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Generalized numerical semigroups

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GENERALIZED NUMERICAL SEMIGROUPS

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"I believe a leaf of grass is no less than the journey work of the stars"
Song of Myself, Walt Whitman

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Introduction

"God made the integers,
all the rest is the work of man"
L. Kronoecker

I like to introduce the topic of this work with the above citation of the mathematician Leopold Kronoecker, since the set $\mathbb{N}$ is always holding hands with the main character of this work and all its related explanations.
In fact this work concerns with *Generalized Numerical Semigroups*, that are submonoids of $\mathbb{N}^d$ with finite complement in $\mathbb{N}^d$. As the name suggests, this subject is thought as a generalization of the notion of *Numerical Semigroups*. Numerical semigroups are simply submonoids of $\mathbb{N}$ with finite complement in $\mathbb{N}$. This mathematical object has been widely studied in many aspects and it is closely related to other fields of mathematics (as Commutative Algebra and Algebraic Geometry). The story of numerical semigroups starts with the so called "Money changing problem", also known as "Frobenius' problem" by the mathematician who first questioned it: consider different coins with different values $a_1, a_2, \ldots, a_n$, suppose that the greatest common divisor of $a_1, a_2, \ldots, a_n$ is 1 and that for each value the availability of coins is very big (in such a way you may think it is infinite). What is the largest amount that cannot be reached using the available coins? Moreover, is it possible to obtain a formula, depending on $a_1, a_2, \ldots, a_n$, that provides such an amount? This problem was described in [43] and its solution in the case $n = 2$ was provided there, that is $a_1a_2 - a_1 - a_2$. This problem can be seen in the context of numerical semigroups asking what is the largest gap (the greatest number not in the semigroup) of the numerical semigroup generated by $a_1, a_2, \ldots, a_n$. Such an element is called the "Frobenius number" of the semigroup, with reference to the Frobenius problem. This problem has been largely studied, for instance it has been proved that a polynomial-type formula for the Frobenius problem does not exist for $n \geq 3$ (see [15]). Numerical semigroups are also related with some issues in commutative algebra, for instance if R is a one dimensional,

analitycally irreducible, local noetherian ring having the same residue class field with its integral closure $\overline{R}$ (these rings are related with monomial curves in algebraic geometry), then $\overline{R}$ is a *discrete valuation ring* (DVR) with valuation v, and the set of values $v(R)$ is a numerical semigroup whose properties are related with properties of the ring R. An example of this link has been shown in [32], in which it is proved that R is Gorenstein if and only if the associated value semigroup is symmetric. It is also possible to do the converse: starting from a fixed numerical semigroup one can produce several examples of a ring with the desidered properties as above. See [4] for a general reference on these arguments. Beyond these applications, numerical semigroups are yet widely studied as monoids: many properties and classes of these algbraic structures have been introduced and there exists a very large literature about them. Very good references for this topic are the monographs [2] and [40].
Submonoids of $\mathbb{N}^d$ viewed as generalizations of the concept of numerical semigroup have been considered from different perspectives, mainly connected to particular topics from commutative algebra. A first approach was to consider finitely generated submonoids of $\mathbb{N}^d$, called *affine semigroups*. In [28] the author describes how to associate a ring to a semigroup, called the *semigroup ring* and studies several properties of it. For an affine semigroup it was questioned in which cases the associated semigroup ring has particular properties, such as Cohen-Macaulayness, Gorensteiness or other. Some related results can be found in [44] and [38]. Other papers concerning affine semigroups are [37], [35] and we can cite also the more general monograph [39]. A second approach that I want to mention is contained in [3] where more general one dimensional rings, whose integral closure is the direct product of DVRs, are considered and in which a value semigroup that is a submonoid of $\mathbb{N}^d$ with peculiar properties can be considered. Such monoids are called *good semigroups* and they are now widely studied (for a general reference see [16]).
Submonoids of $\mathbb{N}^d$ with finite complement in $\mathbb{N}^d$, thought as a straightforward generalization of numerical semigroups, are considered for the first time in [22], where they are called *generalized numerical semigroups*. In that paper some basic features of such monoids are provided: inspired by [5] it is shown how to produce all generalized numerical semigroups with a fixed number of gaps (the elements of $\mathbb{N}^d$ not in the semigroup), called the *genus* of the semigroup, and asymptotic behaviors of the sequence of the number of such semigroups are studied. Furthermore some problems are proposed there. Among them we mention the following:

- produce a table with several data obtained by the given algorithm;

- find a different algorithm depending only on the fixed genus, since the one provided allows to compute all generalized numerical semigroups of genus g if all ones of genus $g-1$ are already computed;

- propose a generalization of a well known conjecture formulated for numerical semigroups, namely *Wilf's conjecture* [45].

The paper [22] represents the beginning of a systematic study of the generalized numerical semigroups: having in account the wide literature about numerical semigroups it is interesting to study which results are well suited to the new general context. This thesis is a possible starting point of this research. Its aim is to gather all the results obtained during the development of my Ph.D research project in this subject. Some of these results have been already published in the last year of this project, but not all. First we consider how to characterize the set of generators of a generalized numerical semigroup. The main tool for this purpose is the following result.

Let $d \geq 2$ and let $S = \langle A \rangle$ be the monoid generated by a set $A \subseteq \mathbb{N}^d$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ be the standard basis vectors of the vector space $\mathbb{R}^d$. Then S is a generalized numerical semigroup if and only if the set A fulfils the following conditions:

1. projections onto the coordinate axes are generators of a numerical semigroup;

2. for every i, k, $1 \leq i < k \leq d$ there exist $\mathbf{x}_{ik}, \mathbf{x}_{ki} \in A$ such that $\mathbf{x}_{ik} = \mathbf{e}_i + n_i^{(k)}\mathbf{e}_k$ and $\mathbf{x}_{ki} = \mathbf{e}_k + n_k^{(i)}\mathbf{e}_i$ with $n_i^{(k)}, n_k^{(i)} \in \mathbb{N}$.

Successively we look for some particular classes of generalized numerical semigroups in this new context. We focus on symmetric and pseudo-symmetric generalized numerical semigroups, since they are nicely characterized for numerical semigroups. For instance we have proven the following result, that allows to characterize those classes by the set of gaps of the semigroup.

Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup such that $|\mathbb{N}^d \setminus S| = g$.

1. S is symmetric if and only if there exists $\mathbf{f} = (f^{(1)}, \ldots, f^{(d)}) \in \mathbb{N}^d \setminus S$ such that $2g = (f^{(1)}+1)(f^{(2)}+1)\cdots(f^{(d)}+1)$.

2. S is pseudo-symmetric if and only if there exists $\mathbf{f} = (f^{(1)}, \ldots, f^{(d)}) \in \mathbb{N}^d \setminus S$ such that $2g-1 = (f^{(1)}+1)(f^{(2)}+1)\cdots(f^{(d)}+1)$.

We introduce in this general context an important tool, very useful for numerical semigroups, that is the *Apéry set*, and some results about it are provided. For instance, related to Apéry set, if S is a generalized numerical semigroup and $\mathbf{n} \in S$ we introduce the set

$$C(S, \mathbf{n}) = \{\mathbf{s} \in S \mid \mathbf{s} - \mathbf{n} \notin S, \mathbf{s} \leq \mathbf{h} + \mathbf{n} \text{ for some } \mathbf{h} \in \mathbb{N}^d \setminus S\}$$

where $\leq$ is the natural partial order in $\mathbb{N}^d$, and the following result is provided.

Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup, $\mathbf{n} \in S$ and $\prec$ a monomial order in $\mathbb{N}^d$. Then S is symmetric if and only if $C(S, \mathbf{n}) = \{\mathbf{a}_0 \prec \mathbf{a}_1 \prec \ldots \prec \mathbf{a}_t\}$ with $\mathbf{a}_i + \mathbf{a}_{t-i} = \mathbf{a}_t$, for $i = 0, 1, \ldots, t$.

The mentioned questions posed in [22] are also studied here:

- algorithms to manage various features of this subject are introduced and implemented in the computer algebra software GAP [24];
- tables with several computational data are provided;
- a generalized Wilf's conjecture is proposed and it is tested on nice examples of semigroups.

We mention also that other interesting researches are yet developing from the paper [22]. We want to cite in particular [26] and the more recent [25] and [19], where new kinds of affine semigroups are defined of which generalized numerical semigroups are a particular case.

We summarize the structure of this thesis. It is structured in seven chapters as follows.
In Chapter 1 we provide a brief collection of definitions and results (without proofs) about numerical semigroups. In particular we show the most important features and we focus mainly on the arguments that we will generalize in $\mathbb{N}^d$.
In Chapter 2 we study the first basic properties of a generalized numerical semigroup: we provide a characterization of its set of generators and we emphasize the most important problem that arises in this new context, that is the definition of a *Frobenius element*. In $\mathbb{N}^d$ there is not a natural total order so it is not immediately clear how to define the Frobenius element for a generalized numerical semigroup (as it is for numerical semigroups). In [22] this aim is reached by defining *relaxed monomial orders*. The Frobenius

element of a generalized numerical semigroup is uniquely determined with respect to the defined relaxed monomial order.

The Frobenius element plays an important role in the class of *irreducible numerical semigroups*.

In Chapter 3 we introduce the concept of irreducibility also for generalized numerical semigroups, showing that in such a case it is possible to have a unique Frobenius element, independent of any total order defined on $\mathbb{N}^d$, and providing some nice generalizations of the existing characterizations for irreducible numerical semigroups.

Among irreducible generalized numerical semigroups there are the *symmetric* generalized numerical semigroups, which suggest us how to generalize Wilf's conjecture. So in Chapter 4 we give the statement of a possible generalized Wilf's conjecture and we start a first general study of it. We compare this generalized Wilf's conjecture with the extension of Wilf's conjecture to affine semigroups given in [26].

Chapter 5 is devoted to describe some classes of generalized numerical semigroups, in which we find minimal generators and test generalized Wilf's conjecture.

In Chapter 6 we introduce the *Apéry set* for generalized numerical semigroups, in particular we focus on irreducible generalized numerical semigroups providing some properties that generalize the analogous for numerical semigroups.

We conclude this thesis with a collection of algorithms in order to do computations with generalized numerical semigroups. Furthermore we provide some computational results obtained by implementation of the previous algorithms in GAP [24]. Some of these implementations are actually included in the GAP package `numericalsgps` [17].

The GAP codes of some implementations, not yet in the package, are provided in the Appendix.

Some of the original results presented in this thesis are contained in the published papers [13] and [12], and in the papers [10] and [11] submitted for publication.

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Chapter 1

Basics on numerical semigroups

Numerical semigroups are the starting point of this research, the basics of our generalizations. There exists a wide literature about this argument and several authors made intensive studies on it. The subject is well-known and here we aim to give the most important features of this argument and the most interesting properties involved in the successive chapters, in particular the semigroup tree, irreducible numerical semigroups and Wilf's conjecture. In this chapter we give a list of results without proofs. The proofs of all cited results can be found in [40] (chapters 1, 2 and 3), a very good reference for numerical semigroups.

1.1 Basic definitions and properties

Definition 1.1.1. Let $S \subseteq \mathbb{N}$. S is a *numerical semigroup* if S is a submonoid of $\mathbb{N}$ and $\mathbb{N} \setminus S$ is a finite set.

If S is a numerical semigroup an element $x \in \mathbb{N} \setminus S$ is called an *hole* (or *gap*) of S, the set $\mathrm{H}(S) = \mathbb{N} \setminus S$ is called the *set of holes* of S.
The *genus* of S is the cardinality of its set of holes, that is $\mathrm{g}(S) = |\mathbb{N} \setminus S|$.

Example 1.1.2. Let $S = \{0, 2, 4, 6, 8, \rightarrow\}$, where the rightarrow means that all integers greater than 8 belong to S. S is a numerical semigroup of genus 4 with $\mathrm{H}(S) = \{1, 3, 5, 7\}$. The sets $\{1, 5, \rightarrow\}$ and $\{2, 3, 5, \rightarrow\}$ are not numerical semigroups.

An useful tool for studying numerical semigroups is the following set, named so in honour of R. Apéry.

Definition 1.1.3. Let S be a numerical semigroup and $n \in S^*$. The *Apéry set* of S with respect to n is:

$$\mathrm{Ap}(S, n) = \{s \in S | s - n \notin S\}$$

Lemma 1.1.4 ([40], Lemma 2.4). *Let S be a numerical semigroup and $n \in S \setminus \{0\}$. Then $\mathrm{Ap}(S, n) = \{0, w(1), w(2), \ldots, w(n-1)\}$. where $w(i)$ is the least element of S congruent with i modulo n, for all $i \in \{1, 2, \ldots, n-1\}$.*

Let $A \subseteq \mathbb{N}$, we define the submonoid of $\mathbb{N}$ generated by A as the set $\langle A \rangle = \{\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n | n \in \mathbb{N}; \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{N}; a_1, a_2, \ldots, a_n \in A\}$. A monoid S is generated by $A \subseteq S$ if $S = \langle A \rangle$, if no proper subset of A generates S then A is called a minimal set of generators.

Lemma 1.1.5 ([40], Lemma 2.6). *Let S be a numerical semigroup and $n \in S^*$. Then for all $s \in S$ there exists an unique element $(k, w) \in \mathbb{N} \times \mathrm{Ap}(S, n)$ such that $s = kn + w$. In particular $\mathrm{Ap}(S, n) \cup \{n\} \setminus \{0\}$ generates S.*

Theorem 1.1.6 ([40], Theorem 2.7). *Every numerical semigroup S admits a unique finite minimal set of generators.*

Example 1.1.7. Let $S = \{0, 5, 7, 9, 10, 12, 14, \rightarrow\}$, we have $\mathrm{Ap}(S, 5) = \{0, 7, 9, 16, 18\}$. $(\mathrm{Ap}(S, 5) \cup \{5\}) \setminus \{0\} = \{5, 7, 9, 16, 18\}$ is a set of generators for S but it is not minimal. The minimal set of generators for S is $\{5, 7, 9\}$.

The following is the main property that characterizes the set of generators of a numerical semigroup. Recall that with $\gcd(A)$ we refer to the greatest common divisor of the elements in A.

Proposition 1.1.8 ([40], Lemma 2.1). *Let S be a submonoid of $\mathbb{N}$ generated by a set A. Then S is a numerical semigroup if and only if $\gcd(A) = 1$.*

A consequence of the previous is the following.

Proposition 1.1.9. *Let M be a submonoid of $\mathbb{N}$. Then M is isomorphic to a numerical semigroup.*

Definition 1.1.10. Let S be a numerical semigroup minimally generated by $\{n_1 < n_2 < \cdots < n_p\}$. The element n_1 is called the *multiplicity* of S and denoted by $\mathrm{m}(S)$. The cardinality of the minimal set of generators of S is called the *embedding dimension* and denoted by $\mathrm{e}(S)$.

By Lemma 1.1.5, $\mathrm{m}(S) \cup \mathrm{Ap}(S, \mathrm{m}(S)) \setminus \{0\}$ is a set of generators of S whose cardinality is $\mathrm{m}(S)$, so we have the following property.

Proposition 1.1.11 ([40], Proposition 2.10)**.** *Let S be a numerical semigroup. Then* $\mathrm{e}(S) \leq \mathrm{m}(S)$*.*

The first appearance of numerical semigroups arises with a problem posed by Frobenius: let $a_1, \ldots, a_n$ be positive integers. Which is the greatest integer that cannot be expressed by a linear combination of those elements with nonnegative integer coefficients? This problem is also known as "the money changing problem" in which we suppose that $a_1, \ldots, a_n$ are the values of a great number of coins ("almost infinity") for each a_i and we wants to know the largest amount that cannot be obtained by the values of those coins. If $\gcd(a_1, \ldots, a_n)$ is 1 then we look for the greatest element in the set of holes of the numerical semigroup generated by $\{a_1, \ldots, a_n\}$.

Definition 1.1.12. Let S be a numerical semigroup. The *Frobenius element* of S, denoted by $\mathrm{F}(S)$, is the greatest element in $\mathbb{N} \setminus S$. The element $\mathrm{c}(S) := \mathrm{F}(S)+1$ is the *conductor* of S. The minimal generators of S greater than $\mathrm{F}(S)$ are called *effective generators*.

The Apéry set with respect to a nonzero element of a numerical semigroup S allows us to compute $\mathrm{F}(S)$ e $\mathrm{g}(S)$.

Proposition 1.1.13 ([40], Proposition 2.12)**.** *Let S be a numerical semigroup and $n \in S \setminus \{0\}$. Then*

1. $\mathrm{F}(S) = max(\mathrm{Ap}(S, n)) - n$*.*
2. $\mathrm{g}(S) = \frac{1}{n}(\sum_{w \in \mathrm{Ap}(S,n)} w) - \frac{n-1}{2}$*.*

If $S = \langle a, b\rangle$, that is $\mathrm{e}(S) = 2$, we have $\mathrm{Ap}(S, a) = \{0, b, 2b, \ldots, (a-1)b\}$ and so:

Corollary 1.1.14. *Let $S = \langle a, b\rangle$ be a numerical semigroup. Then*

1. $\mathrm{F}(S) = ab - a - b$*.*
2. $\mathrm{g}(S) = \frac{ab-a-b+1}{2}$*.*

In particular $\mathrm{g}(S) = \frac{\mathrm{F}(S)+1}{2}$*.*

For all numerical semigroups a more general expression is satisfied:

Proposition 1.1.15 ([40], Proposition 2.14)**.** *Let S be a numerical semigroup. Then*

$$\mathrm{g}(S) \geq \frac{\mathrm{F}(S)+1}{2}$$

.

1.2 The semigroup tree

Let $g \in \mathbb{N}$. We denote S_g the set of all numerical semigroups of genus g. The set S_g of all numerical semigroups of genus g can be generated by the set S_{g-1} of all numerical semigoups of genus $g-1$, in a way that we briefly explain here. In the next chapter we will explain these facts in a more accurate way and in a more general case. The following properties, whose proof are quite easy, are decisive in this sense:

Lemma 1.2.1. *Let S be a numerical semigroup with Frobenius number $\mathrm{F}(S)$. Then $T = S \cup \{\mathrm{F}(S)\}$ is a numerical semigroup, moreover $\mathrm{F}(S)$ is an effective generator in T.*

Lemma 1.2.2. *Let S be a numerical semigroup and $h \in S$ a minimal generator. Then $T = S \setminus \{h\}$ is a numerical semigroup. Furthermore, if h is an effective generator in S, then the Frobenius number of T is $\mathrm{F}(T) = h$.*

Let $\mathcal{S}$ be the set of all numerical semigroups. By the previous results $\mathcal{S}$ can be arranged as a rooted tree, called *semigroup tree*, in the following way: let S be a numerical semigroup of genus $g-1$, we find in S the effective generators, for instance $h_1, \dots, h_m$. From S we produce the numerical semigroups $S \setminus \{h_1\}, \dots, S \setminus \{h_m\}$, that are named the *sons* of S. If we repeat this procedure for all numerical semigroups of genus $g-1$, the two lemmas guarentee that this will produce all numerical semigroups of genus g without redundancy. If we start from the *root*, that is the trivial numerical semigroup $\mathbb{N}$ of genus 0, the structure of the tree is complete.

In the same way, it is possibile to produce algorithmically the set $\mathcal{S}_g$ of all numerical semigroups of genus g, for any g.
An algorithm for computing all numerical semigroups of a given genus is described in [5] and [6], by Maria Bras-Amorós. In particular, let $N_g = |S_g|$, she was able to compute the values of N_g up to $g = 50$. Observing the data, she conjectured that the sequence $\{N_g\}_{g\in\mathbb{N}}$ has a behaviour like a Fibonacci sequence.

Conjecture 1.2.3 ([5])**.** *The following conjectured are proposed:*

1) $\lim_{g\to\infty} \frac{N_{g-1}+N_{g-2}}{N_g} = 1$

2) $\lim_{g\to\infty} \frac{N_g}{N_{g-1}} = \phi = \frac{1+\sqrt{5}}{2}$

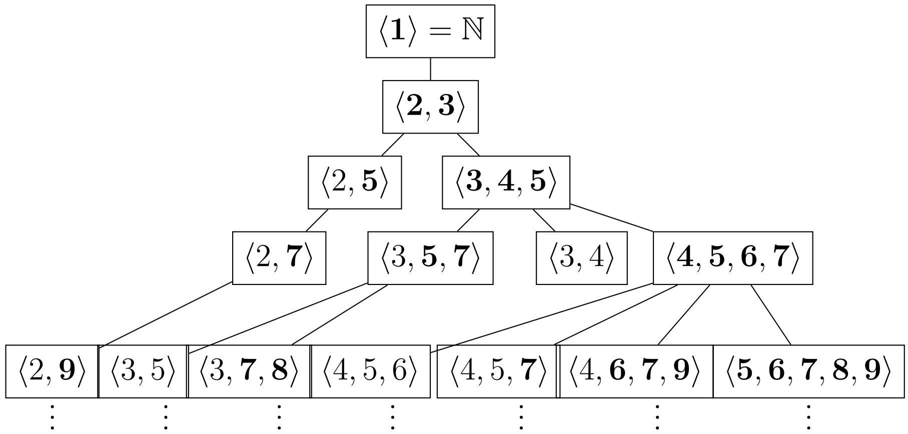


Figure 1.1: The semigroup tree up to genus 4

3) $N_g \geq N_{g-1} + N_{g-2}$ *for every* $g \geq 1$.

It has been developed an intense study about the truth of that conjecture (see for istance [6, 8] and [48]). The most important result is actually that of Alex Zhai:

Theorem 1.2.4 ([47], Theorem 1)**.** *For every* $g \in \mathbb{N}$*, let* N_g *be the number of all numerical semigroups of genus* g *and* $\phi = \frac{1+\sqrt{5}}{2}$ *the golden ratio, then:*

$$\lim_{g \to \infty} \frac{N_g}{\phi^g} = k$$

when k *is a positive constant.*

This result confirms that the first two conjectures are correct and that $N_{g+1} \geq N_g$ for all but finitely many g (conjecture to hold for all g by [30]).

1.3 Symmetric and pseudo-symmetric numerical semigroups

Definition 1.3.1. A numerical semigroup S is called *irreducible* if it cannot be expressed as an intersection of two numerical semigroups properly containing it.

Definition 1.3.2. Let S be an irreducible numerical semigroup. S is called *symmetric* if $\mathrm{F}(S)$ is odd, *pseudo-symmetric* if $\mathrm{F}(S)$ is even.

In the next proposition we show a characterization of symmetric and pseudo-symmetric numerical semigroups.

Proposition 1.3.3 ([40], Proposition 4.4)**.** *Let S a numerical semigroup, the following are verified:*

1. *S is symmetric if and only if $\mathrm{F}(S)$ is odd and for all $x \in \mathbb{Z} \setminus S$ we have $\mathrm{F}(S) - x \in S$.*

2. *S is pseudo-symmetric if and only if $\mathrm{F}(S)$ is even and for all $x \in \mathbb{Z} \setminus S$ we have $\mathrm{F}(S) - x \in S$ or $x = \mathrm{F}(S)/2$.*

A consequence of the previous result is the following corollary.

Corollary 1.3.4 ([40], Corollary 4.5)**.** *Let S be a numerical semigroup. Then*

1. *S is symmetric if and only if $\mathrm{g}(S) = \frac{\mathrm{F}(S)+1}{2}$.*

2. *S is pseudo-symmetric if and only if $\mathrm{g}(S) = \frac{\mathrm{F}(S)+2}{2}$.*

Example 1.3.5. Consider the following:

1. $S = \langle 4, 6, 7 \rangle = \{0, 4, 6, 7, 8, 10, 11, \rightarrow\}$ is symmetric

2. $S = \langle 3, 4, 5 \rangle = \{0, 3, \rightarrow\}$ is pseudo-symmetric.

3. $S = \langle 5, 7, 9 \rangle = \{0, 5, 7, 9, 10, 12, 14, 15, 17, \rightarrow\}$ is not pseudo-symmetric, since $\mathrm{F}(S) = 16$ is even, but $16 - 13 = 3 \notin S$. So it is not irreducible, in fact S is the intersection of $S_1 = \langle 5, 7, 9, 16 \rangle = \{0, 5, 7, 9, 10, 12, 14, 15, 16, 17, \rightarrow\}$ and $S_2 = \langle 5, 7, 9, 13 \rangle = \{0, 5, 7, 9, 10, 12, 13, 14, 15, 17, \rightarrow\}$.

Example 1.3.6. A very useful tool for computations with numericalsemigroup is the `GAP` package `numericalsgps` [17]. It allows to compute all mentioned invariants of a fixed numerical semigroup, to test properties like to be symmetric or pseudo-symmetric, and many other things. We show below and example of GAP session:

```
gap> s:=NumericalSemigroup(4,9,10,19);;
gap> EmbeddingDimension(s);
3
gap> MinimalGenerators(s);
```

```
[ 4, 9, 10 ]
gap> FrobeniusNumber(s);
15
gap> Gaps(s);
[ 1, 2, 3, 5, 6, 7, 11, 15 ]
gap> AperyList(s,4);
[ 0, 9, 10, 19 ]
gap> IsSymmetric(s);
true
gap> IsPseudoSymmetric(s);
false
```

Other characterizations of symmetric and pseudo-symmetric numerical semigroups involve the Apéry set of the semigroup.

Proposition 1.3.7 ([40], Proposition 4.10)**.** *Let S be a numerical semigroup and let n be a positive integer of S. Let $\mathrm{Ap}(S,n) = \{a_0 < a_1 < \cdots < a_{n-1}\}$ be the Apéry set of n in S. Then S is symmetric if and only if $a_i + a_{n-1-i} = a_{n-1}$ for all $i \in \{0, 1, \ldots, n-1\}$.*

Proposition 1.3.8 ([40], Proposition 4.15)**.** *Let S be a numerical semigroup with even Frobenius number and let n be a positive integer of S. Then S is pseudo-symmetric if and only if*

$$\mathrm{Ap}(S,n) = \{a_0 < a_1 < \cdots < a_{n-2} = \mathrm{F}(S) + n\} \cup \{\frac{\mathrm{F}(S)}{2} + n\}$$

and $a_i + a_{n-2-i} = a_{n-2}$ for all $i \in \{0, 1, \ldots, n-2\}$.

If S is a numerical semigroup the following sets are defined:

- $\mathrm{PF}(S) = \{h \in \mathrm{H}(S) \mid h + s \in S \text{ for all } s \in S\}$
- $\mathrm{SG}(S) = \{h \in \mathrm{H}(S) \mid 2h \in S,\ h + s \in S \text{ for all } s \in S\}$

The next properties are known in literature as corollaries of some propositions:

Corollary 1.3.9. *Let S be a numerical semigroup. The following are verified:*

1. *S is irreducible if and only if $|\,\mathrm{SG}(S)| = 1$.*
2. *S is symmetric if and only if $|\,\mathrm{PF}(S)| = 1$.*
3. *S is pseudo-symmetric if and only if $\mathrm{PF}(S) = \{\mathrm{F}(S), \frac{\mathrm{F}(S)}{2}\}$*

1.4 Wilf's conjecture

Another invariant associated to a numerical semigroup S is

$$\mathrm{n}(S) = |\{s \in S | s < \mathrm{F}(S)\}|.$$

It is involved in a famous conjecture about numerical semigroups.

Conjecture 1.4.1 (**Wilf's conjecture** [45]). *Let S be a numerical semigroup. Then*

$$\mathrm{e}(S)\,\mathrm{n}(S) \geq \mathrm{F}(S) + 1$$

The only known examples of numerical semigroups S for which Wilf's conjecture is satisfied as an equality are those ones with $\mathrm{e}(S) = 2$ and those described in the following:

Example 1.4.2. Let $m, g \geq 1$ be positive integers. Consider the semigroup:

$$S = m\mathbb{N} \cup (qm + \mathbb{N}) = \{0, m, 2m, \ldots, (q-1)m, qm, qm+1, qm+2, \rightarrow\}$$

In this case $\mathrm{c}(S) = qm$, $\mathrm{n}(S) = q$ and $\mathrm{e}(S) = m$ since S is minimally generated by $\{m, qm+1, qm+2, \ldots, qm+m-1\}$.
Observe that if $q = 1$ then $S = \{0, m, m+1, m+2, \ldots\}$. In this case we call S an *ordinary* numerical semigroup.

It has been proved that Wilf's conjecure is satisfied by several classes of numerical semigroups, but it has not been proved to be true for every numerical semigroup. In this section we gather some of the known properties that allow the affermative answer to Wilf's conjecture for a numerical semigroup. For a more complete and exhaustive survey about the study of Wilf's conjecture see [18]. In the first proposition we define $\mathrm{t}(S) = |\operatorname{PF}(S)|$.

Proposition 1.4.3 ([40, 20]). *Let S be a numerical semigroup. Then*

$$\mathrm{F}(S) + 1 \leq \mathrm{n}(S)(\mathrm{t}(S) + 1).$$

In particular if $\mathrm{t}(S) + 1 \leq \mathrm{e}(S)$ then S satisfies Wilf's conjecture.

Proposition 1.4.4 ([20]). *Let S be a numerical semigroup. Then S satisfies Wilf's conjecture in the following cases:*

- *S is symmetric or pseudo-symmetric.*
- *S is of* maximal embedding dimension*, that is* $\mathrm{e}(S) = \mathrm{m}(S)$*.*
- $\mathrm{e}(S) \leq 3$*.*
- $\mathrm{F}(S) \leq 20$*.*
- $\mathrm{n}(S) \leq 4$*.*
- $\mathrm{n}(S) \geq \frac{\mathrm{F}(S)+1}{4}$*.*

Proposition 1.4.5 ([42])**.** *Let S be a numerical semigroup such that $2\,\mathrm{e}(S) \geq \mathrm{m}(S)$. Then S satisfies Wilf's conjecture.*

The next result is an asymptotic version of Wilf's conjecture. It was given and proved by Alex Zhai.

Proposition 1.4.6 ([46])**.** *Fix a positive integer k. Then for every $\epsilon > 0$ we find that*

$$\frac{\mathrm{n}(S)}{\mathrm{c}(S)} > \frac{1}{k} - \epsilon$$

for all but finitely many numerical semigroups S satisfing $\mathrm{e}(S) = k$.

This result says in particular that among all numerical semigroups with fixed embedding dimension, there is only a finite number of possible numerical semigroups that do not satisfy Wilf's conjecture.
Another important result is the following:

Theorem 1.4.7 ([21])**.** *Let S be a numerical semigroup. Then S satisfies Wilf's conjecture in each of the following cases:*

- $\mathrm{c}(S) \leq 3\,\mathrm{m}(S)$*.*
- $\gcd(\{s \in S \mid s < \mathrm{F}(S)\}) \geq 2$

The importance of the previous result arises from the fact, proved by A. Zhai ([47]), that as g goes to infinity, the proportion of numerical semigroups S of genus g satisfying $\mathrm{c}(S) \leq 3\,\mathrm{m}(S)$ tends to 1. So Wilf's conjecture is asymptotically true as $g \to \infty$.
We conclude providing the computational approach to verify Wilf's conjecture: as expressed in a previous section, it is possible to produce all numerical semigroups of a given genus, so it is possible to verify Wilf's conjecture for all

semigroups up to a given genus. In [5] it is verified that all numerical semigroups up to genus $g = 50$ satisfies Wilf's conjecture and this bound has been improved by [23]:

Proposition 1.4.8 ([23]). *Every numerical semigroup S of genus $\mathrm{g}(S) \leq 60$ satisfies Wilf's conjecture.*

Another very recent computational result, in which a different tecnique is considered, is the following:

Proposition 1.4.9 ([9]). *Every numerical semigroup S of multiplicity $\mathrm{m}(S) \leq 17$ satisfies Wilf's conjecture.*

We rember that in [18] it is provided a more complete and exhaustive survey about the study of Wilf's conjecture.

Chapter 2

Generalized numerical semigroups

Now we introduce the core of this work: generalized numerical semigroups. This chapter is devoted to provide the basic facts about them. In particular we show that they are finitely generated as monoids, and which are their possible sets of generators. The results contained in Section 2.1 are included in the paper [13]. Moreover, we focus on the first main difference with respect to numerical semigroups that naturally appears, that is the missing of a natural total order in $\mathbb{N}^d$, and we generalize the building of the semigroup tree considered in the second section of the previous chapter. The chapter ends with a first brief discussion on how it would be possible to choose an anologous of the Frobenius number in this more general context, and when this choice could be unique.

In the following we denote by $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$ the standard basis vectors in $\mathbb{R}^d$ (that is, for $i = 1, \dots, d$, $\mathbf{e}_i$ is the vector whose i-th component is 1 and the other components are zero). Furthermore, if $A \subseteq \mathbb{N}^d$, we denote $\langle A \rangle = \{\lambda_1 \mathbf{a}_1 + \cdots + \lambda_n \mathbf{a}_n \mid \lambda_1, \dots, \lambda_n \in \mathbb{N}, \mathbf{a}_1, \dots, \mathbf{a}_n \in A\}$, that is, the submonoid of $\mathbb{N}^d$ generated by the set A. Moreover, if $\mathbf{t} \in \mathbb{N}^d$, its i-th component is usually denoted by $t^{(i)}$. We denote by $\leq$ the natural partial order on $\mathbb{N}^d$, that is if $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$, $\mathbf{x} \leq \mathbf{y}$ if and only if $x^{(i)} \leq y^{(i)}$ for every $i = 1, \dots, d$.

2.1 Generalized numerical semigroups and its generators

Definition 2.1.1. Let $S \subseteq \mathbb{N}^d$ be a monoid. S is a *Generalized Numerical Semigroup* if the set $\mathrm{H}(S) = \mathbb{N}^d \setminus S$ is finite. The elements in $\mathrm{H}(S)$ are called *holes* (or *gaps*) of S and the number $\mathrm{g}(S) = |\,\mathrm{H}(S)|$ is called the *genus* of S.

