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Interacting Diffusion Models: Scaling Limit and Numerical Analysis

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INTERACTING DIFFUSION MODELS: SCALING LIMIT AND NUMERICAL ANALYSIS

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Chapter 1

Introduction

This thesis deals with the analysis of interacting particle systems and with related numerical topics. It is focused on the so-called interacting diffusion models, i.e. we always consider continuous on-site variables. This work is based on the papers [3], [21], [47], [48] and [51] as described in more detail below.

Many of the cases we are going to discuss arise naturally from other fields of study like physics, biology, population dynamics, chemistry or social sciences. Often, some phenomena coming from different of the above disciplines present some similarities in their behavior. These similarities can be captured through the use of mathematics. Many of the models that we currently use to understand such systems represent a simplified version of the reality. These simplifications are made, not only to obtain a model that is easier to study, but also in order to have a better understanding of the primary mechanisms involved. The balance between accuracy and approximation is the key to obtain a model which is well suited for applications to real world problems.

Particle models, or *Individual Based Model*, are a good instrument to obtain a precise description of the physical process involved. Very often in the field of studies mentioned above many phenomena are nothing else that the repercussion in the macroscopic world of what is happening on a smaller scale. Think the dynamics of a fluid or the formation of tissue in a living being. They are all the reflection on the outer world of more than billions of interactions and processes happening underneath. Here we are using the word "particle" in a broad sense. In the case of fluids particles represents water molecules interacting through chemical forces. In tissue formation the situation is more complicated since particles, now representing cells, can affect each other in many ways such as tactile or contact forces, long or short range chemical signaling et cetera. Being able to describe precisely this heterogeneity of interactions can be a major advantage when trying to model living complex systems.

However individual based models are not directly suitable to be used for applications. In practice complex systems such as those introduced above, are made of an enormous amount of individuals, far from what is possible to treat mathematically or to store in a computer. The number of cells in a human tissue is around 10^9 for cm^3, which is already out of our capabilities if one considers an entire organism. In the case of fluids like water the situations is even more complicated since the number of molecules in a single drop is about 10^{21}. Hence a statistical approach is usually preferred where, instead of considering the position of each particle, we try to describe a probability density function $f(t, x)$ for the particles positions at time t around site x. To do so we perform the so called *many particle limit* where we assume to have an infinite number of particles, and derive a deterministic Partial Differential Equation (PDE) for the density f. Some times in this work (Chapters 2, 3) we will deal with the kinetic formulation for particle systems. All particles will hence be defined by position and velocity at each time. Consequently in the many particle limit the equation for the density of the particles is in the kinetic form $f(t, x, v)$. An example of such equation is the well known Boltzmann equation for the distribution of thermodynamic system in a non equilibrium state.

Nonetheless in concrete applications, dealing with nonlinear PDEs especially in high dimension is not an easy task. Imagine a numerical method which aims at approximating the solution of a PDE in a grid made of N points per dimension. If we assume to have a d-dimensional problem the resulting number of points would be of order N^d. Assuming to work in dimension $d = 10$ (proportional to the degrees of freedom of the problem), by taking simply $N = 10$ would result in 10^{10} points to compute the approximation, a number which is already difficult to treat numerically. Not to mention the poorness of the approximation obtained by computing the solution only in ten points per dimension. For this reason sometimes methods based directly on particle simulation are preferred. Of course the setting of PDEs deriving from particle system is not the only one where high dimensionality poses significant problems to the numerical approximation of the solution. Many numerical methods, usually grouped in the huge class of *Monte-Carlo Methods*, are developed exactly for this purpose, where other classical approaches usually fail. We analyze some topics related to Monte-Carlo methods, specifically deepening the analysis of the variance of the estimation, in Chapter 4. A specific subclass of problems, for which I have developed a specific interest, is the computation of rare-events and extreme probabilities. This family of problems requires particular attention even when applying Monte-Carlo methods. In fact average based methods need to be revisited in order to work when the number of successful observables is limited. To make the idea more clear imagine to estimate the probability of some rare events by performing independent iterations and by averaging the number of

successes by the number of iterations. If the number of successes is infinitesimal compared to the total number (imagine of order ten over a million of tries), then the resulting approximation of the probability of the event will be very poor. On this matter, an approach based directly on the Fokker-Planck equation associated to the process should be more feasible. Sometimes the theory of the Kolmogorov equation is easier. One issue which affects the study of Fokker-Planck equations is that in general solutions are measure valued functions instead of functions of space-time. This approach is analyzed in Chapter 5.

This introductory chapter is structured as follows. In Sections 1.1 and 1.2 we develop more details concerning Chapters 2 and 3 of this Thesis. Both these chapters deal with some particle modeling problem, presenting a scaling limit result. In Chapter 2 we work with the so-called Vlasov-Navier-Stokes-Fokker-Planck model which describes the collective behavior of infinitely many particles suspended in a fluid. Chapter 3 instead focuses on a biological problem: the growth of the Hyphae of the fungus, i.e. the microscopic branching filaments which collectively form the mycelium. In both cases the dynamic of the particles is of second order, hence will will always deal with kinetic limiting equations. Moreover in Sections 1.3 and 1.4 we introduce the problematics analyzed in Chapters 4 and 5, included in Part II. Here the focus is more on the numerics. Chapter 4 deals with particle systems in the form of McKean-Vlasov SDE, specifically in the ergodic case. Instead in Chapter 5 we work on the approximation of Kolmogorov Equation in high dimensional setting. The theoretical analysis is performed in infinite dimension. The approach proposed here, even if particle systems are not directly involved, relies on some Gaussian analysis techniques due to the strong connection of such equations witch stochastic analysis. Since chapters are not directly connected, even if they often deal with close topics, we won't make a detailed review of the previous works here. In this introductory chapter we will only make a brief introduction of what is the problem that we analyzed in the corresponding chapter, and point out the main problem. Therefore an appropriate review of the literature is included at the beginning of each chapter.

1.1 Particle-fluid interaction: the Vlasov-Fokker-Planck-Navier-Stokes system

The results of this section appeared in the works [47], [48]. In Chapter 2 we deal with the so called *Vlasov-Navier-Stokes-Fokker-Planck* system:

$$\begin{cases} \partial_t u = \Delta u - u \cdot \nabla u - \nabla \pi - \int_{\mathbb{R}^d} (u - v) F \, dv; \\ \operatorname{div}(u) = 0; \\ \partial_t F + v \cdot \nabla_x F + \operatorname{div}_v((u - v)F) = \frac{\sigma^2}{2} \Delta_v F. \end{cases} \tag{1.1}$$

This system aims to model the interaction between a large amount of particles suspended in a fluid. The variable $u = u(t, x)$ is a vector field, and represents the velocity of the fluid, while π stands for the pressure. The scalar function $F = F(t, x, v)$ stands for the density of the particles suspended, and is defined on the phase space i.e. is a function of the space and velocity variables. This description is well-founded only when one looks at a large amount of particles. In this chapter we aim to prove rigorously that this description is correct. We introduce a system of interacting particles, that are treated as individuals instead of as an ensemble described by a density, and prove the convergence in the many-particle limit to system (1.1).

Before introducing the particle model let us now go a bit more into the details of the system of PDEs. As we said the fluid is described by its velocity field $u(t, x) \in \mathbb{R}^d$. Of course the natural setup is to have $d = 3$, however in what follows we will consider mainly the case $d = 2$. We assume that particles are diluted enough, so that the density of the fluid remains constant $\rho_f > 0$. Hence, calling G the term concerning the interaction with the particles, the function u obeys the following equation

$$\begin{cases} \rho_f(\partial_t + \operatorname{Div}(u \otimes u) + \nabla P) - \nu \Delta u = G, \\ \operatorname{div}(u) = 0 \end{cases}$$

where ν is the viscosity of the fluid. The condition $\operatorname{div}(u) = 0$ is assumed by our hypothesis about the spreading of the particles. Here, since u is a vector field, we denoted by $u \otimes u$ the matrix with component (i, j) equal to $u_i u_j$, and with $\operatorname{Div}(u \otimes u)_i = \sum_{j=1}^d \partial_{x_j}(u_i u_j)$. Due to the incompressibility conditions one also has $\operatorname{Div}(u \otimes u) = (u \cdot \nabla)u$. In system (1.1) we set all the constants identically to one for the sake of simplicity.

On the other hand in the equation for the particle density, by denoting with H the acceleration of the fluid on the particles, the function F satisfies the following

Vlasov-type equation

$$\partial_t F + v \cdot \nabla_x F + \mathrm{div}_v(HF) = \frac{\sigma^2}{2}\Delta_v F. \tag{1.2}$$

The right hand side represents the diffusivity of the system with velocity σ, corresponding the Brownian motion of the particles. Here we will treat always the case $\sigma > 0$. The case $\sigma = 0$ is usually named in the literature simply as Vlasov-Navier-Stokes. In this equation, if one assume particles have mass m, then $m \cdot H$ is the force exerted on the particles by the fluid. Note also that due to the differential structure of the equation, it enjoys the property of conservation of mass

$$\frac{d}{dt}\int\int F(t,x,v)\,dx\,dv = 0.$$

Assuming that particles are not affected by gravity, which corresponds to assuming the density of the particles ρ_p is comparable to that of the fluid ρ_f, the force of the fluids on the particles reduces to the *Stokes drag force*

$$m \cdot H = m(u(t,x) - v).$$

This force is proportional to the relative velocity of each particle with respect to that of the fluid in its position. Therefore equation (1.2) can be seen as a Vlasov-type equation with friction and with forcing term proportional to $u(t,x)$

$$\partial_t F + v \cdot \nabla_x F - \mathrm{div}_v(vF) + \mathrm{div}_v(uF) = \frac{\sigma^2}{2}\Delta_v F.$$

Moreover, the opposed force exerted by all the particles on the fluid is obtained by Newtons Third Law an by summing up all the contributions

$$G = -\int mHF\,dv = -m\int (u(t,x) - v)F\,dv.$$

Again in system (1.1) all the constants, apart from the diffusion σ that is left explicit, are taken to be identically one.

Thanks to the counter effect structure given by Newton Law, this system enjoys an energy condition: setting

$$\mathcal{E}(t) = \frac{1}{2}\left(\int |u(t,x)|^2\,dx + \int\int |v|^2\,F\,dx\,dv\right)$$

one has

$$\frac{d}{dt}\mathcal{E}(t) + \int |\nabla u(t,x)|^2\,dx + \int\int |u-v|^2\,F\,dx\,dv = \frac{\sigma^2}{2}$$

In the case $\sigma = 0$ this implies the dissipation of energy property. The case where $\sigma \neq 0$ is more intricate. In this case, by talking into account also the entropy of the particle ensemble one has

$$\frac{d}{dt}\left(\mathcal{E}(t) + \int\int F \log F \, dx\, dv\right) + \int\int \frac{\left|(u-v)F - \frac{\sigma^2}{2}\nabla_v F\right|^2}{F} dx\, dv + \int |\nabla u(t,x)|^2 \, dx = 0$$

This property does not directly imply any entropy inequality since the term $F \log F$ has an indefinite sign. However the negative components of the latter can be controlled (we omit some details here) leading to the following entropy conservation law:

$$\frac{d}{dt}\left(\mathcal{E}(t) + \frac{1}{2}\int\int |x|^2 F \, dx\, dv + \int\int F\,|\log F| \, dx\, dv\right) + \int\int \frac{|(u-v)F - \nabla_v F|^2}{F} dx\, dv + \int |\nabla u(t,x)|^2 \, dx = 0.$$

Moving on to the particle model, let us start by introducing the discrete-continuous dynamical system:

$$\begin{cases} \partial_t u^N = \Delta u^N - u^N \cdot \nabla u^N - \nabla \pi^N - \frac{1}{N}\sum_{i=1}^N (u^N_{\varepsilon_N}(X_t^{i,N}) - V_t^{i,N})\theta_{\varepsilon_N}(x - X_t^{i,N}) \\ \operatorname{div}(u^N) = 0, \\ \begin{cases} dX_t^{i,N} = V_t^{i,N}\, dt \\ dV_t^{i,N} = ((\theta_{\varepsilon_N} * u^N)(X_t^{i,N}) - V_t^{i,N})\, dt + \sigma dB_t^i \end{cases} \quad i = 1, \dots, N. \end{cases} \tag{1.3}$$

We say this system is discrete-continuous since, while particles are treated as individuals, the fluid is still described by a continuous quantity. In the previous system we assume to have N identically distributed particles that are determined by finite dimensional SDEs. Each particle is identified by the pair $(X_t^{i,N}, V_t^{i,N})$ which stands for position and velocity. The equation for the fluid is the classical Navier-Stokes type but with the addition of an interaction term with the particles, which this time is of discrete type.

Let us go a bit more into the details of the discrete model. Starting from the equation for the particles we see that the dynamics is of second order. In the expression for the velocity note that particles are affected by a random perturbation in the form of Brownian motion. The diffusion coefficient σ is directly related to the term $\frac{\sigma^2}{2}\Delta_v F$ appearing in equation (1.1). Moreover, the drift is given by the

difference between the relative velocity of the particle and a local average of the fluid. This idea is embedded in the expression

$$(\theta_{\varepsilon_N} * u^N)(X_t^{i,N})$$

where $\theta_{\varepsilon_N} = \varepsilon_N^{-d}\theta(\varepsilon_N^{-1}x)$ is a mollifier (ideally with compact support) which expresses the local area where the velocity of the fluid is averaged. The opposed mechanism of particles on the fluid is given, as in the PDE case, by Newton Third Law by summing all the contributions. Each particle affects the fluid in a neighborhood of its position, with an intensity that is proportional to the relative velocity, leading to the interaction term

$$\frac{1}{N}\sum_{i=1}^{N}(u_{\varepsilon_N}^N(X_t^{i,N}) - V_t^{i,N})\theta_{\varepsilon_N}(x - X_t^{i,N}).$$

The area of interaction is the same as the one in the expression for the velocity, i.e. is given by the mollifier θ_{ε_N}. Here the factor $\frac{1}{N}$ in front of the summation stands for the mass of each particle. By this mean the total mass of the particle ensemble is preserved as in the PDE case. Concerning physical quantities related to the mass, let us look more closely at the volume occupied by each particle. This size is strictly related to the scaling factor introduced in the mollifier θ_{ε_N}. The subject of the proper scaling is difficult topic. In this chapter we will assume that ε_N satisfies

$$\varepsilon_N = N^{-\beta} \quad \beta \leq \frac{1}{4} \tag{1.4}$$

in the 2-dimensional case which is the case that we will treat more rigorously (the factor 1/4 is strictly related to the dimension of the problem). This hypothesis which may appear irrelevant is really crucial when dealing with interacting particle system. It is connected with the topic of *long* or *short-range* interactions. This matter is very extensive and would need a separate discussion by itself, but let us try to explain it briefly. As we said we assumed our particles to have each mass $\frac{1}{N}$. At the same time we imposed our particle to have a volume (the local area around which they interact with the fluid) which is given by the scaled mollifier θ_{ε_N}. Assuming by simplicity that θ is of compact support of radius one, the radius of θ_{ε_N} is ε_N^{-1} hence the volume of each particle is of order $\varepsilon_N^{-2} = N^{-2\beta}$. If we want to have particles of constant density, we see that the volume of each should be of order $\frac{1}{N}$, i.e. the volume should shrink at the same rate of the mass. Hence we should have $\beta = 1/2$ so that $\varepsilon_N^{-2} = \frac{1}{N}$. However this choice is prohibited by assumption (1.4). The difficulties behind hypothesis (1.4) are very deep (see Lemma 2.5.5 in Chapter 2) and we will not go further into more details here. Let us only mention that the arguments about rescaling made here fall into the topic of *moderate interactions*, that is a theory developed to be in between of short and long range interactions.

By the discussion made above we see that the description of a microscopic model for particle-fluid interaction is far from being completely rigorous. In fact the proper behavior of a solid object which is suspended in a fluid is a very difficult topic by itself. Including these difficulties in a model which consider infinitely many particles would have been of a tremendous intricacy. Hence in this first work we decided to adopt a more phenomenological description of the phenomenon. Therefore we neglected some of the difficulties strictly related to fluid-dynamics and focus more on other technical problems associated to the macroscopic limit.

1.2 Self interacting network: the case of the mycelium

What is presented in this section, and in the corresponding chapter, is based on the work [21]. In Chapter 3 we are interested in the study of a biological problem: the growth of the *hyphae*. Hyphae are a long filamentous structure that contribute to make the vegetative structure of a fungus. Collectively many hyphae form the so-called *mycelium*. In this chapter we deal with a system of kinetic particles as in the previous chapter. However here we focus more on the modeling side, since we don't have in mind a specific macroscopic model to approximate. Hence our aim is to analyze all the phenomena involved in the growth and propose a microscopic model, together with a scaling limit result. Moreover, since we have in mind to continue this study with the aid of some numerical simulations, we also propose a regularity result for the solution of macroscopic PDE that describes the density of the hyphae.

The growth of the hyphae, as well as that of any complex network, is a very intricate process. Think for example about a system of neurons interacting through electrical signals. In order to develop the graph, neurons needs to increase their connectivity by either connecting with new nodes or by forming new relations with other nodes. In both cases the development is lead by individuals, and modifications happening between them affect the global behavior. The same mechanism works for the hyphae, which is why an individual based model can be meaningful to investigate the phenomenon. For this reason we decided to adopt this strategy by describing the motion of the *tips* of each hypha. By *tip*, we mean the head of any filament that compose the mycelium, and whose growth is responsible for the expansion. In fact branches tend to increase their length by expanding in search for nutrients, allowing the mushroom to sustain. Moreover, as seen in Figure 1.1 the linear expansion of filaments is not the only mechanism involved in the growth. The most relevant tool that the mycelium has to expand is the formation of new tips by a mechanism of branching. By *branching* we mean the creation of a new tip from an existing one, or by creating a ramification from a filament. One can also observe experimentally that branches can also coalesce. In fact when two moving

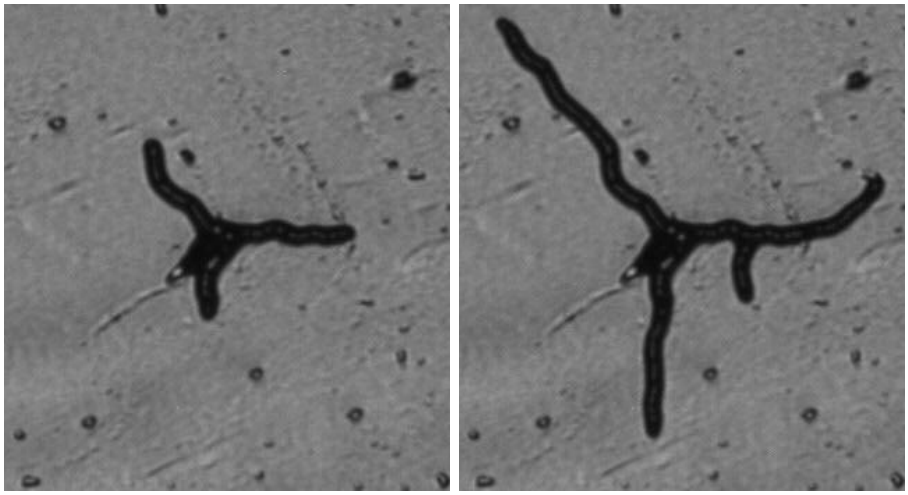

Figure 1.1: Consecutive snapshot of the growth of multiple tips from a single spore. See the increase in length of branches and the bifurcation into additional ones.

filaments meet they merge, forming a loop. With respect to Chapter 2 we see that here the total mass is not conserved since proliferation can occur. The same behavior will be observed at the PDE-level where proliferation will appear as zero order term in the Fokker-Planck equation.

We start by describing the motion of the tips. These are modeled by a second order differential equation:

$$\begin{cases} dX_t^{i,N} = V_t^{i,N} dt \\ dV_t^{i,N} = -\lambda V_t^{i,N} dt + \nabla C^N(t, X_t^{i,N}) dt + \sigma dB_t^i \end{cases} \qquad t \in [T^{i,N}, \Theta^{i,N}) \tag{1.5}$$

where the pair $(X_t^{i,N}, V_t^{i,N})$ stands for position and velocity. We study this system in $\mathbb{R}^d$, even if the main case that we have in mind is $d = 2, 3$. Here B_t^i are independent Brownian motions, and the diffusion coefficient $\sigma > 0$ takes into account the wobbling that we observe experimentally in the growth of the hyphae. The factor λ is the friction coefficient, that expresses the effort that hyphae have to make to move into a solid medium. The motion of particles is driven by the term $\nabla C^N(t, X_t^{i,N})$ which represent the concentration of nutrients that is needed for the expansion and that couples the equations. As we aforementioned one key factor to include in the model is branching, i.e. proliferation of particles. As we see from system (1.5) each particles is active for a certain life-span $[T^{i,N}, \Theta^{i,N})$. The random time $T^{i,N}$ expresses the birth time of the i-th tip. We assume to start at time zero by N individual particles ($T^{i,N} = 0$ for $i = 1, \dots, N$). Afterwards that may increase after a bifurcation event. Hence we consider the total number of

particles which lived up to time t, call it N_t^N. Note that we specifically consider individuals which have been alive up to a certain time since, as mentioned above, coalescence of branches can occur. Living particles representing moving tips can disappear from the system after a coalescence event, as specified by the random time $\Theta^{i,N}$.

The mathematics of proliferation is far from being new in the field of interacting diffusions. We will not report here a detailed survey on the mathematical techniques used in that field, to give more attention to some less known issues. Let us only mention that, as done in the literature, we can handle the mathematics of proliferation by means of Poisson Point processes. To do so one has to specify what is the distribution of points generated by the point process. In our case, since we have to include the possibility to have branching from the present filament, we introduce $\delta_{\mathbb{X}_t^N}$ the uniform measure on the trajectory of the particles up to time t:

$$\delta_{\mathbb{X}_t^N} = \frac{1}{N}\int_0^t \sum_{i=1}^{N_s^N} \mathbb{1}_{[T^{i,N},\Theta^{i,N})}(s)\left|V_s^{i,N}\right|\delta_{X_s^{i,N}}(dx)ds. \tag{1.6}$$

This is the main tool that we will use in Chap 3. For any given t this is a measure over the space $\mathbb{R}^d$ which behaves like a Dirac delta on particles, but instead is spread along all the trajectories. It gives mass only to points that belong to some of the particles trajectories. More than that, it is corresponds to the *uniform measure* on trajectories, i.e. the 1-d Hausdorff measure. The fact that (1.6) corresponds to the uniform measure is not trivial. The key point is the presence of the velocity $\left|V_s^{i,N}\right|$ inside the formulation. As one does when computing the length of a curve $\gamma : [0,t] \to \mathbb{R}^d$

$$L(\gamma) = \int_0^t |\gamma'(s)|\ ds$$

where the length needs to be computed through the use of the velocity, the same happens here. Note that in case of system (1.5) this was possible only due to the Langevin dynamic. By using a first order equation that would not have been possible due to the lack of regularity of the Brownian paths that doesn't allow to take derivatives in the classical sense.

We finally also introduce the equation for the concentration C^N that couples the particle equations

$$\partial_t C^N = \frac{\sigma_C^2}{2}\Delta C^N - (K_C * \delta_{\mathbb{X}_t^N})C^N. \tag{1.7}$$

The rationale behind equation (1.7) is the following: the hyphae in order to expand and to sustain themselves need to absorb some of the nutrients that permeate the environment. Hence the term $(K_C * \delta_{\mathbb{X}_t^N})$ with a negative sign expresses this

absorption in an local area around the trajectories of the tips, whose range is prescribed by the kernel K_C.

Here, we highlight that one of the main difficulties that arises during the study of this system is related to proliferation. Usually with proliferating processes one needs to have an estimate on the average number of individuals that appear in any finite time horizon, $\mathbf{E}\left[N_T^N\right]$. In the case of duplicating cells usually this is achieved by considering a Yule process that dominates the branching process under study. It is enough to consider a process with a rate which is bigger than the random rate of proliferation of the cells considered. However here this strategy was not feasible due to branching happening along all the filaments. This prevents us from establishing an upper-bound on the rate of proliferations as in the classical case and required the development of new techniques that where suitable in this framework.

1.3 A numerical method for the ergodic McKean-Vlasov SDE

In this chapter, based on the work [3], we focus on some numerical aspects related to interacting particle systems. Systems of interacting diffusions and their corresponding Mean-Field limits have been intensively studied since the works of McKean. These interacting diffusions pave a way to probabilistic representations for many important nonlinear or/and nonlocal PDEs, but provide a great challenge for Monte Carlo simulations. The non-linear dependence of the approximation bias due to the approximation of the dynamics and the statistical error due to the approximation of the measure renders classical variance reduction techniques not directly applicable and consequently simulations become numerically prohibitive. Usually the main difficulty is that, especially in the high dimensional setting, to obtain a good approximation one needs to consider a very big number N of particles. Consequently, since all particles can interact with each other, the cost of simulating grows quadratically in the number N, making the computational cost prohibitive. The high computational cost is even more pronounced when the aim is to simulate particle systems over a long-time horizon. The interest in the long-time simulations arises due to study of the equilibrium/invariant measures for associated McKean-Vlasov SDEs. In this chapter we specifically tackle the problem of approximating the McKean-Vlasov equation for large times.

Let us start by introducing the main subject of our study i.e. the McKean-Vlasov

SDE (McV-SDE)

$$\begin{cases} X_t = X_0 + \int_0^t b(X_s, \mu_s)\, ds + \int_0^t \sigma(X_s, \mu_s)\, dB_s \\ \mu_t \text{ is the law of } X_t. \end{cases} \tag{1.8}$$

The peculiarity of this equation is the dependence of b and σ on the law μ_t of the solution. Existence and uniqueness of solutions, also for infinite times, has been widely studied under different regularity assumptions for the coefficients. In the previous chapters of this thesis we worked with limiting PDEs for particle system in the form of Fokker-Planck equations. Here, even if we devote more attention at the McV-SDE, the problems are strictly connected with what we have done before. We recall that, with proper hypothesis on the coefficients, the law of the process X_t satisfies the following nonlinear Fokker-Planck equation

$$\frac{\partial}{\partial t} f(t,x) = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} \big(a_{i,j}(x, f(t,x)) f(x)\big) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \big(b(x, f(t,x)) f(t,x)\big), \tag{1.9}$$

where $a(x,y) = \sigma(x,y)\sigma(x,y)^t$. Here we don't aim to present a complete discussion on the relation between systems (1.8) and (1.9). We only highlight that in this chapter we preferred to give more relevance to the McV-SDE. All the results obtained here can hence be transposed with the suitable caution to the study of the non-linear Fokker-Planck equation (1.9).

As we mentioned before we are particularly interested in the study of the invariant distribution for equations (1.8). We will always consider coefficients b and σ for which the solution converges to the stationary distribution π. One of such examples is the following specific case of the McV-SDE

$$X_t = X_0 - \int_0^t \alpha X_s\, ds + \int_0^t \beta \mathbf{E}\left[X_s\right]\, ds + B_t.$$

We highlight that the dependence on the law of X_t is present only in the drift part. Moreover the dissipative term $-\alpha X_t dt$ is crucial in order to have convergence to equilibrium. In fact, calling μ_t the law of X_t and π the invariant distribution, we have

$$\mathcal{W}_2(\mu_t, \pi) \leq e^{-2(\alpha-\beta)} \mathcal{W}_2(\mu_0, \pi),$$

where $\mathcal{W}_2$ is the 2-Wasserstein distance. If $\alpha > \beta$, i.e. if the dissipation is stronger than the McKean-Vlasov part of the drift, we have convergence to the equilibrium exponentially fast. Exponential type of convergence has been proved on more general assumptions, see Chapter 4 for more extensive references on the topic. The

aim of this chapter is to analyze multiple particle representations that will allow us to approximate

$$\int_{\mathbb{R}^d} f(x)\pi(x)\,dx$$

where $f : \mathbb{R}^d \to \mathbb{R}$ is a suitable observable, for a general McV-SDE with invariant distribution π.

Let us consider the following particle system, approximating (1.8)

$$\begin{cases} X_t^{i,N} = X_0^{i,N} + \int_0^t b(X_s^{i,N}, S_s^N)\,ds, + \int_0^t \sigma(X_s^{i,N}, S_s^N)\,dB_s^i \\ S_t^N = \dfrac{1}{N}\displaystyle\sum_{i=1}^N \delta_{X_t^{i,N}}. \end{cases} \tag{1.10}$$

The measure S^N is called the empirical measure of the particle system. The expression in the drift $b(X_t^{i,N}, S_t^N)$ can be made more explicit in those cases where the drift b (analogue for the diffusion coefficient) has the specific form

$$b(x, \mu) = \int_{\mathbb{R}^d} \tilde{b}(x, y)\,\mu(dy)$$

for some $\tilde{b} \to \mathbb{R}^d \to \mathbb{R}$. In fact one has

$$b(X_t^{i,N}, S_t^N) = \frac{1}{N}\sum_{j=1}^N \tilde{b}(X_t^{i,N}, X_t^{j,N}).$$

Under some regularity assumptions on the coefficients (e.g. Lipschitzianity) one can prove that the law of any subset of size k of $\{X^{i,N}\}$ converges as a probability measure, as N tends to infinity, to the product measure $\mu^{\otimes k}$ where μ is defined as in (1.8). This property is called *Propagation of chaos* and holds even with more general assumptions on the coefficients.

Particle system (1.10) represents the classical approach to the problem and that we will use as a comparison. We now introduce a new particle representation where we consider particle grouped in M *ensembles* or *clouds*, each one made of N particles

$$\begin{cases} X_t^{(i,N),(j,M)} = X_0^{(i,N),(j,M)} + \int_0^t b\big(X_s^{(i,N),(j,M)}, S_s^{N,j}\big)\,ds + \int_0^t \sigma\big(X_s^{(i,N),(j,M)}, S_s^{N,j}\big)\,dB_s^{i,j}, \\ S_t^{N,j} = \dfrac{1}{N}\displaystyle\sum_{i=1}^N \delta_{X_t^{(i,N),(j,M)}}, \quad i = 1, \dots, N, \quad j = 1, \dots, M. \end{cases} \tag{1.11}$$

Here $B^{i,j}$ are independent Brownian Motions for each i, j. The idea is to have ensembles made of N particles that interact with each other, but that are independent from all the other particles that belong to different clouds. Hence for j fixed, particles $(X^{(i,N),(j,M)})_{i=1,\dots,N}$ interact, and they are independent any of the $(X^{(i,N),(j',M)})_{i=1,\dots,N}$ for $j' \neq j$. Moreover we will also follow an alternative approach based on ergodic average. This second idea is based on the analysis of the following interacting diffusion

$$Z_t = X_0 + \int_0^t \left(\frac{1}{s} \int_0^s b(Z_s, Z_r) dr \right) ds + \int_0^t \left(\frac{1}{s} \int_0^s \sigma(Z_s, Z_r) dr \right) dB_s. \qquad (1.12)$$

In fact, we expect the process Z_t to be a good approximation of X_t when t is large. This intuition is due to the ergodicity in the form of exponential convergence to the equilibrium measure. Moreover we will apply the ensembles of particle strategy also to equation (1.12). In this chapter we will perform the analysis of the cost of each method, by computing the mean-squared error of our approximation, and compare the results with the naive particle approximation (1.10).

1.4 A numerical method for Kolmogorov equation in high dimension

In this chapter, based on the work [51], we perform some analysis on Kolmogorov equation in infinite dimension, and propose a numerical method for the finite but high dimensional setting. The numerical analysis of PDEs in high dimension remains one of the most challenging problems which is still to be solved. In many disciplines, such as geophysics or climate study, the number of degrees of freedom to be considered is enormous. In fact a large number of phenomena is involved in these studies, leading to an incredible intricate contraption. Moreover the use of many parameters is needed in order to obtain the desired accuracy in these field. Of course this requirement of accuracy is counterbalanced by the incredible difficulty of dealing with such an intricate system. For this reason these and many other fields of study have the need of solving very high dimensional problems. Specifically in all those frameworks where the evaluation of the parameters of the problem is impossible, it becomes more and more relevant to quantify how the uncertainty on the parameters affect the uncertainty of the results. For these reasons the study Fokker-Planck or Kolmogorov equations becomes significant to tackle these problems.

In this short introduction, to make the discussion more clear, we will treat all

the examples in finite dimension. Our starting point is the Kolmogorov equation

$$\begin{cases} \partial_t u(t,x) = \dfrac{\sigma^2}{2}\mathrm{Tr}\left(QD^2u(t,x)\right) + \langle Ax + B\left(x\right), Du(t,x)\rangle\,, \\ \quad u(0,x) = u_0(x). \end{cases} \tag{1.13}$$

Since we treat the problem in finite dimension here $u(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a scalar function over $\mathbb{R}^d$, $A, Q \in \mathbb{R}^{d\times d}$ are matrices, Q is positive definite and symmetric. B is a non-linear vector field that we will always assume to be bounded measurable. Kolmogorov equation is usually written in backward formulation. In this chapter we will always use the forward formulation to make the numerical analysis more clear. We used the notation D and D^2 to denote the first and second spatial partial derivatives and by $\langle\cdot,\cdot\rangle$ the scalar product. By this choice the notation is consistent with the infinite dimensional case where the euclidean space $\mathbb{R}^d$ is replaced by a Hilbert space H. The SDE associated with equation (1.13) is the following

$$\begin{cases} \mathrm{d}X_t^x = \left(AX_t^x + B\left(X_t^x\right)\right)\mathrm{d}t + \sigma\sqrt{Q}\,\mathrm{d}W_t, \\ \quad X_0^x = x. \end{cases} \tag{1.14}$$

where W_t is a Brownian motion over $\mathbb{R}^d$ and $\sqrt{Q}$ is the standard square root of Q, since Q is a symmetric positive definite matrix.

The relation between the previous SDE and equation (1.13) is

$$u(t,x) = \mathbf{E}\left[u_0(X_t^x)\right]. \tag{1.15}$$

This connection gives the idea to compare Kolmogorov equation to probabilities related to the stochastic process X_t^x. By taking $u_0(x) = \mathbb{1}_{\{||x||\geq R\}}$ one has

$$\mathbf{E}\left[u_0(X_t^x)\right] = \mathbf{P}\left(||X_t^x|| \geq R\right).$$

Moreover, by equation (1.15), we can also deduce the naive Monte-Carlo method that can be used to solve Kolmogorov equation

$$u(t,x) = \mathbb{E}\left[u_0(X_t^x)\right] \approx \frac{1}{N}\sum_{i=1}^{N} u_0\big(X_t^{x,i}\big), \tag{1.16}$$

where N is the number of samples considered, and the processes $X_t^{x,i}, i = 1,\ldots,N$ are independent copies of X_t^x

Here we propose a new numerical method to improve the results of classical Monte-Carlo method, that we will use as a comparison. It is based on some

Gaussian analysis technique, trying to see the solution of the non-linear problem (1.14) as a perturbation of the linear equation

$$\begin{cases} \mathrm{d}Z_t^x = AZ_t^x\,\mathrm{d}t + \sqrt{Q}\,\mathrm{d}W_t, \\ Z_0 = x. \end{cases} \tag{1.17}$$

Considering the semigroup $S_t f(x) = \mathbf{E}\left[f(Z_t^x)\right]$ the mild formulation of Kolmogorov equation (1.13) is the following

$$u(t,x) = (S_t u_0)(x) + \int_0^t \big(S_{t-s}\langle B, Du(s)\rangle\big)(x)\,\mathrm{d}s. \tag{1.18}$$

This suggests us to consider the iterative scheme:

$$u^{n+1}(t,x) = (S_t u_0)(x) + \int_0^t \big(S_{t-s}\langle B, Du^n(s)\rangle\big)(x)\,\mathrm{d}s$$

with $u^0(t,x) = (S_t u_0)(x) = \mathbb{E}u_0(Z_t^x)$ and to take $u^n(t,x)$ as an approximation of the solution for n large enough. Of course to apply this strategy one needs to be able to compute efficiently the expression $\big(S_{t-s}\langle B, Du^n(s)\rangle\big)(x)$ which is far from obvious. The way we are able to compute this expression is quite intricate and we do not discuss it here entirely. We only present the following equation, which is fully explained in Chapter 5

$$I^n(t,x) = \int_0^t \mathrm{d}r_n \int_0^{r_n} \mathrm{d}r_{n-1} \cdots \int_0^{r_2} \mathrm{d}r_1 \prod_{i=1}^{n} \Big\langle \Lambda(r_{i+1}-r_i) B\big(Z_{r_i}^x\big), Q_{r_{i+1}-r_i}^{-1/2}\big(Z_{r_{i+1}}^x - e^{(r_{i+1}-r_i)A} Z_{r_i}^x\big)\Big\rangle,$$

$$u^{n+1}(t,x) = u^n(t,x) + \mathbb{E}\big[u_0(Z_t^x) I^{n+1}(t,x)\big].$$

The previous formula has a very important feature that we now try to explain here. To compute iterates $u^n(t,x)$ one only needs to compute the process Z_t^x and never the original SDE (1.14). This poses a significant advantage when one has to approximate the solution $u(t,x)$ in several spatial points $x \in \mathbb{R}^d$. With the naive Monte-Carlo method, to change the point x, one has to recompute all the samples on all X_t^x and then average. Here instead it is enough to store the samples obtained from the process Z_t^0 (i.e. Z_t^x starting from $x = 0$), compute Z_t^x by linearity and then compute the approximations $u^n(t,x)$. This improvement is highlighted more explicitly in Chapter 5 where we compare the computational time required to compute the solution $u(t,x)$ for many different values of x. Of course, the same reasoning on changing x holds by changing any of the other parameters that are not related to the Gaussian process Z_t^x, such as the non-linearity B.

The gain obtained when changing parameters and avoiding to recompute the stochastic process X_t^x is even more underlined when the dimension of the problem d is large. In fact in this case the cost of re-sampling from X_t^x is even higher than it would have been for smaller values of d. Moreover as we mentioned above, the theoretical analysis that we performed concerning the convergence of u^n to u is performed in infinite dimension on Hilbert spaces. For this reason the rate of convergence of iterates u^n to the real solution is independent of the dimension of the problem. This fact support the numerical method and emphasize the independence from the many issues related to dimensionality.

Part I

Kinetic Particle systems: scaling limit

Chapter 2

Particle-fluid interaction: the Vlasov-Fokker-Planck-Navier-Stokes system

2.1 Introduction

The results of this chapter are included in the works [47], [48]. In the theory of multiphase flows, the coupled PDE system called Navier-Stokes-Vlasov-Fokker-Planck is a way of modeling the behavior of a large number of particles immersed into a fluid. It is made by two major components: a vector field u, representing the velocity of the fluid at a given time and position, and a scalar valued function F, representing the density on phase space of the particles immersed in the fluid. In the incompressible case, when the interaction between particles and fluid is modelled by Stokes drag force, the system is given by the following equations

$$\begin{cases} \partial_t u = \Delta u - u \cdot \nabla u - \nabla \pi - \int_{\mathbb{R}^d} (u - v) F \, dv; \\ \operatorname{div}(u) = 0; \\ \partial_t F + v \cdot \nabla_x F + \operatorname{div}_v((u - v)F) = \frac{\sigma^2}{2} \Delta_v F. \end{cases} \tag{2.1}$$

Often the case $\sigma = 0$ is considered in the literature. Here we deal with the case $\sigma > 0$ because of technical reasons. The case $\sigma = 0$ is usually called Vlasov-Navier-Stokes (VNS); the case $\sigma > 0$, Navier-Stokes-Vlasov-Fokker-Planck. In the sequel, for simplicity of notations, we will often call VNS also the system above with $\sigma > 0$.

The PDE description for the density of particles is reasonable when the number of particles is very large and overcomes the problem of describing the details of

each single particle. The aim of this chapter is to prove that this simplification is correct: we prove that a system composed by Newtonian particles and fluid converges to the PDE system when the number of particles tends to infinity.

The mathematical analysis of the coupled system (2.1) in dimension $d = 2, 3$ has received much attention in the past years. A first result of global existence of weak solutions and large asymptotic for Stokes-Vlasov system in a bounded domain appeared in [61]. Existence of weak solutions has been extended to the Navier-Stokes case, hence including the convection term in the equation for the fluid, in a periodic domain in [16]. Global existence of smooth solutions with small data for Navier-Stokes-Vlasov-Fokker-Planck was obtained first in [57]. In [94] global existence for smooth solutions is generalized for large data. Recent results on the topic of uniqueness have been obtained in the case $\sigma = 0$ in [63]. We shall prove a variant of these results adapted to the regularity of our solutions. Uniqueness plays a fundamental role in the mathematical problem we are interested in; existence is less relevant because it is obtained as a byproduct of our convergence result.

As said above, the aim of this work is to investigate a coupling between the fluid and a particle system, which converges, in the limit of large number of particles, to system (2.1). The literature on this topic is still fragmentary. The works [58] [59], present results of PDE to PDE convergence, only implicitly motivated by particle arguments. The works [1], [2], [8], [37], [38], [44], [64] aim to treat links between particles and fluid but, in the trade-off between different levels of mathematical complexity and physical realism: there in a simplified fluid regime, the correct boundary condition for the interaction between finite size particle and the fluid is included. Compared to these works, our choice here is a sort of phenomenological description of interaction between particles and fluid, that keeps the structure of Stokes drag force and that maintains the usual Navier-Stokes regime. Our attention is devoted to others technical problems related to the macroscopic limit, instead of the very difficult problem of the precise boundary conditions between particles and fluid. The microscopic system considered here has the form

$$\begin{cases} \frac{\partial u^N}{\partial t} = \Delta u^N - u^N \cdot \nabla u^N - \nabla \pi^N - \frac{1}{N} \sum_{i=1}^N \left(u^N_{\varepsilon_N}(t, X^{i,N}_t) - V^i_t \right) \delta^{\varepsilon_N}_{X^{i,N}_t}, \\ \operatorname{div}(u^N) = 0, \\ \begin{cases} dX^{i,N}_t = V^{i,N}_t dt, \\ dV^{i,N}_t = \left(u^N_{\varepsilon_N} \left(t, X^{i,N}_t \right) - V^{i,N}_t \right) dt + \sigma dB^i_t \end{cases} \end{cases}$$

where N is the number of particles and $(X^{i,N}_t, V^{i,N}_t)$ are position and velocity of the particles. The equations for the fluid velocity and pressure (u^N, π^N) are given by the classical Navier-Stokes equations for an incompressible Newtonian fluid with an interaction with particles of discrete type. We choose a phenomenological

description of the interaction:
i) the intensity of the force exerted by the fluid on each single particle is given by the difference between the particle velocity and a local average of fluid velocity around particle position

$$u^N_{\varepsilon_N}(t, X^{i,N}_t) = (\theta^{0,\varepsilon_N} * u^N_t)(X^{i,N}_t);$$

ii) viceversa the force exerted by each single particle on the fluid is given by Newton's third law: the intensity of the force is the same in (i), but with the opposite sign. Moreover we impose an action distributed in a small neighbor of particle position, as described by the mollified delta Dirac function

$$\delta^{\varepsilon_N}_{X^{i,N}_t}(x) = \theta^{0,\varepsilon_N}(x - X^{i,N}_t).$$

The choice to use local averages and locally distributed action is obviously an artefact, convenient for the mathematical investigation; still it preserves the idea that particles are not just points but finite objects, or at least objects with a finite action radius, a sort of small boundary layer of interaction with the fluid.

Finally, let us comment on our previous work [48], and also on [45]. They both deal with a similar particle system coupled with the fluid and the question of its scaling limit. However, they are affected by important restrictions. The paper [45] discusses only the so called two steps approach. In this setting one keeps ε fixed when $N \to \infty$ and removes ϵ only later, as a second step. As usual, the analysis of such disjoint limits is much simpler: the first step is a classical mean field problem (opposite to the problem considered here, see the next section on the technical difficulties), the second step is a question of convergence of PDEs to PDEs (essentially a repetition of schemes known from the proofs of existence theorems for the limit system). One can mix the parameters a posteriori, taking subsequences, but the conditions on the link are quite unrealistic and restrictive. As in the present work, the paper [48] treats the joint limit in the two parameters, but a special bounded modification of Stokes law is required and due to lack of a suitable uniqueness result, we prove only convergence of subsequences. Compared to [45], [48], the result proved here is complete, without the main restrictions of those works. For future research, however, it would be interesting to extend further the range of the parameter β that quantifies the radius of interaction between a particle and the surrounding fluid. See below and in the same vein how to treat more realistic boundary conditions between particles and fluid.

2.1.1 Difficulties

In this subsection we aim to highlight the difficulties we met in proving the convergence from the discrete to the continuous model. Apparently it looks a mean field result but several aspects are far from standard, as we now describe.

Uniform control on velocity and vorticity creation by particles

The rough structure of the particle approximation used here is of a mean field type. The empirical measure S_t^N of the particles

$$S_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{(X_t^{i,N}, V_t^{i,N})}$$

(see also Section 2.2) will be proved to converge to the solution $F_t(x,v)$ of the Vlasov component of our system (in parallel, the approximation of the velocity field will converge to the limit velocity field). However in classical mean field problems, first it is proved that S_t^N converges to $F_t(x,v)$ in the weak sense of measures, then one can pass to the limit, thanks to the non-local structure of the nonlinear terms. In our problem, there is a main difficulty: S_t^N is coupled with the approximation $u_{\epsilon_N}^N$ of the Navier-Stokes component, in a local way. The term in the Navier-Stokes equation takes the form (see system of equations $(PS-NS)$ in Section 2.2)

$$\theta^{0,\epsilon_N} * \left(\left(u_{\epsilon_N}^N - v\right) S_t^N\right)$$

and the corresponding term in the identity satisfied by the empirical measure S_t^N (Lemma 2.3.2) has the form

$$\left\langle S_t^N, \left(u_{\epsilon_N}^N - v\right) \nabla_v \varphi \right\rangle.$$

In order to pass to the limit in the previous terms we need uniform convergence of $u_{\epsilon_N}^N$ to u.

This is a demanding property that we approach, using Sobolev embedding theorem, by controlling the first derivatives of $u_{\epsilon_N}^N$. We approach it by means of the equation for the vorticity ω^N. This strategy reveals a conceptual problem with physical content: the presence of particles in the fluid may produce vorticity. The estimates on the vorticity are far from being obvious, due to the interaction with the particles. The equation for the vorticity contains the interaction term

$$\frac{1}{N}\sum_{i=1}^{N}\left(u_{\epsilon_N}^N\left(X_t^{i,N}\right) - V_t^{i,N}\right)\nabla^{\perp}\cdot\delta_{X_t^{i,N}}^{\epsilon_N}$$

where $\delta_{X_t^{i,N}}^{\epsilon_N}$ is a smooth approximation of the delta Dirac $\delta_{X_t^{i,N}}$. Hence the term $\nabla^{\perp}\cdot\delta_{X_t^{i,N}}^{\epsilon_N}$ may induce a blow-up in the estimates, a priori. This is a key conceptual difficulty we had to overcome, among others of more technical nature. The fact that an infinitesimal particle in a fluid may produce vorticity is the topic of recent research, see [55]. These works are restricted to single particle for very difficult

technical reasons; it may be that some link with the present research will be possible in the future after due progresses.

Thanks to the fact that $\nabla\omega^N$ has a control due to the viscous term, the energy type estimate leads to control the term

$$\left\| \frac{1}{N} \sum_{i=1}^{N} \left(u^N_{\epsilon_N} \left(X^{i,N}_t \right) - V^{i,N}_t \right) \delta^{\epsilon_N}_{X^{i,N}_t} \right\|_{L^2(\mathbb{T}^2)} . \tag{2.2}$$

This is not a simple task; just to mention, the trivial estimate

$$\leq \frac{1}{N} \sum_{i=1}^{N} \left(u^N_{\epsilon_N} \left(X^{i,N}_t \right) - V^{i,N}_t \right) \left\| \delta^{\epsilon_N}_{X^{i,N}_t} \right\|_{L^2(\mathbb{T}^2)}$$

leads to diverging quantities. This introduces a new ingredient with its own difficulties, as explained in the next subsection.

The regularized empirical measure

We control the term (2.2) by introducing the regularized empirical measure $F^N_t(x,v)$

$$F^N_t(x,v) = \theta^{\varepsilon_N} * S^N_t$$

(see details in Section 2.2), inspired by works of Karl Oelschleger, see for instance [85]. It allows us to write

$$\begin{aligned}
&\left| \frac{1}{N} \sum_{i=1}^{N} \left(u^N_{\epsilon_N} \left(t, X^{i,N}_t \right) - V^{i,N}_t \right) \delta^{\epsilon_N}_{X^{i,N}_t}(x) \right| \\
&\leq \left\| u^N_{\epsilon_N}(t,\cdot) \right\|_\infty \frac{1}{N} \sum_{i=1}^{N} \delta^{\epsilon_N}_{X^{i,N}_t}(x) + \left| \frac{1}{N} \sum_{i=1}^{N} V^{i,N}_t \delta^{\epsilon_N}_{X^{i,N}_t}(x) \right| \\
&= \left\| u^N_{\epsilon_N}(t,\cdot) \right\|_\infty \int_{\mathbb{R}^2} F^N_t(x,v)\, dv + \left| \int_{\mathbb{R}^2} v F^N_t(x,v)\, dv \right| .
\end{aligned}$$

The proof of the last line is given in Lemma 2.5.3.

Now the problem is to prove suitable estimates on the regularized empirical measure $F^N_t(x,v)$. Controls on S^N_t essentially amounts to suitable estimates on the SDEs satisfied by particles, while a full treatment of $F^N_t(x,v)$ requires both SDEs properties and PDEs arguments applied to the identity satisfied by $F^N_t(x,v)$ (Lemma 2.3.2). This identity however is not closed; commutators appear and several technical difficulties arise, which perhaps are new here with respect to previous literature.

The cut-off and its removal

We are able to perform the estimates outlined above only when a suitable cut-off on velocity is introduced; see $\chi_R(u)$ introduced in Section 2.3 and appearing in the rest of the chapter. The idea is to use this truncated system as a bridge to the original one. By using the truncation in the interaction between particles and fluid we managed to produce an a priori bound independently on the number of particles N

$$\left|\left|u^{N,R}\right|\right|_\infty \leq C_R, \qquad \text{(Lemma 2.5.9)} \tag{2.3}$$

which we used to obtain a suitable tightness criterion, needed for the convergence. We remark that this bound was only possible due to the presence of the cut-off, since the constant provided in (2.3) depends on the threshold R of the truncation.

Therefore the preliminary result is that the PDE-particle system with cut-off converges to the PDE system with cut-off. However, by showing that the velocity field of the PDE system with the cut-off satisfies

$$\left|\left|u^R\right|\right| \leq C \qquad \text{(Proposition 2.5.13)}$$

independently on R, it is possible to prove that the PDE system with cut-off is also solution without cut-off. In summary we can prove that the PDE-particle system with cut-off converges to the PDE system without cut-off, see Proposition 2.5.1. The proof of this step is organized differently from the previous description but here we have explained the concept behind the proof.

The final problem is to prove that the cut-off can be removed also from the approximating PDE-particle system. This seems to be a difficult question. Here we use a special trick.

To appreciate the difficulty and the trick, think for a second to a different problem where the approximations are not random. Assume we have proved that $u^{N,R}$ converges uniformly to the limit u. Since u is uniformly bounded by a constant R_0 we deduce that, eventually in N, also $u^{N,R}$ is bounded, say, by $R_0 + 1$. Hence eventually in N, the function $u^{N,R}$ solves the equation without cut- off, hence it is equal to the unique solution u^N of such equation. Next, consider the full approximating sequence $(u^N)_{N\geq 1}$ solving the equations without cut-off; this sequence converges uniformly to u, because the property of limit involves only the tail of the sequence and the tail coincides with the tail of the sequence $\left(u^{N,R}\right)_{N\geq 1}$, which we know to converge to u. This idea resemble us the method used to prove well-posedness of 3D Navier-Stokes equations with strong rotation, see for instance [52].

Unfortunately this simple idea does not work when the approximations are random. Forget about the fact that our convergence is in law; go to another probability space where it is almost sure. Thus, almost surely, eventually we may

transfer the uniform bound R_0 of the limit solution to a bound $R_0 + 1$ for the approximations. But this time the "eventually" qualification is random! Hence, given a large N, we cannot claim that the stochastic process $u^{N,R}$ coincides with the unique solution u^N of the equation without cut-off, because the bound on $u^{N,R}$ is true only for certain ω's.

So the problem is that we have two families of stochastic processes, $\left(u^{N,R}\right)_{N\geq 1}$ and $\left(u^N\right)_{N\geq 1}$ and we know that for a.e. ω there is $N_0(\omega)$ such that for $N \geq N_0(\omega)$ the paths of the sequence $\left(u^{N,R}(\omega)\right)_{N\geq N_0(\omega)}$ are bounded, say, by $R_0 + 1 < R$. We want to deduce a relation between $\left(u^{N,R}\right)_{N\geq 1}$ and $\left(u^N\right)_{N\geq 1}$ from this property. To this aim we invoke a property of *path-by-path uniqueness* (see [46]) opposite to the usual concept of pathwise uniqueness: given ω, for $N \geq N_0(\omega)$ the path $u^{N,R}(\omega)$ satisfies the equation without cut-off (formulated for that single ω) and by path-by-path uniqueness it coincides with $u^N(\omega)$. The conclusion is the same as in the deterministic case: consider the sequence of processes $\left(u^N\right)_{N\geq 1}$; for a.e. ω, the sequence of functions $\left(u^N(\omega)\right)_{N\geq 1}$ converges to u because it coincides, eventually, with the sequence $\left(u^{N,R}(\omega)\right)_{N\geq 1}$.The first major result of path-by-path uniqueness for SDEs has been proved by [13] and it is a very sophisticated result; however, here we have additive noise and relatively smooth coefficients, hence path-by-path uniqueness in our case is not difficult. We isolated the idea behind this reasoning into a general criterion, that we applied to transfer the convergence from the particle system where the cut-off is present, to the system without the cut-off.

The structure of this chapter is the following: In Section 2.2 we introduce all the notation that we will use and we present our main result, Theorem 2.2.3. In Section 2.3 we collect some preliminary result that will be needed in the rest of the chapter, while Section 2.4 is devoted to a theorem of uniqueness for the Vlasov-Navier-Stokes system. In Section 2.5 we prove a first intermediate result, that is the convergence of the particle system with the cut-off to the Vlasov-Navier-Stokes system without the cut off. Finally, in Section 2.6 we manage to remove the cut-off also from the approximating system, ending the proof of Theorem 2.2.3.

2.2 Notation and Main Results

We begin this section by introducing rigorously the Vlasov-Navier-Stokes system and its associated particle model. We will always assume the framework of a filtered probability space, denoted by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$. For the whole chapter we will also work on the two dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The case of other bounded domains is more delicate due to creation of vorticity at the boundaries. Some of the intermediate results stated here will work also in higher dimension. However, to obtain the full result, due to uniqueness and smoothness obstacles, dimension $d = 2$

is needed, so we will always keep the dimension fixed for a matter of simplicity.

