \otimes H^l(X,\mathbb{C}) \xrightarrow{\cup} H^{k+l}(X,\mathbb{C})$ is bigraded with respect to the bigraduation given by the Hodge decomposition, i.e. $\alpha^{p,q} \cup \beta^{p'+q'} \in H^{p+p',q+q'}(X) \cap H^{k+l}(X,\mathbb{C})$ with $p+q=k$, $p'+q'=l$.*

Moreover, the pairing given by Poincaré duality (A.1.21) respects the bigraduation given by the Hodge decomposition, i.e. the pairing

$$\begin{aligned} H^{p,q}(X) \otimes H^{n-p,n-q}(X) &\to \mathbb{C} \\ (\alpha,\beta) &\mapsto \int_X \alpha \cup \beta \end{aligned}$$

is non-degenerate.

A.2.3 Lefschetz theorems

Here we present some classical results by Lefschetz that will be extremely useful.

Theorem A.2.14 (Lefschetz' Theorem on $(1,1)-$classes)**.** *Let X be a compact Kähler manifold, then the cycle map* $\mathrm{Pic}(X) \to H^{1,1}(X,\mathbb{Z})$ *is surjective.*

Proof. We consider the exponential short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0,$$

and the induced long exact sequence in cohomology

$$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \xrightarrow{i_*} H^2(X,\mathcal{O}_X).$$

Corollary A.2.11 implies $H^2(X,\mathcal{O}_X) \cong H^{0,2}(X)$. Let us consider now $\alpha \in H^{1,1}(X) \cap H^2(X,\mathbb{Z})$, then $i_*(\alpha) = 0$. Since the sequence is exact, $\alpha \in \mathrm{Im}\, c_1$, hence there exists a line bundle $L \in \mathrm{Pic}(X)$ such that $\alpha = c_1(L) = \eta_D$, where $L = [D]$ for a divisor D on X. □

Since we are considering a smooth variety X, the Picard group is isomorphic to the divisor group modulo linear equivalence: $\mathrm{Pic}(X) = Z^1(X)/\sim$. Hence we have that any class in $H^{1,1}(X)$ has an algebraic representative.

Now we introduce the *Lefschetz operator* L by using the Kähler form $\omega = [h] \in H^{1,1}(X) \cap H^2(X,\mathbb{Q})$, with $h \in |\mathcal{O}_X(1)|$ hyperplane section:

$$\begin{aligned} L\colon H^k(X,\mathbb{Q}) &\to H^{k+2}(X,\mathbb{Q}) \\ \alpha &\mapsto \quad \alpha \cup \omega. \end{aligned} \tag{A.2.15}$$

In particular, the Lefschetz operator restricts to the bigraduate components of the Hodge decomposition, i.e.

$$L\colon H^{p,q}(X) \to H^{p+1,q+1}(X).$$

Definition A.2.16. *By means of the Lefschetz operator we can define the* primitive cohomology groups *for all $k \leq n$ and for all p, q such that $p+q = k \leq n$*

$$H^k(X,\mathbb{Q})_{\mathrm{prim}} = \ker\left(L^{n-k+1}\colon H^k(X,\mathbb{Q}) \to H^{2n-k+2}(X,\mathbb{Q})\right);$$
$$H^{p,q}(X)_{\mathrm{prim}} = \ker\left(L^{n-p-q+1}\colon H^{p,q}(X) \to H^{n-q+1,n-p+1}(X)\right).$$

Finally we state two other theorems due to Lefschetz.