The previous definition is clearly a generalization of the corresponding for numerical semigroup and it has been provided for the first time, in this shape, in [22]. The aim of this section is to study basic properties of a generalized numerical semigroup in order to characterize its minimal system of generators. Some of these properties generalize analogous ones of a classical numerical semigroup. At first we prove that every generalized numerical semigroup in $\mathbb{N}^d$ has a unique minimal system of generators. Then we prove that a finite set $A \subseteq \mathbb{N}^d$ generates a generalized numerical semigroup if and only if the elements in A satisfy certain conditions.

Lemma 2.1.2. [40, Lemma 2.3] *Let S be a submonoid of $\mathbb{N}^d$. Then $S^* \setminus (S^* + S^*)$ is a system of generators for S. Moreover, every system of generators of S contains $S^* \setminus (S^* + S^*)$.*

Lemma 2.1.3. *Let S be a generalized numerical semigroup of genus g with $\mathrm{H}(S) = \{\boldsymbol{h}_1, \boldsymbol{h}_2, \dots, \boldsymbol{h}_{g-1}, \boldsymbol{h}\}$. Let $\boldsymbol{h}$ be a maximal element in $\mathrm{H}(S)$ with respect to the natural partial order in $\mathbb{N}^d$. Then $S' = S \cup \{\boldsymbol{h}\}$ is a generalized numerical semigroup, in particular $\mathrm{H}(S') = \{\boldsymbol{h}_1, \boldsymbol{h}_2, \dots, \boldsymbol{h}_{g-1}\}$ and S' has genus $g-1$.*

Proof. Let $S' = \langle S \cup \{\mathbf{h}\}\rangle$. S' is a generalized numerical semigroup since $S \subseteq S' = \langle S \cup \{\mathbf{h}\}\rangle$, in particular $\mathrm{H}(S) \supsetneq \mathrm{H}(S')$. Let us prove that S' has genus $g-1$. We suppose there exists $\mathbf{h}_j \in \mathrm{H}(S)$, $j \in \{1, \dots, g-1\}$, such that $\mathbf{h}_j \in S' = \langle S \cup \{\mathbf{h}\}\rangle$. Then $\mathbf{h}_j = \sum_k \mu_k \mathbf{g}_k + \lambda \mathbf{h}$, with $\mathbf{g}_k \in S$. If $\lambda = 0$ then $\mathbf{h}_j \in S$, contradiction. If $\lambda \neq 0$ then $\mathbf{h}_j \geq \mathbf{h}$ contradicting the maximality of $\mathbf{h}$ in $\mathrm{H}(S)$. So $\mathbf{h}_j \notin S'$ for $j \in \{1, \dots, g-1\}$, hence $\mathrm{H}(S') = \{\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_{g-1}\}$. ☐

Proposition 2.1.4. *Every generalized numerical semigroup admits a finite system of generators.*

Proof. Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup. We prove the statement by induction on the genus g of S. If $g = 0$ then $S = \mathbb{N}^d$, that is generated by the standard basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d\}$.
Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup of genus $g+1$ and let $\mathbf{h}$ be a

maximal element in $\mathrm{H}(S)$ with respect to the natural partial order in $\mathbb{N}^d$. By Lemma 2.1.3 $S' = S \cup \{\mathbf{h}\}$ is a generalized numerical semigroup in $\mathbb{N}^d$ of genus g, that is finitely generated by induction hypothesis. Hence let $\mathrm{G}(S')$ be a finite system of generators for S'. We have $\mathbf{h} \in \mathrm{G}(S')$ because $\mathbf{h}$ cannot belong to S. So $\mathrm{G}(S') \subset S \cup \{\mathbf{h}\}$ and we can denote $\mathrm{G}(S') = \{\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_s, \mathbf{h}\}$ with $\mathbf{g}_i \in S$ for every $i = 1, 2, \dots, s$. Let $\mathcal{B} = \{\mathbf{g}_1, \dots, \mathbf{g}_s, \mathbf{h}+\mathbf{g}_1, \mathbf{h}+\mathbf{g}_2, \dots, \mathbf{h}+\mathbf{g}_s, 2\mathbf{h}, 3\mathbf{h}\}$. By the maximality of $\mathbf{h}$ in $\mathrm{H}(S)$ we have $\mathcal{B} \subset S$ and furthermore it is easy to prove that $\mathcal{B}$ is a system of generators for S. Hence S is finitely generated. $\square$

Corollary 2.1.5. *Every generalized numerical semigroup admits a unique finite system of minimal generators.*

Proof. By lemma 2.1.2 every generalized numerical semigroup admits a unique system of minimal generators, that is $S^* \setminus (S^* + S^*)$, which is contained in every system of generators. By Proposition 2.1.4 such a system of generators is finite. $\square$

Definition 2.1.6. Let $\mathbf{t} \in \mathbb{N}^d$, we define the set $\pi(\mathbf{t}) = \{\mathbf{n} \in \mathbb{N}^d \mid \mathbf{n} \leq \mathbf{t}\}$ where $\leq$ is the natural partial order defined in $\mathbb{N}^d$.

Remark 2.1.7. Notice that for every $\mathbf{t} \in \mathbb{N}^d$ the set $\pi(\mathbf{t})$ is finite and it represents the set of integer points of the hyper-rectangle whose vertices are $\mathbf{t}$, its projections on the coordinate planes, the origin of axes, and the points in the coordinate axes $(t^{(1)}, 0, \dots, 0), (0, t^{(2)}, 0, \dots, 0), \dots, (0, \dots, 0, t^{(d)})$. If $\mathbf{s} \notin \pi(\mathbf{t})$ then $\mathbf{s}$ has at least one component larger than the respective of $\mathbf{t}$.

Lemma 2.1.8. *Let $S \subseteq \mathbb{N}^d$ be a monoid. Then S is a generalized numerical semigroup if and only if there exists $\boldsymbol{t} \in \mathbb{N}^d$ such that for all elements $\boldsymbol{s} \notin \pi(\boldsymbol{t})$ then $\boldsymbol{s} \in S$.*

Proof. Let S be a generalized numerical semigroup in $\mathbb{N}^d$ whose hole set is $\mathrm{H}(S) = \{\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_g\}$. Let $t^{(i)} \in \mathbb{N}$ be the largest number appearing in the i-th coordinate of elements in $\mathrm{H}(S)$ for $i \in \{1, \dots, d\}$, in other words $t^{(i)} = \max\{h_1^{(i)}, h_2^{(i)}, \dots, h_g^{(i)}\}$. It is easy to see that $\mathbf{t} = (t^{(1)}, t^{(2)}, \dots, t^{(d)}) \in \mathbb{N}^d$ fulfils the thesis.
Conversely, let $\mathbf{t} \in \mathbb{N}^d$ be an element such that for every $\mathbf{s} \notin \pi(\mathbf{t})$ it is $\mathbf{s} \in S$. Therefore if $\mathbf{h} \in \mathbb{N}^d \setminus S$ then $\mathbf{h} \in \pi(\mathbf{t})$, that is $(\mathbb{N}^d \setminus S) \subseteq \pi(\mathbf{t})$ and since $\pi(\mathbf{t})$ is a finite set then S is a generalized numerical semigroup. $\square$

The previous lemma provides an useful tool to prove a characterization of a system of generators for a generalized numerical semigroup in $\mathbb{N}^d$. For the sake of clearness we give first a proof in the particular case $d = 2$, which is simpler. For the proof of the next two theorems we consider that the Frobenius number of $\mathbb{N}$ (the trivial numerical semigroup) is 0, altough it is usually defined to be -1 in the existing literature.

Theorem 2.1.9. *Let $S = \langle A \rangle \subseteq \mathbb{N}^2$ be the monoid generated by a set A. Then S is a generalized numerical semigroup if and only if the set A fulfils the following conditions:*

1. *There exist $(0, a_1), (0, a_2), \ldots, (0, a_n) \in A$, for some $n \in \mathbb{N}\setminus\{0\}$, such that $\gcd(a_1, \ldots, a_n) = 1$ (that is $a_1, \ldots, a_n$ generate a numerical semigroup).*
2. *There exist $(b_1, 0), (b_2, 0), \ldots, (b_m, 0) \in A$, for some $m \in \mathbb{N} \setminus \{0\}$, such that $\gcd(b_1, \ldots, b_m) = 1$ (that is $b_1, \ldots, b_m$ generate a numerical semigroup).*
3. *There exist $r_1, r_2 \in \mathbb{N}$ such that $(r_1, 1), (1, r_2) \in A$.*

Proof. $\Rightarrow$) If A does not contain elements like $(0, a_i)$, for $i = 1, \ldots, n$, such that $a_1, \ldots, a_n$ generate a numerical semigroup then (in y axis) there are infinite elements $(0, h_j)$ not belonging to S, the same argument holds for the elements $(b_i, 0)$ in the x-axis. Furthermore if A does not contain an element like $(r_1, 1)$, then for all $n \in \mathbb{N}$ we have $(n, 1) \notin S$. Moreover if A does not contain an element like $(1, r_2)$, then for all $n \in \mathbb{N}$ we have $(1, n) \notin S$.
$\Leftarrow$) Let S_1, S_2 be the numerical semigroups generated respectively by $\{b_1, \ldots, b_m\}$ and $\{a_1, \ldots, a_n\}$, and let F_1, F_2 be the respective Frobenius numbers. Notice that for all $n \in \mathbb{N} \setminus \{0\}$ we have $(F_1 + n, 0), (0, F_2 + n) \in S$.
Let $\mathbf{v} = (F_2 r_1 + F_1, F_1 r_2 + F_2) \in \mathbb{N}^2$. We prove that for every $(x, y) \notin \pi(\mathbf{v})$ we have $(x, y) \in S$, hence S is a generalized numerical semigroup by Lemma 2.1.8. So, let $(x, y) \in \mathbb{N}^2$. Suppose at first that $x > F_2 r_1 + F_1$, then there exists $n_x \in \mathbb{N} \setminus \{0\}$ such that $x = F_2 r_1 + F_1 + n_x$, in particular $(x, 0) \in S$. We distinguish the following two cases:

a) If $y > F_2$ then $(0, y) \in S$ and $(x, y) = (x, 0) + (0, y) \in S$.

b) If $y \leq F_2$ then $y r_1 \leq F_2 r_1$, so there exists $p \in \mathbb{N}$ such that $F_2 r_1 = y r_1 + p$. Therefore we have $(x, y) = (y r_1 + p + F_1 + n_x, y) = (p + F_1 + n_x, 0) + y(r_1, 1) \in S$

If $y > F_1 r_2 + F_2$ the assertion follows by the same argument. We conclude that $(x, y) \in S$, so S is a generalized numerical semigroup. $\square$

Theorem 2.1.10. *Let $d \geq 2$ and let $S = \langle A \rangle$ be the monoid generated by a set $A \subseteq \mathbb{N}^d$. Then S is a generalized numerical semigroup if and only if the set A fulfils each one of the following conditions:*

1. *For all $j = 1, 2, \ldots, d$ there exist $a_1^{(j)}\boldsymbol{e}_j, a_2^{(j)}\boldsymbol{e}_j, \ldots, a_{r_j}^{(j)}\boldsymbol{e}_j \in A$, $r_j \in \mathbb{N} \setminus \{0\}$, such that $\gcd(a_1^{(j)}, a_2^{(j)}, \ldots, a_{r_j}^{(j)}) = 1$ (that is, the elements $a_i^{(j)}$, $1 \leq i \leq r_j$, generate a numerical semigroup).*

2. *For every i, k, $1 \leq i < k \leq d$ there exist $\boldsymbol{x}_{ik}, \boldsymbol{x}_{ki} \in A$ such that $\boldsymbol{x}_{ik} = \boldsymbol{e}_i + n_i^{(k)}\boldsymbol{e}_k$ and $\boldsymbol{x}_{ki} = \boldsymbol{e}_k + n_k^{(i)}\boldsymbol{e}_i$ with $n_i^{(k)}, n_k^{(i)} \in \mathbb{N}$.*

Proof. $\Rightarrow$) If A does not satisfy the first condition for some j then there exist infinite elements $a\mathbf{e}_j$, $a \in \mathbb{N} \setminus \{0\}$, which do not belong to S. If A does not satisfy the second condition for some $i \neq j$, then there are infinite elements $\mathbf{e}_i + n\mathbf{e}_k$ with $n \in \mathbb{N} \setminus \{0\}$ which do not belong to S.
$\Leftarrow$) For every $j = 1, 2, \ldots, d$, let S_j be the numerical semigroup generated by $\{a_1^{(j)}, a_2^{(j)}, \ldots, a_{r_j}^{(j)}\}$. We denote with $F^{(j)}$ the Frobenius number of S_j. It is easy to verify that for all $n \in \mathbb{N} \setminus \{0\}$, the element $(F^{(j)} + n)\mathbf{e}_j \in \mathbb{N}^d$ belongs to S. Let $\mathbf{v} = (v^{(1)}, v^{(2)}, \ldots, v^{(d)}) \in \mathbb{N}^d$ be the element defined by

$$v^{(j)} = \sum_{\substack{i=1 \\ i \neq j}}^{d} F^{(i)} n_i^{(j)} + F^{(j)}$$

for any $j = 1, 2, \ldots, d$. Let us prove that $\mathbf{x} \in S$ for all $\mathbf{x} \notin \pi(\mathbf{v})$ so, by Lemma 2.1.8, S is a generalized numerical semigroup.
Let $\mathbf{x} = (x^{(1)}, x^{(2)}, \ldots, x^{(d)}) \in \mathbb{N}^d$ such that $x^{(j)} > v^{(j)}$ for some $j \in \{1, \ldots, d\}$. Then there exists $m_j \in \mathbb{N} \setminus \{0\}$ such that $x^{(j)} = v^{(j)} + m_j$.
If $k_1, k_2, \ldots, k_r \in \{1, 2, \ldots, d\} \setminus \{j\}$ are the components of $\mathbf{x}$ such that $x^{(k_i)} \leq F^{(k_i)}$ for every $i \in \{1, 2, \ldots, r\}$, so $x^{(k_i)} n_{k_i}^{(j)} \leq F^{(k_i)} n_{k_i}^{(j)}$ for every $i = 1, \ldots, r$, then for every i there exists $p_i \in \mathbb{N}$ such that $F^{(k_i)} n_{k_i}^{(j)} = x^{(k_i)} n_{k_i}^{(j)} + p_i$.
Moreover let $h_1, \ldots, h_s \in \{1, \ldots, d\} \setminus \{j\}$ be the components of $\mathbf{x}$ such that $x^{(h_i)} > F^{(h_i)}$ for every $i \in \{1, \ldots, s\}$, hence $x^{(h_i)}\mathbf{e}_{h_i} \in S$, for all i.
Then we consider the following equalities:

$$
\begin{aligned}
\mathbf{x} &= \sum_{i=1}^{d} x^{(i)}\mathbf{e}_i = \sum_{i=1}^{r} x^{(k_i)}\mathbf{e}_{k_i} + \sum_{i=1}^{s} x^{(h_i)}\mathbf{e}_{h_i} + x^{(j)}\mathbf{e}_j \\
&= \sum_{i=1}^{r} x^{(k_i)}\mathbf{e}_{k_i} + \sum_{i=1}^{s} x^{(h_i)}\mathbf{e}_{h_i} + \left(\sum_{i\neq j}^{d} F^{(i)}n_i^{(j)} + F^{(j)} + m_j\right)\mathbf{e}_j \\
&= \sum_{i=1}^{r} \left(x^{(k_i)}\mathbf{e}_{k_i} + F^{(k_i)}n_{k_i}^{(j)}\mathbf{e}_j\right) + \sum_{i=1}^{s} x^{(h_i)}\mathbf{e}_{h_i} + \left(\sum_{i=1}^{s} F^{(h_i)}n_{h_i}^{(j)} + F^{(j)} + mj\right)\mathbf{e}_j \\
&= \sum_{i=1}^{r} \left(x^{(k_i)}\mathbf{e}_{k_i} + (x^{(k_i)}n_{k_i}^{(j)} + p_i)\mathbf{e}_j\right) + \sum_{i=1}^{s} x^{(h_i)}\mathbf{e}_{h_i} + \left(\sum_{i=1}^{s} F^{(h_i)}n_{h_i}^{(j)} + F^{(j)} + m_j\right)\mathbf{e}_j \\
&= \sum_{i=1}^{r} x^{(k_i)}\left(\mathbf{e}_{k_i} + n_{k_i}^{(j)}\mathbf{e}_j\right) + \sum_{i=1}^{s} x^{(h_i)}\mathbf{e}_{h_i} + \left(\sum_{i=1}^{s} F^{(h_i)}n_{h_i}^{(j)} + \sum_{i=1}^{r} p_i + F^{(j)} + m_j\right)\mathbf{e}_j
\end{aligned}
$$

Therefore $\mathbf{x}$ is a sum of elements in S (note that the first sum is a linear combination of elements in A, whose coefficients are non negative integers). So S is a generalized numerical semigroup. $\square$

Corollary 2.1.11. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and let A be a finite system of generators of S. With the notation of the previous theorem for the elements in A, let S_j be the numerical semigroup generated by $\{a_1^{(j)}, a_2^{(j)}, \ldots, a_{r_j}^{(j)}\}$ and $F^{(j)}$ the Frobenius number of S_j, for $j = 1, \ldots, d$. Let $\boldsymbol{v} = (v^{(1)}, v^{(2)}, \ldots, v^{(d)}) \in \mathbb{N}^d$ defined by:*

$$
v^{(j)} = \sum_{i\neq j}^{d} F^{(i)}n_i^{(j)} + F^{(j)}
$$

Then $\mathrm{H}(S) \subseteq \pi(\boldsymbol{v})$.

Proof. It easily follows from the proof of Theorem 2.1.10. $\square$

Example 2.1.12. Let $S \subseteq \mathbb{N}^4$ be the generalized numerical semigroup generated by $A = \{(1,0,0,0), (1,0,0,1), (0,1,0,0), (0,1,0,1), (0,0,1,0), (0,0,2,1), (0,0,0,2), (0,0,1,3), (0,0,0,5)\}$.
Actually S is a generalized numerical semigroup and its hole set is $\mathrm{H}(S) = \{(0,0,0,1), (0,0,0,3), (0,0,1,1)\}$. Let us verify that the conditions of theorem 2.1.10 are satisfied.
The generators described in condition 1) of the previous theorem are

$\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,2),(0,0,0,5)\}$.
About the condition 2) we have to verify that A contains at least one element of the following shapes:

(1) $(n_2^{(1)},1,0,0)$

(2) $(1,n_1^{(2)},0,0)$

(3) $(1,0,n_1^{(3)},0)$

(4) $(n_3^{(1)},0,1,0)$

(5) $(1,0,0,n_1^{(4)})$

(6) $(n_4^{(1)},0,0,1)$

(7) $(0,1,n_2^{(3)},0)$

(8) $(0,n_3^{(2)},1,0)$

(9) $(0,1,0,n_2^{(4)})$

(10) $(0,n_4^{(2)},0,1)$

(11) $(0,0,1,n_3^{(4)})$

(12) $(0,0,n_4^{(3)},1)$

The generators described in condition 2) of the previous theorem are $\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(1,0,0,1),(0,1,0,1),(0,0,2,1)\}$.
Observe that the set $A' = A \setminus \{(0,0,1,3)\}$ is a set of generators of a generalized numerical semigroup S', different from S, with a greater number of holes.

Example 2.1.13. Let $S \subseteq \mathbb{N}^2$ be the generalized numerical semigroup whose hole set is $\mathrm{H}(S) = \{(1,0),(2,0),(2,1)\}$. The set of minimal generators of S is $\{(0,1),(1,1),(3,0),(4,0),(5,0)\}$. We can identify $F^{(1)} = 2$, $F^{(2)} = 0$, $n_2^{(1)} = 0$, $n_1^{(2)} = 1$ so $\mathbf{v} = (F^{(2)} n_2^{(1)} + F^{(1)}, F^{(1)} n_{(1)}^{(2)} + F^{(2)}) = (2,2)$.

In Figure 2.1 the point $\mathbf{v}$ is marked in red, the couples of nonnegative integers in the red area represent the elements in $\pi(\mathbf{v})$. The holes of S are marked in black and we can see that they are all in the red area, that is $\pi(\mathbf{v})$. Moreover all the points overside the red area are in S. Indeed $\mathbf{v}' = (2,1)$ satisfies Lemma 2.1.8 too and $|\pi(\mathbf{v}')| < |\pi(\mathbf{v})|$. Anyway this fact does not always occur, as we will see in the next example.

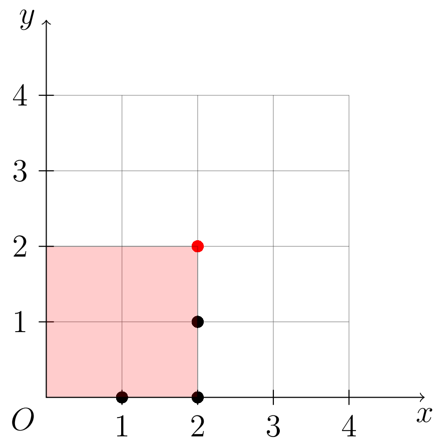


Figure 2.1:

Example 2.1.14. Let $S \subseteq \mathbb{N}^2$ be the monoid generated by $\mathrm{G}(S) = \{(2,0),(0,2),(3,0),(0,3),(1,4),(4,1)\}$.
By Theorem 2.1.10 S is a generalized numerical semigroup. Actually the hole set of S is $\mathrm{H}(S) = \{(0,1),(1,0),(1,1),(1,2),(1,3),(1,5),(2,1),(3,1),(5,1)\}$. We have $F^{(1)} = 1$, $F^{(2)} = 1$, $n_1^{(2)} = 4$, $n_2^{(1)} = 4$, so we consider $\mathbf{v} = (F^{(2)}n_2^{(1)} + F^{(1)}, F^{(1)}n_{(1)}^{(2)} + F^{(2)}) = (5,5)$. The set $\mathrm{H}(S)$ is contained in $\pi(\mathbf{v})$:

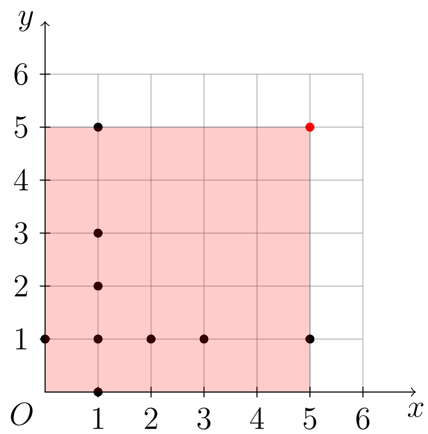


Figure 2.2:

In this case we can argue that it does not exist an element $\mathbf{w} \in \mathbb{N}^2$ such that $\pi(\mathbf{w})$ contains every hole of S and $|\pi(\mathbf{w})| < |\pi(\mathbf{v})|$ (see Figure 2.2).

Remark 2.1.15. Let $S = \langle A \rangle$ be a monoid generated by $A \subseteq \mathbb{N}^d$. For every $j = 1, 2, \dots, n$ we denote with $A_j \subseteq \mathbb{N}^{d-1}$ the set of the elements in $\mathbb{N}^{d-1}$, obtained from the elements in A removing the j-th component. Then the condition 2) of Theorem 2.1.10 is equivalent to the following statement: for every $j = 1, 2, \dots, d$, $\langle A_j \rangle = \mathbb{N}^{d-1}$.

In this section we have seen that all generalized numerical semigroups are finitely generated submnoid of $\mathbb{N}^d$. By this fact we can highlight a very big difference from submonoid of $\mathbb{N}$ and submonid of $\mathbb{N}^d$ for $d > 1$. We mentioned in Proposition 1.1.9 that every submonoid of $\mathbb{N}$ is isomorphic to a numerical semigroup, in particular all submonoids of $\mathbb{N}$ are finitely generated. This is very far from what happens in $\mathbb{N}^d$, because for $d > 1$ we have:

- not all submonoids of $\mathbb{N}^d$ are finitely generated.
- not all finitely generated submonoids of $\mathbb{N}^d$ (called *affine semigroups*) are generalized numerical semigroups.

Example 2.1.16. Let $S = \{(x, y) \in \mathbb{N}^2 \mid x \geq 2, y \in \mathbb{N}\}$. S is a submnoid of $\mathbb{N}^2$ but it is not finitely generated, since $\{(2, y) \in \mathbb{N}^2 \mid y \in \mathbb{N}\} \subseteq S^* \setminus (S^* + S^*)$. Let $T \subseteq \mathbb{N}^2$ be the submonoid generated by $\{(2, 0), (0, 3), (1, 4)\}$. T is a finitely generated submonoid of $\mathbb{N}^2$ but it is not a generalized numerical semigroup, since $\{(2n + 1, 0) \in \mathbb{N}^2 \mid n \in \mathbb{N}\} \subseteq \mathbb{N}^2 \setminus T$.

2.2 Relaxed monomial orders and the semigroup tree

The most important difference between numerical semigroups and generalized numerical semigroups arises immediately when we think about Frobenius number and effective generators: $\mathbb{N}$ has a natural total order, while $\mathbb{N}^d$ has only a partial natural order (that it is induced by the total order in $\mathbb{N}$). The notions of Frobenius number and effective generators, for instance, are important for numerical semigoups in order to provide the techniques to build the semigroup tree and an algorithm generating all numerical semigroups of a given genus. This problem is discussed in [22] where a special type of total order in $\mathbb{N}^d$, that is a *relaxed monomial order*, is introduced.

Definition 2.2.1. A total order, $\prec$, on the elements of $\mathbb{N}^d$ is called a *relaxed monomial order* if it satisfies:

i) If $\mathbf{v}, \mathbf{w} \in \mathbb{N}^d$ and if $\mathbf{v} \prec \mathbf{w}$ then $\mathbf{v} \prec \mathbf{w} + \mathbf{u}$ for any $\mathbf{u} \in \mathbb{N}^d$.

ii) If $\mathbf{v} \in \mathbb{N}^d$ and $\mathbf{v} \neq \mathbf{0}$ then $\mathbf{0} \prec \mathbf{v}$.

Among relaxed monomial orders there are monomial orders induced in $\mathbb{N}^d$ by the well-known monomial orders in the set of monomials of a given polynomial ring (used to define Gröbner basis, for instance) by the identification of a monomial $M = x_1^{v_1} x_2^{v_2} \cdots x_d^{v_d} \in K[x_1, \ldots, x_2]$ with $(v_1, v_2, \ldots, v_d) \in \mathbb{N}^d$ (a good reference for this topic is [14]). The following definition arises.

Definition 2.2.2. A total order $<_m$ in $\mathbb{N}^d$ is called a *monomial order* if it satisfies:

1) If $\mathbf{v}, \mathbf{w} \in \mathbb{N}^d$ with $\mathbf{v} <_m \mathbf{w}$ then $\mathbf{v} + \mathbf{u} <_m \mathbf{w} + \mathbf{u}$ for every $\mathbf{u} \in \mathbb{N}^d$.

2) If $\mathbf{v} \in \mathbb{N}^d$ and $\mathbf{v} \neq \mathbf{0}$ then $\mathbf{0} <_m \mathbf{v}$.

Example 2.2.3. Examples of relaxed monomial orders are the following:

- Let $\alpha, \beta \in \mathbb{N}^d$, we define $\alpha \prec \beta$ if and only if the first nonzero coordinate of $\beta - \alpha$ is positive. $\prec$ is called *lexicographic order* and it is a relaxed monomial order (it is also a monomial order).

- Let $<_m$ be a monomial order and if $\mathbf{u} = (u_1, \ldots, u_d)$, let $min(\mathbf{u}) = min\{u_1, \ldots, u_d\}$. Define $\mathbf{u} \prec \mathbf{v}$ if

  i) $\min(\mathbf{u}) < \min(\mathbf{v})$ or if

  ii) $\min(\mathbf{u}) = \min(\mathbf{v})$ and $\mathbf{u} <_m \mathbf{v}$.
  The order $\prec$ is a relaxed monomial order. One can prove that in general $\prec$ is not a monomial order. For instance, if $<_m$ is the lexicografic order then in $\mathbb{N}^3$ we have $(3,1,1) \prec (2,7,8)$ but $(3,1,1) + (0,2,2) \not\prec (2,7,8) + (0,2,2)$

- Any monomial order on the elements of $\mathbb{N}^d$ can be defined in terms of dot products via an ordered collection of d linearly independent *weight vectors* in $\mathbb{R}^d_{\geq 0}$ [14]. More precisely, if $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d$ are linearly independent vectors in $\mathbb{R}^d_{\geq 0}$ then one can define a monomial order $<_m$ on the elements of $\mathbb{N}^d$ by

  $$\mathbf{u} <_m \mathbf{v} \iff \text{the smallest } i \text{ for which } \mathbf{w}_i \cdot \mathbf{u} \neq \mathbf{w}_i \cdot \mathbf{v} \text{ has } \mathbf{w}_i \cdot \mathbf{u} < \mathbf{w}_i \cdot \mathbf{v}.$$

Definition 2.2.4. Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup. Given a relaxed monomial order $\prec$ in $\mathbb{N}^d$ we define:

1. $\mathbf{F}_{\prec}(S)$ the greatest element in $\mathrm{H}(S)$ with respect to $\prec$, called the *Frobenius element* of S, with respect to $\prec$.
2. $\mathbf{E}_{\prec}(S)$ the set of minimal generators greater than $\mathbf{F}_{\prec}(S)$, called the effective generators of S, with respect to $\prec$.
3. $\mathbf{m}_{\prec}(S)$ the smallest nonzero element of S with respect to $\prec$, called the *multiplicity* of S with respect to $\prec$.

These definitions allow to extend analogous results for numerical semigroups.

Lemma 2.2.5. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and $\prec$ a relaxed monomial order in $\mathbb{N}^d$. Then $T = S \cup \{\mathbf{F}_{\prec}(S)\}$ is a generalized numerical semigroup, moreover $\mathbf{F}_{\prec}(S)$ is an effective generator of T with respect to $\prec$.*

Proof. Obviously $|\mathbb{N}^d \setminus T| = g-1$, let us prove that T is a monoid. Let $\mathbf{u}, \mathbf{v} \in T$. If $\mathbf{u}, \mathbf{v} \in S$ then it is trivial that $\mathbf{u}+\mathbf{v} \in T$. Suppose that $\mathbf{v} = \mathbf{F}(S)$ then, since $\mathbf{F}(S) \preceq \mathbf{v}$ (from reflexivity of order $\prec$), it follows that $\mathbf{F}(S) \prec \mathbf{F}(S)+\mathbf{u} \preceq \mathbf{v}+\mathbf{u}$, then $\mathbf{v}+\mathbf{u} \in S \subset T$, hence we deduce that $\mathbf{v}+\mathbf{u}$ is not a hole in S.
Furthermore $\mathbf{F}(S)$ is a minimal generator of T, if not it is generated by elements of S. Moreover the Frobenius element of T with respect to $\prec$ is a hole of S, since $\mathrm{H}(T) \subset \mathrm{H}(S)$, and for all $\mathbf{h} \in \mathrm{H}(S)$ we have $\mathbf{h} \prec \mathbf{F}(S)$, therefore $\mathbf{F}(S)$ is an effective generator of T. $\square$

Lemma 2.2.6. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and $\boldsymbol{h}$ a minimal generator. Then $T = S \setminus \{\boldsymbol{h}\}$ is a generalized numerical semigroup. Furthermore if $\prec$ is a relaxed monomial order in $\mathbb{N}^d$ and $\boldsymbol{h}$ is an effective generator of S with respect to $\prec$, then $\mathbf{F}_{\prec}(T) = \boldsymbol{h}$.*

Proof. T is trivially a generalized numerical semigroup. Suppose that $\mathbf{h}$ is an effective generator of S with respect to a relaxed monomial order $\prec$. If $\mathbf{u}$ is a hole of T, different from $\mathbf{h}$, then $\mathbf{u}$ is an hole of S, in particular $\mathbf{u} \prec \mathbf{h}$. $\square$

Recall that an *oriented graph* (or *directed graph*) G is a pair (V, E), where V is a nonempty set whose elements are called *vertices*, and E is a subset of $\{(v, w) \in V \times V \mid v \neq w\}$. The elements of E are called *edges* of G. A *path* connecting the vertices x and y of G is a sequence of distinct edges of the form $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ with $v_0 = x$ and $v_n = y$. An oriented graph G is a *rooted tree* if there exists a vertex r (known as the *root* of G) such that for every other vertex x of G, there exists a unique path connecting x and r. If E is the set of edges of a rooted tree and $(x, y) \in E$ we say that x is a *son* of y.

Definition 2.2.7. Let $\mathcal{S}_d$ be the set of all generalized numerical semigroups in $\mathbb{N}^d$ and $\prec$ a relaxed monomial order in $\mathbb{N}^d$. We define the oriented graph $\mathcal{T}_\prec = (\mathcal{S}_d, \mathcal{E})$ whose vertices are the elements in $\mathcal{S}_d$ and edges are the pairs $(S,T) \in \mathcal{E}$ where $T = S \cup \{\mathbf{F}(S)\}$, since $\mathbf{F}(S)$ is the Frobenius number of S with respect to $\prec$.