We start by recalling the Vlasov-Navier-Stokes **PDE-system**

$$\begin{cases} \partial_t u = \Delta u - u \cdot \nabla u - \nabla \pi - \int_{\mathbb{R}^2} (u - v) F(x, v)\, dv & (t, x) \in [0, T] \times \mathbb{T}^2 \\ \partial_t F + v \cdot \nabla_x F + \mathrm{div}_v((u - v)F) = \frac{\sigma^2}{2} \Delta_v F & (t, x, v) \in [0, T] \times \mathbb{T}^2 \times \mathbb{R}^2 \\ \mathrm{div}(u) = 0, \end{cases} \tag{VNS}$$

$\sigma > 0$, with initial condition $u(0, \cdot) = u_0$ and $F(0, \cdot, \cdot) = F_0$. We also introduce the **continuous-discrete Particle System** approximating (VNS):

$$\begin{cases} \partial_t u^N = \Delta u^N - u^N \cdot \nabla u^N - \nabla \pi^N - \frac{1}{N} \sum_{i=1}^N (u^N_{\varepsilon_N}(X^{i,N}_t) - V^{i,N}_t) \delta^{\varepsilon_N}_{X^{i,N}_t} \\ \mathrm{div}(u^N) = 0, \\ \begin{cases} dX^{i,N}_t = V^{i,N}_t\, dt \\ dV^{i,N}_t = (u^N_{\varepsilon_N}(X^{i,N}_t) - V^{i,N}_t)\, dt + \sigma dB^i_t \end{cases} \quad i = 1, \dots, N \end{cases} \tag{PS - NS}$$

with initial condition

$$u^N(0, \cdot) = u_0, \quad (X^{i,N}_0, V^{i,N}_0) \overset{\mathcal{L}aw}{\sim} F(0, \cdot, \cdot)\, dx\, dv \ \ i.i.d$$

namely the random variables $(X^{i,N}_0, V^{i,N}_0)$ are independent and identically distributed with density $F(0, x, v)$. In the previous equations, $(B^i_t)_{t \geq 0}$ is a sequence of independent Brownian motions, θ^0 is a mollifier over the torus, $\varepsilon_N \in \mathbb{R}^+$ is a sequence converging to zero, and

$$\theta^{0,\varepsilon_N}(x) := \varepsilon_N^{-2} \theta^0 \left(x / \varepsilon_N \right), \quad u^N_{\varepsilon_N} := u * \theta^{0,\varepsilon_N}, \quad \delta^{\varepsilon_N}_{X^{i,N}_t}(x) := \theta^{0,\varepsilon_N}(x - X^{i,N}_t),$$

All the hypothesis and requirements on the objects introduced above are collected in subsection 2.2.3.

2.2.1 Definition of weak solutions

Definition 2.2.1 (Definition of weak solution of (VNS))**.** We say a pair (u, F) is a weak solution of (VNS) if the following conditions are satisfied:

a)

$$u \in L^\infty([0, T]; L^2(\mathbb{T}^2)) \cap L^2([0, T]; H^1(\mathbb{T}^2));$$

$$F \in L^\infty([0, T]; L^1(\mathbb{T}^2 \times \mathbb{R}^2) \cap L^\infty(\mathbb{T}^2 \times \mathbb{R}^2)), \quad F \geq 0;$$

$$F\, |v|^2 \in L^\infty([0, T]; L^1(\mathbb{T}^2 \times \mathbb{R}^2));$$

b) for all $\varphi \in C^\infty([0,T]\times\mathbb{T}^2;\mathbb{R}^2)$ with $\mathrm{div}\varphi = 0$ we have

$$\langle u_t,\varphi_t\rangle = \langle u_0,\varphi_0\rangle + \int_0^t \langle u_s, \frac{\partial\varphi_s}{\partial s}\rangle ds + \int_0^t \langle u_s, \Delta\varphi_s\rangle ds + \int_0^t \langle u_s\cdot\nabla\varphi_s, u_s\rangle ds - \int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{T}^2} \varphi_s(x)(u_s(x)-v)F_s(x,v)\,dx\,dv\,ds,$$

c) for all $\psi \in C^\infty([0,T]\times\mathbb{T}^2\times\mathbb{R}^2;\mathbb{R})$ with compact support in v we have

$$\langle F_t,\psi_t\rangle = \langle F_0,\psi_0\rangle + \int_0^t \langle F_s, \frac{\partial\psi_s}{\partial s}\rangle ds + \frac{\sigma^2}{2}\int_0^t \langle F_s,\Delta_v\psi_s\rangle ds + \int_0^t \langle F_s, v\cdot\nabla_x\psi_s\rangle ds + \int_0^t \langle F_s,(u_s-v)\cdot\nabla_v\psi_s\rangle ds;$$

Definition 2.2.2 (Definition of Bounded weak solution of (VNS))**.** We say a pair (u,F) is a bounded weak solution of (VNS) if it is a weak solution and

$$u \in L^\infty([0,T]\times\mathbb{T}^2).$$

We refer to Theorem 2.4.1 for an uniqueness result for bounded weak solutions. Existence of bounded weak solutions for system (VNS) will be obtained as a consequence of our main convergence result.

2.2.2 The Empirical measure of the particle system

Before stating our main result we introduce some function spaces defined as follows. Given the space $E = \mathbb{T}^2\times\mathbb{R}^2$ we introduce

$$\mathbf{P}_1(E) = \left\{\mu \text{ probability measure on } (E,\mathcal{B}(E)) \mid \int_E |x|\ \mu(dx) < \infty\right\}$$

the space of all probability measure on the Borel sets of E, with finite first moment. We endow this space with the Wasserstein-1 metric, that can be defined equivalently as

$$\mathcal{W}_1(\mu,\nu) = \sup_{[\varphi]_{Lip}\le 1}\left|\int_E \varphi\,d\mu - \int_E \varphi\,d\nu\right|$$

where $[\varphi]_{Lip}$ is the usual Lipschitz seminorm. Endowed with this metric the space $\mathbf{P}_1$ becomes a complete separable metric space, whose convergence implies the weak convergence of probability measures.
From now on, when μ is a measure and f is a function, we will denote by $\langle f,\mu\rangle$ the integration in full space of f with respect to μ.

We now introduce the empirical measure of the particle system

$$S_t^N = \sum_{i=1}^{N} \delta_{(X_t^{i,N}, V_t^{i,N})}, \tag{2.4}$$

which is random measure on $(\Omega, \mathcal{F}, \mathbf{P})$, on the space $C([0,T]; \mathbf{P}_1(\mathbb{T}^2 \times \mathbb{R}^2))$. We will consider a smoothed version of the empirical measure: let us introduce two functions $\theta^0 : \mathbb{T}^2 \to \mathbb{R}$ and $\theta^1 : \mathbb{R}^2 \to \mathbb{R}$ which are C^∞, non negative and integrate to one. Introduce also

$$\theta(x,v) := \theta^0(x)\theta^1(v)$$

which is a function on the product space $\mathbb{T}^2 \times \mathbb{R}^2$. Consider then

$$\theta^{\varepsilon_N}(x,v) = \varepsilon_N^{-2}\theta^0(\varepsilon_N^{-1}x)\varepsilon_N^{-2}\theta^1(\varepsilon_N^{-1}v) = \theta^{0,\varepsilon_N}(x)\theta^{1,\varepsilon_N}(v) \tag{2.5}$$

and let us define

$$F_t^N(x,v) := \theta^{\varepsilon_N} * S_t^N = \frac{1}{N}\sum_{i=1}^{N} \theta^{0,\varepsilon_N}(x - X_t^{i,N})\theta^{1,\varepsilon_N}(v - V_t^{i,N})$$

the mollified empirical measure.

Remark 1. Note that the function θ^{0,ε_N} in the previous equation, appear in system $(PS - NS)$ in the coupling term.

In what follows and in the rest of the chapter we will adopt the following notation for the moments on the v component for the function F:

$$m_k F(x) := \int_{\mathbb{R}^2} |v|^k F(x,v)\, dv, \quad M_k F := \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} |v|^k F(x,v)\, dv\, dx.$$

where $m_k F(x)$ is function over $\mathbb{T}^2$ while $M_k F \in \mathbb{R}$.

2.2.3 Main Result

We summarize all the main hypotheses of our framework:

1) $u_0 \in H^2(\mathbb{T}^2)$;
2) $F_0 \in (L^1 \cap L^\infty)(\mathbb{T}^2 \times \mathbb{R}^2)$ and $M_6 F_0 < \infty$;
3) $\theta(x,v) = \theta^0(x)\theta^1(v)$, θ^0 and θ^1 mollifiers on $\mathbb{T}^2$ and $\mathbb{R}^2$ respectively, such that $|\nabla\theta^0(x)| \le \theta^0(x)$ and $\mathrm{supp}(\theta^1) \subseteq B(0,1)$. Moreover $\theta^1(v)$ satisfies the following symmetry assumption $\int_{\mathbb{R}^2} \theta^1(v) v\, dv = 0$;
4) The scaling factor ε_N satisfies $\varepsilon_N = N^{-\beta}$ with $\beta \le 1/4$;

Remark 2. We remark that hypothesis (3) is needed in Lemma 2.5.5 to obtain the first a priori estimate on the mollified empirical measure. Regarding the scaling factor in (4), this hypothesis is also needed for Lemma 2.5.5: the bound on β is strictly related to the space dimension and to the L^p norm that is computed. In our case, we will compute the L^4 norm, and the general requirement in dimension d is

$$\beta \leq \frac{d}{3d+2}.$$

In what follows we will always use the notation $\lesssim$ to indicate that the inequality holds, up to a multiplicative constant that does not depend on any of the key parameters involved. To emphasize the dependence on one of those parameter we will adopt the convention $\lesssim_X$ to denote the dependence on the parameter X. Moreover we will make use of the letter C to mark a constant, whose value does not matter for the argument.

We are finally able to present our main result:

Theorem 2.2.3. *Under hypothesis of subsection 2.2.3, the family of laws* $\left\{Q^N\right\}_{N\in\mathbb{N}}$ *of the couple* $(u^N, S^N)_{N\in\mathbb{N}}$ *is tight on* $C([0,T]\times\mathbb{T}^2)\times C([0,T];\mathbf{P}_1(\mathbb{T}^2\times\mathbb{R}^2))$. *Moreover* $\left\{Q^N\right\}_{N\in\mathbb{N}}$ *converges weakly to* $\delta_{(u,F)}$, *where the couple* (u,F) *is the unique bounded weak solution of system of equation* (VNS).

2.3 Preliminary results.

In this section we collect the basic results about our particle systems, and all the technical inequalities that will be used in the rest of the chapter.

In order to obtain Theorem 2.2.3, it is necessary to introduce another coupled system of PDE-SDE, where the interaction between the particles and fluid is truncated. Introduce for $R>0$ the cut-off function $\chi_R^0:\mathbb{R}\to[0,1]$ defined as

$$\chi_R^0(x) = \begin{cases} 1 & \text{if } x \leq R-1 \\ 0 & \text{if } x \geq R \end{cases}$$

and that is $C^\infty(\mathbb{R})$. Define also $\chi_R(u) = \chi_R^0(||u||_{L^\infty(\mathbb{T}^2)})$. With this choice of notation one has

$$||u\chi_R(u)||_{L^\infty(\mathbb{T}^2)} \leq R.$$

Introduce now the **truncated PDE-system**:

$$\begin{cases} \partial_t u^R = \Delta u^R - u^R\cdot\nabla u^R - \nabla\pi - \int_{\mathbb{R}^2}(u^R - v)\chi_R(u_t^R)F^R(x,v)\,dv \\ \partial_t F^R = \frac{\sigma^2}{2}\Delta_v F^R - v\cdot\nabla_x F^R - \mathrm{div}_v((u^R\chi_R(u^R)-v)F^R) \\ \mathrm{div}(u^R) = 0, \end{cases} \qquad (VNS^R)$$

with the same initial conditions as system (VNS). We also introduce the **continuous-discrete truncated Particle System** approximating (VNS^R):

$$\begin{cases} \partial_t u^{N,R} = \Delta u^{N,R} - u^{N,R}\cdot\nabla u^{N,R} - \nabla \pi^{N,R} \\ \qquad\qquad -\frac{1}{N}\sum_{i=1}^{N}(u^{N,R}_{\varepsilon_N}(X^{i,N,R}_t) - V^{i,N,R}_t)\chi_R(u^{N,R}_t)\,\delta^{\varepsilon_N}_{X^{i,N,R}_t} \\ \operatorname{div}(u^{N,R}) = 0, \\ \begin{cases} dX^{i,N,R}_t = V^{i,N,R}_t\,dt \\ dV^{i,N,R}_t = (u^{N,R}_{\varepsilon_N}(X^{i,N,R}_t)\chi_R(u^{N,R}_t) - V^{i,N;R}_t)\,dt + \sigma dB^i_t \end{cases} \quad i = 1,\dots,N \end{cases} \tag{$PS^R - NS^R$}$$

using the same notation and initial condition as $(PS - NS)$.

Definition 2.3.1 (Definition of bounded weak solution of (VNS^R))**.** We say a pair (u^R, F^R) is a bounded weak solution of (VNS^R) if the following condition are satisfied:

a)

$$u^R \in L^\infty([0,T]\times\mathbb{T}^2)\cap L^2([0,T];H^1(\mathbb{T}^2));$$
$$F^R \in L^\infty([0,T];L^1(\mathbb{T}^2\times\mathbb{R}^2)\cap L^\infty(\mathbb{T}^2\times\mathbb{R}^2)), \quad F^R \geq 0;$$
$$F\,|v|^2 \in L^\infty([0,T];L^1(\mathbb{T}^2\times\mathbb{R}^2));$$

b) for each divergence free, C^∞ vector field $\varphi : [0,T]\times\mathbb{T}^2 \to \mathbb{R}^2$ we have

$$\langle u^R_t,\varphi_t\rangle = \langle u^R_0,\varphi_0\rangle + \int_0^t \langle u^R_s, \frac{\partial\varphi_s}{\partial s}\rangle ds + \int_0^t \langle u^R_s, \Delta\varphi_s\rangle ds + \int_0^t \langle u^R_s\cdot\nabla\varphi_s, u^R_s\rangle ds$$
$$- \int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{T}^2} \varphi_s(x)(u^R_s(x) - v)\chi_R(u^R_s)F^R_s(x,v)\,dx\,dv\,ds,$$

c) for each C^∞ function $\psi : [0,T]\times\mathbb{T}^2\times\mathbb{R}^2 \to \mathbb{R}$, we have

$$\langle F^R_t,\psi_t\rangle = \langle F^R_0,\psi_0\rangle + \int_0^t \langle F^R_s, \frac{\partial\psi_s}{\partial s}\rangle ds + \frac{\sigma^2}{2}\int_0^t \langle F^R_s, \Delta_v\psi_s\rangle ds +$$
$$\int_0^t \langle F^R_s, v\cdot\nabla_x\psi_s\rangle ds + \int_0^t \langle F^R_s, (u^R_s\chi_R(u^R_s) - v)\cdot\nabla_v\psi_s\rangle ds;$$

Applying the maximum principle to system of equation (VNS^R) we have

$$\left|\left|F^R(t,x,v)\right|\right|_{L^p(\mathbb{T}^2\times\mathbb{R}^2)} \leq C_T\,||F_0(x,v)||_{L^p(\mathbb{T}^2\times\mathbb{R}^2)} \quad \forall p > 1$$

so that

$$\left|\left|F^R(t,x,v)\right|\right|_{L^\infty(\mathbb{T}^2\times\mathbb{R}^2)} \leq C$$

independently on R. We now introduce the empirical measure of the truncated particle system

$$S_t^{N,R} = \sum_{i=1}^N \delta_{(X_t^{i,N,R},V_t^{i,N,R})}$$

and its associated mollified empirical measure

$$F_t^{N,R}(x,v) = \theta^{\varepsilon_N} * S_t^{N,R}.$$

We now recall the identity satisfied by the empirical measure S_t^N.

Lemma 2.3.2. *For every test function $\varphi : \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{R}$ the empirical measure S_t^N satisfies the following identity*

$$d\langle S_t^N, \varphi\rangle = \langle S_t^N, v\cdot\nabla_x\varphi\rangle\, dt + \langle S_t^N, (u_{\varepsilon_N}^N - v)\cdot\nabla_v\varphi\rangle\, dt + \frac{\sigma^2}{2}\langle S_t^N, \Delta_v\varphi\rangle\, dt + dM_t^{N,\varphi},$$

with

$$M_t^{N,\varphi} = \frac{\sigma}{N}\sum_{i=1}^N \int_0^t \nabla_v\varphi\left(X_t^{i,N}, V_t^{i,N}\right)\cdot dB_t^i.$$

Moreover $F_t^N(x,v) = (\theta^{\varepsilon_N} * S_t^N)(x,v)$ satisfies:

$$dF_t^N = \frac{\sigma^2}{2}\Delta_v F_t^N - \mathrm{div}_v(\theta^{\varepsilon_N} * (u_{\varepsilon_N}^N - v)S_t^N)\, dt - \mathrm{div}_x(\theta^{\varepsilon_N} * vS_t^N)\, dt + dM_t^{N,\varepsilon_N},$$

with $M_t^{N,\varepsilon_N} = M_t^{N,\theta^{\varepsilon_N}(x-\cdot,v-\cdot)}$.

Proof. The first part follows easily by applying Itô formula to $\varphi(X_t^{i,N}, V_t^{i,N})$ and using linearity. The second part follows by taking $\varphi(\cdot,\cdot) = \theta^{\varepsilon_N}(x-\cdot, v-\cdot)$. □

The analogue of the previous result holds for the empirical measure of the truncated system $S^{N,R}$, as well as for it mollified version $F^{N,R}$. We now state the kinetic energy balance for the truncated system:

Lemma 2.3.3. *With the previous notation, we denote with $\mathcal{E}^{N,R}$ the kinetic energy of the microscopic system,*

$$\mathcal{E}^{N,R}(t) = \frac{1}{2}\int_{\mathbb{T}^2}\left|u_t^{N,R}(x)\right|^2 dx + \frac{1}{2N}\sum_{i=1}^N\left|V_t^{i,N,R}\right|^2.$$

One has formally

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}^{N,R}(t) + \int_{\mathbb{T}^2} \left|\nabla u_t^{N,R}(x)\right|^2 \, dx\, dt + \\ + \frac{1}{N}\sum_{i=1}^{N} \left| u_{\varepsilon_N}^{N,R}(X_t^{i,N,R})\chi_R(u_t^{N,R}) - V_t^{i,N,R}\right|^2 \, dt \leq \\ \leq \sigma^2\, dt + \frac{\sigma}{N}\sum_{i=1}^{N} V_t^{i,N,R}\cdot dB_t^i.$$

Proof. The lemma follows by Itô formula and by classical energy estimates for $u^{N,R}$. □

Remark 3. The last inequality guarantees that, even if the truncated system has no direct interpretation for the dynamics of particle-fluid, it maintains the basics physical properties.

An analogue of the previous result holds for the limit PDE system (VNS^R), as well as for (VNS). We state it in the case of system (VNS^R) and omit the proof, which is classical.

Lemma 2.3.4. *If (u^R, F^R) is a weak solution of (VNS^R), denoting with $\mathcal{E}$ the kinetic energy of the macroscopic system*

$$\mathcal{E}(t) = \frac{1}{2}\left(\int_{\mathbb{T}^2} \left|u_t^R\right|^2 \, dx + \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} |v|^2 \, F_t^R \, dx\, dv\right),$$

one has

$$\frac{d}{dt}\mathcal{E}(t) + \int_{\mathbb{T}^2}\left|\nabla u_t^R\right|^2 \, dx + \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} F_t^R \left|u_t^R - v\right|^2 \chi_R(u_t^R)\, dx\, dv = \frac{\sigma^2}{2}\,||F_0||_{L^1(\mathbb{R}^2\times\mathbb{T}^2)}.$$

Moreover there exists a constant C, independent on R such that

$$\int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} |v|^2 \, F_t^R \, dx\, dv\, dt \leq C.$$

Remark 4. By the previous lemma we have a bound on u^R in the norm $L^2([0,T]; H^1(\mathbb{T}^2))$ independently on R. By Sobolev embedding in dimension two we also have an uniform bound with respect to R on u^R in the space $L^2([0,T]; L^p(\mathbb{T}^2))$ for all $p < \infty$.

We now collect all the inequalities concerning the marginal distributions of the function F: some of them are classical, see [61], [94], while others have been used in [48].

Lemma 2.3.5. *If F is positive, defined on $\mathbb{T}^2 \times \mathbb{R}^2$ and $\int\int F(x,v)\,dx\,dv = 1$, then the followings hold*

1. $$||m_0 F||^2_{L^2(\mathbb{T}^2)} \lesssim (||F||_{L^\infty(\mathbb{T}^2\times\mathbb{R}^2)} + 1)^2 M_2 F,$$
$$||m_0 F||^4_{L^4(\mathbb{T}^2)} \lesssim (||F||_{L^\infty(\mathbb{T}^2\times\mathbb{R}^2)} + 1)^4 M_6 F;$$
2. $$||m_1 F||^2_{L^2(\mathbb{T}^2)} \lesssim (||F||_{L^\infty(\mathbb{T}^2\times\mathbb{R}^2)} + 1)^2 M_4 F;$$
3. $$||m_0 F||^2_{L^2(\mathbb{T}^2)} \lesssim ||F||^4_{L^4(\mathbb{T}^2\times\mathbb{R}^2)} + M_3 F;$$
4. $$||m_1 F||^2_{L^2(\mathbb{T}^2)} \lesssim ||F||^4_{L^4(\mathbb{T}^2\times\mathbb{R}^2)} + M_6 F;$$
5. *For all $k < k'$*
$$M_k F \lesssim ||F||_{L^1(\mathbb{T}^2\times\mathbb{R}^2)} + M_{k'} F.$$

Proof. All the inequalities are derived through the same strategy: 1. and 2. are classical, see [61], while the proof of 3. can be found in [48], so we only outline the main idea. For inequality 1. and 3. let us consider the following decomposition

$$\int_{\mathbb{R}^2} F\,dv = \int_{|v|\le r(x)} F\,dv + \int_{|v|>r(x)} F\,dv \le \int_{|v|\le r(x)} F\,dv + \frac{1}{r(x)^k}\int_{|v|>r(x)} |v|^k F\,dv$$

where $r(x)$ will be chosen in the next lines. Now one can estimates the integral on the ball of radius $r(x)$ using the infinity norm of F for inequality 1. or using Holder inequality to obtain $||F||_{L^4}$ for inequality 3. To obtain the desired result, one has to take the square both sides, integrate on $\mathbb{T}^2$ and choose $r(x)$ in order to group all the terms. For inequality 2. and 4. one has just to decompose $\int |v|\,F\,dv$ and apply the same strategy, while for 5 is enough to take $r(x) \equiv 1$. □

Remark 5. Inequality 3. and 4. will be used to prove a first tightness result in Section 2.5. Motivated by the fact that the infinity norm is not available on the mollified empirical measure, we propose a variant of 1. and 2., avoiding the use of such norm. Inequalities 1. and 2. will be used in Section 2.5 in order to prove a bound on the infinity norm of u^R, while 5. will be used in the next lemma.

We now state and prove a variant of Lemma 2.1 in [61]. This variation is needed due to the presence of the noise on the diffusion on the particle velocity, i.e. the presence of Δ_v in the equation for F^R.

Lemma 2.3.6. *If (u^R, F^R) is a bounded weak solution of (VNS^R), $k > 2$ and if $M_k F_0$ is finite, then there exists a constant C_k, independent on R, such that*

$$\sup_{t\in[0,T]} M_k F_t^R \leq C_k.$$

The same result holds for any (u, F) weak solutions of (VNS).

Proof. In this proof we omit the superscript R in (u^R, F^R) to short the notation. We start by computing

$$\frac{d}{dt}\int_{\mathbb{R}^2}\int_{\mathbb{T}^2} |v|^k F_t\,dx\,dv \lesssim \int_{\mathbb{T}^2} |u(t,x)| \int_{\mathbb{R}^2} |v|^{k-1} F_t\,dv\,dx + \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} |v|^k F_t\,dx\,dv + \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} |v|^{k-2} F_t\,dx\,dv.$$

Following [61] we have

$$\int_{\mathbb{T}^2} |u(t,x)| \int_{\mathbb{R}^2} |v|^{k-1} F_t\,dv\,dx \lesssim ||u_t||_{L^{k+2}(\mathbb{T}^2)} \left(\int_{\mathbb{R}^2}\int_{\mathbb{T}^2} |v|^k F_t\,dx\,dv\right)^{1-\frac{1}{k+2}},$$

while, using Lemma 2.3.5 inequality 5. we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{T}^2} |v|^{k-2} F_t\,dx\,dv \leq \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} |v|^k F_t\,dx\,dv + ||F_t||_{L^1(\mathbb{T}^2\times\mathbb{R}^2)}.$$

Hence we get

$$M_k F_t \lesssim M_k F_0 + \int_0^t ||u_s||_{L^{k+2}(\mathbb{T}^2)} \left(\int_{\mathbb{R}^2}\int_{\mathbb{T}^2} |v|^k F_s\,dx\,dv\right)^{1-\frac{1}{k+2}} dt + \int_0^t M_k F_s\,ds + C.$$

We now note that

$$\left(\int_{\mathbb{R}^2}\int_{\mathbb{T}^2} |v|^k F_s\,dx\,dv\right)^{1-\frac{1}{k+2}} \leq C\left(\int_{\mathbb{R}^2}\int_{\mathbb{T}^2} |v|^k F_s\,dx\,dv + 1\right),$$

hence we obtain

$$M_k F_t \leq M_k F_0 + C\int_0^t (||u_s||_{L^{k+2}(\mathbb{T}^2)} + 1) M_k F_s\,ds + C\int_0^t (||u_s||_{L^{k+2}(\mathbb{T}^2)} + 1)\,ds \leq$$

$$\leq C(M_k F_0 + ||u||_{L^2([0,T];L^{k+2}(\mathbb{T}^2))}) + C\int_0^t (||u_s||_{L^{k+2}(\mathbb{T}^2)} + 1) M_k F_s\,ds.$$

We conclude by classical Gronwall Lemma applied to the function $M_k F_t$ and by Remark 4. □

2.3.1 Maximum principle for weak solutions of the linear Vlasov-Fokker-Planck equation

We now focus on boundedness of weak solutions for the linear Vlasov-Fokker-Planck equation

$$\partial_t F + v \cdot \nabla_x F + \mathrm{div}_v(a(t,x,v)F) = \Delta_v F.$$

Boundedness of solutions will be fundamental in the latter when we will prove that the limit points, in the appropriate sense, of particle system $(PS^R - NS^R)$ are supported on bounded weak solutions of (VNS).

While this topic is classical in the case of smooth solutions, the case of weak solutions is more delicate. What follows is mainly an adaptation of the work [36], Appendix A, Proposition A.3.
In that work the author assumed the vector field a to be

$$a \in L^\infty([0,T] \times \mathbb{T}^2 \times \mathbb{R}^2), \quad \mathrm{div}_v(a) \in L^\infty([0,T] \times \mathbb{T}^2 \times \mathbb{R}^2),$$

and solutions F are assumed to belong to the set

$$Y := \left\{ F \in L^2([0,T] \times \mathbb{T}^2; H^1(\mathbb{R}^2)) \text{ s.t. } \partial_t F + v \cdot \nabla_x F \in L^2([0,T] \times \mathbb{T}^2; H^{-1}(\mathbb{R}^2)) \right\}.$$

On these solutions, a maximum principle is proved,

$$||F_t||_{L^\infty(\mathbb{T}^2 \times \mathbb{R}^2)} \leq C \, ||F_0||_{L^\infty(\mathbb{T}^2 \times \mathbb{R}^2)} .$$

In our case, we have to consider

$$a(t,x,v) = u(t,x) - v \tag{2.6}$$

hence, we cannot apply directly the result presented in [36] since the function $a(t,x,v)$ is not globally bounded. However, it is possible to recover the same result by considering some estimates on higher moments for the function F. If a satisfies (2.6), where u is uniformly bounded, one can consider

$$\widetilde{Y} := \left\{ F \in L^2([0,T] \times \mathbb{T}^2; H^1(\mathbb{R}^2)) \text{ s.t. } vF \in L^2([0,T] \times \mathbb{T}^2 \times \mathbb{R}^2), \right.$$
$$\left. \partial_t F + v \cdot \nabla_x F \in L^2([0,T] \times \mathbb{T}^2; H^{-1}(\mathbb{R}^2)) \right\}.$$

Namely, in this setup the same result proved in [36] still works, provided that one can consider solutions satisfying

$$\int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} |v|^2 \, F_s^2 \, dx \, dv \, ds < \infty.$$

Without going into the details of this adaptation, we only remark that this additional condition is achievable under our hypothesis, since

$$\int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} |v|^2 F_s^2 \, dx \, dv \, ds = \int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} |v|^2 F_s^{\frac{1}{2}} F_s^{\frac{3}{2}} \, dx \, dv \, ds$$
$$\leq \left(\int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} |v|^4 F_s \, dx \, dv \, ds \right)^{\frac{1}{2}} \left(\int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} F_s^3 \, dx \, dv \, ds \right)^{\frac{1}{2}},$$

and we will show how to control the last two terms when needed.

2.4 Uniqueness for bounded weak solutions of system of equations (VNS)

In this section we isolate a first major result needed to prove Theorem 2.2.3. We preferred to isolate it here, because it has some interest by itself. We present an uniqueness result for (VNS) in the class of bounded weak solutions (Definition (2.2.2)). This result is required in order to prove that converging subsequences of the laws of (u^{N_k}, S^{N_k}) are all supported on the same limit, which are in fact weak solutions of (VNS).

Before going into the details of this Theorem let us make some remark about the hypothesis. We first highlight that the boundedness of solutions on the fluid component is strictly needed: we will make frequent use of the fact that $u \in L^\infty([0,T] \times \mathbb{T}^2)$ in order to close some of the estimates needed to end the proof. We also remark that, even if in the proof we used the uniform bound $||u||_{L^\infty([0,T]\times\mathbb{T}^2)}$, with a bit more effort it is possible to complete the proof using only $u \in L^2([0,T]; L^\infty(\mathbb{T}^2))$. Motivated by the fact that in this work we prove existence of solutions uniformly bounded in time and space, we prefer to choose $u \in L^\infty([0,T] \times \mathbb{T}^2)$. Regarding the assumptions on weak derivatives, we require only

$$u \in L^\infty([0,T] \times \mathbb{T}^2) \cap L^2([0,T]; H^1(\mathbb{T}^2))$$

avoiding any assumption on the second derivative of u.
Also in the following proof we will make frequent use of Gagliardo-Nirenberg inequality in dimension two

$$||u||_{L^p} \lesssim ||u||_{L^2}^{\frac{2}{p}} ||\nabla u||_{L^2}^{\frac{2}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. However, this is only needed to minimize the hypothesis on $M_k F_0$, required to be finite only for some k strictly bigger than 4. One could have used

the classical Ladyzhenskaya's inequality ($p = q = 4$) with the downside of requiring higher-order moments to be finite.
The proof of this result is mainly inspired by the work [24].

Theorem 2.4.1. *Let (u_1, F_1) and (u_2, F_2) be two bounded weak solutions (Definition (2.2.2)) with the same initial conditions, of system (VNS). If*

$$M_{4+\varepsilon} F_i(0) < \infty$$

for some $\varepsilon > 0$, then $u_1 = u_2$ and $F_1 = F_2$.

Proof. We introduce the new variables $F = F_1 - F_2$ and $u = u_1 - u_2$. Then the pair (u, F) satisfies in the weak sense

$$\partial_t u = \Delta u - u \cdot \nabla u_1 - u_2 \cdot \nabla u - \nabla(\pi_1 - \pi_2) - \int_{\mathbb{R}^2} (uF_1 + u_2 F - vF)\, dv,$$

$$\partial_t F = \Delta_v F - v \cdot \nabla_x F - \mathrm{div}_v(uF_1 + u_2 F - vF)$$

with $(u(0,\cdot), F(0,\cdot,\cdot)) = 0$. We prove uniqueness by applying Gronwall Lemma to the quantity

$$||u_t||^2_{L^2(\mathbb{T}^2)} + \left|\left|\langle v\rangle^k F_t\right|\right|^2_{L^2(\mathbb{T}^2\times\mathbb{R}^2)}$$

for some $k > 2$ which will be chosen later and where $\langle v\rangle = (1 + |v|^2)^{\frac{1}{2}}$.

We start by estimating $||u_t||^2_{L^2(\mathbb{T}^2)}$: computing the time derivative we have

$$\begin{aligned}\frac{d}{dt}||u||^2_{L^2(\mathbb{T}^2)} + ||\nabla u||^2_{L^2(\mathbb{T}^2)} \lesssim \\ - \int_{\mathbb{T}^2} u(u \cdot \nabla u_1)\, dx - \int_{\mathbb{T}^2} u(u_2 \cdot \nabla u)\, dx \\ + \int_{\mathbb{T}^2} u \int_{\mathbb{R}^2} vF\, dvdx - \int_{\mathbb{T}^2} u \int_{\mathbb{R}^2} uF_1\, dvdx - \int_{\mathbb{T}^2} u \int_{\mathbb{R}^2} u_2 F\, dvdx. \quad (2.7)\end{aligned}$$

Integrating by parts the term

$$\int_{\mathbb{T}^2} u(u_2 \cdot \nabla u)\, dx = 0$$

vanishes, while the term

$$-\int_{\mathbb{T}^2} u \int_{\mathbb{R}^2} uF_1\, dvdx = -\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} u^2 F_1\, dvdx \leq 0$$

can be neglected due to positivity of F_1. Hence we can estimate the remaining terms as

$$(2.7) \lesssim -\int_{\mathbb{T}^2} u(u\cdot\nabla u_1)\,dx + \int_{\mathbb{T}^2} u\int_{\mathbb{R}^2} vF\,dvdx \\ -\int_{\mathbb{T}^2} u\int_{\mathbb{R}^2} u_2 F\,dvdx = (I)+(II)+(III).$$

where

$$(I) \le \int_{\mathbb{T}^2} |u|\,|\nabla u|\,|u_1|\;dx \le ||u_1||_\infty \int_{\mathbb{T}^2} |u|\,|\nabla u|\;dx \\ \lesssim \frac{1}{\delta}\,||u||^2_{L^2(\mathbb{T}^2)} + \delta\,||\nabla u||^2_{L^2(\mathbb{T}^2)}$$

and $\delta > 0$ can be taken arbitrarily small.

$$(II) \le \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} |u|\,|v|\,F\,dxdv \le \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} \frac{|u|}{\langle v\rangle^{k-1}}\langle v\rangle^k F\,dxdv \\ \le \int_{\mathbb{T}^2} |u|^2\,dx \int_{\mathbb{R}^2} \frac{1}{\langle v\rangle^{2(k-1)}}\,dv + \left|\left|\langle v\rangle^k F\right|\right|^2_{L^2(\mathbb{T}^2\times\mathbb{R}^2)} \\ \lesssim ||u||^2_{L^2(\mathbb{T}^2)} + \left|\left|\langle v\rangle^k F\right|\right|^2_{L^2(\mathbb{T}^2\times\mathbb{R}^2)},$$

because $2(k-1) > 2$ being $k > 2$.

$$(III) \le \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} |u|\,|u_2|\,F\,dxdv \le ||u_2||_\infty \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} \frac{|u|}{\langle v\rangle^k}\langle v\rangle^k F\,dxdv \\ \lesssim \int_{\mathbb{T}^2} |u|^2\,dx \int_{\mathbb{R}^2} \frac{1}{\langle v\rangle^{2k}}dv + \left|\left|\langle v\rangle^k F\right|\right|^2_{L^2(\mathbb{T}^2\times\mathbb{R}^2)} \lesssim ||u||^2_{L^2(\mathbb{T}^2)} + \left|\left|\langle v\rangle^k F\right|\right|^2_{L^2(\mathbb{T}^2\times\mathbb{R}^2)}.$$

This ends the estimate for $||u||^2_{L^2(\mathbb{T}^2)}$. Concerning $\left|\left|\langle v\rangle^k F\right|\right|^2_{L^2(\mathbb{T}^2\times\mathbb{R}^2)}$ we proceed by computing the time derivative

$$\frac{d}{dt}\left|\left|\langle v\rangle^k F\right|\right|^2_{L^2(\mathbb{T}^2\times\mathbb{R}^2)} + \left|\left|\langle v\rangle^k \nabla_v F\right|\right|^2_{L^2(\mathbb{T}^2)} \lesssim \\ + \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} \langle v\rangle^{2k-2} F^2\,dxdv - \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} \langle v\rangle^{2k} Fv\cdot\nabla_x F\,dxdv \\ - \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} \langle v\rangle^{2k} F\mathrm{div}_v(u_2F)\,dxdv - \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} \langle v\rangle^{2k} F\mathrm{div}_v(uF_1)\,dxdv \\ + \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} \langle v\rangle^{2k} F\mathrm{div}_v(vF)\,dxdv. \quad (2.8)$$

The first term on the r.h.s. can be estimated with $\left|\left|\langle v\rangle^k F\right|\right|^2_{L^2(\mathbb{T}^2\times\mathbb{R}^2)}$, being $\langle v\rangle \geq 1$. By a standard integration by parts argument, it is proved that the second term is equal to zero. Hence, what is left from (2.8) is

$$-\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\langle v\rangle^{2k} F\mathrm{div}_v(u_2F)\,dxdv - \int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\langle v\rangle^{2k} F\mathrm{div}_v(uF_1)\,dxdv + \int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\langle v\rangle^{2k} F\mathrm{div}_v(vF)\,dxdv = (IV)+(V)+(VI).$$

Now we proceed by treating each term separately:

$$(IV) = -\frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\langle v\rangle^{2k} u_2\cdot\nabla_v F^2\,dxdv \leq \int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\langle v\rangle^{2k-1}|u_2|\,F^2\,dxdv \lesssim ||u_2||_\infty\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\langle v\rangle^{2k}F^2\,dxdv \lesssim \left|\left|\langle v\rangle^k F\right|\right|^2_{L^2(\mathbb{T}^2\times\mathbb{R}^2)}.$$

$$(V) = \int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\nabla_v(\langle v\rangle^{2k}F)\cdot uF_1\,dxdv \leq \int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\langle v\rangle^{2k-1}F\;|u|\;F_1\,dxdv + \int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\langle v\rangle^{2k}|\nabla_v F|\;|u|\;F_1\,dxdv.$$

The first term on the r.h.s. of the last inequality can be treated in the following way

$$\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\langle v\rangle^{2k-1}F\;|u|\;F_1\,dxdv = \int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\left(\langle v\rangle^k F\right)\frac{|u|}{\langle v\rangle}\left(\langle v\rangle^k F_2\right)dxdv \leq \left|\left|\langle v\rangle^k F\right|\right|_{L^2(\mathbb{T}^2\times\mathbb{R}^2)}||u||_{L^p(\mathbb{T}^2)}\left(\int_{\mathbb{R}^2}\frac{1}{\langle v\rangle^p}\,dv\right)^{\frac{1}{p}}\left|\left|\langle v\rangle^k F_2\right|\right|_{L^q(\mathbb{T}^2\times\mathbb{R}^2)} \tag{2.9}$$

where p and q are such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{2}=1$. Note that $p>2$ so that $1/\langle v\rangle^p$ is integrable in dimension two. Applying Gagliardo-Nirenberg inquality to the previous identity we have

$$(2.9) \leq \left|\left|\langle v\rangle^k F\right|\right|_{L^2(\mathbb{T}^2\times\mathbb{R}^2)}||u||^{\frac{2}{p}}_{L^2(\mathbb{T}^2)}||\nabla u||^{\frac{2}{q}}_{L^2(\mathbb{T}^2)}\left|\left|\langle v\rangle^k F_2\right|\right|_{L^q(\mathbb{T}^2\times\mathbb{R}^2)} \lesssim \left|\left|\langle v\rangle^k F\right|\right|^2_{L^2(\mathbb{T}^2\times\mathbb{R}^2)} + \frac{1}{\delta}||u||^2_{L^2(\mathbb{T}^2)} + \delta\left|\left|\langle v\rangle^k F_2\right|\right|^q_{L^q(\mathbb{T}^2\times\mathbb{R}^2)}||\nabla u||^2_{L^2(\mathbb{T}^2)}$$

where δ can be taken arbitrarily small. In order to control the quantity $\left|\left|\langle v\rangle^k F_2\right|\right|^q_{L^q(\mathbb{T}^2\times\mathbb{R}^2)}$ at the end of the proof we will impose that $kq<4+\varepsilon$. On the

other hand for the second term on the r.h.s. of (V), introduce $\alpha > 0$ such that $\alpha p > 2$ so that

$$\int_{\mathbb{R}^2}\int_{\mathbb{T}^2} \langle v\rangle^{2k} |\nabla_v F|\ |u|\ F_1\, dxdv = \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} \langle v\rangle |\nabla_v F| \frac{|u|}{\langle v\rangle^{\alpha}} \langle v\rangle^{k+\alpha} F_1\, dxdv$$
$$\leq \left|\left|\langle v\rangle^k |\nabla_v F|\right|\right|_{L^2(\mathbb{T}^2\times\mathbb{R}^2)} ||u||_{L^p(\mathbb{T}^2)} \left(\int_{\mathbb{R}^2} \frac{1}{\langle v\rangle^{\alpha p}}\, dv\right)^{\frac{1}{p}} \left|\left|\langle v\rangle^{k+\alpha} F_1\right|\right|_{L^q(\mathbb{T}^2\times\mathbb{R}^2)}.$$

Now we apply Gagliardo-Niremberg inequality and Young inequality, in the same manner as we treated (2.9), obtaining

$$\lesssim\ \delta \left|\left|\langle v\rangle^k |\nabla_v F|\right|\right|^2_{L^2(\mathbb{T}^2\times\mathbb{R}^2)} + \frac{1}{\delta^2} ||u||^2_{L^2(\mathbb{T}^2)} + \delta \left|\left|\langle v\rangle^{k+\alpha} F_1\right|\right|^q_{L^q(\mathbb{T}^2\times\mathbb{R}^2)} ||\nabla u||^2_{L^2(\mathbb{T}^2)}.$$

We require that $(k+\alpha)q < 4+\varepsilon$ in order to match our hypothesis on $M_{4+\varepsilon}F(0)$. This ends the term in (V). For the last one, by the product rule

$$(VI) \lesssim \left|\left|\langle v\rangle^k F\right|\right|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^2)} + \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} \langle v\rangle^{2k+1} \nabla_v(F^2)\, dxdv\ \lesssim\ \left|\left|\langle v\rangle^k F\right|\right|^2_{L^2(\mathbb{R}^2\times\mathbb{T}^2)}.$$

What is left, before applying Gronwall Lemma, is only to find parameters (k, p, q, α) matching all the needed constraints:

$$\begin{cases} k > 2; \\ \frac{1}{p} + \frac{1}{q} + \frac{1}{2} = 1; \\ \alpha p > 2; \\ (k+\alpha)q < 4+\varepsilon. \end{cases}$$

The rationale behind this is the following: k and q can be taken arbitrarily close to 2. Doing so, p will be very large and hence α can be take arbitrarily small preserving the condition $\alpha p > 2$, and having $(k+\alpha)$ close to 2.

These conditions allow us obtain that

$$\left|\left|\langle v\rangle^{k+\alpha} F_1\right|\right|^q_{L^q(\mathbb{T}^2\times\mathbb{R}^2)}, \left|\left|\langle v\rangle^k F_2\right|\right|^q_{L^q(\mathbb{T}^2\times\mathbb{R}^2)}, \int_{\mathbb{R}^2} \frac{1}{\langle v\rangle^{\alpha p}}\, dv \leq C,$$

being

$$\left|\left|\langle v\rangle^{k+\alpha} F_1\right|\right|^q_{L^q(\mathbb{T}^2\times\mathbb{R}^2)} \leq ||F_1||^{q-1}_{L^\infty(\mathbb{T}^2\times\mathbb{R}^2)} \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} \langle v\rangle^{(k+\alpha)q} F_1\, dxdv$$
$$\lesssim ||F_1||^{q-1}_{L^\infty(\mathbb{T}^2\times\mathbb{R}^2)} \left(||F_1||_{L^1(\mathbb{T}^2\times\mathbb{R}^2)} + M_{(k+\alpha)q}F_1\right) \leq C.$$

Summarizing we have obtained

$$\frac{d}{dt} ||u_t||^2_{L^2(\mathbb{T}^2)} + \frac{d}{dt} \left|\left|\langle v\rangle^k F_t\right|\right|^2_{L^2(\mathbb{T}^2\times\mathbb{R}^2)} \leq C_1 ||u_t||^2_{L^2(\mathbb{T}^2)} + C_2 \left|\left|\langle v\rangle^k F_t\right|\right|^2_{L^2(\mathbb{T}^2\times\mathbb{R}^2)},$$

hence by Gronwall Lemma we obtain $u \equiv 0$ and $F \equiv 0$, proving uniqueness. $\square$

2.5 Scaling limit for the truncated system

In this section we focus on the proof of a first tightness result. As remarked in the introduction, first we will prove the convergence of $(PS^R - NS^R)$ to (VNS). To do so, we will show that, if the cutoff threshold R is large enough, then the system (VNS^R) coincides with (VNS). This whole section is devoted to the proof of this intermediate result:

Proposition 2.5.1. *Under hypothesis of subsection 2.2.3 and if $R \geq \mathbf{K}_u + 1$, where the constant $\mathbf{K}_u$ will be specified later (Proposition 2.5.13), the family of laws $\left\{Q^{N,R}\right\}$ of the couple $(u^{N,R}, S^{N,R})_{N\in\mathbb{N}}$ is tight on $C([0,T]\times\mathbb{T}^2)\times C([0,T];\mathbf{P}_1(\mathbb{T}^2\times\mathbb{R}^2))$. Moreover $\left\{Q^{N,R}\right\}_{N\in\mathbb{N}}$ converges weakly to $\delta_{(u,F)}$, where the couple (u,F) is the unique weak solution of system of equation (VNS).*

With a special argument we will be finally able to remove the cut-off also in the approximating system and to get our main result, Theorem 2.2.3.

2.5.1 Tightness

In order to prove Proposition 2.5.1 we have to establish the tightness of the laws of the empirical measure $S^{N,R}$ and that of $u^{N,R}$. First we deal with the empirical measure, the easier of the two. The tightness of $S^{N,R}$ follows easily by a well known criterion, [89], the particles being exchangeable and due to the presence of the cut-off.

Proposition 2.5.2. *The family of laws $\{Q^{N,R,S}\}_{N\in\mathbb{N}}$ of the empirical measure $\{S^{N,R}_\cdot\}_{N\in\mathbb{N}}$ is relatively compact with respect to the weak convergence on $C\left([0,T];\mathbf{P}_1(\mathbb{T}^2\times\mathbb{R}^2)\right)$.*

We now focus on the tightness of the laws of $u^{N,R}$. To get an idea of what is the right topology to work with, we focus on the coupling term that appears in the equation for $u^{N,R}$ in $(PS^R - NS^R)$. The term can be rewritten as

$$\chi_R(u_t^{N,R})\frac{1}{N}\sum_{i=1}^{N}(u_{\varepsilon_N}^{N,R}(X_t^{i,N,R}) - V_t^{i,N,R})\,\delta^{\varepsilon_N}_{X_t^{i,N,R}} =$$
$$\chi_R(u_t^{N,R})\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}(u_{\varepsilon_N}^{N,R}(x') - v')\theta^{0,\varepsilon_N}(x-x')S_t^{N,R}(dx',dv') =$$
$$= \chi_R(u_t^{N,R})(\theta^{0,\varepsilon_N} * (u_{\varepsilon_N}^{N,R} - v)S_t^{N,R})(x).$$

In order to pass to the limit in the previous term, it is required that $u^{N,R}$ is converging uniformly over $\mathbb{T}^2$, since $S^{N,R}$ is converging only weakly as probability

measure. Hence, we look for a tightness criterion for $\{u^{N,R}\}_{N\in\mathbb{N}}$ in $C(\mathbb{T}^2)$. By Sobolev embedding in dimension two we have $H^2(\mathbb{T}^2) \hookrightarrow C(\mathbb{T}^2)$ (and also in the space of Hölder continuous functions). Thus, to get estimates on second derivative of $u^{N,R}$, we start by looking at the equation for $u^{N,R}$ in vorticity form:

$$\partial_t \omega^{N,R} = \Delta\omega^{N,R} - u^{N,R}\cdot\nabla\omega^{N,R} - \frac{\chi_R(u_t^{N,R})}{N}\sum_{i=1}^{N}\big(u_{\varepsilon_N}^{N,R}(X_t^{i,N,R}) - V_t^{i,N,R}\big)\big)\nabla^\perp\cdot\delta^{\varepsilon_N}_{X_t^{i,N,R}}. \quad (2.10)$$

In order to be able to obtain a priori estimates on $\omega^{N,R}$ we need first to rewrite the coupling term in (2.10) as a function of the mollified empirical measure $F^{N,R}$. We highlight that this is one of the most important key steps in this work, that remarks the importance to introduce the mollified empirical measure, and justify all the following computations.

Lemma 2.5.3.

$$\frac{1}{N}\sum_{i=1}^{N} V_t^{i,N,R}\delta^{\varepsilon_N}_{X_t^{i,N,R}}(x) = \int_{\mathbb{R}^2} vF_t^{N,R}(x,v)\,dv = m_1F_t^{N,R}(x)$$

$$\frac{1}{N}\sum_{i=1}^{N} \delta^{\varepsilon_N}_{X_t^{i,N,R}}(x) = \int_{\mathbb{R}^2} F_t^{N,R}(x,v)\,dv = m_0F_t^{N,R}(x)$$

Proof. Notice that

$$\frac{1}{N}\sum_{i=1}^{N} V_t^{i,N,R}\delta^{\varepsilon_N}_{X_t^{i,N,R}}(x) = \int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\theta^{0,\varepsilon_N}(x-x')v'S_t^{N,R}(dx',dv') = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\theta^{0,\varepsilon_N}(x-x')\theta^{1,\varepsilon_N}(v-v')v'S_t^{N,R}(dx',dv')\,dv,$$

and

$$\int_{\mathbb{R}^2} vF_t^{N,R}(x,v)\,dv = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\theta^{0,\varepsilon_N}(x-x')\theta^{1,\varepsilon_N}(v-v')vS_t^{N,R}(dx',dv')dv$$

so that to complete the proof we only need to prove

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\theta^{0,\varepsilon_N}(x-x')\theta^{1,\varepsilon_N}(v-v')(v-v')S_t^{N,R}(dx',dv')dv = 0.$$

However this is true due to

$$\int_{\mathbb{R}^2}\theta^{1,\varepsilon_N}(v-v')(v-v')\,dv = 0$$

by the hypothesis of symmetry (3) in 2.2.3. The second identity of the Lemma follows by the very definition of $\delta^{\varepsilon_N}_{X_t^{i,N,R}}$. This ends the proof. □

As stated above, we look for an estimate in $H^2(\mathbb{T}^2)$ for u^N. This is obtained by energy type estimates for the fluid in the vorticity form.

Lemma 2.5.4.

$$\mathbf{E}\left[\sup_{t\in[0,T]}\left|\left|\omega_t^{N,R}\right|\right|^2_{L^2(\mathbb{T}^2)}+\int_0^T\left|\left|\nabla\omega_s^{N,R}\right|\right|^2_{L^2(\mathbb{T}^2)}\,ds\right]\lesssim\mathbf{E}\left[\left|\left|\omega_0^{N,R}\right|\right|^2_{L^2(\mathbb{T}^2)}\right]$$
$$+\mathbf{E}\left[\left|\left|m_1F^{N,R}\right|\right|^2_{L^2([0,T]\times\mathbb{T}^2)}\right]+R\mathbf{E}\left[\left|\left|m_0F^{N,R}\right|\right|^2_{L^2([0,T]\times\mathbb{T}^2)}\right].$$

Proof. The thesis follows by classical energy inequality for $\omega^{N,R}$ and by using lemma 2.5.3. □

We remark that the previous computation was only possible due to the presence of the cuf-off. The truncation is needed to decouple the fluid and particles in $(PS^R - NS^R)$, hence allowing us to close estimates for fluid and particles separately. From Lemma 2.5.4 it is clear that it is necessary to control the L^2 norm of both $m_1F^{N,R}$ and $m_0F^{N,R}$. To do so we will exploit Lemma 2.3.5 and thus look for an estimate for $M_6F^{N,R}$ and for $(F^{N,R})^4$. This is exactly the goal of the next lemmas.