Theorem A.2.17 (Hard Lefschetz Theorem)**.** *Let X be a compact Kähler manifold of dimension n, then for any $k \leq n$ we have an isomorphism*

$$L^{n-k}\colon H^k(X,\mathbb{Q}) \cong H^{2n-k}(X,\mathbb{Q}).$$

For any k, the so–called Lefschetz decomposition *holds, namely*

$$H^k(X,\mathbb{Q}) = \bigoplus_{i\geq 0} L^i H^{k-2i}(X,\mathbb{Q})_{\mathrm{prim}}. \tag{A.2.18}$$

Moreover, both assertions respect the bigraduation given by the Hodge decomposition, and we have the following induced decomposition on the primitive cohomology

$$H^k(X,\mathbb{Q})_{\mathrm{prim}} \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X)_{\mathrm{prim}}.$$

Theorem A.2.19 (Lefschetz Hyperplane Theorem)**.** *Let X be a projective variety of dimension n, and let $Y \overset{j}{\hookrightarrow} X$ be a hyperplane section such that $U = X - Y$ is smooth. Then the restriction map*

$$j^*\colon H^k(X,\mathbb{Z}) \to H^k(Y,\mathbb{Z})$$

is an isomorphism for $k < n-1$, and it is injective for $k = n-1$.

For the proof of these two theorems we refer the reader to [Huy05, Proposition 3.3.13] and [Voi07b, Theorem 1.23] respectively.

Intersection form

By means of the Kähler form ω, we can define an intersection form $q_\omega^{(k)}$ on $H^k(X,\mathbb{Q})$ as follows

$$q_\omega^{(k)}(\alpha,\beta) = \int_X \alpha\cup\beta\cup\omega^{n-k} = \int_X \alpha\cup L^{n-k}\beta = \int_X L^{n-k}\alpha\cup\beta.$$

Due to Poincaré duality (A.1.21) and Hard Lefschetz Theorem A.2.17, $q_\omega^{(k)}$ is a perfect pairing.
For the cup product it holds $\alpha\cup\beta = (-1)^{kl}\,\beta\cup\alpha$ for any $\alpha\in H^k(X,\mathbb{Q})$ and any $\beta\in H^l(X,\mathbb{Q})$. Hence $q_\omega^{(k)}$ is symmetric if k is even, otherwise it is antisymmetric.
By using $q_\omega^{(k)}$, we can define a sesquilinear hermitian pairing $h_\omega^{(k)}$ as

$$\begin{aligned} h_\omega^{(k)}\colon H^k(X,\mathbb{C})\otimes H^k(X,\mathbb{C}) &\to \mathbb{C}\\ (\alpha,\beta) \quad &\mapsto \quad \mathrm{i}^k q_\omega^{(k)}\left(\alpha,\bar\beta\right). \end{aligned}$$

Finally, we cite a useful result (see [Voi07a, §6.3.2]).

Lemma A.2.20 (Hodge-Riemann bilinear relations)**.** *Both Hodge decomposition* (A.2.12) *and Lefschetz decomposition* (A.2.18) *are orthogonal with respect to $h_\omega^{(k)}$. Moreover, the form $(-1)^{\frac{k(k-1)}{2}} i^{p-q-k} h_\omega^{(k)}$ is positive definite on $H^{p,q}_{\mathrm{prim}} := H^k(X,\mathbb{C})\cap H^{p,q}(X)$ (see Defintion A.2.16).*

A.3 Complexes and spectral sequences

We give a brief presentation of spectral sequences and we introduce the Leray spectral sequence. These are essential tools when dealing with cohomology. For further details, we refer the reader to [GH94, Chapter 3, Section 5].

We recall that a *complex* (C^*, d) is a sequence of algebraic groups

$$C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} \dots,$$

where the maps $d\colon C^i \to C^{i+1}$ are differentials such that $d \circ d = 0$. As we did for singular cohomology we can define the group of *cycles*

$$Z^i := \ker\{d\colon C^i \to C^{i+1}\},$$

and the subgroup of *boundaries*

$$B^i = dC^{i-1} \subset Z^i.$$

Taking the quotient we get the *cohomology* groups

$$H^i(C^*) = Z^i/B^i = \frac{\{i\text{–th cycles}\}}{\{i\text{–th boundaries}\}},$$