Proposition 2.2.8. *The graph $\mathcal{T}_\prec$ is a rooted tree whose root is $\mathbb{N}^d$. Furthermore, if $S \in \mathcal{S}_d$, all the sons of S are the semigroups $S \setminus \{\boldsymbol{h}_1\}, S \setminus \{\boldsymbol{h}_2\}, \dots, S \setminus \{\boldsymbol{h}_l\}$ where $\{\boldsymbol{h}_1, \dots, \boldsymbol{h}_l\}$ is the set of the effective generators of S with respect to $\prec$.*

Proof. Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup. We define the following sequence:

- $S_0 = S$.
- $S_{i+1} = \begin{cases} S_i \cup \{\mathbf{F}_\prec(S_i)\} & \text{if } S_i \neq \mathbb{N}^d \\ \mathbb{N}^d & \text{otherwise} \end{cases}$

Since S has finite complement in $\mathbb{N}^d$ then there exists a nonnegative integer k such that $S_k = \mathbb{N}^d$. So the edges $(S_0, S_1), (S_1, S_2), \dots, (S_{k-1}, S_k)$ provide a path from S to $\mathbb{N}^d$, hence $\mathcal{T}_\prec$ is a rooted tree whose root is $\mathbb{N}^d$.
Furthermore, let $\mathbf{h}_i$ be an effective generator (with respect to $\prec$) of S. By Lemma 2.2.6 $\mathbf{F}_\prec(S \setminus \{\mathbf{h}_i\}) = \mathbf{h}_i$, so $(S \setminus \{\mathbf{h}_i\}, S)$ is an edge of $\mathcal{T}$ for every i, that is $S \setminus \{\mathbf{h}_i\}$ are the sons of S for every $\mathbf{h}_i$ effective generator of S. Moreover, let T be a son of S, then $S = T \cup \{\mathbf{F}_\prec(T)\}$ and by Lemma 2.2.5 $\mathbf{F}_\prec(T)$ is an effective generator of S, that is $T = S \setminus \{\mathbf{h}_i\}$ for some effective generator $\mathbf{h}_i$. □

Given a relaxed monomial order in $\mathbb{N}^d$, it is possible to arrange the set $\mathcal{S}_d$ of all generalized numerical semigroups in $\mathbb{N}^d$ as a rooted tree $\mathcal{T}_\prec$, with root in $\mathbb{N}^d$, in the same way described previously for numerical semigroups. In particular we can write an algorithm that provides all generalized numerical semigroups in $\mathbb{N}^d$ of a given genus g. This is going to be the topic of the last chapter of this work, where we explain this building in more details. An important remark: different monomial orders can define, in the same generalized numerical semigroup, different Frobenius elements and effective generators. So the rooted tree $\mathcal{T}_\prec$ is different if we change the relaxed monomial order. However, with respect to every relaxed monomial order, the sons of all generalized numerical semigroups of genus $g-1$ are going to be all generalized numerical semigroups of genus g, generated without redundancy (for a reference see [22]).

2.3 Uniqueness of the Frobenius element: Frobenius generalized numerical semigroups

In $\mathbb{N}^d$ there is not a natural total order so it is not immediate to define for a generalized numerical semigroup the Frobenius element as for numerical semigroups. In the previous section this aim is reached with the definition of *relaxed monomial order*, whose main purpose is to allow the building of the *generalized numerical semigroup tree*. Now we investigate which element in $\mathrm{H}(S)$ can be a Frobenius element for S. That is: let $\mathbf{h} \in \mathrm{H}(S)$, we look for some instances such that there exists a relaxed monomial order $\prec$ with $\mathbf{F}_\prec = \mathbf{h}$.

Proposition 2.3.1. *Every relaxed monomial order in $\mathbb{N}^d$ extends the natural partial order in $\mathbb{N}^d$.*

Proof. Let $\mathbf{a}, \mathbf{b} \in \mathbb{N}^d$ be distinct elements with $\mathbf{a} \leq \mathbf{b}$, so there exists $\mathbf{c} \in \mathbb{N}^d$ such that $\mathbf{a} + \mathbf{c} = \mathbf{b}$. Furthermore, let $\prec$ be a relaxed monomial ordering in $\mathbb{N}^d$. Suppose that $\mathbf{b} \prec \mathbf{a}$, then $\mathbf{b} \prec \mathbf{a} + \mathbf{c} = \mathbf{b}$ but it is a contradiction. $\square$

Proposition 2.3.2. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and $\boldsymbol{f} \in \mathrm{H}(S)$. Then it is verified that $\mathbf{F}_\prec = \boldsymbol{f}$ for every relaxed monomial order $\prec$ if and only if $\boldsymbol{f}$ is the maximum in $\mathrm{H}(S)$ with respect to the natural partial order in $\mathbb{N}^d$.*

Proof. The sufficiency is trivial considering Proposition 2.3.1. We need to prove the necessary condition. Observe that $\mathbf{f}$ must be maximal in $\mathrm{H}(S)$ with respect to the natural partial order in $\mathbb{N}^d$, because if there exists $\mathbf{h} \in \mathrm{H}(S)$ with $\mathbf{f} \leq \mathbf{h}$ then $\mathbf{f} \preceq \mathbf{h}$ for every relaxed monomial order in $\mathbb{N}^d$, by Proposition 2.3.1. We have to prove that $\mathbf{f}$ is the unique maximal element. If there exists another maximal element $\mathbf{g} \neq \mathbf{f}$ then, since $\mathbf{f} \nleq \mathbf{g}$ and $\mathbf{g} \nleq \mathbf{f}$, $\mathbf{g}$ has at least one component, the j-th for instance, such that it is greater than the j-th component of $\mathbf{f}$. We are going to define a relaxed monomial order, $\prec$, by assigning weight vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_d$ with this order, in a suitable way. Let $\mathbf{e}_i$, for $i = 1, \dots, d$, be the standard basis vectors. Fix $\mathbf{w}_1 = \mathbf{e}_j, \mathbf{w}_j = \mathbf{e}_1$ and $\mathbf{w}_i = \mathbf{e}_i$ for $i \neq 1, j$. The relaxed monomial order defined in such a way leads to $\mathbf{f} \prec \mathbf{g}$, but this is a contradiction. $\square$

Definition 2.3.3. Let S a generalized numerical semigroup such that there exists $\mathbf{f} \in \mathrm{H}(S)$ maximum in $\mathrm{H}(S)$. We call S a *Frobenius* generalized numerical semigroup and we refer to it with the notation $(S, \mathbf{f})$.

By Proposition 2.3.2 Frobenius generalized numerical semigroups are all and only those generalized numerical semigroups which have a unique Frobenius element, independently by the fixed relaxed monomial order. Therefore if $(S, \mathbf{f})$ is a Frobenius generalized numerical semigroup we can refer to $\mathbf{f}$ as the Frobenius element of S without ambiguity.

Remark 2.3.4. Every numerical semigroup is a Frobenius generalized numerical semigroup (S, f), where f is the Frobenius number.

We have discussed the property of uniqueness of the Frobenius element with respect to every fixed relaxed monomial order. Now we investigate about the existence of a relaxed monomial order such that an element $\mathbf{h} \in \mathrm{H}(S)$ is the Frobenius element with respect to it. By Proposition 2.3.1 these elements must be looked for among the maximal elements in $\mathrm{H}(S)$ with respect to the natural partial order in $\mathbb{N}^d$.

Definition 2.3.5. Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and $\mathbf{h} \in \mathrm{H}(S)$. We call $\mathbf{h}$ *Frobenius allowable* if there exists a relaxed monomial order, $\prec$, such that $\mathbf{F}_{\prec} = \mathbf{h}$.

Proposition 2.3.6. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup where $\mathrm{H}(S)$ has exactly two maximal elements, $\boldsymbol{h}_1, \boldsymbol{h}_2$, with respect to the natural partial order in $\mathbb{N}^d$. Then both $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ are Frobenius allowable.*

Proof. It suffices to prove that there exist two relaxed monomial orders, $\prec_1, \prec_2$, such that $\mathbf{h}_1 \prec_1 \mathbf{h}_2$ and $\mathbf{h}_2 \prec_2 \mathbf{h}_1$. Since $\mathbf{h}_1$ and $\mathbf{h}_2$ are distinct maximal elements then $\mathbf{h}_1$ has at least one coordinate, the i-th for instance, greater than the i-th coordinate of $\mathbf{h}_2$, and $\mathbf{h}_2$ has a coordinate, the j-th for instance, greater than the corresponding coordinate of $\mathbf{h}_1$. We can define a relaxed monomial order by weight vectors, like in the proof of Proposition 2.3.2. So we set $\prec_1$ by the assignements $\mathbf{w}_1 = \mathbf{e}_j, \mathbf{w}_j = \mathbf{e}_1, \mathbf{w}_k = \mathbf{e}_k$ for $k \neq 1, j$, for $\prec_2$ we choose $\mathbf{w}_1 = \mathbf{e}_i, \mathbf{w}_i = \mathbf{e}_1, \mathbf{w}_k = \mathbf{e}_k$ for $k \neq 1, i$. In this way, $\mathbf{h}_1 \prec_1 \mathbf{h}_2$ and $\mathbf{h}_2 \prec_2 \mathbf{h}_1$ are both satisfied. $\square$

Example 2.3.7. The same argument does not hold if $\mathrm{H}(S)$ has more then two maximal elements, even if it can occur that every maximal in $\mathrm{H}(S)$ is Frobenius allowable. This example shows these facts.
Let $S = \mathbb{N}^2 \setminus \{(0,1),(1,0),(1,1),(1,2),(1,3),(2,1),(3,0),(5,0),(7,0)\}$. The maximal elements in $\mathrm{H}(S)$ with respect to the natural partial order in $\mathbb{N}^2$ are $(1,3),(2,1),(7,0)$. Let $\mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1)$, then:

- $(1,3)$ is Frobenius allowable, in fact we can define $\prec$, relaxed monomial order by weight vectors $\mathbf{w}_1 = \mathbf{e}_2$ and $\mathbf{w}_2 = \mathbf{e}_1$. It results $(7,0) \prec (2,1) \prec (1,3)$.
- $(7,0)$ is Frobenius allowable, in fact we can define $\prec$, relaxed monomial order by weight vectors $\mathbf{w}_1 = \mathbf{e}_1$ and $\mathbf{w}_2 = \mathbf{e}_2$. We have $(1,3) \prec (2,1) \prec (7,0)$.
- $(2,1)$ is also Frobenius allowable: let $<_m$ be the lexicographic order and define $\prec$ as in the third case in Example 2.2.3. We have $(7,0) \prec (1,3) \prec (2,1)$.

Observe that for the element $(2,1)$ we can not argue as in the proof of the previous proposition because it has not coordinates greater than both the corresponding in $(7,0)$ and $(1,3)$.

It is an open question which are the Frobenius allowable elements in a generalized numerical semigroup. Are they all the maximal elements in $\mathrm{H}(S)$ with respect to the natural partial order in $\mathbb{N}^d$ or only any of that?

Related to the argument of this section is the definition of *Frobenius vectors*, given in [1] for affine semigroups. Let $S \subseteq \mathbb{N}^d$ be an affine semigroup, we define:

- $\mathfrak{G}(S) = \{\sum_{i=1}^{n} a_i \mathbf{s}_i \mid n \in \mathbb{N}, a_i \in \mathbb{Z}, \mathbf{s}_i \in S \text{ for all } i = 1, \dots, n\}$.
- $\mathrm{cone}(S) = \{\sum_{i=1}^{n} q_i \mathbf{s}_i \mid n \in \mathbb{N}, q_i \in \mathbb{Q}_{\geq 0}, \mathbf{s}_i \in S \text{ for all } i = 1, \dots, n\}$.
- $\mathrm{relint}(\mathrm{cone}(S)) = \{\sum_{i=1}^{n} q_i \mathbf{s}_i \mid n \in \mathbb{N}, q_i \in \mathbb{Q}_{>0}, \mathbf{s}_i \in S \text{ for all } i = 1, \dots, n\}$.

Definition 2.3.8. Let $S \subseteq \mathbb{N}^d$ be an affine semigroup. A *Frobenius vector* is an element $\mathbf{f} \in \mathfrak{G}(S) \setminus S$ such that:

$$\mathbf{f} + (\mathrm{relint}(\mathrm{cone}(S)) \cap \mathfrak{G}(S)) \subseteq S \setminus \{\mathbf{0}\}$$

We show that every generalized numerical semigroup has Frobenius vectors, in fact the Frobenius element with respect to some relaxed monomial order is a Frobenius vector.

Proposition 2.3.9. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and $\prec$ a relaxed monomial order. Then $\mathbf{F}_{\prec}(S)$ is a Frobenius vector.*

Proof. Observe that, if $S \subseteq \mathbb{N}^d$ is a generalized numerical semigroup, then $\mathfrak{G}(S) = \mathbb{Z}^d$, $\text{cone}(S) = \mathbb{N}^d$ and $\text{relint}(\text{cone}(S)) = \{(a_1, \ldots, a_d) \in \mathbb{N}^d \mid a_i > 0, \text{ for all } i = 1, \ldots, d\}$. So, if $\mathbf{x} \in \text{relint}(\text{cone}(S))$ then $\mathbf{F}_{\prec}(S) + \mathbf{x} \succ \mathbf{F}_{\prec}(S) \succ \mathbf{0}$, in particular $\mathbf{F}_{\prec}(S) \in S \setminus \{\mathbf{0}\}$. □

The converse of the previous result can be considered as another problem for other researches. That is, we question if every Frobenius vector of a generalized numerical semigroup is the Frobenius element with respect to some relaxed monomial order.

Chapter 3

Irreducible generalized numerical semigroups

In this chapter we extend to generalized numerical semigroups definitions and some properties formulated for symmetric and pseudo-symmetric numerical semigroups. Some results are very similar (see [41] and [40]) but their proofs are obtained in a little different way. The work [12] has been produced for this argument. Some similar results about irreducibility have been obtained later in a more general context in [25].

3.1 Pseudo-Frobenius elements and special gaps

Definition 3.1.1. Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup. We denote by:

$$\mathrm{PF}(S) = \{\mathbf{x} \in \mathrm{H}(S) \mid \mathbf{x} + \mathbf{s} \in S, \text{ for all } \mathbf{s} \in S \setminus \{\mathbf{0}\}\}$$

the set of the pseudo-Frobenius elements of S. The number $|\mathrm{PF}(S)|$ is called the *type* of S.

If $S \subseteq \mathbb{N}^d$ is a monoid, it is possible to define in $\mathbb{Z}^d$ the following relation:

$$\mathbf{a} \leq_S \mathbf{b} \text{ if and only if } \mathbf{b} - \mathbf{a} \in S$$

It is easy to see that $\leq_S$ is a partial order in $\mathbb{Z}^d$.

Proposition 3.1.2. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup. Then $\mathrm{PF}(S)$ is the set of maximal elements in $\mathrm{H}(S)$ with respect to $\leq_S$.*

Proof. We take $\mathbf{x}$ maximal in $\mathbb{N}^d \setminus S$ with respect to $\leq_S$. If there exists $\mathbf{s} \in S$ such that $\mathbf{x}+\mathbf{s} \notin S$ then $\mathbf{x} \leq_S \mathbf{x}+\mathbf{s}$ which contradicts the maximality of $\mathbf{x}$. Conversely let $\mathbf{x} \in \mathrm{PF}(S)$. If there exists $\mathbf{y} \in \mathbb{N}^d \setminus S$ such that $\mathbf{y}-\mathbf{x} = \mathbf{s} \in S$ then $\mathbf{x}+\mathbf{s} \notin S$ which is again a contradiction. $\square$

Since $\mathrm{H}(S)$ is a finite set, the previous proposition implies that $\mathrm{PF}(S)$ is nonempty.

Definition 3.1.3. Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup. We define:

$$\mathrm{SG}(S) = \{\mathbf{x} \in \mathrm{H}(S) \mid 2\mathbf{x} \in S, \mathbf{x}+\mathbf{s} \in S, \text{for all } \mathbf{s} \in S \setminus \{\mathbf{0}\}\}$$

The elements of the set $\mathrm{SG}(S)$ are called *special gaps* of S.

Remark 3.1.4. Obviously $\mathrm{SG}(S) \subseteq \mathrm{PF}(S)$, but the equality is not true in general. For instance, let $S = \mathbb{N}^2 \setminus \{(0,1),(1,0),(2,0)\}$, we have $\mathrm{PF}(S) = \mathrm{H}(S)$ but $\mathrm{SG}(S) = \{(0,1),(2,0)\}$.
Moreover $\mathrm{SG}(S) \neq \emptyset$: if $\mathbf{f}$ is a maximal element in $\mathrm{H}(S)$ with respect to the natural partial order in $\mathbb{N}^d$ (it exists because $\mathrm{H}(S)$ is finite) then $\mathbf{f} \in \mathrm{SG}(S)$. In particular $\mathrm{PF}(S)$ and $\mathrm{SG}(S)$ are always non empty sets.

The elements in $\mathrm{SG}(S)$ characterize the extensions of S, as formulated by the following:

Proposition 3.1.5. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and $\boldsymbol{x} \in \mathrm{H}(S)$. Then $S \cup \{\boldsymbol{x}\}$ is a semigroup if and only if $\boldsymbol{x} \in \mathrm{SG}(S)$.*

Proof. It is an easy consequence of the definition of $\mathrm{SG}(S)$. $\square$

Definition 3.1.6. Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup. S is *irreducible* if it cannot be expressed as the intersection of two generalized numerical semigroups containing it properly.

In this chapter we are going to prove some characterizations of irreducible generalized numerical semigroups. Similar results are formulated in [41] in the case of numerical semigroups. The following proposition was given for numerical semigroups as a corollary of other results. We prove it as a preliminary fact.

Proposition 3.1.7. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup. S is irreducible if and only if $|\mathrm{SG}(S)| = 1$.*

Proof. $\Rightarrow$) Let S be an irreducible generalized numerical semigroup and suppose that there exist two distinct elements $\mathbf{x}, \mathbf{y} \in \mathrm{SG}(S)$. Then $S \cup \{\mathbf{x}\} \supsetneq S$ and $S \cup \{\mathbf{y}\} \supsetneq S$ are generalized numerical semigroups, furthermore $(S \cup \{\mathbf{x}\}) \cap (S \cup \{\mathbf{y}\}) = S$, but this is a contradiction.
$\Leftarrow$) We suppose there exist two different generalized numerical semigroups S_1, S_2, such that $S_1 \supsetneq S$, $S_2 \supsetneq S$ and $S_1 \cap S_2 = S$. Let $\mathbf{x}, \mathbf{y}$ be maximal elements respectively in $S_1 \setminus S$ and in $S_2 \setminus S$ (that are finite sets), with respect to the natural partial order in $\mathbb{N}^d$. Obviously $\mathbf{x}, \mathbf{y} \in \mathrm{H}(S)$. Now we prove that $\mathbf{x}, \mathbf{y} \in \mathrm{SG}(S)$. It is $2\mathbf{x} > \mathbf{x}$, so $2\mathbf{x} \notin S_1 \setminus S$, furthermore $2\mathbf{x} \in S_1$ so $2\mathbf{x} \in S$. Now take $\mathbf{s} \in S \setminus \{\mathbf{0}\}$, it is $\mathbf{x} + \mathbf{s} > \mathbf{x}$, so $\mathbf{x} + \mathbf{s} \notin S_1 \setminus S$, furthermore $\mathbf{x} \in S_1$ and $\mathbf{s} \in S \setminus \{\mathbf{0}\} \subset S_1 \setminus \{\mathbf{0}\}$, therefore $\mathbf{x} + \mathbf{s} \in S_1$ so $\mathbf{x} + \mathbf{s} \in S$. Hence we proved that $\mathbf{x} \in \mathrm{SG}(S)$. In a similar way we can prove that $\mathbf{y} \in \mathrm{SG}(S)$. By hypothesis $|\,\mathrm{SG}(S)| = 1$, then $\mathbf{x} = \mathbf{y}$, that is $\mathbf{x} \in S_1 \setminus S$ and $\mathbf{x} \in S_2 \setminus S$, hence $\mathbf{x} \in S_1 \cap S_2 = S$, a contradiction because $\mathbf{x} \in \mathrm{H}(S)$. $\square$

3.2 Symmetric and pseudo-Symmetric generalized numerical semigroups

In numerical semigroups the Frobenius number plays an important role, in particular for irreducible numerical semigroups. To have a good generalization we wish that for irreducible generalized numerical semigroups there is a unique possible choice for the analogous of Frobenius number. Proposition 3.1.7 and the following one give us an answer to this question.

Proposition 3.2.1. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup. S is irreducible if and only if there exists $\boldsymbol{f} \in \mathrm{H}(S)$ such that for every $\boldsymbol{h} \in \mathrm{H}(S)$ with $2\boldsymbol{h} \neq \boldsymbol{f}$ it is $\boldsymbol{f} - \boldsymbol{h} \in S$.*

Proof. $\Rightarrow$) By Proposition 3.1.7 $\mathrm{SG}(S)$ consists of one element. Let $\mathrm{SG}(S) = \{\mathbf{f}\}$. Let $\mathbf{h} \in \mathrm{H}(S)$ with $\mathbf{h} \neq \mathbf{f}$ and suppose that $2\mathbf{h} \neq \mathbf{f}$. Since $\mathbf{h} \notin \mathrm{SG}(S)$ we have two possibilities.
1) Suppose there exists $\mathbf{s}_1 \in S \setminus \{\mathbf{0}\}$ such that $\mathbf{f}_1 = \mathbf{h} + \mathbf{s}_1 \notin S$, in particular $\mathbf{f}_1 - \mathbf{h} \in S$. If $\mathbf{f}_1 = \mathbf{f}$ we can conclude. If $\mathbf{f}_1 \neq \mathbf{f}$ then $\mathbf{f}_1 \notin \mathrm{SG}(S)$. We show that in any case there exist $\mathbf{s}_2 \in S \setminus \{\mathbf{0}\}$ and $\mathbf{f}_2 \notin S$ with $\mathbf{f}_2 > \mathbf{f}_1$ (where $>$ is the natural partial order) such that $\mathbf{f}_2 = \mathbf{h} + \mathbf{s}_2$. Since $\mathbf{f}_1 \notin \mathrm{SG}(S)$, if there exists $\mathbf{t} \in S \setminus \{\mathbf{0}\}$ such that $\mathbf{f}_1 + \mathbf{t} \notin S$, we consider $\mathbf{f}_2 = \mathbf{f}_1 + \mathbf{t} = \mathbf{h} + (\mathbf{s}_1 + \mathbf{t})$, that is $\mathbf{s}_2 = \mathbf{s}_1 + \mathbf{t}$ and $\mathbf{f}_2 > \mathbf{f}_1$. Otherwise, if $\mathbf{f}_1 + \mathbf{s} \in S$ for every $s \in S \setminus \{\mathbf{0}\}$ we consider $\mathbf{f}_2 = 2\mathbf{f}_1 \notin S$, so $\mathbf{f}_2 = \mathbf{h} + (\mathbf{h} + 2\mathbf{s}_1)$ and $\mathbf{s}_2 = \mathbf{h} + 2\mathbf{s}_1 = \mathbf{f}_1 + \mathbf{s}_1 \in S$. Therefore we have proved that there exist $\mathbf{s}_2 \in S \setminus \{\mathbf{0}\}$ and $\mathbf{f}_2 \notin S$ with

$\mathbf{f}_2 > \mathbf{f}_1$ such that $\mathbf{f}_2 = \mathbf{h} + \mathbf{s}_2$. If $\mathbf{f}_2 = \mathbf{f}$ we can conclude, on the contrary the argument continues in a similar way repeating the same procedure for $\mathbf{f}_2$. In particular we obtain a sequence of elements $\mathbf{f}_i \notin S$ with $\mathbf{f}_i > \mathbf{f}_{i-1}$ for every i and $\mathbf{f}_i = \mathbf{h} + \mathbf{s}_i$ and $\mathbf{s}_i \in S \setminus \{\mathbf{0}\}$. Since $\mathrm{H}(S)$ is a finite set, there exists $k \in \mathbb{N}$ such that $\mathbf{f}_k = \mathbf{f}$, furthermore $\mathbf{f}_k - \mathbf{h} \in S$.
2) Suppose that $\mathbf{h}+\mathbf{s} \in S$ for every $\mathbf{s} \in S\setminus\{\mathbf{0}\}$ and $2\mathbf{h} \notin S$. We will prove that this is a contradiction. Observe that for every $i \in \mathbb{N}$ we have $i\mathbf{h} + \mathbf{s} \in S$ for every $\mathbf{s} \in S \setminus \{\mathbf{0}\}$. Since $\mathrm{H}(S)$ is finite there exists $k = \max\{i \in \mathbb{N} \mid i\mathbf{h} \notin S\}$, in particular $k\mathbf{h} \in \mathrm{SG}(S)$, that is $k\mathbf{h} = \mathbf{f}$. By our assumption $k \geq 3$. Consider the element $\overline{\mathbf{h}} = (k-1)\mathbf{h}$, we have $\overline{\mathbf{h}} + \mathbf{s} \in S$ for every $\mathbf{s} \in S \setminus \{\mathbf{0}\}$ and $2\overline{\mathbf{h}} = 2(k-1)\mathbf{h} \in S$ since $2(k-1) > k$, that is $\overline{\mathbf{h}} \in \mathrm{SG}(S)$. But this is a contradiction since $\overline{\mathbf{h}} \neq \mathbf{f}$.

$\Leftarrow$) By hypothesis, $\mathbf{f}$ is greater than every element in $\mathrm{H}(S)$ with respect to $\leq_S$, except for the element $\mathbf{h} \in \mathrm{H}(S)$ such that $2\mathbf{h} = \mathbf{f}$, if it exists. By Proposition 3.1.2 the possible elements in $\mathrm{PF}(S)$ are $\mathbf{f}$ and $\mathbf{h} = \frac{\mathbf{f}}{2}$. Furthermore $\mathrm{SG}(S) \subseteq \mathrm{PF}(S)$ and $\mathbf{h} \notin \mathrm{SG}(S)$, since $2\mathbf{h} = \mathbf{f} \notin S$, so it must be $\mathrm{SG}(S) = \{\mathbf{f}\}$, hence S is irreducible. □

Lemma 3.2.2. *Let $S \subseteq \mathbb{N}^d$ be an irreducible generalized numerical semigroup with $\mathrm{SG}(S) = \{\boldsymbol{f}\}$. Then one and only one of these conditions is verified:*

1. $\mathrm{PF}(S) = \{\boldsymbol{f}\}$ *if there exists a component of $\boldsymbol{f}$ that is odd.*
2. $\mathrm{PF}(S) = \left\{\boldsymbol{f}, \frac{\boldsymbol{f}}{2}\right\}$ *if all the components of $\boldsymbol{f}$ are even.*

Proof. If $\mathbf{f}$ has an odd component then it does not exist $\mathbf{h} \in \mathrm{H}(S)$ such that $2\mathbf{h} = \mathbf{f}$ and, by Proposition 3.2.1, $\mathbf{f}$ is the maximum in $\mathrm{H}(S)$ with respect to $\leq_S$, so $\mathrm{PF}(S) = \{\mathbf{f}\}$ from Proposition 3.1.2.
If all components of $\mathbf{f}$ are even then $\frac{\mathbf{f}}{2} \in \mathbb{N}^d$ and such an element is in $\mathrm{H}(S)$, since $\mathbf{f} \in \mathrm{H}(S)$. It is $\mathbf{f} - \frac{\mathbf{f}}{2} = \frac{\mathbf{f}}{2} \notin S$, then $\mathbf{f}$ and $\frac{\mathbf{f}}{2}$ are not comparable with respect to $\leq_S$. Furthermore, by Proposition 3.2.1, $\mathbf{f}$ is greater than all elements in $\mathrm{H}(S)$ different by $\frac{\mathbf{f}}{2}$ with respect to $\leq_S$, so $\mathbf{f}$ is a maximal element with respect to that order, that is $\mathbf{f} \in \mathrm{PF}(S)$. Moreover $\frac{\mathbf{f}}{2}$ is maximal in $\mathrm{H}(S)$ with respect to $\leq_S$, because, on the contrary, there exists $\mathbf{h} \in \mathrm{H}(S)$ such that $\frac{\mathbf{f}}{2} \leq_S \mathbf{h} \leq_S \mathbf{f}$, but this is a contradiction. We conclude that $\mathrm{PF}(S) = \{\mathbf{f}, \frac{\mathbf{f}}{2}\}$. □

We can easily gather the previous results in the following theorems.

Theorem 3.2.3. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup. Then the following statements are equivalent:*

1. $|\operatorname{PF}(S)| = 1$.
2. $\operatorname{PF}(S) = \{\boldsymbol{f}\}$ *and* $\boldsymbol{f}$ *has at least one odd component.*
3. *There exists* $\boldsymbol{f} \in \operatorname{H}(S)$ *such that, for all* $\boldsymbol{h} \in \operatorname{H}(S)$ *we have* $\boldsymbol{f} - \boldsymbol{h} \in S$.

Theorem 3.2.4. *Let* $S \subseteq \mathbb{N}^d$ *be a generalized numerical semigroup. Then the following statements are equivalent:*

1. $\operatorname{PF}(S) = \{\boldsymbol{f}, \frac{\boldsymbol{f}}{2}\}$.
2. *There exists* $\boldsymbol{f} \in \operatorname{H}(S)$ *such that its component are all even and for all* $\boldsymbol{h} \in \operatorname{H}(S)$ *with* $\boldsymbol{h} \neq \frac{\boldsymbol{f}}{2}$ *we have* $\boldsymbol{f} - \boldsymbol{h} \in S$.

Definition 3.2.5. A generalized numerical semigroup $S \subseteq \mathbb{N}^d$ is called *symmetric* if it satisfies one of the equivalent statements of Therorem 3.2.3. S is called *pseudo-symmetric* if it satisfies one of the equivalent statements of Theorem 3.2.4. In both cases S is irreducibile.

Example 3.2.6. Let $S = \mathbb{N}^2 \setminus \{(0,1),(1,1),(2,1),(3,1),(4,1),(5,1),(6,1)\}$. By an easy argument we can state that $\operatorname{PF}(S) = \{(6,1)\} = \operatorname{SG}(S)$, so S is a symmetric generalized numerical semigroup.
Let $S' = \mathbb{N}^2 \setminus \{(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(12,0)\}$. In this case it is $\operatorname{PF}(S) = \{(6,0),(12,0)\}$ and $\operatorname{SG}(S) = \{(12,0)\}$, so S' is pseudo-symmetric.
Both S and S' are irreducible generalized numerical semigroups.

Remark 3.2.7. If $S \subseteq \mathbb{N}^d$ is a symmetric generalized numerical semigroup and $d = 1$, in other words S is a symmetric numerical semigroup, we have that S is irreducibile, so $\operatorname{SG}(S) = \{f\}$ and by Theorem 3.2.3 f is odd. Moreover f is the Frobenius number of the numerical semigroup S, because for every numerical semigroup the Frobenius number is trivially an element of $\operatorname{SG}(S)$. Therefore, the definition of symmetric generalized numerical semigroups, provided here, is really a generalization of symmetric numerical semigroups. The same argument holds for pseudo-symmetric generalized numerical semigroups.

For an irreducible generalized numerical semigroup the choice of the Frobenius element as its unique special gap is consistent with previous discussions, as shown by the following:

Proposition 3.2.8. *Let* $S \subseteq \mathbb{N}^d$ *be an irreducibile generalized numerical semigroup with* $\operatorname{SG}(S) = \{\boldsymbol{f}\}$. *Then* $(S, \boldsymbol{f})$ *is a Frobenius generalized numerical semigroup.*

Proof. It suffices to prove that $\mathbf{f}$ is the maximum in $\mathrm{H}(S)$ with respect to the natural partial order in $\mathbb{N}^d$. Let $\mathbf{h} \in \mathrm{H}(S)$, if $\mathbf{h} = \frac{\mathbf{f}}{2}$ then it is tirivially $\mathbf{h} \leq \mathbf{f}$, if $\mathbf{h} \neq \mathbf{f}$ then by Proposition 3.2.1 $\mathbf{f} - \mathbf{h} \in S \subseteq \mathbb{N}^d$, so it is $\mathbf{h} \leq \mathbf{f}$. $\square$

Remark 3.2.9. The converse of Proposition 3.2.8 is not true. Let $S = \mathbb{N}^2 \setminus \{(1,0),(2,0),(3,0),(4,0),(6,0),(7,0),(9,0)\}$. It is easy to see that $(9,0)$ is the maximum in $\mathrm{H}(S)$ with respect to the natural partial order in $\mathbb{N}^2$, but it is $\mathrm{SG}(S) = \{(9,0),(7,0),(6,0),(4,0)\}$. So $(S,(9,0))$ is a Frobenius generalized numerical semigroup but it is not irreducible.

3.3 Relationships on invariants

We know that numerical semigroups satisfy some relationships that involve their invariants, like the Frobenius number, the multiplicity, the genus and others. Some of these relantionships characterize specific classes of numerical semigropus, for instance the symmetric numerical semigroups which fulfil the relation $\mathrm{g}(S) = \frac{F(S)+1}{2}$, where g is the genus and $F(S)$ the Frobenius number. We have obtained similar relationships in the generalized numerical semigroups context too.

Recall that if $\mathbf{t} \in \mathbb{N}^d$ we denote $\pi(\mathbf{t}) = \{\mathbf{n} \in \mathbb{N}^d \mid \mathbf{n} \leq \mathbf{t}\}$. Let us start by giving some notations:

Definition 3.3.1. Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup, $\mathbf{h} \in \mathbb{N}^d$, and $\leq$ the natural partial order in $\mathbb{N}^d$. We define the sets:

- $\mathrm{LH}(\mathbf{h}) = \{\mathbf{g} \in \mathrm{H}(S) \mid \mathbf{g} \leq \mathbf{h}\}$.
- $\mathrm{N}(\mathbf{h}) = \{\mathbf{n} \in \pi(\mathbf{h}) \mid \mathbf{n} \in S\}$.
- $\mathrm{MH}(S)$ the set of maximals in $\mathrm{H}(S)$, with respect to the partial order $\leq$.