Lemma 2.5.5. *There exists a constant $C_{T,R,4}$, independent on N, such that*

$$\sup_{t\in[0,T]}\mathbf{E}\left[\left|\left|F_t^{N,R}\right|\right|^4_{L^4(\mathbb{T}^2\times\mathbb{R}^2)}\right]\leq C_{T,R,4}.$$

Proof. This proof strictly follows the proof of Lemma 3.3 in [48]. By Itô formula and integration by parts we have

$$\frac{1}{4}d\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}(F_t^{N,R})^4\,dxdv+\frac{3\sigma^2}{2}\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}(F_t^{N,R})^2\left|\nabla_vF_t^{N,R}\right|^2\,dxdvdt=$$

$$-\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}(F_t^{N,R})^3\mathrm{div}_x(\theta^{\varepsilon_N}*(vS_t^{N,R}))\,dxdvdt \tag{2.11}$$

$$-\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}(F_t^{N,R})^3\mathrm{div}_v(\theta^{\varepsilon_N}*(u_{\varepsilon_N}^{N,R}(t,x)\chi_R(u_t^{N,R})-v)S_t^{N,R})\,dxdvdt \tag{2.12}$$

$$+\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}(F_t^{N,R})^3\,dM_t^{N,\varepsilon_N}\,dxdv+\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}(F_t^{N,R})^2\,d[M^{N,\varepsilon_N}]_t\,dxdv. \tag{2.13}$$

We estimate each of the terms above separately. Concerning (2.11), we can rewrite the convolution inside the integral as

$$\mathrm{div}_x(\theta^{\varepsilon_N}*(vS_t^{N,R}))=v\cdot\nabla_x(\theta^{\varepsilon_N}*S_t^{N,R})-((\nabla_x\theta^{\varepsilon_N}\cdot v)*S_t^{N,R}).$$

Hence, for the first term on the r.h.s. we have

$$-\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}(F_t^{N,R})^3\nabla_x F_t^{N,R}\cdot v\,dx\,dv\,dt=\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\nabla_x(F_t^{N,R})^4\cdot v\,dx\,dv\,dt=0.$$

For the second one, note that due to our hypothesis on the mollifiers $\theta^0(x)$ and $\theta^1(v)$ we have

$$\begin{aligned}&\left|\nabla_x\theta^{0,\varepsilon_N}(x-x')\right|\theta^{1,\varepsilon_N}(v-v')\,|(v-v')|\\&=\varepsilon_N^{-1}\varepsilon_N^{-2}\left|\nabla_x\theta^0(\varepsilon_N^{-1}(x-x'))\right|\varepsilon_N^{-2}\theta^1(\varepsilon_N^{-1}(v-v'))\,|v-v'|\\&\le\varepsilon_N^{-2}\left|\theta^0(\varepsilon_N^{-1}(x-x'))\right|\varepsilon_N^{-2}\theta^1(\varepsilon_N^{-1}(v-v'))\frac{|v-v'|}{\varepsilon_N}\\&\le\theta^{0,\varepsilon_N}(x-x')\theta^{1,\varepsilon_N}(v-v')2\end{aligned}$$

implying

$$|(2.11)|\lesssim\left|\left|F_t^{N,R}\right|\right|^4_{L^4(\mathbb{R}^2\times\mathbb{T}^2)}.$$

The main differences with respect to the proof of [48] concerns the term (2.12): we split it into two parts. One contains the fluid velocity u and the other one contains the velocity variable: the first one follows easily by the truncation, being

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}(F_t^{N,R})^3\mathrm{div}_v(\theta^{\varepsilon_N}*u^{N,R}_{\varepsilon_N}(t,x)\chi_R(u_t^{N,R})S_t^{N,R})\,dxdv\right|\\&=\left|\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\nabla_v(F_t^{N,R})^3(\theta^{\varepsilon_N}*u^{N,R}_{\varepsilon_N}(t,x)\chi_R(u_t^{N,R})S_t^{N,R})\,dxdv\right|\\&\le\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\left|\nabla_v(F_t^{N,R})^3\right|(\theta^{\varepsilon_N}*\left|u^{N,R}_{\varepsilon_N}(t,x)\chi_R(u_t^{N,R})\right|S_t^{N,R})\,dxdv\end{aligned}$$

$$\le R\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\left|\nabla_v F_t^{N,R}F_t^{N,R}\right|(F_t^{N,R})^2\,dx\,dv\lesssim\frac{1}{\delta}\left|\left|F_t^{N,R}\right|\right|^4_{L^4(\mathbb{T}^2\times\mathbb{R}^2)}+\delta\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}(F_t^{N,R})^2\left|\nabla_v F_t^{N,R}\right|^2\,dxdv$$

and by choosing δ small enough we can take the second term to the l.h.s. maintaining the positivity. For the other one we again split it into a basic term plus a commutator

$$\begin{aligned}\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}(F_t^{N,R})^3\mathrm{div}_v(\theta^{\varepsilon_N}*vS_t^{N,R})\,dxdv=&\\\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}(F_t^{N,R})^3\mathrm{div}_v(v(\theta^{\varepsilon_N}*S_t^{N,R}))\,dxdv&\\-\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}(F_t^{N,R})^3\mathrm{div}_v(\theta^{\varepsilon_N}v*S_t^{N,R})\,dxdv.&\quad(2.14)\end{aligned}$$

The first term on the r.h.s. on (2.14) is easily handled by direct computation

$$= -\int_{\mathbb{R}^2}\int_{\mathbb{T}^2} \nabla_v (F_t^{N,R})^3 \cdot v\, F_t^{N,R}\, dxdv = \\ -\frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{T}^2} \nabla_v (F_t^{N,R})^4 \cdot v\, dxdv = \frac{1}{2}\left|\left|F_t^{N,R}\right|\right|^4_{L^4(\mathbb{T}^2\times\mathbb{R}^2)},$$

while the second one is more tricky: we compute the divergence on v and obtain

$$\int_{\mathbb{R}^2}\int_{\mathbb{T}^2} (F_t^{N,R})^3 \mathrm{div}_v(\theta^{\varepsilon_N} v * S_t^{N,R})\, dxdv = 2\int_{\mathbb{R}^2}\int_{\mathbb{T}^2} (F_t^{N,R})^4\, dxdv$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{T}^2} (F^{N,R})^3 \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} \theta^{0,\varepsilon_N}(x-x')\nabla_v \theta^{1,\varepsilon_N}(v-v')\cdot(v-v')\, S_t^{N,R}(dx',dv')dxdv$$

$$\leq 2\left|\left|F_t^{N,R}\right|\right|^4_{L^4(\mathbb{T}^2\times\mathbb{R}^2)} +$$

$$+\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\left|\nabla_+ v (F^{N,R})^3\right| \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} \theta^{0,\varepsilon_N}(x-x')\theta^{1,\varepsilon_N}(v-v')\,|v-v'| S_t^{N,R}(dx',dv')dxdvdt.$$

Now we just look at the most inner term in the last inequality: using the compact support assumption for $\theta^1(v)$, see 2.2.3 hypothesis (3), we get

$$\theta^{0,\varepsilon_N}(x-x')\theta^{1,\varepsilon_N}(v-v')\,|v-v'| \leq 2\varepsilon_N \theta^{0,\varepsilon_N}(x-x')\theta^{1,\varepsilon_N}(v-v'),$$

which leads to (2.14) being

$$(2.14) \lesssim \left|\left|F_t^{N,R}\right|\right|^4_{L^4(\mathbb{T}^2\times\mathbb{R}^2)} + \varepsilon_N \int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\left|\nabla (F_t^{N,R})^3\right| F_t^{N,R}\, dxdv$$

$$\lesssim \left|\left|F_t^{N,R}\right|\right|^4_{L^4(\mathbb{T}^2\times\mathbb{R}^2)} + 2\varepsilon_N \int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\left|\nabla_v F_t^{N,R}\right|^2 (F_t^{N,R})^2\, dxdv.$$

We now deal with the two last term in (2.13): the integral with respect to the martingale M_t^{N,ε_N} vanishes when computing the expected value, while for the integral with respect to the quadratic variation we have

$$\int_{\mathbb{R}^2}\int_{\Pi^d} (F_t^N)^2\, d[M^{N,\varepsilon_N}]_t\, dxdv = \frac{\sigma^2}{N}\int_{\mathbb{R}^2}\int_{\Pi^d} (F_t^N)^2 (|\nabla_v \theta^{\varepsilon_N}|^2 * S_t^N)\, dxdvdt \leq$$

$$\sigma^4 \left|\left|F_t^N\right|\right|^4_{L^4} dt + \frac{1}{N^2}\int_{\mathbb{R}^2}\int_{\Pi^d} (|\nabla_v \theta^{\varepsilon_N}|^2 * S_t^N)^2\, dxdvdt.$$

The square outside the convolution $(|\nabla_v \theta^{\varepsilon_N}|^2 * S_t^N)^2$ can be troublesome, but we can handle it using the property of compact support of $\theta^1(v)$ and the separation of

variables, in the following way:

$$\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}(|\nabla_v\theta^{\varepsilon_N}|^2 * S_t^N)^2\,dx\,dv \lesssim$$
$$\frac{1}{N}\sum_{i=1}^{N}\left(\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}\left|\nabla_v\theta^{1,\varepsilon_N}(v-V_t^{i,N,R})\right|^2\theta^{0,\varepsilon_N}(x-X_t^{i,N,R})^2\,dx\,dv\right)^2$$

$$\lesssim \frac{1}{N}\sum_{i=1}^{N}\int_{\mathbb{R}^2}\left|\nabla_v\theta^{1,\varepsilon_N}(v-V_t^{i,N,R})\right|^4\,dv\int_{\mathbb{T}^2}\theta^{0,\varepsilon_N}(x-X_t^{i,N,R})^4\,dx.$$

Now we compute

$$\int_{\mathbb{R}^2}\left|\nabla_v\theta^{1,\varepsilon_N}(v-V_t^{i,N})\right|^4\,dv = CN^{5\beta},$$
$$\int_{\Pi^d}\theta^{0,\varepsilon_N}(x-X_t^{i,N})^4\,dx = CN^{3\beta},$$

and substitute into the integral for the quadratic variation

$$\frac{1}{N^2}\int_{\mathbb{R}^2}\int_{\Pi^d}(|\nabla_v\theta^{\varepsilon_N}|^2 * S_t^N)^2\,dxdv \lesssim \frac{1}{N^2}N^{5\beta}N^{3\beta}$$

which is bounded for $\beta \le \frac{1}{4}$.

Summarizing we have obtained

$$d\left|\left|F_t^{N,R}\right|\right|^4_{L^4(\mathbb{T}^2\times\mathbb{R}^2)} + C\int_{\mathbb{R}^2}\int_{\mathbb{T}^2}(F_t^{N,R})^2\left|\nabla_v F_t^{N,R}\right|^2\,dxdvdt \le$$

$$\lesssim C_R\left|\left|F_t^{N,R}\right|\right|^4_{L^4(\mathbb{T}^2\times\mathbb{R}^2)}\,dt + \int_{\mathbb{R}^2}\int_{\mathbb{T}^2}(F_t^{N,R})^3\,dM_t^{N,\varepsilon_N}\,dxdv + Cdt$$

which, after taking the average, ends the proof by standard Gronwall lemma. □

By interpolation between L^p spaces, and the fact that $F_t^{N,R}$ is a probability density function, we obtain the following corollary:

Corollary 2.5.6. *There exists a constant $C_{T,R,2}$, independent on N, such that*

$$\sup_{t\in[0,T]}\mathbf{E}\left[\left|\left|F_t^{N,R}\right|\right|^2_{L^2(\mathbb{T}^2\times\mathbb{R}^2)}\right] \le C_{T,R,2}.$$

We now proceed to bound the moments on the v-component of the mollified empirical measure $F^{N,R}$. The proof of the next Lemma follows by the very definition of $M_k F^{N,R}$ by using change of variable formula.

Lemma 2.5.7. *For all $k \leq 6$ and for all N and R, there exists a constant $C_k^{T,R}$, independent on N such that*

$$\mathbf{E}\left[\sup_{t\in[0,T]} M_k F_t^{N,R}\right] \leq C_k^{T,R}.$$

Proof. The proof follows by expanding $F^{N,R}$ as a summation, and by a change of variables inside the integral with respect to v. This allow to bound the k-th moments along v of $F^{N,R}$ by

$$\mathbf{E}\left[\sup_{t\in[0,T]} \left|V_t^{i,N,R}\right|^k\right].$$

Moreover, we can bound the expected value in the previous formula using the SDEs for the particles velocity, by using the truncation and the hypothesis on the initial conditions. □

Summarizing, up to this point we were able to prove the following bounds, independently on N:

$$\sup_{t\in[0,T]} \mathbf{E}\left[\left|\left|m_0 F_t^{N,R}\right|\right|^2_{L^2(\mathbb{T}^2)}\right] \leq C_{T,R},$$

$$\sup_{t\in[0,T]} \mathbf{E}\left[\left|\left|m_1 F_t^{N,R}\right|\right|^2_{L^2(\mathbb{T}^2)}\right] \leq C_{T,R},$$

by Lemmas 2.5.5, 2.5.7 and inequality 3. and 4. from Lemma 2.3.5. Also

$$\mathbf{E}\left[\sup_{t\in[0,T]} \left|\left|\omega_t^{N,R}\right|\right|^2_{L^2(\mathbb{T}^2)} + \int_0^T \left|\left|\nabla\omega_s^{N,R}\right|\right|^2_{L^2(\mathbb{T}^2)}\, ds\right] \leq C_{T,R}.$$

by Lemma 2.5.4.
Hence we have obtained the desired bound for the fluid in vorticity form. However, in order to obtain convergence, we need to apply an appropriate tightness criterion. Classical Aubin-Lions Lemma states that when $E_0 \subseteq E \subseteq E_1$ are three Banach spaces with continuous embedding, and E_0 compactly embedded into E, then for all $p, q < \infty$ the space $L^p([0,T]; E_0) \cap W^{1,q}([0,T]; E_1)$ is compactly embedded into $L^p([0,T]; E)$. Hence, we can apply this criterion choosing $p = q = 2$ and $E_0 = H^2(\mathbb{T}^2)$, $E = C(\mathbb{T}^2)$ and $E_1 = H^{-1}(\mathbb{T}^2)$ to obtain

$$L^2([0,T]; H^2(\mathbb{T}^2)) \cap W^{1,2}([0,T]; H^{-1}(\mathbb{T}^2)) \hookrightarrow L^2([0,T]; C(\mathbb{T}^2))$$

and the embedding is compact. Thus, in order to obtain the required tightness result, we also need an a priori estimate for the time derivative of $\omega^{N,R}$:

Lemma 2.5.8. *For every $\varepsilon > 0$ there exists $Z > 0$, such that*

$$\mathbf{P}\left(\left|\left|\omega^{N,R}\right|\right|_{W^{1,2}([0,T];H^{-1}(\mathbb{T}^2))} > Z\right) \leq \varepsilon$$

Proof. By Lemma 2.5.4 we already have the result for the $L^2([0,T];L^2(\mathbb{T}^2))$ norm of ω^N. Since $H^1 \hookrightarrow L^2 \hookrightarrow H^{-1}$ we already know that

$$\mathbf{P}\left(\left|\left|\omega^{N,R}\right|\right|_{L^2([0,T];H^{-1}(\mathbb{T}^2))} > Z\right) \leq \varepsilon.$$

Hence we only need to estimate $\left|\left|\partial_t\omega^{N,R}\right|\right|_{L^2([0,T];H^{-1}(\mathbb{T}^2))}$. Thus we compute the H^{-1} norm both sides in the equation for $\omega^{N,R}$, obtaining

$$\begin{aligned}\left|\left|\partial_t\omega_t^{N,R}\right|\right|_{H^{-1}(\mathbb{T}^2)} \lesssim \left|\left|\Delta\omega_t^{N,R}\right|\right|_{H^{-1}(\mathbb{T}^2)} + \left|\left|u_t^{N,R}\cdot\nabla\omega_t^{N,R}\right|\right|_{H^{-1}(\mathbb{T}^2)} \\ + R\left|\left|m_0F_t^{N,R}\right|\right|_{L^2(\mathbb{T}^2)} + \left|\left|m_1F_t^{N,R}\right|\right|_{L^2(\mathbb{T}^2)}\end{aligned}$$

by classical argument and integration by parts. Taking the square and integrating both sides we obtain

$$\begin{aligned}\int_0^T \left|\left|\partial_t\omega_t^{N,R}\right|\right|^2_{H^{-1}(\mathbb{T}^2)} dt \lesssim \int_0^T \left|\left|\nabla\omega_t^{N,R}\right|\right|^2_{L^2(\mathbb{T}^2)} dt + \\ \sup_{t\in[0,T]}\left|\left|\omega_t^{N,R}\right|\right|^2_{L^2(\mathbb{T}^2)}\int_0^T \left|\left|u_t^{N,R}\right|\right|^2_{C(\mathbb{T}^2)} dt \\ + R\int_0^T \left|\left|m_0F_t^{N,R}\right|\right|^2_{L^2(\mathbb{T}^2)} dt + \int_0^T \left|\left|m_1F_t^{N,R}\right|\right|^2_{L^2(\mathbb{T}^2)} dt.\end{aligned}$$

Finally, We compute probability both sides

$$\mathbf{P}\left(\int_0^T \left|\left|\partial_t\omega_t^{N,R}\right|\right|^2_{H^{-1}(\mathbb{T}^2)} dt > Z\right)$$

and use the fact that we can split product term inside probabilities

$$\begin{aligned}&\mathbf{P}\left(\sup_{t\in[0,T]}\left|\left|\omega_t^{N,R}\right|\right|^2_{L^2(\mathbb{T}^2)}\int_0^T \left|\left|u_t^{N,R}\right|\right|^2_{C(\mathbb{T}^2)} dt > Z\right) \\ &\qquad\leq \mathbf{P}\left(\sup_{t\in[0,T]}\left|\left|\omega_t^{N,R}\right|\right|^2_{L^2(\mathbb{T}^2)} > \sqrt{Z}\right) + \mathbf{P}\left(\int_0^T \left|\left|u_t^{N,R}\right|\right|^2_{C(\mathbb{T}^2)} dt > \sqrt{Z}\right).\end{aligned}$$

Since all the terms above are bounded in expected value, we can apply Chebyshev inequality to make each term smaller than ε. This ends the proof. □

At this point, thanks to Aubin's Lemma, we are able to obtain a first tightness result for the law of $u^{N,R}$ in $L^2([0,T];C(\mathbb{T}^2))$. L^2 estimates on time are enough to prove a convergence result (as partially done in [48]), but they are not sufficient to remove the cutoff at the particle level, thus obtaining Theorem 2.2.3. Hence we will have to improve our estimates in order to obtain stronger time convergence. We apply Corollary 8 in [88] by taking

$$X = H^{1+2\alpha}(\mathbb{T}^2), \quad B = H^{1+2\alpha-\varepsilon}(\mathbb{T}^2), \quad Y = H^{-1}(\mathbb{T}^2),$$

where $\varepsilon < 2\alpha$ and where $X \hookrightarrow Y$ is compact. The interpolation inequality between the space B and X, Y, required in Corollary 8, it is an easy result of Fourier analysis since we are on the torus. Hence we have that

$$L^\infty([0,T];H^{1+2\alpha}(\mathbb{T}^2)) \cap W^{1,2}([0,T];H^{-1}(\mathbb{T}^2)) \hookrightarrow C([0,T];H^{1+2\alpha-\varepsilon}(\mathbb{T}^2))$$

with a compact embedding. Hence, by Sobolev embedding in dimension two of $H^{1+2\alpha-\varepsilon}(\mathbb{T}^2)$ into $C(\mathbb{T}^2)$ we also have that

$$L^\infty([0,T];H^{1+2\alpha}(\mathbb{T}^2)) \cap W^{1,2}([0,T];H^{-1}(\mathbb{T}^2)) \hookrightarrow C([0,T]\times\mathbb{T}^2)$$

with a compact embedding. Clearly the result also holds when $H^{1+2\alpha}(\mathbb{T}^2)$ is replaced by $H^2(\mathbb{T}^2)$. However we were not able to obtain a uniform in time result for the H^2 norm and hence we tried to trim our requirements. To do so, we first rewrite the equation for $\omega^{N,R}$ in its mild formulation

$$\begin{aligned}\omega_t^{N,R} = e^{t\Delta}\omega_0^{N,R} &- \int_0^t e^{(t-s)\Delta} u_s^{N,R}\cdot\nabla\omega_s^{N,R} ds \\ &- \int_0^t e^{(t-s)\Delta}\nabla^\perp\cdot\frac{1}{N}\sum_{i=1}^N (u_{\varepsilon_N}^{N,R}(X_s^{i,N,R})\chi_R(u_s^{N,R}) - V_s^{i,N,R})\delta^{\varepsilon_N}_{X_s^{i,N,R}} ds. \quad (2.15)\end{aligned}$$

Lemma 2.5.9. *For all $\alpha < \frac{1}{2}$ and for each ε, there exists Z such that*

$$\mathbf{P}\left(\left|\left|u^{N,R}\right|\right|_{L^\infty([0,T],H^{1+2\alpha})} > Z\right) \leq \varepsilon$$

Proof. We apply a generalized Gronwall Lemma to the function of the time variable only $\left|\left|u_t^{N,R}\right|\right|_{H^{1+2\alpha}(\mathbb{T}^2)}$. Since $\left|\left|u_t^{N,R}\right|\right|_{H^{1+2\alpha}(\mathbb{T}^2)} \sim \left|\left|\omega_t^{N,R}\right|\right|_{H^{2\alpha}(\mathbb{T}^2)}$ we apply the

operator $(I-\Delta)^\alpha$ on the mild formulation of vorticity equation (2.15), obtaining

$$\begin{aligned}\left|\left|(I-\Delta)^\alpha \omega_t^{N,R}\right|\right|_{L^2(\mathbb{T}^2)} &\leq \left|\left|(I-\Delta)^\alpha e^{t\Delta}\omega_0^{N,R}\right|\right|_{L^2(\mathbb{T}^2)} + \\ \int_0^t \left|\left|(I-\Delta)^\alpha e^{(t-s)\Delta}\nabla^\perp \cdot \frac{1}{N}\sum_{i=1}^N (u_{\varepsilon_N}^{N,R}(X_s^{i,N,R})\chi_R(u_s^{N,R}) - V_s^{i,N,R})\delta^{\varepsilon_N}_{X_s^{i,N,R}}\right|\right|_{L^2(\mathbb{T}^2)} ds \\ + \int_0^t \left|\left|(I-\Delta)^\alpha e^{(t-s)\Delta} u_s^{N,R}\cdot\nabla\omega_s^{N,R}\right|\right|_{L^2(\mathbb{T}^2)} ds.\end{aligned} \tag{2.16}$$

We start by estimating the initial conditions:

$$\begin{aligned}\left|\left|(I-\Delta)^\alpha e^{t\Delta}\omega_0^{N,R}\right|\right|_{L^2(\mathbb{T}^2)} \leq \\ \left|\left|e^{t\Delta}\right|\right|_{L^2(\mathbb{T}^2)\to L^2(\mathbb{T}^2)} \left|\left|(I-\Delta)^\alpha\omega_0^{N,R}\right|\right|_{L^2(\mathbb{T}^2)} \lesssim \left|\left|\omega_0^{N,R}\right|\right|_{H^{2\alpha}(\mathbb{T}^2)}.\end{aligned} \tag{2.17}$$

Regarding the second term of the r.h.s. of (2.16)

$$\begin{aligned}&\left|\left|(I-\Delta)^\alpha e^{(t-s)\Delta}\nabla^\perp\cdot\frac{1}{N}\sum_{i=1}^N (u_{\varepsilon_N}^{N,R}(X_s^{i,N,R})\chi_R(u_s^{N,R}) - V_s^{i,N,R})\delta^{\varepsilon_N}_{X_s^{i,N,R}}\right|\right|_{L^2(\mathbb{T}^2)} \\ &\quad\leq \left|\left|\nabla(I-\Delta)^{-1/2}\right|\right|_{L^2(\mathbb{T}^2)\to L^2(\mathbb{T}^2)} \left|\left|(I-\Delta)^{\alpha+1/2}e^{(t-s)\Delta}\right|\right|_{L^2(\mathbb{T}^2)\to L^2(\mathbb{T}^2)} \\ &\qquad\times\left|\left|\frac{1}{N}\sum_{i=1}^N (u_{\varepsilon_N}^{N,R}(X_s^{i,N,R})\chi_R(u_s^{N,R}) - V_s^{i,N,R})\delta^{\varepsilon_N}_{X_s^{i,N,R}}\right|\right|_{L^2(\mathbb{T}^2)} \\ &\quad\leq \frac{C}{(t-s)^{\alpha+1/2}}\left|\left|\frac{1}{N}\sum_{i=1}^N (u_{\varepsilon_N}^{N,R}(X_s^{i,N,R})\chi_R(u_s^{N,R}) - V_s^{i,N,R})\delta^{\varepsilon_N}_{X_s^{i,N,R}}\right|\right|_{L^2(\mathbb{T}^2)} \\ &\qquad\leq \frac{C}{(t-s)^{\alpha+1/2}}\left(R\left|\left|m_0F_t^{N,R}\right|\right|_{L^2(\mathbb{T}^2)} + \left|\left|m_1F_t^{N,R}\right|\right|_{L^2(\mathbb{T}^2)}\right),\end{aligned} \tag{2.18}$$

while for the last one of (2.16) we have

$$\begin{aligned}&\left|\left|(I-\Delta)^\alpha e^{(t-s)\Delta}u_s^{N,R}\cdot\nabla\omega_s^{N,R}\right|\right|_{L^2(\mathbb{T}^2)} \\ &\quad\leq \left|\left|(I-\Delta)^{\alpha+1/2}e^{(t-s)\Delta}\right|\right|_{L^2(\mathbb{T}^2)\to L^2(\mathbb{T}^2)} \left|\left|(I-\Delta)^{-1/2}u_s^{N,R}\cdot\nabla\omega_s^{N,R}\right|\right|_{L^2(\mathbb{T}^2)} \\ &\quad\leq \frac{C}{(t-s)^{\alpha+1/2}}\left|\left|(I-\Delta)^{-1/2}u_s^{N,R}\cdot\nabla\omega_s^{N,R}\right|\right|_{L^2(\mathbb{T}^2)},\end{aligned}$$

and

$$\left|\left|(I-\Delta)^{-1/2}u_s^{N,R}\cdot\nabla\omega_s^{N,R}\right|\right|_{L^2(\mathbb{T}^2)}=\sup_{\varphi\in L^2(\mathbb{T}^2)}\left|\langle(I-\Delta)^{-1/2}u_s^{N,R}\cdot\nabla\omega_s^{N,R},\varphi\rangle\right|.$$

Now, notice that

$$\begin{aligned}\langle u_s^{N,R}\cdot\nabla\omega_s^{N,R},(I-\Delta)^{-1/2}\varphi\rangle&=-\langle\omega_s^{N,R},u_s^{N,R}\cdot\nabla(I-\Delta)^{-1/2}\varphi\rangle\\&\leq\sup_{||\varphi||_{L^2(\mathbb{T}^2)}\leq1}||\varphi||_{L^2(\mathbb{T}^2)}\left|\left|u_s^{N,R}\right|\right|_\infty\left|\left|\omega^{N,R}\right|\right|_{L^2(\mathbb{T}^2)}.\quad(2.19)\end{aligned}$$

Combining (2.17),(2.18),(2.19):

$$\begin{aligned}\left|\left|\omega_t^{N,R}\right|\right|_{H^{2\alpha}(\mathbb{T}^2)}&\lesssim\left|\left|\omega_0^{N,R}\right|\right|_{H^{2\alpha}(\mathbb{T}^2)}\\&\quad+\int_0^t\frac{\left(R\left|\left|m_0F_s^{N,R}\right|\right|_{L^2(\mathbb{T}^2)}+\left|\left|m_1F_s^{N,R}\right|\right|_{L^2(\mathbb{T}^2)}\right)}{(t-s)^{\alpha+1/2}}ds\\&\quad+\int_0^t\frac{\left|\left|u_s^{N,R}\right|\right|_{L^\infty(\mathbb{T}^2)}\left|\left|\omega_s^{N,R}\right|\right|_{L^2(\mathbb{T}^2)}}{(t-s)^{\alpha+1/2}}ds\\&\leq C\left|\left|\omega_0^{N,R}\right|\right|_{H^{2\alpha}(\mathbb{T}^2)}+\int_0^T\frac{\left(R\left|\left|m_0F_s^{N,R}\right|\right|_{L^2(\mathbb{T}^2)}+\left|\left|m_1F_s^{N,R}\right|\right|_{L^2(\mathbb{T}^2)}\right)}{(T-s)^{\alpha+1/2}}ds+\\&\quad+C\left(\sup_{t\in[0,T]}\left|\left|\omega_t^{N,R}\right|\right|_{L^2(\mathbb{T}^2)}\right)\int_0^t\frac{\left|\left|u_s^{N,R}\right|\right|_{H^{1+2\alpha}(\mathbb{T}^2)}}{(t-s)^{\alpha+1/2}}ds.\end{aligned}$$

Finally,

$$\begin{aligned}\left|\left|u_t^{N,R}\right|\right|_{H^{1+2\alpha}(\mathbb{T}^2)}&\lesssim C\left|\left|\omega_0^{N,R}\right|\right|_{H^{2\alpha}(\mathbb{T}^2)}+\int_0^T\frac{R\left|\left|m_0F_s^{N,R}\right|\right|_{L^2(\mathbb{T}^2)}}{(T-s)^{\alpha+1/2}}\,ds+\\&\quad+\int_0^T\frac{\left|\left|m_1F_s^{N,R}\right|\right|_{L^2(\mathbb{T}^2)}}{(T-s)^{\alpha+1/2}}\,ds+\left(\sup_{t\in[0,T]}\left|\left|\omega_t^{N,R}\right|\right|_{L^2(\mathbb{T}^2)}\right)\int_0^t\frac{\left|\left|u_s^{N,R}\right|\right|_{H^{1+2\alpha}(\mathbb{T}^2)}}{(t-s)^{\alpha+1/2}}ds.\end{aligned}$$

Notice that in the expression above the terms involved above are random (for simplicity we have omitted $\omega\in\Omega$). Introduce, to short the notation, the random function

$$\varphi(t):=\left|\left|u_t^{N,R}\right|\right|_{H^{1+2\alpha}(\mathbb{T}^2)}.$$

We have proved that the function φ satisfies

$$\varphi(t)\leq X_1+X_2\int_0^t\frac{\varphi(s)}{(t-s)^{\alpha+1/2}}\,ds$$

where

$$X_1 = \left|\left|\omega_0^{N,R}\right|\right|_{H^{2\alpha}(\mathbb{T}^2)} + \int_0^T \frac{\left(R\left|\left|m_0 F_s^{N,R}\right|\right|_{L^2(\mathbb{T}^2)} + \left|\left|m_1 F_s^{N,R}\right|\right|_{L^2(\mathbb{T}^2)}\right)}{(T-s)^{\alpha+1/2}}\, ds$$

$$X_2 = \sup_{t\in[0,T]} \left|\left|\omega_t^{N,R}\right|\right|_{L^2(\mathbb{T}^2)}.$$

Notice that, by the uniform estimates proved in this section, there exist two constant C_1 and C_2, independent on N, such that

$$\mathbf{E}\left[X_1\right] \leq C_1, \quad \mathbf{E}\left[X_2\right] \leq C_2,$$

so that, for fixed ε we can chose $R_1, R_2 > 0$ in order to have

$$\mathbf{P}(X_1 > R_1) < \frac{\varepsilon}{2}, \quad \mathbf{P}(X_2 > R_2) < \frac{\varepsilon}{2}.$$

For a fixed $\omega \in \Omega$ applying Gronwall Lemma to the function φ we obtain

$$\sup_{t\in[0,T]} \varphi(t)(\omega) \leq f(X_1, X_2)(\omega).$$

We now claim that

$$\mathbf{P}\left(\sup_{t\in[0,T]} \varphi(t) > f(R_1, R_2)\right) < \varepsilon.$$

In fact we have the following chain of inequalities

$$\mathbf{P}\left(\sup_{t\in[0,T]} \varphi(t) \leq f(R_1, R_2)\right) \geq \mathbf{P}\left(\varphi(t) \leq R_1 + R_2 \int_0^t \frac{\varphi(s)}{(t-s)^{\alpha+1/2}}\, ds\right)$$
$$\geq \mathbf{P}\left((X_1 \leq R_1) \cap (X_2 \leq R_2)\right) \geq 1 - \mathbf{P}(X_1 > R_1) - \mathbf{P}(X_2 > R_2) \geq 1 - \varepsilon.$$

We end the proof by taking the complement set both sides. □

We are finally able to obtain the following tightness result:

Lemma 2.5.10. *The family of laws $\{Q^{N,R,u}\}_{N\in\mathbb{N}}$ of $\{u^{N,R}\}_{N\in\mathbb{N}}$, is tight, and hence is relatively compact as a probability measure on $C([0,T]\times\mathbb{T}^2)$.*

Proof. The proof is just an application of Simons embedding in [88]. For each $M, Z > 0$ we can consider the following set, for all $\alpha < 1/2$

$$K_{M,Z} = \Big\{ u \in C([0,T]\times\mathbb{T}^2) \mid ||u||_{L^\infty([0,T];H^{1+2\alpha}(\mathbb{T}^2))} \leq M,$$
$$||u||_{W^{1,2}([0,T];H^{-1}(\mathbb{T}^2))} \leq Z \Big\}.$$

By the Simons Lemma $K_{M,Z}$ is relatively compact in $C([0,T]\times\mathbb{T}^2)$. Notice that

$$Q^{N,R,u}(K^c_{M,Z}) = \mathbf{P}(u^{N,R}\in K^c_{M,Z}) \leq$$
$$\mathbf{P}\left(\left|\left|u^{N,R}\right|\right|_{L^\infty([0,T];H^{1+2\alpha}(\mathbb{T}^2))} > M\right) + \mathbf{P}\left(\left|\left|u^{N,R}\right|\right|_{W^{1,2}([0,T];H^{-1}(\mathbb{T}^2))} > Z\right) \leq$$
$$\leq \frac{\mathbf{E}\left[\left|\left|u^{N,R}\right|\right|_{L^\infty([0,T];H^{1+2\alpha}(\mathbb{T}^2))}\right]}{M} + \varepsilon$$

by lemma 2.5.8. By Lemma 2.5.9 the expected values on the r.h.s. is uniformly bounded with respect to N, hence the sequence $\{Q^{N,R,u}\}_{N\in\mathbb{N}}$ is tight and proof is concluded. □

Combining Proposition 2.5.2 and Lemma 2.5.10 we obtain the following:

Corollary 2.5.11. *The family of laws $\{Q^{N,R}\}_{N\in\mathbb{N}}$ of the couple $(u^{N,R}, S^{N,R})$ is tight, and hence relatively compact as a probability measures on $C([0,T]\times\mathbb{T}^2)\times C([0,T];\mathbf{P}_1(\mathbb{T}^2\times\mathbb{R}^2))$.*

2.5.2 Convergence of $(PS^R - NS^R)$ to (VNS).

We will now prove that, under hypothesis on Section 2.2.3, and if R is large enough, then the solution (u^R, F^R) of (VNS^R) coincide with the solution (u,F) of (VNS). To do so we will prove that u^R is bounded in $L^\infty([0,T]\times\mathbb{T}^2)$, independently on R. First we summarize all the intermediate results needed for the proof. We remark that all the following bounds hold independently on R:

- For all $k\leq 6$
$$\sup_{t\in[0,T]} M_k F^R_t \leq C$$
by Lemma 2.3.6 and hypothesis 2.2.3;
- $$\left|\left|m_0 F^R\right|\right|_{L^\infty([0,T];L^2(\mathbb{T}^2))} \leq C, \quad \text{and} \quad \left|\left|m_1 F^R\right|\right|_{L^\infty([0,T];L^2(\mathbb{T}^2))} \leq C$$
by Lemma 2.3.6 and inequality 1. and 2. in Lemma 2.3.5;
- for all $p>1$
$$\left|\left|u^R\right|\right|_{L^2([0,T];L^p(\mathbb{T}^2))} \leq C_p$$
by Remark 4.

We can now formulate a further result, needed in the proof of Theorem 2.5.14.

Lemma 2.5.12. *There exists a constant C, independent on R, such that*

$$\left|\left|\omega^R\right|\right|_{L^\infty([0,T];L^2(\mathbb{T}^2))} \leq C.$$

Proof. Computing the time derivative of $\int_{\mathbb{T}^2} \left|\omega_t^R\right|^2 dx$ we obtain

$$\left|\left|\omega_t^R\right|\right|^2_{L^2(\mathbb{T}^2)} + \int_0^T \int_{\mathbb{T}^2} \left|\nabla\omega_s^R\right|^2 dx\, ds \lesssim ||\omega_0||^2_{L^2(\mathbb{T}^2)} + \int_0^t \int_{\mathbb{T}^2} \omega_s^R \nabla^\perp \cdot \int_{\mathbb{R}^2} (u_s^R - v)\chi_R(u^R) F_s^R\, dv\, dx\, ds. \quad (2.20)$$

Focusing only on the last term of the previous inequality we have

$$(2.20) \lesssim \int_0^t \int_{\mathbb{T}^2} \left|\nabla\omega_s^R\right| \left|u_s^R\right| \int_{\mathbb{R}^2} F_s^R\, dv\, dx\, ds + \int_0^t \int_{\mathbb{T}^2} \left|\nabla\omega_s^R\right| \int_{\mathbb{R}^2} |v|\, F_s^R\, dv\, dx\, ds$$

$$\lesssim \int_0^T \int_{\mathbb{T}^2} \left|\nabla\omega_s^R\right|^2 dx\, ds + \int_0^T \int_{\mathbb{T}^2} \left|u_s^R\right|^2 \left(\int_{\mathbb{R}^2} F_s^R\, dv\right)^2 dx\, ds + \int_0^T \int_{\mathbb{T}^2} \left|\nabla\omega_s^R\right|^2 dx\, ds + \int_0^T \int_{\mathbb{T}^2} \left(\int_{\mathbb{R}^2} |v|\, F_s^R\, dv\right)^2 dx\, ds.$$

Let us notice that

$$\int_0^T \int_{\mathbb{T}^2} \left|u_s^R\right|^2 \left(\int_{\mathbb{R}^2} F_s^R dv\right)^2 dx ds \leq \int_0^T \left|\left|u_s^R\right|\right|^2_{L^4(\mathbb{T}^2)} \left(\int_{\mathbb{T}^2} \left(\int_{\mathbb{R}^2} F_s^R\, dv\right)^4 dx\right)^{\frac{1}{2}} ds$$
$$\leq \sup_{t\in[0,T]} \left|\left|m_0 F_t^R\right|\right|^2_{L^4(\mathbb{T}^2)} \left|\left|u^R\right|\right|_{L^2([0,T];L^4(\mathbb{T}^2))}$$
$$\lesssim \sup_{t\in[0,T]} (M_6 F_t^R)^{\frac{1}{2}} \left|\left|u^R\right|\right|_{L^2([0,T];L^4(\mathbb{T}^2))} \leq C$$

and

$$\int_0^T \int_{\mathbb{T}^2} \left(\int_{\mathbb{R}^2} |v|\, F_s^R\, dv\right)^2 dx\, ds \lesssim_T \sup_{t\in[0,T]} M_4 F_t^R \leq C$$

again by Lemma 2.3.5. We conclude the proof by classical Gronwall Lemma. □

We emphasize that, even if it is possible to prove the uniform bound with respect the parameter R, it is not possible to obtain the same result directly at the particle level. In other terms, we were not able to obtain directly any bound on the vorticity in the particle system $(PS - NS)$

$$\mathbf{E}\left[\left|\left|\omega^N\right|\right|_{L^\infty([0,T];L^2(\mathbb{T}^2))}\right]$$

without using the cut off. This result would have allowed us to remove the cut off directly at the particle level, without any further complication.
We finally prove the uniform bound on u^R:

Proposition 2.5.13. *There exists a constant $\mathbf{K}_u$, independent on R, such that*

$$\left|\left|u^R\right|\right|_\infty \leq \mathbf{K}_u.$$

Proof. In order to produce the required bound we bound uniformly the norm of u^R in the space $L^\infty([0,T];H^{1+2\alpha}(\mathbb{T}^2))$ for any $\alpha < 1/2$. Hence we use the mild formulation for the vorticity equation associated with u^R:

$$\partial_t \omega^R = \Delta\omega^R - u^R \cdot \nabla\omega^R - \nabla^\perp \cdot \int_{\mathbb{R}^2} (u^R - v)\chi_R(u^R) F^R \, dv.$$

Following the same argument of Lemma 2.5.9 we get

$$\begin{aligned}
\left|\left|u_t^R\right|\right|_{H^{1+2\alpha}(\mathbb{T}^2)} &\lesssim \left|\left|\omega_t^R\right|\right|_{H^{2\alpha}(\mathbb{T}^2)} \lesssim \left|\left|\omega_0^R\right|\right|_{H^{2\alpha}(\mathbb{T}^2)} \\
&\quad + \int_0^t \frac{\left|\left|u_s^R\right|\right|_{L^\infty(\mathbb{T}^2)} \left|\left|\omega_s^R\right|\right|_{L^2(\mathbb{T}^2)}}{(t-s)^{\alpha+1/2}} ds \\
&\quad + \int_0^t \frac{\left|\left|u_s^R\right|\right|_{L^\infty(\mathbb{T}^2)} \left|\left|m_0 F_s^R\right|\right|_{L^2(\mathbb{T}^2)}}{(t-s)^{\alpha+1/2}} ds + \int_0^t \frac{\left|\left|m_1 F_s^R\right|\right|_{L^2(\mathbb{T}^2)}}{(t-s)^{\alpha+1/2}} \, ds \\
&\lesssim \left|\left|\omega_0^R\right|\right|_{H^{2\alpha}(\mathbb{T}^2)} + \left|\left|\omega^R\right|\right|_{L^\infty([0,T];L^2(\mathbb{T}^2))} \int_0^t \frac{\left|\left|u_s^R\right|\right|_{H^{1+2\alpha}(\mathbb{T}^2)}}{(t-s)^{\alpha+1/2}} ds \\
&\quad + \left(\sup_{t\in[0,T]} M_2 F_t^R\right)^{\frac{1}{2}} \int_0^t \frac{\left|\left|u_s^R\right|\right|_{H^{1+2\alpha}(\mathbb{T}^2)}}{(t-s)^{\alpha+1/2}} ds + \left(\sup_{t\in[0,T]} M_4 F_t^R\right)^{\frac{1}{2}},
\end{aligned}$$

by neglecting the cutoff function χ_R which is bounded by one. By using the uniform bound described at the beginning of Section 2.5.2, Lemma 2.5.12 and Lemma 2.3.5 inequality 1. and 2. we see that all the expression above are bounded independently on R and we conclude by a Gronwall type argument applied to the function $\left|\left|u_t^R\right|\right|_{H^{1+2\alpha}(\mathbb{T}^2)}$. □

In conclusion we have the following Theorem:

Theorem 2.5.14. *If $R \geq \mathbf{K}_u + 1$, then any weak solution (u^R, F^R) of system of PDE (VNS^R) coincide with the unique bounded weak solutions of system of equations (VNS).*

Proof. By proposition 2.5.13, taking $R \geq \mathbf{K}_u + 1$ we have that the function $\chi_R(u^R) \equiv 1$, hence system of equation (VNS^R) reduce to (VNS). Hence, we obtain that the couple (u^R, F^R) satisfies system of equation (VNS). By the uniqueness of solution for system of equations (VNS), we obtain $u = u^R$ and $F = F^R$. □

In order to complete the proof of Proposition 2.5.1 we need only to verify that limit points of the sequence $\{Q^{N,R}\}_{N\in\mathbb{N}}$ are supported on weak solutions of system of equations (VNS).

Proposition 2.5.15. *If $R \geq \mathbf{K}_u + 1$ limit points of subsequences of $\{Q^{N,R}\}_{N\in\mathbb{N}}$ are supported on the bounded weak solutions of system of PDE (VNS) (see Definition 2.2.2).*

Proof. In order to prove that weak limits are supported on weak solutions, we have to prove that those objects satisfies equation (VNS) in the weak sense, and that they have the correct regularity. The fact that limit objects satisfy system of equations (VNS) is classical, see [68]. Let us focus on the regularity issue.

First, by By Lemma 2.5.4 together with Lemma 2.3.5 inequality 3. and 4, limit points on the component corresponding to $u^{N,R}$ satisfy the regularity properties of Definition 2.2.2. Similarly from a priori estimates in Corollary 2.5.6 limit points of subsequences have a density on their particle component (corresponding to $S^{N,R}$) which is also in $L^2([0,T]\times\mathbb{T}^2\times\mathbb{R}^2)$. In order to complete the proof we need to verify that such density is uniformly bounded, as required in Definition 2.2.2. This follows by the maximum principle argued in Section 2.3.1. Namely, the fact that the limit points along the particles component satisfies system of equations (VNS), where u is uniformly bounded, yields an uniform bound for the density in $L^\infty([0,T]\times\mathbb{T}^2\times\mathbb{R}^2)$. Denoting by F one of the limit points, we only need to verify that

$$\int_0^T \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} |v|^2 F_s^2 \, dx \, dv \, ds < \infty. \tag{2.21}$$

By using Lemma 2.5.5 F is in $L^4([0,T]\times\mathbb{T}^2\times\mathbb{R}^2)$. By interpolation inequality of L^p spaces we also have $F \in L^3([0,T]\times\mathbb{T}^2\times\mathbb{R}^2)$. Also, the uniform bound on the v-moments of $F^{N,R}$, provided in Lemma 2.5.7, grants also M_4F to be finite. Hence, by an easy computation (see Section 2.3.1), we see that (2.21) is satisfied. Thus by the maximum principle we have $F \in L^\infty([0,T]\times\mathbb{T}^2\times\mathbb{R}^2)$, hence ending the proof. □

Combining Proposition 2.5.15 with Theorem 2.4.1 we complete the proof of Proposition 2.5.1.

2.6 Scaling limit for the full system

The aim of this section is to prove that the cut-off can be removed also in the approximating system $(u^{N,R}, S^{N,R})$: the uniform convergence result proved in the previous section, Proposition 2.5.1, gives a simple but relevant hint to prove the final result of convergence. We expect that the converging object $(u^{N,R}, S^{N,R})$

inherits the property of boundedness, independently on the parameter R, that holds for the limit object. If so, we can remove the cut-off, choosing R large enough from the beginning. The first difficulty in the realization of this intuition is given by the type of convergence which we are dealing with: convergence in law. We will overcome this technicality, appealing to the Skorohod's Theorem to strengthen the convergence.

We will first state and prove a general result for almost sure convergence of random variables. Then, in order to utilize such criterion, we will make use of Skorohod's Theorem and we will understand our particle systems in a *path-by-path* sense: we will give a precise definition of path-by-path solutions and prove a uniqueness result for such kind of solutions. The application of the above mentioned criterion to our case will let us transfer the property of convergence from the sequence $(u^{N,R}, S^{N,R})$ to (u^N, S^N).

In the rest of the section we will always assume to have taken

$$R = \max(\mathbf{K}_u + 1, ||u||_\infty + 1)$$

where the constant $\mathbf{K_u}$ has been defined in Proposition 2.5.13. This choice will assure that Proposition 2.5.1 is verified. The condition that R is greater than $||u||_\infty + 1$ is needed in order to let the sequence of $u^{N,R}$ to inherit the uniform boundedness of the limit u. This process will be clarified later.

2.6.1 Convergence criterion

We now present the general criterion that we will use to obtain the convergence of the sequence $(u^N, S^N)_{N\in\mathbb{N}}$ from that of $(u^{N,R}, S^{N,R})_{N\in\mathbb{N}}$. The framework of this criterion is pretty general. We preferred to isolate it and state it in its general form, rather than in our specific case, in order to make the underlying idea more evident.

Theorem 2.6.1 (General Principle)**.** *Let $(\Omega, \mathcal{F}, \mathbf{P})$ a probability space and let (E, d_E) a separable metric space. Let $\{X_N\}_{N\in\mathbb{N}}$ and $\{Y_N\}_{N\in\mathbb{N}}$ two sequences of random variables taking values in E and let x be a point in E. Moreover, suppose that for each $N \in \mathbb{N}$, there exist two collections of subset $S_N^X(\omega) \subseteq E$ and $S_N^Y(\omega) \subseteq E$, indexed by $\omega \in \Omega$. Assume further that the following conditions are satisfied:*

1. $$X_N \xrightarrow{N\to\infty} x \in E \quad \mathbf{P}\text{-}a.s.;$$

2. *denoting*
$$\Omega_S = \left\{\omega \in \Omega \,|\, \sharp S_N^Y(\omega) \leq 1 \quad \forall N \in \mathbb{N}\right\}$$
where by $\sharp A$ we mean the cardinality of the set A, we have
$$\mathbf{P}(\Omega_S) = 1;$$

3. denoting

$$\Omega_X = \left\{\omega \in \Omega \,|\, X_N(\omega) \in S_N^X(\omega)\ \forall N \in \mathbb{N}\right\},$$
$$\Omega_Y = \left\{\omega \in \Omega \,|\, Y_N(\omega) \in S_N^Y(\omega)\ \forall N \in \mathbb{N}\right\},$$

we have

$$\mathbf{P}(\Omega_X) = \mathbf{P}(\Omega_Y) = 1;$$

4.

$$B_E(x,1) \cap S_N^X(\omega) \subseteq S_N^Y(\omega) \quad \forall N \in \mathbb{N},\, \forall \omega \in \Omega.$$

Then the sequence $\{Y_N\}_{N\in\mathbb{N}}$ converges in E to the same limit of the sequence $\{X_N\}_{N\in\mathbb{N}}$

$$Y_N \xrightarrow{N\to\infty} x \in E \quad \mathbf{P}\text{-}a.s.$$

Proof. Consider the set

$$\Omega_{C,X} := \left\{\omega \in \Omega \,|\, d(X^N(\omega), x)_E \xrightarrow{N} 0\right\}$$

and

$$\Omega_{C,Y} := \left\{\omega \in \Omega \,|\, d(Y^N(\omega), x)_E \xrightarrow{N} 0\right\}$$

Note that, by property 1. the set $\Omega_{C,X}$ has full measure $\mathbf{P}(\Omega_{C,X}) = 1$.
We will prove that

$$\Omega_S \cap \Omega_{C,X} \cap \Omega_X \cap \Omega_Y \subseteq \Omega_{C,Y} \tag{2.22}$$

thus implying the thesis being $\mathbf{P}(\Omega_S) = \mathbf{P}(\Omega_X) = \mathbf{P}(\Omega_Y) = 1$ by property 2. and 3. To do so let us consider the set

$$\Omega_1 = \{\omega \in \Omega \,|\, \exists N(\omega)\, d(X_N(\omega), x) \leq 1\, \forall N > N(\omega)\}$$

and note that

$$\Omega_{X,C} \subseteq \Omega_1.$$

Now define

$$\Omega_2 = \{\omega \in \Omega \,|\, X_N(\omega) = Y_N(\omega)\, \forall N > N(\omega)\}$$

where $N(\omega)$ is defined for each ω, in the set Ω_1. We claim that

$$\Omega_S \cap \Omega_{X,C} \cap \Omega_X \cap \Omega_Y \subseteq \Omega_2. \tag{2.23}$$

Take $\omega \in \Omega_S \cap \Omega_{X,C} \cap \Omega_X \cap \Omega_Y$. Hence if $N > N(\omega)$, given that ω lies in $\Omega_{X,C}$, it also lies in Ω_1, thus we have $X_N(\omega) \in B_E(x,1)_E$. Moreover, ω lies also in Ω_X, hence $X_N(\omega) \in S_N^X(\omega)$. By property 4. we conclude $X_N(\omega) \in S_N^Y(\omega)$. Furthermore $\omega \in \Omega_Y$ implies $Y_N(\omega) \in S_N^Y(\omega)$, but ω is also in Ω_S hence by property 2. $S_N^Y(\omega)$ is a singleton, hence $S_N^Y(\omega) = \{Y_N(\omega)\}$. Since $X_N(\omega) \in S_N^X(\omega)$ and $S_N^Y(\omega) = \{Y_N(\omega)\}$

we obtain $X_N(\omega) = Y_N(\omega)$ and we have proven condition (2.23).
Finally, we can prove condition (2.22): taking $\omega \in \Omega_S \cap \Omega_{X,C} \cap \Omega_X \cap \Omega_Y$, we have that $\forall \varepsilon > 0$ there exists $N_\varepsilon(\omega)$, such that

$$d(X_N(\omega), x)_E < \varepsilon \quad \forall N > N_\varepsilon(\omega)$$

By condition (2.23) ω lies also in Ω_2, hence

$$X_N(\omega) = Y_N(\omega) \quad \forall N > N(\omega).$$

Calling $\overline{N}(\omega) = \max(N_\varepsilon(\omega), N(\omega))$ we conclude

$$d(Y_N(\omega), x)_E < \varepsilon \quad \forall N > \overline{N}_\varepsilon(\omega)$$

and hence $\omega \in \Omega_{Y,C}$. Thus the proof is concluded.

□

2.6.2 Path by Path solutions for $(PS - NS)$

We will now focus on the problem of uniqueness for *path-by-path* solutions. The issue of uniqueness for this class of solutions is very difficult: very few result are know before the work of [35]. The analysis of such kind of problem for $(PS - NS)$ will be a key point of the proof of Theorem 2.2.3. In fact, to apply Theorem 2.6.1 to our case, we will see that strong uniqueness in the sense of SDEs, which is more classical than that path-by-path, will not be enough. We now recall the concept of path-by-path solutions and uniqueness in this class. We will discuss this topic in the specific case that is needed here, the system of PDE-SDEs $(PS - NS)$.

Recall system of equation $(PS - NS)$ and note that, in the equation for the particle position and velocity $(X_t^{i,N}, V_t^{i,N})$ the noise is pure additive Brownian motion, i.e. the diffusion coefficient is constant. For this reason Itô integral is not involved into the equations and one can understand system of equations $(PS - NS)$ in its integral form as a coupling PDE-ODEs, where the Brownian motions plays the role of a given external force. This perspective is outlined in the following system

$$\begin{cases} \partial_t u^N = \Delta u^N - u^N \cdot \nabla u^N - \nabla \pi^N - \frac{1}{N} \sum_{i=1}^N (u^N_{\varepsilon_N}(X_t^{i,N}) - V_t^{i,N}) \delta^{\varepsilon_N}_{X_t^{i,N}} \\ \operatorname{div}(u^N) = 0, \\ \begin{cases} X_t^{i,N} = X_0^i + \int_0^t V_s^{i,N}\, ds \\ V_t^{i,N} = V_0^i + \int_0^t (u^N_{\varepsilon_N}(X_s^{i,N}) - V_s^{i,N})\, ds + \sigma B_t^i(\omega) \end{cases} \qquad i = 1, \dots, N \end{cases} \tag{2.24}$$

where $B_t^i(\omega)$ stands for a single realization of a Brownian path for fixed $\omega \in \Omega$. We now introduce the set of path-by-path solutions for a given realization of $\omega \in \Omega$ and for fixed $N \in \mathbb{N}$:

$$S_N(\omega) = \Bigg\{ \Big(w, \big(x_\cdot^i, v_\cdot^i\big)_{i=1,\dots,N} \Big) \in C([0,T] \times \mathbb{T}^2) \times C([0,T]; \mathbb{T}^2 \times \mathbb{R}^2)^N \text{ s.t.} \\ \Big(w, \big(x_\cdot^i, v_\cdot^i\big)_{i=1,\dots,N} \Big) \text{ solves (2.24) with additive noise } (B_t^i(\omega))_{i=1,\dots,N} \Bigg\}. \quad (2.25)$$

Roughly speaking $S_N(\omega)$ is the set of curves that solves (2.24) in a deterministic setting for a prescribed realization of a Brownian path (identified by ω). We do not give a precise definition of existence of path-by-path solutions. We remark that existence of weak or strong solutions in an SDE settings imply that the set $S_N(\omega)$ is non empty with probability one. We now focus our attention to the topic of uniqueness.

Definition 2.6.2 (Uniqueness of path-by-path solutions)**.** Given a natural number N we say that there is path-by-path uniqueness for system of equations $(PS - NS)$ with N particles, if there exist a set $\Omega_S \subseteq \Omega$ with probability one $\mathbf{P}(\Omega_S) = 1$ such that

$$\sharp S_N(\omega) \leq 1 \quad \forall \omega \in \Omega_S$$

where $\sharp A$ stands for the cardinality of the set A.

Opposite to the case of existence, uniqueness of path-by-path solutions is a much more difficult topic: uniqueness in this class is a stronger notion that weak or strong uniqueness for SDE. In Definition 2.6.2 no measurability with respect to the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ is required. In case of uniqueness for SDE a much richer structure is available, given that solutions are required at least to be adapted to the filtration $\mathcal{F}_t$.

We now prove a path-by-path uniqueness result for system of equation $(PS - NS)$. Some results about path-by-path uniqueness for SDEs are already known: Davie in [35] proved the result for a single SDE with pure additive Brownian noise and only bounded measurable drift. This type of result for low regularity drift functions, less than locally Lipschitz, are very difficult. In our case, the drift appearing into the particle equations $(X_t^{i,N}, V_t^{i,N})$ is even more regular than Lipschitz: in fact the function $u_{\varepsilon_N}^N(t,x)$ is C^∞ in the space variable due to the convolution with the C^∞ function $\theta^{\varepsilon_N}(x)$. However, the case here is slightly different from the case of a single SDE due to the strong coupling with the Navier-Stokes equation that introduces additional difficulty.