and the cohomology of the complex $H^*(C^*) = \bigoplus_{i\geq 0} H^i(C^*)$. A *subcomplex* (S^*, d) is a complex given by subgroups $S^i \subset C^i$, closed under the differentials, i.e. such that $dA^* \subset A^*$. The *quotient complex* (Q^*, d) is defined by the quotient subcomplexes $Q^* = C^*/S^*$. A *filtered complex* (F^iC^*, d) is a decreasing sequence of subcomplexes

$$C^* = F^0C^* \supset F^1C^* \supset \cdots \supset F^nC^* \supset F^{n+1}C^* = \{0\},$$

where F is said to be a decreasing *filtration* on the complex (C^*, d). Given a filtered complex we can define the *graded pieces*

$$\mathrm{Gr}^iC^* = \frac{F^iC^*}{F^{i+1}C^*},$$

and the *associated graded complex* $\mathrm{Gr}C^* = \bigoplus_{i\geq 0} \mathrm{Gr}^iC^*$. We have also an induced filtration on cohomology given by

$$F^iH^j(C^*) = \frac{F^iZ^j}{F^iB^j},$$

and we can consider the *associated graded cohomology*

$$\mathrm{Gr}H^*(C^*) = \bigoplus_{i,j\geq 0} \mathrm{Gr}^iH^j(C^*),$$

which is given by the graded pieces

$$\mathrm{Gr}^i H^j(C^*) = \frac{F^i H^j(C^*)}{F^{i+1} H^j(C^*)}.$$

Given a short exact sequence of complexes $0 \to S^* \to C^* \to Q^* \to 0$, we have an induced long exact sequence in cohomology

$$\dots H^i(S^*) \to H^i(C^*) \to H^i(Q^*) \to H^{i+1}(S^*) \to \dots$$

For filtered complexes the role of long cohomology exact sequence is played by spectral sequences.

Definition A.3.1. *Let $\{E_p, d_p\}$ with $p \geq 0$ be a sequence of bigraded groups*

$$E_p = \bigoplus_{i,j} E_p^{i,j},$$

with differentials $d_p \colon E_p^{i,j} \to E^{i+p,j-p+1}$ such that $d_p \circ d_p = 0$. If $H^*(E_p) = E_{p+1}$, then $\{E_p\}$ is said to be a spectral sequence.

One can see that there exists $\bar{p}$ such that $E_\infty := E_p = E_{p+1} = \dots$ for every $p \geq \bar{p}$. We say that the spectral sequence $\{E_p\}$ *converges* to E_∞.

Given a filtered complex $(F^i C^*, d)$, there exists a spectral sequence $\{E_p\}$ such that

$$E_0^{i,j} = \frac{F^i C^{i+j}}{F^{i+1} C^{i+j}};$$
$$E_1^{i,j} = H^{i+j}(\mathrm{Gr} H^{i+j}(C^*));$$
$$E_\infty^{i,j} = \mathrm{Gr}(H^{i+j}(C^*)).$$

We say that the spectral sequence *abuts* to $H^*(C^*)$, and we denote this by $E_p \Rightarrow H^*(C^*)$. For a proof of this result we refer the reader to [GH94, Proposition pag. 440].

Definition A.3.2. *A* double complex $(C^{*,*}, d, \delta)$ *is given by a bigraded group*

$$C^{*,*} = \bigoplus_{i,j \geq 0} C^{i,j}$$

and differentials $d \colon C^{i,j} \to C^{i+1,j}$, $\delta \colon C^{i,j} \to C^{i,j+1}$ such that $d \circ d = \delta \circ \delta = 0$ and $d\delta + \delta d = 0$. The associated single complex (C^*, D) *is the complex defined by $C^p = \bigoplus_{i+j=p} C^{i,j}$ and $D = d + \delta$.*

We have two filtrations on the associated single complex

$$'F^q C^p = \bigoplus_{i+j=p, i \geq q} C^{i,j};$$
$$''F^q C^p = \bigoplus_{i+j=p, j \geq q} C^{i,j}.$$