Morevore we denote by $h^{(i)}$ the i-th component of $\mathbf{h}$, for any $i \in \{1,\ldots,d\}$.

By a simple argument one can be convinced that the next proposition is true.

Proposition 3.3.2. *Let $\boldsymbol{h} \in \mathbb{N}^d$ and let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup. Then the following hold:*

1. $|\pi(\boldsymbol{h})| = (h^{(1)}+1)(h^{(2)}+1)\cdots(h^{(d)}+1)$.
2. $|\,\mathrm{N}(\boldsymbol{h})| + |\,\mathrm{LH}(\boldsymbol{h})| = (h^{(1)}+1)(h^{(2)}+1)\cdots(h^{(d)}+1)$.

where $|A|$ denotes the cardinality of the set A.

Proof. The first is quite easy, the second follows from the remark that $\pi(\mathbf{h}) = \mathrm{N}(\mathbf{h}) \cup \mathrm{LH}(\mathbf{h})$ for all $\mathbf{h} \in \mathbb{N}^d$, moreover $\mathrm{N}(\mathbf{h})$ and $\mathrm{LH}(\mathbf{h})$ are disjoint. $\square$

Definition 3.3.3. Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and $\mathbf{h} \in \mathrm{H}(S)$. Then we define the following map:

$$\Psi_{\mathbf{h}} : \mathrm{N}(\mathbf{h}) \to \mathrm{LH}(\mathbf{h}),\ \mathbf{s} \longmapsto \mathbf{h} - \mathbf{s}$$

It is easy to see that the map is well defined and it is injective.

Lemma 3.3.4. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup of genus g and $\boldsymbol{h} \in H(S)$. Then*

$$|\,\mathrm{N}(\boldsymbol{h})| \leq |\,\mathrm{LH}(\boldsymbol{h})| \leq |\,\mathrm{H}(S)| = g.$$

Proof. It follows easily since the map $\Psi_{\mathbf{h}}$ is injective. $\square$

Now we provide new characterizations for symmetric and pseudo-symmetric generalized numerical semigroups.

Theorem 3.3.5. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup of genus g. Then S is symmetric if and only if there exists $\boldsymbol{f} \in \mathrm{H}(S)$ with $2g = (f^{(1)} + 1)(f^{(2)} + 1)\cdots(f^{(d)} + 1)$. Furthermore $\boldsymbol{f}$ is the Frobenius element of S.*

Proof. ($\Rightarrow$) We suppose S is symmetric. Then $\mathrm{SG}(S) = \mathrm{PF}(S) = \{\mathbf{f}\}$, so $\mathrm{LH}(\mathbf{f}) = \mathrm{H}(S)$ by Proposition 3.2.8. Let us prove that the map $\Psi_{\mathbf{f}}$ is bijective, for this it suffices to prove it is surjective. If $\mathbf{h} \in \mathrm{LH}(\mathbf{f})$, since S is symmetric, then $\mathbf{s} = \mathbf{f} - \mathbf{h} \in S$, therefore $\Psi_{\mathbf{f}}(\mathbf{s}) = \mathbf{h}$ so the map is surjective. It follows that $|\,\mathrm{N}(\mathbf{f})| = |\,\mathrm{LH}(\mathbf{f})| = g$ and

$$2g = |\,\mathrm{N}(\mathbf{f})| + |\,\mathrm{LH}(\mathbf{f})| = (f^{(1)} + 1)(f^{(2)} + 1)\cdots(f^{(d)} + 1)$$

from Proposition 3.3.2.

($\Leftarrow$) Let $\mathbf{f} \in \mathrm{H}(S)$ be such that $2g = (f^{(1)}+1)(f^{(2)}+1)\cdots(f^{(d)}+1)$. Therefore from Lemma 3.3.4 and Proposition 3.3.2 it follows that

$$2g = |\,\mathrm{N}(\mathbf{f})| + |\,\mathrm{LH}(\mathbf{f})| \leq 2|\,\mathrm{LH}(\mathbf{f})| \leq 2g.$$

So $|\,\mathrm{LH}(\mathbf{f})| = g$ and also $|\,\mathrm{N}(\mathbf{f})| = g$, hence the map $\Psi_{\mathbf{f}}$ is bijective. Now we prove that for every $\mathbf{h} \in \mathrm{H}(S)$ we have $\mathbf{f} - \mathbf{h} \in S$. Since $|\,\mathrm{LH}(\mathbf{f})| = g$ and $\Psi_{\mathbf{f}}$ is surjective, then $\mathrm{LH}(\mathbf{f}) = \mathrm{H}(S)$ and if $\mathbf{h} \in \mathrm{H}(S)$ there exists $\mathbf{s} \in S$ such that $\Psi_{\mathbf{f}}(\mathbf{s}) = \mathbf{f} - \mathbf{s} = \mathbf{h}$, in other words $\mathbf{f} - \mathbf{h} = \mathbf{s} \in S$. From Theorem 3.2.3 it follows that S is symmetric, in particular $\mathbf{f} \in \mathrm{MH}(S)$, so it is the Frobenius element. $\square$

Theorem 3.3.6. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup of genus g. Then S is pseudo-symmetric if and only if there exists $\boldsymbol{f} \in H(S)$ with $2g-1 = (f^{(1)}+1)(f^{(2)}+1)\cdots(f^{(d)}+1)$. Furthermore $\boldsymbol{f}$ is the Frobenius element of S*

Proof. ($\Rightarrow$) We suppose S is pseudo-symmetric, so $\mathrm{PF}(S) = \{\mathbf{f}, \frac{f}{2}\}$, $\mathrm{SG}(S) = \{\mathbf{f}\}$ and $\mathrm{LH}(\mathbf{f}) = \mathrm{H}(S)$. Moreover for all $\mathbf{h} \in \mathrm{H}(S)$ with $\mathbf{h} \neq \frac{\mathbf{f}}{2}$ we have $\mathbf{f} - \mathbf{h} \in S$, so arguing as in the proof of Theorem 3.3.5 we can prove that $|\,\mathrm{N}(\mathbf{f})| = |\,\mathrm{LH}(\mathbf{f}) \setminus \{\frac{\mathbf{f}}{2}\}| = g-1$. It follows that $(f^{(1)}+1)(f^{(2)}+1)\cdots(f^{(d)}+1) = |\,\mathrm{N}(\mathbf{f})| + |\,\mathrm{LH}(\mathbf{f})| = g+g-1 = 2g-1$.
($\Leftarrow$) Let $\mathbf{f} \in \mathrm{H}(S)$ be such that $2g-1 = (f^{(1)}+1)(f^{(2)}+1)\cdots(f^{(d)}+1)$, in particular every component of $\mathbf{f}$ is an even number and $2g-1 = |\,\mathrm{N}(\mathbf{f})| + |\,\mathrm{LH}(\mathbf{f})| \leq 2|\,\mathrm{LH}(\mathbf{f})| \leq 2g$. Therefore $|\,\mathrm{LH}(\mathbf{f})| = g$ (it is impossibile $2g-1 = 2|\,\mathrm{LH}(\mathbf{f})|$) and, as a consequence, $|\,\mathrm{N}(\mathbf{f})| = g-1$. Furthermore $\frac{\mathbf{f}}{2} \in \mathrm{H}(S)$ because $\mathbf{f} \in \mathrm{H}(S)$, so the map $\overline{\Psi}_{\mathbf{f}} : \mathrm{N}(\mathbf{f}) \to \mathrm{LH}(\mathbf{f}) \setminus \{\frac{\mathbf{f}}{2}\}$, induced by $\Psi_{\mathbf{f}}$, is bijective. This implies that for all $\mathbf{h} \in \mathrm{LH}(\mathbf{f}) \setminus \{\frac{\mathbf{f}}{2}\}$, in other words $\mathbf{h} \in \mathrm{H}(S)$ and $\mathbf{h} \neq \frac{\mathbf{f}}{2}$, there exists $\mathbf{s} \in S$ such that $\mathbf{f} - \mathbf{s} = \mathbf{h}$, that is $\mathbf{f} - \mathbf{h} \in S$. Hence S is pseudo-symmetric by Theorem 3.2.4, in particular $\mathbf{f} \in \mathrm{MH}(S)$ and it is the Frobenius element. ∎

Example 3.3.7. Let $S = \mathbb{N}^2 \setminus \{(0,1),(1,1),(2,1),(3,1),(4,1),(5,1),(6,1)\}$. S is a generalized numerical semigroup of genus $g = 7$ and for $\mathbf{f} = (6,1) \in \mathrm{H}(S)$ the equality $2g = (6+1)(1+1)$ holds, so S is symmetric. Indeed it is $\mathrm{PF}(S) = \{(6,1)\}$.
Let $S' = \mathbb{N}^2 \setminus \{(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(12,0)\}$. S' is a generalized numerical semigroup of genus $g = 7$ and for the element $(12,0)$, it holds $2g-1 = (12+1)(0+1)$, so S' is pseudo-symmetric. Indeed $\mathrm{PF}(S') = \{(12,0),(6,0)\}$.

Remark 3.3.8. If $d = 1$ Theorem 3.3.5 becomes: S is symmetric if and only if $2g = F(S)+1$, where $F(S)$ is the Frobenius number of S. Hence this theorem is really a generalization of the corresponding result for numerical semigroups. We can say the analogous for Theorem 3.3.6, about pseudo-symmetric numerical semigroups.

Proposition 3.3.9. *Let $(S, \boldsymbol{f})$ be a Frobenius generalized numerical semigroup of genus g in $\mathbb{N}^d$. Then $2g \geq (f^{(1)}+1)\cdots(f^{(d)}+1)$.*

Proof. In this case it is $g = |\,\mathrm{H}(S)| = |\,\mathrm{LH}(\mathbf{f})|$, moreover $|\,\mathrm{LH}(\mathbf{f})| \geq |\,\mathrm{N}(\mathbf{f})|$. Therefore

$$(f^{(1)}+1)\cdots(f^{(d)}+1) = |\,\mathrm{LH}(\mathbf{f})| + |\,\mathrm{N}(\mathbf{f})| \leq 2g.$$

∎

Example 3.3.10. Let $S = \mathbb{N}^3 \setminus \{(1,0,0),(1,0,1),(2,0,0),(2,0,1)\}$. S is a Frobenius generalized numerical semigroup with Frobenius element $\mathbf{f} = (2,0,1)$. Indeed $2g = 8 > (2+1)(0+1)(1+1) = 6$.
Let $S = \mathbb{N}^3 \setminus \{(1,0,0),(1,1,0),(3,0,0),(3,1,0)\}$. In this case the Frobenius element is $\mathbf{f} = (3,1,0)$ and $2g = 8 = (3+1)(1+1)(0+1)$, in particular S is symmetric.

Every numerical semigroup is a Frobenius generalized numerical semigroup and the previous proposition provides the well konwn inequality $g \geq \frac{F(S)+1}{2}$, where $F(S)$ and g are respectively the Frobenius number and the genus of the given numerical semigroup.

3.4 Decomposition of a generalized numerical semigroup as an intersection of finitely many irreducible ones

It is known that every numerical semigroup can be expressed as an intersection of a finite number of irreducible numerical semigroups. A decomposition with the least number of irreducible numerical semigroups involved can be obtained algorithmically (see [41]). Analogous results can be obtained in our context and this is the aim of this section.

Definition 3.4.1. Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup. We define the sets:

1. $\mathcal{O}(S) = \{T \subseteq \mathbb{N}^d \mid T \text{ is a generalized numerical semigroup, } S \subseteq T\}$, named the set of the *oversemigroups* of S.

2. $\mathcal{I}(S) = \{T \in \mathcal{O}(S) \mid T \text{ is irreducible}\}$.

Observe that $\mathcal{O}(S)$ is a finite set since S has finite complement in $\mathbb{N}^d$, moreover $\mathcal{I}(S) \subseteq \mathcal{O}(S)$.
We know that if $S \subseteq \mathbb{N}^d$ is a generalized numerical semigroup and $\mathbf{x} \notin S$ then $S \cup \{\mathbf{x}\}$ is a generalized numerical semigroup if and only if $\mathbf{x}$ is a special gap of S. In particular, in order to obtain the set of oversemigroups of S it suffices to compute the set $\mathrm{SG}(S)$, then for all $\mathbf{x} \in \mathrm{SG}(S)$ we compute $S_{\mathbf{x}} = S \cup \{\mathbf{x}\}$ and perform the procedure for all semigroups $S_{\mathbf{x}}$ to obtain $\mathbb{N}^d$.

Proposition 3.4.2. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup. If S is not irreducible then $S = S_1 \cap \cdots \cap S_n$ with $S_1, \ldots, S_n \in \mathcal{I}(S)$.*

Proof. If S is not irreducible then $S = S_1 \cap S_2$, since S_1 and S_2 are generalized numerical semigroups such that $S \subset S_1$ and $S \subset S_2$. If S_1 and S_2 are not irreducible then we can repeat for them the same argument of S. Finally, we obtain S as an intersection of a finitely many irreducible generalized numerical semigroups, since $\mathcal{I}(S)$ is a a finite set. $\square$

Let $\mathrm{Minimals}_{\subseteq}\mathcal{I}(S)$ be the set of elements in $\mathcal{I}(S)$ that are minimal with respect to set inclusion. A decomposition $S = S_1 \cap \cdots \cap S_n$ of S, with $S_i \in \mathcal{I}(S)$ for every i, is called *minimal* (or *not refinable*) if $S_1, \ldots, S_n \in \mathrm{Minimals}_{\subseteq}\mathcal{I}(S)$.

Proposition 3.4.3. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and $S = S_1 \cap \cdots \cap S_n$ with $S_1, \ldots, S_n \in \mathcal{I}(S)$. Then there exist $S'_1, \ldots, S'_n \in \mathrm{Minimals}_{\subseteq}\mathcal{I}(S)$ such that $S = S'_1 \cap \cdots S'_n$.*

Proof. If $S = S_1 \cap \cdots \cap S_n$ and there exists $i \in \{1, \ldots, n\}$ such that $S_i \notin \mathrm{Minimals}_{\subseteq}\mathcal{I}(S)$ then we can choose $S'_i \subset S_i$ with $S'_i \in \mathrm{Minimals}_{\subseteq}\mathcal{I}(S)$. $\square$

Lemma 3.4.4. *Let S and T be two generalized numerical semigroups in $\mathbb{N}^d$ such that $S \subsetneq T$. Let $\boldsymbol{h} \in \mathrm{Maximals}(T \setminus S)$ (maximal with respect to the natural partial order in $\mathbb{N}^d$). Then $\boldsymbol{h} \in \mathrm{SG}(S)$.*

Proof. Let us denote with $\leq$ the natural partial ordering in $\mathbb{N}^d$ and let $\mathbf{h} \in \mathrm{Maximals}(T \setminus S)$. Then $\mathbf{h} \in \mathrm{H}(S)$ and for all $\mathbf{s} \in S \setminus \{\mathbf{0}\}$ we have that $\mathbf{h}+\mathbf{s} \in T$ and $\mathbf{h}+\mathbf{s} > \mathbf{h}$, so $\mathbf{h}+\mathbf{s} \in S$. Analogously $2\mathbf{h} \in T$ and $2\mathbf{h} > \mathbf{h}$ so $2\mathbf{h} \in S$. Thus $\mathbf{h} \in \mathrm{SG}(S)$. $\square$

Definition 3.4.5. Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and $T \in \mathcal{O}(S)$. We define:

$$\mathcal{C}(T) = \{\mathbf{h} \in \mathrm{SG}(S) \mid \mathbf{h} \notin T\}$$

Proposition 3.4.6. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and let $S_1, \ldots, S_n \in \mathcal{O}(S)$. Then the following conditions are equivalent:*

1. $S = S_1 \cap \cdots \cap S_n$
2. *For all $\boldsymbol{h} \in \mathrm{SG}(S)$ there exists $i \in \{1, \ldots, n\}$ such that $\boldsymbol{h} \notin S_i$.*
3. $\mathcal{C}(S_1) \cup \cdots \cup \mathcal{C}(S_i) = \mathrm{SG}(S)$.

Proof. 1. $\Rightarrow$ 2. Let $\mathbf{h} \in \mathrm{SG}(S)$. Then $\mathbf{h} \notin S = S_1 \cap \cdots \cap S_n$, that is $\mathbf{h} \notin S_i$ for some $i \in \{1, \ldots, d\}$.
2. $\Rightarrow$ 1. Suppose $S \subsetneq S_1 \cap \cdots \cap S_n$. Then let $\mathbf{h} \in \mathrm{Maximals}((S_1 \cap \cdots \cap S_n) \setminus S)$, by Lemma 3.4.4 we have $\mathbf{h} \in \mathrm{SG}(S)$ and $\mathbf{h} \in S_1 \cap \cdots \cap S_n$, that is a contradiction.
2. $\Leftrightarrow$ 3. It is trivial. $\square$

As in the case of numerical semigroups it is possible to consider a minimal decomposition into irreducibles and to produce an algorithm to compute such a decomposition.

Algorithm 3.4.7. Let $S \subseteq \mathbb{N}^d$ be a not irreducible generalized numerical semigroup.

1. Compute the set $\mathrm{SG}(S)$.
2. Set $I = \emptyset$ and $C = \{S\}$.
3. For all S' in C let B be the set of generalized numerical semigroups $\overline{S}$ such that $|\overline{S} \setminus S'| = 1$.
4. Remove from B the generalized numerical semigroups S' such that $\mathrm{SG}(S) \subseteq S'$.
5. Remove from B the generalized numerical semigroups S' such that there exists $T \in I$ with $T \subseteq S'$.
6. Set $C = \{S' \in B \mid S' \text{ is not irreducible}\}$.
7. Set $I = \{S' \in B \mid S' \text{ is irreducible}\}$.
8. If $C \neq \emptyset$ go to Step 3.
9. For every $S' \in I$, compute $\mathcal{C}(S')$.
10. Return a set of semigroups $S'_1, \ldots, S'_r$ that are minimal elements in I and

$$\mathcal{C}(S'_1) \cup \cdots \cup \mathcal{C}(S'_r) = \mathrm{SG}(S)$$

We explain briefly some lines of the previous algorithm:

- Step 3: The semigroups $\overline{S}$ are obtained as $S' \cup \{\mathbf{x}\}$ with $\mathbf{x} \in \mathrm{SG}(S')$.
- Step 4: If $\mathrm{SG}(S) \subseteq S'$ by Proposition 3.4.6 S' does not occur in a representation of S as an intersection of generalized numerical semigroups.
- Step 5: Since we want to compute a minimal decomposition of S as an intersection of irreducible semigroups we do not need the oversemigroups of a computed irreducible generalized numerical semigroup.

- Step 8: By step 4 and step 5 it will occur that C will be empty at a certain iteration.
- Step 10: Since $\mathrm{Minimals}_{\subseteq}\mathcal{I}(S) \subseteq I$ we can obtain a minimal decomposition as in Proposition 3.4.3

It is known that for numerical semigroups a minimal decomposition as defined in Proposition 3.4.3 is not unique and it is not always minimal with respect to the number of the semigroups that appear in the decomposition. The same occurs for generalized numerical semigroups. In Step 10 of Algorithm 3.4.7 we can produce also a decomposition of a generalized numerical semigroup containing the minimum number of irreducible components.
In the following example we obtain two not refinable decompositions.

Example 3.4.8. Let $S = \mathbb{N}^2 \setminus \{(0,1),(0,2),(0,4),(0,5),(0,8),(0,11),(1,0),$ $(1,2),(1,3),(1,5),(1,6),(1,9),(1,12),(2,1),(2,4),(3,0),(3,2),(3,5),(4,1),(5,2)\}$
We consider

- $S_1 = S \cup \{(2,1),(2,4),(3,0),(3,2),(3,5),(4,1),(5,2)\}$.
- $S_2 = S \cup \{(0,8),(0,11),(1,6),(1,9),(1,12),(3,0),(4,1),(4,2)\}$.
- $S_3 = S \cup \{(0,4),(0,5),(0,8),(0,11),(1,6),(1,9),(1,12),(2,4),(3,5)\}$.
- $S_4 = S \cup \{(0,5),(0,8),(0,11),(1,6),(1,9),(1,12),(4,1),(4,2)\}$.

S can be minimally decomposed as $S = S_1 \cap S_2 \cap S_3$ or $S = S_1 \cap S_3 \cap S_4$.

The set of special gaps of a generalized numerical semigroup allows to obtain some properties on maximality of a given generalized numerical semigroup in the set of all generalized numerical semigroups.

Proposition 3.4.9. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and $\{\boldsymbol{h}_1, \dots, \boldsymbol{h}_t\} \subseteq \mathrm{H}(S)$. Then the following are equivalent:*

1. *S is maximal with respect to inclusion among the generalized numerical semigroups T such that $T \cap \{\boldsymbol{h}_1, \dots, \boldsymbol{h}_t\} = \emptyset$.*
2. *$\mathrm{SG}(S) \subseteq \{\boldsymbol{h}_1, \dots, \boldsymbol{h}_t\}$.*

Proof. 1. $\Rightarrow$ 2. Let $\mathbf{h} \in \mathrm{H}(S)$ and suppose that $\mathbf{h} \notin \{\mathbf{h}_1, \dots, \mathbf{h}_t\}$, then $S \subseteq S \cup \{\mathbf{h}\}$ and $(S \cup \{\mathbf{h}\}) \cap \{\mathbf{h}_1, \dots, \mathbf{h}_t\} = \emptyset$, a contradiction.
2. $\Rightarrow$ 1. Let T be a generalized numerical semigroup such that $T \cap \{\mathbf{h}_1, \dots, \mathbf{h}_t\} = \emptyset$ and suppose that $S \subsetneq T$. Then consider $\mathbf{h} \in \mathrm{Maximals}_{\leq}(T \setminus S)$. By Proposition 3.4.4, $\mathbf{h} \in \mathrm{SG}(S)$ but $\mathbf{h} \notin \{\mathbf{h}_1, \dots, \mathbf{h}_t\}$, that is a contradiction. $\square$

This means that if S and T are two generalized numerical semigroups such that $\mathrm{SG}(S) = \mathrm{SG}(T)$ but $\mathrm{H}(S) \neq \mathrm{H}(T)$ then $S \nsubseteq T$ and $T \nsubseteq S$.

Corollary 3.4.10. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup. Then S is irreducible with Frobenius element $\boldsymbol{f}$ if and only if it is maximal in the set of all generalized numerical semigroups not containing $\boldsymbol{f}$.*

Proof. ($\Rightarrow$) If S is irreducible with Frobenius element $\mathbf{f}$ then $\mathrm{SG}(S) = \{\mathbf{f}\}$ so the assertion follows from the previous proposition.
($\Leftarrow$) Suppose that $S = S_1 \cap S_2$ with $S \subset S_1$ and $S \subset S_2$. Then $\mathbf{f} \in S_1 \cap S_2 = S$, that is a contradiction. $\square$

Chapter 4

A generalized Wilf's conjecture

The Wilf's conjecture is one of the most intriguing problems for numerical semigroups. We recall it:

Conjecture 4.0.1 (**Wilf's conjecture** [45]). *Let S be a numerical semigroup. Then*

$$\mathrm{e}(S)\,\mathrm{n}(S) \geq \mathrm{F}(S) + 1$$

where $\mathrm{e}(S)$ *denotes the embedding dimension of* S *and* $\mathrm{n}(S) = |\{s \in S | s < \mathrm{F}(S)\}|$.

It is natural to ask what is a possible generalization for generalized numerical semigroups and here we propose such a conjecture. Our idea of such a conjecture originates from the attempt to generalize it for symmetric generalized numerical semigroups. Successively we looked for a consistent definition that could include all generalized numerical semigroups. In this chapter, besides to provide the first peculiarities of such a conjecture we examine a comparison with another extension of this conjecture given in [26]. In the last part we prove that our version of the conjecture holds for all irreducible generalized numerical semigroups. In the successive chapter we study this conjecture for some different classes of generalized numerical semigroups.

4.1 Basic definitions and the conjecture

Every generalized numerical semigroup has a unique finite minimal system of generators (Corollary 2.1), then the following definition is justified.

Definition 4.1.1. Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and $\mathrm{G}(S)$ be the minimal system of generators of S. The number $\mathrm{e}(S) = |\,\mathrm{G}(S)|$

is the *embedding dimension* of S. It is known that that $\mathrm{e}(S) \geq 2d$ for every generalized numerical semigroup with $S \neq \mathbb{N}^d$ (this can be easily proved from Theorem 2.1.10 or see [26], Theorem 11).

Definition 4.1.2. Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup. We define

$$\mathrm{n}(S) = \left| \bigcup_{\mathbf{h} \in \mathrm{H}(S)} \mathrm{N}(\mathbf{h}) \right|$$

Note that if S is a $(S, \mathbf{f})$ Frobenius generalized numerical semigroup then $\mathrm{n}(S) = |\,\mathrm{N}(\mathbf{f})|$ and $\mathrm{n}(S) \leq g = |\,\mathrm{LH}(\mathbf{f})|$, where g is the genus of S.

Proposition 4.1.3. *Let $S \subseteq \mathbb{N}^d$ be a symmetric generalized numerical semigroup with Frobenius element $\boldsymbol{f}$. Then*

$$\mathrm{e}(S)\,\mathrm{n}(S) \geq d(f^{(1)} + 1) \cdots (f^{(d)} + 1).$$

Proof. Since S is a Frobenius generalized numerical semigroup then $|\,\mathrm{LH}(\mathbf{f})| = g$, where g is the genus of S. If S is symmetric, by the map $\Psi_{\mathbf{f}}$ in Definition 3.3.3 we have $g = |\,\mathrm{LH}(\mathbf{f})| = |\,\mathrm{N}(\mathbf{f})| = \mathrm{n}(S)$, so

$$\mathrm{e}(S)\,\mathrm{n}(S) \geq 2dg = d(f^{(1)} + 1) \cdots (f^{(d)} + 1),$$

by Theorem 3.3.5. □

Let S be a symmetric numerical semigroup with Frobenius number $\mathrm{F}(S)$, then by Proposition 4.1.3, S satisfies $\mathrm{e}(S)\,\mathrm{n}(S) \geq \mathrm{F}(S) + 1$. This inequality for irreducibile numerical semigroups is Wilf's conjecture which is satisfied by several classes of numerical semigroups but it has not yet proved for all numerical semigroups. In a first step we propose a straightforward generalization for Frobenius generalized numerical semigroups.

Conjecture 4.1.4. *(Wilf' s conjecture for Frobenius generalized numerical semigroups) Let $(S, \boldsymbol{f})$ be a Frobenius generalized numerical semigroup in $\mathbb{N}^d$. Then*

$$\mathrm{e}(S)\,\mathrm{n}(S) \geq d(f^{(1)} + 1) \cdots (f^{(d)} + 1).$$

The key idea in the previous results is to substitute the integer $\mathrm{F}(S)+1$ in the conjecture for numerical semigroups with the cardinality of the hyper-rectangle $\pi(\mathbf{f})$, if $(S,\mathbf{f})$ is a Frobenius generalized numerical semigroup. However, in more general situations than the case of Frobenius generalized numerical semigroups, $\mathrm{F}(S)+1$ can be modified in a different way.

Definition 4.1.5. Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup, we define

$$\mathrm{c}(S) = \left| \bigcup_{\mathbf{h}\in \mathrm{H}(S)} \pi(\mathbf{h}) \right|.$$

Note that if $(S,\mathbf{f})$ is a Frobenius generalized numerical semigroup then

$$\mathrm{c}(S) = |\pi(\mathbf{f})| = (f^{(1)}+1)\cdots(f^{(d)}+1)$$

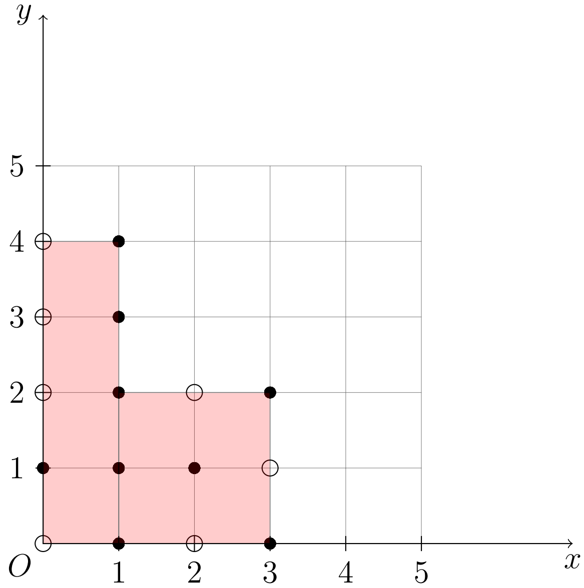
and it is the conductor for a numerical semigroup S.

Corollary 4.1.6. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup of genus $\mathrm{g}(S)$. Then*

1. $\mathrm{c}(S) = \mathrm{g}(S) + \mathrm{n}(S)$.
2. $(h^{(1)}+1)\cdots(h^{(d)}+1) \leq \mathrm{c}(S)$ *for every* $\boldsymbol{h} \in \mathrm{H}(S)$.

Proof. Trivial. □

Example 4.1.7. Let $S = \mathbb{N}^2 \setminus \{(0,1),(1,0),(1,1),(1,2),(1,3),(1,4),(2,1),(3,0),(3,2)\}$. S is a generalized numerical semigroup and its minimal system of generators is $\{(2,0),(5,0),(0,2),(0,3),(1,5),(1,6),(3,1),(4,1)\}$. Let us use a "graphical help":



The holes of S (marked in black in figure above) characterize the set $\bigcup_{\mathbf{h}\in \mathrm{H}(S)} \pi(\mathbf{h})$ (the red area in figure). The elements of S in that area are those of $\bigcup_{\mathbf{h}\in \mathrm{H}(S)} \mathrm{N}(\mathbf{h}) = \{(0,0),(2,0),(3,1),(0,2),(2,2),(0,3),(0,4)\}$ (marked by a white point). We have $\mathrm{n}(S) = 7$, $\mathrm{e}(S) = 8$, $\mathrm{c}(S) = \mathrm{g}(S) + \mathrm{n}(S) = 16$ and for any $\mathbf{h} \in \mathrm{H}(S)$ $(h^{(1)} + 1)(h^{(2)} + 1) \leq 16$.

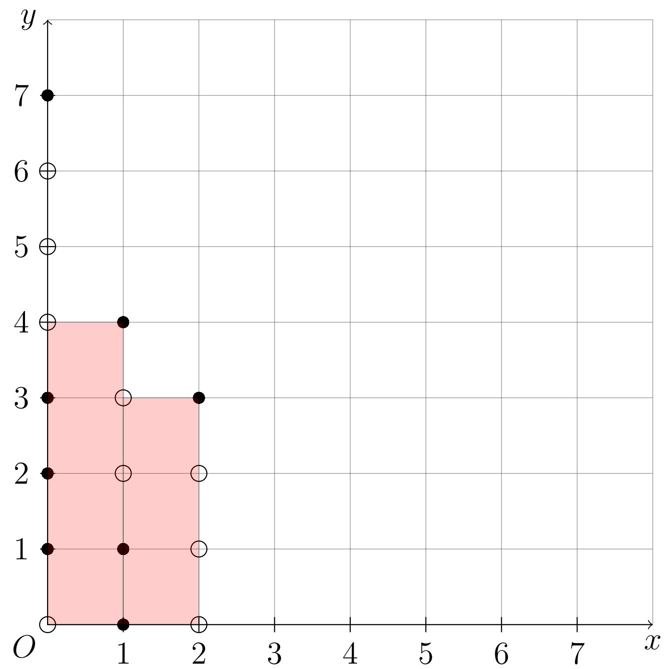
If we suppose that $\mathrm{c}(S)$ is for generalized numerical semigroups the analogous of $\mathrm{F}(S)+1$ in numerical semigroups, we can generalize the conjecture 4.1.4 to all generalized numerical semigroups as follows:

Conjecture 4.1.8. *(Wilf's generalized conjecture) Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup with genus $\mathrm{g}(S)$. Then*

$$\mathrm{e}(S)\,\mathrm{n}(S) \geq d\ \mathrm{c}(S) \quad \textit{or equivalently} \quad \mathrm{n}(S)(\mathrm{e}(S) - d) \geq d\,\mathrm{g}(S).$$

We know that every numerical semigroup of genus g satisfy $2g \geq \mathrm{F}(S) + 1$, this fact can be thought as a consequence of Proposition 3.3.9. But, in general, if S is a generalized numerical semigroup of genus g, then $2g \not\geq \mathrm{c}(S)$, as next example shows.

Example 4.1.9. Let $S = \mathbb{N}^2 \setminus \{(0,1),(0,2),(0,3),(0,7),(1,0),(1,1),(1,4),(2,3)\}$. The minimal system of generators is $\{(0,4),(0,5),(0,6),(2,0),(3,0),$
$(1,2),(1,3),(1,5),(2,1),(2,2),(3,1)\}$.



We can see that $\mathrm{g}(S) = 8$ and $\mathrm{n}(S) = 9$, so $2g = 16 < 17 = \mathrm{c}(S)$, in particular $\mathrm{n}(S) > \mathrm{g}(S)$. However $\mathrm{e}(S) = 11$ hence $\mathrm{n}(S)(\mathrm{e}(S) - 2) \geq 2\,\mathrm{g}(S)$.

Remark 4.1.10. If S is a Frobenius generalized numerical semigroup of genus g then $2g \geq \mathrm{c}(S)$. The converse is not true, that is: if $S \subseteq \mathbb{N}^d$ is a generalized numerical semigroup of genus g such that $2g \geq \mathrm{c}(S)$, it does not imply that S is a Frobenius generalized numerical semigroup. See for instance Example 4.1.7.