Proposition 2.6.3. *Let us consider on the probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$

$$\Omega_{\mathcal{B}} = \left\{ \omega \in \Omega \,|\, B_t^i(\omega) \text{ is continuous on } [0,T] \, \forall i \in \mathbb{N} \right\} \subseteq \Omega$$

the set where all the Brownian motion $(B^i)_{i\in\mathbb{N}}$ are continuous, which is of full measure with respect to $\mathbf{P}$. Then, for all $N \in \mathbb{N}$ we have uniqueness path-by-path for system of equation $(PS-NS)$ with N particles, namely

$$\sharp S_N(\omega) \leq 1 \quad \forall \omega \in \Omega_{\mathcal{B}}.$$

Proof. For a matter of simplicity we prove the result in the case $N = 1$: the generalization for general N, is straightforward. Moreover, to make the notation less heavy, we will omit the dependence on N and ω indicating with u_t the variable $u_t^N(\omega)$ and with (X_t, V_t) the couple of variables $(X_t^{1,1}, V_t^{1,1})(\omega)$. Also the mollifier θ^{0,ε_N} will be labeled simply by θ. In our simplification, the system becomes:

$$\begin{cases} \partial_t u = \Delta u - u \cdot \nabla u - \nabla \pi - ((\theta * u)(X_t) - V_t)\, \theta(x - X_t) \\ \operatorname{div}(u) = 0 \\ \begin{cases} \dot{X}_t &= V_t \\ \dot{V}_t &= ((\theta * u_t)(X_t) - V_t) + B_t. \end{cases} \end{cases}$$

Now we consider two solutions (u, X, V) and (u', X', V'), with $(u_0, X_0, V_0) = (u_0', X_0', V_0')$, and we apply Gronwall Lemma to the quantity

$$|X_t - X_t'| + |V_t - V_t'| + ||u_t - u_t'||_{H^{1+2\alpha}(\mathbb{T}^2)},$$

for $\alpha < \frac{1}{2}$.
We start by computing the distance of velocities, recalling that $V_0 = V_0'$ and B_t is the same given function for the two solutions

$$\begin{aligned} |V_t - V_t'| &\leq \int_0^t [(\theta * u_s)(X_s) - (\theta * u_s')(X_s')]\, ds + \int_0^t |V_s - V_s'|\, ds \\ &\leq \int_0^t [(\theta * u_s)(X_s) - (\theta * u_s')(X_s)]\, ds + \int_0^t [(\theta * u_s')(X_s) - (\theta * u_s')(X_s')]\, ds \\ &\qquad + \int_0^t |V_s - V_s'|\, ds \\ &\qquad \lesssim \int_0^t ||u_s - u_s'||_{H^{1+2\alpha}(\mathbb{T}^2)}\, ds + \int_0^t |X_s - X_s'|\, ds + \int_0^t |V_s - V_s'|\, ds \end{aligned}$$

where we have used both the Lipschitzianity and boundedness in $L^\infty(\mathbb{T}^2)$ of $\theta * u_s$, as well as the embedding $H^{1+2\alpha}(\mathbb{T}^2) \hookrightarrow C(\mathbb{T}^2)$.
Regarding the X component we simply have

$$|X_t - X_t'| \leq \int_0^t |V_s - V_s'|\, ds.$$

The main difficulty consists in estimating $||u_t - u'_t||_{H^{1+2\alpha}(\mathbb{T}^2)}$. As done in previous sections we approach the problem through the vorticity formulation. Call ω and ω' the vorticity associated to u and u'. As in Lemma 2.5.9, by the mild formulation of $\omega - \omega'$ we have

$$||\omega_t - \omega'_t||_{H^{2\alpha}(\mathbb{T}^2)} \leq \int_0^t \left|\left|(I-\Delta)^\alpha e^{(t-s)\Delta} u_s \cdot \nabla(\omega_s - \omega'_s)\right|\right|_{L^2(\mathbb{T}^2)} ds \tag{2.26}$$

$$+ \int_0^t \left|\left|(I-\Delta)^\alpha e^{(t-s)\Delta} (u_s - u'_s) \cdot \nabla \omega'_s\right|\right|_{L^2(\mathbb{T}^2)} ds \tag{2.27}$$

$$+ \int_0^t \left|\left|(I-\Delta)^\alpha e^{(t-s)\Delta} \nabla^\perp \cdot \Lambda_{u,X,V}(s)\right|\right|_{L^2(\mathbb{T}^2)} ds \tag{2.28}$$

where

$$\Lambda_{u,X,V}(s) := \Big[((\theta * u)(X_s) - V_s)\, \theta(x - X_s) - ((\theta * u')(X'_s) - V'_s)\, \theta(x - X'_s) \Big].$$

We now deal with each of the terms above separately. We strictly follow the same computation of Lemma 2.5.9, starting from (2.26):

$$(2.26) \lesssim \int_0^t \frac{||u_s||_{C(\mathbb{T}^2)}\, ||\omega_s - \omega'_s||_{L^2(\mathbb{T}^2)}}{|t-s|^{\alpha+1/2}} ds \lesssim$$

$$\lesssim ||u||_\infty \int_0^t \frac{||u_s - u'_s||_{H^{1+2\alpha}(\mathbb{T}^2)}}{|t-s|^{\alpha+1/2}} ds,$$

$$(2.27) \lesssim \int_0^t \frac{||u_s - u'_s||_{C(\mathbb{T}^2)}\, ||\omega'_s||_{L^2(\mathbb{T}^2)}}{|t-s|^{\alpha+1/2}} ds \lesssim$$

$$\lesssim ||\omega'||_{L^\infty([0,T];L^2(\mathbb{T}^2))} \int_0^t \frac{||u_s - u'_s||_{H^{1+2\alpha}(\mathbb{T}^2)}}{|t-s|^{\alpha+1/2}} ds.$$

In the same way we have

$$(2.28) \lesssim \int_0^t \frac{||\Lambda_{u,X,V}(s)||_{L^2(\mathbb{T}^2)}}{|t-s|^{\alpha+1/2}} ds$$

We proceed now by adding and subtracting the right quantities from $\Lambda_{u,X,V}(s)$ obtaining

$$\Big|\Big[(\theta * u_s)(X_s) - V_s\Big]\theta(x - X_s) - \Big[(\theta * u'_s)(X'_s) - V'_s\Big]\theta(x - X'_s)\Big| \leq$$

$$\begin{aligned}
&\theta(x-X_s)\,|(\theta*u_s)(X_s)-(\theta*u'_s)(X_s)|\\
&+\theta(x-X_s)\,|(\theta*u'_s)(X_s)-(\theta*u'_s)(X'_s)|\\
&+(\theta*u_s)'(X'_s)\,|\theta(x-X_s)-\theta(x-X'_s)|\\
&+\theta(x-X_s)\,|V_s-V'_s|\\
&+|V'_s|\,|\theta(x-X_s)-\theta(x-X'_s)|
\end{aligned}$$

$$\lesssim ||u_s-u'_s||_{H^{1+2\alpha}(\mathbb{T}^2)}+|X_s-X'_s|+|V_s-V'_s|$$

by using the boundedness of u and u', the Lipschitzianity of $(\theta*u)$, the boundedness of $|V_s|$ and that of θ. Hence we obtained

$$(2.28)\lesssim\int_0^t\frac{||u_s-u'_s||_{H^{1+2\alpha}(\mathbb{T}^2)}+|X_s-X'_s|+|V_s-V'_s|}{|t-s|^{1/2+\alpha}}\,ds$$

We conclude by a standard Gronwall type inequality. □

2.6.3 Proof of Theorem 2.2.3

We finally have all the ingredients to prove Theorem 2.2.3. Since the proof is quite technical we first outline the general strategy.
From Proposition 2.5.1 we have obtained convergence in Law of the sequence $(u^{N,R},S^{N,R})$ to the unique weak solution of (VNS), call it (u,F). We aim to obtain the same result for the sequence (u^N,S^N), namely to prove Theorem 2.2.3. To do so, we will apply the convergence criterion stated in Theorem 2.6.1, to transfer the convergence from one sequence to another. However, Theorem 2.6.1 requires the sequences of random variables involved, to converge almost surely in the appropriate topology, while Proposition 2.5.1 grants us only convergence in law. Hence, to overcome this problem, we will first appeal to a slight variation of Skorohod representation Theorem, Lemma 2.6.4, applied to the sequence $(u^{N,R},S^{N,R})_{N\in\mathbb{N}}$ in order to obtain almost sure convergence from convergence in law. Let us omit some technicalities concerning Skorohod Theorem, whose details will be clarified later, and assume now that the sequence $(u^{N,R},S^{N,R})$ is converging almost surely to (u,F). We will apply Theorem 2.6.1 by taking

$$X_N=(u^{N,R},S^{N,R}),\quad Y_N=(u^N,S^N),\quad x=(u,F).$$

Still avoiding some technicalities we will chose

$$S_N^X(\omega)=\text{the set of path-by-path solutions of }(PS^R-NS^R)$$

and

$$S_N^Y(\omega) = \text{the set of path-by-path solutions of } (PS - NS).$$

With this choice we will see that conditions [1-4] stated in Theorem 2.6.1 will be satisfied. We can now outline the reasoning behind the hypotheses of Theorem 2.6.1 in the following scheme:

Condition 1. corresponds to Proposition 2.5.1, that is the convergence of $(u^{N,R}, S^{N,R})$ to the limit point (u, F);

Condition 2. resembles to the path-by-path uniqueness result, Proposition 2.6.3;

Condition 3. states that $(u^{N,R}, S^{N,R})$ is a path-by-path solution of $(PS^R - NS^R)$ and the analogue for (u^N, S^N);

Condition 4. expresses the fact that path-by-path solutions of $(PS^R - NS^R)$ which are also bounded from above, also satisfies $(PS - NS)$ if the parameter R is large enough. This is the same idea used to prove Theorem 2.5.14 when we proved that two PDE systems coincide for large R.

We now remark the importance of dealing with path-by-path uniqueness. Imagine to replace condition 2. in Theorem 2.6.1, with some condition that mimics the idea of strong uniqueness for SDE, instead of that for path-by-path uniqueness. A possible modification is the following:
Condition 2.bis: For all $N \in \mathbb{N}$ and for every Z E-valued random variable, if

$$\mathbf{P}(Z(\omega) \in S_N^Y(\omega)) = 1$$

then

$$\mathbf{P}(Z(\omega) = Y_N(\omega)) = 1.$$

Now, following the proof of Theorem 2.6.1, we can proceed into the proof up to a certain point. Specifically we can prove that the set

$$\left\{X_N(\omega) \in S_N^Y(\omega) \quad \forall N > N(\omega)\right\}$$

is of full measure with respect to $\mathbf{P}$. However, there is no way to apply condition *2.bis*, to conclude that

$$\mathbf{P}(X_N(\omega) = Y_N(\omega) \quad \forall N > N(\omega)) = 1$$

as it would be needed to end the proof.
We now recall and prove a small variation of Skorohod's Theorem, that we will need in the proof of Theorem 2.2.3.

Lemma 2.6.4. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $(X_N, Y_N)_{N\in\mathbb{N}}$ be a sequence of random variables defined on Ω, taking values on a separable metric space $E \times F$. Assume that F is also a Banach space and $Y_N \in L^1(\Omega; F)$ for each $N \in \mathbb{N}$. Let also $X : \Omega \to E$ be a random variable and assume further that*

$$X_N \overset{\mathcal{L}aw}{\to} X.$$

Then, there exist a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}})$ and random variables defined on the new probability space $(\widetilde{X}_N, \widetilde{Y}_N)_{N\in\mathbb{N}}$, $\widetilde{X}$ such that

$$(\widetilde{X}_N, \widetilde{Y}_N) \overset{\mathcal{L}aw}{\sim} (X_N, Y_N), \quad \widetilde{X} \overset{\mathcal{L}aw}{\sim} X$$

and

$$\widetilde{X}_N \to \widetilde{X} \quad \widetilde{\mathbf{P}}\text{-almost-surely.}$$

Proof. The proof relies on the classical Skorohod's Theorem, see [12].
Call $c_N := \mathbf{E}\left[||Y_N||_F\right]$, and introduce $a_N = Nc_N$. Consider now the sequence $Z_N := Y_N/a_N$ and notice that

$$Z_N \overset{\mathcal{L}aw}{\to} 0$$

since the convergence also holds in probability. Now, applying Skorohod's Theorem to the sequence (X_N, Z_N) we obtain that there exist a new probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}})$ and random variables $(\widetilde{X}_N, \widetilde{Z}_N)_{N\in\mathbb{N}}$, $\widetilde{X}$ such that

$$(\widetilde{X}_N, \widetilde{Z}_N) \overset{\mathcal{L}aw}{\sim} (X_N, Z_N), \quad \widetilde{X} \overset{\mathcal{L}aw}{\sim} X$$

and

$$\widetilde{X}_N \to \widetilde{X} \quad \widetilde{\mathbf{P}}\text{-almost-surely.}$$

Introduce $\widetilde{Y}_N := a_N \widetilde{Z}_N$ and observe that $(\widetilde{X}_N, \widetilde{Y}_N) \overset{\mathcal{L}aw}{\sim} (X_N, Y_N)$. This concludes the proof. □

Proof of Theorem 2.2.3. As explained the above the proof is divided into three steps: first we apply Lemma 2.6.4 to the sequence $(u^{N,R}, S^{N,R})$ to obtain almost sure convergence on a new probability space. Second, we will see that the new random variables obtained, on the new probability space satisfies the same equations as the original one. Lastly, we apply the general principle Theorem 2.6.1 to transfer the convergence from $(u^{N,R}, S^{N,R})$ to (u^N, S^N).
Step 1: Let us first introduce some notation. For each $N \in \mathbb{N}$ we first introduce $\mathbf{X}^{N,R} \in C([0,T]; \mathbb{T}^2)^{\mathbb{N}}$ and $\mathbf{V}^{N,R} \in C([0,T]; \mathbb{R}^2)^{\mathbb{N}}$ defined as

$$\mathbf{X}^{N,R}(i) = \begin{cases} X_\cdot^{i,N,R} & \text{if } i \le N, \\ 0 \quad \text{otherwise,} \end{cases} \qquad \mathbf{V}^{N,R}(i) = \begin{cases} V_\cdot^{i,N,R} & \text{if } i \le N, \\ 0 \quad \text{otherwise.} \end{cases}$$

where 0 stands for the function which is identically zero. Roughly speaking $\mathbf{X}^{N,R}$ (respectively $\mathbf{V}^{N,R}$) is the sequence of functions, where the first N elements are the particles trajectories $X_\cdot^{i,N,R}$, and all the others are identically zero. Now we apply Lemma 2.6.4 to the sequence

$$(u^{N,R}, S^{N,R}, (B^i)_{i\in\mathbb{N}}, \mathbf{X}^{N,R}, \mathbf{V}^{N,R})_{N\in\mathbb{N}}$$

where

$$(u^{N,R}, S^{N,R}, (B^i)_{i\in\mathbb{N}}) \xrightarrow{\mathcal{L}aw} (u, F, (B^i)_{i\in\mathbb{N}})$$

and we just need to verify that $(\mathbf{X}^{N,R}, \mathbf{V}^{N,R})$ is integrable with respect to $\mathbf{P}$ for each $N \in \mathbb{N}$. However this is true because

$$\mathbf{E}\left[\left|\left|\mathbf{X}^{N,R}\right|\right|_{L^\infty([0,T];\mathbb{T}^2)^\mathbb{N}}\right] = \mathbf{E}\left[\max_{i\le N}\sup_{t\in[0,T]}\left|\left|X_t^{i,N,R}\right|\right|\right]$$
$$\le N\mathbf{E}\left[\sup_{t\in[0,T]}\left|\left|X_t^{i,N,R}\right|\right|\right] \le C\cdot N$$

by using exchangeability and by the fact that $\mathbf{E}\left[\sup_{t\in[0,T]}\left|\left|X_t^{i,N,R}\right|\right|\right] \le C$ due to the presence of the cutoff in system of equations $PS^R - NS^R$. The same result holds for every $\mathbf{V}^{N,R}$ by using the same argument.
We can now apply Lemma 2.6.4. Hence there exists a new filtered probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \{\widetilde{\mathcal{F}}_t\}, \widetilde{\mathbf{P}})$ and new sequences of random variables

$$(\widetilde{u}^{N,R}, \widetilde{S}^{N,R}, (\widetilde{B}^{i,N})_{i\le N}, \widetilde{\mathbf{X}}^{N,R}, \widetilde{\mathbf{V}}^{N,R})_{N\in\mathbb{N}}$$

that shares the same laws of the initial sequences

$$(\widetilde{u}^{N,R}, \widetilde{S}^{N,R}, (\widetilde{B}^{i,N})_{i\le N}, \widetilde{\mathbf{X}}^{N,R}, \widetilde{\mathbf{V}}^{N,R}) \overset{\mathcal{L}aw}{\sim} (u^{N,R}, S^{N,R}, (B^i)_{i\le N}, \mathbf{X}^{N,R}, \mathbf{V}^{N,R})$$

for all $N \in \mathbb{N}$, and that satisfies

$$\left(\widetilde{u}^{N,R}, \widetilde{S}^{N,R}\right) \xrightarrow{N\to\infty} (u, F) \quad \widetilde{\mathbf{P}}\text{-a.s.}$$

Step 2: We now verify that the new random variables satisfies the same equations as the original ones, namely system of equations $(PS^R - NS^R)$. Moreover, in order to apply Theorem 2.6.1 we also need to have on the new probability space an analogue of (u^N, S^N), that still satisfies system of equations $(PS - NS)$ and of which we will prove the convergence. Namely:

1. Denoting by $\widetilde{X}^{i,N,R}$ and $\widetilde{V}^{i,N,R}$ the first N components of $(\widetilde{\mathbf{X}}^{N,R}, \widetilde{\mathbf{V}}^{N,R})$, corresponding to those that are non zero, we need to check that, for every $N \in \mathbb{N}$

$$\widetilde{S}_t^{N,R} = \frac{1}{N}\sum_{i=1}^{N} \delta_{(\widetilde{X}_t^{i,N,R}, \widetilde{V}_t^{i,N,R})}. \tag{2.29}$$

To prove this, consider the functional $\Phi^{S,N}$ defined as:

$$\Phi^{S,N}(S^{N,R}, (X^{i,N,R})_{i\leq N}, (V^{i,N,R})_{i\leq N}) \\ := \sup_{\varphi\in C_b(\mathbb{T}^2\times\mathbb{R}^2)} \sup_{t\in[0,T]} \left| \left\langle S^{N,R}, \varphi \right\rangle - \frac{1}{N}\sum_{i=1}^{N} \varphi(X_t^{i,N,R}, V_t^{i,N,R}) \right|$$

which is a measurable functional, and note that this is identically zero by definition of $S^{N,R}$. Now, by the fact that the random vectors $(S^{N,R}, (X^{i,N,R})_{i\leq N}, (V^{i,N,R})_{i\leq N})$ and $(\widetilde{S}^{N,R}, (\widetilde{X}^{i,N,R})_{i\leq N}, (\widetilde{V}^{i,N,R})_{i\leq N})$ share the same law, we have

$$\mathbb{E}^{\widetilde{\mathbf{P}}}\left[\Phi^{S,N}(\widetilde{S}^{N,R}, (\widetilde{X}^{i,N,R})_{i\leq N}, (\widetilde{V}^{i,N,R})_{i\leq N})\right] \\ = \mathbb{E}^{\mathbf{P}}\left[\Phi^{S,N}(S^{N,R}, (X^{i,N,R})_{i\leq N}, (V^{i,N,R})_{i\leq N})\right] = 0.$$

Hence, we conclude that $\Phi^{S,N}(\widetilde{S}^{N,R}, (\widetilde{X}^{i,N,R})_{i\leq N}, (\widetilde{V}^{i,N,R})_{i\leq N})$ is identically zero $\widetilde{\mathbf{P}}$-almost surely, which implies (2.29).

2. To prove that the new object satisfies the same equation as the initial one, for each $N \in \mathbb{N}$ we consider a bounded and measurable functional Φ^N taking as argument the function $u^{N,R}$, the particles $(X^{i,N,R})_{i\leq N}$, $(V^{i,N,R})_{i\leq N}$ and the Brownian motions $(B^i)_{i\leq N}$, that vanishes in expected value on solutions of system of equation $(PS^R - NS^R)$. The measurability of Φ^N follows by the path-by-path formulation while the boundedness requirement can be dealt with by considering a sequence $\Phi^{M,N} := \Phi^N \wedge M$ and passing to the limit in M inside the expected value by monotone convergence. By the equality in law of the new sequences with respect to the initial one, we have that the functional Φ^N vanishes also on the new objects, when averaged with respect to $\widetilde{\mathbf{P}}$, namely (we omit some technical details of integrability)

$$\mathbb{E}^{\widetilde{\mathbf{P}}}\left[\Phi^N\left(\widetilde{u}^{N,R}, (\widetilde{X}^{i,N,R})_{i\leq N}, (\widetilde{V}^{i,N,R})_{i\leq N}, (\widetilde{B}^{i,N})_{i\leq N}\right)\right] = \\ \mathbb{E}^{\mathbf{P}}\left[\Phi^N\left(u^{N,R}, (X^{i,N,R})_{i\leq N}, (V^{i,N,R})_{i\leq N}), (B^i)_{i\leq N}\right)\right] = 0.$$

Hence, $(\widetilde{u}^{N,R}, (\widetilde{X}^{i,N,R})_{i\leq N}, (\widetilde{V}^{i,N,R})_{i\leq N}), (\widetilde{B}^{i,N})_{i\leq N})$ satisfies system of equation $(PS^R - NS^R)$ in the new probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \{\widetilde{\mathcal{F}}_t\}, \widetilde{\mathbf{P}})$ which ends this part.

3. Consider now the sequence $(u^N, (X^{i,N})_{i\leq N}, (V^{i,N})_{i\leq N}))_{N\in\mathbb{N}}$, associated with system of equations $(PS - NS)$, that is the particle system without the cut-off. On the new probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \{\widetilde{\mathcal{F}}_t\}, \widetilde{\mathbf{P}})$ consider the same system of equations $(PS - NS)$, i.e. the system of equation where the Brownian motions $(B^i)_{i\leq N}$ are replaced by the Brownian motions $(\widetilde{B}^{i,N})_{i\leq N}$ introduced in Step 1. Call $(\widetilde{u}^N, (\widetilde{X}^{i,N})_{i\leq N}, (\widetilde{V}^{i,N})_{i\leq N}))_{N\in\mathbb{N}}$ the solution of such system, which can be seen as a random variable on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \{\widetilde{\mathcal{F}}_t\}, \widetilde{\mathbf{P}})$. Since solutions of system $(PS - NS)$ are unique in law we conclude that for all $N \in \mathbb{N}$

$$(\widetilde{u}^N, (\widetilde{X}^{i,N})_{i\leq N}, (\widetilde{V}^{i,N})_{i\leq N})) \overset{\mathcal{L}aw}{\sim} (u^N, (X^{i,N})_{i\leq N}, (V^{i,N})_{i\leq N})).$$

Also introduce the analogue of the empirical measure S^N on the new space

$$\widetilde{S}^N_t := \frac{1}{N}\sum_{i=1}^N \delta_{(\widetilde{X}^{i,N}_t, \widetilde{V}^{i,N}_t)}.$$

By the previous definition and by construction of $((X^{i,N})_{i\leq N}, (V^{i,N})_{i\leq N})$ we immediately have

$$(u^N, S^N) \overset{\mathcal{L}aw}{\sim} (\widetilde{u}^N, \widetilde{S}^N), \quad \forall N \in \mathbb{N}.$$

Step 3: We can now apply Theorem 2.6.1. We have to define all the objects needed in the Theorem and verify all the four conditions required. Let $E = C([0,T]\times\mathbb{T}^2) \times C([0,T]; \mathbf{P}_1(\mathbb{T}^2\times\mathbb{R}^2)$ and let $x \in E$ be the couple (u, F). Now we take

$$X_N := (\widetilde{u}^{N,R}, \widetilde{S}^{N,R}), \quad Y_N := (\widetilde{u}^N, \widetilde{S}^N).$$

Now define, for $\widetilde{\omega} \in \widetilde{\Omega}$

$$S^R_N(\widetilde{\omega}) = \Big\{(w, (x^i)_{i\leq N}, (v^i)_{i\leq N}) \in C([0,T]\times\mathbb{T}^2) \times C([0,T]; \mathbb{T}^2\times\mathbb{R}^2)^N \text{ s.t.}$$

$$(w, (x^i)_{i\leq N}, (v^i)_{i\leq N}) \text{ solves } (PS^R - NS^R) \text{ with additive noise } (B^i_t(\widetilde{\omega}))_{i\leq N}\Big\},$$

the set of path-by-path solutions for system of equations $(PS^R - NS^R)$. We also introduce the analogue for $(PS - NS)$: call it $S_N(\widetilde{\omega})$. Now we consider

$$S^X_N(\widetilde{\omega}) := \Big\{(w,\mu) \in E \,|\, \mu = \frac{1}{N}\sum_{i=1}^N \delta_{(x^i,v^i)}, (w, (x^i)_{i\leq N}, (v^i)_{i\leq N}) \in S^R_N(\widetilde{\omega})\Big\},$$

and

$$S_N^Y(\widetilde{\omega}) := \left\{ (w,\mu) \in E \,|\, \mu = \frac{1}{N}\sum_{i=1}^{N} \delta_{(x^i,v^i)}\,,\, (w,(x^i)_{i\leq N},(v^i)_{i\leq N}) \in S_N(\widetilde{\omega}) \right\}.$$

Roughly speaking, S_N^X (resp. S_N^Y) are the set of couples (w,μ) where u is a function and μ is a measure, such that μ is the empirical measure of a set of particles which, together with u, are path-by-path solutions of $(PS^R - NS^R)$. This is just a way of rewriting sets of path-by-path solutions, which match the structure of the space E where the converging objects belong.
Now we just need to verify rigorously all the four conditions stated in this Theorem:

1. In the first Step of this proof, we saw that

$$\left(\widetilde{u}^{N,R}, \widetilde{S}^{N,R}\right) \xrightarrow{N\to\infty} (u,F) \quad \widetilde{\mathbf{P}}\text{-a.s.}$$

   which correspond exactly to condition 1.

2. Introduce

$$\Omega_B = \left\{ \widetilde{\omega} \in \widetilde{\Omega} \,|\, (\widetilde{B}^{i,N}(\widetilde{\omega}))_{i\leq N, N\in\mathbb{N}} \text{ are continuous } \right\}$$

   and note that, since we are considering a countable set of Brownian Motions, this set is of full measure with respect to $\widetilde{\mathbf{P}}$. Then, by Proposition 2.6.3, we have that

$$\sharp S_N(\widetilde{\omega}) \leq 1 \quad \forall \widetilde{\omega} \in \Omega_B.$$

   Hence, the same result holds for the set $S_N^Y(\widetilde{\omega})$.

3. Condition 3. states that $(\widetilde{u}^{N,R}, \widetilde{S}^{N,R})$ belongs to the set S_N^X almost surely. However, in Step 2. of this proof we have verified that on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \{\widetilde{\mathcal{F}}_t\}, \widetilde{\mathbf{P}})$ $(\widetilde{u}^{N,R}, (\widetilde{X}^{i,N,R})_{i\leq N}, (\widetilde{V}^{i,N,R})_{i\leq N}), (\widetilde{B}^{i,N})_{i\leq N})$ satisfies system of equation $(PS^R - NS^R)$ in the sense of SDEs. This condition implies that for fixed $\widetilde{\omega} \in \widetilde{\Omega}$ the vector $(\widetilde{u}^{N,R}(\widetilde{\omega}), (\widetilde{X}^{i,N,R}(\widetilde{\omega}))_{i\leq N}, (\widetilde{V}^{i,N,R}(\widetilde{\omega}))_{i\leq N}) \in S_N^R(\widetilde{\omega})$. Since in Step 2. we verified that $\widetilde{S}^{N,R}$ is in fact an empirical measures on particle solutions of $(PS^R - NS^R)$ and by the definition of S_N^X, this imply the first part of condition 3. The same result holds for $(\widetilde{u}^N, \widetilde{S}^N)$ and S_N^Y by an analogous argument.

4. Condition 4. is the most delicate. Take a couple $(w,\mu) \in S_N^X(\widetilde{\omega}) \cap B_E((u,F),1)$. Since $(w,\mu) \in B_E((u,F),1)$ we have that

$$||w||_{C([0,T]\times\mathbb{T}^2)} \leq ||u||_{C([0,T]\times\mathbb{T}^2)} + 1.$$

The couple (w, μ) also lies in $S_N^X(\widetilde{\omega})$, hence there exist $((x^i), (v^i))_{i\leq N} \in C([0,T]; \mathbb{T}^2 \times \mathbb{R}^2)$ such that $(w, (x^i)_{i\leq N}, (v^i)_{i\leq N}) \in S_N^R(\widetilde{\omega})$, which means that is a path-by-bath solutions of $(PS^R - NS^R)$. However, since w is uniformly bounded by $||u||_{C([0,T]\times\mathbb{T}^2)} + 1$, which corresponds exactly to our choice of R (see at the beginning of this section), we see that the cut off function $\chi_R(w) \equiv 1$ is identically one. Hence system of equation $(PS^R - NS^R)$ reduce to $(PS - NS)$, which is the particle system without the cut-off. This implies that $(w, (x^i)_{i\leq N}, (v^i)_{i\leq N})$ solves also $(PS - NS)$, hence $(w, \mu) \in S_N^Y(\widetilde{\omega})$.

Since we verified all the necessary conditions, we can apply Theorem 2.6.1, obtaining

$$\left(\widetilde{u}^N, \widetilde{S}^N\right) \xrightarrow{N\to\infty} (u, F) \quad \widetilde{\mathbf{P}}\text{-a.s.}$$

Since almost sure convergence implies convergence in law, and since we verified in Step 2. that

$$(u^N, S^N) \overset{\mathcal{L}aw}{\sim} (\widetilde{u}^N, \widetilde{S}^N), \quad \forall N \in \mathbb{N}.$$

we can transport this type of convergence to the original probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$, hence the proof is ended. □

Chapter 3

Self interacting network: the case of the mycelium

3.1 Introduction

What is presented in this chapter is based on the work [21]. The study of complex networks has seen a growing interest in the last few years. The nature of such networks is not uniquely defined: some examples are informational networks, (of relation between individuals, citation graphs,...), technological (power grids, public transportation, computer network,...), or biological (vascular, biochemical, neural network,...). In all the above mentioned phenomena, transformations arise from individuals: being it the development of a new connection between existing entities, as it often appears in neurons, or the introduction of a new individual into the system. All these contributions sum up to the evolution of the network as a unit at the macroscopic level. The mathematics of such intricate process, which needs to be able to capture modifications at different scales, can be achieved by linking microscopic objects, which describe individuals, with their collective mean behavior.

Here we focus on the case of the spatial evolution of a complex biological network, which evolves by means of the motions of its nodes. We consider a system of second order SDEs

$$\begin{cases} dX_t^{i,N} = V_t^{i,N} dt \\ dV_t^{i,N} = -\lambda V_t^{i,N} dt + \nabla C^N(t, X_t^{i,N}) dt + \sigma dB_t^i \end{cases} \quad t \in [T^{i,N}, \Theta^{i,N}), \tag{3.1}$$

where $(X_t^{i,N}, V_t^{i,N}) \in \mathbb{R}^d \times \mathbb{R}^d$ represents the position of the nodes of the network on the phase space. The processes $(B_t^i)_{t\geq 0}$ are independent Brownian motions, which affects the dynamics of the nodes at the microscopic level. Here C^N represents a

sort of "resource" expandable for the development, taking into account the cost the network has to pay to expand towards a specific direction. The motion of the particles is driven by the term ∇C^N which couples the equations and describes the fact that particles tend to move where more resources are available, and by the term $-\lambda V_t^{i,N}$ which represents the friction. Each of the equations in (3.1), describing position and velocity of a node, is valid only for a limited life span $[T^{i,N}, \Theta^{i,N})$, denoting the time interval where the node can contribute to the evolution of the network. Outside of this interval the particle is not moving.

We thus consider the network skeleton made of all the trajectories of the spatial components of the nodes, i.e.

$$\mathcal{N}_t := \bigcup_{i=1}^{N_t^N} \{X_s^{i,N} \mid s \leq t,\ s \in [T^{i,N}, \Theta^{i,N})\} \subseteq \mathbb{R}^d.$$

We allow the number of living nodes in the network to increase after a bifurcation event, or to decrease when the node ceases to exist. Creation and destruction of particles are provided according to Poisson point process: modifications in the number of particles at time t are affected by the configuration of the system at all times $s \leq t$. Since the number of individuals is changing in time, we introduce the total number of particles that are alive or appeared up to time t: call it N_t^N. We introduce the empirical measure of the particle system (3.1)

$$S_t^N = \frac{1}{N} \sum_{i=1}^{N_t^N} \mathbb{1}_{[T^{i,N}, \Theta^{i,N})}(t) \delta_{(X_t^{i,N}, V_t^{i,N})}$$

which is a (random) finite positive measure. We allow a branching event to appear at time t, in any point on the trajectory of the particles $X_s^{i,N}$ for $s \leq t$, with a uniform spatial distribution on the trajectory. Introduce $\delta_{\mathbb{X}_t^N}$ the uniform measure on the trajectory of the particles up to time t:

$$\delta_{\mathbb{X}_t^N} = \frac{1}{N} \int_0^t \sum_{i=1}^{N_s^N} \mathbb{1}_{[T^{i,N}, \Theta^{i,N})}(s) \left| V_s^{i,N} \right| \delta_{X_s^{i,N}}(dx) ds. \tag{3.2}$$

Note the presence of the term $|V_t^{i,N}|$ in the measure $\delta_{\mathbb{X}_t^N}$: the scaling by the velocity of each particle is of crucial importance in order to obtain the uniform measure on the trajectory, see also appendix 3.A. The possibility to handle the uniform measure on the particles path, as well as the coalescence events, was opened by the choice of Langevin dynamics thanks to the more regularity of the spatial trajectory. In the same spirit as branching, when a moving node hits the network structure (i.e. they superimpose) coalescence occurs: the node stops moving and a loop is created in the network.

The function C^N is itself influenced by particles, leading to a coupled (random) PDE-SDEs system

$$\partial_t C^N = \frac{\sigma_C^2}{2}\Delta C^N - (K_C * \delta_{\mathbb{X}_t^N})C^N. \tag{3.3}$$

The rationale behind the previous equation is the following: to contribute to the development of the network, the particles need to spend (consume) some resources. This absorption mechanism affects not only the sites where the network is expanding, i.e. corresponding to the particles position, but is present along the entire trajectory. Having in mind a biological framework we can imagine that the network absorbs nutrients in order to sustain itself, along its entire length. Moreover, if we want to consider the network as a solid structure, we cannot use directly the uniform measure $\delta_{\mathbb{X}_t^N}$, since the trajectories of the particles are one-dimensional objects and hence are negligible for the d-dimensional Lebesgue measure. For this reason, we introduce the convolution kernel K_C into equation (3.3). The drift $\nabla C^N(t, X_t^{i,N})$ in the particles equations encodes one very important features of our model: self avoidance, namely particles tend to avoid visiting sites which are close to their past trajectories. The values of C^N decrease in correspondence of the network structure $\mathcal{N}_t$, due to the term $-(K_C * \delta_{\mathbb{X}_t^N})C^N$ in (3.3). Hence $\nabla C^N(t, x)$ is pointing towards the region of the space which are free, i.e. those which are far from $\mathcal{N}_t$.
We are interested in the mean behavior of system of equation (3.1)-(3.3) when N is large, and on the propagation of chaos property:

Theorem (Theorem 3.3.2)**.** *The couple (S^N, C^N) converges as N goes to infinity to the unique measure solution of the following system of PDEs*

$$\partial_t u + v\cdot\nabla_x u - \lambda \mathrm{div}_v(vu) = \frac{\sigma^2}{2}\Delta_v u - \nabla C\cdot\nabla_v u + G(v)\overline{u} + G(v)(K*\rho) - (K*\rho)u, \tag{3.4}$$

$$\partial_t \rho(t,x) = \int_{\mathbb{R}^d} |v| u(t,x,v)dv, \quad \overline{u}(t,x) = \int_{\mathbb{R}^d} u(t,x,v)dv,$$

$$\partial_t C = \frac{\sigma_C^2}{2}\Delta C - (K_C * \rho)C. \tag{3.5}$$

The strong coupling between all the elements, especially the interaction with the past configuration, was the main issue when dealing with a priori estimates, see section 3.1.1.

The literature of interacting diffusion is very extensive, ranging from the earlier results [81], [84], [86], [89]. Many works, mostly applied to population dynamics or more generally to biology, are devoted to the interplay between different species, and to the discontinuity arising from creation or destruction of individuals. The spatial component of such discontinuity plays a major role in the progression of the system, and has been widely studied, [25], [26], [29], [83]. Because of its different

features, for the analysis of a network organization it is essential to combine all the effects of the existing connections with the evolution of the individuals: in some cases the connections can also assume a physical meaning, affecting the structural transformation. In [6], [7], [42] the case of self-interaction is analyzed, considering the interaction of a stochastic process $(X_t)_{t\geq 0}$ with his trajectory for $s \leq t$. More precisely in [42] self avoidance, which is also a key feature of our work, is treated.

We are also interested in a regularity result, showing that measure solutions of (3.4) are actually regular function solutions.

Theorem (Theorem 3.3.3)**.** *If u is a measure solution of equation* (3.4) $u \in C([0,T];\mathbf{P}_r(\mathbb{R}^d \times \mathbb{R}^d))$ *then, for all $t_0 > 0$ u lies in $C([t_0,T];C_b^\infty(\mathbb{R}^d \times \mathbb{R}^d))$.*

Following [39] we proved this result under some minor additional assumption with respect to our convergence result, see Section 3.2 for the detailed hypothesis.

3.1.1 Difficulties

In this subsection we aim to highlight the difficulties we met in proving the convergence from the discrete to the continuous model and the regularity of solutions. The first problem we had to solve is the convergence of the empirical measure S_t^N. In fact, since the total number of living particles changes over time, S^N is not a probability measure but only a finite positive measure. A proper tightness criterion in the space of finite positive measure is thus required, [68]. In order to prove tightness an a priori bound on the total mass, i.e. on the ratio

$$\mathbf{E}\left[\frac{N_t^N}{N}\right]$$

is needed. This is not a simple task. Since proliferation can occur with an uniform distribution at any point along the network $\mathcal{N}_t$, the rate of proliferation depends on the total length. However the rate of growth of the network, which corresponds to how much the rate of proliferation increases, is influenced by the other elements of the system. The velocity of each living particles $V_t^{i,N}$, which affect the expansion, is driven by the term ∇C^N. Moreover particles velocity is also affected by the noise, hence it has arbitrarily large fluctuations and can be controlled only in the average. This intricacy leads to a very difficult coupled problem.
Note that the high complexity of the system is all due to the scaling term $|V_s^{i,N}|$ in the uniform measure over $\mathcal{N}_t$ (3.2). In order to make this difficulty more clear consider the following

$$\widetilde{\delta_{\mathbb{X}_t^N}} = \frac{1}{N}\int_0^t \sum_{i=1}^{N_s^N} \mathbb{1}_{[T^{i,N},\Theta^{i,N})}(s)\delta_{X_s^{i,N}}(dx)ds,$$

that is the same as $\delta_{\mathbb{X}_t^N}$ where the velocity term is neglected (hence is not uniform on $\mathcal{N}_t$). Integrating the function 1 and computing the expected value leads to

$$\mathbf{E}\left[\left\langle \widetilde{\delta_{\mathbb{X}_t^N}}, 1\right\rangle\right] \leq \int_0^t \mathbf{E}\left[\frac{N_s^N}{N}\right] ds.$$

It is now clear that in this case it can be possible to obtain a close equation for the average total mass, independently on C^N or the particle velocity. By considering $\delta_{\mathbb{X}_t^N}$ this last inequality is not immediate, since it involves the term $|V_s^{i,N}|$ that has to be controlled separately.
We managed to solve this issue by closing a first a priori estimate independently on the others (Lemma 3.4.3)

$$\mathbf{E}\left[\sup_{t\in[0,T]} \left|\left|\nabla C^N(t,\cdot)\right|\right|_\infty\right] \leq C. \tag{3.6}$$

Thanks to the previous estimate it is possible to obtain a control on the particle velocity, that leads to the required bound on the total mass (Lemma 3.4.6).

Let us now focus on the tightness of C^N. The coupling with the particles system in the equation for C^N has the form

$$(K_C * \delta_{\mathbb{X}_t^N}) C^N. \tag{3.7}$$

The analogous term in the equation satisfied by the empirical measure S^N in its weak formulation, see equation (3.12), takes the form

$$\langle S_s^N, \nabla_v f \cdot \nabla C^N \rangle,$$

where f is a test function. Since S^N is converging only weakly as a probability measure we see that is required uniform convergence of ∇C^N to ∇C. The bound in (3.6) is not enough to prove the convergence of the first derivatives of C^N, hence we had to refine this result. Thanks to the control on the total mass previously discussed we prove (Corollary 3.4.8)

$$\mathbf{E}\left[\sup_{t\in[0,T]} \left|\left|D^2 C^N(t,\cdot)\right|\right|_\infty\right] \leq C.$$

Moreover, from equation (3.7) we also understand that it is required to prove the weak convergence in the sense of finite measure of $\delta_{\mathbb{X}^N}$. The tightness of the sequence $\{\delta_{\mathbb{X}^N}\}_{N\in\mathbb{N}}$ is proved in the same manner of that of $\{S^N\}_{N\in\mathbb{N}}$ (Theorem 3.4.9).

Uniqueness is another difficult topic. Since we aim to prove the propagation of chaos property at the level of bounded measures, it is required to prove uniqueness at this level of regularity. We will first derive a formulation for system (3.4)-(3.5) in Fourier space, and we will understand the solution in Fourier space in its Mild formulation. Using the technique developed in [39] we will prove some hypoelliptic estimates for the Fourier multiplier involved in the Fourier formulation. These estimates will also be used when dealing with the regularity of solutions. Moreover, we will us the fact that if u has $(1+\beta)$ moments along the velocity component, for some $\beta > 0$ and uniformly in time

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v|^{1+\beta}\ u(t, dx, dv) \leq C,$$

then we can have a control on the Fourier transform of u in the space of $(1+\beta)$-Holder continuous functions. By this last remark we will produce a Gronwall type estimate in Fourier space, proving uniqueness.

In the following, we will specialize our work to the modeling of the *Podospora anserina*, a filamentous fungus which has been widely used as a model organism of research. We focused on the development of the *Hyphae* of the fungus, i.e. the microscopic branching filaments which collectively form the mycelium. Many works focused on the growth of single Tip of an Hypha, [56], while others focused on the collective evolution of the mycelium using PDE approach, [13],[14],[15]. In our work we directed our attention first on a proper description of the network on the microscopic scale, taking into account the formation of new individuals (creation of new Tips for the filament), the possibility of coalescence of existing branches, and linking the above mentioned phenomena with the collective behavior at the macroscopic scale.

This chapter is structured as follows: Section 3.2 is devoted to the rigorous presentation of the model and to all the hypothesis we will need in order to prove our main results. In Section 3.3 we introduce the mathematical tools that we will need in the rest of the chapter and identify heuristically the limiting equation of the microscopic model. We also give a precise statements of our main results Theorem 3.3.2 and Theorem 3.3.3. In Section 3.4 we isolate the a priori estimates needed for the tightness of the particle system, that we will need in order to prove our propagation of chaos results. Section 3.5 is devoted to the Fourier formulation for the limiting equation and to a semi explicit formulation for the solution in Fourier coordinates. In Section 3.6 we produce some uniform in time Hypoelliptic estimates on the solution, while in Section 3.7 we prove uniqueness of measure solutions. Finally in section 3.8 we prove our regularity result, applying the results developed in the previous sections.

3.2 Rigorous description of the model

Let us now introduce the model in detail: as stated in the introduction the case we have in mind is the growth of hyphae. All our discussion is made in an arbitrary dimension d, even if we are more interested in $d = 2, 3$.

The fundamental components of our model are the following:

- *Tips*: we describe the growth of a branch by the motion of its *Tip*, i.e. the final portion of the branch, taken in the direction of growth. The motion is described by a second order SDE, driven by a field of nutrients: position and velocity of the i-th particle at time t is denoted by $(X_t^{i,N}, V_t^{i,N})$;

- *Branching and coalescing distributions*: At time $t = 0$ we assume to have $N_0 = N$ particles: however the total number of tips can change, at random, due to *branching* or *coalescing* events. Each tip is considered "active" for $t \in [T^{i,N}, \Theta^{i,N})$ denoting its time of birth and death. Branching can appear either in the position of a tip (tip-branching), either uniformly on the set of trajectories of the particles (network-branching). These events happen accordingly to Poisson point process, specified below;

- *Concentration of Nutrients*: Tips move according to a field of nutrients, shifting in the direction where the concentration is higher. The concentration is absorbed, in the regions of space that are occupied by the path of the Tips (i.e. where the filament is present). We indicate the field of nutrients by the function $C^N(t, x)$ that is expressed by a PDE, coupled with the particles SDEs.

Let us now detail all of the elements introduced above: We denote by N_t^N the total number of particle who are alive or lived up to time t. Note that with notation we have $N_0^N = N$ and that N_t^N is an increasing process. Each of the particles satisfies a second order SDE, valid for a limited time between its birth time (which is zero if the particle is one of the N ancestors) and to its death:

$$\begin{cases} dX_t^{i,N} = V_t^{i,N} dt \\ dV_t^{i,N} = -\lambda V_t^{i,N} dt + \nabla C^N(t, X_t^{i,N}) dt + \sigma dB_t^i \end{cases} \qquad t \in [T^{i,N}, \Theta^{i,N}),$$

with $(X_0^{i,N}, V_0^{i,N})$ according to a *pdf* u_0 on $\mathbb{R}^d \times \mathbb{R}^d$, and where $(B_t^i)_{t \geq 0}$ are independent Brownian motions on $\mathbb{R}^d$. In the previous $\lambda > 0$ represents the friction coefficient, σ is the diffusion and C^N is the concentration of nutrients introduced above. Here the sign in front of the gradient of the nutrient field denotes the tendency of the particle of *leaving and avoiding* the places where the potential field is low. This will allow us to obtain the self-avoiding behavior, that is present in spatial exploration phenomena.

In order to specify the Point process for the branching and coalescence we have to introduce the following empirical measures:

$$S_t^N(dx,dv) = \frac{1}{N}\sum_{k=1}^{N_t^N} \mathbb{1}_{[T^{i,N},\Theta^{i,N})}(t)\delta_{X_t^{i,N},V_t^{i,N}}(dx,dv),$$

$$\bar{S}_t^N(dx) = \frac{1}{N}\sum_{k=1}^{N_t^N} \mathbb{1}_{[T^{i,N},\Theta^{i,N})}(t)\delta_{X_t^{i,N}}(dx),$$

$$\delta_{\mathbb{X}_t^N}(dx) = \int_0^t \frac{1}{N}\sum_{k=1}^{N_s^N} \mathbb{1}_{[T^{i,N},\Theta^{i,N})}(s)|V_t^{i,N}|\delta_{X_s^{i,N}}(dx).$$

The measure $\delta_{\mathbb{X}_t^N}(dx)$ is the uniform measure on the network renormalized by the number of initial particles (see curvilinear abscissa (3.48) in Appendix 3.A). A new tip is created according to a Compound Poisson Point process $\Phi(dx \times dv \times dt)$ defined by its compensator

$$G(v)dv\bar{S}_t^N(dx)dt + G(v)dv(K * \delta_{\mathbb{X}_t^N})(x)dxdt$$

where $G(v)$ is a *pdf* on $\mathbb{R}^d$ and K is a mollifier with compact support. The first part of the previous correspond to tip-branching while the second to network-branching. The reasoning is the following: a new tip can appear either on the spatial position of an existing particle, with a starting velocity specified by the density G, or it can appear, with an uniform distribution, on the network of the trajectories. However, trajectories are 1-dimensional object: therefore the convolution of $\delta_{\mathbb{X}_t^N}(dx)$ with the mollifier K has the purpose to take into account the (nonzero) thickness of the hypha.

Concerning the coalescence: we introduce the Compound Poisson Point process $\Psi(dx \times dv \times dt)$, specified by the compensator

$$(K * \delta_{\mathbb{X}_t^N})(x)S_t^N(dx,dv)dt.$$

As for the network-branching the rationale behind this is the following: coalescence between a tip and an existing branch can only happen when the two superimpose. The convolution with the kernel K is again to take into account for the thickness of the branch.

Finally we discuss the equation for the potential field C^N:

$$\partial_t C^N = \frac{\sigma_C^2}{2}\Delta C^N - K_C * \delta_{\mathbb{X}_t^N}(x)C^N$$

with $C^N(0,x) = C_0(x)$ and K_C is mollifier with the property $|\nabla K_C(x)| \leq K_C(x)$. The network of hyphae, consumes the nutrient field in order to stay alive and

receive the proper sustain, hence the term $-K_C * \delta_{\mathbb{X}_t^N}(x)C^N$. Since the hyphae are not curves (as in the modeling) but have a dimension, the kernel K_C reintroduce it.

Remark 3.2.1. Here the choice of K_C is really the crux of the matter: the specific hypothesis on K_C will allow some a priori estimates to follows easily without interlacing to each other. For a discussion without this hypothesis see [19].

We summarize all the hypothesis on the model: we split our hypothesis into three blocks, separating the hypothesis needed for the tightness, uniqueness, and the additional hypothesis needed for the regularity theorem 3.3.3.

Hypothesis 3.2.2. (Tightness and passage to the limit, Corollary 3.4.13, Theorem 3.4.14):

1. $K_C \in C_b^2$ with the properties that $|\nabla K_C(x)| \leq K_C(x)$. Note that this prevents to take K_C with compact support;
2. $K \in C_b^2$ with compact support;
3. u_0 probability density function on $\mathbb{R}^d \times \mathbb{R}^d$;
4. G *pdf* on $\mathbb{R}^d$ s.t. $G(v)dv \sim u_0(\mathbb{R}^d, dv)$;
5. G and $u_0(\mathbb{R}^d, dv)$ have finite $1 + \overline{\beta}$ moments for some $\overline{\beta} > 0$;
6. $C_0 \in C_b^2$.

Hypothesis 3.2.3. (Uniqueness, Theorem 3.7.1)

- The function $C_0 \in H^{m+2}$ for some $m > \frac{d}{2}$;
- The function $u_0 \in H^m$ for some $m > \frac{d}{2}$;
- The Kernel $K_C \in C_b^{m+2} \cap H^{2m+2}$ for some $m > \frac{d}{2}$.

Hypothesis 3.2.4. (Smoothness, Theorem 3.3.3): Denoting by $\hat{f}$ the Fourier transform of f

- For all $N \geq 0$, $\sup_{\xi \in \mathbb{R}^d} |\hat{G}(\xi)\big|\big(1 + |\xi|^2\big)^N < \infty$.
- For all $N \geq 0$, $\sup_{\xi \in \mathbb{R}^d} |\hat{K}(k)\big|\big(1 + |\xi|^2\big)^N < \infty$.
- For all $m \geq 0$, $C_0, K_C \in H^m$ and $K_C \in C_b^m$.

3.3 Limiting Fluid equations

3.3.1 Notation and function spaces

From now on we will denote by $\mathcal{M}_f^+(\mathbb{R}^d \times \mathbb{R}^d)$, the space of all the finite positive measures over $\mathbb{R}^d \times \mathbb{R}^d$. Given $(\varphi_k)_{k\in\mathbb{N}}$ a countable dense subset of $C_b(\mathbb{R}^d \times \mathbb{R}^d)$ we define

$$\delta(\mu,\nu) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|\langle \mu, \varphi_k\rangle - \langle \nu, \varphi_k\rangle|}{1 + |\langle \mu, \varphi_k\rangle - \langle \nu, \varphi_k\rangle|}$$

which makes $\mathcal{M}_f^+(\mathbb{R}^d \times \mathbb{R}^d)$ into a complete metric space, whose topology correspond to weak convergence of measure. We will then denote by $\mathcal{D}([0,T]; \mathcal{M}_f^+(\mathbb{R}^d \times \mathbb{R}^d))$ the space of all the càdlàg functions from $[0,T]$ to $\mathcal{M}_f^+(\mathbb{R}^d \times \mathbb{R}^d)$, endowed with the Skorohod topology.

We also denote by $C^1(\mathbb{R}^d)$ the space of all $C^1(\mathbb{R}^d)$ functions, endowed with the topology of uniform convergence over compact sets. We recall that this topology is generated by the following metric

$$d(f,g) = \sum_{N=1}^{\infty} 2^{-N} \left|\left|f - g\right|\right|_{C^1\left(B(0,N)\right)} \wedge 1.$$

We will also introduce the spaces

$$\widetilde{X} = \mathcal{D}\Big([0,T]; \mathcal{M}_f^+(\mathbb{R}^d \times \mathbb{R}^d)\Big) \times \mathcal{D}\Big([0,T]; \mathcal{M}_f^+(\mathbb{R}^d)\Big) \times C\Big([0,T]; C^1(\mathbb{R}^d)\Big)$$

endowed with the product metric introduced above, and

$$X = C\Big([0,T]; \mathcal{M}_f^+(\mathbb{R}^d \times \mathbb{R}^d)\Big) \times C\Big([0,T]; \mathcal{M}_f^+(\mathbb{R}^d)\Big) \times C\Big([0,T]; C_b^1(\mathbb{R}^d)\Big)$$

again with the product metric.

3.3.2 Itô formula and limiting equation

Let us remind that we are interesting by the convergence of the following empirical measures :

$$S_t^N(dx,dv) = \frac{1}{N} \sum_{k=1}^{N_t^N} \mathbb{1}_{[T^{i,N},\Theta^{i,N})}(t) \delta_{X_t^{i,N}, V_t^{i,N}}(dx,dv), \tag{3.8}$$

$$\bar{S}_t^N(dx) = \frac{1}{N} \sum_{k=1}^{N_t^N} \mathbb{1}_{[T^{i,N},\Theta^{i,N})}(t) \delta_{X_t^{i,N}}(dx), \tag{3.9}$$

$$\delta_{\mathbb{X}_t^N}(dx) = \int_0^t \frac{1}{N} \sum_{k=1}^{N_s^N} \mathbb{1}_{[T^{i,N},\Theta^{i,N})}(s) |V_t^{i,N}| \delta_{X_s^{i,N}}(dx), \tag{3.10}$$

for $\beta \leq \overline{\beta}$

$$\delta^{\beta}_{\mathbb{X}_t^N}(dx) = \int_0^t \frac{1}{N} \sum_{k=1}^{N_s^N} \mathbb{1}_{[T^{i,N},\Theta^{i,N})}(s) |V_t^{i,N}|^{1+\beta} \delta_{X_s^{i,N}}(dx) \tag{3.11}$$

and by the convergence of the function C^N.

For every C^2 function $f : \mathbb{R}^{2d} \to \mathbb{R}$, we finally have, by Itô formula, used for the $X^{i,N}, V^{i,N}$ and using the fact that branching and merging are encoded thanks to Poisson Point Processes,

$$\begin{aligned}
\langle S_t^N, f\rangle - \langle u_0, f\rangle = & \underbrace{\int_0^t \langle S_s^N, v \cdot \nabla_x f\rangle ds}_{\text{kinetic equation}} - \underbrace{\int_0^t \langle S_s^N, \lambda v \cdot \nabla_v f\rangle ds}_{\text{friction}} \\
& + \underbrace{\int_0^t \langle S_s^N, \nabla_v f \cdot \nabla C^N\rangle ds}_{\text{potential}} + \underbrace{\frac{\sigma^2}{2} \int_0^t \langle S_s^N, \Delta_v f\rangle ds}_{\text{noise on the velocity}} \\
& + \underbrace{\int_0^t \int_{\mathbb{R}^{2d}} f(x,v) G(v) dv \overline{S}_s^N(dx) ds}_{\text{creation at the Apex}} \\
& + \underbrace{\int_0^t \int_{\mathbb{R}^{2d}} f(x,v) G(v) (K * \delta_{\mathbb{X}_s^N})(x) dx dv ds}_{\text{creation on the network}} \\
& - \underbrace{\int_0^t \int_{\mathbb{R}^{2d}} f(x,v) (K * \delta_{\mathbb{X}_s^N})(x) S_s^N(dx,dv) ds}_{\text{anastomosis}} \\
& + \underbrace{M_t^{1,N,f}}_{\substack{\text{martingale remainder} \\ \text{of motion}}} + \underbrace{M_t^{2,N,f}}_{\substack{\text{martingale remainder} \\ \text{of branching}}} + \underbrace{M_t^{3,N,f}}_{\substack{\text{martingale remainder} \\ \text{of anastomosis}}}
\end{aligned} \tag{3.12}$$

Where the explicit martingale terms are the following :

$$M_t^{1,N,f} = \int_0^t \frac{\sigma}{N} \sum_{i=1}^{N^N(s)} \mathbb{1}_{[T^{i,N},\Theta^{i,N})}(s) \nabla_v f(X_s^{i,N}, V_s^{i,N}) \cdot dB_s^i$$

$$M_t^{2,N,f} = \int_0^t \int_{\mathbb{R}^{2d}} f(x,v) \Big[\Phi(dx \times dv \times ds) - G(v) \overline{S}_s^N(dx) dv ds - G(v) (K * \delta_{\mathbb{X}_s^N})(x) dx dv ds \Big]$$

$$M_t^{3,N,f} = -\int_0^t \int_{\mathbb{R}^{2d}} f(x,y)\Big[\Psi(dx\times dv\times ds) - (K*\delta_{\mathbb{X}_s^N})(x)S_s^N(dx,dv)ds\Big]$$

and

$$\partial_t C^N = \frac{\sigma_C^2}{2}\Delta C^N - K_C * \delta_{\mathbb{X}^N} C^N$$

This gives us the wanted limiting equation :

$$\begin{cases} \partial_t u + v\cdot\nabla_x u - \lambda \mathrm{div}_v(vu) = \dfrac{\sigma^2}{2}\Delta_v u - \nabla C\cdot\nabla_v u \\ \qquad\qquad + G(v)\overline{u} + G(v)(K*\rho) - (K*\rho)u \\ \partial_t \rho(t,x) = \displaystyle\int_{\mathbb{R}^d} |v| u(t,x,v)dv \\ \overline{u}(t,x) = \displaystyle\int_{\mathbb{R}^d} u(t,x,v)dv \\ \partial_t C = \dfrac{\sigma_C^2}{2}\Delta C - (K_C*\rho)C \\ u(0,\cdot,\cdot) = u_0, \quad C(0,\cdot) = C_0, \quad \rho(0,\cdot) = 0. \end{cases} \tag{3.13}$$

3.3.3 Definitions of measure solutions and main Theorem

Definition 3.3.1. A measure solution of system of equation (3.13) is a triple (u,ρ,C), where $u \in C([0,T];\mathcal{M}_f^+(\mathbb{R}^d\times\mathbb{R}^d))$, $\rho \in C([0,T];\mathcal{M}_f^+(\mathbb{R}^d))$ and $C \in C([0,T];C_b^1(\mathbb{R}^d))$ are such that, for every test function $\varphi \in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ one has

$$\begin{aligned} &\int_{\mathbb{R}^d\times\mathbb{R}^d}\varphi(x,v)u(t,dx,dv) - \int_{\mathbb{R}^d\times\mathbb{R}^d}\varphi(x,v)u(0,dx,dv) \\ &-\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d} v\cdot\nabla_x\varphi(x,v)u(s,dx,dv)ds + \lambda\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d}\nabla_v\varphi(x,v)\cdot vu(s,dx,dv)ds \\ &= \frac{\sigma^2}{2}\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d}\Delta_v\varphi(x,v)u(s,dx,dv)ds + \int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d}\nabla_v\varphi(x,v)\cdot\nabla C(s,x)u(s,dx,dv)ds \\ &+\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d}\varphi(x,v)G(v)\overline{u}_s(dx)dvds + \int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d}\varphi(x,v)G(v)(K*\rho(s,\cdot))(x)dxdvds \\ &\qquad - \int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d}\varphi(x,v)(K*\rho(s,\cdot))(x)u(s,dx,dv)ds, \end{aligned}$$

$$\overline{u}(t,dx) = u(t,dx,\mathbb{R}^d),$$

the measure ρ satisfies, for every test function $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d}\varphi(x)\rho(t,dx) = \int_{\mathbb{R}^d}\varphi(x)\rho(0,dx) + \int_0^t\int_{\mathbb{R}^d}\varphi(x)\,|v|\,u(s,dx,dv)ds,$$

and the function C satisfies

$$\int_{\mathbb{R}^d} C(t,x)\varphi(x)dx = \int_{\mathbb{R}^d} C(0,x)\varphi(x)dx + \int_0^t \int_{\mathbb{R}^d} \frac{\sigma_c^2}{2} C(s,x)\Delta\varphi(x)dxds - \int_0^t \int_{\mathbb{R}^d} C(s,x)(K_C * \rho(t,\cdot))(x)\varphi(x)dxds. \tag{3.14}$$

We end up this section by properly stating our main results:

Theorem 3.3.2 (Propagation of chaos)**.** *If the hypothesis 3.2.2 are satisfied, the family of laws $\{Q^N\}_{N\in\mathbb{N}}$ of the triple $(S^N, \delta_{\mathbb{X}^N}, C^N)_{N\in\mathbb{N}}$ is tight on the space $\widetilde{X}$. Moreover $\{Q^N\}_{N\in\mathbb{N}}$ converges weakly in $\widetilde{X}$ to $\delta_{(u,\rho,C)}$, where the triple $(u,\rho,C) \in X$ is the unique measure solution of system of equation* (3.13)*.*

We will prove the tightness of the sequences in Section 3.4. The proof concludes in Section 3.7 by proving uniqueness for the limit system.