When considering a bigraded complex $(C^{*,*}, d, \delta)$, there are two spectral sequences $\{'E_p\}$ and $\{''E_p\}$ both abutting to the cohomology of the total complex, such that

$$'E_2^{i,j} \cong H_d^i(H_\delta^j(C^{*,*}));$$
$$''E_2^{i,j} \cong H_\delta^j(H_d^i(C^{*,*})).$$

Example A.3.3. *The main example we are interested in is the case of a complex manifold M, where we have that $C^{i,j} = \Omega^{i,j}(M)$, $d = \partial$ and $\delta = \bar{\partial}$. The associated single complex is the De Rham complex (Ω^*, d). In this case the filtration is easy to understand: $'F^q\Omega^p$ gives the p–forms which contain at least q times dz, and $''F^q\Omega^p$ is the anti–holomorphic version of it. The two spectral sequences which abut to $H^*_{DR}(M)$ are called* Frölicher spectral sequences*, though not much is known about these objects.*
If moreover M is a compact Kähler manifold, we have that

$$'E_1 \cong' E_2 \cong \ldots \cong' E_\infty,$$

and the filtration on the De Rham cohomology is the Hodge filtration

$$F^p H^i_{DR}(M) \cong H^{i,0}(M) \oplus \cdots \oplus H^{p,i-p}(M).$$

A.3.1 The Leray spectral sequence and the Leray filtration

We give a brief introduction to this spectral sequence which is very useful when dealing with cohomology. For the proofs of the next results we refer the reader to [GH94, Chapter 3, Section 5] or [Voi07b, Chapter I, Section 4]. First of all we recall some definitions.

Definition A.3.4. *[Har77, Chapter III, Section 8, Proposition 8.1] Let $f\colon X \to Y$ be a continuous map of topological spaces, and let $\mathcal{F}$ be a sheaf of abelian groups on X. The i–th* higher direct image sheaf *is the sheaf $R^i f_*(\mathcal{F})$ associated to the presheaf*

$$U \mapsto H^i\left(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)}\right).$$

We recall a useful characterization of the stalks.

Lemma A.3.5. *Let $f\colon X \to Y$ be a proper map, and let $\mathcal{F}$ be a sheaf on X. Let $y \in Y$ be a point, and we consider the natural map on the stalks $\left(R^i f_*(\mathcal{F})\right)_y \to H^i\left(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)}\right)$. Then this map is an isomorphism.*

We introduce the following result due to Leray ([Voi07b, Theorem 4.11]).

Theorem A.3.6 (Leray)**.** *Let $f\colon X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathcal{F}$ on X, there exists a filtration L on $H^i(X, \mathcal{F})$ and there exists a spectral sequence $E_p^{i,j} \Rightarrow H^{i+j}(X, \mathcal{F})$ such that*

$$E_\infty^{i,j} = Gr_L^i H^{i+j}(X, \mathcal{F});$$
$$E_2^{i,j} = H^i\left(Y, R^j f_*(\mathcal{F})\right).$$

The filtration L is called the *Leray filtration*, and the spectral sequence $\{E_p\}$ is the *Leray spectral sequence.*

If we consider a fiber bundle $\pi \colon X \to B$ with compact fiber F and the constant sheaf $\mathbb{Q}$ on X, we get that

$$H^i\left(\pi^{-1}(U), \mathbb{Q}\right) \cong H^i(F, \mathbb{Q}),$$

for a small enough open set $U \subset B$ such that $\pi^{-1}(U) \cong U \times F$. Then it is possible to prove that $E_2^{i,j} = H^i_{DR}\left(B, H^j_{DR}(F)\right)$.

Finally, we present a useful result due to Deligne ([Voi07b, Theorem 4.15])

Theorem A.3.7 (Deligne, 1968)**.** *Let $f \colon X \to Y$ be a submersive projective morphism. Then the Leray spectral sequence of f with rational coefficients degenerates at E_2, i.e. $E_2 \cong E_\infty$ and $H^*(X, \mathbb{Q}) \cong H^*(Y, R^*f_*(\mathbb{Q}))$.*

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