The following property, derived from similar concepts on numerical semigroups, allows to give a sufficient condition for a generalized numerical semigroup to satisfy the generalized Wilf's conjecture.

Proposition 4.1.11. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and let $\mathrm{t}(S) = |\,\mathrm{PF}(S)|$. Then*

$$\mathrm{g}(S) \leq \mathrm{t}(S)\,\mathrm{n}(S).$$

Proof. Consider in $\mathbb{N}^d$ a monomial order $\prec$. Let $\mathbf{x} \in \mathrm{H}(S)$, we define $\mathbf{f_x} = \min_{\prec}\{\mathbf{f} \in \mathrm{PF}(S) \mid \mathbf{x} \leq_S \mathbf{f}\}$. Observe that the previous set is not empty by Proposition 3.1.2. So we can consider the function

$$\phi : \mathrm{H}(S) \longrightarrow \mathrm{PF}(S) \times \left(\bigcup_{\mathbf{h} \in \mathrm{H}(S)} \mathrm{N}(\mathbf{h}) \right), \qquad \mathbf{x} \longmapsto (\mathbf{f_x}, \mathbf{f_x} - \mathbf{x})$$

It is easy to see that ϕ is injective so $\mathrm{g}(S) \leq \mathrm{t}(S)\,\mathrm{n}(S)$. $\square$

Since $\mathrm{c}(S) = \mathrm{g}(S) + \mathrm{n}(S)$, then $\mathrm{c}(S) \leq (\mathrm{t}(S)+1)\,\mathrm{n}(S)$. So we can state the following:

Corollary 4.1.12. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and $\mathrm{t}(S) = |\,\mathrm{PF}(S)|$. If $\mathrm{e}(S) \geq d(\mathrm{t}(S)+1)$ then S satisfies generalized Wilf's conjecture 4.1.8.*

4.2 Comparison with a different extension of Wilf's conjecture

In [26] another generalization of Wilf's conjecture is given. Actually that generalization involves a larger class of affine semigroup, called $\mathcal{C}$-semigroups, and generalized numerical semigroups lay among them. In particular the following definitions are introduced for a generalized numerical semigroup $S \subseteq \mathbb{N}^d$ (with reference also in [22]): let $\prec$ be a monomial order satisfying that every monomial is preceded only by a finite number of monomials. The maximum of $\mathrm{H}(S)$ with respect to $\prec$ is the Frobenius element of S, denoted by $\mathrm{Fb}(S)$. By convention, $\mathrm{Fb}(\mathbb{N}^d)$ is the vector $(-1,\ldots,-1) \in \mathbb{Z}^d$. Denote by $\mathrm{n}_\prec(S)$ the cardinality of the finite set $\{\mathbf{x} \in S \mid \mathbf{x} \prec \mathrm{Fb}(S)\}$. The Frobenius number of a S is defined as $\mathrm{n}_\prec(S) + \mathrm{g}(S)$ and denoted by $\mathrm{N}(\mathrm{Fb}(S))$. So the following conjecture is stated:

Conjecture 4.2.1. *(Extended Wilf's conjecture) Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup. Then $\mathrm{n}_\prec(S)\,\mathrm{e}(S) \geq \mathrm{N}(\mathrm{Fb}(S)) + 1$, for every monomial order satisfying that every monomial is preceded only by a finite number of monomials.*

We would like to compare the generalized Wilf's conjecture 4.1.8 and the extended Wilf's conjecture 4.2.1. First of all, we can remark that the first cojecture has the good property that does not depend on the choice of a monomial order. In order to provide a simple link between the two conjectures, we recall the following property (Proposition 2.3.1):
Every monomial order in $\mathbb{N}^d$ extends the natural partial order in $\mathbb{N}^d$.

Proposition 4.2.2. *If $S \subseteq \mathbb{N}^d$ is a generalized numerical semigroup that satisfies the generalized Wilf's conjecture 4.1.8 then S satisfies the extended Wilf's conjecture 4.2.1.*

Proof. If $d = 1$ it is clear that the two inequalities are the same, so we supoose that $d > 1$. Fix a monomial order $\prec$ in $\mathbb{N}^d$. Let $\mathbf{s} \in \bigcup_{\mathbf{h}\in \mathrm{H}(S)} \mathrm{N}(\mathbf{h})$, then $\mathbf{s} \leq \mathbf{h}$ for some $\mathbf{h} \in \mathrm{H}(S)$, with respect to the natural partial order in $\mathbb{N}^d$. By Proposition 2.3.1 $\mathbf{s} \prec \mathbf{h} \prec \mathrm{Fb}(S)$, so $\mathbf{s} \in \{\mathbf{x} \in S \mid \mathbf{x} \prec \mathrm{Fb}(S)\}$. Therefore $\mathrm{n}(S) \leq \mathrm{n}_\prec(S)$. Consider the generalized Wilf's conjecture 4.1.8 in the form $\mathrm{n}(S)(\mathrm{e}(S) - d) \geq d\,\mathrm{g}(S)$. Hence $\mathrm{n}_\prec(S)(\mathrm{e}(S) - 1) \geq \mathrm{n}(S)(\mathrm{e}(S) - 1) \geq \mathrm{n}(S)(\mathrm{e}(S) - d) \geq d\ \mathrm{g}(S) \geq \mathrm{g}(S) + 1$, in particular

$$\mathrm{n}_\prec(S)\,\mathrm{e}(S) \geq \mathrm{n}_\prec(S) + \mathrm{g}(S) + 1 = \mathrm{N}(\mathrm{Fb}(S)) + 1.$$

$\square$

In [26] some classes of generalized numerical semigroups are described for which the extended Wilf's conjecture 4.2.1 is satisfied. Now we want to see the behaviour of those classes with respect to the generalized Wilf's conjecture 4.1.8.

The first class is that of generalized numerical semigroups in $\mathbb{N}^d$ generated by

$$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \ldots, \mathbf{e}_d, 2\mathbf{e}_i, 3\mathbf{e}_i\} \\ \cup \{\mathbf{e}_i + h\mathbf{e}_k\} \cup \{\mathbf{e}_i + \mathbf{e}_j \mid j \in \{1, 2, \ldots, d\} \setminus \{k, i\}\},$$

$i \in \{1, 2, \ldots, d\}$, $k \in \{1, 2, \ldots, d\} \setminus \{i\}$, $h > 1$ a positive integer ([26], Lemma 15)

Proposition 4.2.3. *Let $h > 1$ be a positive integer, $i \in \{1, 2, \ldots, d\}$, $k \in \{1, 2, \ldots, d\} \setminus \{i\}$. Consider the generalized numerical semigroup $S \subseteq \mathbb{N}^d$ generated by:*

$$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{i-1}, \boldsymbol{e}_{i+1}, \ldots, \boldsymbol{e}_d, 2\boldsymbol{e}_i, 3\boldsymbol{e}_i\} \\ \cup \{\boldsymbol{e}_i + h\boldsymbol{e}_k\} \cup \{\boldsymbol{e}_i + \boldsymbol{e}_j \mid j \in \{1, 2, \ldots, d\} \setminus \{k, i\}\}$$

Then S satisfies the generalized Wilf's conjecture 4.1.8.

Proof. First we have $\mathrm{e}(S) = 2d$. The set of holes of S is $\mathrm{H}(S) = \{\mathbf{e}_i, \mathbf{e}_i + \mathbf{e}_k, \mathbf{e}_i + 2\mathbf{e}_k, \ldots, \mathbf{e}_i + (h-1)\mathbf{e}_k\}$, so S is a Frobenius generalized numerical semigroup with Frobenius element $\mathbf{f} = \mathbf{e}_i + (h-1)\mathbf{e}_k$. Therefore $\mathrm{c}(S) = |\pi(\mathbf{f})| = 2h$. Furthermore

$$\bigcup_{\mathbf{h}\in \mathrm{H}(S)} \mathrm{N}(\mathbf{h}) = \{\mathbf{0}, \mathbf{e}_k, 2\mathbf{e}_k, \ldots, (h-1)\mathbf{e}_k\}$$

so $\mathrm{n}(S) = h$. Finally
$$d\ \mathrm{c}(S) = 2d\,h = \mathrm{n}(S)\ \mathrm{e}(S).$$
$\square$

The second class contains semigroups $S = \mathbb{N}^d \setminus \{\mathbf{e}_i, 2\mathbf{e}_i, \ldots, (q-1)\mathbf{e}_i\}$, with $i \in \{1, 2, \ldots, d\}$ and $k \in \mathbb{N}\setminus\{0\}$ ([26], Lemma 16). In this case $S = (\mathbb{N}^d \setminus \pi((q-1)\mathbf{e}_i)$ and S satisfies the generalized Wilf's conjecture by Proposition 5.1.12 that we will see in a successive section.
The last class of generalized numerical semigroups is the following: let T be a numerical semigroup minimally generated by $\{\lambda_1, \lambda_2\}$, $j \in \{1, 2, \ldots, d\}$ and a set $\{q_i \in \mathbb{N} \mid i \in \{1, 2, \ldots, d\} \setminus \{j\}\}$, we can consider $S = \mathbb{N}^d \setminus \{(x_1, \ldots, x_d) \mid x_j \notin T, x_i < q_i, i \in \{1, 2, \ldots, d\} \setminus \{j\}\}$ ([26], Lemma 17). Now we prove that a more general class than S satisfies the generalized Wilf's conjecture 4.1.8. We recall that a Frobenius generalized numerical semigroup, with Frobenius element $\mathbf{f} = (f^{(1)}, f^{(2)}, \ldots, f^{(d)})$, is *symmetric* if and only if $2\,\mathrm{g}(S) = (f^{(1)} + 1)(f^{(2)}+1)\cdots(f^{(d)}+1)$. From Proposition 4.1.3 and the fact that $\mathrm{c}(S) = \pi(\mathbf{f}) = (f^{(1)} + 1)(f^{(2)} + 1)\cdots(f^{(d)} + 1)$ it follows that every symmetric generalized numerical semigroup satisfies the generalized Wilf's conjecture 4.1.8. Hence we can state the following:

Proposition 4.2.4. *Let $T \subseteq \mathbb{N}$ be a symmetric numerical semigroup, $j \in \{1, 2, \ldots, d\}$ and a set $\{q_i \in \mathbb{N} \mid i \in \{1, 2, \ldots, d\} \setminus \{j\}\}$. Then $S = \mathbb{N}^d \setminus \{(x_1, \ldots, x_d) \in \mathbb{N}^d \mid x_j \notin T, x_i < q_i, i \in \{1, 2, \ldots, d\} \setminus \{j\}\}$ is a generalized numerical semigroup and it satisfies the generalized Wilf's conjecture 4.1.8.*

Proof. Observe that S is a Frobenius generalized numerical semigroup with Frobenius element $\mathbf{f} = (q_1 - 1, q_2 - 1, \ldots, q_{j-1} - 1, \mathrm{F}(T), q_{j+1} - 1), \ldots, q_d - 1)$, where $\mathrm{F}(T)$ is the Frobenius number of T. Moreover $\mathrm{g}(S) = q_1 q_2 \cdots q_{j-1}\,\mathrm{g}(T) q_{j+1} \cdots q_d$, $\mathrm{g}(T)$ the genus of the numerical semigroup T. Since T is a symmetric numerical semigroup then $2\,\mathrm{g}(T) = \mathrm{F}(T) + 1$. It follows $2\,\mathrm{g}(S) = (f^{(1)} + 1)(f^{(2)} + 1)\cdots(f^{(d)} + 1)$, so S is a symmetric generalized numerical semigroup, then S satisfies the generalized Wilf's conjecture 4.1.8. $\square$

4.3 A "reduction" on generalized Wilf's conjecture

Let A be a subset of $\mathbb{N}^d$, we denote by $\mathrm{Span}_{\mathbb{R}}(A)$ the $\mathbb{R}$-vector subspace of $\mathbb{R}^d$ generated by the elements of A. Recall that a vector subspace of $\mathbb{R}^d$ is a *coordinate linear space* if it is spanned by a subset of $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d\}$. The results in this section are inspired by the following proposition.

Proposition 4.3.1 ([22], Proposition 5.2)**.** *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and $\mathrm{H}(S)$ the set of its holes. Then $\mathrm{Span}_{\mathbb{R}}(\mathrm{H}(S))$ is a coordinate linear space.*

Notations 4.3.2. We will use the following notations:

- $S_{g,d}$ is the set of all generalized numerical semigroup with genus g in $\mathbb{N}^d$
- $S_{g,d}^{(r)} = \{S \in S_{g,d} \mid \dim(\mathrm{Span}_{\mathbb{R}}(\mathrm{H}(S))) = r\}$.
- $N_{g,d}$ and $N_{g,d}^{(r)}$ denote respectively the cardinalities of $S_{g,d}$ and $S_{g,d}^{(r)}$.

We want to involve these notions in the generalized Wilf's conjecture.

Definition 4.3.3. :

1. Let $S \in S_{g,d}^{(r)}$. We define the set $Axes(S) = \{k \in \{1, 2, \ldots, d\} \mid \text{for all } \mathbf{h} \in \mathrm{H}(S), h^{(k)} = 0\}$, where $h^{(k)}$ is the k-th coordinate of $\mathbf{h} \in \mathbb{N}^d$. By Proposition 5.2 ([22]) it is $\mid Axes(S) \mid = d - r$.
2. Let $S \in S_{g,d}^{(r)}$, $\{i_1, i_2, \ldots, i_r\} = \{1, 2, \ldots, d\} \setminus Axes(S)$ and $\{\overline{\mathbf{e}}_1, \overline{\mathbf{e}}_2, \ldots, \overline{\mathbf{e}}_r\}$ the standard basis of $\mathbb{R}^r$. If $\mathbf{h} \in \mathrm{H}(S)$ it is possible to write $\mathbf{h} = \sum_{j=1}^{r} h^{(i_j)} \mathbf{e}_{i_j}$.We consider the element $\overline{\mathbf{h}} = \sum_{j=1}^{r} h^{(i_j)} \overline{\mathbf{e}}_j \in \mathbb{N}^r$ and define the set $\overline{H}(S) = \{\overline{\mathbf{h}} \mid \mathbf{h} \in \mathrm{H}(S)\}$.
3. We define $\overline{S} := \mathbb{N}^r \setminus \overline{H}(S)$.

We remark that if $S \in S_{g,d}^{(r)}$ then there exist r coordinates such the coordinates different from those r are zero in every hole of S (the elements in $Axes(S)$). The set $\overline{H}(S)$ is obtained projecting in $\mathbb{N}^r$ every element in $\mathrm{H}(S)$ with respect to those r coordinates. This set has the following property.

Lemma 4.3.4. *Let $S \in S_{g,d}^{(r)}$ and $\overline{S} := \mathbb{N}^r \setminus \overline{H}(S)$. Then $\overline{S} \in S_{g,r}^{(r)}$, in particular $\mathrm{H}(\overline{S}) = \overline{H}(S)$.*

Proof. Obviously $\overline{S}$ has finite complement in $\mathbb{N}^r$, $\overline{H}(S)$, whose cardinality is g and $\langle \overline{H}(S) \rangle$ is a coordinate linear space of dimension r (in $\mathbb{R}^r$). It suffices to prove that $\overline{S}$ is a semigroup. Without loss of generality suppose $Axis(S) = \{r+1, r+2, \ldots, d\}$, that is $\langle \mathrm{H}(S) \rangle = \langle \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_r \rangle$. Let $\mathbf{s}_1, \mathbf{s}_2 \in \overline{S}$. Then $\begin{bmatrix} \mathbf{s}_1 \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{s}_2 \\ \mathbf{0} \end{bmatrix} \in S$ (with $\mathbf{0} \in \mathbb{N}^{d-r}$). Therefore $\begin{bmatrix} \mathbf{s}_1 + \mathbf{s}_2 \\ \mathbf{0} \end{bmatrix} \in S$. It follows that $\mathbf{s}_1 + \mathbf{s}_2 \notin \overline{H}(S)$, in particular $\mathbf{s}_1 + \mathbf{s}_2 \in \overline{S}$. The last statement is trivial. $\square$

Observe that if $Axis(S) = \emptyset$ then $\mathrm{H}(S) = \overline{H}(S)$ and $S = \overline{S}$. We suppose in the following that $Axis(S) \neq \emptyset$.

Lemma 4.3.5. *If $S \in S_{g,d}^{(r)}$ then $\mathrm{c}(\overline{S}) = \mathrm{c}(S)$ and $\mathrm{n}(\overline{S}) = \mathrm{n}(S)$.*

Proof. Let $\overline{\mathbf{s}} = \sum_{j=1}^{r} s^{(i_j)} \overline{\mathbf{e}}_j \in \mathbb{N}^r$, where we have labeled the j-th component of $\overline{\mathbf{s}}$ with i_j. Suppose that $\overline{\mathbf{s}} \in \pi(\overline{\mathbf{h}})$, that is $\overline{\mathbf{s}} \leq \overline{\mathbf{h}}$, where $\overline{\mathbf{h}} \in \mathrm{H}(\overline{S})$. In $\mathbb{N}^d$ we consider the element $\mathbf{s} = \sum_{j=1}^{r} s^{(i_j)} \mathbf{e}_{i_j}$. If $\overline{\mathbf{h}} = \sum_{j=1}^{r} h^{(i_j)} \overline{\mathbf{e}}_j \in \mathbb{N}^r$ and $\mathbf{h} = \sum_{j=1}^{r} h^{(i_j)} \mathbf{e}_{i_j}$, we have $\mathbf{s} \leq \mathbf{h}$. It means that the map

$$\bigcup_{\overline{\mathbf{h}} \in \mathrm{H}(\overline{S})} \pi(\overline{\mathbf{h}}) \to \bigcup_{\mathbf{h} \in \mathrm{H}(S)} \pi(\mathbf{h}) \quad \text{such that} \quad \overline{\mathbf{s}} \mapsto \mathbf{s}$$

is injective, in particular $\mathrm{c}(\overline{S}) \leq \mathrm{c}(S)$. Moreover if $\mathbf{s} \in \mathbb{N}^d$ and there exists $\mathbf{h} \in \mathrm{H}(S)$ with $\mathbf{s} \leq \mathbf{h}$, since the coordinates in $Axes(S)$ of $\mathbf{h}$ are zero, then also the coordinates in $Axes(S)$ of $\mathbf{s}$ are zero and it is possible to consider $\overline{\mathbf{s}} \geq \overline{\mathbf{h}}$, so also $\mathrm{c}(\overline{S}) \geq \mathrm{c}(S)$. By similar argument it can be proved that $\mathrm{n}(\overline{S}) = \mathrm{n}(S)$. $\square$

Lemma 4.3.6. *Let $S \in S_{g,d}^{(r)}$, then the minimal generators of $\overline{S}$ are exactly the elements $\overline{\boldsymbol{g}} = \sum_{j=1}^{r} g^{(i_j)} \overline{\mathbf{e}}_j \in \mathbb{N}^r$ such that $\boldsymbol{g} = \sum_{j=1}^{r} g^{(i_j)} \mathbf{e}_{i_j} \in S$ is a minimal generator of S. In particular $\mathrm{e}(\overline{S}) < \mathrm{e}(S)$.*

Proof. Let $\overline{\mathbf{g}} = \sum_{j=1}^{r} g^{(i_j)} \overline{\mathbf{e}}_j$ be a minimal generator of $\overline{S}$ and consider the element $\mathbf{g} = \sum_{j=1}^{r} g^{(i_j)} \mathbf{e}_{i_j} \in S$. Suppose that $\mathbf{g}$ is not a minimal generator of S, then $\mathbf{g} = \sum_{k=1}^{n} \lambda_k \mathbf{g}_k$, where $\mathbf{g}_k \in S$ and $\lambda_k \in \mathbb{N} \setminus \{0\}$ for all k. Observe that, for every $k = 1, 2, \ldots, n$ all components in $Axes(S)$ of $\mathbf{g}_k$ must be zero since $\lambda_k > 0$. Therefore one can define the elements $\overline{\mathbf{g}}_k$ as in the definition of $\overline{\mathbf{h}} \in \overline{H}(S)$ from $\mathbf{h} \in \mathrm{H}(S)$. In such a way we have $\overline{\mathbf{g}}_k \in \overline{S}$ and $\overline{\mathbf{g}} = \sum_{k=1}^{n} \lambda_k \overline{\mathbf{g}}_k$, but this is a contradiction because $\overline{\mathbf{g}}$ is a minimal generator of $\overline{S}$. Therefore $\mathrm{e}(\overline{S}) \leq \mathrm{e}(S)$.
Now let $\mathbf{g}$ be a minimal generator of S, $\mathbf{g} = \sum_{j=1}^{r} g^{(i_j)} \mathbf{e}_{i_j} \in S$. By a similar argument it can be proved that $\overline{\mathbf{g}} = \sum_{j=1}^{r} g^{(i_j)} \overline{\mathbf{e}}_j \in \mathbb{N}^r$ is a minimal generator of $\overline{S}$.
Finally, if $k \in Axes(S)$ then there exists at least a minimal generator of S whose k-th component is nonzero, on the contrary we would have infinite elements $\lambda \mathbf{e}_k \notin S$. It follows that $\mathrm{e}(\overline{S}) < \mathrm{e}(S)$. $\square$

Example 4.3.7. Let $S = \mathbb{N}^5 \setminus \{(0,0,0,1,0), (0,0,0,2,0), (0,1,0,0,0), (0,1,0,3,0)\}$. The set of minimal generators of S is $\mathrm{G}(S) = \{(1,0,0,0,0), (0,0,1,0,0), (0,0,0,0,1), (1,0,0,1,0), (0,1,0,1,0), (0,0,1,1,0),$

$(0,0,0,1,1),(0,0,0,3,0),(1,0,0,2,0),(0,1,0,2,0),(0,0,1,2,0),(0,0,0,2,1),$
$(0,0,0,5,0),(0,0,0,4,0),(1,1,0,0,0),(0,1,1,0,0),(0,1,0,0,1),(0,2,0,0,0),$
$(0,2,0,1,0),(0,3,0,0,0)\}$. $\mathrm{e}(S)=20$, $\mathrm{g}(S)=4$.
In this case $Axes(S)=\{1,3,5\}$ and $i_1=2$, $i_2=4$.
With the previous construction we have $\overline{S}=\mathbb{N}^2\setminus\{(0,1),(0,2),(1,0),(1,3)\}$.
The set of minimal generators of $\overline{S}$ is $\mathrm{G}(\overline{S})=\{(1,1),(1,2),(0,3),(0,4),(0,5),$
$(2,1),(2,0),(3,0)\}$. So $\mathrm{e}(\overline{S})=8$.

In order to prove the main result we need to estimate the difference between $\mathrm{e}(S)$ and $\mathrm{e}(\overline{S})$.

Lemma 4.3.8. *Let $S\in S^{(r)}_{g,d}$, $s=|Axes(S)|=d-r$ and $t=\mathrm{e}(S)-\mathrm{e}(\overline{S})$. Then $s\,\mathrm{c}(\overline{S})\le t\,\mathrm{n}(\overline{S})$.*

Proof. Let $Axes(S)=\{j_1,j_2,\dots,j_s\}$ and $\{i_1,i_2,\dots,i_r\}=\{1,2,\dots,d\}\setminus Axes(S)$. By Lemma 4.3.6 the minimal set of generators of S can be write as $A\cup\mathcal{B}$ where A contains the minimal generators of S whose expression is $\mathbf{g}=\sum_{j=1}^{r}g^{(i_j)}\mathbf{e}_{i_j}\in S$, in particular $|A|=\mathrm{e}(\overline{S})$ and the set $\overline{A}$ (containing the corresponding $\overline{\mathbf{g}}$) is the minimal set of generators for $\overline{S}$. So $|\mathcal{B}|=t$ and observe that $\{\mathbf{e}_{j_1},\mathbf{e}_{j_2},\dots,\mathbf{e}_{j_s}\}\subseteq\mathcal{B}$. Consider the following function:

$$\Psi:\{\mathbf{e}_{j_1},\mathbf{e}_{j_2},\dots,\mathbf{e}_{j_s}\}\times\left(\bigcup_{\overline{\mathbf{h}}\in\mathrm{H}(\overline{S})}\pi(\overline{\mathbf{h}})\right)\longrightarrow\mathcal{B}\times\left(\bigcup_{\overline{\mathbf{h}}\in\mathrm{H}(\overline{S})}N(\overline{\mathbf{h}})\right)$$

defined as follows: let $\overline{\mathbf{x}}=\sum_{n=1}^{r}x^{(n)}\overline{\mathbf{e}}_n\in\left(\bigcup_{\overline{\mathbf{h}}\in\mathrm{H}(\overline{S})}\pi(\overline{\mathbf{h}})\right)$, in particular we consider $\mathbf{x}=\sum_{n=1}^{r}x^{(n)}\mathbf{e}_{i_n}\in\mathbb{N}^d$. Observe that $\mathbf{e}_{j_k}+\mathbf{x}\in S$ for every $k\in\{1,\dots,s\}$. Then we define:

- If $\mathbf{e}_{j_k}+\mathbf{x}\in\mathcal{B}$ then $\Psi(\mathbf{e}_{j_k},\overline{\mathbf{x}})=(\mathbf{e}_{j_k}+\mathbf{x},\mathbf{0})$.
- If $\mathbf{e}_{j_k}+\mathbf{x}\notin\mathcal{B}$ and $\mathbf{x}\in S$ then $\Psi(\mathbf{e}_{j_k},\overline{\mathbf{x}})=(\mathbf{e}_{j_k},\overline{\mathbf{x}})$.
- If $\mathbf{e}_{j_k}+\mathbf{x}\notin\mathcal{B}$ and $\mathbf{x}\notin S$ then it is possible to write $\mathbf{e}_{j_k}+\mathbf{x}=\mathbf{g}+\mathbf{a}$ with $\mathbf{g}\in\mathcal{B}$ having nonzero j_k-th coordinate and $\mathbf{a}\in S$ with $\mathbf{a}<\mathbf{x}$ (with respect to the partial order). Among all such decomposition of $\mathbf{e}_{j_k}+\mathbf{x}$ we can choose that with $\mathbf{g}$ minimum with respect to any fixed total order in $\mathbb{N}^d$. So it is uniquely defined $\Psi(\mathbf{e}_{j_k},\overline{\mathbf{x}})=(\mathbf{g},\overline{\mathbf{a}})$ with $\overline{\mathbf{a}}$ defined as usual from $\mathbf{a}$.

Observe that if $\Psi(\mathbf{e}_{j_k}, \overline{\mathbf{x}}) = \Psi(\mathbf{e}_{j_n}, \overline{\mathbf{y}})$ then $\mathbf{e}_{j_k} + \mathbf{x} = \mathbf{e}_{j_n} + \mathbf{y}$ in S. It follows $k = n$ and in particular $\mathbf{x} = \mathbf{y}$. Therefore the map Ψ is injective and this concludes the proof. $\square$

Theorem 4.3.9. *Let $S \in \mathcal{S}^{(r)}_{g,d}$ and suppose that $\overline{S} \in \mathcal{S}^{(r)}_{g,r}$ satisfies the generalized Wilf's conjecture. Then also S satisfies the generalized Wilf's conjecture 4.1.8.*

Proof. Let $t = \mathrm{e}(S) - \mathrm{e}(\overline{S})$ and $s = |Axes(S)| = d - r$. Then, using Lemma 4.3.5 and Lemma 4.3.8 we obtain the following chain of inequalities:

$$\mathrm{e}(S)\,\mathrm{n}(S) = (\mathrm{e}(\overline{S}) + t)\,\mathrm{n}(\overline{S}) = \mathrm{e}(\overline{S})\,\mathrm{n}(\overline{S}) + t\ \mathrm{n}(\overline{S}) \geq r\ \mathrm{c}(\overline{S}) + t\ \mathrm{n}(\overline{S}) \geq$$
$$r\ \mathrm{c}(\overline{S}) + s\ \mathrm{c}(\overline{S}) = (r+s)\,\mathrm{c}(\overline{S}) = d\ \mathrm{c}(\overline{S}) = d\ \mathrm{c}(S),$$

and this concludes the proof. $\square$

Remark 4.3.10. Theorem 4.3.9 states that in order to prove the truth of the generalized Wilf's conjecture 4.1.8 it suffices to prove it only for the generalized numerical semigroups belonging in $\mathcal{S}^{(d)}_{g,d}$.

4.4 Generalized Wilf's conjecture for irreducible generalized numerical semigroups

In Proposition 4.1.3 we show that all symmetric generalized numerical semigroups satisfy generalized Wilf's conjecture. Now we want to show that actually this occurs for all irreducible generalized numerical semigroups. The proof of the conjecture for pseudo-symmetric generalized numerical semigroups requires some preliminary results and Theorem 4.3.9 of the previous section.

Lemma 4.4.1. *Let $S \subseteq \mathbb{N}^d$ be an irreducible generalized numerical semigroup such that* $\mathrm{e}(S) = 2d$*. Then S is symmetric.*

Proof. If $\mathrm{e}(S) = 2d$ then by [13, Theorem 2.8] it follows that $S = \langle A \rangle$ with $A = \{\mathbf{e}_1, \ldots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \ldots, \mathbf{e}_d, a\mathbf{e}_i, b\mathbf{e}_i \mid i \in \{1, \ldots, d\}, 1 < a < b \in \mathbb{N} \setminus \{0\}, \mathrm{GCD}(a,b) = 1\} \cup \{\mathbf{e}_i + h^{(j)}\mathbf{e}_j \mid j \in \{1, \ldots, d\} \setminus \{i\}, h^{(j)} \in \mathbb{N} \setminus \{0\}\}$. Observe that a and b generate a numerical semigroup in the i-th axe. We distinguish two cases:
1) $a = 2$. In such a case $\mathrm{H}(\langle 2, b \rangle) = \{1, 3, 5, \ldots, b-2\}$ and with a simple

argument we see that

$$\mathrm{H}(S)=\{h\mathbf{e}_i+\sum_{j\neq i} i_j\mathbf{e}_j \mid h\in \mathrm{H}(\langle 2,b\rangle), i_j\in\{0,1,\dots,h^{(j)}-1\},$$
$$j\in\{1,\dots,d\}\setminus\{i\}\}.$$

Moreover S is a Frobenius generalized numerical semigroup with Frobenius element $\mathbf{f}=(b-2)\mathbf{e}_i+\sum_{j\neq i}(h^{(j)}-1)\mathbf{e}_j$ and genus $\mathrm{g}(S)=\frac{b-1}{2}\prod_{j\neq i}h^{(j)}$. By Theorem 3.3.5 S is symmetric.
2) $a>2$. In such a case we show that S is not a Frobenius generalized numerical semigroup, so it is not irreducible. This will prove the claim of this lemma. Let $F=ab-a-b$ be the Frobenius number of $\langle a,b\rangle$ and consider the element $\mathbf{h}=F\mathbf{e}_i+\sum_{j\neq i}(h^{(j)}-1)\mathbf{e}_j$. We show that $\mathbf{h}$ is a maximal element in $\mathrm{H}(S)$ with respect to the natural partial order in $\mathbb{N}^d$. First we show that $\mathbf{h}\in\mathrm{H}(S)$. If not, $\mathbf{h}\in\langle A\rangle$ and since $\mathbf{h}-(\mathbf{e}_i+h^{(j)}\mathbf{e}_j)\notin\mathbb{N}^d$ for all $j\in\{1,\dots,d\}\setminus\{i\}$ then $\mathbf{h}=\lambda_1 a\mathbf{e}_i+\lambda_2 b\mathbf{e}_i+\sum_{j\neq i}\mu_j\mathbf{e}_j$, with $\lambda_1,\lambda_2,\mu_j\in\mathbb{N}$. But this implies $F=\lambda_1 a+\lambda_2 b$ that is a contradiction, so $\mathbf{h}\in\mathrm{H}(S)$. In order to prove that it is a maximal hole it suffices to prove that $\mathbf{h}+\mathbf{e}_k\in S$ for all $k\in\{1,\dots,d\}$. It is obvious that $\mathbf{h}+\mathbf{e}_i\in S$. So let $k\neq i$, then $\mathbf{h}+\mathbf{e}_k=(F-1)\mathbf{e}_i+\sum_{j\neq i,k}(h^{(j)}-1)\mathbf{e}_k+\mathbf{e}_i+h^{(k)}\mathbf{e}_k$. Since $\langle a,b\rangle$ is a symmetric numerical semigroup, $F-1\in\langle a,b\rangle$, hence $(F-1)\mathbf{e}_i\in S$. Therefore $\mathbf{h}+\mathbf{e}_k\in S$ and $\mathbf{h}$ is maximal in $\mathrm{H}(S)$. It remains to prove that there exists an element in $\mathrm{H}(S)$ not comparable with $\mathbf{h}$. Consider $\mathbf{x}=2\mathbf{e}_i+h^{(k)}\mathbf{e}_k$ with $k\neq i$. Obviously $\mathbf{x}\nleq\mathbf{h}$, moreover one can see by a simple argument that $\mathbf{x}\in\mathrm{H}(S)$. This concludes the proof. □

Remark 4.4.2. The proof of the previous lemma shows actually a stronger result: If $S\subseteq\mathbb{N}^d$ is a generalized numerical semigroup with $\mathrm{e}(S)=2d$, then S is Frobenius if and only if S is symmetric.

For the claim and the proof of the following lemma we use the same notation of the previous section.