Theorem 3.3.3 (Smoothness of solutions)**.** *Under Hypothesis 3.2.2, 3.2.3 and 3.2.4, for all $t_0 > 0$, u lies in $L^\infty([t_0,T]; C^\infty(\mathbb{R}^d \times \mathbb{R}^d))$.*

The proof of this result can be found in Section 3.8.

Remark 3.3.4. The proof of Theorem 3.3.3 gives us a bit better, actually denoting by $\hat{u}$ the Fourier transform of u, $\hat{u}(t,\cdot,\cdot)$ is decaying faster than any polynomials, impliying for example the $\hat{u}(t,\cdot,\cdot) \in H^m$ for all $m \in \mathbb{N}$.

3.4 A priori estimates on microscopic model and tightness

We start by some a priori estimates on the microscopic model, needed for the tightness, and thus the propagation of chaos.

3.4.1 A priori estimates

Lemma 3.4.1. *Suppose that $w(t,x)$ is of class $C^{1,2}([0,T]\times\mathbb{R}^d)$ and satisfies the following PDE:*

$$\begin{cases} \partial_t w(t,x) = \dfrac{a^2}{2}\Delta w(t,x) - (K * f(t,\cdot))(x)w(t,x) \quad (t,x) \in [0,T]\times\mathbb{R}^d \\ w(0,x) = w_0(x) \end{cases} \tag{3.15}$$

where $f \in L^\infty([0,T];\mathcal{M}_f^+(\mathbb{R}^d))$, $K \in C_b^2(\mathbb{R}^d)$ *is non negative and satisfies* $|\nabla K| \lesssim K$. *Assume also that* $w_0 \in C_b^2(\mathbb{R}^d)$. *Then*

$$\sup_{t\in[0,T]} ||w(t,\cdot)||_{C_b^1} \leq C\,||w_0||_{C_b^1} \tag{3.16}$$

and

$$\sup_{t\in[0,T]} ||w(t,\cdot)||_{C_b^2} \lesssim_T (1+||w_0||_{C_b^2})\,||K||_{C_b^2} \sup_{t\in[0,T]} f(t,\mathbb{R}^d) \tag{3.17}$$

Proof. By Feynman-Kac representation we have

$$w(t,x) = \mathbf{E}\left[\exp\left(-\int_0^t (K*f(s,\cdot))(x+aW_t-aW_s)ds\right) w_0(x+aW_t)\right], \tag{3.18}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion. Notice that, for every $g \in C_b^2(\mathbb{R}^d)$ with $|\nabla g| \lesssim g$ one has

$$\nabla e^{-g} = -\nabla g e^{-g}$$

so that, using the hypothesis on the first derivative we have

$$\left|\nabla e^{-g}\right| \leq 1.$$

Moreover for each i,j

$$\left|\partial_i\partial_j e^{-g}\right| \leq |\partial_i g| + |\partial_i\partial_j g| \lesssim ||g||_{C_b^2}.$$

Applying the previous computation in equation (3.18) we get to (3.16). Finally

$$\left|\left|D^2 w(t,\cdot)\right|\right|_\infty \leq ||w_0||_{C_b^2}\,||K||_{C_b^2} \sup_{t\in[0,T]} f(t,\mathbb{R}^d)$$

which leads to (3.17).

Remark 3.4.2. Note that, thanks to the assumptions, the bound (3.16) of the previous lemma is independent of $K*f$. This uniform bound is in fact the main reason for hypothesis (1) in 3.2.2.

Lemma 3.4.3. *Under assumptions 3.2.2 we have*

$$\sup_{t\in[0,T]} \left|\left|C^N(t,\cdot)\right|\right|_{C_b^1} \leq C\,||C_0||_{C_b^1} \quad a.s. \tag{3.19}$$

Proof. By the hypothesis 3.2.2 we have $K*\delta_{\mathbb{X}^N} \in C([0,T];C^\infty(\mathbb{R}^d))$ and $|\nabla(K*\delta_{\mathbb{X}^N})|$ $= |(\nabla K)*\delta_{\mathbb{X}^N}| \lesssim K*\delta_{\mathbb{X}^N}$ almost surely. We can then apply the same strategy as Lemma 3.4.1 and obtain the desired result. □

Lemma 3.4.4. *There exist a sequence of i.i.d. random variables χ_i satisfying $\mathbf{E}\big[\exp(a\chi_i^2)\big] < \infty$ for some $a > 0$, such that, for all $\beta \leq \overline{\beta}$*

$$\sup_{t\in[0,T]} \left|V_t^{i,N}\right|^{1+\beta} \lesssim_T Z^{i,\beta} := \left|V_{T^{i,N}}^{i,N}\right|^{1+\beta} + \chi_i^{1+\beta} + ||C_0||_{C_b^1}^{1+\beta} \tag{3.20}$$

and

$$\left|X_t^{i,N} - X_s^{i,N}\right| + \left|V_t^{i,N} - V_s^{i,N}\right| \lesssim Z^{i,0}\,|t-s|^{\frac{1}{2}-\varepsilon} \tag{3.21}$$

for all $0 < \varepsilon < \frac{1}{2}$. Moreover for all β the variables $(Z^{i,\beta})_{i\in\mathbb{N}}$ are i.i.d r.v.

Proof. For $t \in [T^{i,N}, \Theta^{i,N})$

$$V_t^{i,N} = V_{T^{i,N}}^{i,N} e^{-\lambda(t-T^{i,N})} + \int_{T^{i,N}}^t e^{-\lambda(t-r)} \nabla C^N(r, X_r^{i,N}) dr + \sigma \int_{T^{i,N}}^t e^{-\lambda(t-r)} dB_r^i. \tag{3.22}$$

Call

$$U_{T^{i,N},t}^i = \int_{T^{i,N}}^t e^{-\lambda(t-r)} dB_r^i$$

and notice that these are Gaussian random variables. Moreover there is constant b_λ depending on λ such that

$$\mathrm{Var}(|U_{t,t'}^i - U_{s,s'}^i|) \lesssim b_\lambda\,|t-s| + |t'-s'|$$

and, since the variables are Gaussian, we have

$$\mathbf{E}\left[\left(\frac{|U_{t,t'}^i - U_{s,s'}^i|}{|t-s| + |t'-s'|}\right)^{2k}\right] \lesssim \frac{b_\lambda^k (2k)!}{2^k k!}.$$

Hence, by the proof Kolmogorov regularity Theorem, there exists a random variable χ_i, $\mathbf{E}\big[\exp(a\chi_i^2)\big] < \infty$ for some $a > 0$ depending on λ, such that

$$|U_{t,t'}^i - U_{s,s'}^i| \lesssim \chi_i (|t-s| + |t'-s'|)^{\frac{1}{2}-\varepsilon} \tag{3.23}$$

a.s. for every $\varepsilon \in (0, \frac{1}{2})$. Note that the variables χ_i depends only on B_t^i and thus are i.i.d. Plugging the last inequality into (3.22), and using (3.19) we obtain the first part of the lemma. The fact that the variables $Z^{i,\beta}$ are independent is a consequence of the independence of χ_i and of $V_{T^{i,N}}^{i,N}$. □

Remark 3.4.5. Notice that for all $i \in \mathbb{N}$ the random variable $Z^{i,\beta}$ is independent on $T^{i,N}$: in fact $Z^{i,\beta}$ depends only on the variables χ_i, which are independent on $T^{i,N}$, and on $V_{T^{i,N}}^{i,N}$ which is distributed as $G(v)dv$. This fact will be crucial in the proof of Lemma 3.4.6.

Lemma 3.4.6. *With the same notation of Lemma, 3.4.4 for all* $\beta \leq \overline{\beta}$

$$\mathbf{E}\left[\frac{1}{N}\sum_{i=1}^{N_t^N}\left|V_t^{i,N}\right|^{1+\beta}\right] \lesssim_T \mathbf{E}\left[\sum_{i=1}^{N_t^N} Z^{i,\beta}\right] = \mathbf{E}\left[Z^{1,\beta}\right]\mathbf{E}\left[\frac{N_t^N}{N}\right]. \tag{3.24}$$

Moreover, recall that N_t^N *is non decreasing, it holds*

$$\mathbf{E}\left[\frac{N_T^N}{N}\right] \lesssim_T 1. \tag{3.25}$$

Proof. □

Let us first start by proving the first part of the lemma. We can control

$$\mathbf{E}\left[\sum_{i=1}^{N_t^N}\left|V_t^{i,N}\right|^{1+\beta}\right] \lesssim_T \mathbf{E}\left[\sum_{i=1}^{N_t^N} Z^{i,\beta}\right]$$

using inequality (3.20) in Lemma 3.4.4. Recall that $Z^{i,\beta}$ is independent on $T^{i,N}$: we can now prove a variant of Wald identity.

$$\mathbf{E}\left[\sum_{i=1}^{N_t^N} Z^{i,\beta}\right] = \sum_{k=1}^{\infty}\sum_{i=1}^{k}\mathbf{E}\left[Z^{i,\beta}\mathbb{1}_{N_t^N=k}\right] = \sum_{i=1}^{\infty}\sum_{k=i}^{\infty}\mathbf{E}\left[Z^{i,\beta}\mathbb{1}_{N_t^N=k}\right] = \sum_{i=1}^{\infty}\mathbf{E}\left[Z^{i,\beta}\mathbb{1}_{N_t^N>i}\right]$$

$$= \sum_{i=1}^{\infty}\mathbf{E}\left[Z^{i,\beta}\mathbb{1}_{T^{i,N}\leq t}\right] = \sum_{i=1}^{\infty}\mathbf{E}\left[Z^{i,\beta}\right]\mathbf{E}\left[\mathbb{1}_{T^{i,N}\leq t}\right] = \mathbf{E}\left[Z^{i,\beta}\right]\mathbf{E}\left[N_t^N\right].$$

This proves the first part. Concerning the second, by applying Itô formula (3.12) with $f \equiv 1$ we get

$$\frac{N_t^N}{N} = \frac{N_0^N}{N} + \int_0^t \frac{N_s^N}{N}ds + \int_0^t\int_0^s \frac{1}{N}\sum_{i=1}^{N_r^N}\mathbb{1}_{[T^{i,N},\Theta^{i,N}]}\left|V_r^{i,N}\right|drds$$
$$- \int_0^t\int_{\mathbb{R}^d}(K * \delta_{\mathbb{X}_s^N})(x)S_s^N(dx,dv)ds + M_t^{2,N,1} + M_t^{3,N,1}.$$

Taking the expected value and neglecting the indicator function, we obtain (3.25) by applying the generalized Grönwall lemma of Appendix 3.B.

□

Remark 3.4.7. This part of the proof follows the same strategy as [19] regarding the variant of Wald identity. The r.v. $Z^{i,\beta}$ are obtained by a different type argument, using the a priori bound on ∇C^N available in this case.

Corollary 3.4.8.

$$\mathbf{E}\left[\sup_{t\in[0,T]}\left|\left|D^2C^N(t,\cdot)\right|\right|_\infty\right]\lesssim_T ||C_0||_{C_b^2}\,||K||_{C_b^2}\,\mathbf{E}\left[\sup_{t\in[0,T]}\frac{N_t^N}{N}\right]$$

Proof. By Lemma 3.4.1 it's enough to verify that $\delta_{\mathbb{X}^N}(\cdot,dx)\in L^\infty([0,T];\mathcal{M}_f^+(\mathbb{R}^d))$ almost surely. Note that $\delta_{\mathbb{X}^N}(\cdot,\mathbb{R}^d)$ is increasing in time, so that it's enough to bound $\delta_{\mathbb{X}^N}(T,\mathbb{R}^d)$. Using Lemma 3.4.6 we have

$$\mathbf{E}\left[\delta_{\mathbb{X}^N}(T,\mathbb{R}^d)\right]\leq\int_0^T\mathbf{E}\left[\frac{1}{N}\sum_{i=1}^{N_s^N}\left|V_s^{i,N}\right|\right]ds\lesssim_T C\mathbf{E}\left[\frac{N_T^N}{N}\right],$$

which ends the proof. □

3.4.2 Tightness of the laws and passage to the limit

Theorem 3.4.9. *The sequence $\{Q_S^N\}_{N\in\mathbb{N}}$ of the laws of the empirical measure $\{S_\cdot^N\}_{N\in\mathbb{N}}$ is tight on $\mathcal{D}([0,T];\mathcal{M}_f^+(\mathbb{R}^d\times\mathbb{R}^d))$. Moreover, the sequence $\{Q^N_{\delta^\beta_{\mathbb{X}}}\}_{N\in\mathbb{N}}$ of the laws of the empirical measure $\{\delta^\beta_{\mathbb{X}_\cdot^N}\}_{N\in\mathbb{N}}$, defined in* (3.11)*, is tight on $\mathcal{D}([0,T];\mathcal{M}_f^+(\mathbb{R}^d))$ for every $\beta\leq\overline{\beta}$.*

Proof. In order to prove the tightness of the laws of Q_S^N on $\mathcal{D}([0,T];\mathcal{M}_f^+(\mathbb{R}^d\times\mathbb{R}^d))$ we will show that $Q_S^N\circ\Phi_k$ is tight on $\mathcal{D}([0,T];\mathbb{R})$, for every function Φ_k in a dense subfamily of $C_b(\mathbb{R}^d\times\mathbb{R}^d)$ (recall the definition of M_f^+ in the introduction). In particular the functions Φ_k can be taken to be Lipschitz continuous. Thanks to Aldous criterion ([68]), to prove the tightness of $Q_S^N\circ\Phi_k$ it sufficient to verify the following conditions:

$$\forall t\in[0,T],\forall\varepsilon>0,\exists R\text{ s.t. }\sup_{N\in\mathbb{N}}Q_S^N\left(|\langle\pi_t,\Phi_k\rangle|>R\right)<\varepsilon\tag{3.26}$$

and

$$\forall\delta,\lim_{\gamma\to0}\limsup_{N\in\mathbb{N}}\sup_{\substack{\tau\in\mathcal{L}_T\\\theta\leq\gamma}}Q_S^N\left(|\langle\pi_{\tau+\theta},\Phi_k\rangle-\langle\pi_\tau,\Phi_k\rangle|>\delta\right)=0\tag{3.27}$$

where $\mathcal{L}_T$ is the family of all stopping times bounded by T. Notice that

$$Q_S^N\left(|\langle\pi_t,\Phi_k\rangle|>R\right)=\mathbf{P}\left(\left|\left\langle S_t^N,\Phi_k\right\rangle\right|>R\right)\leq\frac{1}{R}\mathbf{E}\left[\left|\left\langle S_t^N,\Phi_k\right\rangle\right|\right]\leq\frac{1}{R}\mathbf{E}\left[\frac{N_t^N}{N}\right]||\Phi_k||_\infty$$

so that condition (3.26) follows by Lemma 3.4.6. Concerning condition (3.27) we have

$$Q_S^N\left(|\langle\pi_{\tau+\theta},\Phi_k\rangle-\langle\pi_\tau,\Phi_k\rangle|>\delta\right)=\mathbf{P}\left(\left|\left\langle S_{\tau+\theta}^N,\Phi_k\right\rangle-\left\langle S_\tau^N,\Phi_k\right\rangle\right|>\delta\right)$$

furthermore

$$\left|\left\langle S^N_{\tau+\theta},\Phi_k\right\rangle-\left\langle S^N_{\tau},\Phi_k\right\rangle\right|\leq\frac{1}{N}\sum_{i=N^N_\tau+1}^{N^N_{\tau+\theta}}\left|\Phi_k(X^{i,N}_{\tau+\theta},V^{i,N}_{\tau+\theta})-\Phi_k(X^{i,N}_{\tau},V^{i,N}_{\tau})\right|$$

$$\leq||\Phi_k||_{Lip}\frac{1}{N}\sum_{i=N^N_\tau+1}^{N^N_{\tau+\theta}}\left(\left|X^{i,N}_{\tau+\theta}-X^{i,N}_{\tau}\right|+\left|V^{i,N}_{\tau+\theta}-V^{i,N}_{\tau}\right|\right)$$

$$\leq||\Phi_k||_{Lip}\frac{1}{N}\sum_{i=N^N_\tau+1}^{N^N_{\tau+\theta}}Z^{i,0}\,\theta^{\frac{1}{2}-\varepsilon}$$

by Lemma 3.4.4. By Lemma 3.4.6 we have

$$\mathbf{E}\left[\frac{1}{N}\sum_{i=N^N_\tau+1}^{N^N_{\tau+\theta}}Z^{i,0}\right]\leq\mathbf{E}\left[\frac{1}{N}\sum_{i=1}^{N^N_T}Z^{i,0}\right]=\mathbf{E}\left[\frac{N^N_T}{N}\right]\mathbf{E}\left[Z^{1,0}\right].$$

and thus, by using Markov inequality

$$\mathbf{P}\left(\left|\left\langle S^N_{\tau+\theta},\Phi_k\right\rangle-\left\langle S^N_{\tau},\Phi_k\right\rangle\right|>\delta\right)\leq||\Phi_k||_{Lip}\frac{C}{\delta}\theta^{\frac{1}{2}-\varepsilon}.$$

Letting $\gamma\to 0$ ends the first part of the proof. For the second part, to prove the tightness of $Q^N_{\delta^\beta_{\mathbb{X}}}$, we apply the same strategy: Thus we need to proove that for every Lipschitz function Φ_k, in dense subset of $C_b(\mathbb{R}^d)$, we have

$$\forall t\in[0,T],\forall\varepsilon>0\exists R\text{ s.t. }\sup_{N\in\mathbb{N}}Q^N_{\delta^\beta_{\mathbb{X}}}\left(|\langle\pi_t,\Phi_k\rangle|>R\right)<\varepsilon \tag{3.28}$$

and

$$\forall\delta,\lim_{\gamma\to 0}\limsup_{N\in\mathbb{N}}\sup_{\substack{\tau\in\mathcal{L}_T\\ \theta\leq\gamma}}Q^N_{\delta^\beta_{\mathbb{X}}}\left(|\langle\pi_{\tau+\theta},\Phi_k\rangle-\langle\pi_\tau,\Phi_k\rangle|>\delta\right)=0 \tag{3.29}$$

For the first condition we have

$$Q^N_{\delta^\beta_{\mathbb{X}}}\left(|\langle\pi_t,\Phi_k\rangle|>R\right)=\mathbf{P}\left(\left|\left\langle\delta^\beta_{\mathbb{X}^N_t},\Phi_k\right\rangle\right|>R\right)$$

and

$$\mathbf{E}\left[\left|\left\langle\delta^\beta_{\mathbb{X}^N_t},\Phi_k\right\rangle\right|\right]=\mathbf{E}\left[\int_0^t\frac{1}{N}\sum_{i=1}^{N^N_s}\left|V^{i,N}_s\right|^{1+\beta}\Phi_k(X^{i,N}_s)ds\right]$$

$$\leq ||\Phi_k||_\infty \int_0^t \mathbf{E}\left[\frac{1}{N}\sum_{i=1}^{N_s^N}\left|V_s^{i,N}\right|^{1+\beta}\right] ds \leq ||\Phi_k||_\infty \mathbf{E}\left[Z^{i,\beta}\right] \int_0^t \mathbf{E}\left[\frac{N_s^N}{N}\right] ds$$
$$\leq ||\Phi_k||_\infty \mathbf{E}\left[Z^{1,\beta}\right] T \cdot C$$

so that we can obtain condition (3.28) by Markov inequality and Lemma 3.4.6. Furthermore

$$\mathbf{E}\big[\,\Big|\Big\langle \delta^\beta_{\mathbb{X}_t^N}(\tau+\theta), \Phi_k\Big\rangle - \Big\langle \delta^\beta_{\mathbb{X}_t^N}(\tau), \Phi_k\Big\rangle\Big|\,\big] \leq ||\Phi_k||_\infty \int_\tau^{\tau+\theta} \mathbf{E}\left[\frac{1}{N}\sum_{i=N_\tau^N+1}^{N_{\tau+\theta}^N}\left|V_s^{i,N}\right|^{1+\beta}\right] ds.$$

Again, using Lemma 3.4.6

$$\mathbf{E}\left[\frac{1}{N}\sum_{i=N_\tau^N+1}^{N_{\tau+\theta}^N}\left|V_s^{i,N}\right|^{1+\beta}\right] \leq \mathbf{E}\left[\frac{1}{N}\sum_{i=1}^{N_T^N} Z^{i,\beta}\right] = \mathbf{E}\left[\frac{N_T^N}{N}\right]\mathbf{E}\left[Z^{1,\beta}\right].$$

leading to

$$\mathbf{E}\left[\Big|\Big\langle \delta^\beta_{\mathbb{X}_t^N}(\tau+\theta), \Phi_k\Big\rangle - \Big\langle \delta^\beta_{\mathbb{X}_t^N}(\tau), \Phi_k\Big\rangle\Big|\right] \leq ||\Phi_k||_\infty \mathbf{E}\left[\frac{N_T^N}{N}\right]\mathbf{E}\left[Z^{1,\beta}\right]\theta$$

This ends the proof of condition (3.29) and thus the proof of the lemma. □

Lemma 3.4.10. *For every $\varepsilon > 0$ there exists R such that*

$$\mathbf{P}\Big(\,\big|\big|C^N\big|\big|_{W^{1,\infty}\left([0,T];C_0(\mathbb{R}^d)\right)} > R\Big) < \varepsilon \tag{3.30}$$

Proof. Let us first notice that

$$\mathbf{P}\Big(\,\big|\big|C^N\big|\big|_{W^{1,\infty}\left([0,T];C_0(\mathbb{R}^d)\right)} > R\Big) \leq \mathbf{P}\Big(\sup_{t\in[0,T]}\big|\big|\partial_t C^N(t,\cdot)\big|\big|_\infty > \frac{R}{2}\Big)$$
$$+\,\mathbf{P}\Big(\sup_{t\in[0,T]}\big|\big|C^N(t,\cdot)\big|\big|_\infty > \frac{R}{2}\Big)$$

and that the second term can be made arbitrary small by Markov inequality and Lemma 3.4.3.

By a direct computation we also have

$$\big|\big|\partial_t C^N(t,\cdot)\big|\big|_\infty \lesssim_T \frac{\sigma_C^2}{2}\big|\big|D^2 C^N(t,\cdot)\big|\big|_\infty + \frac{N_t^N}{N}||K_C||_\infty\big|\big|C^N(t,\cdot)\big|\big|_\infty.$$

Taking the supremum in time we obtain

$$\sup_{t\in[0,T]} \left|\left|\partial_t C^N(t,\cdot)\right|\right|_\infty \lesssim_T \frac{\sigma_C^2}{2} \sup_{t\in[0,T]} \left|\left|D^2 C^N(t,\cdot)\right|\right|_\infty + \frac{N_T^N}{N} ||K_C||_\infty \sup_{t\in[0,T]} \left|\left|C^N(t,\cdot)\right|\right|_\infty$$

thus

$$\mathbf{P}\Big(\sup_{t\in[0,T]} \left|\left|\partial_t C^N(t,\cdot)\right|\right|_\infty > \frac{R}{2}\Big) \lesssim_T \frac{2\sigma_C^2}{R}\mathbf{E}\left[\sup_{t\in[0,T]} \left|\left|D^2 C^N(t,\cdot)\right|\right|_\infty\right] +$$

$$\frac{2\,||K_C||_\infty}{\sqrt{R}}\mathbf{E}\left[\frac{N_T^N}{N}\right] + \frac{2}{\sqrt{R}}\mathbf{E}\left[\sup_{t\in[0,T]} \left|\left|C^N(t,\cdot)\right|\right|_\infty\right].$$

By taking R sufficiently large we can conclude by Lemmas 3.4.3, 3.4.6 and Corollary 3.4.8. □

Theorem 3.4.11. *Denoting by B_M the d-dimensional ball of radius M we have that $\forall M \in \mathbb{N}$ the sequence $\{Q_C^{N,M}\}_{N\in\mathbb{N}}$ of the laws of the function $\{C_\cdot^N\}_{N\in\mathbb{N}}$ restricted to B_M, is tight on $C\Big([0,T];C^1(B_M)\Big)$.*

Proof. By Simon's lemma we have that

$$W^{1,\infty}\Big([0,T];C(B_M)\Big) \cap L^\infty\Big([0,T];C^2(B_M)\Big)$$

is compactly embedded into

$$C\Big([0,T];C^1(B_M)\Big).$$

Thus, the set

$$K_{M,R,S} := \left\{ f \;\middle|\; ||f||_{W^{1,\infty}\left([0,T];C_0(B_M)\right)} \le R, ||f||_{L^\infty\left([0,T];C^2(B_M)\right)} \le S \right\}$$

is a compact subset of $C\Big([0,T];C^1(B_M)\Big)$ with respect to the strong topology. We have

$$Q_C^{N,M}\left(K_{M,R,S}^c\right) \le \mathbf{P}\left(\left|\left| C^N_{|_{B_M}} \right|\right|_{W^{1,\infty}\left([0,T];C_0(B_M)\right)} > R \right) + \mathbf{P}\left(\left|\left| C^N_{|_{B_M}} \right|\right|_{L^\infty\left([0,T];C^2(B_M)\right)} > S \right)$$

$$\leq \mathbf{P}\left(\left|\left|C^N\right|\right|_{W^{1,\infty}\left([0,T];C_0(\mathbb{R}^d)\right)} > R\right) + \frac{1}{S}\mathbf{E}\left[\sup_{t\in[0,T]}\left|\left|D^2C^N(t,\cdot)\right|\right|_{C(\mathbb{R}^d)}\right]. \tag{3.31}$$

By Lemma 3.4.10 and Corollary 3.4.8, choosing R and S big enough, we can make (3.31) arbitrary small, ending the proof. □

From the previous lemma we immediately get the following theorem:

Theorem 3.4.12. *The sequence $\{Q_C^N\}_{N\in\mathbb{N}}$ of the laws of the function $\{C_\cdot^N\}_{N\in\mathbb{N}}$ is tight on $C\Big([0,T];C^1(\mathbb{R}^d)\Big)$.*

Proof. The thesis follows by the tightness of the sequence of the laws $\{Q_C^{N,M}\}_{N\in\mathbb{N}}$. □

With a little abuse of notation we will refer to the laws of $\delta_{\mathbb{X}^N}$ as $Q^N_{\delta_{\mathbb{X}}} = Q^N_{\delta^0_{\mathbb{X}}}$. We will also denote by Q^N the measure

$$Q^N = Q_S^N \otimes Q^N_{\delta_{\mathbb{X}}} \otimes Q_C^N$$

defined on the product space X.

By Theorems 3.4.9 and 3.4.12 we immediately get the following corollary:

Corollary 3.4.13. *The sequence $\{Q^N\}_{N\in\mathbb{N}}$ of probability measure is tight on the space $\widetilde{X}$.*

Theorem 3.4.14. *Any limit point of any subsequence of the sequence $\{Q^N\}_{N\in\mathbb{N}}$, is supported on the measure solutions of system of equation* (3.13).

Sketch of the proof. The fact that limit objects satisfy system of equations (3.13) is classical, see [68]. Hence we highlight only the main difficulties. Let us show that all the reminders in the Itô formulations, the martingales $M_t^{k,N,f}$ for $k = 1, 2, 3$, vanish when N tends to infinity. For every test function $\varphi \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ we have to check that (recall Itô formula in section 3.3.2)

$$\mathbf{E}\left[\sup_{t\in[0,T]}\left|M_t^{k,N,f}\right|^2\right] \overset{N\to\infty}{\longrightarrow} 0 \tag{3.32}$$

for $k = 1, 2, 3$. By using Burkholder inequality we will conclude by showing that the quadratic variation of all these martingale goes to zero in $L^1(\Omega)$. The first martingale, coming from the Brownian motion is classical. Concerning the

martingale $M^{2,N,f}$, deriving from the branching process, we first note that we can rewrite

$$M_t^{2,N,f} = \frac{1}{N}\sum_{i=1}^{N_t^N} M_t^{i,2,N,f}$$

where $M_t^{i,2,N,f}$ are martingales defined as the integral of f with respect to a compensated Poisson point process Φ^i, whose compensator is the random measure

$$G(v)\mathbb{1}_{s\in[T^{i,N},\theta^{i,N})}\delta_{X_s^{i,N}}(dx)dvds + \int_0^s G(v)\mathbb{1}_{s\in[T^{i,N},\theta^{i,N})}\left|V_r^{i,N}\right|\delta_{X_r^{i,N}}(dx)dvdrds.$$

It follows that

$$\begin{aligned}\left[M_\cdot^{2,N,f}\right]_T &= \frac{1}{N^2}\sum_{i,j=1}^{N_T^N}\left[M_\cdot^{i,2,N,f}, M_\cdot^{j,2,N,f}\right]_T \\ &= \frac{1}{N}\int_0^T\int_{\mathbb{R}^d} f(x,v)^2\Big[G(v)\overline{S}_s^N(dx)dvds - G(v)\delta_{\mathbb{X}_s^N}(dx)dvds\Big] \\ &\qquad + \frac{1}{N^2}\sum_{i\neq j}\left[M_\cdot^{i,2,N,f}, M_\cdot^{j,2,N,f}\right]_T\end{aligned}$$

It is now clear that, if we show that the terms corresponding to $i \neq j$ vanish, we will obtain the desired result. To do so, observe that for $i \neq j$, shortening the notation to M_t^i and M_t^j

$$\left[M_\cdot^i, M_\cdot^j\right]_T = [(M_\cdot^i),(M_\cdot^j)]_T^c + \Delta\left[M_\cdot^i, M_\cdot^j\right]_T = [(M_\cdot^i)^c,(M_\cdot^j)^c]_T + \Delta M_T^i \Delta M_T^j$$

where $(M^i)^c$ denotes the continuous part of the variation. The continuous part obviously vanish, being the motion of particles for $i \neq j$ driven by independent Brownian motions. For the jump part, notice that the probability of two birth or coalescence events, corresponding to jumps, to happen at the same time is zero: this is a consequence of the conditional independence with respect to $\mathcal{F}_t$ of $M^{2,N,f}$ and $M^{3,N,f}$, as well as that of the martingales M^i and M^j. The proof for the martingale $M^{3,N,f}$ follows in the same manner.

Concerning the time regularity of the limit points we just remark the fact that limits point are probability measures on X, thus are continuous in time, is a consequence of the tightness criterion in the space $\mathcal{D}$, [68]. □

Remark 3.4.15. In Theorem 3.4.9 we have seen that the for every $\beta \leq \overline{\beta}$ the sequence of the laws of $\delta^\beta_{\mathbb{X}^N}$ is tight. Starting from this fact, we consider the following equation

$$\begin{cases}\partial_t\rho^\beta(t,x) = \displaystyle\int_{\mathbb{R}^d}|v|^{1+\beta}u(t,x,v)dv \\ \rho^\beta(0,x) = 0.\end{cases} \tag{3.33}$$

We can then prove with small modification from Theorem 3.4.14, that the sequence of laws of $\delta^{\beta}_{\mathbb{X}^N}$ is supported on the weak solutions of the previous PDE, and thus that the whole sequence of laws $\{Q^N_{\delta^{\beta}_{\mathbb{X}}}\}_{N\in\mathbb{N}}$ is converging in $\mathcal{D}([0,T];\mathcal{M}_f^+(\mathbb{R}^d))$ to the unique solution of (3.33). Hence we deduce the following fact:

Corollary 3.4.16. *If the function u_0 starting condition of the system* (3.13) *satisfying hypothesis* (3),(4)*, satisfies also hypothesis* (5)*, then the solution of system* (3.13) *has the property*

$$\sup_{t\in[0,T]} \rho^{\beta}(t,\mathbb{R}^d) = \int_{\mathbb{R}^d} |v|^{1+\beta} u(t,\mathbb{R}^d,dv) < \infty$$

for all $\beta \leq \overline{\beta}$.

3.5 Fourier formulation

3.5.1 The coupled PDEs in Fourier space

In the previous section, we proved existence of measure solution of the system of equations, when u_0, ρ_0 are bounded measures with moments of sufficiently high order and C_0 is regular enough :

$$\begin{cases} \partial_t u(t,x,v) + v\cdot\nabla_x u(t,x,v) - \lambda \mathrm{div}_v\big(v u(t,x,v)\big) \\ \qquad = \dfrac{\sigma^2}{2}\Delta_v u(t,x,v) + \nabla C(t,x).\nabla_v u(t,x,v) \\ \qquad\quad + G(v)\bar{u}(t,x) + G(v)K*\rho(t,x) - u(t,x,v)K*\rho(t,x) \\ \rho(t,x) = \displaystyle\int_0^t \int_{\mathbb{R}^d} |v| u(s,x,v) dv ds \\ \bar{u}(t,x) = \displaystyle\int_{\mathbb{R}^d} u(t,x,v) dv \\ \partial_t C(t,x) = \dfrac{\sigma_C^2}{2}\Delta C(t,x) - K_C * \rho(t,x) C(t,x). \end{cases} \tag{3.34}$$

The aim of this section is to prove uniqueness of such solutions. We will use Fourier techniques from [39] and hypoelliptic estimates. The proof of uniqueness will be decomposed in several steps. The first one is an a priori bound on C in Sobolev spaces. The second one will be a reminder of the derivation of the formulation of the system (3.34) in Fourier space, as long as a mild formulation in the Fourier space. The third step will be the proof of hypoelliptic estimates (following [39]) for the Fourier multiplier involved in the Fourier expression of

(3.34). Finally, by using a Grönwall type argument, we will conclude about the uniqueness of measure solutions. The hypoelliptic bound will also be useful in the last subsection 3.8 to prove smoothness of the solutions.

3.5.2 A priori bound for C

Let us suppose that (u, ρ, C) is a measure solution of Equation (3.13), as in Definition 3.3.1. To prove uniqueness, we will strongly use the smoothness of the potential field C. This lemma is obviously an extension of Lemma 3.4.1, we state it and prove it for the sake of the comprehension.

Lemma 3.5.1. *Let $m \in \mathbb{N}$, $f \in L^\infty\big([0,T]; \mathcal{M}_f^+(\mathbb{R}^d)\big)$, $g \in L^\infty\big([0,T]; H^m(\mathbb{R}^d, \mathbb{R})\big)$ and $K \in C_b^m(\mathbb{R}^d; \mathbb{R}_+)$ and $w_0 \in H^m(\mathbb{R}^d; \mathbb{R}_+)$. let w be the solution of the heat equation*

$$\partial_t w = \frac{a^2}{2}\Delta w - (K * f)w + g, \quad w(0,\cdot) = w_0.$$

Then $w \in L^\infty\big([0,T]; H^m(\mathbb{R}^d; \mathbb{R})\big)$, and one has for all $t \in [0,T]$,

$$\|w(t,\cdot)\|_{H^m} \lesssim \big(1 + \|K\|_{C_b^m}\big)^m \big(1 + \sup_{s\in[0,T]} f(s, \mathbb{R}^d)\big)^m \left(\|w_0\|_{H^m} + \int_0^t \|g(s,\cdot)\|_{H^m} ds \right)$$

Proof. Let W be a standard Brownian motion on $[0,T]$, as in Lemma 3.4.1, we have the explicit formula

$$w(t,x) = \mathbf{E}\Big[e^{-\int_0^t K*f\left(r, x+a(W_t - W_r)\right)dr} w_0(x + aW_t) + \int_0^t g\big(s, x + a(W_t - W_s)\big) e^{-\int_s^t K*f(r, x+aW_t - aW_r)dr} ds. \Big].$$

Hence, thanks to Faá di Bruno formula, for all multi-index α with $|\alpha| \le m$, we have

$$\partial_\alpha w(t,x) = \sum_{|\beta|+|\gamma|\le|\alpha|} c_{\alpha,\beta,\gamma} \mathbf{E}\Big[\partial_\beta w_0(x + aW_t) \partial_\gamma \Big(e^{-\int_0^t K*f\left(r,\cdot+a(W_t - W_r)\right)dr} \Big)(x) + \int_0^t \partial_\beta g\big(s, x + a(W_t - W_s)\big) \partial_\gamma \Big(e^{-\int_s^t K*f(r,\cdot+aW_t - aW_r)dr} \Big)(x) ds \Big]$$

Remark also that

$$\left| \partial_\gamma \Big(e^{-\int_0^t K*f\left(r,\cdot+a(W_t - W_r)\right)dr} \Big)(x) \right| \lesssim \big(1 + \|K\|_{C_b^m}\big)^m \big(1 + \sup_{s\in[0,T]} f(s, \mathbb{R}^d)\big)^m.$$

Hence,

$$\int_{\mathbb{R}^d} \big|\partial_\alpha w(t,x)\big|^2 dx \lesssim \big(1+\|K\|_{C_b^m}\big)^m \big(1+\sup_{s\in[0,T]} f(s,\mathbb{R}^d)\big)^m$$
$$\sum_{|\beta|+|\gamma|\le|\alpha|} \mathbf{E}\bigg[\int_{\mathbb{R}^d} \big|\partial_\beta w_0(x+aW_t)\big|^2 dx + \int_0^t \int_{\mathbb{R}^d} \Big|\partial_\beta g\big(s,x+a(W_t-W_s)\big)\Big|^2 dxds\bigg]$$

which gives, by summing on α,

$$\|w(t,\cdot)\|_{H^m} \lesssim \big(1+\|K\|_{C_b^m}\big)^m \big(1+\sup_{s\in[0,T]} f(s,\mathbb{R}^d)\big)^m \bigg(\|w_0\|_{H^m} + \int_0^t \|g(s,\cdot)\|_{H^m} ds\bigg),$$

which is the wanted result. □

3.5.3 The equation in Fourier space

Note that when u (and its associated ρ, see Definition 3.3.1) are finite positive measure solutions, their associated Fourier transforms (respectively in space and velocity) exist as bounded functions. Furthermore we have

$$\hat{u}(t,k,\xi) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-ik\cdot x} e^{-i\xi\cdot v} u(t,dx,dv),$$

$$\hat{\hat{u}}(t,k) = \int_{\mathbb{R}^d\times\mathbb{R}^d} e^{-ik\cdot x} u(t,dx,dv) = \hat{u}(t,k,0).$$

and

$$\hat{\rho}(t,k) = \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-ik\cdot x}|v|u(t,dx,dv)ds.$$

Hence, as Schwartz distributions,

$$\hat{\rho}(t,k) = \int_0^t (-\Delta)^{\frac{1}{2}}\hat{u}(s,k,0)ds = \int_0^t \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}}\, P.V. \int_{\mathbb{R}^d} \frac{\hat{u}(s,k,0)-\hat{u}(s,k,\xi)}{|\xi|^{d+1}} d\xi ds,$$

where *P.V.* denotes the principal value. By using Lemma 3.C.1 of the Appendix, a way of controlling the sup norm of $\hat{\rho}(t,k)$ is to control the $(1+\beta)$-Hölder norm of $\xi\to\hat{u}(t,k,\xi)$, uniformly in t,k. Finally, in Fourier space (as Schawrtz distributions), Equation (3.34) becomes

$$\begin{cases} \partial_t \hat{u}(t,k,\xi) - k\cdot\nabla_\xi \hat{u}(t,k,\xi) + \lambda\xi\cdot\nabla_\xi\hat{u}(t,k,\xi) \\ \qquad\qquad = -|\xi|^2\dfrac{\sigma^2}{2}\hat{u}(t,k,\xi) + i\xi\cdot\widehat{\nabla C} * \hat{u}(t,k,\xi) \\ \qquad\qquad\quad + \hat{G}(\xi)\big(\hat{u}(t,k,0) + \hat{K}\hat{\rho}(t,k)\big) - \Big(\hat{K}\hat{\rho}\Big) * \hat{u}(t,k,\xi). \\ \qquad \hat{\rho}(t,k) = \displaystyle\int_0^t (-\Delta)^{\frac{1}{2}}\hat{u}(s,k,0)ds \\ \partial_t C(t,x) = \dfrac{\sigma_C^2}{2}\Delta C(t,x) - \mathcal{F}^{-1}\Big(\hat{K}_C\hat{\rho}\Big)(t,x)C(t,x). \end{cases} \tag{3.35}$$

We know that there exists a solution of the previous equation, seen as Schwarz distribution $(\hat{u},\hat{\rho},C)$. We can rewrite the previous system in its Mild formulation (see Appendix 3.D). For $k,\xi\in\mathbb{R}^d$ let us define $\xi(t) = \big(\xi - \frac{k}{\lambda}\big)e^{-\lambda t} + \frac{k}{\lambda}$, and one have the following semi-explicit formula for the first equation of (3.35) :

$$\begin{aligned} \hat{u}(t,k,\xi) =& \hat{u}_0\big(k,\xi(t)\big)\exp\Big(-\frac{\sigma^2}{2}\int_0^t |\xi(r)|^2 dr\Big) \\ &+ \int_0^t i\xi(t-s)\cdot\big(\widehat{\nabla C}*\hat{u}\big)\big(s,k,\xi(t-s)\big)e^{-\frac{\sigma^2}{2}\int_0^{t-s}|\xi(r)|^2dr}ds \\ &+ \int_0^t G\big(\xi(t-s)\big)\big(\hat{u}(s,k,0)+\hat{K}\hat{\rho}(s,k)\big)e^{-\frac{\sigma^2}{2}\int_0^{t-s}|\xi(r)|^2dr}ds \\ &- \int_0^t \Big(\big(\hat{K}\hat{\rho}\big)*\hat{u}\Big)\big(s,k,\xi(t-s)\big)e^{-\frac{\sigma^2}{2}\int_0^{t-s}|\xi(r)|^2dr}ds. \end{aligned} \tag{3.36}$$

3.6 Hypoelliptic estimates

Proposition 3.6.1. *There exists two universal constants $c>0$ such that for all $t\geq 0$ and all $k,\xi\in\mathbb{R}^d$, we have*

$$e^{-\frac{\sigma^2}{2}\int_0^t|\xi(r)|^2dr} \leq e^{-c\frac{\sigma^2}{2}\left(\int_0^t\left(\frac{1-e^{-\lambda r}}{\lambda}\right)^2 dr|k|^2 + \int_0^t e^{-2\lambda r}dr|\xi|^2\right)} \tag{3.37}$$

and for all $n\geq 0$,

$$|\xi(t)|\Big(\int_0^t\big|\xi(r)\big|dr\Big)^n e^{-\frac{\sigma^2}{2}\int_0^t|\xi(r)|^2dr} \lesssim \frac{t^{\frac{n-1}{2}}}{\sigma^n}.$$

Proof. Let us remark that

$$\begin{aligned}\int_0^t |\xi(r)|^2 dr =& |\xi|^2 \int_0^t e^{-2\lambda r} dr + |k|^2 \int_0^t \left(\frac{1-e^{-\lambda r}}{\lambda}\right)^2 dr \\ &+ 2\xi \cdot k \int_0^t e^{-\lambda r} \left(\frac{1-e^{-\lambda r}}{\lambda}\right) dr \\ =& \big(A(t)\Xi(t)\big) \cdot \Xi(t)\end{aligned}$$

where

$$\Xi(t) = \begin{pmatrix} \left(\int_0^t e^{-2\lambda r} dr\right)^{\frac{1}{2}} \xi \\ \left(\int_0^t \left(\frac{1-e^{-\lambda r}}{\lambda}\right)^2 dr\right)^{\frac{1}{2}} k \end{pmatrix}$$

and

$$A(t) = \begin{pmatrix} Id & a(t) I_d \\ a(t) Id & Id \end{pmatrix}$$

and

$$a(t) = \frac{\int_0^t e^{-\lambda r} \left(\frac{1-e^{-\lambda r}}{\lambda}\right) dr}{\left(\int_0^t \left(\frac{1-e^{-\lambda r}}{\lambda}\right)^2 dr \int_0^t e^{-2\lambda r} dr\right)^{\frac{1}{2}}}.$$

The matrix $A(t)$ has two eigenvalues : $1 + a(t)$ and $1 - a(t)$, hence

$$\big(A(t)\Xi(t)\big) \cdot \Xi(t) \geq \big(1 - a(t)\big)|\Xi(t)|^2.$$

Furthermore, by taking

$$g(x) = \frac{\int_x^1 1 - u du}{\left(\int_x^1 u du \int_x^1 \frac{(1-u)^2}{u} du\right)^{1/2}} = \frac{(1-x)^{\frac{3}{2}}}{\Big((1+x)\big(-2\log(x) - (1-x)(3-x)\big)\Big)^{1/2}},$$

one has $a(t) = g(e^{-\lambda t})$. Furthermore $g(0) = 0$, $g(x) \to_{x\to 1} \frac{\sqrt{3}}{2}$, and

$$g'(x) = \frac{2(x+2)(1-x)^{\frac{1}{2}} h(x)}{\Big((1+x)\big(-2\log(x) - (1-x)(3-x)\big)\Big)^{3/2}},$$

with $h(x) = \frac{(1-x)(1+5x)}{2x(x+2)} + \log(x)$. Finally $h'(x) = -\frac{(1-x)^3}{x^2(x+2)^2}$, and $h(1) = 0$, hence $h \geq 0$ and so does $g'(x)$. Hence g is non-decreasing and $a(t) = g(e^{-\lambda t}) \leq \frac{\sqrt{3}}{2}$, and

the smallest egeinvalue of the matrix is greater than $1 - \frac{\sqrt{3}}{2}$. Hence one have

$$e^{-\frac{\sigma^2}{2}\int_0^t |\xi(r)|^2 dr} \leq e^{-\frac{(1-a(t))\sigma^2}{2}|\Xi(t)|^2}$$
$$\leq \exp\left(-\frac{1-\sqrt{3}}{2}\frac{\sigma^2}{2}\left(\int_0^t e^{-\lambda 2r} dr|\xi|^2 + \int_0^t (\frac{1-e^{-\lambda r}}{\lambda}) dr|k|^2 \right) \right)$$

and (3.37) holds true with $c = 1 - \frac{\sqrt{3}}{2}$.

For the second inequality, notice that

$$|\xi(t)| \leq |\xi| e^{-\lambda t} + |k| \frac{1-e^{-\lambda t}}{\lambda}.$$

Furthermore, $\sqrt{\int_0^t e^{-2\lambda r} dr} \geq \sqrt{t} e^{-\lambda t}$ and by convexity,

$$\sqrt{\int_0^t \Big(\frac{1-e^{-\lambda r}}{\lambda}\Big)^2 dr} \geq \frac{1-e^{-\lambda t}}{\lambda} \sqrt{\int_0^t \frac{r^2}{t^2} dr} = \sqrt{t} \frac{1-e^{-\lambda t}}{\sqrt{3}\lambda}.$$

Hence, there exists a constant such that

$$|\xi(t)| \lesssim \frac{1}{\sigma\sqrt{t}} \Big(\sigma^2 |\Xi(t)|^2 \Big)^{1/2}.$$

Using the fact that $|\Xi|$ is increasing in time, we have by using Equation (3.37)

$$\big|\xi(t)\big| \left(\int_0^t |\xi(s)| ds \right)^n e^{-\frac{\sigma^2}{2}\int_0^t |\xi(s)|^2 ds}$$
$$\lesssim \frac{1}{\sigma^{n+1}\sqrt{t}} \sigma^{n+1} |\Xi(t)|^{n+1} \left(\int_0^t \frac{1}{\sqrt{s}} ds \right)^n e^{-\frac{c\sigma^2}{2}|\Xi(t)|^2}$$

which allows us to conclude easily. □

3.7 Uniqueness of solutions

In order to prove uniqueness, we will use a Gronwall type argument.

Theorem 3.7.1. *Let G, K_C, K, C_0 and u_0 which satisfies the Hypothesis 3.2.2 and 3.2.3. Then there exists a unique measure solution for equation* (3.34).

The proof is decomposed into three parts. We first obtain a priori bounds for C, ρ and u for any measure solution. We then show how to control the difference of the ρ and the C parts of two solutions by the difference of the u parts. Finally we control the difference of the u part and use a Gronwall type argument to have uniqueness.

Proof. A priori bounds for the solutions:

Note first that when (u, ρ, C) is a measure solution, since ρ is a finite measure and $C_0 \in H^{m+2}$ (thanks to Lemma 3.5.1) , $C \in C\big([0,T]; H^{m+2}(\mathbb{R}^d)\big)$ (take $g = 0$ and $f = \rho$) with a bound which depends only on $\sup_{t\in[0,T]} \rho(t, \mathbb{R}^d)$ and of $\|K_C\|_{C_b^{m+2}}$, and since $m > \frac{d}{2}$,

$$\|\widehat{\nabla C}(t,\cdot)\|_{L^1} \lesssim \|C(t,\cdot)\|_{H^{m+2}} \lesssim_{T,C_0,K_C} \big(1 + \sup_{s\in[0,T]} \rho(s,\mathbb{R}^d)\big)^{m+2}.$$

Finally, $\rho(s,\mathbb{R}^d) = \int_0^s \int_{\mathbb{R}^d\times\mathbb{R}^d} |v| u(r, dx, dv)dr$ and u is a non negative measure for all time, hence ρ is non-decreasing in time and $\sup_{s\in[0,T]} \rho(s,\mathbb{R}^d) = \rho(T,\mathbb{R}^d)$. Thanks to Young inequality, one have

$$\|\widehat{\nabla C} * \hat\rho(s,\cdot)\|_{L^\infty} \lesssim \big(1 + \rho(T,\mathbb{R}^d)\big)^{m+2} \|\hat\rho(s,\cdot)\|_{L^\infty}.$$

Furthermore, thanks to the hypothesis and Corollary 3.4.16, u has $(1+\beta)$ moments (in v, uniformely in time and the x variable) which implies that $\xi \to \hat u(t,k,\xi)$ is a bounded and $(1+\beta)$-Hölder continuous function, uniformely in t, k, and we have thanks to Appendix 3.C.

$$\big|\hat\rho(t,k)\big| \le \int_0^t \sup_{k\in\mathbb{R}^d} \|\hat u(s,k,\cdot)\|_{\mathcal{C}_b^{1+\beta}} ds.$$

Note also that

$$\|\hat u(t,k,\cdot)\|_{\mathcal{C}_b^{1+\beta}} \lesssim \sup_{t\in[0,T]} \int_{\mathbb{R}^d\times\mathbb{R}^d} \big(1+|v|\big)^{1+\beta} u(t, dv, dx),$$

and thanks to Remark 3.4.15 and Corollary 3.4.16 this is finite, as soon as the main Hypothesis is satisfied.

Control of the second and third coordinates of the solutions by the first one. Now, take two measures solutions (u_1, ρ_1, C_1) and (u_2, ρ_2, C_2) with the same initial conditions u_0 and C_0. For now, we allow our bound in the $\lesssim$ to depends on $\|C_0\|_{H^{m+2}}$, $\|K_C\|_{H^{\frac{3m}{2}}}$, $\rho_i(T,\mathbb{R}^d)$, $\sup_{t\in[0,T]} u_i(t,\mathbb{R}^d,\mathbb{R}^d)$ and $\sup_{t\in[0,T]} \|C_i(t,\cdot)\|_{H^m}$.

Remark that thanks to the previous discussion, we have

$$\big\|(\hat\rho_1 - \hat\rho_2)(s,\cdot)\big\|_\infty \lesssim \int_0^t \sup_k \|(\hat u_1 - \hat u_2)(s,k,\cdot)\|_{\mathcal{C}_b^{1+\beta}} ds. \tag{3.38}$$

Remark that $w = C_1 - C_2$ satisfies the following equation :

$$\partial_t w = \frac{\sigma_C^2}{2}\Delta w - (K_C * \rho_1)w + \big(K_C * (\rho_1 - \rho_2)\big)C_2, \quad w_0 = 0.$$

Thanks to Lemma 3.5.1, we have

$$\big\|C_1(t,\cdot)-C_2(t,\cdot)\big\|_{H^{m+2}} \lesssim \big(1+\|K_C\|_{C_b^{m+2}}\big)^{m+2}\big(1+\rho_1(T,\mathbb{R}^d)\big)^{m+2} \times \int_0^t \Big\|\big(K_C*(\rho_1-\rho_2)\big)(s,\cdot)C_2(s,\cdot)\Big\|_{H^{m+2}}ds.$$

Furthermore

$$\Big\|\big(K_C*(\rho_1-\rho_2)\big)(s,\cdot)C_2(s,\cdot)\Big\|^2_{H^{m+2}} = \int_{\mathbb{R}^d}\big(1+|k|^2\big)^{m+2}\Big|\big(\hat{K}_C(\hat{\rho}_1-\hat{\rho}_2)\big)*\hat{C}_2(s,k)\Big|^2dk,$$

and by using the fact that for all k,k', $\big(1+|k|^2\big)^{m+2}\le\big(1+|k'|^2\big)^{m+2}\big(1+|k-k'|^2\big)^{m+2}$, by Young inequality,

$$\begin{aligned}\Big\|\big(K_C*(\rho_1-\rho_2)\big)(s,\cdot)C_2(s,\cdot)\Big\|_{H^{m+2}} \lesssim& \|C_2(s,\cdot)\|_{H^{m+2}}\int_{\mathbb{R}^d}|\hat{K}_C(k)|\big(1+|k|^2\big)^{\frac{m+2}{2}}dk \\ &\times\big\|\hat{\rho}_1(s,\cdot)-\hat{\rho}_2(s,\cdot)\big\|_\infty \\ \lesssim& \sup_{t\in[0,T]}\|C_2(t,\cdot)\|_{H^{m+2}}\|K_C\|_{H^{2m+2}}\big\|(\hat{\rho}_1-\hat{\rho}_2)(s,\cdot)\big\|_\infty.\end{aligned}$$

Finally, we have the following bound for C_1-C_2,

$$\big\|C_1(t,\cdot)-C_2(t,\cdot)\big\|_{H^m}\lesssim\int_0^t\int_0^s\sup_{k\in\mathbb{R}^d}\big\|(\hat{u}_1-\hat{u}_2)(s,k,\cdot)\big\|_{\mathcal{C}_b^{1+\beta}}drds. \tag{3.39}$$

Gronwall type argument :
Thanks to Equation (3.36), we have

$$\begin{aligned}\hat{u}_1(t,k,\xi)-\hat{u}_2(t,k,\xi) =& \\ &\int_0^t i\xi(t-s)\cdot\big((\widehat{\nabla C_1}-\widehat{\nabla C_2})*\hat{u}_1\big)\big(s,k,\xi(t-s)\big)e^{-\frac{\sigma^2}{2}\int_0^{t-s}|\xi(r)|^2dr}ds \qquad (3.40)\\ &+\int_0^t i\xi(t-s)\cdot\big(\widehat{\nabla C_2}*(\hat{u}_1-\hat{u}_2)\big)\big(s,k,\xi(t-s)\big)e^{-\frac{\sigma^2}{2}\int_0^{t-s}|\xi(r)|^2dr}ds \qquad (3.41)\\ &+\int_0^t\hat{G}\big(\xi(t-s)\big)\big(\hat{u}_1(s,k,0)-\hat{u}_2(s,k,0)\big)e^{-\frac{\sigma^2}{2}\int_0^{t-s}|\xi(r)|^2dr}ds \qquad (3.42)\\ &+\int_0^t\hat{G}\big(\xi(t-s)\big)\hat{K}(k)\big(\hat{\rho}_1(s,k)-\hat{\rho}_2(s,k)\big)e^{-\frac{\sigma^2}{2}\int_0^{t-s}|\xi(r)|^2dr}ds \qquad (3.43)\\ &-\int_0^t\Big(\big(\hat{K}(\hat{\rho}_1-\hat{\rho}_2)*\hat{u}_1\big)\big(s,k,\xi(t-s)\big)e^{-\frac{\sigma^2}{2}\int_0^{t-s}|\xi(r)|^2dr}ds \qquad (3.44)\\ &-\int_0^t\Big(\big(\hat{K}\hat{\rho}_2\big)*(\hat{u}_1-\hat{u}_2)\big)\big(s,k,\xi(t-s)\big)e^{-\frac{\sigma^2}{2}\int_0^{t-s}|\xi(r)|^2dr}ds. \qquad (3.45)\end{aligned}$$

Let us recall a basic inequality on Hölder norms : if f_1 and f_2 are two $(1+\beta)$-Hölder continuous functions from $\mathbb{R}^d$ to $\mathbb{R}^d$, then

$$\|f_1 \cdot f_2\|_{\mathcal{C}_b^{1+\beta}} \lesssim \|f_1\|_{\mathcal{C}_b^{1+\beta}} \|f_2\|_{\mathcal{C}_b^{1+\beta}}.$$

When dealing with (3.40), by take $f_1(\xi) = \xi(t-s)e^{-\int_0^{t-s} \left|\xi(r)\right|^2 dr}$, one has (since $D_\xi \xi(r) = e^{-\lambda r} Id$),

$$Df_1(\xi) = e^{-\lambda(t-s)} e^{-\int_0^{t-s} \left|\xi(r)\right|^2 dr} Id + \sigma \xi(t-s) \otimes \int_0^{t-s} \xi(r) dr e^{-\int_0^{t-s} \left|\xi(r)\right|^2 dr}$$

and

$$\begin{aligned} D^2 f_1(\xi) &= \sigma^2 e^{-\lambda(t-s)} e^{-\int_0^{t-s} \left|\xi(r)\right|^2 dr} \int_0^{t-s} \xi(r) \otimes Id dr \\ &+ \sigma^2 \int_0^{t-s} I_d \otimes \xi(r) dr e^{-\int_0^{t-s} \left|\xi(r)\right|^2 dr} + \sigma^2 \frac{1 - e^{-\lambda(t-s)}}{\lambda} \xi(t-s) \otimes Id\, e^{-\int_0^{t-s} \left|\xi(r)\right|^2 dr} \\ &+ \sigma^4 \xi(t-s) \otimes \left(\int_0^{t-s} \xi(r) dr \right)^{\otimes 2} e^{-\int_0^{t-s} \left|\xi(r)\right|^2 dr}. \end{aligned}$$

Hence, thanks to the second inequality of Proposition 3.6.1 with $n = 0, 1, 2$, we have

$$\|f\|_{\mathcal{C}_b^{1+\beta}} \lesssim \frac{1}{\sqrt{t-s}}.$$

Take

$$f_2(\xi) = \big((\widehat{\nabla C_1} - \widehat{\nabla C_2}) * \hat{u}_1 \big)\big(s, k, \xi(t-s)\big),$$

by using the previous bound for $C^1 - C^2$, we have

$$\begin{aligned} \|f_2\|_{\mathcal{C}_b^{1+\beta}} &\lesssim \sup_k \|\hat{u}_1(s,k,\cdot)\|_{\mathcal{C}_b^{1+\beta}} \int_0^s \int_0^r \sup_k \|\hat{u}_1(\tau,k,\cdot) - \hat{u}_2(\tau,k,\cdot)\|_{\mathcal{C}_b^{1+\beta}} d\tau dr \\ &\lesssim \int_0^s \int_0^r \sup_k \|\hat{u}_1(\tau,k,\cdot) - \hat{u}_2(\tau,k,\cdot)\|_{\mathcal{C}_b^{1+\beta}} d\tau dr \end{aligned}$$

Hence, we have the following bound for (3.40) :

$$\sup_k \big\|(3.40)\big\|_{\mathcal{C}_b^{1+\beta}} \lesssim \int_0^t \frac{1}{\sqrt{t-s}} \int_0^s \int_0^r \sup_k \|(\hat{u}_1 - \hat{u}_2)(\tau,k,\cdot)\|_{\mathcal{C}_b^{1+\beta}} d\tau dr ds.$$

The proof for (3.41) when taking f_1 to be the same and $f_2 = \big(\widehat{\nabla C_2} * (\hat{u}_1 - \hat{u}_2)\big)\big(s, k, \xi(t-s)\big)$, since $\|f_2\|_{\mathcal{C}_b^{1+\beta}} \lesssim \sup_k \big\|\hat{u}_1(s,k,\cdot) - \hat{u}_2(s,k,\cdot)\big\|_{\mathcal{C}_b^{1+\beta}}$, and

$$\sup_k \big\|(3.41)\big\|_{\mathcal{C}_b^{1+\beta}} \lesssim \int_0^t \frac{1}{\sqrt{t-s}} \sup_k \big\|(\hat{u}_1 - \hat{u}_2)(s,k,\cdot)\big\|_{\mathcal{C}_b^{1+\beta}} ds.$$

In (3.42) and (3.43), since G has finite $(1+\beta)$ moments, $\hat{G}$ is $(1+\beta)$-Hölder continuous. Take $f_1(\xi) = \widehat{G}\big(\xi(t-s)\big)e^{-\frac{\sigma^2}{2}\int_0^{t-s}\left|\xi(r)\right|^2 dr}$, we have $\|f_1\|_{\mathcal{C}_b^{1+\beta}} \lesssim 1$. Furthermore take $f_2 = \hat{u}_1(s,k,0) - \hat{u}_2(s,k,0) + \hat{K}(\xi)\big(\hat{\rho}_1(s,k) - \hat{\rho}_2(s,k)\big)$. Thanks to (3.38) we have $\|f_2\|_{\mathcal{C}_b^{1+\beta}} \lesssim \sup_k \|\hat{u}_1(s,k,\cdot)\|_{\mathcal{C}_b^{1+\beta}} + \int_0^s \sup_k \|\hat{u}_1(s,k,\cdot)\|_{\mathcal{C}_b^{1+\beta}} dr$, and we have

$$\sup_{k\in\mathbb{R}^d} \|(3.42)+(3.43)\|_{\mathcal{C}_b^{1+\beta}} \lesssim \int_0^t \sup_k \|\hat{u}_1(s,k,\cdot) - \hat{u}_2(s,k,\cdot)\|_{\mathcal{C}_b^{1+\beta}} ds + \int_0^t \int_0^s \sup_k \|\hat{u}_1(s,k,\cdot) - \hat{u}_2(s,k,\cdot)\|_{\mathcal{C}_b^{1+\beta}} dr ds.$$

In (3.44), take $f_1 = e^{-\frac{\sigma^2}{2}\int_0^{t-s}|\xi(r)|^2 dr}$ and $f_2 = \Big(\big(\hat{K}(\hat{\rho}_1 - \hat{\rho}_2) * \hat{u}_1\big)\Big)\big(s,k,\xi(t-s)\big)$.Remind that the convolution is in the k variable, hence thanks to Young inequality,

$$\begin{aligned}\|f_2\|_{\mathcal{C}_b^{1+\beta}} \lesssim & \|\hat{K}_m(\hat{\rho}_1 - \hat{\rho}_2)(s,\cdot)\|_{L^1} \sup_{k\in\mathbb{R}^d} \|\hat{u}_1(s,k,\cdot)\|_{\mathcal{C}_b^{1+\beta}} \\ \lesssim & \|K\|_{H^m} \int_0^s \sup_k \|\hat{u}_1(r,k,\cdot) - \hat{u}_2(r,k,\cdot)\|_{\mathcal{C}_b^{1+\beta}} dr.\end{aligned}$$

The same holds for (3.45) with $f_2 = \Big(\big(\hat{K}\hat{\rho}_2\big) * (\hat{u}_1 - \hat{u}_2)\Big)$, and we have

$$\|f_2\|_{\mathcal{C}_b^{1+\beta}} \lesssim \sup_k \|\hat{u}_1(r,k,\cdot) - \hat{u}_2(r,k,\cdot)\|_{\mathcal{C}_b^{1+\beta}}.$$

Hence, one have

$$\sup_{k\in\mathbb{R}^d} \|(3.44)+(3.45)\|_{\mathcal{C}_b^{1+\beta}} \lesssim \int_0^t \sup_k \|\hat{u}_1(s,k,\cdot) - \hat{u}_2(s,k,\cdot)\|_{\mathcal{C}_b^{1+\beta}} ds + \int_0^t \int_0^s \sup_k \|\hat{u}_1(r,k,\cdot) - \hat{u}_2(r,k,\cdot)\|_{\mathcal{C}_b^{1+\beta}} dr ds,$$

and finally

$$\begin{aligned}\sup_{k\in\mathbb{R}^d} \|\hat{u}_1(t,k,\cdot) - \hat{u}_2(t,k,\cdot)\|_{\mathcal{C}_b^{1+\beta}} \lesssim & \int_0^t \Big(1 + \frac{1}{\sqrt{t-s}}\Big) \sup_k \|\hat{u}_1(s,k,\cdot) - \hat{u}_2(s,k,\cdot)\|_{\mathcal{C}_b^{1+\beta}} ds \\ & + \int_0^t \int_0^s \sup_k \|\hat{u}_1(r,k,\cdot) - \hat{u}_2(r,k,\cdot)\|_{\mathcal{C}_b^{1+\beta}} dr ds \\ & + \int_0^t \frac{1}{\sqrt{t-s}} \int_0^s \int_0^r \sup_k \|\hat{u}_1(\tau,k,\cdot) - \hat{u}_2(\tau,k,\cdot)\|_{\mathcal{C}_b^{1+\beta}} d\tau dr ds.\end{aligned}$$

We can conclude by using the Gronwall type lemma of Appendix 3.B, with $A_0 = 0$, and we have $\hat{u}_1 = \hat{u}_2$ for all t, k, ξ. using the bound for $C_1 - C_2$ and $\hat{\rho}_1 - \hat{\rho}_2$, we can conclude that $(u_1, \rho_1, C_1) = (u_2, \rho_2, C_2)$, which ends the proof. □

Proof of Theorem 3.3.2. From the tightness of $\{Q^N\}_{N\in\mathbb{N}}$ and Theorem 3.4.14 we obtain the convergence of subsequences. By Theorem 3.7.1 we obtain the convergence of the full sequence and thus the desired result. □

3.8 Smoothness of the solution

We end by recalling the hypothesis and proving the theorem for the smoothness of the solutions.

- For all $N \geq 0$, $\sup_{\xi\in\mathbb{R}^d} |\hat{G}(\xi)|\big(1 + |\xi|^2\big)^N < \infty$.
- For all $N \geq 0$, $\sup_{\xi\in\mathbb{R}^d} |\hat{K}(k)|\big(1 + |\xi|^2\big)^N < \infty$.
- For all $m \geq 0$, $C_0, K_C \in H^m$ and $K_C \in C_b^m$.

Proof of Theorem 3.3.3. Following Desvillette and Villani [39], one only has to prove that if there exists $n \geq 0$ such that

$$\sup_{t\in[0,T]} \sup_{k,\xi\in\mathbb{R}^d} \big(1 + |k|^2 + |\xi|^2\big)^{\frac{n}{6}} \big|\hat{u}(t, k, \xi)\big| < +\infty, \tag{3.46}$$

then for all $t_0 > 0$,

$$\sup_{t\in[t_0,T]} \sup_{k,\xi\in\mathbb{R}^d} \big(1 + |k|^2 + |\xi|^2\big)^{\frac{n+1}{6}} \big|\hat{u}(t, k, \xi)\big| < +\infty,$$

and then conclude by induction. Note first that

$$|k|^2 + |\xi(r)|^2 = \left(\begin{pmatrix} e^{-2\lambda r} & e^{-\lambda r}\frac{1-e^{-\lambda r}}{\lambda} \\ e^{-\lambda r}\frac{1-e^{-\lambda r}}{\lambda} & 1 + \left(\frac{1-e^{-\lambda r}}{\lambda}\right)^2 \end{pmatrix} \begin{pmatrix} \xi \\ k \end{pmatrix} \right) \cdot \begin{pmatrix} \xi \\ k \end{pmatrix}.$$

Furthermore, the determinant of the previous matrix is equal to $e^{-\lambda r}$, hence there exists a constant $c > 0$ (depending on T) such that for all $r \in [0, T]$,

$$\frac{1}{c}\big(|k|^2 + |\xi|^2\big) \leq |k|^2 + |\xi(r)|^2 \leq c\big(|\xi|^2 + |k|^2\big).$$

Since (3.46) is true for $t = 0$, one have

$$\Big|\hat{u}_0\big(t, k, \xi(t)\big)\Big| \lesssim \Big(1 + |k|^2 + \big|\xi(r)\big|^2\Big)^{-\frac{n}{6}} \lesssim \big(1 + |k|^2 + |\xi|^2\big)^{-\frac{n}{6}}.$$

Furthermore, thanks to the hypoelliptic estimates, we have

$$\exp\left(-\frac{\sigma^2}{2}\int_0^t|\xi(r)|^2dr\right)\lesssim\frac{1}{t^{\frac{2}{3}}}\Big(1+|k|^2+\big|\xi\big|^2\Big)^{-\frac{1}{6}} \tag{3.47}$$

and this close the bound for the first term of (3.36). This also allows us to bound the second last two terms of (3.36). Indeed, since G and K are decaying faster than any polynomials, one has

$$\Big|G\big(\xi(t-s)\big)\Big|\big|\hat{u}(s,k,0)\big|\lesssim\big(1+|\xi(t-s)|^2\big)^{-\frac{n}{6}}\big(1+|k|^2\big)^{-\frac{n}{6}}\lesssim\big(1+|k|^2+|\xi(t-s)|^2\big)^{-\frac{n}{6}}$$

and

$$\begin{aligned}\Big|G\big(\xi(t-s)\big)\Big|\big|\hat{K}(k)\hat{\rho}(t,k)\big|\lesssim_{\rho(T,\mathbb{R}^d)}&\big(1+|\xi(t-s)|^2\big)^{-\frac{n}{6}}\big(1+|k|^2\big)^{-\frac{n}{6}}\\&\lesssim\big(1+|k|^2+|\xi(t-s)|^2\big)^{-\frac{n}{6}}\lesssim\big(1+|k|^2+|\xi|^2\big)^{-\frac{n}{6}}.\end{aligned}$$

Thanks to the bound

$$\big(1+|k|^2+|\xi|^2\big)\lesssim\big(1+|k-k'|^2+|\xi|^2\big)\big(1+|k'|^2+|\xi(t-s)|^2\big),$$

we also have the following bound,

$$\begin{aligned}\Big|\big(\hat{K}\hat{\rho}\big)*\hat{u}(s,\xi(t-s))\Big|\big(1+|k|^2+|\xi|^2\big)^{\frac{n}{6}}\lesssim&\int\big|\hat{K}(k)\big|\big|\hat{\rho}(s,k)\big|\big(1+|k|^2\big)^{\frac{n}{6}}dk\\\lesssim&1.\end{aligned}$$

Hence using (3.47), one can bound the last to lines of (3.36) by a constant times

$$\big(1+|\xi|^2+|k|^2\big)^{-\frac{n+1}{6}}\int_0^t(t-s)^{-\frac{2}{3}}ds.$$

Finally C_0 and K_C are regular enough, $C\in H^m$ for all $m\in\mathbb{N}$. One has

$$\begin{aligned}\Big|\widehat{\nabla C}*\hat{u}(s,k,\xi)\big|\big(1+|k|^2+|\xi|^2\big)^{\frac{n}{6}}\lesssim&\int_{\mathbb{R}^d}\big|\widehat{\nabla C}(t,k)\big|\big(1+|k|^2\big)dk\\\lesssim&\|C(t,\cdot)\|_{H^{2+m+\frac{n}{3}}}\end{aligned}$$

for some $m>\frac{d}{2}$, and one can conlcude by using Lemma 3.5.1. Finally, since $\xi(r)\leq|\xi|+r|k|$ One can bound the last line of (3.36) by

$$\begin{aligned}\big(1+|k|^2+|\xi|^2\big)^{-\frac{n}{6}}&\int_0^t\big(|\xi|+r|k|\big)e^{-\frac{\sigma^2}{2}\int_0^s|\xi(r)|dr}ds\\&\lesssim\big(1+|k|^2+|\xi|^2\big)^{-\frac{n}{6}}\int_0^t\big(|\xi|+r|k|\big)e^{-cs^3|k|^2+s|\xi|^2}ds.\end{aligned}$$

Following Desvillette and Villani [39] Lemma 5.3, one can conclude that the latter is no greater than a constant times

$$\big(1+|k|^2+|\xi|^2\big)^{-\frac{n}{6}},$$

which ends the proof. □

3.A Reminder on curvilinear abscissa

Let us remind that if X is a C^1 curve in $\mathbb{R}^d$, parametrized by $t \in [0,T]$

$$s(t) = \int_0^t \big|X'(r)\big| dr,$$

is the curvilinear abscissa of X. Hence, let us define $\tilde{X}$ such that

$$\tilde{X}\big(s(t)\big) = X(t),$$

where $\tilde{X}$ is parametrized by $t \in \big[0, s(T)\big]$. Hence, we have

$$s'(t)\tilde{X}'\big(s(t)\big) = X'(t),$$

and finally for all $t \in [0, s(T)]$,

$$|\tilde{X}'(t)| = 1,$$

and norm of the velocity of $\tilde{X}$ is 1. Finally, it is possible to parameterize the following spatial set (independently of the speed of the curve) :

$$\mathbb{X}_t = \{X(s), s \in [0,t]\} = \{\tilde{X}\big(s(r)\big), r \in [0,t]\} = \{\tilde{X}(r), r \in [0, s(t)]\}.$$

Hence,

$$\begin{aligned}
\big\langle f, \delta_{\mathbb{X}_t} \big\rangle &= \int_0^{s(t)} f\big(\tilde{X}(r)\big) dr \\
&= \int_0^{s(t)} f\Big(X\big(s^{-1}(r)\big)\Big) dr \\
&= \int_0^t f\big(X(r)\big)\big|X'(r)\big| dr \\
&= \left\langle f, \int_0^t \big|X'(r)\big| \delta_{X(r)} dr \right\rangle.
\end{aligned}$$

Which gives

$$\delta_{\mathbb{X}_t}(dx) = \int_0^t \big|X'(r)\big| \delta_{X(r)}(dx) dr \tag{3.48}$$

3.B A generalized Gronwall Lemma

Lemma 3.B.1. *Let $n \geq 1$. Let $A_0, A_1, a_1, \cdots, A_n, a_n \in \mathbb{R}_+$, $q_1, \cdots, q_n > 1$ and let f be a positive measurable function such that for all $t \in [0,T]$,*

$$f(t) \leq A_0 + A_1 \int_0^t \big(1 + a_1(t-s_1)^{-\frac{1}{q_1}}\big) f(s_1) ds_1 + \cdots \\ + A_n \int_0^t \Big(1 + a_n(t-s_1)^{-\frac{1}{q_n}}\Big) \int_0^{s_1} \cdots \int_0^{s_{n-1}} f(s_n) ds_n \cdots ds_1.$$

There exists a constant $C > 0$ which may depend on all the parameters such that for all $t \in [0,T]$,

$$f(t) \lesssim A_0 e^{Ct}.$$

Proof. Let $q < \min_{1 \leq i \leq n} q_i$ and let $p > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. By using Hölder inequality, and Jensen Inequality, we have, for all $1 \leq k \leq n$

$$\int_0^t \Big(1 + a_k(t-s_1)^{-\frac{1}{q_n}}\Big) \int_0^{s_1} \cdots \int_0^{s_{k-1}} f(s_n) ds_k \cdots ds_1 \\ \lesssim \bigg(\int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} f(s_k)^p ds_k \cdots ds_2 ds_1 \bigg)^{\frac{1}{p}}$$

There exists some constants $B > 0$ depending on T, $A_1, \cdots, A_n$ and $a_1, \cdots, a_n$, $q, q_1, \cdots, q_n$ and n such that

$$f(t) \leq A_0 + B \bigg(\int_0^t g_0(s) + g_1(s)^p + \cdots g_{n-1}(s)^p ds \bigg)^{\frac{1}{p}},$$

where $g_0(t) = f(t)^p$ et $g_k'(t) = g_{k-1}(t)$. Finally, since $g_1(t) + \cdots + g_{n-1}(t) = \int_0^t g_1(s) + \cdots + g_{n-2}(s) ds$, there exists a constant c and a constant $C > 0$ such that

$$g_0(t) + \cdots + g_{n-1}(t) \leq cA_0^p + pC \int_0^t g_0(s) + \cdots + g_{n-1}(s) ds.$$

We conclude by the classical Hölder inequality, and we have

$$f(t) \lesssim A_0 e^{Ct}.$$

□

3.C Toolbox on fractional Laplacian

Let us recall that we define the non local operator $(-\Delta)^{\frac{1}{2}}$ for sufficient regular functions f by the formula

$$(-\Delta)^{\frac{1}{2}} f(x) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}} V.P. \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x-y|^{d+1}} dy,$$

where *V.P.* denotes the principal value. We then have the following lemma :

Lemma 3.C.1. *Let $\beta > 0$. The operator $(-\Delta)^{\frac{1}{2}}$ is well-defined on the space of Hölder continuous functions $\mathcal{C}_b^{1+\beta}(\mathbb{R}^d;\mathbb{R})$ to the space of bounded functions, and for $f, g \in \mathcal{C}_b^{1+\beta}(\mathbb{R}^d;\mathbb{R})$ we have*

$$\|(-\Delta)^{\frac{1}{2}} f - (-\Delta)^{\frac{1}{2}} g\|_{\infty,\mathbb{R}^d} \lesssim \|f-g\|_{\mathcal{C}_b^{1+\beta}}.$$

Proof. First, let us remark that

$$\left| \int_{\mathbb{R}^d \setminus B(x,1)} \frac{f(x) - f(y)}{|x-y|^{d+1}} dy \right| \lesssim \|f\|_\infty$$

since f is a bounded function. Furthermore, for all $1 > \varepsilon > 0$, we have

$$\int_{B(x,1)\setminus B(x,\varepsilon)} Df(x) \frac{y-x}{|x-y|^{d+1}} dy = 0.$$

Finally, remark that

$$\begin{aligned}\int_{B(x,1)\setminus B(x,\varepsilon)} \frac{f(x) - f(y)}{|x-y|^{d+1}} dy &= \int_{B(x,1)\setminus B(x,\varepsilon)} \int_0^1 Df(l(y-x)+x) dl \frac{x-y}{|x-y|^{d+1}} dy \\ &= \int_{B(x,1)\setminus B(x,\varepsilon)} \int_0^1 \big(Df(l(y-x)+x) - Df(x)\big) dl \frac{x-y}{|x-y|^{d+1}} dy.\end{aligned}$$

Hence, one can use the fact the Df is β-Hölder continuous, and one have

$$\left| \int_{B(x,1)\setminus B(x,\varepsilon)} \frac{f(x) - f(y)}{|x-y|^{d+1}} dy \right| \lesssim \|f\|_{\mathcal{C}_b^{1+\beta}}.$$

By using the dominated convergence theorem, $(-\Delta)^{\frac{1}{2}} f$ is well-defined, and we have the wanted bound, since the operator is linear. □

3.D Duhamel formulation of kinetic transport equation

Let $f \in L^\infty\big([0,T]\times\mathbb{R}^d\times\mathbb{R}^d\big)$, us look at the equations for the characteristics lines starting from $\tilde\xi, \tilde k \in \mathbb{R}^d$ of the first order equation

$$\partial_t h(t,k,\xi) - k\cdot\nabla_\xi h(t,k,\xi) + \lambda\xi\cdot\nabla_\xi h(t,k,\xi) = -\frac{|\xi|^2\sigma^2}{2}h(t,k,\xi) + f(t,k,\xi).$$

We have

$$\begin{cases} \tilde\xi'(t) &= -\tilde k'(t) + \lambda\tilde\xi'(t) \\ \tilde k'(t) &= 0 \end{cases}$$

Hence, $\tilde\xi(t) = \Big(\tilde\xi - \frac{\tilde k}{\lambda}\Big)e^{\lambda t} + \frac{\tilde k}{\lambda}$ and $\tilde k(t) = \tilde k$ Finally, one have

$$\partial_t h\big(t,\tilde k,\tilde\xi(t)\big) = -\frac{|\tilde\xi(t)|^2\sigma^2}{2}h\big(t,\tilde k,\tilde\xi(t)\big) + f\big(t,\tilde k,\tilde\xi(t)\big)$$

and by solving this ordinary differential equation, one have

$$h\big(t,\tilde k,\tilde\xi(t)\big) = h_0(\tilde k,\tilde\xi)e^{-\frac{\sigma^2}{2}\int_0^t|\tilde\xi(r)|^2dr} + \int_0^t f\big(s,\tilde k,\tilde\xi(s)\big)e^{-\frac{\sigma^2}{2}\int_s^t|\tilde\xi(r)|^2dr}ds.$$

Now, let us fix $t\in[0,T]$ and let us take $\tilde k = k$ and $\tilde\xi = \Big(\xi - \frac{k}{\lambda}\Big)e^{-\lambda t} + \frac{k}{\lambda}$, such that $\tilde\xi(t) = \xi$ and we have

$$\begin{aligned} h(t,k,\xi) &= h_0\bigg(k, \Big(\xi - \frac{k}{\lambda}\Big)e^{-\lambda t} + \frac{k}{\lambda}\bigg)\exp\bigg(-\frac{\sigma^2}{2}\int_0^t\Big|\Big(\xi-\frac{k}{\lambda}\Big)e^{-\lambda r} + \frac{k}{\lambda}\Big|^2dr\bigg) \\ &+ \int_0^t f\bigg(s,k,\Big(\xi-\frac{k}{\lambda}\Big)e^{-\lambda(t-s)} + \frac{k}{\lambda}\bigg)\exp\bigg(-\frac{\sigma^2}{2}\int_0^{t-s}\Big|\Big(\xi-\frac{k}{\lambda}\Big)e^{-\lambda r} + \frac{k}{\lambda}\Big|^2dr\bigg)ds \\ &= h_0\big(k,\xi(t)\big)e^{-\frac{\sigma^2}{2}\int_0^t|\xi(r)|^2dr} + \int_0^t f\big(s,k,\xi(t-s)\big)e^{-\frac{\sigma^2}{2}\int_0^{t-s}|\xi(r)|^2dr} \end{aligned}$$

with $\xi(r) = \big(\xi - \frac{k}{\lambda}\big)\,e^{-\lambda r} + \frac{k}{\lambda}$.

Part II

Numerical analysis for particle systems and Kolmogorov equation

Chapter 4

The ergodic McKean-Vlasov SDE

4.1 Introduction

The content of this chapter is based on [3]. Diffusion processes are at the core of many algorithms used in statistics to sample from, typically high-dimensional, distributions. These algorithms are often based on some variant of Langevin stochastic dynamics [79]. Given a probability measure π (possibly known up to a normalising constant), the key idea is to construct a diffusion process which admits π as its invariant measure. Then one can run long-time simulations of that diffusion to obtain samples from π. This ideas has been extensively studied in the context of classical SDEs [92, 91, 74, 75, 80, 87, 34, 41].

Recently new promising classes of algorithms based on the theory of gradient flows takes the form of McKean-Vlasov ODEs or SDEs [78, 9, 77]. To turn them into practical algorithms one needs to approximate them with systems of interacting diffusions also called stochastic interacting particle systems. The key challenge is that, typically, with the increase of the dimension of the problem one needs to consider large number N of particles. Because, for most models, the cost of particle samples growths as N^2 (as each particle interacts with others), the computational cost for simulating the particle systems is prohibitive. Another complication is that when using a single ensemble of particles the statistical error due to the approximation of the measure creates biased dynamics. Put differently bias is a non-linear function of the statistical error. In addition particles are not independent. All of that renders classical variance reduction techniques not directly applicable and consequently simulations of particle systems challenging. The high computational cost is even more pronounced when the aim is to simulate particle systems over a long-time horizon. This should not come as a surprise as particle systems give rise to probabilistic numerical methods for highly non-linear PDEs e.g Burgers or Navier-Stokes PDEs.

In this work we leverage recent progress in weak convergence analysis of interacting diffusions [28, 27, 71, 20]. With this new insight we propose several new algorithms and analyse their errors and costs. The emphasis of the work is on algorithmic side and we gloss over some theoretical bounds that will require further research in future. As such we see this work as beacon that helps to identify the most promising research directions in the area of simulations of the ergodic particle systems.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space endowed with an $\mathbb{R}^k$-valued Wiener process $w = (B_t)_{t\geq 0}$. Let $b : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d\times k}$. We consider, for $t \geq 0$, the McKean–Vlasov SDEs (McKV-SDE)

$$\begin{cases} x_t = x_0 + \int_0^t b(x_s, \mu_s)\, ds + \int_0^t \sigma(x_s, \mu_s)\, dB_s \\ \mu_t \text{ is the law of } x_t, \end{cases} \tag{4.1}$$

where x_0 is distributed according to a given $\mathbb{R}^d$-measure μ_0. The nonlinearity in the McKean-Vlasov SDEs (4.1) appears through the dependence of its coefficients on the law of the process. Existence of the unique solution to (4.1) has been established under various conditions on (b, σ). See [89, 82] classical results on that topic that mainly cover the case of finite time interval. For the infinite time horizon we refer to [93, 62].

Furthermore, [53] gives conditions for the existence and uniqueness of the invariant measure π for the equation (4.1). We refer to [43] for more complete theory. In particular [43] gives fairly general conditions that guarantee that the convergence to the invariant measure in the L^2-Wasserstein distance is exponentially fast for some $\lambda > 0$, i.e

$$W_2(\mu_t, \pi) \leq \exp(-\lambda t) W_2(\mu_0, \pi). \tag{4.2}$$

One can also control the bias of ergodic averages

$$\left| \mathbb{E}\left[\frac{1}{t} \int_0^t f(x_s) ds - \int_{\mathbb{R}^d} f(x)\pi(dx) \right] \right| \lesssim t^{-1}. \tag{4.3}$$

Consider the following system of N particles $(x^{1,N}, x^{2,N}, ..., x^{N,N})$ defined as

$$\begin{cases} x_t^{i,N} = x_0^{i,N} + \int_0^t b(x_s^{i,N}, \mu_s^N)\, ds, + \int_0^t \sigma(x_s^{i,N}, \mu_s^N)\, dB_s^i \\ \mu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{x_t^{i,N}}, \end{cases} \tag{4.4}$$

where $(B^i)_i$ are independent k-dimensional Brownian motions, $(x_0^i)_i$ are initial i.i.d. variables independent of $(B_t^i)_i$. The measure valued random variable μ_t^N is an empirical measure of the system at time t. For the purpose of computer simulations one needs to introduce time discretisation to simulate (4.4). We will do that in the forthcoming section. Under classical Lipschitz continuity conditions the law, seen as element of $\mathcal{P}([0,T],\mathbb{R}^d)$, of every fixed subsystem of k particles from $(x^{i,N})$ converges, when N tends to infinity, to the law $\mu^{\otimes k}$. This property is called propagation of chaos phenomenon. Under strong convexity of the drift, a time-uniform version of propagation of chaos has been established in [53]. In a recent work, [40] it has been shown that in general only convexity at infinity is needed.

Let $f:\mathbb{R}^d\to\mathbb{R}$. The objective of this work is to derive, analyse and numerically investigate, several novel particle representations that will allow to approximate:

$$\int_{\mathbb{R}^d} f(x)\pi(dx). \tag{4.5}$$

To motivate our work, let's temporarily assume that (b,σ) do not depend on measure, i.e we are dealing with a classical SDEs. Then a typical strategy in obtaining an approximation to (4.5) would be to set a finite time t and take N i.i.d. trajectories to compute $\frac{1}{t}\int_0^t \frac{1}{N}\sum_{i=1}^N f(x_s^i)ds$ ($N=1$ corresponds to ergodic estimator). The error of this estimator can be decomposed as follows

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{\mathbb{R}^d} f(x)\pi(dx) - \frac{1}{t}\int_0^t \frac{1}{N}\sum_{i=1}^N f(x_s^i)ds\right)^2\right]^{1/2} \\
&\qquad\leq \mathbb{E}\left[\left(\int_{\mathbb{R}^d} f(x)(\pi(dx)-\mu_t(dx))\right)^2\right]^{1/2} \\
&\qquad\quad + \mathbb{E}\left[\left(\int_{\mathbb{R}^d} f(x)\mu_t(dx) - \frac{1}{t}\int_0^t \frac{1}{N}\sum_{i=1}^N f(x_s^i)ds\right)^2\right]^{1/2}.
\end{aligned}$$

The first term in the right hand side is the (weak) error of approximating the invariant measure which decays to zero as $\exp(-\lambda t)$ due to (4.2). The second term is CLT type result and can be shown to decay to zero as $(N\cdot t)^{-1/2}$, see e.g [69, 22]). We see that both t and N have the same impact on the variance. In the case of SDEs the cost of simulations is linear in t and N and hence one may be indifferent whether to simulate one long trajectory (ergodic estimator) and many shorter ones (space average). Of course if one uses parallel computer architecture, taking more samples is much more efficient.

The situation of McKean-Vlasov SDE (4.1) is dramatically different. The cost of simulating interacting particles (4.4) is N^2 while it still increases linearly with time. As we will show it is possible to construct estimator that has one-order of magnitude lower cost while maintains the same accuracy. Furthermore, we will investigate *ensemble* version of interacting diffusions where we generate M independent systems of particles with N particles in each system (ensemble). More precisely we define

$$\begin{cases} X_t^{(i,N),(j,M)} = X_0^{(i,N),(j,M)} + \int_0^t b\big(X_s^{(i,N),(j,M)}, S_s^{N,j}\big)\, ds + \int_0^t \sigma\big(X_s^{(i,N),(j,M)}, S_s^{N,j}\big)\, dB_s^{i,j}, \\ S_t^{N,j} = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \delta_{X_t^{(i,N),(j,M)}}, \quad i = 1, \dots, N, \quad j = 1, \dots, M. \end{cases} \tag{4.6}$$

where $(B^{i,j}, i, j)$ are independent Brownian motions. That way particles within each ensemble j^* driven by $(B^{i,j^*})_{i,j^*}$ are interacting and are not independent. The particle systems j and j', $j \neq j'$, driven by $(B^{i,j})_{i,j}$ and $(B^{i,j'})_{i,j}$ respectively, are independent. This idea for finite time simulations has been proposed in [60]. Another approach that we investigate are self-interacting diffusion

$$z_t = x_0 + \int_0^t \left(\frac{1}{s}\int_0^s b(z_s, z_r)dr\right) ds + \int_0^t \left(\frac{1}{s}\int_0^s \sigma(z_s, z_r)dr\right) dB_s \tag{4.7}$$

We expect that the law of z_t approximates the law of x_t for large t due to ergodic property (4.2,4.3). This gives an alternative to the particle system. We will show that the structure of the equation seems to play a crucial role in this set up.

The rest of the chapter is organised as follows. In Section 4.2, we recall some classical methods to approximate (4.5) and give error estimations and computational costs of the associated algorithms. In Section 4.3 we study several variants of algorithms for ergodic interacting particle systems In Section 4.4, we present the ensemble algorithm with ergodic average particle system. We end this chapter with a general conclusion and some perspectives in Section 4.5.

4.2 Setup

4.2.1 Algorithms

As in this work we are interested in designing implementable algorithms for (4.1), we need to introduce time discretisation for (4.4). Let us define $t_k^n := \frac{k}{n}$,

$k = 0, 1, \ldots$ and $\kappa_n(t) = t_k^n$ for $t \in [t_k^n, t_{k+1}^n)$. We introduce (continuous time) Euler approximations for each $i \leq N$, $(y_t^{i,N}, t \geq 0)$, $n \in \mathbb{N}$,

$$\begin{cases} y_t^{i,N} = y_0^{i,N} + \int_0^t b(y_{\kappa_n(s)}^{i,N}, \bar{\mu}_{\kappa_n(s)}^N)\, ds + \int_0^t \sigma(y_{\kappa_n(s)}^{i,N}, \bar{\mu}_{\kappa_n(s)}^N)\, dB_s^i, \\ \bar{\mu}_{\kappa_n(t)}^N = \frac{1}{N} \sum_{i=1}^N \delta_{y_{\kappa_n(t)}^{i,N}}. \end{cases} \tag{4.8}$$

One may approximate (4.5) by ergodic average estimator with fixed t,

$$\frac{1}{t} \int_0^t f(y_{\kappa_n(s)}^{1,N}) ds\,. \tag{EA}$$

For error analysis one needs to choose N (number of particles) and n (number of timesteps) to control the bias of the approximation of (4.1). While (EA) estimator is a reasonable choice for computing approximation of (4.5) for the invariant measures induced by classical SDEs as we already argued, the case of McKean-Vlasov SDEs approximated with a particle system, (EA) estimator does not seem to be the best choice. This is because when using particle system (4.4), one computes N particles and therefore calculating ergodic average along one trajectory is not efficient. See sections 4.3.1, 4.3.2 for more details. Hence, improvement can be obtained by computing, averaged ergodic average estimator

$$\frac{1}{N} \sum_{i=1}^N \left(\frac{1}{t'} \int_0^{t'} f(y_{\kappa_n(s)}^{i,N}) ds \right). \tag{AEA}$$

Of course one expects that t' in (AEA) to be smaller than t in (EA) for fixed accuracy. Alternative strategy for approximating (4.5) is to resort to the standard Monte Carlo estimator where the average is taken only "over the space". More precisely we compute Monte Carlo average

$$\frac{1}{N} \sum_{i=1}^N f(y_t^{i,N}). \tag{MCA}$$

Of course, the above estimator is less efficient than (AEA), as we explain in the coming section. The only reason we study it here is to warn practitioners that if not careful with setting up particle estimators the cost might be huge.

Computational Cost.

By $\mathcal{A}(\eta)$ we denote an algorithm that outputs the approximation for the quantity (4.5), where η denotes the set of all the parameters we need to choose to implement

it. To be able to compare the algorithms we need to fix a measure of error. For simplicity, we resort to the mean-square-error:

$$mse(\mathcal{A}(\eta)) := \mathbb{E}\left[\left(\int_{\mathbb{R}^d} f(x)\pi(dx) - \mathcal{A}(\eta)\right)^2\right]^{1/2}. \tag{4.9}$$

With the measure of error of a given estimator set up, the second equally important quantity is the computation cost of algorithm $\mathcal{A}$, denoted by $cost(\mathcal{A})$. With both quantities in place we can wonder about the optimal choice of parameters achieving a prescribed tolerance. More precisely, for fixed error tolerance $\varepsilon > 0$, we need to solve the following optimisation problem:

$$\begin{cases} mse(\mathcal{A}(\eta)) < \epsilon\,, \\ \min\limits_{\eta} cost(\mathcal{A}(\eta))\,. \end{cases}$$

4.2.2 Assumptions

In this section we list all the assumptions needed for our considerations. The only assumption that has not been yet established in the literature is uniform in time particle error (HW) estimation below. To the best of authors knowledge only finite time weak particle error has been studied. It is clear how to extend the weak convergence to be uniform in time but this would require lengthy introduction of heavy machinery of PDEs on measure spaces. This falls outside this chapter. All other assumptions are established in literature under various level of generality and we point out reader to the corresponding papers.

We label by x^i a McKean-Vlasov SDE driven by ith Brownian motion, that is

$$x_t^i = x_0^i + \int_0^t b(x_s^i, \mu_s)ds + \int_0^t \sigma(x_s^i, \mu_s)dB_s^i\,. \tag{4.10}$$

Assumption 4.2.1. Convergence rate to ergodic measure: there exists $\lambda > 0$ such that

$$W_2(\mu_t, \pi) \lesssim \exp(-\lambda t) W_2(\mu_0, \pi). \tag{HE}$$

As we already mentioned this has been proved in [53] and [43] under fairly general conditions.

Assumption 4.2.2. Convergence rate of ergodic average

$$\left(\mathbb{E}\left[\left(\int_{\mathbb{R}^d} f(x)\mu_t(dx) - \frac{1}{t}\int_0^t f(x_s)ds\right)^2\right]\right)^{1/2} \lesssim \frac{(\sup_{s\in[0,t]} \mathbb{V}ar[x_s])^{1/2}}{\sqrt{t}}. \tag{HEA}$$

This is classical CLT result. See [22].

Assumption 4.2.3. Uniform in time weak convergence of the particle system: for sufficiently smooth f,

$$\sup_{t\geq 0}|\mathbb{E}f(x_t^1)-\mathbb{E}f(x_t^{1,N})|\lesssim\frac{1}{N}. \tag{HW}$$

This type of bound is new in the literature. We refer a reader to [28, 27, 71, 20] for more details.

Assumption 4.2.4. Uniform in time strong propagation of chaos

$$\sup_{t\geq 0}(\mathbb{E}|x_t^1-x_t^{1,N}|^2)^{1/2}\lesssim\frac{1}{\sqrt{N}}. \tag{HS}$$

See [40] for details.

Assumption 4.2.5. Uniform in time weak discretisation error: for sufficiently smooth f,

$$\sup_{t\geq 0}|\mathbb{E}f(x_t^1)-\mathbb{E}f(y_t^{1,N})|\lesssim\frac{1}{N}. \tag{HDW}$$

Uniform in time strong discretisation error

$$\sup_{t\geq 0}(\mathbb{E}|x_t^1-y_t^{1,N}|^2)^{1/2}\lesssim\frac{1}{\sqrt{N}}. \tag{HDS}$$

One can refer to [53], where such results are proved.

4.3 Algorithms for Ergodic Interacting Particle systems

4.3.1 Monte Carlo Average

For the Monte Carlo Average estimator we introduce the following notation $\mathcal{A}^{MCA}(t,n,N)$. The aim is to find the optimal allocation of the parameters (t,n,N)

for fixed mean-square-error. We have

$$
\begin{aligned}
mse(\mathcal{A}^{MCA}(t,n,N)) &= \mathbb{E}\left[\left(\int_{\mathbb{R}^d} f(x)\pi(dx) - \frac{1}{N}\sum_{i=1}^{N} f(y_t^{i,N})\right)^2\right]^{1/2} \\
&\lesssim \left|\int_{\mathbb{R}^d} f(x)(\pi(dx) - \mu_t(dx))\right| + \left|\mathbb{E}[f(x_t)] - \mathbb{E}[f(x_t^{1,N})]\right| \\
&+ \left|\mathbb{E}[f(x_t^{1,N})] - \mathbb{E}[f(y_t^{1,N})]\right| + \mathbb{E}\left[\left(\mathbb{E}[f(y_t^{1,N})] - \frac{1}{N}\sum_{i=1}^{N} f(y_t^{i,N})\right)^2\right]^{1/2}.
\end{aligned}
$$

The four error terms are in order: bias (due to finite time simulation); weak particle approximation error; weak time discretisation error; variance/propagation of chaos. The first three error terms can be estimated directly from the Assumptions in Section 4.2.2. The last variance error term requires extra comment, as the fact that particles are not i.i.d does not allow to use classical central limit theorem (CLT). Indeed

$$
\begin{aligned}
\mathbb{E}[(\mathbb{E}[f(y_t^{1,N})] - \frac{1}{N}\sum_{i=1}^{N} f(y_t^{i,N}))^2]^{1/2} &= \mathbb{E}[(\mathbb{E}[\frac{1}{N}\sum_{i=1}^{N} f(y_t^{i,N})] - \frac{1}{N}\sum_{i=1}^{N} f(y_t^{i,N}))^2]^{1/2} \\
&\leq \mathbb{E}[(\frac{1}{N}\sum_{i=1}^{N} f(y_t^{i,N}))^2]^{1/2}.
\end{aligned}
$$

Next, define $(\tilde{y}^i)_i$ as the solution of the continuous Euler scheme:

$$
\begin{cases}
\tilde{y}_t^i = \tilde{y}_0^i + \displaystyle\int_0^t b(\tilde{y}_{\kappa_n(s)}^i, \mathscr{L}(\tilde{y}_{\kappa_n(s)}))\, ds + \int_0^t \sigma(\tilde{y}_{\kappa_n(s)}^i, \mathscr{L}(\tilde{y}_{\kappa_n(s)}))\, dB_s^i\,, \\
\mathscr{L}(\tilde{y}_t) = Law(\tilde{y}_t)\,.
\end{cases}
$$

It is an easy exercise to show that strong propagation of chaos (HS) can be established on the level of Euler discretisation. This together with Cauchy-Schwarz

inequality gives

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N} f(y_t^{i,N}) - f(\tilde{y}_t^i)\right)^2\right] \\
&= \mathbb{E}\left[\frac{1}{N^2}\sum_{i=1}^{N}(f(y_t^{i,N}) - f(\tilde{y}_t^i))^2 + \frac{1}{N^2}\sum_{i<j}^{N}(f(y_t^{i,N}) - f(\tilde{y}_t^i))f(y_t^{j,N}) - f(\tilde{y}_t^j))\right] \\
&\leq \frac{1}{N^2}\sum_{i=1}^{N}\mathbb{E}\left[(f(y_t^{i,N}) - f(\tilde{y}_t^i))^2\right] \\
&\qquad + \frac{1}{N^2}\sum_{i<j}^{N}(\mathbb{E}[(f(y^{i,N}) - f(\tilde{y}_t^{i,N}))^2])^{1/2}(\mathbb{E}[(f(y_t^j) - f(\tilde{y}_t^j))^2])^{1/2} \lesssim \frac{1}{N}\,.
\end{aligned} \tag{4.11}$$

This, and the fact that $(\tilde{y}_t^i)_i$ are i.i.d. allows to conclude that

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N} f(y_t^{i,N})\right)^2\right] &= \mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N} f(\tilde{y}_t^i) + (f(y_t^{i,N}) - f(\tilde{y}_t^i)\right)^2\right] \\
&\lesssim \mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N} f(\tilde{y}_t^i)\right)^2\right] + \mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N} f(y_t^{i,N}) - f(\tilde{y}_t^i)\right)^2\right] \lesssim \frac{1}{N}.
\end{aligned}$$

From here and Assumptions in Section 4.2.2 we have

$$mse(\mathcal{A}^{MCA}(t,n,N)) \lesssim e^{-\lambda t} + \frac{1}{N} + \frac{1}{n} + \frac{1}{\sqrt{N}}.$$

Notice that because of the term $1/\sqrt{N}$, it is not clear how we can take advantage of the assumption (HW). Fix $\epsilon > 0$ and set $mse(\mathcal{A}^{MCA}(t,n,N)) \lesssim \epsilon$. This leads to the following choice of the parameters $t \approx \lambda^{-1}\log(\epsilon^{-1}), N \approx \epsilon^{-2}, n \approx \epsilon^{-1}$. As the cost of simulating particle system at every step of the Euler scheme is N^2 we have

$$cost(\mathcal{A}^{MCA}(t,n,N)) = tnN^2 \approx \log(\epsilon^{-1})\epsilon^{-5}.$$

This should be compared to $tnN = \log(\epsilon^{-1})\epsilon^{-3}$ for the simulation of standard SDEs.

4.3.2 Averaged Ergodic Average

As before, we denote by $\mathcal{A}^{AEA}(t,n,N)$ the averaged ergodic average estimator in (AEA). We have

$$mse(\mathcal{A}^{AEA}(t,n,N)) = \mathbb{E}\left[\left(\int_{\mathbb{R}^d} f(x)\pi(dx) - \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{t}\int_0^t f(y^{i,N}_{\kappa_n(s)})ds\right)\right)^2\right]^{1/2}$$
$$\lesssim \left|\int_{\mathbb{R}^d} f(x)(\pi(dx)-\mu_t(dx))\right| + \left|\mathbb{E}[f(x_t)] - \mathbb{E}[f(x_t^{1,N})]\right|$$
$$+\left|\mathbb{E}[f(x_t^{1,N})] - \mathbb{E}[f(y_t^{1,N})]\right| + \mathbb{E}\left[\left(\mathbb{E}[f(y_t^{1,N})] - \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{t}\int_0^t f(y^{i,N}_{\kappa_n(s)})ds\right)\right)^2\right]^{1/2}.$$

To estimate the variance term we note that

$$\mathbb{E}[f(y_t^{1,N})] - \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{t}\int_0^t f(y^{i,N}_{\kappa_n(s)})ds\right)$$
$$= \mathbb{E}[\frac{1}{N}\sum_{i=1}^{N}(y_t^{i,N})] - \left(\frac{1}{t}\int_0^t \frac{1}{N}\sum_{i=1}^{N} f(y^{i,N}_{\kappa_n(s)})ds\right).$$

Hence to estimate the variance we use (HEA) combined with the computation (4.11). Therefore, by the Assumptions in Section 4.2.2, we have

$$mse(\mathcal{A}^{AEA}(t,n,N)) \lesssim e^{-\lambda t} + \frac{1}{N} + \frac{t}{n} + \frac{1}{\sqrt{tN}}.$$

We notice that comparing to the (MCA) case, the last term is multiplied by $1/\sqrt{t}$. The (asymptotic) cost of the algorithm is the same as before. Again we fix ϵ. The following choice of the parameters ensures that $mse(\mathcal{A}^{AEA}(t,n,N)) \lesssim \epsilon^2$ is $t \approx \epsilon^{-1}$, $N \approx \epsilon^{-1}$, $n \approx \epsilon^{-1}$. The cost consists of two parts: the cost of simulating particle system and the cost of computing averaged ergodic estimator. We have

$$cost(\mathcal{A}^{AEA}(t,n,N)) = tnN^2 + tN \approx \epsilon^{-4}\,.$$

Which is an order of magnitude lower than for Monte Carlo average!

Notice that similar computation for ergodic average estimator gives $mse(\mathcal{A}^{EA}(t,n,N)) \lesssim e^{-\lambda t} + \frac{1}{N} + \frac{1}{n} + \frac{1}{\sqrt{t}}$, leading to the same cost than Monte Carlo average.

4.3.3 ensemble AEA

For the ensemble version of the algorithm we generate M independent systems of particles with N particles in each system. More precisely we define,

$$\begin{cases} x_t^{(i,N),(j,M)} = x_0^{(i,N),(j,M)} + \int_0^t b(x_s^{(i,N),(j,M)}, \mu_s^{N,j})\, ds + \int_0^t \sigma(x_s^{(i,N),(j,M)}, \mu_s^{N,j})\, dB_s^{i,j}, \\ \mu_t^{N,j} = \frac{1}{N}\sum_i^N \delta_{x_t^{(i,N),(j,M)}}, \end{cases} \tag{4.12}$$

where $(B^{i,j}, i, j)$ are independent Brownian motions. That way particle within each cloud j^* driven by $(B^{i,j^*})_{i,j^*}$ are interacting and are not independent. The particle systems j and j', $j \neq j'$, driven by $(B^{i,j})_{i,j}$ and $(B^{i,j'})_{i,j}$ respectively, are independent. This idea has been proposed in [60] for the finite time simulations.

We consider ensemble version of AEA.

$$\frac{1}{M}\sum_{j=1}^M \frac{1}{N}\sum_{i=1}^N \left(\frac{1}{t}\int_0^t f(y_{\kappa_n(s)}^{(i,N),(j,M)})ds\right). \tag{C-AEA}$$

In fact, all algorithms that we study can have their ensemble versions. By denoting $\mathcal{A}^{C\text{-}AEA}$ the new method

$$\begin{aligned} &mse(\mathcal{A}^{C\text{-}AEA}(t,n,N,M)) = \\ &\quad \mathbb{E}\left[\left(\int_{\mathbb{R}^d} f(x)\pi(dx) - \frac{1}{M}\sum_{j=1}^M \frac{1}{N}\sum_{i=1}^N \left(\frac{1}{t}\int_0^t f(y_{\kappa_n(s)}^{(i,N),(j,N)})ds\right)\right)^2\right]^{1/2} \\ &\lesssim \left|\int_{\mathbb{R}^d} f(x)(\pi(dx) - \mu_t(dx))\right| + \left|\mathbb{E}[f(x_t)] - \mathbb{E}[f(x_t^{1,N})]\right| + \left|\mathbb{E}[f(x_t^{1,N})] - \mathbb{E}[f(y_t^{1,N})]\right| \\ &\qquad + \mathbb{E}\left[\left(\mathbb{E}[f(y_t^{1,N})] - \frac{1}{M}\sum_{j=1}^M \frac{1}{N}\sum_{i=1}^N \left(\frac{1}{t}\int_0^t f(y_{\kappa_n(s)}^{(i,N),(j,M)})ds\right)\right)^2\right]^{1/2}. \end{aligned}$$

The Assumptions in Section 4.2.2 yield

$$mse(\mathcal{A}^{C\text{-}AEA}(t,n,N,M)) \lesssim e^{-\lambda t} + 1/N + 1/n + \frac{1}{\sqrt{tNM}}.$$

We notice that comparing to the previous case the last term is multiplied by $1/\sqrt{M}$. Crucially, the cost of the algorithm growths linearly in M. As the cost growths as

N^2, we are better off taking $M \approx N$ to balance the error in the last term (instead of taking $M = 1$ and N^2). To make it precise we fix ϵ. The following choice of the parameters ensures that $mse(\mathcal{A}^{C\text{-}AEA}(t, n, N)) \lesssim \epsilon$.

$$t \approx \lambda^{-1} \log(\epsilon^{-1}),\ N \approx \epsilon^{-1},\ n \approx \epsilon^{-1},\ M = (\lambda^{-1} \log(\epsilon^{-1}))^{-1} \epsilon^{-1}.$$

The cost of simulating particles and computing the estimator is

$$cost(\mathcal{A}^{C\text{-}AEA}(t, n, N)) = tnN^2M + tNM \approx \epsilon^{-4}.$$

This is the same as for averaged ergodic estimator. However the above computations do not take under consideration the fact that ensemble algorithms can take full advantage from the parallel computer architecture and therefore will be superior in practice.

4.4 Algorithms for Sef-interacting Particle systems

In this section, we present the key ideas of improvement in the definition of more efficient algorithm. From the decomposition of the mean square error, we see that different algorithms that we considered only affected the "variance" of the final estimator. Therefore, to improve the efficiency of the algorithm we need to either modify particle system itself or consider different simulation strategies such as Multilevel-Monte Carlo. Here we focus on the former.