Lemma 4.4.3. *Let $S\in\mathcal{S}_{g,d}^{(r)}$ and $\overline{S}\in\mathcal{S}_{g,r}^{(r)}$ be as in Definition 4.3.3. Then the following hold:*

1. *If S is symmetric then $\overline{S}$ is symmetric.*
2. *If S is pesudo-symmetric then $\overline{S}$ is pseudo-symmetric.*

Proof. Let $\{1,2,\dots,d\} \setminus Axes(S) = \{i_1, i_2, \dots, i_r\}$. Suppose S is symmetric or pseudo-symmetric and let $\mathbf{f} = (f^1, \dots, f^{(d)})$ be the Frobenius element of S. Then $\prod_{i=1}^{d}(f^{(i)}+1) = \prod_{k=1}^{r}(f^{(i_k)}+1)$ since for $j \in Axes(S)$ we have $f^{(j)}+1 = 1$. But $\bar{\mathbf{f}} = (f^{(i_1)}, \dots, f^{(i_r)})$ is the Frobenius element of $\overline{S}$. So both the statement follow easily from Theorem 3.3.5 and Theorem 3.3.6, since $\mathrm{g}(S) = \mathrm{g}(\overline{S})$. □

Lemma 4.4.4. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup. Then the following hold:*

1. *If $\mathrm{g}(S) < d$ then $S \in S_{g,d}^{(r)}$ with $r < d$. In particular $\mathrm{g}(S) \geq r$.*
2. *If $\mathrm{g}(S) = d$ and $S \in S_{g,d}^{(d)}$ then S is not pseudo-symmetric.*

Proof. The first statement is quite easy, considering that a vector space of dimension r is spanned by exactly r independent vectors. To prove the second statement, suppose that $(S, \mathbf{f})$ is a pseudo-symmetric generalized numerical semigroup. Then $\mathbf{f}/2, \mathbf{f} \in \mathrm{H}(S)$, so S must have at least $d+1$ holes to have d linearly independent holes. It follows that if $\mathrm{g}(S) = d$ then S is not pseudo-symmetric. □

Theorem 4.4.5. *Let $S \subseteq \mathbb{N}^d$ be a pseudo-symmetric generalized numerical semigroup. Then S satisfies generalized Wilf's congecture 4.1.8.*

Proof. Let $g = \mathrm{g}(S)$. We know that S has Frobenius element $\mathbf{f} = (f^{(1)}, \dots, f^{(d)})$ and by Theorem 3.3.6 (2) we have $2g - 1 = (f^{(1)}+1)\cdots(f^{(d)}+1) = \mathrm{c}(S)$. So it suffices to prove that $\mathrm{e}(S)\,\mathrm{n}(S) \geq d(2g-1)$. If S is pseudo-symmetric then, by the map $\Psi_{\mathbf{f}}$, $g - 1 = |\,\mathrm{LH}(\mathbf{f})| - 1 = |\,\mathrm{N}(\mathbf{f})| = \mathrm{n}(S)$. Furthermore $\mathrm{e}(S) \geq 2d+1$ by Lemma 4.4.1. So $\mathrm{e}(S)\,\mathrm{n}(S) \geq (2d+1)(g-1) = d(2g-1) + g - (d+1)$, in particular if $g \geq d+1$ we conclude. Now consider that $S \in S_{g,d}^{(r)}$ with $r \leq d$. If $r = d$ by Lemma 4.4.4 we have that S is not peudo-symmetric. If $r < d$ then we can consider $\overline{S} \in S_{g,r}^{(r)}$ and by Lemma 4.4.3 it is pseudo-symmetric. Moreover $\mathrm{g}(\overline{S}) \geq r$ so by a similar argument we have that $\overline{S}$ satisfies the conjecture. By Theorem 4.3.9 the same holds for S. □

Put together the previous theorem with Proposition 4.1.3 we can state the following general result:

Theorem 4.4.6. *Let $S \subseteq \mathbb{N}^d$ be an irreducible generalized numerical semigroup. Then S satisfies generalized Wilf's conjecture 4.1.8.*

Generalized Wilf's conjecture for generalized numerical semigroups is introduced and studied in the work in progress [11].

Chapter 5

Some classes of generalized numerical semigroups

Once the definitions for generalized numerical semigroups are given and new properties are provided, it could be interesting to introduce new classes of semigroups for which it is possible to verify and test the arguments so far discussed. In this chapter we introduce three classes of generalized numerical semigroups, each one of these classes contains an infinite number of semigroups. We focus our attention on the study of the minimal systems of generators and on the generalized Wilf's conjecture.

5.1 Ordinary generalized numerical semigroups

Definition 5.1.1. Let $\mathbf{t} \in \mathbb{N}^d$. We define the set $\pi(\mathbf{t}) = \{\mathbf{n} \in \mathbb{N}^d \mid \mathbf{n} \leq \mathbf{t}\}$ where $\leq$ is the natural partial order defined in $\mathbb{N}^d$.

Proposition 5.1.2. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and let $i \in \{1, 2, \dots, d\}$. Then the set $S_i = \{s \in \mathbb{N} \mid s\boldsymbol{e}_i \in S\}$ is a numerical semigroup.*

Proof. S_i is trivially a monoid, moreover it has finite complement in $\mathbb{N}$ otherwise there are infinitely many elements not in S. $\square$

Definition 5.1.3. Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup. The numerical semigroup S_i defined in Proposition 5.1.4 is called the *i-axis semigroup* of S. We denote with c_i the conductor of S_i. We define also the element $w(S) = \sum_{i=1}^{d} c_i\mathbf{e}_i$, and call it the *weak conductor* of S.

Lemma 5.1.4. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup. Then for every $\boldsymbol{s} \in \mathbb{N}^d$ such that $w(S) \leq \boldsymbol{s}$, with respect to the natural partial order in $\mathbb{N}^d$, it is $\boldsymbol{s} \in S$ and $\boldsymbol{s}$ is not a minimal generator of S.*

Proof. Let $\mathbf{s} = \sum_{i=1}^{d} s^{(i)}\mathbf{e}_i$ with $w(S) \leq \mathbf{s}$. Then $s^{(i)} \geq c_i$ for every $i \in \{1, 2, \ldots, d\}$, in particular $s^{(i)} \in S_i$ so $s^{(i)}\mathbf{e}_i \in S$ for every i. Eventually $\mathbf{s}$ is not a minimal generator of S since it is a sum of elements in S. $\square$

Proposition 5.1.5. *Let $\boldsymbol{f} \in \mathbb{N}^d$, then $\left(\mathbb{N}^d \setminus \pi(\boldsymbol{f})\right) \cup \{\boldsymbol{0}\}$ is a generalized numerical semigroup.*

Proof. Let $S = \left(\mathbb{N}^d \setminus \pi(\mathbf{f})\right) \cup \{\mathbf{0}\}$. We want to prove that S is a monoid. Trivially $\mathbf{0} \in S$, so let $\mathbf{x}, \mathbf{y} \in S$ be nonzero elements. If $\mathbf{x} + \mathbf{y} \notin S$ then $\mathbf{x} + \mathbf{y} \in \pi(\mathbf{f})$, in particular we have, with respect to the partial order in $\mathbb{N}^d$, $\mathbf{x} \leq \mathbf{x}+\mathbf{y} \leq \mathbf{f}$ and $\mathbf{y} \leq \mathbf{x}+\mathbf{y} \leq \mathbf{f}$, that is $\mathbf{x}, \mathbf{y} \in \pi(\mathbf{f})$, but this is a contradiction. Therefore $\mathbf{x} + \mathbf{y} \in S$, that is S is a monoid. We have $\mathbb{N}^d \setminus S = \pi(\mathbf{f})$ and it is a finite set. $\square$

Definition 5.1.6. Let $\mathbf{f} \in \mathbb{N}^d$, then we define $S = \left(\mathbb{N}^d \setminus \pi(\mathbf{f})\right) \cup \{\mathbf{0}\}$ an *ordinary generalized numerical semigroup*. The number $\mathrm{c}(S) = |\pi(\mathbf{f})|$ is the *conductor* of S.

Remark 5.1.7. Let $S = \left(\mathbb{N}^d \setminus \pi(\mathbf{f})\right) \cup \{\mathbf{0}\}$ be an ordinary generalized numerical semigroup. Then S is a Frobenius generalized numerical semigroup where the Frobenius element is $\mathbf{f}$. Furthermore for every $i \in \{1, 2, \ldots, d\}$ the i-axis semigroup is an ordinary numerical semigroup with conductor $f^{(i)} + 1$, that is $S_i = \{0, f^{(i)} + 1, \rightarrow\}$, in particular S_i is minimally generated by the set $\{f^{(i)} + 1, f^{(i)} + 2, \ldots, 2f^{(i)} + 1\}$.

Theorem 5.1.8. *Let $S = \left(\mathbb{N}^d \setminus \pi(\boldsymbol{f})\right) \cup \{\boldsymbol{0}\}$ be an ordinary generalized numerical semigroup. For every $i \in \{1, 2, \ldots, d\}$ let*

$$A_i = \left\{ k_i\boldsymbol{e}_i + \sum_{i \neq j} n_j\boldsymbol{e}_j \mid k_i \in \{f^{(i)} + 1, f^{(i)} + 2, \ldots, 2f^{(i)} + 1\}, n_j \in \{0, 1, \ldots, f^{(j)}\} \right\}$$

Then $G = \bigcup_{i=1}^{d} A_i$ is the minimal set of generators for S.

Proof. Let $\mathbf{s} = \sum_{i=1}^{d} s^{(i)}\mathbf{e}_i$ be a minimal generator of S. If we suppose that $\mathbf{s} \notin A_i$ for all $i = \{1, 2, \ldots, d\}$ then we have the following possibilities:

- $s^{(k)} \leq f^{(k)}$ for every $k \in \{1, 2, \ldots, d\}$, so $\mathbf{s} \in \pi(\mathbf{f})$, that is $\mathbf{s} \notin S$.

- $s^{(k)} \geq f^{(k)} + 1$ for every $k \in \{1, 2, \ldots, d\}$.
In this case $\mathbf{s} \geq w(S)$ so it is not a minimal generator of S by Lemma 5.1.4.

- There exists $k \in \{1, 2, \ldots, d\}$ such that $s^{(k)} > 2f^{(k)} + 1$.
Let $z \in \mathbb{N} \setminus \{0\}$ such that $s^{(k)} = 2f^{(k)} + 1 + z = (f^{(k)} + 1) + (f^{(k)} + z)$, then $\mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2$ with $s_1 = (f^{(k)} + 1)\mathbf{e}_k + \sum_{i \neq k} s^{(i)}\mathbf{e}_i$ and $\mathbf{s}_2 = (f^{(k)} + z)\mathbf{e}_k$. Moreover $\mathbf{s}_1, \mathbf{s}_2 \notin \pi(\mathbf{f})$, so they belong to S. This fact implies that $\mathbf{s}$ is not a minimal generator of S.

Therefore we have proved that if $\mathbf{s}$ is a minimal generator of S then $\mathbf{s} \in A_i$ for some i. Now we prove that every element in A_i is a minimal generator of S, for all $i \in \{1, 2, \ldots, d\}$. Observe that, for every $\mathbf{x} \in A_i$, it is $\mathbf{x} \notin \pi(\mathbf{f})$, so $A_i \subset S$. Let $\mathbf{s} \in A_i$ and suppose that $\mathbf{s}$ is not a minimal generator of S. Then $\mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2$ with $\mathbf{s}_1, \mathbf{s}_2 \in S$. Arguing on the components of $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}$ one can be aware that $s_1^{(j)} \leq s^{(j)} \leq f^{(j)}$ and $s_2^{(j)} \leq s^{(j)} \leq f^{(j)}$ for all $j \in \{1, 2, \ldots, d\} \setminus \{i\}$. Hence $s_1^{(i)} > f^{(i)}$ and $s_2^{(i)} > f^{(i)}$, otherwise $\mathbf{s}_1, \mathbf{s}_2 \in \pi(\mathbf{f})$ that is they do not belong to S. In particular $s_1^{(i)} \geq f^{(i)} + 1$ and $s_2^{(i)} \geq f^{(i)} + 1$, so $2f^{(i)} + 2 \leq s_1^{(i)} + s_2^{(i)} = s^{(i)} \leq 2f^{(i)} + 1$ that is a contradiction. Therefore $\mathbf{s}$ is a minimal generator of S. □

Remark 5.1.9. Consider the sets A_i of the previous theorem. It is not difficult to check that

$$|A_i| = (f^{(1)} + 1)(f^{(2)} + 1) \cdots (f^{(d)} + 1) = |\pi(\mathbf{f})|,$$

for all $i \in \{1, 2, \ldots, d\}$. Visually, the minimal generators of S breaks into d copies of $\pi(\mathbf{f})$, in particular $A_i = (f^{(i)} + 1)\mathbf{e}_i + \pi(\mathbf{f})$, see the next example.

Example 5.1.10. Let $\mathbf{f} = (2, 3) \in \mathbb{N}^2$. Then $\pi((2, 3)) = \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 1), (2, 2), (2, 3)\}$. The set of minimal generators of $S = (\mathbb{N}^2 \setminus \pi(\mathbf{f})) \cup \{(0, 0)\}$ is $G = A_1 \cup A_2$ with:

- $A_1 = \{(3, 0), (4, 0), (5, 0), (3, 1), (4, 1), (5, 1), (3, 2), (4, 2), (5, 2), (3, 3), (4, 3), (5, 3)\}$

- $A_2 = \{(0, 4), (0, 5), (0, 6), (0, 7), (1, 4), (1, 5), (1, 6), (1, 7), (2, 4), (2, 5), (2, 6), (2, 7)\}$

In figure 5.1 the holes of S are marked in black (they are the elements of $\pi((2, 3))$ except for $(0, 0)$), the minimal generators of S are marked in red. The minimal generators correspond to the set $(3, 0) + \pi((2, 3))$ and $(0, 4) + \pi((2, 3))$. The elements in the shaded part belong to S by Lemma 5.1.4.

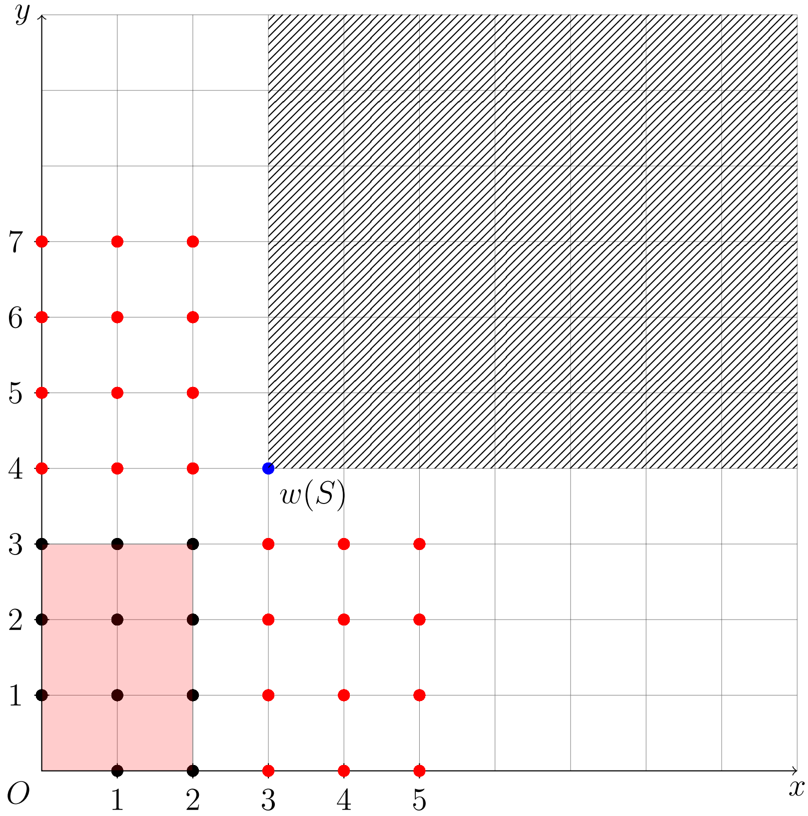


Figure 5.1:

Example 5.1.11. Let $\mathbf{f} = (2, 1, 1) \in \mathbb{N}^3$.
So $\pi(\mathbf{f}) = \{(0, 0, 0), (0, 0, 1), (1, 0, 0), (1, 0, 1), (2, 0, 0), (2, 0, 1), (0, 1, 0), (0, 1, 1), (1, 1, 0), (1, 1, 1), (2, 1, 0), (2, 1, 1)\}$.
The set of minimal generators of $S = (\mathbb{N}^3 \setminus \pi(\mathbf{f})) \cup \{(0, 0)\}$ is $G = A_1 \cup A_2 \cup A_3$ with:

- $A_1 = \{(3, 0, 0), (3, 1, 0), (3, 0, 1), (3, 1, 1), (4, 0, 0), (4, 1, 0), (4, 0, 1), (4, 1, 1), (5, 0, 0), (5, 1, 0), (5, 0, 1), (5, 1, 1)\}$
- $A_2 = \{(0, 2, 0), (1, 2, 0), (2, 2, 0), (0, 2, 1), (1, 2, 1), (2, 2, 1), (0, 3, 0), (1, 3, 0), (2, 3, 0), (0, 3, 1), (1, 3, 1), (2, 3, 1)\}$
- $A_3 = \{(0, 0, 2), (1, 0, 2), (2, 0, 2), (0, 1, 2), (1, 1, 2), (2, 1, 2), (0, 0, 3), (1, 0, 3), (2, 0, 3), (0, 1, 3), (1, 1, 3), (2, 1, 3)\}$

We prove that the generalized Wilf's conjecture is satisfied for ordinary generalized numerical semigroups.

Proposition 5.1.12. *Let* $S = \big(\mathbb{N}^d \setminus \pi(\boldsymbol{f})\big) \cup \{\boldsymbol{0}\}$ *be an ordinary generalized numerical semigroup. Then* S *satisfies the generalized Wilf's conjecture. In particular the equality* $\mathrm{e}(S)\ \mathrm{n}(S) = d\ \mathrm{c}(S)$ *is true.*

Proof. If S is an ordinary generalized numerical semigroup with Frobenius element $\mathbf{f}$ then

$$\mathrm{c}(S) = |\pi(\mathbf{f})| = (f^{(1)}+1)(f^{(2)}+1)\cdots(f^{(d)}+1)$$

and

$$\mathrm{n}(S) = |N(\mathbf{f})| = |\{\mathbf{0}\}| = 1.$$

By Theorem 5.1.8 $\mathrm{e}(S) = \sum_{i=1}^{d} |A_i|$, since the sets A_i are disjoint. By Remark 5.1.9

$$|A_i| = (f^{(1)}+1)(f^{(2)}+1)\cdots(f^{(d)}+1) = |\pi(\mathbf{f})|,$$

for all $i \in \{1, 2, \ldots, d\}$. Hence $\mathrm{e}(S)\ \mathrm{n}(S) = d\ \mathrm{c}(S)$. □

Remark 5.1.13. Let $S = \{0, f, \rightarrow\} \subseteq \mathbb{N}$ be an ordinary numerical semigroup. In particular it is well known that $\mathrm{e}(S)\ \mathrm{n}(S) = \mathrm{c}(S) = f$. The semigroup S can be seen as $S = \big(\mathbb{N}^d \setminus \pi(f)\big) \cup \{0\}$ with $d = 1$ and we obtain $\mathrm{e}(S)\ \mathrm{n}(S) = \mathrm{c}(S)$ also by Proposition 5.1.12. Therefore the ordinary generalized numerical semigroups just introduced are a nice generalization of ordinary numerical semigroups. Furthermore, ordinary numerical semigroups are one of known classes of numerical semigroups for which Wilf's conjecture is satisfied as an equality (see Example 1.4.2). Our generalizations of ordinary generalized numerical semigroups and generalized Wilf's conjecture preserve the equality for the conjecture.

5.2 Symplectic generalized numerical semigroups

This new class of generalized numerical semigroups has appeared in [34], but it is not analyzed as a class of monoids there. It can be seen as a different generalization of ordinary numerical semigroups.

Definition 5.2.1. Let $g \in \mathbb{N}$ and $S = \{(\alpha^{(1)}, \ldots, \alpha^{(d)}) \in \mathbb{N}^d \mid \alpha^{(1)}+\ldots+\alpha^{(d)} > g\} \cup \{\mathbf{0}\}$. It is simple to verify that S is a generalized numerical semigroup. We call it a g-symplectic generalized numerical semigroup.

Remark 5.2.2. If $S \subseteq \mathbb{N}^d$ is a g-symplectic generalized numerical semigroup then

$$\mathrm{H}(S) := \mathbb{N}^d \setminus S = \{(\alpha^{(1)}, \ldots, \alpha^{(d)}) \in \mathbb{N}^d \mid \alpha^{(1)} + \ldots + \alpha^{(d)} \leq g\} \setminus \{\mathbf{0}\}.$$

If $G_i = \{(\alpha^{(1)}, \ldots, \alpha^{(d)}) \in \mathbb{N}^d \mid \alpha^{(1)} + \cdots + \alpha^{(d)} = i\}$, we can verify that $|G_i| = \binom{i+d-1}{d-1}$, in particular $\mathrm{H}(S) = \bigcup_{k=1}^{g} G_i$. Therefore

$$|\mathrm{H}(S)| = \sum_{k=1}^{g} \binom{k+d-1}{d-1} = \binom{g+d}{d} - 1$$

is the genus of S (the last equality is easily proved by induction).
Moreover $\mathrm{H}(S)$ is the set of lattice points in $\mathbb{N}^d$ in the simplex of vertices $\{g\mathbf{e}_1, \ldots, g\mathbf{e}_d, \mathbf{0}\}$. This feature has motivated the name given to this class.

We want to investigate the minimal set of generators of a g-symplectic generalized numerical semigroup. To characterize it completely we begin by proving the following general result.

Lemma 5.2.3. *Let $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d$ be elements in $\mathbb{N}$ such that $\sum_{i=1}^{d} \alpha_i > g$ with g nonzero integer. Then there exist $\beta_1, \beta_2, \ldots, \beta_d$ such that $\alpha_i - \beta_i \in \mathbb{N}$ and $\sum_{i=1}^{d} \beta_i = g$.*

Proof. If there exists $j \in \{1, 2, \ldots, d\}$ such that $\alpha_j \geq g$ then we can consider $\beta_j = g$ and $\beta_i = 0$ for every $i \in \{1, 2, \ldots, d\} \setminus \{j\}$. If $\alpha_i < g$ for every $i = 1, 2, \ldots, d$ then let $r^{(1)} \in \{1, \ldots, d\}$ such that $\alpha_i = 0$ for every $i = r^{(1)} + 1, r^{(1)} + 2, \ldots, d$. If $r^{(1)} \geq g$ then we fix $\beta_i = 1$ for $i = 1, \ldots, g$ and $\beta_j = 0$ for $j = g + 1 \ldots, d$. So $\alpha_i - \beta_i \geq 0$ for all $i \in \{1, \ldots, d\}$ and $\sum_{i=1}^{d} \beta_i = g$. If $r^{(1)} < g$ we consider the following steps:
<u>First step</u>. Put $\gamma_i^{(1)} = 1$ for $i \in \{1, \ldots, r^{(1)}$ and $\gamma_j^{(1)} = 0$ for $j \in \{r^{(1)}+1, \ldots, d\}$. Let $\alpha_i^{(1)} = \alpha_i - \gamma_i^{(1)} \geq 0$ for $i = 1, \ldots, d$ and $\triangle^{(1)} = g - r^{(1)}$. Observe that

$$\sum_{i=1}^{d} \alpha_i^{(1)} = \sum_{i=1}^{d} \alpha_i - \sum_{i=1}^{d} \gamma_i^{(1)} > g - r^{(1)} > 0,$$

in particular there exists $j \in \{1, \ldots, d\}$ such that $\alpha_j^{(1)} \neq 0$.
<u>Second step</u>. Let $r^{(2)} \in \{1, \ldots, r^{(1)}\}$ such that $\alpha_i^{(1)} = 0$ for $i \in \{r^{(2)}+1, \ldots, d\}$.

If $r^{(2)} \geq \triangle^{(1)}$ then we fix $\gamma_i^{(2)} = 1$ for $i = 1, \ldots, \triangle^{(1)}$ and $\gamma_j^{(2)} = 0$ for $j = \triangle^{(1)} + 1, \ldots, d$. We consider $\beta_i = \gamma_i^{(1)} + \gamma_i^{(2)}$ for $i = 1, \ldots, d$ and we have $\alpha_i - \beta_i = \alpha_i^{(1)} - \gamma_i^{(2)} \geq 0$ for every $i \in \{1, \ldots, d\}$ and $\sum_i^d \beta_i = r^{(1)} + \triangle^{(1)} = g$. If $r^{(2)} < \triangle^{(1)}$ we define $\gamma_i^{(2)} = 1$ for $i = 1, \ldots, r^{(2)}$, $\gamma_j^{(2)} = 0$ for $j = r^{(2)}+1, \ldots, d$ and $\alpha_i^{(2)} = \alpha_i^{(1)} - \gamma_i^{(2)} \geq 0$ for $i = 1, \ldots, d$. We define also $\triangle^{(2)} = \triangle^{(1)} - r^{(2)} = g - r^{(1)} - r^{(2)} > 0$. Observe that $\sum_i^d \alpha_i^{(2)} > g - r^{(1)} - r^{(2)} > 0$ so there exists $j \in \{1, \ldots, d\}$ such that $\alpha_j^{(2)} \neq 0$. Therefore we can repeat the procedure from the beginning of the second step, defining the greatest index $r^{(3)} \in \{1, \ldots, r^{(2)}\}$ such that $\alpha_i^{(2)} = 0$ for $i \in \{r^{(3)} + 1, \ldots, d\}$ and considering the two cases $r^{(3)} \geq \triangle^{(2)}$ (and in this case we conclude) or $r^{(3)} < \triangle^{(2)}$, and so on. After a finite number of steps h, it occurs that $r^{(h)} \geq \triangle^{(h-1)}$ (because it is impossible to obtain $g - r^{(1)} - \cdots - r^{(h)} > 0$ for infinitely many steps) since $r^{(j)} > 0$ for every j. For such h, we obtain $r^{(h)} \geq \triangle^{(h-1)}$, then we define $\beta_i = \sum_{j=1}^h \gamma_i^{(j)}$ for every $i = 1, \ldots, d$ and these elements satisfy the requested condition.

$\square$

We explain the procedure in the proof of the previous lemma with an example.

Example 5.2.4. Let $d = 4$, $g = 10$ and consider $\alpha_1 = 8, \alpha_2 = 7, \alpha_3 = 3, \alpha_4 = 2$. We have $\sum_{i=1}^4 \alpha_i = 20 > g$. Moreover $\alpha_i < g$ for $i = 1, 2, 3, 4$.
We have $r^{(1)} = 4 < g$. So we define $\gamma_i^{(1)} = 1$ for $i = 1, 2, 3, 4$ and consider the following positive integers:

- $\alpha_1^{(1)} = \alpha_1 - \gamma_1^{(1)} = 7$
- $\alpha_2^{(1)} = \alpha_2 - \gamma_2^{(1)} = 6$
- $\alpha_3^{(1)} = \alpha_3 - \gamma_3^{(1)} = 2$
- $\alpha_4^{(1)} = \alpha_4 - \gamma_4^{(1)} = 1$.

We have $\triangle^{(1)} = g - r^{(1)} = 6$ and define $r^{(2)} = 4 < \triangle^{(1)}$. So in the second step we define $\gamma_i^{(2)} = 1$ for $i = 1, 2, 3, 4$ and the following:

- $\alpha_1^{(2)} = \alpha_1^{(1)} - \gamma_1^{(2)} = 6$
- $\alpha_2^{(2)} = \alpha_2^{(1)} - \gamma_2^{(2)} = 5$
- $\alpha_3^{(2)} = \alpha_3^{(1)} - \gamma_3^{(2)} = 1$
- $\alpha_4^{(2)} = \alpha_4^{(1)} - \gamma_4^{(2)} = 0$.

We have $\triangle^{(2)} = \triangle^{(1)} - r^{(2)} = g - r^{(1)} - r^{(2)} = 2$ and define $r^{(3)} = 3 > \triangle^{(2)}$. So the successive step is the last, in which $\gamma_1^{(3)} = 1$, $\gamma_2^{(3)} = 1$, $\gamma_3^{(3)} = 0$, $\gamma_4^{(3)} = 0$. We conclude defining:

- $\beta_1 = \gamma_1^{(1)} + \gamma_1^{(2)} + \gamma_1^{(3)} = 3$
- $\beta_2 = \gamma_2^{(1)} + \gamma_2^{(2)} + \gamma_2^{(3)} = 3$
- $\beta_3 = \gamma_3^{(1)} + \gamma_3^{(2)} + \gamma_3^{(3)} = 2$
- $\beta_4 = \gamma_4^{(1)} + \gamma_4^{(2)} + \gamma_4^{(3)} = 2$.

Let $\mathbf{x} = (x^{(1)}, x^{(2)}, \ldots, x^{(d)}) \in \mathbb{N}^d$. We define $|\mathbf{x}| = \sum_{i=1}^{d} x^{(i)}$. In particular if $S \subseteq \mathbb{N}^d$ is a g-symplectic generalized numerical semigroup then $\mathbf{x} \in S$ if and only if $|\mathbf{x}| > g$.

Theorem 5.2.5. *Let $S \subseteq \mathbb{N}^d$ be the g-symplectic generalized numerical semigroup and let $G_k = \{\boldsymbol{a} \in \mathbb{N}^d \mid |\boldsymbol{a}| = k\}$. Then the set $G = \bigcup_{k=g+1}^{2g+1} G_k$ is the minimal set of generators of S.*

Proof. If $g = 0$ then $S = \mathbb{N}^d$ so it is trivial. We suppose $g > 0$.
Let $\mathbf{a} \in S$. If $|\mathbf{a}| \leq 2g+1$ then $\mathbf{a} \in G$. If $|\mathbf{a}| > 2g+1 > g+1$, by Lemma 5.2.3 there exists $\mathbf{b} \in \mathbb{N}^d$ such that $|\mathbf{b}| = g+1$ and $\mathbf{a} - \mathbf{b} \in \mathbb{N}^d$. In particular $\mathbf{b} \in G$ and $\mathbf{a} = \mathbf{b} + \mathbf{c}$ where $|\mathbf{c}| > 2g+1-(g+1) = g$, so $\mathbf{c} \in S$. If $|\mathbf{c}| \leq 2g+1$ then $\mathbf{c} \in G$, otherwise we apply to $\mathbf{c}$ the previous procedures. Then we find $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k$, such that $|\mathbf{c}_k| \leq 2g+1$ and $\mathbf{a}$ can be expressed as a sum of elements of G. The computation will finish because $|\mathbf{c}_{j+1}| < |\mathbf{c}_j| < |\mathbf{c}| < |\mathbf{a}|$, for every $j = 1, \ldots, k-1$. So every element of S can be expressed as a sum of elements in G.
If $\mathbf{a}, \mathbf{b} \in G$ then $|\mathbf{a} + \mathbf{b}| \geq 2g+1$, that is $\mathbf{a} + \mathbf{b} \notin G$. So the set G of the generators of S is minimal. $\square$

Example 5.2.6. Let $g = 3$ and $d = 2$. The 3-symplectic generalized numerical semigroup in $\mathbb{N}^2$ is

$$\mathbb{N}^2 \setminus \{(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(2,0),(2,1),(3,0)\}.$$

The minimal set of generators is $G(S) = \{(0,4),(0,5),(0,6),(0,7),(4,0),$ $(5,0),(6,0),(7,0),(1,3),(1,4),(1,5),(1,6),(2,2),(2,3),(2,4),(2,5),(3,1),$ $(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(5,1),(5,2),(6,1)\}$.

Figure 5.2 provides a graphical view of the generalized numerical semigroup: black points are the holes of the generalized numerical semigroup, while the red points are the minimal generators.

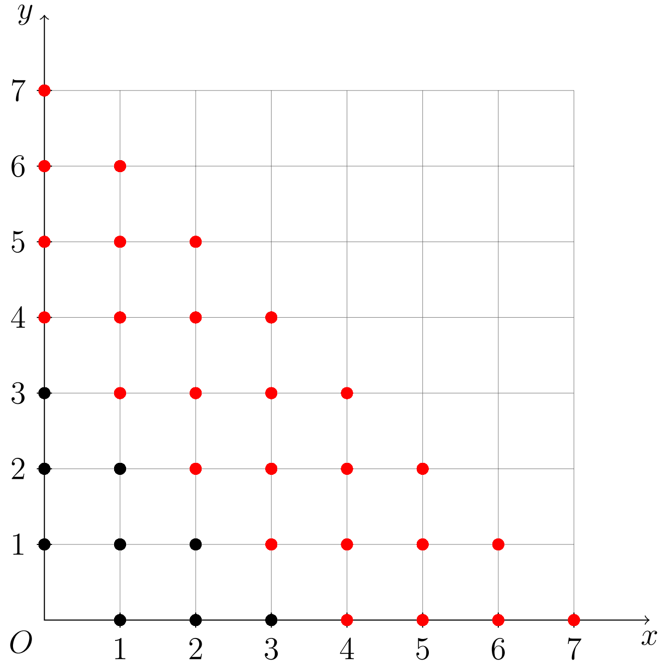


Figure 5.2:

Corollary 5.2.7. *Let $S \subseteq \mathbb{N}^d$ be a g-symplectic generalized numerical semigroup. Then*

$$\mathrm{e}(S) = \sum_{k=g+1}^{2g+1} \binom{k+d-1}{d-1} = \binom{2g+1+d}{d} - \binom{g+d}{d}.$$

Proof. It easily follows from Theorem 5.2.5. $\square$

Example 5.2.8. Let $g = 3$ and $d = 3$. Let S be the 3-symplectic generalized numerical semigroup in $\mathbb{N}^3$, then:

- $|\,\mathrm{H}(S)| = \binom{3}{2} + \binom{4}{2} + \binom{5}{2} = \binom{6}{3} - 1 = 19.$
- $\mathrm{e}(S) = \binom{6}{2} + \binom{7}{2} + \binom{8}{2} + \binom{9}{2} = \binom{10}{3} - \binom{6}{3} = 100.$

Observe that $\mathrm{e}(S)$ becomes quite large with increasing d or g.