Significance of the ergodic theorem is that one can approximate the integral (4.5) by simulating only one path of the process (4.1) rather then the whole particle system. From now on, we will keep the structural assumptions on the coefficients of (4.1), namely

$$\begin{cases} x_t = x_0 + \displaystyle\int_0^t b(x_s, \mathscr{L}(x_s))\, ds + \int_0^t \sigma(x_s, \mathscr{L}(x_s))\, dB_s \\ \mathscr{L}(x_t) \text{ is the law of } x_t. \end{cases}$$

One may consider

$$z_t = x_0 + \int_0^t \left(\frac{1}{s} \int_0^s b(z_s, z_r) dr \right) ds + \int_0^t \left(\frac{1}{s} \int_0^s \sigma(z_s, z_r) dr \right) dB_s\,. \tag{4.13}$$

We expect that the law of z_t approximates the law of x_t for large t due to ergodic property (4.2)

Processes of the form (4.13) are known in literature as self interacting diffusions. We refer to [70] where the convergence to the invariant measure has been established.

Notice that there is no need for the particle system any more as one could simply simulate one path of the process to calculate ergodic integral (4.5).

However, motivated by computations in the previous section where mixed ergodic/Monte Carlo average we introduce the corresponding mean self-interacting SDE

$$z_t = x_0 + \int_0^t \left(\frac{1}{s} \int_0^s b(z_s, \mathscr{L}(z_r))dr \right) ds + \int_0^t \left(\frac{1}{s} \int_0^s \mathbb{E}\sigma(z_s, \mathscr{L}(z_r))dr \right) dB_s \quad (4.14)$$

and its independent copies (z^i) driven by Brownian motion (B^i). Note that "one-particle" approximation of (4.14) is precisely a self-interacting diffusion.

To gain better insight, into the idea of using self-interacting diffusions to approximate McKean-Vlasov SDEs we consider a simple example first.

Example 4.4.1. Consider a simple scalar McKean-Vlasov SDE x, together with its mean self-integrated version y, and its self-interaction motion SDE z,

$$\begin{aligned} x_t &= x_0 - \int_0^t \alpha x_s ds + \int_0^t \beta \mathbb{E}[x_s] ds + B_t, \\ z_t &= x_0 - \int_0^t \alpha z_s ds + \int_0^t \frac{1}{s} \left(\int_0^s \beta \mathbb{E}[z_\theta] d\theta \right) ds + B_t. \end{aligned}$$

We stress out that dissipativity comes from the part of the drift that does not depend on measure. We assume $\alpha > \beta$. To estimate the convergence rate to the invariant measure we analyse the evolution of the difference of two solutions to the above x SDE initiated at L^2 random variables ξ_1 and ξ_2. With

$$\mathbb{E}[(x_t^{\xi_1} - x_t^{\xi_2})] = e^{-(\alpha-\beta)t} \mathbb{E}[(\xi_1 - \xi_2)], \quad (4.15)$$

we have

$$\begin{aligned} e^{2\alpha t} \mathbb{E}[(x_t^{\xi_1} - x_t^{\xi_2})^2] =& \mathbb{E}[(\xi_1 - \xi_2)^2] + 2\beta \int_0^t e^{2\alpha s} (\mathbb{E}[x_s^{\xi_1} - x_s^{\xi_2}])^2 ds \\ =& \mathbb{E}[(\xi_1 - \xi_2)^2] + 2(\mathbb{E}[\xi_1 - \xi_2])^2 \beta \int_0^t e^{2\alpha s} e^{-2(\alpha-\beta)s} ds \\ =& \mathbb{E}[(\xi_1 - \xi_2)^2] + (\mathbb{E}[\xi_1 - \xi_2])^2 (e^{2\beta t} - 1). \end{aligned}$$

Due to properties of W_2 distance and the fact that the above calculation does not depend on a particular choice of random variables ξ_1 and ξ_2, we have

$$W_2^2(\mathscr{L}(x_t^{\xi_1}), \mathscr{L}(x_t^{\xi_2})) \lesssim e^{-2(\alpha-\beta)t} W_2^2(\mathscr{L}(\xi_1), \mathscr{L}(\xi_2)) .$$

Furthermore, in this simple example we can take advantage from the explicit solutions to calculate

$$\mathbb{E}[x_t] - \frac{1}{t} \int_0^t \mathbb{E}[x_s] ds = \mathbb{E}[x_0] \left(e^{-(\alpha-\beta)t} - \frac{1}{t(\alpha-\beta)} (1 - e^{-(\alpha-\beta)t}) \right) .$$

Hence if $\log t < (\alpha - \beta)t$ then we have that

$$|\mathbb{E}[x_t] - \frac{1}{t}\int_0^t \mathbb{E}[x_s]ds| \leq c1/t\,.$$

Let us consider now process z. By integration by part we have that

$$\mathbb{E}[z_t] = \mathbb{E}[z_0] - \int_0^t \alpha\mathbb{E}[z_\theta]d\theta + \beta\int_0^t \log(t/\theta)\mathbb{E}z_\theta d\theta$$

and

$$e^{\alpha t}\mathbb{E}[z_t] = \mathbb{E}[z_0] + \int_0^t e^{\alpha s}\frac{1}{s}\int_0^s \mathbb{E}z_\theta d\theta\,.$$

Then we observe that if $\mathbb{E}[z_0] > 0$, then $\mathbb{E}[z_t]$ stays non-negative and do not cross 0. We observe also that $t \mapsto \mathbb{E}[z_t]$ is decreasing as for all $t \geq 0$, for all $s \in [te^{-\alpha/\beta}, t]$

$$\mathbb{E}[z_t] - \mathbb{E}[z_s] = \int_s^t (\beta\log(t/\theta) - \alpha)\,\mathbb{E}z_\theta\,d\theta \leq 0.$$

In particular

$$\frac{\mathbb{E}[z_t] - \mathbb{E}[z_s]}{t-s} = \frac{1}{t-s}\int_s^t (\beta\log(t/\theta) - \alpha)\,\mathbb{E}z_\theta d\theta \leq \frac{1}{t-s}\int_s^t (\beta\log(t/\theta) - \alpha)\,d\theta.$$

Taking limit $s \to t$, we obtain after integration that

$$\mathbb{E}[z_t] \leq \mathbb{E}[z_0]\exp(-(\alpha-\beta)t)\,. \tag{4.16}$$

The following computation on z is a tentative to evidence the rate's gain in the convergence rate to equilibrium. To this aim, we analyse the evolution of the difference of two solutions z initiated at L^2 random variables ξ_1 and ξ_2.

Repeating the previous computation for $\mathbb{E}[\xi_1 - \xi_2] \geq 0$, we also obtain

$$0 \leq \mathbb{E}[(z_t^{\xi_1} - z_t^{\xi_2})] \leq e^{-(\alpha-\beta)t}\mathbb{E}[(\xi_1 - \xi_2)],$$

and we use this weak estimation to derive the L^2-norm bound. We have first that

$$\begin{aligned}
e^{2\alpha t}\mathbb{E}[(z_t^{\xi_1} - z_t^{\xi_2})^2] &= \mathbb{E}[(\xi_1-\xi_2)^2] + 2\beta\int_0^t e^{2\alpha s}\mathbb{E}[z_s^{\xi_1} - z_s^{\xi_2}]\frac{1}{s}\left(\int_0^s \mathbb{E}[z_\theta^{\xi_1} - z_\theta^{\xi_2}]d\theta\right)ds \\
&\leq \mathbb{E}[(\xi_1-\xi_2)^2] + 2\beta(\mathbb{E}[\xi_1-\xi_2])^2\int_0^t e^{2\alpha s}e^{-(\alpha-\beta)s}\frac{1}{(\alpha-\beta)s}(1-e^{-(\alpha-\beta)s})ds \\
&\leq \mathbb{E}[(\xi_1-\xi_2)^2] + 2\beta(\mathbb{E}[\xi_1-\xi_2])^2\int_0^t \frac{1}{(\alpha-\beta)s}(e^{(\alpha+\beta)s} - e^{2\beta s})ds\,.
\end{aligned}$$

As $\alpha > \beta$, it is not difficult to check that for t big enough,

$$0 \leq \int_0^t \frac{1}{s}(e^{(\alpha+\beta)s} - e^{2\beta s})ds \leq \frac{2}{t}\left(\frac{e^{(\alpha+\beta)t}}{\alpha+\beta} - \frac{e^{2\beta t}}{2\beta}\right) - \frac{2}{t}\left(\frac{1}{\alpha+\beta} - \frac{1}{2\beta}\right).$$

And we obtain the contraction inequality,

$$\mathbb{E}[(z_t^{\xi_1} - z_t^{\xi_2})^2] \leq e^{-2\alpha t}\mathbb{E}[(\xi_1 - \xi_2)^2] + c\frac{e^{-(\alpha-\beta)t}}{t}(\mathbb{E}[\xi_1 - \xi_2])^2,$$

which means that after a time $t \geq t_0$, the convergence in L^2 is exponentially fast with a rate $\alpha \wedge (\alpha + \log(t) - \beta)/2$, that accelerates and becomes with time better than the rate for process x (in $\alpha - \beta$).

We have also to show that x and z have the same equilibrium measure. A sufficient condition is the L^2-convergence for z toward x in time. To not spend to much time on this example, we make the following assumption from the previous Wasserstein contraction for z and from what we obtain for x, that

$$|\mathbb{E}[z_t] - \frac{1}{t}\int_0^t \mathbb{E}[z_s]ds| \leq c1/t\,.$$

We consider now

$$e^{2(\alpha-\beta)t}\mathbb{E}(x_t - z_t)^2 \leq e^{2(\alpha-\beta)}\mathbb{E}(x_1 - z_1)^2 - 2\beta\int_1^t e^{2(\alpha-\beta)s}\mathbb{E}(x_s - z_s)^2 ds$$
$$+ 2\beta\int_1^t e^{2(\alpha-\beta)s}\mathbb{E}(x_s - z_s)\left(\mathbb{E}[x_s] - \frac{1}{s}\int_0^s \mathbb{E}[z_\theta]d\theta\right)ds.$$

But since $|\mathbb{E}(x_s - z_s)| \leq |\mathbb{E}(x_s)| + |\mathbb{E}(z_s)| \leq ce^{-(\alpha-\beta)s}$ from (4.15) and (4.16),

$$\int_1^t e^{2(\alpha-\beta)s}\mathbb{E}(x_s - z_s)\left(\mathbb{E}[x_s] - \frac{1}{s}\int_0^s \mathbb{E}[z_\theta]d\theta\right)ds$$
$$\leq \int_1^t e^{2(\alpha-\beta)s}\mathbb{E}(x_s - z_s)^2 + \int_1^t e^{2(\alpha-\beta)s}|\mathbb{E}(x_s - z_s)|\frac{c}{s}ds$$
$$\leq \int_1^t e^{2(\alpha-\beta)s}\mathbb{E}(x_s - z_s)^2 + \int_1^t e^{(\alpha-\beta)s}\frac{c'}{s}ds$$

and we have obtained

$$e^{2(\alpha-\beta)t}\mathbb{E}(x_t - z_t)^2 \leq e^{2(\alpha-\beta)}\mathbb{E}(x_1 - z_1)^2 + 2\beta\int_1^t e^{(\alpha-\beta)s}\frac{c'}{s}ds$$
$$\mathbb{E}(x_t - z_t)^2 \leq e^{-2(\alpha-\beta)(t-1)}\mathbb{E}(x_1 - z_1)^2 + Ce^{-(\alpha-\beta)t}\log(t)\,.$$

The convergence is then ensured. Hence, mean self-interacting version z of x is converging to x in L^2, which means that z convergence to the equilibrium measure π and hence can used as an alternative model for sampling.

Self-averaged ergodic averaged algorithm

By considering error decomposition studies in the previous section, we see that the key ingredient that we ought to understand is the following rate of convergence

Key ideas: here we want to take part of a special structure, i.e

$$\begin{cases} x_t = x_0 + \int_0^t V(x_s) + W(x_s, \mathscr{L}(x_s))\, ds + \int_0^t \sigma\, dB_s \\ \mathscr{L}(x_t) \text{ is the law of } x_t, \end{cases}$$

such that potential V is say convex and W has a small Lipschitz constant we should obtain exponential "forgetting property", as we observed in Example 4.4.1.

For small time this will be bad approximation and because we average the error the bad approximation from the initial time will prevail.

Further, we consider particle system of the form

$$z_t^{i,N} = x_0^{i,N} + \int_0^t \frac{1}{N} \sum_{j=1}^{N} \left(\frac{1}{s} \int_0^s b(z_s^{i,N}, z_r^{j,N}) dr \right) ds + \int_0^t \frac{1}{N} \sum_{j=1}^{N} \left(\frac{1}{s} \int_0^s \sigma(z_s^{i,N}, z_r^{j,N}) dr \right) dB_s^i \quad (4.17)$$

for which the error to be investigated is

$$\left| \mathbb{E}\left[f(z_t^1) \right] - \mathbb{E}\left[f(z_t^{1,N}) \right] \right|,$$

to have in place of (HW). According to our computation on example 4.4.1, we impose the following assumption

The following bound gives a leading error term. We chose λ as a exponent for simplicity as it particular value does not affect asymptotic cost/error analysis.

Assumption 4.4.2.

$$\left| \mathbb{E} f(z_t^1) - \mathbb{E}[f(z_t^{1,N_t})] \right| \lesssim \frac{e^{-\lambda t}}{N_t}. \quad \text{(HEW)}$$

The time-discretisation of the equation (4.17) reads as

$$r_t^{i,N_t} = x_0^{i,N_t} + \int_0^t \frac{1}{N_t} \sum_{j=1}^{N_t} \left(\frac{1}{s} \int_0^s b(r_{\kappa_n(s)}^{i,N_t}, r_{\kappa_n(\theta)}^{j,N_t}) d\theta \right) ds + \int_0^t \frac{1}{N_t} \sum_{j=1}^{N_t} \left(\frac{1}{s} \int_0^s \sigma(r_{\kappa_n(s)}^{i,N_t}, r_{\kappa_n(\theta)}^{j,N_t}) d\theta \right) dB_s^i. \quad (4.18)$$

Let us introduce the corresponding estimator, an ensemble self-integrated version of AEA.

$$\frac{1}{M}\sum_{j=1}^{M}\frac{1}{N_t}\sum_{i=1}^{N_t}\left(\frac{1}{t}\int_0^t f(r_{\kappa_n(s)}^{(i,N_t),(j,M)})ds\right). \tag{CS-AEA}$$

In what follows, we test (HEW) in the cost analysis. Notice that we cannot test it directly, as we need to do Monte-Carlo approximation for the expectation. We first observe that by exchangeability of the law of the particle systems,

$$\mathbb{E}\left[f(z_t^1)\right]-\mathbb{E}\left[f(z_t^{1,N})\right]=\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}f(z_t^i)\right]-\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}f(z_t^{i,N})\right].$$

Next we consider M independent ensembles to approximate expectations that is

$$\begin{aligned}
&\left|\frac{1}{M}\sum_{j=1}^{M}\frac{1}{N}\sum_{i=1}^{N}f(z_t^{ij})-\frac{1}{M}\sum_{j=1}^{M}\frac{1}{N}\sum_{i=1}^{N}f(z_t^{(i,N),(j,M)})]\right| \\
&\quad\leq\left|\frac{1}{M}\sum_{j=1}^{M}\frac{1}{N}\sum_{i=1}^{N}f(z_t^{ij})-\mathbb{E}[f(z_t^1)]\right| \\
&\qquad+\left|\mathbb{E}[f(z_t^1)]-\mathbb{E}[f(z_t^{1,N})]\right|+\left|\mathbb{E}[f(z_t^{1,N})]-\frac{1}{M}\sum_{j=1}^{M}\frac{1}{N}\sum_{i=1}^{N}f(z_t^{(i,N),(j,M)})\right| \\
&\quad\lesssim(\sqrt{MN})^{-1}+(Nt)^{-1}+(\sqrt{MN})^{-1}
\end{aligned}$$

where the first and last error are standard MC estimates while the middle one is given by (HEW).

4.4.1 Cost analysis

We analyse the cost of the self-averaged ergodic averaged estimator

$$mse(\mathcal{A}^{ES\text{-}AEA}(t,n,N))=\mathbb{E}\left[\left(\int_{\mathbb{R}^d}f(x)\pi(dx)-\frac{1}{N_t}\sum_{i=1}^{N_t}\left(\frac{1}{t}\int_0^t f(r_{\kappa_n(s)}^{i,N_t})ds\right)\right)^2\right]^{1/2}.$$

The mean-square error decomposition reads

$$\begin{aligned} &mse(\mathcal{A}^{ES\text{-}AEA}(t,n,N_t)) \\ &\qquad \lesssim \left| \int_{\mathbb{R}^d} f(x)(\pi(dx) - \mu_t(dx)) \right| + \left| \mathbb{E}[f(y_t)] - \mathbb{E}[f(z_t^{1,N_t})] \right| \\ &\qquad\qquad + \left| \mathbb{E}[f(z_t^{1,N_t})] - \mathbb{E}[f(r_t^{1,N_t})] \right| \\ &\qquad + \left(\mathbb{E}\left[\left(\mathbb{E}[f(r_t^{1,N_t})] - \frac{1}{N_t} \sum_{i=1}^{N_t} \left(\frac{1}{t} \int_0^t f(r^{i,N_t}_{\kappa_n(s)}) ds \right) \right)^2 \right] \right)^{1/2} \end{aligned}$$

Reasoning as before, with (HW) replaced by (HEW), we have

$$mse(\mathcal{A}^{ES\text{-}AEA}(t,n,N_t)) \lesssim e^{-\lambda t} + e^{-\lambda t}(N_t)^{-1} + 1/n + 1/\sqrt{tN_t}.$$

Note that because of the variance (last term) there is no benefit of exponential decay in t assumed in HEW. Again we fix ϵ. The following choice of the parameters ensures that $mse(\mathcal{A}^{ES\text{-}AEA}(t,n,N_t)) \lesssim \epsilon$:

$$t \approx \lambda^{-1} \log(\epsilon^{-1}), \quad tN_t \approx \epsilon^{-2}, \quad n \approx \epsilon^{-1}.$$

Notice that the choice implies that tN_t to be "constant". Hence we chose $N_t = Nt^{-1}$. Which in the case of $t \approx \lambda^{-1} \log(\epsilon^{-1})$ implies that $N \approx \epsilon^{-2}/\log(\epsilon^{-1})$.

Now we study the computational cost of simulating self-interacting diffusions (for the nonlinear interacting kernel). Note that, due nonlinear interactions at every step of the Euler scheme, we have N_t particles and each particle interacts with itself from all the past times-steps. Recall also that we take n time steps in each unit time interval so that overall number of steps on the interval $[0,t]$ is tn. With that in mind and the fact that we take $N_t = Nt^{-1}$ we have

$$\begin{aligned} &cost(\mathcal{A}^{ES\text{-}AEA}(t,n,N)) = N_{1/n}^2 1 + N_{2/n}^2 2 + N_{3/n}^2 3 + \ldots + N_{tn/n}^2 tn \\ &= (nN)^2(1 + 1/2 + 1/3 + \ldots + 1/tn) = (nN)^2 \sum_{k=1}^{tn} 1/k \approx (nN)^2 \log(1+tn). \end{aligned}$$

Hence the cost for the set of parameters $\{t \approx \lambda^{-1} \log(\epsilon^{-1}),\ N \approx \epsilon^{-2}/\log(\epsilon^{-1}),\ n \approx \epsilon^{-1}\}$ is

$$cost(\mathcal{A}^{ES\text{-}AEA}(t,n,N)) \approx \epsilon^{-6}$$

Note also that one can also take

$$t \approx \epsilon^{-2}, \quad (N_t = 1 \implies tN_t = \epsilon^{-2}), \quad n \approx \epsilon^{-1}$$

to ensure $mse(\mathcal{A}^{ES\text{-}AEA}(t,n,N_t)) \lesssim \epsilon$. This second choice, with N_t not varying with time, leads to

$$\begin{aligned} &cost(\mathcal{A}^{ES\text{-}AEA}(t,n,N)) \\ &= N^2 1 + N^2 2 + N^2 3 + \ldots + N^2 tn = N^2 \frac{1}{2} tn(1+tn) \approx \epsilon^{-6}. \end{aligned}$$

Let us consider ensemble implementation of the above algorithm. Reasoning as before, by the Assumption HEW, we have

$$mse(\mathcal{A}^{ES\text{-}AEA}(t,n,N_t)) \lesssim e^{-\lambda t} + e^{-\lambda t}(N_t)^{-1} + 1/n + 1/\sqrt{tN_tM}.$$

To balance the first two terms on the right hand side we take $N_t = 1$. With that choice we then chose M so that $e^{-\lambda t} = (tM)^{-1/2}$, i.e $M = e^{2\lambda t}t^{-1}$. With this choices we have

$$mse(\mathcal{A}^{ES\text{-}AEA}(t,n,N_t)) \lesssim e^{-\lambda t} + 1/n,$$

to make that error to be less then ϵ we take $t = \lambda^{-1}\log\epsilon^{-1}$ and $n = \epsilon^{-1}$. Note that this leads to $M = e^{\log(\epsilon^{-2})}(\log(\epsilon^{-1}))^{-1} = \epsilon^{-2}(\log(\epsilon^{-1}))^{-1}$. Reasoning as before the cost with $N_t = 1$ is

$$cost(\mathcal{A}^{CS\text{-}AEA}(t,n,N_t,M)) \approx M(tn)^2 \approx \epsilon^{-4}\log(\epsilon^{-1}).$$

Hence one more time we achieved order better computational cost in comparison to naive estimator. Note that the presented analysis implies that ensemble of M independent self-interacting diffusions yields best result.

4.5 Conclusion and perspectives

We presented a number of different algorithms for the approximation of the invariant measure of the McKean-Vlasov SDE. We have achieved one order better computational costs comparing to naive particle based estimator. On algorithmic side possible extensions are that may consider fixed length window for self-interacting diffusion and it also possible to study Multilevel Monte Carlo strategies in this setup. Overall we anticipate that it will be possible to bring the cost of simulating particle system to the same level as for standard independent copies of SDEs.

Chapter 5

A numerical approach to Kolmogorov equation in high dimension based on Gaussian analysis

5.1 Introduction

The content of this chapter is based on the work [51]. Kolmogorov equations are parabolic equations with a structure directly related to stochastic differential equations (SDEs). The SDEs considered here are in a finite dimensional space but they are inspired by the spatial discretization of stochastic Partial Differential Equations (SPDEs). When the noise is additive and the nonlinearity is time-independent, a general form of such SDEs is

$$\begin{cases} \mathrm{d}X_t = (AX_t + B(X_t))\,\mathrm{d}t + \sigma\sqrt{Q}\,\mathrm{d}W_t, \\ X_0 = x, \end{cases} \tag{5.1}$$

where $x \in \mathbb{R}^d$, $(W_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^d$ (namely $W_t = \left(W_t^1, \ldots, W_t^d\right)$ where the W_t^i's are independent real valued Brownian motions), defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$, σ is a positive real number measuring the strength of the noise, Q is a $d \times d$ positive definite symmetric matrix (the so called covariance matrix of the noise) describing the spatial structure of the noise and $\sqrt{Q}$ is its square root, A is a $d \times d$ matrix and $B : \mathbb{R}^d \to \mathbb{R}^d$ is a function with the degree of regularity specified below. Obviously we could include the scalar σ^2 inside the matrix Q but for certain practical arguments it is useful to

distinguish between them. The solution X_t is a continuous adapted process in $\mathbb{R}^d$. The associated Kolmogorov equation is

$$\begin{cases} \partial_t u(t,x) = \dfrac{\sigma^2}{2}\mathrm{Tr}\left(QD^2u(t,x)\right) + \langle Ax + B\left(x\right), Du(t,x)\rangle\,, \\ u(0,x) = u_0(x), \end{cases} \tag{5.2}$$

where $u : [0,T]\times\mathbb{R}^d \to \mathbb{R}$, $Du(t,x)$ and $D^2u(t,x)$ denote respectively the vector of first partial derivatives and the matrix of second partial derivatives, $\mathrm{Tr}\left(QD^2u(t,x)\right)$ is the trace of the $d\times d$ matrix $QD^2u(t,x)$ and $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^d$. Both for the SDE and the Kolmogorov equation we have used notations which may be adapted to the infinite dimensional case, when $\mathbb{R}^d$ is replaced by a Hilbert space (see Section 5.2 for the general theory); however, the aim of this work is numerical and all objects in the introduction will belong to $\mathbb{R}^d$. The link between the Kolmogorov equation and the SDE is

$$u(t,x) = \mathbb{E}\left[u_0(X_t^x)\right],$$

where $\mathbb{E}$ denotes the mathematical expectation on $(\Omega,\mathcal{F},\mathbb{P})$ and X_t^x is the solution of the SDE above, where the initial condition x is explicitly indicated. Several elements of theory both in finite and infinite dimensions for SDEs and associated Kolmogorov equations can be found in many books, like [23, 31, 33, 67, 72].

Solving the Kolmogorov equation with suitable initial condition u_0 is a way to compute relevant expected values and probabilities associated to the solution of an SDE. For instance, when $u_0(x) = 1_{\{\|x\|>R\}}$, $u(t,x)$ is the probability that the solution exceeds a threshold R:

$$u(t,x) = \mathbb{E}\left[1_{\{\|x\|>R\}}\left(X_t^x\right)\right] = \mathbb{P}\left(\|X_t^x\| > R\right).$$

The classical method of computing these expected values is the Monte-Carlo method (with important variants, see for instance [54, 90, 17, 76]): several realizations of the process X_t^x are simulated by solving the SDE – typically by Euler method – and then the corresponding values of $u_0(X_t^x)$ are averaged. Going beyond this strategy is a fundamental issue, due to its limitations in relevant applications like Geophysics and Climate change projections [66], especially concerning extreme events. The question is whether Kolmogorov equation can be efficiently solved numerically without using the simulation of the SDE. But the problem is that the dimension d is extremely high in these examples and common numerical methods for solution of parabolic equations already require strong computational power when $d = 3$, [18, 73]. A grid of N points in $\mathbb{R}$, repeated for all dimensions, give rise to N^d grid points, numerically impossible when, for instance, $N = 10$, $d = 10$ (which still would be an extremely poor approximation). Spectral methods seem

to meet the same restrictions: N^d is the cardinality of basis elements obtained by tensorization of N basis elements for each space variable.

The problem of dimensionality, the limitations of present methodologies and several motivations are recalled in two recent works [5, 65] which also aim to go beyond Monte-Carlo and propose a method based on deep artificial neural networks. We address to these brilliant works for other comments on the problem, see also [30, Introduction]. The approach developed here is however completely different.

Our aim is to take advantage of the probabilistic structure of the problem to devise numerical schemes for the Kolmogorov equation, in particular using Gaussian analysis. We implement a perturbative scheme which links the solution of Kolmogorov equation to a Gaussian process, the solution Z_t of the linear stochastic equation

$$\begin{cases} \mathrm{d}Z_t = AZ_t\,\mathrm{d}t + \sqrt{Q}\,\mathrm{d}W_t, \\ Z_0 = 0. \end{cases}$$

The idea comes from the theoretical investigations of infinite dimensional Kolmogorov equations associated to SPDEs, see for instance [33, 32]. We modify and adapt that idea giving an explicit formula in terms of a series of Gaussian integrals. We provide here a first glance at the strategy by writing the final formula:

$$u(t,x) = \sum_{n=0}^{\infty} v^n(t,x),$$

where

$$v^0(t,x) = \mathbb{E}\left[u_0\left(e^{tA}x + \sigma Z_t\right)\right]$$

and for $n \geq 1$

$$v^n(t,x) = \int_0^t \mathrm{d}r_n \int_0^{r_n} \mathrm{d}r_{n-1} \cdots \int_0^{r_2} \mathrm{d}r_1$$
$$\mathbb{E}\left[u_0\big(e^{tA}x + \sigma Z_t\big) \prod_{i=1}^{n} \left\langle \Xi_\sigma(r_{i+1} - r_i) B\big(e^{r_i A}x + \sigma Z_{r_i}\big), Z_{r_{i+1}} - e^{(r_{i+1}-r_i)A} Z_{r_i} \right\rangle\right].$$

The matrix $\Xi_\sigma(t)$ will be defined in the next sections, see (5.5); it is easily computed by A and Q, and it depends on the parameters t and σ. A theoretical analysis of this series is made, proving the following result.

Theorem 5.1.1. *Assume that u_0 and B are bounded. Then, under suitable conditions on A and Q (see Hypothesis 5.2.1 for details), we have the following uniform estimate:*

$$\|v^n(t)\|_\infty \leq \|u_0\|_\infty \|B\|_\infty^n C_\delta^n\, t^{n(1-\delta)} \frac{\Gamma(1-\delta)^n}{\Gamma(1+n(1-\delta))}, \quad t > 0,$$

where $\Gamma(\cdot)$ *is the Gamma function,* $C_\delta > 0$ *is a constant and* $\delta \in (0,1)$ *the parameter in (iv) of Hypothesis 5.2.1.*

This theorem sustains the numerical method and stresses the independence on the dimension of certain issues of the method (obviously others, like getting a sample of Z, have a cost which increases with d). When $\mathbb{R}^d$ is replaced by a Hilbert space H (and below we shall formulate the theorem with assumptions in a Hilbert space) it contains also some theoretical novelties with respect to the literature, especially because it provides an explicit formula.

The numerical evaluation of the terms $v^n(t,x)$ is made here, in this chapter, by Monte-Carlo method based on a sample of the process Z_t obtained by solving the linear SDE by Euler method. These are the most obvious choices, but other possibilities exist, since $(Z_t)_{t\geq 0}$ is a centered Gaussian process with known covariance function. A main strategy invoked here is to store once for ever a large and accurate sample of $(Z_t)_{t\geq 0}$ (this requires the pair (A,Q) to be given) and use it later in the formula for different values of the other parameters, t, x, σ, u_0 and even B.

This new method is aimed to replace direct Monte-Carlo simulations. We should therefore accurately compare them. If the purpose is to make one single computation, classical Monte-Carlo wins: the Gaussian method above still requires Monte-Carlo simulations of the linear problem, which is less expensive than the nonlinear one but then one has to compute possibly several terms $v^n(t,x)$; some experiments clearly show that classical Monte-Carlo is less expensive for a comparable degree of precision. The advantage comes when we want to vary parameters, since the Gaussian method for given (A,Q) allows to store a possibly expensive sample of the process Z_t and reuse it for several values of the parameters, just having to compute the averages over the Gaussian sample which give us the terms $v^n(t,x)$. On the contrary, classical Monte-Carlo method requires to repeat the simulation of the nonlinear problem for each new value of the parameters. By "parameters", as we have already mentioned above, we mean t, x, σ, u_0, B. Let us comment on the interest in changing them.

The interest in changing t is obvious. In certain applications it is necessary to change the initial condition x and compare or collect the results. We have in mind for instance the ensemble methods used in weather prediction where the initial condition is uncertain, a first guess is made on the basis of physical observations, but then the initial condition is perturbed in various directions and the final results averaged by suitable methods. See also [5, 65], where the need to change (t,x) is stressed.

Changing the strength σ of the noise is a very important issue, related also to Large Deviation Theory. We have to advise that the precision of our simulations degenerates as $\sigma \to 0$, or the number of iterates needed to maintain a reasonable precision blows-up, but at least one can detect some tendency by moving σ in a

finite range without arriving to too small values.

Concerning the change of function u_0, unfortunately the main comment is in favor of Monte-Carlo: having at disposal a sample of the process X_t^x immediately gives a way to compute $\mathbb{E}\left[u_0(X_t^x)\right]$ for different functions u_0. Hence the best we can say on this issue is that our formula allows for such computations with a moderate additional effort – but not with an improvement over Monte-Carlo.

Finally, changing the nonlinearity B is of theoretical interest for the investigation of the performances of the method, and in applications it may be of interest in those – very common – cases when some parameters of B are not precisely known and different simulations may be useful for comparison or for ensemble averaging methods performed over the range of those parameters.

Let us finally come to a brief description of numerical results. In Section 3, we present some numerical results based on the method proposed here in the finite dimensional settings with $d \geq 10$. The results, even if not fully satisfactory yet, should be compared with the fact that the innovative attempts to solve the Kolmogorov equation in $d > 3$ by direct methods, see [30], are often restricted to dimensions smaller than 10. Large dimension is therefore a very difficult problem that deserves strong effort for improvement, and some of our results – although not in all examples – are quite promising.

As a final comment, let us explicitly mention that the class of Kolmogorov equations studied here is particular, because of the additive and very non-degenerate noise and because we have treated only relatively mild nonlinearities. We have not considered relevant cases from fluid mechanics which have more severe nonlinearities and activation of more scales; after a few initial tests on dyadic models – we point in particular to the recent models on trees which may be very relevant for turbulence theory, see [4, 10, 11] – it was clear that covering these examples with this approach requires further research and improvements. Extension to multiplicative transport noises [49, 50] is another challenging open question.

5.2 The iteration scheme for Kolmogorov equations on Hilbert spaces

In this section we work in an infinite dimensional separable Hilbert space H and study the iteration scheme for the Kolmogorov equation:

$$\partial_t u(t,x) = \frac{1}{2}\mathrm{Tr}\big(QD^2u(t,x)\big) + \big\langle Ax + B(x), Du(t,x)\big\rangle, \quad u(0,\cdot) = u_0. \tag{5.3}$$

Here $A : D(A) \subset H \to H$ is an unbounded linear operator, Q is a nonnegative self-adjoint bounded linear operator on H, $B : D(B) \subset H \to H$ is a nonlinear

measurable mapping and $u_0 : H \to \mathbb{R}$ is a real valued measurable function. In this section Q plays the role of $\sigma^2 Q$ to simplify notation. In the following we write $\mathcal{L}(H, H)$ for the Banach space of bounded linear operators on H with the norm $\|\cdot\|_{\mathcal{L}(H)}$.

Throughout this section we assume the following conditions:

Hypothesis 5.2.1. (i) $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup e^{tA}.

(ii) Q is a nonnegative self-adjoint operator in $\mathcal{L}(H, H)$ satisfying $\mathrm{Ker}(Q) = \{0\}$, and for any $t > 0$ the linear operator

$$Q_t = \int_0^t e^{sA} Q e^{sA^*} \, \mathrm{d}s \tag{5.4}$$

is of trace class.

(iii) We have $e^{tA}(H) \subset Q_t^{1/2}(H)$ for any $t > 0$.

(iv) Letting $\Lambda(t) = Q_t^{-1/2} e^{tA}$, we assume there exist $\delta \in (0, 1)$ and $C_\delta > 0$ such that

$$\|\Lambda(t)\|_{\mathcal{L}(H)} \leq C_\delta / t^\delta, \quad t > 0.$$

The assumptions (i)–(iii) are quite standard in the literature, see for instance [31, Hypothesis 2.1 and 2.24]. The operator $\Xi_\sigma(t)$ appeared in the introduction has the form

$$\Xi_\sigma(t) = \sigma Q_t^{-1/2} \Lambda(t) = \sigma Q_t^{-1} e^{tA}; \tag{5.5}$$

we remark that, in the setting of the introduction, the operator Q in (5.4) should be replaced by $\sigma^2 Q$ when computing Q_t. The following example is taken from [31, Example 2.5] which verifies all the assumptions.

Example 5.2.2. Let $\mathcal{O} = [0, \pi]^d$ with $d \in \mathbb{N}$. We choose $H = L^2(\mathcal{O})$, and

$$Ax = \Delta x, \quad x \in D(A) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}),$$

where Δ is the Laplacian operator with Dirichlet boundary condition. A is a self-adjoint negative operator in H, and

$$Ae_k = -|k|^2 e_k, \quad k \in \mathbb{N}^d,$$

where for $k \in \mathbb{N}^d$, $|k|^2 = k_1^2 + \cdots + k_d^2$ and

$$e_k(\xi) = (2/\pi)^{d/2} \sin(k_1 \xi_1) \cdots \sin(k_d \xi_d), \quad \xi \in [0, \pi]^d.$$

Choose $Q = (-A)^{-\alpha}$, $\alpha \in [0, 1)$, so that

$$Qx = \sum_{k \in \mathbb{N}^d} |k|^{-2\alpha} \langle x, e_k \rangle e_k, \quad x \in H.$$

For any $t > 0$, if $\alpha > d/2 - 1$, then

$$\mathrm{Tr}(Q_t) = \sum_{k \in \mathbb{N}^d} \frac{1}{2|k|^{2+2\alpha}} \Big(1 - e^{-2t|k|^2}\Big) < \infty.$$

So (ii) is satisfied.

Next, (iii) can be checked by explicit computations. Moreover,

$$\Lambda(t)x = \sum_{k \in \mathbb{N}^d} \frac{\sqrt{2}\,|k|^{1+\alpha}}{\sqrt{e^{2t|k|^2} - 1}} \langle x, e_k \rangle e_k, \quad x \in H.$$

From this we deduce that

$$\|\Lambda(t)\|_{\mathcal{L}(H)} \leq \frac{\sqrt{2C_\alpha}}{t^{(1+\alpha)/2}},$$

where

$$C_\alpha = \sup_{\theta > 0} \frac{\theta^{1+\alpha}}{e^{2\theta} - 1} < +\infty.$$

Thus (iv) holds with $\delta = (1 + \alpha)/2 \in [1/2, 1)$.

We also need the following technical conditions.

Hypothesis 5.2.3. The initial datum $u_0 : H \to \mathbb{R}$ and the nonlinear part $B : H \to H$ in (5.3) are bounded and measurable.

This section is organized as follows. In Subsection 5.2.1, we recall some basic facts in Gaussian analysis on Hilbert space and give the formula for the first term $v^1(t, x)$ of the iteration (5.10). We give in Section 5.2.2 the details for calculating the second term $v^2(t, x)$, which will help us to guess and prove the formula for general terms $v^n(t, x)$ in Section 5.2.3. In the last part, we estimate the uniform norm of $v^n(t, x)$ and show the convergence of the iteration scheme. The limit is the unique mild solution of (5.3), see Theorem 5.2.14.

5.2.1 Some preparations

Let W be a cylindrical Brownian motion on H:

$$W_t = \sum_{k=1}^{\infty} W_t^k e_k, \quad t \geq 0,$$

where $\{e_k\}_{k\geq 1}$ is a complete orthonormal basis of H and $\{W^k\}_{k\geq 1}$ is a family of independent one dimensional standard Brownian motions defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Under the conditions (i) and (ii) in Hypothesis 5.2.1, the linear SDE

$$\mathrm{d}Z_t^x = AZ_t^x\,\mathrm{d}t + \sqrt{Q}\,\mathrm{d}W_t, \quad Z_0^x = x \in H \tag{5.6}$$

has a unique solution with the expression

$$Z_t^x = e^{tA}x + W_A(t), \quad t > 0,$$

where $W_A(t)$ is the stochastic convolution:

$$W_A(t) = \int_0^t e^{(t-s)A}\sqrt{Q}\,\mathrm{d}W_s.$$

For any $t > 0$, $W_A(t)$ is a centered Gaussian variable on H with covariance operator Q_t. We denote its law by $N_{Q_t}(\mathrm{d}y)$. Accordingly, the law of Z_t^x is denoted as $N_{e^{tA}x,Q_t}(\mathrm{d}y)$. Recall that for any $h \in H$, $\langle h, Q_t^{-1/2}W_A(t)\rangle$ is a centered real Gaussian variable with variance

$$\mathbb{E}\langle h, Q_t^{-1/2}W_A(t)\rangle^2 = |h|_H^2.$$

We shall write $\mathcal{B}(H)$ for the space of bounded measurable functions on H and $C_b^1(H)$ the space of Fréchet differentiable functions, bounded with bounded derivatives. When $f \in C_b^1(H)$, its Fréchet derivative will be denoted by Df. For any $f \in \mathcal{B}(H)$ and $t \geq 0$, let

$$S_tf(x) := \mathbb{E}f(Z_t^x) = \int_H f(y)\,N_{e^{tA}x,Q_t}(\mathrm{d}y) = \int_H f\big(e^{tA}x + y\big)\,N_{Q_t}(\mathrm{d}y).$$

This defines a Markov semigroup on H. We have the following important result which implies S_t is strong Feller (see [31, Proposition 2.28] for a proof).

Proposition 5.2.4. *Assume the conditions (i)–(iii) in Hypothesis 5.2.1. Then for all $f \in \mathcal{B}(H)$ and $t > 0$, we have $S_tf \in C_b^1(H)$ and for any $h \in H$,*

$$\langle h, DS_tf(x)\rangle = \mathbb{E}\big[f(Z_t^x)\big\langle \Lambda(t)h, Q_t^{-1/2}\big(Z_t^x - e^{tA}x\big)\big\rangle\big]. \tag{5.7}$$

Moreover,

$$\|DS_tf\|_\infty \leq \|f\|_\infty \|\Lambda(t)\|_{\mathcal{L}(H)}. \tag{5.8}$$

Using the semigroup S_t, the *mild* formulation of the Kolmogorov equation (5.3) is

$$u(t,x) = (S_tu_0)(x) + \int_0^t \big(S_{t-s}\langle B, Du(s)\rangle\big)(x)\,\mathrm{d}s. \tag{5.9}$$

This suggests us to consider the iterative scheme:

$$u^{n+1}(t,x) = (S_t u_0)(x) + \int_0^t \big(S_{t-s}\langle B, Du^n(s)\rangle\big)(x)\,\mathrm{d}s$$

with $u^0(t,x) = (S_t u_0)(x) = \mathbb{E}u_0(Z_t^x)$. We define $v^0(t,x) = u^0(t,x)$ and

$$v^n(t,x) = u^n(t,x) - u^{n-1}(t,x), \quad n \geq 1,$$

then the new functions satisfy the iteration procedure:

$$\begin{cases} v^{n+1}(t,x) = \displaystyle\int_0^t (S_{t-s}k_s^n)(x)\,\mathrm{d}s, \\ k_s^n(y) = \langle B(y), Dv^n(s,y)\rangle, \\ v^0(t,x) = \mathbb{E}u_0(Z_t^x). \end{cases} \tag{5.10}$$

Before concluding this section, we show how to obtain the first term $v^1(t,x)$. Since $u_0 \in \mathcal{B}(H)$, Proposition 5.2.4 implies $v^0(t) \in C_b^1(H)$ for any $t > 0$, and thus $\langle B, Dv^0(t)\rangle \in \mathcal{B}(H)$. Denote by $\mathcal{F}_t$ the filtration generated by the cylindrical Brownian motion W_t.

Lemma 5.2.5. *It holds that*

$$\big(S_{t-s}k_s^0\big)(x) = \mathbb{E}\Big[u_0(Z_t^x)\big\langle \Lambda(s)B(Z_{t-s}^x), Q_s^{-1/2}\big(Z_t^x - e^{sA}Z_{t-s}^x\big)\big\rangle\Big].$$

Proof. Use the property of conditional expectation:

$$\begin{aligned} &\mathbb{E}\Big[u_0(Z_t^x)\big\langle \Lambda(s)B(Z_{t-s}^x), Q_s^{-1/2}(Z_t^x - e^{sA}Z_{t-s}^x)\big\rangle\Big] \\ &= \mathbb{E}\Big\{\mathbb{E}\Big[u_0(Z_t^x)\big\langle \Lambda(s)B(Z_{t-s}^x), Q_s^{-1/2}(Z_t^x - e^{sA}Z_{t-s}^x)\big\rangle\big|\mathcal{F}_{t-s}\Big]\Big\} \\ &= \mathbb{E}\Big\{\mathbb{E}\Big[u_0(Z_t^x)\big\langle \Lambda(s)B(Z_{t-s}^x), Q_s^{-1/2}(Z_t^x - e^{sA}Z_{t-s}^x)\big\rangle\big|Z_{t-s}^x\Big]\Big\}, \end{aligned}$$

where the second step follows from the Markov property. Again by the Markov property,

$$\begin{aligned} &\mathbb{E}\Big[u_0(Z_t^x)\big\langle \Lambda(s)B(Z_{t-s}^x), Q_s^{-1/2}(Z_t^x - e^{sA}Z_{t-s}^x)\big\rangle\big|Z_{t-s}^x\Big] \\ &= \mathbb{E}\Big[u_0(Z_s^y)\big\langle \Lambda(s)B(y), Q_s^{-1/2}(Z_s^y - e^{sA}y)\big\rangle\Big]_{y=Z_{t-s}^x} \\ &= k_s^0(y)\big|_{y=Z_{t-s}^x} = k_s^0(Z_{t-s}^x), \end{aligned}$$

where the second step is due to (5.7). Substituting this equality into the previous one we obtain the identity. □

The above lemma implies

Corollary 5.2.6. *For any $t > 0$ and $x \in H$,*

$$v^1(t,x) = \int_0^t \mathbb{E}\Big[u_0(Z_t^x)\big\langle \Lambda(s)B(Z_{t-s}^x), Q_s^{-1/2}\big(Z_t^x - e^{sA}Z_{t-s}^x\big)\big\rangle\Big]\,\mathrm{d}s. \tag{5.11}$$

Moreover,

$$\|v^1(t)\|_\infty \le \|u_0\|_\infty \|B\|_\infty \int_0^t \|\Lambda(s)\|_{\mathcal{L}(H)}\,\mathrm{d}s$$

and

$$\|Dv^1(t)\|_\infty \le \|u_0\|_\infty \|B\|_\infty \int_0^t \|\Lambda(t-s)\|_{\mathcal{L}(H)}\|\Lambda(s)\|_{\mathcal{L}(H)}\,\mathrm{d}s.$$

Proof. The formula (5.11) follows directly from Lemma 5.2.5. Next, by the definition (5.10) of the iteration, for any $s > 0$ and $y \in H$,

$$\big|k_s^0(y)\big| \le |B(y)|\,|Dv^0(s,y)| \le \|B\|_\infty |DS_s u_0(y)| \le \|B\|_\infty \|u_0\|_\infty \|\Lambda(s)\|_{\mathcal{L}(H)}, \tag{5.12}$$

where the last inequality follows from (5.8). Therefore,

$$|v^1(t,x)| \le \int_0^t \big|\big(S_{t-s}k_s^0\big)(x)\big|\,\mathrm{d}s \le \int_0^t \big\|k_s^0\big\|_\infty\,\mathrm{d}s \le \|u_0\|_\infty\|B\|_\infty \int_0^t \|\Lambda(s)\|_{\mathcal{L}(H)}\,\mathrm{d}s$$

which yields the estimate on $\|v^1(t)\|_\infty$. The inequality (5.12) implies that $k_s^0 \in \mathcal{B}(H)$ for all $s > 0$, hence by Proposition 5.2.4, $S_{t-s}k_s^0 \in C_b^1(H)$ and

$$Dv^1(t,x) = \int_0^t D\big(S_{t-s}k_s^0\big)(x)\,\mathrm{d}s.$$

Finally, by (5.8),

$$\|Dv^1(t)\|_\infty \le \int_0^t \big\|D\big(S_{t-s}k_s^0\big)\big\|_\infty\,\mathrm{d}s \le \int_0^t \big\|k_s^0\big\|_\infty \|\Lambda(t-s)\|_{\mathcal{L}(H)}\,\mathrm{d}s,$$

which, together with (5.12), gives us the last estimate. □

5.2.2 The term $v^2(t,x)$

In this part, we compute the second term in the iteration to illustrate the ideas. First we prove

Lemma 5.2.7. *One has*

$$\begin{aligned} k_t^1(x) = \int_0^t \mathbb{E}\Big[&u_0(Z_t^x)\big\langle \Lambda(s)B(Z_{t-s}^x), Q_s^{-1/2}(Z_t^x - e^{sA}Z_{t-s}^x)\big\rangle \\ &\times \big\langle \Lambda(t-s)B(x), Q_{t-s}^{-1/2}(Z_{t-s}^x - e^{(t-s)A}x)\big\rangle\Big]\,\mathrm{d}s. \end{aligned}$$

Proof. By Corollary 5.2.6, for any $t > 0$, $v^1(t) \in C_b^1(H)$ and

$$k_t^1(x) = \big\langle B(x), Dv^1(t,x)\big\rangle = \int_0^t \big\langle B(x), D\big(S_{t-s}k_s^0\big)(x)\big\rangle \,\mathrm{d}s.$$

Recall that (5.12) implies $k_s^0 \in \mathcal{B}(H)$, thus by Proposition 5.2.4,

$$k_t^1(x) = \int_0^t \mathbb{E}\Big[k_s^0(Z_{t-s}^x)\big\langle \Lambda(t-s)B(x), Q_{t-s}^{-1/2}\big(Z_{t-s}^x - e^{(t-s)A}x\big)\big\rangle\Big] \,\mathrm{d}s.$$

According to the proof of Lemma 5.2.5, we have

$$k_s^0(Z_{t-s}^x) = \mathbb{E}\Big[u_0(Z_t^x)\big\langle \Lambda(s)B(Z_{t-s}^x), Q_s^{-1/2}(Z_t^x - e^{sA}Z_{t-s}^x)\big\rangle\big|\mathcal{F}_{t-s}\Big].$$

Note that $\big\langle \Lambda(t-s)B(x), Q_{t-s}^{-1/2}\big(Z_{t-s}^x - e^{(t-s)A}x\big)\big\rangle$ is $\mathcal{F}_{t-s}$-measurable. Substituting this equality into the one above and using the property of conditional expectation, we obtain the desired result. □

Now we are ready to present the expression and estimates for the second iteration.

Proposition 5.2.8. *For any $t > 0$ and $x \in H$,*

$$v^2(t,x) = \int_0^t \int_0^s \mathbb{E}\Big[u_0(Z_t^x)\big\langle \Lambda(r)B(Z_{t-r}^x), Q_r^{-1/2}(Z_t^x - e^{rA}Z_{t-r}^x)\big\rangle \\ \times \big\langle \Lambda(s-r)B(Z_{t-s}^x), Q_{s-r}^{-1/2}(Z_{t-r}^x - e^{(s-r)A}Z_{t-s}^x)\big\rangle\Big] \,\mathrm{d}r\mathrm{d}s.$$

Furthermore,

$$\|v^2(t)\|_\infty \le \|u_0\|_\infty \|B\|_\infty^2 \int_0^t \int_0^s \|\Lambda(s-r)\|_{\mathcal{L}(H)} \|\Lambda(r)\|_{\mathcal{L}(H)} \,\mathrm{d}r\mathrm{d}s$$

and

$$\|Dv^2(t)\|_\infty \le \|u_0\|_\infty \|B\|_\infty^2 \int_0^t \int_0^s \|\Lambda(t-s)\|_{\mathcal{L}(H)} \|\Lambda(s-r)\|_{\mathcal{L}(H)} \|\Lambda(r)\|_{\mathcal{L}(H)} \,\mathrm{d}r\mathrm{d}s.$$

Proof. By Lemma 5.2.7, for any $s > 0$ and $y \in H$,

$$k_s^1(y) = \int_0^s \mathbb{E}\Big[u_0(Z_s^y)\big\langle \Lambda(r)B(Z_{s-r}^y), Q_r^{-1/2}(Z_s^y - e^{rA}Z_{s-r}^y)\big\rangle \\ \times \big\langle \Lambda(s-r)B(y), Q_{s-r}^{-1/2}(Z_{s-r}^y - e^{(s-r)A}y)\big\rangle\Big] \,\mathrm{d}r.$$

We have

$$\begin{aligned}\mathbb{E}\big[k_s^1(Z_{t-s}^x)\big] &= \mathbb{E}\bigg\{\int_0^s \mathbb{E}\Big[u_0(Z_s^y)\big\langle\Lambda(r)B(Z_{s-r}^y), Q_r^{-1/2}(Z_s^y - e^{rA}Z_{s-r}^y)\big\rangle \\ &\qquad\qquad \times\big\langle\Lambda(s-r)B(y), Q_{s-r}^{-1/2}(Z_{s-r}^y - e^{(s-r)A}y)\big\rangle\Big]_{y=Z_{t-s}^x}\mathrm{d}r\bigg\} \\ &= \int_0^s \mathbb{E}\Big[u_0(Z_t^x)\big\langle\Lambda(r)B(Z_{t-r}^x), Q_r^{-1/2}(Z_t^x - e^{rA}Z_{t-r}^x)\big\rangle \\ &\qquad\qquad \times\big\langle\Lambda(s-r)B(Z_{t-s}^x), Q_{s-r}^{-1/2}(Z_{t-r}^x - e^{(s-r)A}Z_{t-s}^x)\big\rangle\Big]\,\mathrm{d}r,\end{aligned}$$

where the second step follows from the Markov property. Therefore,

$$\begin{aligned}v^2(t,x) &= \int_0^t \big(S_{t-s}k_s^1\big)(x)\,\mathrm{d}s = \int_0^t \mathbb{E}\big[k_s^1(Z_{t-s}^x)\big]\mathrm{d}s \\ &= \int_0^t\int_0^s \mathbb{E}\Big[u_0(Z_t^x)\big\langle\Lambda(r)B(Z_{t-r}^x), Q_r^{-1/2}(Z_t^x - e^{rA}Z_{t-r}^x)\big\rangle \\ &\qquad\qquad \times\big\langle\Lambda(s-r)B(Z_{t-s}^x), Q_{s-r}^{-1/2}(Z_{t-r}^x - e^{(s-r)A}Z_{t-s}^x)\big\rangle\Big]\,\mathrm{d}r\mathrm{d}s.\end{aligned}$$

Next, by the definition of k_s^1 and the last inequality in Corollary 5.2.6,

$$\big\|k_s^1\big\|_\infty \le \|B\|_\infty\|Dv^1(s)\|_\infty \le \|u_0\|_\infty\|B\|_\infty^2\int_0^s \|\Lambda(s-r)\|_{\mathcal{L}(H)}\|\Lambda(r)\|_{\mathcal{L}(H)}\,\mathrm{d}r. \tag{5.13}$$

This immediately implies

$$|v^2(t,x)| \le \int_0^t \big\|k_s^1\big\|_\infty\,\mathrm{d}s \le \|u_0\|_\infty\|B\|_\infty^2\int_0^t\int_0^s \|\Lambda(s-r)\|_{\mathcal{L}(H)}\|\Lambda(r)\|_{\mathcal{L}(H)}\,\mathrm{d}r\mathrm{d}s,$$

and we obtain the estimate on $\|v^2(t)\|_\infty$. Moreover, by Proposition 5.2.4,

$$|Dv^2(t,x)| \le \int_0^t \big|D\big(S_{t-s}k_s^1\big)(x)\big|\,\mathrm{d}s \le \int_0^t \big\|k_s^1\big\|_\infty\|\Lambda(t-s)\|_{\mathcal{L}(H)}\,\mathrm{d}s,$$

which, combined with (5.13), gives us the second estimate. □

5.2.3 The general terms $v^n(t,x)$

In order to do further iteration, we rewrite the formula in Proposition 5.2.8 as

$$\begin{aligned}v^2(t,x) = \int_0^t \mathrm{d}s_2\int_0^{s_2}\mathrm{d}s_1\,\mathbb{E}\Big[&u_0(Z_t^x)\big\langle\Lambda(s_1)B(Z_{t-s_1}^x), Q_{s_1}^{-1/2}(Z_t^x - e^{s_1A}Z_{t-s_1}^x)\big\rangle \\ &\times\big\langle\Lambda(s_2-s_1)B(Z_{t-s_2}^x), Q_{s_2-s_1}^{-1/2}(Z_{t-s_1}^x - e^{(s_2-s_1)A}Z_{t-s_2}^x)\big\rangle\Big].\end{aligned}$$

Moreover, denoting by $s_0 = 0$, then we have

$$v^2(t,x) = \int_0^t \mathrm{d}s_2 \int_0^{s_2} \mathrm{d}s_1 \\ \mathbb{E}\left[u_0(Z_t^x) \prod_{i=1}^{2} \left\langle \Lambda(s_i - s_{i-1}) B(Z^x_{t-s_i}), Q^{-1/2}_{s_i - s_{i-1}} \left(Z^x_{t-s_{i-1}} - e^{(s_i - s_{i-1})A} Z^x_{t-s_i} \right) \right\rangle \right].$$

From this we can guess the general formulae.