Proposition 5.2.9. *Let $S \subseteq \mathbb{N}^d$ be a g-symplectic generalized numerical semigroup. Then S satisfies the generalized Wilf's conjecture 4.1.8. Furtermore, if $d > 1$ the equality does not hold.*

Proof. Consider that $\mathrm{n}(S) = 1$. In fact if $\mathbf{h} \in \mathbb{N}^d \setminus S$ then $N(\mathbf{h}) = \{\mathbf{0}\}$, because the sum of coordinates of a nonzero element in $N(\mathbf{h})$ is smaller then the sum of coordinates of $\mathbf{h}$. We have $\mathrm{c}(S) = \mathrm{g}(S)+1 = \binom{g+d}{d}$ and $\mathrm{e}(S) = \binom{2g+1+d}{d} - \binom{g+d}{d}$. Therefore the inequality 4.1.8 is equivalent to $\binom{2g+1+d}{d} \geq (d+1)\binom{g+d}{d}$. Handling the binomial coefficients we see that the previous expression is equivalent to the following inequality:

$$(2g+1+d)(2g+d)\cdots(2g+1-g+d) \geq (d+1)\cdot(2g+1)(2g)\cdots(2g+1-g)$$

It can be seen in the form $p(d) \geq 0$ where $p(d) = d^{g+1} + a_g d^g + \cdots a_2 d^2 + a_1 d$ where $a_i > 0$ for every $i = 2, \ldots, g+1$. In particular, for $d \geq 0$, we have that $p(d)$ is an increasing function of d and furthermore $p(1) = 0$. So it is $p(d) \geq 0$ for every $d > 0$ and equality does not hold for $d > 1$.

□

From the previous proposition it follows that a g-symplectic generalized numerical semigroup satisfies also the extended Wilf's conjecture 4.2.1. Furhermore we remark that, in a g-symplectic generalized numerical semigroup, all the elements $\mathbf{h} \in \mathbb{N}^d$ such that $|\mathbf{h}| = g$ are maximal in the set of holes with respect to the natural partial order, so a g-symplectic generalized numerical semigroup is not a Frobenius generalized numerical semigroup. Moreover the set of holes $\mathrm{H}(S)$ generates a coordinate linear space of dimension d.

Remark 5.2.10. Observe that ordinary and symplectic generalized numerical semigroups have the property $\mathrm{n}(S) = 1$. We let for future research the aim of study the behaviour of all generalized numerical semigroups S with $\mathrm{n}(S) = 1$.

5.3 Stripe generalized numerical semigroups

In order to provide another class of generalized numerical semigroups, we give in this section a straightforward generalization of a g-symplectic generalized numerical semigroup, that we call *stripe* generalized numerical semigroup by considering its graphical profile. Looking at the next definition we see that a stripe generalized numerical semigroup is associated to a numerical semigroup T and such a numerical semigroup shares some information with the associated generalized numerical semigroup, as we are going to show in the first properties. At first glance the features of this class are not so trivial like the previous two ones, in particular for the generalized Wilf's conjecture. So, for what concerns this thesis, we provide only some peculiarity of this class.

Definition 5.3.1. Let T be a numerical semigroup. We define $S = \{\mathbf{a} \in \mathbb{N}^d \mid |\mathbf{a}| \in T\}$. It is easy to prove that S is a generalized numerical semigroup in $\mathbb{N}^d$. We call S a T-stripe generalized numerical semigroup.

Remark 5.3.2. If $S \subseteq \mathbb{N}^d$ is a T-stripe generalized numerical semigroup then

$$\mathrm{H}(S) := \mathbb{N}^d \setminus S = \{\mathbf{a} \in \mathbb{N}^d \mid |\mathbf{a}| \notin T\}.$$

Moreover if $\mathbf{x} \in \mathbb{N}^d$ then $\mathbf{x} \in S$ if and only if $|\mathbf{x}| \in T$.

Theorem 5.3.3. *Let T be a numerical semigroup and $S \subseteq \mathbb{N}^d$ be a T-stripe generalized numerical semigroup. Consider the minimal set of generators of T, $G(T) = \{n_1, \dots, n_r\}$, and for each $n_i \in G(T)$, the set $G_{n_i} = \{\boldsymbol{a} \in \mathbb{N}^d \mid |\boldsymbol{a}| = n_i\}$. Then $G = \bigcup_{i=1}^{r} G_{n_i}$ is the set of minimal generators of S.*

Proof. Let $\mathbf{a} \in S$. In particular $|\mathbf{a}| = \sum_{j=1}^{r} \lambda_j n_j$, with $\lambda_j \in \mathbb{N}$ for every $j = 1, \dots, r$. If $|\mathbf{a}| = n_k$ for some $k \in \{1, \dots, r\}$ there is nothing to prove. Otherwise there exists $k \in \{1, \dots, r\}$ such that $|\mathbf{a}| > n_k$ and $\lambda_k > 0$. In such a case, by Lemma 5.2.3, there exists $\mathbf{b} \in \mathbb{N}^d$ such that $|\mathbf{b}| = n_k$ and $\mathbf{a} - \mathbf{b} \in \mathbb{N}^d$. So there exists $\mathbf{c} \in \mathbb{N}^d$ such that $\mathbf{a} = \mathbf{b} + \mathbf{c}$, moreover $|\mathbf{c}| = \sum_{j \neq k} \lambda_j n_j + (\lambda_k - 1) n_k$, in particular $\mathbf{c} \in S$. Now, if $|\mathbf{c}| = n_h$ for some $h \in \{1, \dots, r\}$ then $\mathbf{c} \in G$, otherwise there exists $h \in \{1, \dots, r\}$ such that $|\mathbf{c}| > n_h$ and we can repeat for $\mathbf{c}$ the procedure above. The computation will finish because $|\mathbf{c}| < |\mathbf{a}|$. So every element of S can be expressed as sum of elements in G.
Finally, if $\mathbf{a}, \mathbf{b} \in G$ then the sum of coordinates of $\mathbf{a} + \mathbf{b}$ is sum of at least two elements in T, hence $|\mathbf{a} + \mathbf{b}|$ is not a minimal generator of T. So the set of generators G is minimal for S.

$\square$

Example 5.3.4. Let $T = \langle 4, 6, 7 \rangle = \mathbb{N} \setminus \{1, 2, 3, 5, 9\}$ and let S be the T-stripe generalized numerical semigroup in $\mathbb{N}^2$. Then S is generated by the set $G = \{(4,0), (3,1), (2,2), (1,3), (0,4), (6,0), (5,1), (4,2), (3,3), (2,4), (1,5), (0,6), (7,0), (6,1), (5,2), (4,3), (3,4), (2,5), (1,6), (0,7)\}$, and $\mathrm{H}(S) = \{(1,0), (0,1), (2,0), (1,1), (0,2), (3,0), (2,1), (1,2), (0,3), (5,0), (4,1), (3,2), (2,3), (1,4), (0,5), (9,0), (8,1), (7,2), (6,3), (5,4), (4,5), (3,6), (2,7), (1,8), (0,9)\}$.

Figure 5.3 provides a graphical view of S: black points are the holes of S, while the red points are the minimal generators. The other points are all elements in S.

For pseudo-Frobenius elements and special gaps we note a similar behavior.

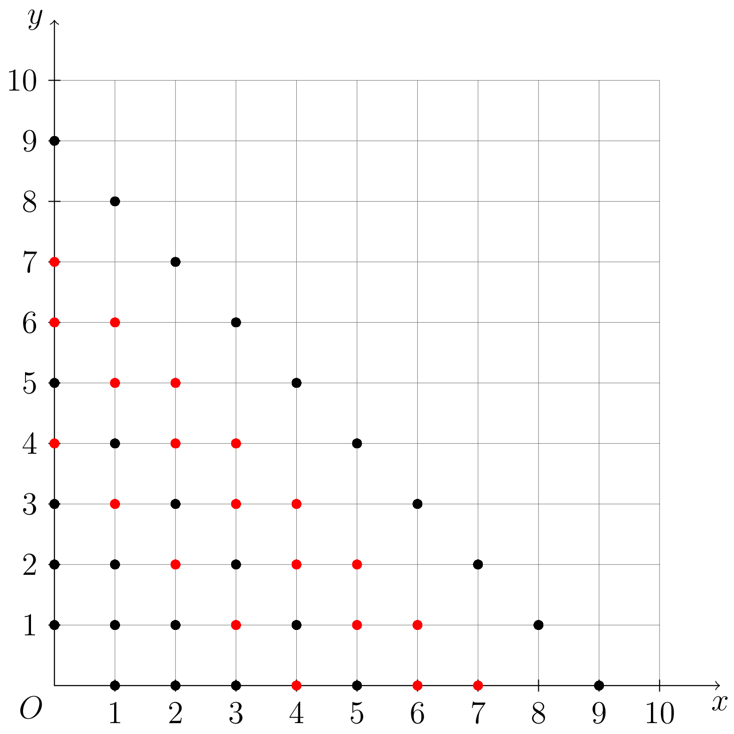


Figure 5.3:

Proposition 5.3.5. *Let T be a numerical semigroup and $S \subseteq \mathbb{N}^d$ be a T-stripe generalized numerical semigroup. For every $i \in \mathbb{N}$ consider $G_i = \{\boldsymbol{a} \in \mathbb{N}^d \mid |\boldsymbol{a}| = i\}$. Then:*

1. *$PF(S) = \bigcup_{i \in PF(T)} G_i$.*

2. *$SG(S) = \bigcup_{i \in SG(T)} G_i$.*

Proof. 1) Let $\mathbf{x} \in G_i$ with $i \in PF(T)$, in particular $\mathbf{x} \in \mathrm{H}(S)$. Let $\mathbf{s} \in S$, then $|\mathbf{x}+\mathbf{s}| = i + |\mathbf{s}| \in T$ since $|\mathbf{s}| \in T$, so $\mathbf{x}+\mathbf{s} \in S$. Otherwise, let $\mathbf{x} \in PF(S)$ and $i = |\mathbf{x}|$. In particular $\mathbf{x} \in G_i$ and $i \in \mathrm{H}(T)$. We prove that $i \in PF(T)$. Let $t \in T \setminus \{0\}$, then $t\mathbf{e}_j \in S$ and $\mathbf{x} + t\mathbf{e}_j \in S$ for any $j \in \{1, \ldots, d\}$. This means that $|\mathbf{x} + t\mathbf{e}_j| \in T$, that is $i + t \in T$.
2) From 1) we know that $\mathbf{x} \in PF(S)$ if and only if $|\mathbf{x}| \in PF(T)$. So $\mathbf{x} \in SG(S) \Leftrightarrow 2\mathbf{x} \in S \Leftrightarrow |2\mathbf{x}| \in T \Leftrightarrow 2|\mathbf{x}| \in T \Leftrightarrow |\mathbf{x}| \in SG(T)$.

$\square$

Observe that if T is the ordinary numerical semigroup $\{0, g+1, \to\}$ then S is the g-symplectic generalized numerical semigroup. The simplest examples of stripe generalized numerical semigroups are those associated to numerical semigroups generated by two elements. For them we can compute the embedding dimension and the type.

Corollary 5.3.6. *Let $T = \langle m, n\rangle$ be a numerical semigroup of embedding dimension 2 and $S \subseteq \mathbb{N}^d$ be the T-stripe generalized numerical semigroup. Then:*

a) $\mathrm{e}(S) = \binom{m+d-1}{d-1} + \binom{n+d-1}{d-1}$.

b) $t(S) = \binom{mn-m-n+d-1}{d-1}$

Proof. From Remark 5.2.2 and Theorem 5.3.3 the first statement easily follows. Since $PF(T) = \{F(T)\}$ and $F(T) = mn - m - n$ the second statement follows from Proposition 5.3.5. $\square$

If $T = \langle m, n\rangle$ and $S \subseteq \mathbb{N}^2$ is the T stripe generalized numerical semigroup in $\mathbb{N}^2$ then the inequality $\mathrm{e}(S) \geq 2\,(t(S)+1)$ is equivalent to $2mn \leq 3(m+n)-2$ and it is true only in the case $m = 2, n = 3$. So in $\mathbb{N}^2$ it is not possible to use Corollary 4.1.12 in order to verify Generalized Wilf's conjecture. In general it does not seem so easy to study the generalized Wilf's conjecture for T-stripe generalized numerical semigroups, also in the case T is generated by two elements. So this could be the subject for further researches.

Chapter 6

The Apéry set

The Apéry set is an important tool for numerical semigroups and it is involved in many properties of them. It is not quite easy to use Apéry sets in more general contexts such as for submoid of $\mathbb{N}^d$, even if they can provide some useful characterizations also in that case (see for instance [38] and [25]). The generalization of properties of the Apéry set from numerical semigroups is not straightforward in some cases, anyway we have tried to study the behaviour of such a set also for generalized numerical semigroups. We first introduce the Apéry set in finitely generated submonoids of $\mathbb{N}^d$, called also *affine semigroups*, considering some general properties. Succesively we focus on generalized numerical semigroups, in particular in symmetric and pseudo-symmetric ones.

6.1 The Apéry set in submonoids of $\mathbb{N}^d$

Definition 6.1.1. Let $S \subseteq \mathbb{N}^d$ be a monoid and $\mathbf{n} \in S$. The *Apéry set* of S with respect to $\mathbf{n}$ is the set:

$$\mathrm{Ap}(S, \mathbf{n}) = \{\mathbf{s} \in S | \ \mathbf{s} - \mathbf{n} \notin S\},$$

where $\mathbf{s} - \mathbf{n}$ is the difference in $\mathbb{Z}^d$.

Example 6.1.2. Let $S = \mathbb{N}^2 \setminus \{(1,0),(1,1)\} = \langle (0,1),(1,2),(2,0),(3,0) \rangle$. We can compute:
$\mathrm{Ap}((1,3)) = \{(0,0),(0,1),(0,2),(0,3),(0,m),(1,2),(2,0),(2,1),(2,2), (2,3), (2,4),(m,0),(m,1),(m,2) | \ m \geq 3\}$.
$\mathrm{Ap}((0,9)) = \{(0,0),(0,1),(0,2),\dots,(0,8),(1,2),(1,3),(1,4),\dots,(1,10),(2,0), (2,1),(2,2),\dots,(2,8),(m,0),(m,1),\dots,(m,8) | \ m \geq 3\}$,
$\mathrm{Ap}((0,2)) = \{(0,0),(0,1),(1,2),(1,3),(m,0),(m,1) | \ m \geq 2\}$.

$\mathrm{Ap}((0,1)) = \{(0,0),(1,2),(n,0) \mid n \geq 2\}$,

Proposition 6.1.3. *Let $S \subseteq \mathbb{N}^d$ be a monoid and $\boldsymbol{n} \in S \backslash \{\boldsymbol{0}\}$. Then $\mathrm{Ap}(S,\boldsymbol{n}) \cup \{\boldsymbol{n}\}$ is a set of generators for S.*

Proof. Let $\mathbf{s} \in S$ and $\overline{k} = max\{k \in \mathbb{N} \mid \mathbf{s} - k\mathbf{n} \in S\}$. If $\overline{k} = 0$ then $\mathbf{s} - \mathbf{n} \notin S$, that is $\mathbf{s} \in \mathrm{Ap}(S,\mathbf{n})$. If $\overline{k} \neq 0$ then, put $\mathbf{t} = \mathbf{s} - \overline{k}\mathbf{n}$, we have $\mathbf{t} - \mathbf{n} \notin S$ that is $\mathbf{t} \in \mathrm{Ap}(S,\mathbf{n})$ and $\mathbf{s} = \mathbf{t} + \overline{k}\mathbf{n}$. □

Observe that even if $\mathrm{Ap}(S,\mathbf{n}) \cup \{\mathbf{n}\}$ is a system of generators, in general it is not the minimal system of generators as we have shown in Example 6.1.2. Morevorer the Apéry set is an infinite set in case of submonoids of $\mathbb{N}^d$ with $d > 1$.

Proposition 6.1.4. *Let $S \subseteq \mathbb{N}^d$ be a monoid and let A be the minimal system of generators of S. If $B \subset A$ then $(A \setminus B) \subseteq \bigcap_{\boldsymbol{n} \in B} \mathrm{Ap}(S,\boldsymbol{n})$.*

Proof. Let $\mathbf{x} \in A \setminus B$ and $\mathbf{n} \in B$. Since $\mathbf{x}$ and $\mathbf{n}$ are both minimal generators of S then $\mathbf{x} - \mathbf{n} \notin S$. □

Corollary 6.1.5. *Let $S \subseteq \mathbb{N}^d$ be a monoid, let A be the minimal system of generators of S and $B \subset A$. Then $\left(\bigcap_{\boldsymbol{n} \in B} \mathrm{Ap}(S,\boldsymbol{n})\right) \cup B$ is a system of generators of S.*

The previuos result may be useful in the case that the monoid $S \subseteq \mathbb{N}^d$ is finitely generated. In fact even if a single Apéry set is an infinite set for $d > 1$, the intersection of some Apéry sets can be a finite set for some classes of monoids, as we are going to show. In particular from the previous corollary we obtain a finite system of generators. We remind that a finitely generated monoid $S \subseteq \mathbb{N}^d$ is called an *affine semigroup*.

Let $A \subseteq \mathbb{N}^d$, the set $\mathfrak{G}(A) = \{\lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2 + \ldots + \lambda_n\mathbf{a}_n \mid n \in \mathbb{N}, \lambda_i \in \mathbb{Z}, \mathbf{a}_i \in A, i \in \{1,2,\ldots,n\}\}$ is the subgroup of $\mathbb{Z}^d$ generated by A. If we consider the affine semigroup $S = \langle A \rangle$ then $\mathfrak{G}(S) = \mathfrak{G}(A)$ and it is called the *quotient group* of S.

Definition 6.1.6. Let $S \subseteq \mathbb{N}^d$ be an affine semigroup and let $\mathfrak{G}(S)$ be the quotient group of S. S is *simplicial* if $S = \langle n_1,\ldots,n_d,n_{d+1},\ldots,n_{d+m}\rangle$ with $\mathrm{rank}(\mathfrak{G}(S)) = d$ and $L_{\mathbb{Q}^+}(S) = L_{\mathbb{Q}^+}(\{n_1,\ldots,n_d\})$, where $L_{\mathbb{Q}^+}(B) = \{\sum_{i=1}^n q_i b_i \mid q_i \in \mathbb{Q}^+, b_i \in B\}$. The elements $n_1,\ldots,n_d$ are called *extremal rays* of S.

By the next results (that are inspired by [38]) we show that in simplicial affine semigroups the Apéry sets with respect to extremal rays have a good behaviour.

Definition 6.1.7. Let $A \subseteq \mathbb{N}^d$, $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$. We define the following relation: $\mathbf{x} \equiv_d \mathbf{y} \mod A$ if and only if $\mathbf{x} - \mathbf{y} \in \mathfrak{G}(A)$.
It is easy to prove that $\equiv_d$ is an equivalence relation in $\mathbb{Z}^d$. Moreover $\mathbf{x} \equiv_d \mathbf{0} \mod A$ if and only if $\mathbf{x}$ is spanned by a finite subset of A.

Remark 6.1.8. Let $\mathbf{x} = \sum_i^d x^{(i)}\mathbf{e}_i \in \mathbb{Z}^d$, $\mathbf{m}_i = m_i\mathbf{e}_i$, $m_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, d$. Consider $q_i, r_i \in \mathbb{Z}$, $0 \leq r_i < |m_i|$ such that $x^{(i)} = q_i m_i + r_i$ for all $i = 1, 2, \ldots, d$. Then $\mathbf{r} = \sum_i^d r_i\mathbf{e}_i$ has the property that $\mathbf{x} \equiv_d \mathbf{r} \mod (\{\mathbf{m}_1, \ldots, \mathbf{m}_d\})$. In particular $[\mathbf{x}]_{\equiv_d} = [\mathbf{r}]_{\equiv_d}$ and we can choose $\mathbf{r}$ as a representative of $[\mathbf{x}]_{\equiv_d}$.
Moreover, let $\mathbf{x}, \mathbf{r} \in \mathbb{Z}^d$ with $\mathbf{x} = \sum_i^d x^{(i)}\mathbf{e}_i$, $\mathbf{r} = \sum_i^d r^{(i)}\mathbf{e}_i$, then $\mathbf{x} \equiv_d \mathbf{r} \mod (\{\mathbf{m}_1, \ldots, \mathbf{m}_d\})$ if and only if $x^{(i)} \equiv r^{(i)} \mod m_i$ (the usual congruence in $\mathbb{Z}$).
So, if $\mathbf{m}_i = m_i\mathbf{e}_i$, $m_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, d$ and consider the equivalence relation $\equiv_d$ modulo the set $\{\mathbf{m}_1, \ldots, \mathbf{m}_d\}$ then $\mathbb{Z}^d / \equiv_d \cong \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \mathbb{Z}_{m_d}$, where $\mathbb{Z}_{m_i}$ is the set of remainders of the division by m_i, for $i = 1, \ldots, d$.

Proposition 6.1.9. *Let $S \subseteq \mathbb{N}^d$ be a simplicial affine semigroup, and let $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_d$ be extremal rays of S, with $\boldsymbol{m}_i = m_i\boldsymbol{e}_i$, $m_i \in \mathbb{N} \setminus \{0\}$ for $i = 1, \ldots, d$. Then $\boldsymbol{x} \in \bigcap_{i=1}^d \mathrm{Ap}(S, \boldsymbol{m}_i)$ if and only if $\boldsymbol{x}$ is minimal, with respect to the natural partial order in $\mathbb{N}^d$, in the set $\{\boldsymbol{r} \in S \mid \boldsymbol{r} \equiv_d \boldsymbol{x} \mod (\{\boldsymbol{m}_1, \ldots, \boldsymbol{m}_d\})\}$.*

Proof. $\Rightarrow$) Suppose $\mathbf{x} \in \bigcap_{i=1}^d \mathrm{Ap}(S, \mathbf{m}_i)$ and there exists $\mathbf{r} \equiv_d \mathbf{x} \mod (\{\mathbf{m}_1, \ldots, \mathbf{m}_d\})$ such that $\mathbf{r} < \mathbf{x}$. Then $\mathbf{x} = \mathbf{r} + \sum_{i=1}^d k_i\mathbf{m}_i = \mathbf{r} + \sum_{i=1}^d k_i m_i\mathbf{e}_i$, with $k_i \in \mathbb{Z}$ for $i = 1, \ldots, d$. Since $\mathbf{r} < \mathbf{x}$, if $x^{(i)}$ denotes i-th coordinate of $\mathbf{x}$ and $r^{(i)}$ denotes i-th coordinate of $\mathbf{r}$, then $r^{(i)} \leq x^{(i)}$ for all $i \in \{1, 2, \ldots, d\}$ and there exists $j \in \{1, 2, \ldots, d\}$ such that $r^{(j)} < x^{(j)}$. It follows that $k_i \geq 0$ for all $i \in \{1, 2, \ldots, d\}$ and $k_j > 0$. Therefore

$$\mathbf{x} - \mathbf{m}_j = \mathbf{r}_j + \sum_{\substack{i=1 \\ i \neq j}}^d k_i\mathbf{m}_i + (k_j - 1)\mathbf{m}_j \in S$$

that is a contradiction. So $\mathbf{x}$ is minimal.
$\Leftarrow$) Let $\mathbf{x}$ be a minimal element, with respect to the natural partial order in $\mathbb{N}^d$, in the set $\{\mathbf{r} \in S \mid \mathbf{r} \equiv_d \mathbf{x} \mod (\{\mathbf{m}_1, \ldots, \mathbf{m}_d\})\}$. If $\mathbf{x} - \mathbf{m}_i \in S$ for some $i \in \{1, 2, \ldots, d\}$ then $\mathbf{x} - \mathbf{m}_i < \mathbf{x}$ and $\mathbf{x} \equiv_d \mathbf{x} - \mathbf{m}_i \mod (\{\mathbf{m}_1, \ldots, \mathbf{m}_d\})$, contradicting minimality of $\mathbf{x}$. $\square$

The last good result we mention is the following:

Proposition 6.1.10 ([38]). *Let $S \subseteq \mathbb{N}^d$ be a simplicial affine semigroup, and let $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_d$ be extremal rays of S. Then $\bigcap_{i=1}^d \mathrm{Ap}(S, \boldsymbol{m}_i)$ is a finite set.*

6.2 Apéry set in generalized numerical semigroups

Definition 6.2.1. Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup. For all $j \in \{1, 2, \ldots, d\}$ we consider $m_j = \min\{m \in \mathbb{N} \mid m\mathbf{e}_j \in S\}$. We call $\mathbf{m}_1 := m_1\mathbf{e}_1, \ldots, \mathbf{m}_d := m_d\mathbf{e}_d$ the *multiplicities* of S.

Remark 6.2.2. As shown in Chapter 2 every generalized numerical semigroup is finitely generated. Moreover its multiplicities belong to the minimal set of generators, so it is easy to see that a generalized numerical semigroup is a simplicial affine semigroup in which the multiplicities are extremal rays.

Proposition 6.2.3. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup with multiplicities $\boldsymbol{m}_1 = m_1\boldsymbol{e}_1, \ldots, \boldsymbol{m}_d = m_d\boldsymbol{e}_d$. For all $\boldsymbol{r} \in \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_d}$ let*

$$M_{\boldsymbol{r}} = \{\boldsymbol{x} \in \bigcap_{i=1}^{d} \mathrm{Ap}(S, \boldsymbol{m}_i) \mid \boldsymbol{x} \equiv_d \boldsymbol{r} \mod (\{\boldsymbol{m}_1, \ldots, \boldsymbol{m}_d\})\}.$$

If $\boldsymbol{r} \in S$ then $M_{\boldsymbol{r}} = \{\boldsymbol{r}\}$, otherwise $|M_{\boldsymbol{r}}| \geq d$.

Proof. Let $\mathbf{r} \in \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_d}$. Suppose $\mathbf{r} \in S$, if $\mathbf{x} \in S \setminus \{\mathbf{r}\}$ and $\mathbf{x} \equiv_d \mathbf{r} \mod (\{\mathbf{m}_1, \ldots, \mathbf{m}_d\})$ then $\mathbf{x} = \mathbf{r} + \lambda_1\mathbf{m}_1 + \cdots + \lambda_d\mathbf{m}_d$ with $\lambda_i \in \mathbb{N}$ for all $i = 1, \ldots, d$ (since $\mathbf{r} - \mathbf{m}_i \notin \mathbb{N}^d$ for all i) and there exists $j \in \{1, \ldots, d\}$ such that $\lambda_j > 0$, in particular $\mathbf{x} - \mathbf{m}_j \in S$, that is $\mathbf{x} \notin M_{\mathbf{r}}$. Now suppose $\mathbf{r} \notin S$. We define $k_j = \min\{k \in \mathbb{N} \mid \mathbf{r} + k\mathbf{m}_j \in S\}$, for all $j = 1, \ldots, d$. Observe that such k_j exists since $\mathbb{N}^d \setminus S$ is a finite set. Let us show that $\mathbf{r} + k_j\mathbf{m}_j \in \bigcap_{i=1}^d \mathrm{Ap}(S, \mathbf{m}_i)$, for all $j = 1, \ldots, d$. Fix $j \in \{1, \ldots, d\}$, by definition of k_j we have $\mathbf{r} + k_j\mathbf{m}_j - \mathbf{m}_j \notin S$ and moreover $\mathbf{r} + k_j\mathbf{m}_j - \mathbf{m}_i \notin S$ for all $i \neq j$, since the i-th coordinate of $\mathbf{r}$ is less than m_i. $\square$

Corollary 6.2.4. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup with multiplicities $\boldsymbol{m}_1 = m_1\boldsymbol{e}_1, \ldots, \boldsymbol{m}_d = m_d\boldsymbol{e}_d$. Then, for $d \geq 2$ and $S \neq \mathbb{N}^d$, $|\bigcap_{i=1}^d \mathrm{Ap}(S, \boldsymbol{m}_i)| > m_1 m_2 \cdots m_d$.*

Proof. From Proposition 6.2.3 we have:

$$\left|\bigcap_{i=1}^{d} \mathrm{Ap}(S, \mathbf{m}_i)\right| = \sum_{\mathbf{r}\in \mathbb{Z}_{m_1}\oplus\mathbb{Z}_{m_2}\oplus\cdots\oplus\mathbb{Z}_{m_d}} |M_{\mathbf{r}}|$$

If $S \neq \mathbb{N}^d$ and $d \geq 2$ then there exists at least one $\mathbf{r} \in \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_d}$ such that $|M_{\mathbf{r}}| \geq 2$. □

Observe that if S is a numerical semigroup we know that $|M_{\mathbf{r}}| = 1$ for all $\mathbf{r}$. In general if $d \geq 2$ and $S \subseteq \mathbb{N}^d$ is a simplicial affine semigroup in [38] it is shown that $|M_{\mathbf{r}}|$ can be also less than d, in particular simplicial affine semigroups such that $|M_{\mathbf{r}}| \leq 1$, for all $\mathbf{r}$, are Cohen Macaulay. Therefore, by the previous corollary, if S is a generalized numerical semigroup with $S \neq \mathbb{N}^d$ then S is not Cohen Macaulay. Moreover it can occur that $|M_{\mathbf{r}}| > d$.

Example 6.2.5. Let $S \subseteq \mathbb{N}^2$ generated by {(3,0),(0,4),(4,0),(0, 5),(1,4),(5,1)}. We can consider that $(9, 6), (18, 2), (0, 10) \in \mathrm{Ap}(S, (3, 0)) \cap \mathrm{Ap}(S, (0, 4)$ and each one belongs to equivalence class of $(0, 2)$.

Proposition 6.2.6. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and let $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_d$ be its multiplicities. Then $\left(\bigcap_{i=1}^{d} \mathrm{Ap}(S, \boldsymbol{m}_i)\right) \cup \{\boldsymbol{m}_1, \ldots, \boldsymbol{m}_d\}$ is a finite system of generators of S.*

Proof. It easily follows from Remark 6.2.2, Corollary 6.1.5 and Proposition 6.1.10. □

By the previous proposition we have that: if $S \subseteq \mathbb{N}^d$ is a generalized numerical semigroup with multiplicities $\mathbf{m}_1, \ldots, \mathbf{m}_d$ then

$$\mathrm{e}(S) \leq \left| \left(\bigcap_{i=1}^{d} \mathrm{Ap}(S, \mathbf{m}_i)\right) \setminus \{\mathbf{0}\} \cup \{\mathbf{m}_1, \ldots, \mathbf{m}_d\} \right| \tag{6.1}$$

In the next example we show that the inequality can be both strict or sharp. For $d = 1$, the numerical semigroups for which 6.1 is an equality are called of maximal embedding dimension (see [40]).

Definition 6.2.7. Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup with multiplicities $\mathbf{m}_1, \ldots, \mathbf{m}_d$. We call S a generalized numerical semigroup of *maximal embbedding dimension* if $\mathrm{e}(S) = \left|\left(\bigcap_{i=1}^{d} \mathrm{Ap}(S, \mathbf{m}_i)\right) \setminus \{\mathbf{0}\} \cup \{\mathbf{m}_1, \ldots, \mathbf{m}_d\}\right|$. In this case Proposition 6.2.6 provides the minimal set of generators for S.

Example 6.2.8. Consider the following:

1. Let $S = \mathbb{N}^2 \setminus \{(1,0),(1,1)\} = \langle (0,1),(2,0),(3,0),(1,2)\rangle$. We have computed some Apéry sets of S in Example 6.1.2. In particular $\mathrm{Ap}((0,1)) \cap \mathrm{Ap}((2,0)) = \{(0,0),(3,0),(1,2)\}$, that is S is of maximal embedding dimension.

2. Let $S = \langle (0,3),(3,0),(4,0),(5,0),(0,4),(0,5),(1,1)\rangle \subseteq \mathbb{N}^2$.
   S is a generalized numerical semigroup whose hole set is $\mathrm{H}(S) = \{(0,1),(0,2),(1,0),(2,0),(1,2),$
   $(1,3),(2,1),(2,3),(2,4),(3,1),(3,2),(4,2)\}$.
   We have that $(2,2) \in \mathrm{Ap}((3,0)) \cap \mathrm{Ap}((0,3))$ but $(2,2)$ is not a minimal generator of S. In particular $\mathrm{Ap}((3,0)) \cap \mathrm{Ap}((0,3)) = \{(0,0),(4,0),(5,0),(0,4),(0,5),(1,1),(2,2),(1,5),(5,1),(1,6),(6,1),$
   $(2,6),(6,2),(2,7),(7,2)\}$ so S is not of maximal embedding dimension.

generalized numerical semigroups of maximal embedding dimension are not sporadic, in fact all ordinary generalized numerical semigroups are among these.