Theorem 5.2.9. *Let $s_0 = 0$. For any $n \geq 1$,*

$$v^n(t,x) = \int_0^t \mathrm{d}s_n \int_0^{s_n} \mathrm{d}s_{n-1} \cdots \int_0^{s_2} \mathrm{d}s_1 \\ \mathbb{E}\left[u_0(Z_t^x) \prod_{i=1}^{n} \left\langle \Lambda(s_i - s_{i-1}) B(Z^x_{t-s_i}), Q^{-1/2}_{s_i - s_{i-1}} \left(Z^x_{t-s_{i-1}} - e^{(s_i - s_{i-1})A} Z^x_{t-s_i} \right) \right\rangle \right]. \tag{5.14}$$

Moreover,

$$\|v^n(t)\|_\infty \leq \|u_0\|_\infty \|B\|_\infty^n \int_0^t \mathrm{d}s_n \int_0^{s_n} \mathrm{d}s_{n-1} \cdots \int_0^{s_2} \mathrm{d}s_1 \prod_{i=1}^{n} \|\Lambda(s_i - s_{i-1})\|_{\mathcal{L}(H)}$$

and, letting $s_{n+1} = t$,

$$\|Dv^n(t)\|_\infty \leq \|u_0\|_\infty \|B\|_\infty^n \int_0^t \mathrm{d}s_n \int_0^{s_n} \mathrm{d}s_{n-1} \cdots \int_0^{s_2} \mathrm{d}s_1 \prod_{i=1}^{n+1} \|\Lambda(s_i - s_{i-1})\|_{\mathcal{L}(H)}.$$

Proof. We proceed by induction. Indeed, in view of the proofs in Section 5.2.2, we shall also prove inductively the formula

$$k_t^n(x) = \int_0^t \mathrm{d}s_n \int_0^{s_n} \mathrm{d}s_{n-1} \cdots \int_0^{s_2} \mathrm{d}s_1 \\ \mathbb{E}\left[u_0(Z_t^x) \prod_{i=1}^{n+1} \left\langle \Lambda(s_i - s_{i-1}) B(Z^x_{t-s_i}), Q^{-1/2}_{s_i - s_{i-1}} \left(Z^x_{t-s_{i-1}} - e^{(s_i - s_{i-1})A} Z^x_{t-s_i} \right) \right\rangle \right],$$

where $s_0 = 0$ and $s_{n+1} = t$. The discussions in Sections 5.2.1 and 5.2.2 show that the assertions on v hold for $n = 1$, 2, and the above formula of k holds with $n = 1$. Now we assume the assertions on v (resp. on k) hold for n (resp. for $n-1$), and try to prove them in the next iteration.

By the induction hypotheses, we have $v^n(s) \in C_b^1(H)$ for all $s > 0$ and thus, by the definition of the iteration (5.10), $k_s^n \in \mathcal{B}(H)$ with

$$\begin{aligned}\big\|k_s^n\big\|_\infty &\le \|B\|_\infty \|Dv^n(s)\|_\infty \\ &\le \|u_0\|_\infty \|B\|_\infty^{n+1} \int_0^s \mathrm{d}s_n \int_0^{s_n} \mathrm{d}s_{n-1} \cdots \int_0^{s_2} \mathrm{d}s_1 \prod_{i=1}^{n+1} \|\Lambda(s_i - s_{i-1})\|_{\mathcal{L}(H)},\end{aligned}$$

where $s_{n+1} = s$. Proposition 5.2.4 implies $S_{t-s}k_s^n \in C_b^1(H)$ for all $s \in (0,t)$, and from the formula

$$v^{n+1}(t,x) = \int_0^t \big(S_{t-s}k_s^n\big)(x)\,\mathrm{d}s$$

we deduce readily the estimates on $\|v^{n+1}(t)\|_\infty$ and $\|Dv^{n+1}(t)\|_\infty$.

Next we prove the formula for $k_t^n(x)$ (note that the induction hypothesis gives us the expression of $k_t^{n-1}(x)$). We have

$$\begin{aligned}k_t^n(x) = \langle B(x), Dv^n(t,x)\rangle &= \int_0^t \big\langle B(x), D\big(S_{t-s}k_s^{n-1}\big)(x)\big\rangle\,\mathrm{d}s \\ &= \int_0^t \mathbb{E}\Big[k_s^{n-1}(Z_{t-s}^x)\big\langle \Lambda(t-s)B(x), Q_{t-s}^{-1/2}(Z_{t-s}^x - e^{(t-s)A}x)\big\rangle\Big]\,\mathrm{d}s, \quad (5.15)\end{aligned}$$

where we used Proposition 5.2.4 in the last step. By the induction hypothesis,

$$\begin{aligned}k_s^{n-1}(y) = &\int_0^s \mathrm{d}s_{n-1} \int_0^{s_{n-1}} \mathrm{d}s_{n-2} \cdots \int_0^{s_2} \mathrm{d}s_1 \\ &\mathbb{E}\left[u_0(Z_s^y) \prod_{i=1}^n \Big\langle \Lambda(s_i - s_{i-1})B(Z_{s-s_i}^y), Q_{s_i-s_{i-1}}^{-1/2}\big(Z_{s-s_{i-1}}^y - e^{(s_i-s_{i-1})A}Z_{s-s_i}^y\big)\Big\rangle\right], \quad (5.16)\end{aligned}$$

where $s_0 = 0$ and $s_n = s$. Therefore, by the Markov property,

$$\begin{aligned}k_s^{n-1}(Z_{t-s}^x) = &\int_0^s \mathrm{d}s_{n-1} \int_0^{s_{n-1}} \mathrm{d}s_{n-2} \cdots \int_0^{s_2} \mathrm{d}s_1 \\ \mathbb{E}\left[u_0(Z_t^x) \prod_{i=1}^n \Big\langle \Lambda(s_i - s_{i-1})B(Z_{t-s_i}^x), Q_{s_i-s_{i-1}}^{-1/2}\big(Z_{t-s_{i-1}}^x - e^{(s_i-s_{i-1})A}Z_{t-s_i}^x\big)\Big\rangle \,\Big|\, \mathcal{F}_{t-s}\right].\end{aligned}$$

Inserting this identity into (5.15) and noticing that $\big\langle \Lambda(t-s)B(x), Q_{t-s}^{-1/2}(Z_{t-s}^x -$

$e^{(t-s)A}x)\big\rangle$ is measurable with respect to $\mathcal{F}_{t-s}$, we obtain

$$k_t^n(x) = \int_0^t \mathrm{d}s \int_0^s \mathrm{d}s_{n-1} \cdots \int_0^{s_2} \mathrm{d}s_1 \, \mathbb{E}\Bigg\{ \big\langle \Lambda(t-s)B(x), Q_{t-s}^{-1/2}(Z_{t-s}^x - e^{(t-s)A}x) \big\rangle \\ \times u_0(Z_t^x) \prod_{i=1}^n \Big\langle \Lambda(s_i - s_{i-1})B(Z_{t-s_i}^x), Q_{s_i-s_{i-1}}^{-1/2}\big(Z_{t-s_{i-1}}^x - e^{(s_i-s_{i-1})A} Z_{t-s_i}^x\big) \Big\rangle \Bigg\}.$$

Renaming s as s_n gives us the formula of $k_t^n(x)$ in the new iteration for all $t > 0$ and $x \in H$.

Finally we prove the expression for $v^{n+1}(t,x)$. We have

$$v^{n+1}(t,x) = \int_0^t \big(S_{t-s}k_s^n\big)(x)\,\mathrm{d}s = \int_0^t \mathbb{E}\big[k_s^n(Z_{t-s}^x)\big]\,\mathrm{d}s.$$

Using the formula we have just proved for $k_s^n(y)$ and the Markov property, we can obtain the expression for $v^{n+1}(t,x)$ in a similar way as above. □

We give a slightly different formula which is more appropriate for numerical purpose.

Corollary 5.2.10. *For any $n \geq 1$,*

$$v^n(t,x) = \int_0^t \mathrm{d}r_n \int_0^{r_n} \mathrm{d}r_{n-1} \cdots \int_0^{r_2} \mathrm{d}r_1 \\ \mathbb{E}\Bigg[u_0(Z_t^x) \prod_{i=1}^n \Big\langle \Lambda(r_{i+1} - r_i)B\big(Z_{r_i}^x\big), Q_{r_{i+1}-r_i}^{-1/2}\big(Z_{r_{i+1}}^x - e^{(r_{i+1}-r_i)A} Z_{r_i}^x\big) \Big\rangle \Bigg], \tag{5.17}$$

where $r_{n+1} = t$. Accordingly,

$$\|v^n(t)\|_\infty \leq \|u_0\|_\infty \|B\|_\infty^n \int_0^t \mathrm{d}r_n \int_0^{r_n} \mathrm{d}r_{n-1} \cdots \int_0^{r_2} \mathrm{d}r_1 \prod_{i=1}^n \|\Lambda(r_{i+1} - r_i)\|_{\mathcal{L}(H)}$$

and, setting $r_0 = 0$,

$$\|Dv^n(t)\|_\infty \leq \|u_0\|_\infty \|B\|_\infty^n \int_0^t \mathrm{d}r_n \int_0^{r_n} \mathrm{d}r_{n-1} \cdots \int_0^{r_2} \mathrm{d}r_1 \prod_{i=0}^n \|\Lambda(r_{i+1} - r_i)\|_{\mathcal{L}(H)}.$$

Proof. We change variables as follows:

$$r_i = t - s_{n+1-i}, \quad 1 \leq i \leq n.$$

The domain of integration becomes

$$\big\{(r_1,\cdots,r_n): 0<r_1<\cdots<r_n<t\big\};$$

and $s_i - s_{i-1} = r_{n+2-i} - r_{n+1-i}$, $1 \le i \le n$. Therefore, by (5.14),

$$\begin{aligned} v^n(t,x) = \int_0^t \mathrm{d}r_n \int_0^{r_n} \mathrm{d}r_{n-1} \cdots \int_0^{r_2} \mathrm{d}r_1 \\ \mathbb{E}\Bigg[u_0(Z_t^x) \prod_{i=1}^n \Big\langle \Lambda(r_{n+2-i} - r_{n+1-i}) B\big(Z^x_{r_{n+1-i}}\big), \\ Q^{-1/2}_{r_{n+2-i}-r_{n+1-i}}\big(Z^x_{r_{n+2-i}} - e^{(r_{n+2-i}-r_{n+1-i})A} Z^x_{r_{n+1-i}}\big)\Big\rangle \Bigg]. \end{aligned}$$

In the product, letting $j = n+1-i$, we get the desired formula (5.17). The proofs of the two estimates are similar. □

Remark 6. Due to the convolution structure (5.10), it seems that (5.17) is not suitable for the induction argument in the proof of Theorem 5.2.9.

5.2.4 Convergence of the iteration scheme (5.10)

We need the following technical result, where we use the Gamma function $\Gamma(\alpha)$:

$$\Gamma(\alpha) = \int_0^\infty \theta^{\alpha-1} e^{-\theta}\, \mathrm{d}\theta, \quad \alpha > 0.$$

Lemma 5.2.11. *Assume $\delta \in (0,1)$ and $n \ge 1$. Let $r_0 = 0$ and $r_{n+1} = t$. One has*

$$\int_0^t \mathrm{d}r_n \int_0^{r_n} \mathrm{d}r_{n-1} \cdots \int_0^{r_2} \mathrm{d}r_1 \prod_{i=1}^n \frac{1}{(r_{i+1}-r_i)^\delta} = \frac{\Gamma(1-\delta)^n}{\Gamma(1+n(1-\delta))} t^{n(1-\delta)}$$

and

$$\int_0^t \mathrm{d}r_n \int_0^{r_n} \mathrm{d}r_{n-1} \cdots \int_0^{r_2} \mathrm{d}r_1 \prod_{i=0}^n \frac{1}{(r_{i+1}-r_i)^\delta} = \frac{\Gamma(1-\delta)^{n+1}}{\Gamma((n+1)(1-\delta))} t^{n(1-\delta)-\delta}.$$

Proof. First we prove

$$\int_0^t \mathrm{d}r_n \int_0^{r_n} \mathrm{d}r_{n-1} \cdots \int_0^{r_2} \mathrm{d}r_1 \prod_{i=1}^n \frac{1}{(r_{i+1}-r_i)^\delta} = t^{n(1-\delta)} \prod_{i=1}^n B\big(1-\delta, 1+(i-1)(1-\delta)\big), \tag{5.18}$$

where $B(\alpha,\beta)$ is the Beta function:

$$B(\alpha,\beta)=\int_0^1 \theta^{\alpha-1}(1-\theta)^{\beta-1}\,\mathrm{d}\theta,\quad \alpha,\beta>0.$$

We proceed by induction. For $n=1$, noting that $r_2=t$, we change the variable $\theta=r_1/t$ and get

$$\int_0^t \frac{\mathrm{d}r_1}{(t-r_1)^\delta}=t^{1-\delta}\int_0^1\frac{\mathrm{d}\theta}{(1-\theta)^\delta}=t^{1-\delta}\int_0^1\theta^0(1-\theta)^{-\delta}\,\mathrm{d}\theta=t^{1-\delta}B(1-\delta,1).$$

Therefore the equality holds when $n=1$. Now suppose the equality holds for $n-1$, we prove it for n. By the induction hypothesis,

$$\int_0^{r_n}\mathrm{d}r_{n-1}\cdots\int_0^{r_2}\mathrm{d}r_1\prod_{i=1}^{n-1}\frac{1}{(r_{i+1}-r_i)^\delta}=r_n^{(n-1)(1-\delta)}\prod_{i=1}^{n-1}B\big(1-\delta,1+(i-1)(1-\delta)\big),$$

thus, noticing that $r_{n+1}=t$,

$$\begin{aligned}\int_0^t\mathrm{d}r_n\int_0^{r_n}\mathrm{d}r_{n-1}\cdots\int_0^{r_2}\mathrm{d}r_1\prod_{i=1}^{n}\frac{1}{(r_{i+1}-r_i)^\delta}&\\ =\prod_{i=1}^{n-1}B\big(1-\delta,1+(i-1)(1-\delta)\big)\int_0^t\frac{r_n^{(n-1)(1-\delta)}}{(t-r_n)^\delta}\,\mathrm{d}r_n.&\end{aligned}$$

We have, by changing variable $\theta=r_n/t$,

$$\int_0^t\frac{r_n^{(n-1)(1-\delta)}}{(t-r_n)^\delta}\,\mathrm{d}r_n=t^{n(1-\delta)}\int_0^1\theta^{(n-1)(1-\delta)}(1-\theta)^{-\delta}\,\mathrm{d}\theta=t^{n(1-\delta)}B\big(1-\delta,1+(n-1)(1-\delta)\big).$$

Substituting this result into the previous one gives us the identity (5.18).

Next, it is well known that

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Therefore,

$$\prod_{i=1}^{n}B\big(1-\delta,1+(i-1)(1-\delta)\big)=\prod_{i=1}^{n}\frac{\Gamma(1-\delta)\Gamma(1+(i-1)(1-\delta))}{\Gamma(1+i(1-\delta))}=\frac{\Gamma(1-\delta)^n}{\Gamma(1+n(1-\delta)}.$$

Combining this with (5.18) we obtain the desired formula.

The proof of the second identity is similar, by first establishing the identity

$$\int_0^t\mathrm{d}r_n\int_0^{r_n}\mathrm{d}r_{n-1}\cdots\int_0^{r_2}\mathrm{d}r_1\prod_{i=1}^{n}\frac{1}{(r_{i+1}-r_i)^\delta}=t^{n(1-\delta)-\delta}\prod_{i=1}^{n}B\big(1-\delta,i(1-\delta)\big).$$

We omit the details here. □

As a consequence, we have the following estimates.

Corollary 5.2.12. *Under the Hypotheses 5.2.1 and 5.2.3, for any $n \geq 0$ and $t > 0$,*

$$\|v^n(t)\|_\infty \leq \|u_0\|_\infty \|B\|_\infty^n C_\delta^n t^{n(1-\delta)} \frac{\Gamma(1-\delta)^n}{\Gamma(1+n(1-\delta))} \tag{5.19}$$

and

$$\|Dv^n(t)\|_\infty \leq \|u_0\|_\infty \|B\|_\infty^n C_\delta^{n+1} t^{n(1-\delta)-\delta} \frac{\Gamma(1-\delta)^{n+1}}{\Gamma((n+1)(1-\delta))}.$$

Proof. The case $n = 0$ follows directly from (5.8). Combining Lemma 5.2.11 and Corollary 5.2.10, we obtain the general cases. □

Now we can prove the existence of limit for the iteration scheme (5.10).

Proposition 5.2.13. *Assume the Hypotheses 5.2.1 and 5.2.3. For any $T > 0$, the series*

$$\sum_{n=0}^{\infty} v^n(t,x)$$

converge uniformly on $[0,T] \times H$. Moreover, for any $t_0 \in (0,T)$, the series

$$\sum_{n=0}^{\infty} Dv^n(t,x)$$

converge uniformly on $[t_0,T] \times H$.

Proof. We only prove the first assertion; the proof of the second one is similar. By Corollary 5.2.12 and using the ratio test, it is sufficient to show that

$$\lim_{n\to\infty} \frac{\Gamma(1+n(1-\delta))}{\Gamma(1+(n+1)(1-\delta))} = 0.$$

This follows from elementary calculations. Indeed, setting $\alpha = 1-\delta$ for simplicity of notation,

$$\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)} = \frac{n\alpha}{(n+1)\alpha} \cdot \frac{n\alpha-1}{(n+1)\alpha-1} \cdots \frac{1+(n\alpha)}{1+\alpha+(n\alpha)} \cdot \frac{\Gamma((n\alpha))}{\Gamma(\alpha+(n\alpha))},$$

where $(n\alpha)$ is the decimal part of $n\alpha$. Using the simple inequality $\log(1+x) < x$ for all $x \in (-1,0)$, we have

$$\log\left(\frac{n\alpha-k}{(n+1)\alpha-k}\right) = \log\left(1 - \frac{\alpha}{(n+1)\alpha-k}\right) < -\frac{\alpha}{(n+1)\alpha-k}.$$

Hence,

$$\log\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)} < \\ -\alpha\left(\frac{1}{(n+1)\alpha}+\frac{1}{(n+1)\alpha-1}+\cdots+\frac{1}{1+\alpha+(n\alpha)}\right)+\log\frac{\Gamma((n\alpha))}{\Gamma(\alpha+(n\alpha))}.$$

Note that the first part on the right hand side tends to $-\infty$ as $n\to\infty$, while the last part is uniformly bounded in n, thus we conclude the result. □

Thanks to Proposition 5.2.13, we can define the limit

$$u(t,x)=\lim_{n\to\infty}u^n(t,x)=\lim_{n\to\infty}\sum_{i=0}^{n}v^i(t,x);$$

moreover, for any $t>0$, one has $u(t)\in C_b^1(H)$ and

$$Du(t,x)=\lim_{n\to\infty}Du^n(t,x)=\lim_{n\to\infty}\sum_{i=0}^{n}Dv^i(t,x)$$

which holds uniformly on $[t_0,T]\times H$ for any $0<t_0<T$. Finally we can prove the main result.

Theorem 5.2.14. *The limit $u(t,x)$ is the unique solution to the Kolmogorov equation* (5.3) *in the following sense:*

(a) *for any $T>0$, $u(t,x)$ is uniformly bounded for $(t,x)\in[0,T]\times H$, and $u(t)\in C_b^1(H)$ for any $t>0$;*

(b) *for any $T>0$, one has $\int_0^T\|Du(t)\|_\infty\,\mathrm{d}t<\infty$;*

(c) *it satisfies the mild formulation* (5.9) *for any $t>0$ and $x\in H$.*

Proof. Obviously our limit verifies (a). Next,

$$\|Du(t)\|_\infty\le\sum_{n=0}^{\infty}\left\|Dv^n(t)\right\|_\infty\le\|u_0\|_\infty\sum_{n=0}^{\infty}\|B\|_\infty^n C_\delta^{n+1}t^{n(1-\delta)-\delta}\frac{\Gamma(1-\delta)^{n+1}}{\Gamma((n+1)(1-\delta))}.$$

Therefore,

$$\begin{aligned}\int_0^T\|Du(t)\|_\infty\,\mathrm{d}t&\le\|u_0\|_\infty\sum_{n=0}^{\infty}\|B\|_\infty^n C_\delta^{n+1}\frac{\Gamma(1-\delta)^{n+1}}{\Gamma((n+1)(1-\delta))}\int_0^T t^{n(1-\delta)-\delta}\,\mathrm{d}t\\&=\|u_0\|_\infty\sum_{n=0}^{\infty}\|B\|_\infty^n C_\delta^{n+1}\frac{\Gamma(1-\delta)^{n+1}}{\Gamma((n+1)(1-\delta))}\frac{T^{(n+1)(1-\delta)}}{(n+1)(1-\delta)},\end{aligned}\tag{5.20}$$

which shows that (b) is also satisfied. Moreover, for any $t > 0$ and $x \in H$,

$$\left| \int_0^t \big(S_{t-s}\langle B, Du(s)\rangle\big)(x)\,\mathrm{d}s \right| \le \int_0^t \|\langle B, Du(s)\rangle\|_\infty \,\mathrm{d}s \le \|B\|_\infty \int_0^t \|Du(s)\|_\infty \,\mathrm{d}s.$$

This implies the integral in the signs of absolute value makes sense.

It remains to check that $u(t,x)$ verify (5.9). By the iteration scheme (5.10), one has, for any $n > 1$,

$$u^n(t,x) = u^0(t,x) + \int_0^t \big(S_{t-s}\big\langle B, Du^{n-1}(s)\big\rangle\big)(x)\,\mathrm{d}s \quad \text{for all } t > 0,\ x \in H. \tag{5.21}$$

The left hand side converges uniformly to $u(t,x)$ on $[0,T] \times H$ for any $T > 0$. It suffices to show the uniform convergence of the right hand side. We have

$$\begin{aligned} &\left| \int_0^t \big(S_{t-s}\big\langle B, Du^{n-1}(s)\big\rangle\big)(x)\,\mathrm{d}s - \int_0^t \big(S_{t-s}\langle B, Du(s)\rangle\big)(x)\,\mathrm{d}s \right| \\ \le\ & \int_0^t \big\|\big\langle B, Du^{n-1}(s) - Du(s)\big\rangle\big\|_\infty \,\mathrm{d}s \le \|B\|_\infty \sum_{i=n}^\infty \int_0^t \big\|Dv^i(s)\big\|_\infty \,\mathrm{d}s. \end{aligned}$$

Similarly to the calculations in (5.20), we can show that the right hand side vanishes as n goes to infinity. Therefore we let $n \to \infty$ on both sides of (5.21) and conclude that $u(t,x)$ satisfies (5.9) uniformly in $(t,x) \in [0,T] \times H$.

Finally we prove the uniqueness of solutions. Suppose $u(t,x)$ and $\tilde{u}(t,x)$ are two solutions to (5.3) with the properties (a)–(c). Then, for any $t > 0$ and $x \in H$,

$$u(t,x) - \tilde{u}(t,x) = \int_0^t S_{t-s}\big(\langle B, D(u(s) - \tilde{u}(s))\rangle\big)(x)\,\mathrm{d}s.$$

Therefore,

$$|u(t,x) - \tilde{u}(t,x)| \le \|B\|_\infty \int_0^t \|Du(s) - D\tilde{u}(s)\|_\infty \,\mathrm{d}s. \tag{5.22}$$

Moreover, by Proposition 5.2.4,

$$\begin{aligned} |D(u(t,x) - \tilde{u}(t,x))| &\le \int_0^t \big|DS_{t-s}\big(\langle B, D(u(s) - \tilde{u}(s))\rangle\big)(x)\big|\,\mathrm{d}s \\ &\le \int_0^t \big\|\langle B, D(u(s) - \tilde{u}(s))\rangle\big\|_\infty \|\Lambda(t-s)\|_{\mathcal{L}(H)}\,\mathrm{d}s \\ &\le \|B\|_\infty \int_0^t \|Du(s) - D\tilde{u}(s)\|_\infty \|\Lambda(t-s)\|_{\mathcal{L}(H)}\,\mathrm{d}s. \end{aligned}$$

Hence,

$$\begin{aligned}\int_0^t \|Du(s) - D\tilde{u}(s)\|_\infty \,\mathrm{d}s &\le \|B\|_\infty \int_0^t \int_0^s \|Du(r) - D\tilde{u}(r)\|_\infty \|\Lambda(s-r)\|_{\mathcal{L}(H)} \,\mathrm{d}r\mathrm{d}s \\ &= \|B\|_\infty \int_0^t \|Du(r) - D\tilde{u}(r)\|_\infty \,\mathrm{d}r \int_r^t \|\Lambda(s-r)\|_{\mathcal{L}(H)} \,\mathrm{d}s \\ &\le \left[\|B\|_\infty \int_0^t \|\Lambda(s)\|_{\mathcal{L}(H)} \,\mathrm{d}s\right] \int_0^t \|Du(r) - D\tilde{u}(r)\|_\infty \,\mathrm{d}r.\end{aligned}$$

Under Hypothesis 5.2.1-(iv), there is some $t_1 > 0$ such that $\|B\|_\infty \int_0^{t_1} \|\Lambda(s)\|_{\mathcal{L}(H)} \,\mathrm{d}s < 1$. Then

$$\int_0^t \|Du(s) - D\tilde{u}(s)\|_\infty \,\mathrm{d}s = 0 \quad \text{for all } t \le t_1.$$

Combining this with (5.22) we see that $u(t,x) = \tilde{u}(t,x)$ for any $(t,x) \in [0,t_1] \times H$. Next, by the semigroup property, it is easy to show that, for $t \in (0,t_1]$,

$$u(t+t_1,x) = S_t u_{t_1}(x) + \int_0^t S_{t-s}\big(\langle B, Du(t_1+s)\rangle\big)(x) \,\mathrm{d}s.$$

Repeating the above procedure we can prove the uniqueness on the interval $[t_1, 2t_1]$ and so on. Thus we complete the proof. □

5.3 Numerical Simulations

In this section we propose some experiment of the iteration scheme (5.10) studied in Section 5.2 in the finite dimensional setting. We have in mind the framework of Example 5.2.2, i.e. $A = \Delta$. Since we are in the finite dimensional setting this choice corresponds to take $A \in \mathbb{R}^d \otimes \mathbb{R}^d$ as the diagonal matrix where $A_{k,k} = -k^2$, $k = 1, \dots, d$. Moreover we consider the matrix $Q = \sigma^2 I_{d\times d}$ where $I_{d\times d}$ is the identity matrix over $\mathbb{R}^d$, and the parameter σ will be specified below (see Table 5.1 for reference parameters).

We will consider two main classes of examples as a benchmark for our approximation scheme. First, we consider the nonlinear vector field

$$B(x)_i = \sin(x_i), \quad i = 1, \dots, d, \tag{5.23}$$

i.e. we apply the sine function to all the components. This nonlinearity will be the easier one of our examples since it is close to linear, at least for small values of x. We will also consider some variation of the previous example, made by

$$B(x)_i = \sin(x_i)(B_m x)_i, \quad i = 1, \dots, d, \tag{5.24}$$

where $B_m \in \mathbb{R}^d \times \mathbb{R}^d$ is the skew symmetric matrix

$$(B_m)_{i,j} = \begin{cases} 1 & \text{if } i < j; \\ -1 & \text{if } i > j; \\ 0 & \text{if } i = j, \end{cases}$$

i.e. the Toeplitz matrix with all one above the diagonal and minus one below. This example is more complex than the previous one. It is significant since it deals with skew symmetric matrices, inducing rotations, which are a first simple step in the direction of fluid dynamics. The vector field $B_m x$ is also multiplied by the function $\sin(x)$ in order to make example (5.24) nonlinear. Notice that this last example is not covered by our present theory, since it does not satisfy Hypothesis 5.2.3. However, even if (5.24) is not bounded, it satisfies a linear growth condition. We hope to improve our theory and the generality of the assumptions in such direction in a future research, and limit ourselves to some numerical experiments for the present work.

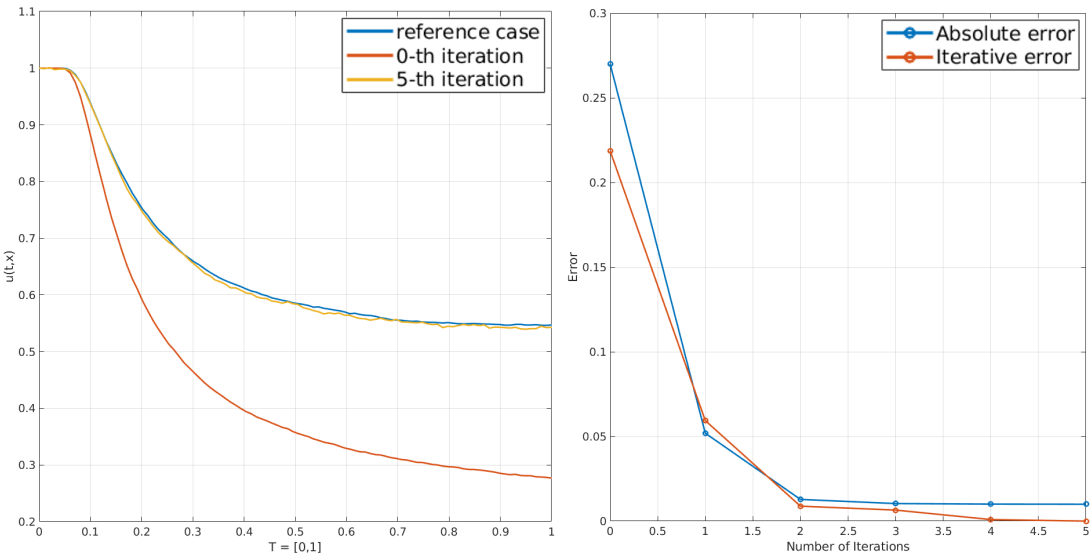

Figure 5.1: Sine case (5.23). Left: trajectory of $u(t,x)$ for $t \in [0,T]$, $d = 10$. Right: difference between consecutive iterations and error with respect to the reference case, X-axis number of iterations.

Second, we consider the following class of polynomial nonlinearities

$$B(x)_i = ||\overline{y}|| \, \frac{(\overline{y}_i - x_i)\,|\overline{y}_i - x_i|^{p-1}}{||\overline{y}|| + ||\overline{y} - x||^p}, \quad i = 1, \dots, d \tag{5.25}$$

where $\overline{y} \in \mathbb{R}^d$ is fixed. Note that this example appeals to the one dimensional case

$$B(x) = (\overline{y} - x) \, |\overline{y} - x|^{p-1} \, ,$$

for which the dynamical system

$$\dot{x}(t) = B(x(t))$$

has the singleton $\{\overline{y}\}$ as a stable attractor. The reason behind the example (5.25) is the following: it is close to a polynomial nonlinearity, so that it makes a significant test case; at the same time, the normalization by the factor $||\overline{y}|| \, / (||\overline{y}|| + ||\overline{y} - x||^p)$ makes it a bounded operator, so that it fulfills Hypothesis 5.2.3.

In all the examples above we adopt the following choice of initial condition

$$u_0(x) = \mathbb{1}_{\{||x|| \geq H\}},$$

where the parameter H is set to 1 (see Table 5.1).

5.3.1 Approximation schemes

Standard Monte-Carlo approach. Since an explicit solution for Equation (5.2) is not available we will always compare to the solution obtained by means of Monte-Carlo simulation of the nonlinear process X_t^x:

$$u(t,x) = \mathbb{E}\left[u_0(X_t^x)\right] \simeq \frac{1}{N_s} \sum_{i=1}^{N_s} u_0\big(X_t^{x,i}\big), \tag{5.26}$$

where N_s is the number of samples considered, and the processes $X_t^{x,i}, i = 1, \ldots, N_s$ are independent copies of X_t^x. To compute samples of the process $X_t^{x,i}$ we use the Euler-Maruyama scheme with a very fine time step in order to get a good approximation to be used as a comparison. The solution computed by (5.26) will always be referred to in what follows as the reference case.

Numerical iteration scheme. Under our assumption, since A and Q are diagonal, we can rewrite the equations for the processes Z_t^x and Z_t in a simple way: for $k = 1, \ldots, d$,

$$\begin{cases} \mathrm{d}Z_t^k = -k^2 Z_t^k \, \mathrm{d}t + \mathrm{d}W_t^k, \\ Z_0^k = 0 \end{cases} \tag{5.27}$$

and

$$Z_t^{x,k} = e^{-k^2 t} x_k + \sigma Z_t^k.$$

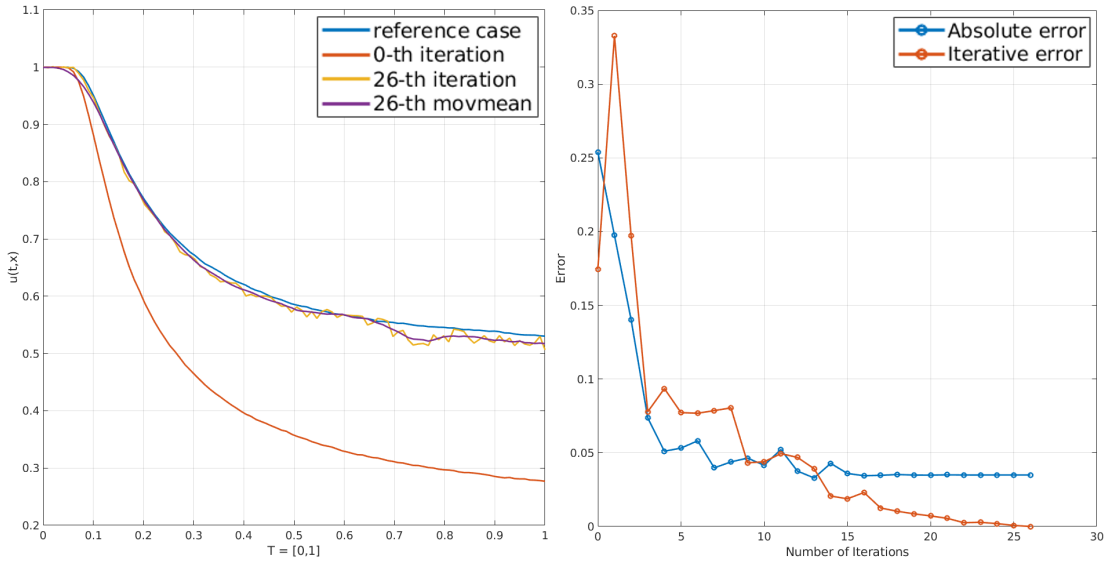

Figure 5.2: Polynomial bounded quadratic case (5.25) with $p = 2$, $d = 10$. Left: trajectory of $u(\cdot, x)$ for $t \in [0, T]$. The purple line is obtained by applying a moving average smoothing to the last iteration. Right: difference between consecutive iterations and error with respect to the reference case, X-axis number of iterations.

We remark that, differently from Z_t, the process Z_t^x depends also on the parameter σ, but we do not explicitly write $Z_t^{x,\sigma}$ for ease of notation. Note that the process Z_t depends only on the operators A. This opens the possibility of computing Z_t^x, and hence also $u(t, x)$, for many values of x without repeating the computations for Z_t. The same reasoning holds for different values of σ, see Figure 5.6. Note also that this strategy cannot be applied to the process X_t^x since in that case the problem is nonlinear.

Once realizations of the process Z_t^x are computed, we can proceed with the iteration algorithm (5.10). In order to compute numerically the quantity $v^n(t, x)$ appearing in Theorem 5.2.9 one needs to be able to compute first

$$\left\langle \Lambda(s)\, B\left(Z_{t-s}^x\right), Q_s^{-1/2}\left(Z_t^x - e^{sA} Z_{t-s}^x\right)\right\rangle. \tag{5.28}$$

Since A and Q are diagonal and explicit (see the beginning of this section), one has

$$(Q_t)_{k,k} = \int_0^t \left(e^{sA}\right)_{k,k} Q_{k,k} \left(e^{sA^*}\right)_{k,k} \mathrm{d}s = \int_0^t e^{-2sk^2}\sigma^2 \,\mathrm{d}s = \frac{\sigma^2}{2k^2}\left(1 - e^{-2tk^2}\right),$$

$$\left(Q_t^{-1/2}\right)_{k,k} = \frac{\sqrt{2}k}{\sigma\sqrt{1 - e^{-2tk^2}}}, \quad (\Lambda(t))_{k,k} = \frac{\sqrt{2}ke^{-tk^2}}{\sigma\sqrt{1 - e^{-2tk^2}}},$$

and thus,

$$\begin{aligned}&\left\langle \Lambda\left(s\right) B\left(Z_{t-s}^{x}\right), Q_{s}^{-1/2}\left(Z_{t}^{x}-e^{sA}Z_{t-s}^{x}\right)\right\rangle \\ &=\sum_{k=1}^{d}\frac{2k^{2}e^{-sk^{2}}}{\sigma^{2}(1-e^{-2sk^{2}})}B\left(Z_{t-s}^{x}\right)_{k}\left(Z_{t}^{x,k}-e^{-sk^{2}}Z_{t-s}^{x,k}\right).\end{aligned}$$

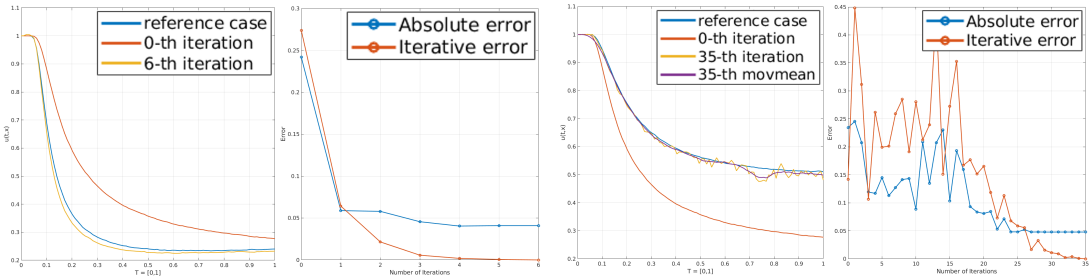

Figure 5.3: Left block: Sine times skew-symmetric case (5.24) with $d = 10$. Right block: Polynomial bounded cubic case (5.25) $p = 3$, $d = 10$. The purple line is obtained by applying a moving average smoothing to the last iteration.

Hence, when integrating expression (5.28), by change of variable we have

$$\begin{aligned}&\int_{0}^{t}\left\langle \Lambda\left(s\right) B\left(Z_{t-s}^{x}\right), Q_{s}^{-1/2}\left(Z_{t}^{x}-e^{sA}Z_{t-s}^{x}\right)\right\rangle \mathrm{d}s\\ &=\int_{0}^{t}\left\langle \Lambda\left(t-s\right) B\left(Z_{s}^{x}\right), Q_{t-s}^{-1/2}\left(Z_{t}^{x}-e^{(t-s)A}Z_{s}^{x}\right)\right\rangle \mathrm{d}s\\ &=\int_{0}^{t}\sum_{k=1}^{d}\frac{2k^{2}e^{-(t-s)k^{2}}}{\sigma^{2}(1-e^{-2(t-s)k^{2}})}B\left(Z_{s}^{x}\right)_{k}\left(Z_{t}^{x,k}-e^{-(t-s)k^{2}}Z_{s}^{x,k}\right)\mathrm{d}s. \qquad (5.29)\end{aligned}$$

Changing variable provides a significant advantage when performing numerical integration. In fact it is more complex to compute Z_{t-s}^{x} than $\Lambda(t-s)$ (resp. $Q_{t-s}^{1/2}$) since Z^x is random and hence we would have been obliged to reverse the time for every sample of the process. On the other hand the matrix $\Lambda(t-s)$ (resp. $Q_{t-s}^{1/2}$) is deterministic so that changing time $s \mapsto t-s$ can be done once only.

Moreover, thanks to Corollary 5.2.10, it is possible to compute $v^n(t,x)$ with a single time integration from the previous step. Introduce

$$\begin{aligned}&I^{n}(t,x)=\\ &\int_{0}^{t}\mathrm{d}r_{n}\int_{0}^{r_{n}}\mathrm{d}r_{n-1}\cdots\int_{0}^{r_{2}}\mathrm{d}r_{1}\prod_{i=1}^{n}\left\langle \Lambda(r_{i+1}-r_{i})B\big(Z_{r_{i}}^{x}\big), Q_{r_{i+1}-r_{i}}^{-1/2}\big(Z_{r_{i+1}}^{x}-e^{(r_{i+1}-r_{i})A}Z_{r_{i}}^{x}\big)\right\rangle,\end{aligned}$$

and notice that, due to Equation (5.17), we have

$$v^n(t,x) = \mathbb{E}\big[u_0(Z_t^x) I^n(t,x)\big].$$

Since

$$I^{n+1}(t,x) = \int_0^t \left\langle \Lambda(t-s)\, B(Z_s^x), Q_{t-s}^{-1/2}\left(Z_t^x - e^{(t-s)A} Z_s^x\right)\right\rangle I^n(s,x)\,\mathrm{d}s,$$

once we have computed I^n, computing I^{n+1} is a matter of a single integration. This is really crucial because, otherwise, by using the direct expression (5.14) in Theorem 5.2.9, to compute $v^n(t,x)$ one should have done an n-dimensional numerical integration, independently on the previous iteration.

Stopping conditions. Since the numerical scheme is iterative and since an exact solution is not available, we adopt a consecutive-iterations stopping condition. At every step we measure the difference between consecutive iterations and stop when this difference is below a certain threshold *tol*. Specifically we adopt two strategies in different situations: when we compute the entire trajectory of $u(t,x)$ for $t \in [0,T]$, we measure

$$err(n) := \sup_{t\in[0,T]} ||v^n(t,x)||$$

and stop the iterations if $err(n) < tol$ (see Figures 5.1 and 5.2); when we are interested only in $u(T,x)$ for a fixed T, then

$$err(n) := |v^n(T,x)|$$

and adopt the same stopping rule (Figures 5.6 and 5.7).

The entire procedure can be summarized in the following scheme:

Result: $u^n(t,x)$ approximating solution after n iterations
Compute N_s samples of the process Z_t;
Compute N_s samples for Z_t^x starting from Z_t;
Compute $u^0(t,x) = v^0(t,x) = \mathbb{E}[u_0(Z_t^x)]$ by Monte-Carlo average;
Set $err = 1$, $n = 0$;
while $err > tol$ **do**
  Compute $v^{n+1}(t,x)$ as in Corollary 5.2.10;
  Set $u^{n+1}(t,x) = u^n(t,x) + v^{n+1}(t,x)$;
  Set $err = |v^{n+1}(t,x)|$;
  Set $n = n + 1$;
end

Algorithm 1: Iteration Scheme

5.3.2 Examples

Here we collect the results obtained, and all the parameters involved in the simulations. Parameters are divided into two categories: those related to the mathematical problem, and those strictly related to the numerical approximations, see Tables 5.1 and 5.2. Those are our reference parameters: we will specify each time any modifications.

In all the figures below, when showing the entire trajectory of the solution $u(t, x)$ for $t \in [0, T]$, we also plot the 0-th order iteration. This corresponds to the solution of the linear case for (5.2), i.e. the Kolmogorov equation with $B \equiv 0$. This will allow us to compare with the linear case, in order to be sure to have introduced a significant nonlinearity into the problem.

Parameter	Value	Description
d	10	dimension of the problem
y_0	2**e**	parameter of the nonlinearity B, Polynomial case
x	**e**	values where the solution $u(t, x)$ is computed
σ	1	noise
T	1	final time of computation for $u(t, x_0)$
H	1	threshold for the initial condition $u_0(x)$

Table 5.1: Model parameters, **e** stands for the vector with all components identically 1.

Parameter	Value	Description
Δt	10^{-4}	time step for Euler scheme
dt	10^{-2}	time step for numerical integration
N_s	10^5	number of samples averages
tol	10^{-3}	tolerance for stopping iterations

Table 5.2: Numerical parameters.

Mixed-time-step strategy. To perform numerical simulation of SDEs and numerical integration we adopt a mixed-time-step strategy. When we compute the reference solution, through the simulation of the process X_t^x, as well as when computing samples of the linear process Z_t, we adopt a time step Δt. On the other hand when we perform numerical integration, to compute successive iterations, we adopt a time discretization parameter $dt \gg \Delta t$, see Table 5.2. This is due to the fact that, in equation (5.1), as well as in (5.27), a coefficient $-k^2$ is present in the k-th component of the drift of the equation. This coefficient, and hence the Lipschitz constant of the drift, is growing as the square of the dimension d of the problem. This is caused by the intrinsic exponential decay of equation (5.27), which require a high level of precision in computation. Differently, in equation (5.29), part of this exponential decay is absorbed by the convolutional structure of the integration. The limits and what is the proper ratio between Δt and dt is a difficult topic. A more precise investigation is needed: for the present chapter we only highlight the numerical result obtained, and hope to improve the theoretical counterpart in a future work.

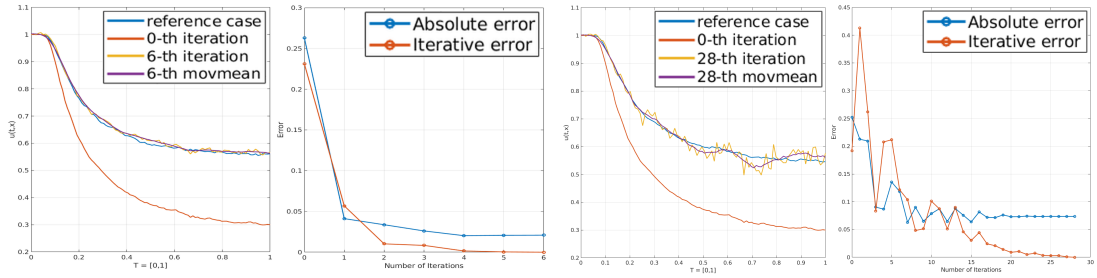

Figure 5.4: Left block: Sine case (5.23) in dimension $d = 50$, $N_s = 10^4$. The purple line is obtained by applying a moving-average smoothing to the last iteration. Right block: Polynomial bounded quadratic case (5.25) $p = 2$ in dimension $d = 50$, $N_s = 10^4$.

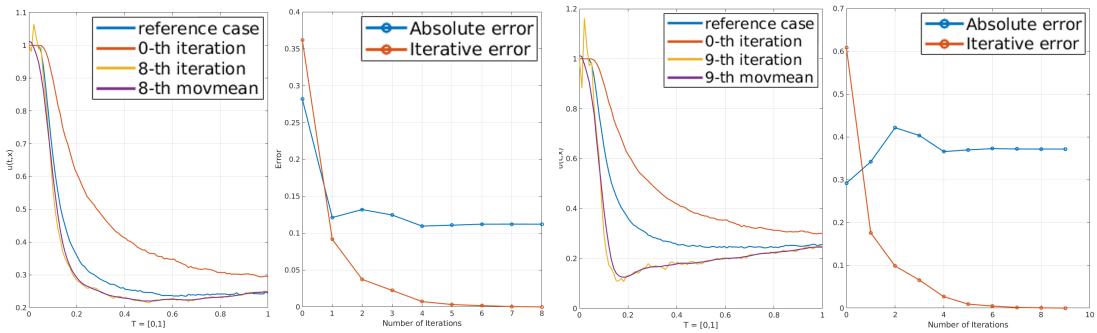

Figure 5.5: Sine times skew symmetric matrix (5.24). Left dimension $d = 20$, $N_s = 10^4$. Right dimension $d = 50$, $N_s = 10^4$.

Positive results. For the simpler test case, the sine case (5.23), see Figure 5.1, convergence is obtained in five iterations. This is due to the simplicity of the example, as $\sin(x)$ is almost linear near the origin. The situation is different when dealing with some more concrete examples like the polynomial case. In Figure 5.2, where we use formula (5.25) with $p = 2$, we see that the number of iterations to convergence is much bigger (26 in our example). At the same time the difference between the last iteration and the reference case is quite small, comparable to the sine case. However, we notice that the oscillation of the solution computed via our iteration scheme, related to the variance of the estimator, is a bit bigger than that of the reference case. This discrepancy is not completely clear yet, even if we expect it to be due to the low number of samples used to compute averages. In Figure 5.2 we also add a moving-average smoothing of the solution, to make more

perceivable this last intuition.

The same behavior is obtained in the variations of the previous examples. In Figure 5.3 we see that the same fast convergence as in the sine case, is obtained also in the sine times skew-symmetric case (5.24). The Polynomial cubic case (5.25) with $p = 3$ has the same level of complexity as the case with $p = 2$, even if it requires a higher number of iterations to obtain convergence, and presents the same type of oscillations.

We also perform the same tests in much higher dimension. In Figure 5.4 we show the results of the same examples, performed in dimension $d = 50$ with $N_s = 10^4$ samples. We see that the number of iterations required to convergence are comparable with result in $d = 10$: this confirms the estimate of Corollary 5.2.12 which is in the infinite dimensional framework and hence is independent of any dimension. The small variations in the number of iterations, as well as the slight increase of the oscillations in the quadratic case, can be explained by the reduction in the number of samples used to compute empirical averages. It is also important to remark that, in the current example, the estimate of Corollary 5.2.12 is still too rough: by computing the right-hand side of (5.19) one finds that the number n of iterations needed to have $|v^n(t, x)| < tol$ is far bigger than what we find in the numerical test (in fact it should be bigger than one hundred).

In Figure 5.6 we followed a different approach: we fix the test case as the sine times skew-symmetric matrix (5.24), and analyze what is the limit of $u(1, x)$ as σ goes to zero. Also in this case the solution computed through the iteration scheme is quite close to the reference case. At the same time, on the right side of Figure 5.6, we can appreciate the great advantage in time-saving of the iteration scheme. We remark that the plot on the right side is cumulative, meaning that it takes into account the time spent to compute the solution multiple times. In particular, we note that the reference case is a straight line, since the computational time does not depend on the different values of σ. On the other hand, for the iteration scheme there is a change in the number of iterations for different values of σ that justifies the nonlinear shape. Moreover we see that, even including the time of computing samples of the process Z_t that can be done only once (since σ does not appear in (5.27)), we still have a great advantage in time.

As remarked in the introduction, this kind of advantage is a main feature of the new method proposed here and applies also to the variation of other parameters than σ. In particular, it applies to the change of initial conditions x, one of the most fundamental problems in weather and climate prediction, related to the ensemble forecasting method, see [66, Chapter 6]. Again, Monte-Carlo pays linearly with the number of variations of x, while our method pays the bulk (i.e. Z_t in (5.27)) only once and then (here for the initial conditions) roughly linearly in the number of different x's, but with a linear slope much smaller than the one of Monte-Carlo,

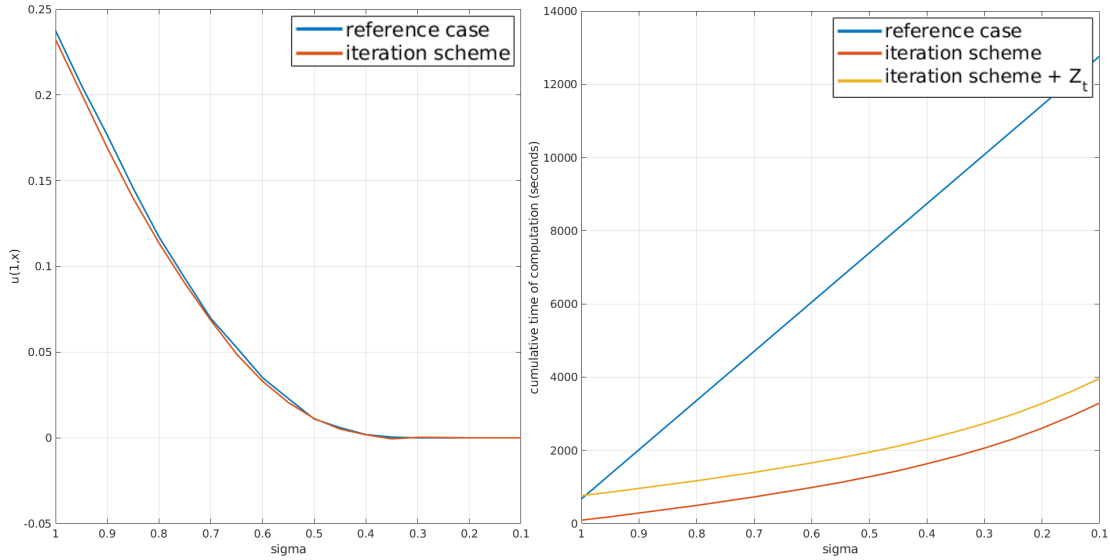

Figure 5.6: Sine times skew symmetric matrix (5.24), $d = 10$. Left: Y-axis value of $u(1, x)$ for different values of σ. X-axis different values of σ in the reverse order. Right: Y-axis computational time measured in seconds to compute the solution $u(1, x)$ for various values of σ. The measurement of time is cumulative: we give the cost of computing $u(1, x)$ for several values of σ, starting from $\sigma = 1$ in decreasing order. X-axis different values of σ in the reverse order. The red line refers only to the time to compute iterations. The yellow line includes also the time to compute samples of Z_t one time at the beginning of the simulation.

similarly to the initial slope of Figure 5.6 right side. We illustrate the interest in varying x by Figure 5.7 right side, where it is illustrated the relative importance of different variations.

Difficulties with small σ and high dimension. However, not every situation is well behaved as those presented above: in Figure 5.7 left side, we present the plot for different values of σ in the polynomial quadratic case. Here the approximation tends to degenerate for smaller values of σ (already around 0.5). This is due to the higher level of nonlinearity of the polynomial case with respect to (5.24). It is also important to mention that the number of iterations to convergence is really important for what concerns the computational time. In the polynomial quadratic (and also cubic) case, since the number of iterations to convergence is much higher than in the simpler case, the advantage in the computational time is less relevant. Still for what concerns negative results we also show in Figure 5.5 that, when the dimension grows (left $d = 20$, right $d = 50$), the sine times skew-symmetric case

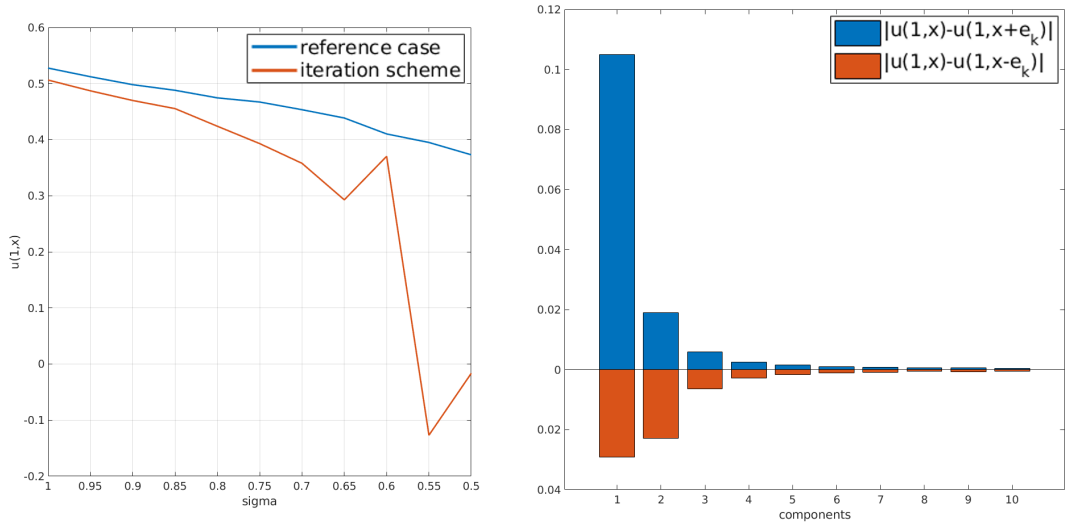

Figure 5.7: Polynomial quadratic bounded case (5.25), $p = 2$, $d = 10$. Left: Y-axis Value of $u(1, x)$ computed by the iteration scheme, with different values of σ. X-axis Different values of σ in the reverse order. Right: Sine times skew symmetric matrix (5.24), $d = 10$. Difference of $u(1, x)$ with respect to $u(1, x \pm e_k)$ for $k = 1, \ldots, 10$. Blue positive values are obtained by comparing with $u(1, x + e_k)$, orange negative by comparing with $u(1, x - e_k)$. X-axis different values of $k = 1, \ldots, 10$.

(5.24) tends to degenerate. Iterations are still converging but the limit is far from the reference solution. This is definitively the most difficult of our examples since it is the only one which mixes strongly all the components and produces a strong energy flux between them. However we also remark that at present time this case is not covered by our theory, but is still relevant since it has the rotational behavior which appeals to fluid dynamics.

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