Proposition 6.2.9. *Let $S = (\mathbb{N}^d \setminus \pi(\boldsymbol{f})) \cup \{\boldsymbol{0}\}$, with $\boldsymbol{f} \in \mathbb{N}^d$, be an ordinary generalized numerical semigroup. Then S has maximal embedding dimension.*

Proof. The multiplicities are $\mathbf{m}_1 = (f^{(1)}+1)\mathbf{e}_1, \mathbf{m}_2 = (f^{(2)}+1)\mathbf{e}_2, \ldots, \mathbf{m}_d = (f^{(d)}+1)\mathbf{e}_d$. Let $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, with $\mathbf{x}_1, \mathbf{x}_2 \in S$, that is $\mathbf{x}$ is not a minimal generator. In particular $\mathbf{x}_1 \not\leq \mathbf{f}$, $\mathbf{x}_2 \not\leq \mathbf{f}$. So there exist $j,k \in \{1,\ldots,d\}$ such that $x_1^{(j)} > f^{(j)}$ and $x_2^{(k)} > f^{(k)}$. Consider the element $\mathbf{x} - (f^{(j)}+1)\mathbf{e}_j$ and we focus on its k-th coordinate, if $j \neq k$ it is $x^{(k)} = x_1^{(k)} + x_2^{(k)} > f^{(k)}$, if $j = k$ it is $x^{(k)} - f^{(k)} - 1 = x_1^{(k)} + x_2^{(k)} - f^{(k)} - 1 \geq 2f^{(k)} + 2 - f^{(k)} - 1 = f^{(k)} + 1$. In both cases $\mathbf{x} - (f^{(j)}+1)\mathbf{e}_j \in S$, that is $\mathbf{x} \notin \mathrm{Ap}(S, \mathbf{m}_j)$. □

The g-symplectic generalized numerical semigroups are not of maximal embedding dimension. The set $\left(\bigcap_{i=1}^d \mathrm{Ap}(S,\mathbf{m}_i)\right) \setminus \{\mathbf{0}\} \cup \{\mathbf{m}_1,\ldots,\mathbf{m}_d\}$ is easy to compute in this case.

Proposition 6.2.10. *Let $S \subseteq \mathbb{N}^d$ be a g-symplectic generalized numerical semigroup. Then*

$$\left(\bigcap_{i=1}^{d} \mathrm{Ap}(S,\boldsymbol{m}_i)\right) \setminus \{\boldsymbol{0}\} \cup \{\boldsymbol{m}_1,\ldots,\boldsymbol{m}_d\} =$$

$$= G(S) \cup \{\boldsymbol{x} \in \mathbb{N}^d \mid x^{(1)} + \cdots + x^{(d)} > 2g+1, x^{(i)} < g+1 \text{ for all } i \in \{1,\ldots,d\}\}$$

where $G(S)$ is the minimal set of generators of S.

Proof. The multiplicities are $\mathbf{m}_1 = (g+1)\mathbf{e}_1, \mathbf{m}_2 = (g+1)\mathbf{e}_2, \dots, \mathbf{m}_d = (g+1)\mathbf{e}_d$. Let $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, with $\mathbf{x}_1, \mathbf{x}_2 \in S$, that is $\mathbf{x}$ is not a minimal generator. Then $\sum_{i=1}^d x^{(i)} = \sum_{i=1}^d x_1^{(i)} + \sum_{i=1}^d x_2^{(i)} \geq 2g+2$. Let $j \in \{1, 2, \dots, d\}$ and consider the element $\mathbf{x} - \mathbf{m}_j$, the sum of its coordinates is $\sum_{i=1}^d x^{(i)} - (g+1) \geq 2g+2-(g+1) = g+1 > g$. So if $\mathbf{x}$ is not a minimal generator it belongs to $\mathrm{Ap}(S, \mathbf{m}_j)$ if and only if $\mathbf{x} - \mathbf{m}_j \notin \mathbb{N}^d$. In particular $\mathbf{x} - \mathbf{m}_j \notin \mathbb{N}^d$ for all $j = 1, \dots, d$ if and only if $x^{(j)} < g+1$ for all $j \in \{1, \dots, d\}$. $\square$

Example 6.2.11. Let $S = \mathbb{N}^3 \setminus \{(1,0,0), (0,1,0), (0,0,1), (2,0,0), (1,1,0), (1,0,1), (0,1,1), (0,2,0), (0,0,2), (3,0,0), (0,3,0), (0,0,3), (1,1,1), (2,1,0), (2,0,1), (0,2,1), (1,2,0), (0,1,2), (1,0,2)\}$. S is the 3-symplectic generalized numerical semigroup in $\mathbb{N}^3$. The element $(3,3,3)$ is not a minimal generator but it is an element of $\left(\bigcap_{i=1}^d \mathrm{Ap}(S, \mathbf{m}_i)\right) \setminus \{\mathbf{0}\}$, where $\mathbf{m}_i = 4\mathbf{e}_i$ for all i.

The following properties about Apéry sets in a generalized numerical semigroup are also useful.

Proposition 6.2.12. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and $\boldsymbol{n} \in S$ a nonzero element. Then*

$$\mathrm{PF}(S) = \{\boldsymbol{w} - \boldsymbol{n} \mid \boldsymbol{w} \in \mathit{Maximals}_{\leq_S} \mathrm{Ap}(S, \boldsymbol{n})\}$$

Proof. Let $\mathbf{x} \in \mathrm{PF}(S)$, then $\mathbf{x} \notin S$ and $\mathbf{x} + \mathbf{n} \in S$, that is $\mathbf{x} + \mathbf{n} \in \mathrm{Ap}(S, \mathbf{n})$. If we consider $\mathbf{w} \in \mathrm{Ap}(S, \mathbf{n})$ with $\mathbf{x} + \mathbf{n} \leq_S \mathbf{w}$ it follows that $\mathbf{y} = \mathbf{w} - \mathbf{x} - \mathbf{n} \in S$ and $\mathbf{w} - \mathbf{n} = \mathbf{x} + \mathbf{y} \notin S$. So, since $\mathbf{x} \in \mathrm{PF}(S)$ it follows that $\mathbf{y} = \mathbf{0}$ and that $\mathbf{w} = \mathbf{x} + \mathbf{n}$. Now if $\mathbf{w} \in \mathrm{Maximals}_{\leq_S} \mathrm{Ap}(S, \mathbf{n})$ then $\mathbf{w} - \mathbf{n} \notin S$. If $\mathbf{w} - \mathbf{n} \notin \mathbb{N}^d$ then there exists $i \in \{1, \dots, d\}$ such that the i-th coordinate of $\mathbf{w}$, namely $w^{(i)}$, is smaller than the i-th coordinate of $\mathbf{n}$. Since $\mathbb{N}^d \setminus S$ is finite, it is not difficult to find an element $\mathbf{z} \in S$ such that $\mathbf{z} - \mathbf{w} \in S$ and the i-th coordinate of $\mathbf{z}$ is $w^{(i)}$ (and the other coordinates are quite large), so $\mathbf{z} \in \mathrm{Ap}(S, \mathbf{n})$ but this contradicts the maximality of $\mathbf{w}$ with respect to $\leq_S$. So $\mathbf{w} - \mathbf{n} \in \mathrm{H}(S)$. If $\mathbf{s} \in S$ and $\mathbf{w} - \mathbf{n} + \mathbf{s} \notin S$ then it follows that $\mathbf{w} + \mathbf{s} \in \mathrm{Ap}(S, \mathbf{n})$ which contradicts the maximality of $\mathbf{w}$. So $\mathbf{w} - \mathbf{n} \in \mathrm{PF}(S)$. $\square$

Proposition 6.2.13. *Let $S \subseteq \mathbb{N}^d$ be a monoid and $\boldsymbol{s}_1, \boldsymbol{s}_2 \in S$ with $\boldsymbol{s}_1 + \boldsymbol{s}_2 \in \mathrm{Ap}(S, \boldsymbol{n})$. Then $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathrm{Ap}(S, \boldsymbol{n})$.*

Proof. If $\mathbf{s}_1 - \mathbf{n} \in S$, then $\mathbf{s}_1 + \mathbf{s}_2 - \mathbf{n} \in S$, that is a contradiction. $\square$

6.3 A "reduced" Apéry set for irreducible generalized numerical semigroups

Definition 6.3.1. Let $\mathrm{MH}(S)$ the set of maximal elements in $\mathrm{H}(S)$ with respect to the natural partial order in $\mathbb{N}^d$. Let $\mathbf{n} \in S$, we define the set

$$\mathrm{C}(S, \mathbf{n}) = \{\mathbf{s} \in \mathrm{Ap}(S, \mathbf{n}) \mid \mathbf{s} \leq \mathbf{h} + \mathbf{n},\ \mathbf{h} \in \mathrm{MH}(S)\}$$

Observe that $\mathrm{C}(S, \mathbf{n})$ is a finite set, in particular it is obtained from $\mathrm{Ap}(S, \mathbf{n})$ "splitting" the infinite chains considering the partial order $\leq$. Moreover, if S is a numerical semigroup then $\mathrm{Ap}(S, n) = \mathrm{C}(S, n)$.

Proposition 6.3.2. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and $\boldsymbol{n} \in S$. The following assertions are verified:*

1) $\mathrm{Maximals}_{\leq} \mathrm{Ap}(S, \boldsymbol{n}) = \mathrm{Maximals}_{\leq} \mathrm{C}(S, \boldsymbol{n}) = \{\boldsymbol{h} + \boldsymbol{n} \mid \boldsymbol{h} \in \mathrm{MH}(S)\}$

2) $\mathrm{Maximals}_{\leq_S} \mathrm{Ap}(S, \boldsymbol{n}) \subseteq \mathrm{Maximals}_{\leq_S} \mathrm{C}(S, \boldsymbol{n})$

3) $\mathrm{Maximals}_{\leq} \mathrm{Ap}(S, \boldsymbol{n}) \subseteq \mathrm{Maximals}_{\leq_S} \mathrm{Ap}(S, \boldsymbol{n})$.

Proof. 1) Let $\mathbf{a} \in S$ be maximal in $\mathrm{Ap}(S, \mathbf{n})$ with respect to $\leq$. Since $\mathbf{a} - \mathbf{n} \notin S$, there exists $\mathbf{h} \in \mathrm{MH}(S)$ such that $\mathbf{a} - \mathbf{n} \leq \mathbf{h}$ (even if $\mathbf{a} - \mathbf{n} \notin \mathbb{N}^d$). Then $\mathbf{a} \leq \mathbf{h} + \mathbf{n}$ and $\mathbf{h} + \mathbf{n} \in \mathrm{Ap}(S, \mathbf{n})$. By maximality of $\mathbf{a}$ it is verified that $\mathbf{a} = \mathbf{h} + \mathbf{n}$. It follows by definition that $\mathrm{C}(S, \mathbf{n})$ has the same maximal elements.
2) Let $\mathbf{s}$ be maximal in $\mathrm{Ap}(S, \mathbf{n})$ with respect to $\leq_S$. It suffices to prove that $\mathbf{s} - \mathbf{n} \in \mathrm{H}(S)$, in such a case there exists $\mathbf{h} \in \mathrm{H}(S)$ and $\mathbf{s} = \mathbf{h} + \mathbf{n}$, in particular $\mathbf{s} \in \mathrm{C}(S, \mathbf{n})$ and if there exists $\mathbf{t} \in \mathrm{C}(S, \mathbf{n})$ such that $\mathbf{s} \leq_S \mathbf{t}$ it is in contradiction with maximality of $\mathbf{s}$ in $\mathrm{Ap}(S, \mathbf{n})$, because $\mathbf{t} \in \mathrm{Ap}(S, \mathbf{n})$.
Suppose that $\mathbf{s} - \mathbf{n} \notin \mathbb{N}^d$, then there exists $i \in \{1, 2, \ldots, d\}$ such that it is possible to write $\mathbf{s} = \mathbf{s}' + s^{(i)}\mathbf{e}_i$ and $\mathbf{n} = \mathbf{n}' + n^{(i)}\mathbf{e}_i$, where $\mathbf{s}', \mathbf{n}' \in \mathbb{N}^d$ whose i-th component is zero, and $s^{(i)} < n^{(i)}$. Since $\mathrm{H}(S)$ is finite the components of all element in $\mathrm{H}(S)$ are limited, then there exists $\mathbf{t} = \sum_{j=1}^{d} t^{(j)}\mathbf{e}_j$ such that $t^{(i)} = s^{(i)}$ and for all $j \neq i$, $t^{(j)}$ is sufficient larger in order to $\mathbf{t} - \mathbf{s} \in S$. In particular $\mathbf{s} \leq_S \mathbf{t}$ and $\mathbf{t} \in \mathrm{Ap}(S, \mathbf{n})$, that is a contradiction for the maximality of $\mathbf{s}$ in $\mathrm{Ap}(S, \mathbf{n})$.
3) Let $\mathbf{s} = \mathbf{h} + \mathbf{n}$ with $\mathbf{h} \in \mathrm{MH}(S)$. If there exists $\mathbf{t} \in \mathrm{Ap}(S, \mathbf{n})$ such that $\mathbf{s} \leq_S \mathbf{t}$, then $\mathbf{t} - \mathbf{n} \notin S$ and $\mathbf{t} - \mathbf{s} \in S \subseteq \mathbb{N}^d$, in particular $\mathbf{h} \leq \mathbf{t} - \mathbf{n}$. This is a contradiction by maximality of $\mathbf{h}$ in $\mathrm{H}(S)$.

∎

Example 6.3.3. Let $S = \mathbb{N}^2 \setminus \{(0,1),(0,3),(1,0),(1,1),(1,3),(2,1),(2,0),(3,0)\}$. Let us compute the sets $\mathrm{Ap}(S,\mathbf{n})$ and $\mathrm{C}(S,\mathbf{n})$, for $\mathbf{n} = (4,0)$.

$\mathrm{Ap}(S,(4,0)) = \{(0,0),(0,2),(0,n),(1,2),(1,n),(2,2),(2,3),(2,n),(3,1),(3,2),(3,3),(3,n),(4,1),(4,3),(5,0),(5,1),(5,3),(6,1),(6,0),(7,0) \mid n \geq 4\}$

$\mathrm{C}(S,(4,0)) = \{(0,0),(0,2),(1,2),(2,2),(2,3),(3,1),(3,2),(3,3),(4,1),(4,3),(5,0),(5,1),(5,3),(6,1),(6,0),(7,0)\}$

We can see that:

- $\mathrm{Maximals}_{\leq} \mathrm{Ap}(S,\mathbf{n}) = \{(7,0),(6,1),(5,3)\}$
- $\mathrm{Maximals}_{\leq} \mathrm{C}(S,\mathbf{n}) = \{(7,0),(6,1),(5,3)\}$
- $\mathrm{Maximals}_{\leq_S} \mathrm{Ap}(S,\mathbf{n}) = \{(4,3),(5,0),(5,3),(6,1),(6,0),(7,0)\}$
- $\mathrm{Maximals}_{\leq_S} \mathrm{C}(S,\mathbf{n}) = \{(4,3),(5,0),(5,3),(6,1),(6,0),(7,0),(3,3),(3,2),(2,3)\}$

It seems in general $\mathrm{Maximals}_{\leq_S} \mathrm{Ap}(S,\mathbf{n}) \subsetneq \mathrm{Maximals}_{\leq_S} \mathrm{C}(S,\mathbf{n})$.
For instance $(3,3)$ is not maximal in $\mathrm{Ap}(S,(4,0))$ with respect to $\leq_S$, in particular $(3,5)-(3,3) = (0,2) \in S$ and $(3,5) \in \mathrm{Ap}(S,(4,0)$.

Remark 6.3.4. If $\mathbf{s}$ is a minimal generator of S it is possible that $\mathbf{s} \notin \mathrm{C}(S,\mathbf{n})$, for instance in Example 6.3.3 the element $(0,5)$ is a minimal generator of S but $(0,5) \notin \mathrm{C}(S,(4,0))$. So the set $\mathrm{C}(S,\mathbf{n})$ loses an important property of $\mathrm{Ap}(S,\mathbf{n})$. Furthermore in numerical semigroup $\mathrm{Ap}(S,\mathbf{n})$ and $\mathrm{C}(S,\mathbf{n})$ are equal. It is possible to avoid this problem considering $\mathrm{C}(S,\mathbf{n})$ for an opportune element $\mathbf{n}$.

Proposition 6.3.5. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup with multiplicities $\boldsymbol{m}_1,\ldots,\boldsymbol{m}_d$. Let $\boldsymbol{m} = \sum_{i=1}^{d} \boldsymbol{m}_i$ and A be the minimal set of generators of S. Then $A \subseteq \mathrm{C}(S,\boldsymbol{m})$.*

Proof. Let $\mathbf{x} \in A$. First observe that $\mathbf{x} \in \mathrm{Ap}(S,\mathbf{m})$, otherwise $\mathbf{x} = \mathbf{m}+\mathbf{s}$ with $\mathbf{s} \in S$. In order to prove that $\mathbf{x} \in \mathrm{C}(S,\mathbf{m})$ we distinguish two cases. If there exists $i \in \{1,\ldots,d\}$ such that $\mathbf{x}-\mathbf{m}_i \in \mathrm{H}(S)$ then $\mathbf{x} = \mathbf{h}+\mathbf{m}_i \leq \mathbf{h}+\mathbf{m}$ for some $\mathbf{h} \in \mathrm{H}(S)$. In the other case, if $\mathbf{x}-\mathbf{m}_i \notin \mathbb{N}^d$ for all $i \in \{1,\ldots,d\}$ then $\mathbf{x} \leq \mathbf{m} \leq \mathbf{h}+\mathbf{m}$ for all $\mathbf{h} \in \mathrm{MH}(S)$. $\square$

We have proved that, if $S \subseteq \mathbb{N}^d$ is an irreducible generalized numerical semigroup (that is symmetric or pseudo-symmetric) with Frobenius element $\mathbf{f}$, then for all $\mathbf{x} \in \mathrm{H}(S)$ with $\mathbf{x} \neq \frac{\mathbf{f}}{2}$ it is $\mathbf{f}-\mathbf{x} \in S$. If $d=1$ the same statement can be easily extended to all $\mathbf{x} \in \mathbb{Z} \setminus S$, not only in $\mathrm{H}(S)$. In the case $d>1$ we can not consider all elements $\mathbf{x} \in \mathbb{Z}^d \setminus S$, but only some of them.

Lemma 6.3.6. *Let $S \subseteq \mathbb{N}^d$ be an irreducible generalized numerical semigroup with Frobenius element $\boldsymbol{f}$ and let $\boldsymbol{x} \in \mathbb{Z}^d \setminus S$ with $\boldsymbol{x} \neq \frac{\boldsymbol{f}}{2}$ and $\boldsymbol{f} - \boldsymbol{x} \in \mathbb{N}^d$. Then $\boldsymbol{f} - \boldsymbol{x} \in S$.*

Proof. Let $\mathbf{x} \in \mathbb{Z}^d \setminus S$. If $\mathbf{x} \in \mathbb{N}^d$ then $\mathbf{x} \in \mathrm{H}(S)$ and the result follows from the property of irreducible generalized numerical semigroups. Suppose $\mathbf{x} \notin \mathbb{N}^d$ and $\mathbf{f} - \mathbf{x} = \mathbf{y} \in \mathbb{N}^d$. Then $\mathbf{f} - \mathbf{y} \notin \mathbb{N}^d$ that is $\mathbf{y} \not\leq \mathbf{f}$ with respect to the natural partial order in $\mathbb{N}^d$. Since $\mathbf{f}$ is the maximum in the set $\mathrm{H}(S)$ and $\mathbf{y} \in \mathbb{N}^d$, then $\mathbf{y} \in S$. □

Example 6.3.7. Consider the symmetric generalized numerical semigroup $S = \mathbb{N}^2 \setminus \{(0,1),(1,1),(2,1),(3,1),(4,1),(5,1),(6,1)\}$. If we consider $(-2,1) \in \mathbb{Z}^2 \setminus S$, then $(6,1) - (-2,1) = (8,0) \in \mathbb{N}^d$ and it belongs to S.

The following theorem is a generalization of a property of the Apéry set for symmetric numerical semigroups (Proposition 1.3.7).

Theorem 6.3.8. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup, $\boldsymbol{n} \in S$ and $\prec$ a monomial order in $\mathbb{N}^d$. Then S is symmetric if and only if $\mathrm{C}(S, \boldsymbol{n}) = \{\boldsymbol{a}_0 \prec \boldsymbol{a}_1 \prec \ldots \prec \boldsymbol{a}_t\}$ with $\boldsymbol{a}_i + \boldsymbol{a}_{t-i} = \boldsymbol{a}_t$, for $i = 0, 1, \ldots, t$.*

Proof. $\Rightarrow$) If S is symmetric it has a maximum hole $\mathbf{f}$ with respect to natural partial order in $\mathbb{N}^d$. Let $\mathrm{C}(S, \mathbf{n}) = \{\mathbf{a}_0 \prec \mathbf{a}_1 \prec \ldots \prec \mathbf{a}_t\}$, then by Proposition 6.3.2, since every monomial order extends the natural partial order in $\mathbb{N}^d$, $\mathrm{Maximals}_{\leq} \mathrm{C}(S, \mathbf{n}) = \{\mathbf{f} + \mathbf{n}\}$, then $\mathbf{a}_t = \mathbf{f} + \mathbf{n}$. Let $\mathbf{a}_i \in \mathrm{C}(S, \mathbf{n})$, then $\mathbf{a}_i - \mathbf{n} \notin S$ and $\mathbf{f} - (\mathbf{a}_i - \mathbf{n}) = \mathbf{f} + \mathbf{n} - \mathbf{a}_i \in \mathbb{N}^d$. Since S is symmetric, by Lemma 6.3.6 $\mathbf{f} - (\mathbf{a}_i - \mathbf{n}) = \mathbf{a}_t - \mathbf{a}_i \in S$. In particular, there exists $\mathbf{s} \in S$ such that $\mathbf{a}_t = \mathbf{a}_i + \mathbf{s} \in \mathrm{Ap}(S, \mathbf{n})$ and, by Proposition 6.2.13, $\mathbf{s} \in \mathrm{Ap}(S, \mathbf{n})$. Furthermore $\mathbf{s} < \mathbf{a}_t$, so $\mathbf{s} \in \mathrm{C}(S, \mathbf{n})$. Then for every $i = 0, 1, \ldots, t$ there exists $j \in \{0, 1, \ldots, t\}$ such that $\mathbf{a}_i + \mathbf{a}_j = \mathbf{a}_t$. If $i = 0$ then $\mathbf{a}_0 + \mathbf{a}_{t-0} = \mathbf{a}_t$ is true, since $\mathbf{a}_0 = \mathbf{0}$. Now let $i \in \{1, 2, \ldots, t\}$ and suppose the assertion $\mathbf{a}_k + \mathbf{a}_{t-k} = \mathbf{a}_t$ is true for every $k < i$, we want to prove that $\mathbf{a}_i + \mathbf{a}_{t-i} = \mathbf{a}_t$. We know there exists $j \in \{0, 1, \ldots, t\}$ such that $\mathbf{a}_i + \mathbf{a}_j = \mathbf{a}_t$. Suppose that $j \neq t - i$, we want to show it is contradiction.

- If $j > t - i$ then there exists $0 < k \leq t$ such that $j = t - (i - k)$. Furthermore $i - k < i$, and by induction we have $\mathbf{a}_{i-k} + \mathbf{a}_{t-(i-k)} = \mathbf{a}_t = \mathbf{a}_i + \mathbf{a}_j$. They imply $\mathbf{a}_i = \mathbf{a}_{i-k}$, that is a contradiction.

- If $j < t - i$, consider that $\mathbf{a}_{t-i} + \mathbf{a}_k = \mathbf{a}_t$ for some $k \in \{0, 1, \ldots, t\}$. If $k < i$ then, by hypothesis, $\mathbf{a}_k + \mathbf{a}_{t-k} = \mathbf{a}_{t-i} + \mathbf{a}_k = \mathbf{a}_t$ that implies $\mathbf{a}_{t-k} = \mathbf{a}_{t-i}$, in particular $t - k = t - i$, that is the contradiction $k = i$. So $k \geq i$ and $\mathbf{a}_t = \mathbf{a}_i + \mathbf{a}_j \prec \mathbf{a}_i + \mathbf{a}_{t-i} \prec \mathbf{a}_k + \mathbf{a}_{t-i} = \mathbf{a}_t$, that is a contradiction.

It follows that $j = t - i$ and this part is proved.
($\Leftarrow$) Let $\mathrm{C}(S, \mathbf{n}) = \{\mathbf{a}_0 \prec \mathbf{a}_1 \prec \ldots \prec \mathbf{a}_t\}$ with $\mathbf{a}_i + \mathbf{a}_{t-i} = \mathbf{a}_t$, for $i = 0, 1, \ldots, t$. Then $\mathrm{Maximals}_{\leq} \mathrm{Ap}(S, \mathbf{n}) = \mathrm{Maximals}_{\leq} \mathrm{C}(S, \mathbf{n}) = \{\mathbf{a}_t\}$. Moreover $\mathbf{a}_i \leq_S \mathbf{a}_t$, for every $i \in \{0, 1, 2, \ldots, t\}$, that is $\mathrm{Maximals}_{\leq_S} \mathrm{C}(S, \mathbf{n}) = \{\mathbf{a}_t\}$. By Proposition 6.3.2, it follows that $\mathrm{Maximals}_{\leq_S} \mathrm{Ap}(S, \mathbf{n}) = \{\mathbf{a}_t\}$, in particular $\mathrm{PF}(S) = \{\mathbf{a}_t - \mathbf{n}\}$. Then S is symmetric. $\square$

Example 6.3.9. Let $S = \mathbb{N}^2 \setminus \{(0,1), (0,2), (1,0), (1,1), (2,0), (2,1), (3,0), (4,0), (5,2)\}$. S has Frobenius element $(5,2)$ and it is symmetric because $2 \cdot g(S) = 18 = (5+1) \cdot (2+1)$.

$\mathrm{Ap}(S, (0,3)) = \{(0,0), (0,4), (0,5), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (5,0), (5,1), (5,5), (n,0), (n,1), (n,2) \mid n \geq 6\}$

Now we compute $\mathrm{C}(S, (0,3))$ and we arrange its elements using two different monomial orders.

- Lexicographic order:
  $\mathrm{C}(S, (0,3)) = \{(0,0), (0,4), (0,5), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (5,0), (5,1), (5,5)\}$
- Graded Reverse Lexicographic order:
  $\mathrm{C}(S, (0,3)) = \{(0,0), (1,2), (3,1), (2,2), (1,3), (0,4), (5,0), (4,1), (3,2), (2,3), (1,4), (0,5), (5,1), (4,2), (3,3), (2,4), (4,3), (5,5)\}$

Theorem 6.3.8 is satisfied in both monomial orders. We can choose another Apéry set:

$\mathrm{Ap}(S, (5,0)) = \{(0,0), (0,n), (1,n), (2,n), (3,1), (4,1), (3,n), (4,n), (5,1), (6,0), (6,1), (7,0), (7,1), (8,0), (9,0), (10,2) \mid n \geq 2\}$

The set $\mathrm{C}(S, (5,0))$, arranged with respect to the lexicographic order is :

$\mathrm{C}(S, (5,0)) = \{(0,0), (1,2), (2,2), (3,1), (3,2), (4,1), (4,2), (5,1), (6,0), (6,1), (7,0), (7,1), (8,0), (9,0), (10,2)\}$

The analogous of Theorem 6.3.8 in the pseudo-symmetric case is the following:

Theorem 6.3.10. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup, $\boldsymbol{n} \in S$ and $\prec$ a monomial order in $\mathbb{N}^d$. Then S is pseudo-symmetric if and only if $\mathrm{C}(S, \boldsymbol{n}) = \{\boldsymbol{a}_0 \prec \boldsymbol{a}_1 \prec \ldots \prec \boldsymbol{a}_t = \boldsymbol{f} + \boldsymbol{n}\} \cup \left\{\frac{\boldsymbol{f}}{2} + \boldsymbol{n}\right\}$ where $\boldsymbol{f} \in \mathrm{MH}(S)$ and $\boldsymbol{a}_i + \boldsymbol{a}_{t-i} = \boldsymbol{a}_t$, for $i = 0, 1, \ldots, t$.*

Proof. $\Rightarrow$) Suppose S is pseudo-symmetric with Frobenius element $\mathbf{f}$. Let us show that $\frac{\mathbf{f}}{2} + \mathbf{n} \in \mathrm{C}(S, \mathbf{n})$. Obviously $\frac{\mathbf{f}}{2} \notin S$ and $\frac{\mathbf{f}}{2} \in \mathrm{PF}(S)$, so $\frac{\mathbf{f}}{2} + \mathbf{n} \in S$, that is $\frac{\mathbf{f}}{2} + \mathbf{n} \in \mathrm{Ap}(S, \mathbf{n})$. Moreover $\frac{\mathbf{f}}{2} + \mathbf{n} < \mathbf{f} + \mathbf{n}$, then $\frac{\mathbf{f}}{2} + \mathbf{n} \in \mathrm{C}(S, \mathbf{n})$. Since $\mathbf{f}$ is maximum in the set of holes, $\mathbf{a}_t = \mathbf{f} + \mathbf{n}$. Let $\mathbf{a}_i \in \mathrm{C}(S, \mathbf{n}) \setminus \left\{\frac{\mathbf{f}}{2} + \mathbf{n}\right\}$, then $\mathbf{a}_t - \mathbf{a}_i \neq \frac{\mathbf{f}}{2} + \mathbf{n}$, $\mathbf{a}_i - \mathbf{n} \notin S$ and $\mathbf{f} - (\mathbf{a}_i - \mathbf{n}) = \mathbf{f} + \mathbf{n} - \mathbf{a}_i \in \mathbb{N}^d$. Since S is pseudo-symmetric, by Lemma 6.3.6 $\mathbf{f} - (\mathbf{a}_i - \mathbf{n}) = \mathbf{a}_t - \mathbf{a}_i \in S$. In particular, there exists $\mathbf{s} \in S$ such that $\mathbf{a}_t = \mathbf{a}_i + \mathbf{s} \in \mathrm{Ap}(S, \mathbf{n})$ and from Proposition 6.2.13, $\mathbf{s} \in \mathrm{Ap}(S, \mathbf{n})$. Furthermore $\mathbf{s} < \mathbf{a}_t$, so $\mathbf{s} \in \mathrm{C}(S, \mathbf{n})$. Then for every $i = 0, 1, \ldots, t$ there exists $j \in \{0, 1, \ldots, t\}$ such that $\mathbf{a}_i + \mathbf{a}_j = \mathbf{a}_t$. The last part of the proof is similar to the proof of Theorem 6.3.8.
$\Leftarrow$) Suppose that $\mathrm{C}(S, \mathbf{n}) = \{\mathbf{a}_0 \prec \mathbf{a}_1 \prec \ldots \prec \mathbf{a}_t = \mathbf{f} + \mathbf{n}\} \cup \left\{\frac{\mathbf{f}}{2} + \mathbf{n}\right\}$ where $\mathbf{f} \in \mathrm{MH}(S)$ and $\mathbf{a}_i + \mathbf{a}_{t-i} = \mathbf{a}_t$, for $i = 0, 1, \ldots, t$. Since $\frac{\mathbf{f}}{2} + \mathbf{n} \in S$ then $\mathbf{f}$ has all even coordinates. Furthermore let $\mathbf{h} \in \mathrm{H}(S)$ and $\mathbf{h} \neq \frac{\mathbf{f}}{2}$, we want to prove that $\mathbf{f} - \mathbf{h} \in S$ so S is pseudo-symmetric. Consider $k = \min\{j \in \mathbb{N} \mid \mathbf{h} + j\mathbf{n} \in S\}$, since S is a generalized numerical semigroup there exists such $k \geq 1$. Obviously $\mathbf{h} + k\mathbf{n} \in \mathrm{Ap}(S, \mathbf{n})$. Moreover, by Proposition 6.3.2, we obtain $\{\mathbf{f} + \mathbf{n}\} = \mathrm{Maximals}_{\leq} \mathrm{C}(S, \mathbf{n}) = \mathrm{Maximals}_{\leq} \mathrm{Ap}(S, \mathbf{n})$, in particular $\mathbf{f}$ is maximum in $\mathrm{H}(S)$. Since $\mathbf{h} + (k-1)\mathbf{n} \in \mathrm{H}(S)$, we have $\mathbf{h} + (k-1)\mathbf{n} < \mathbf{f}$, hence $\mathbf{h} + k\mathbf{n} < \mathbf{f} + \mathbf{n}$, that is $\mathbf{h} + k\mathbf{n} \in \mathrm{C}(S, \mathbf{n})$. If there exists $i \in \{1, 2, \ldots, t\}$ such that $\mathbf{a}_i = \mathbf{h} + k\mathbf{n}$ then $\mathbf{a}_t - \mathbf{a}_i = \mathbf{f} - \mathbf{h} - (k-1)\mathbf{n} = \mathbf{a}_{t-i} \in S$, in particular $\mathbf{f} - \mathbf{h} \in S$. If $\mathbf{h} + k\mathbf{n} = \frac{\mathbf{f}}{2} + \mathbf{n}$, consider that $k \geq 2$ since $\mathbf{h} \neq \frac{\mathbf{f}}{2}$, so $\mathbf{f} - \mathbf{h} = \frac{\mathbf{f}}{2} + (k-1)\mathbf{n} \in S$. $\square$

Chapter 7

Algorithms and computational results

The aim of this chapter is to provide a brief collection of the first basic algorithms concerning generalized numerical semigroups, in order to do computations in this subject. Such algorithms are very useful to test properties, conjectures and to produce useful examples. A lot of examples in this work were produced by implementation of such algorithms in the computer algebra software GAP [24] and the GAP package `numericalsgps` [17]. The provided algorithms concern in computing the set of holes from the set of generators of a given generalized numerical semigroup and viceversa. Moreover we provide two procedures to generate all generalized numerical semigroups in $\mathbb{N}^d$ of a given genus (the first based on the semigroup tree introduced in Chapter 2) and a combinatorial tool for it. The last section is devoted to give some computational results. Much of the contents of this chapter is part of the paper [10].

7.1 Useful properties

In this section we provide some useful properties in order to have a complete exposition of the successive algorithms. They can be useful from the theoretical point of view and for computations.

Proposition 7.1.1. [22, Proposition 4.3] *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup, $G(S)$ its minimal set of generators and $\boldsymbol{v} \in G(S)$. Then $S \setminus \{\boldsymbol{v}\}$ is generated by $\mathcal{G}(G, \boldsymbol{v}) = (G(S) \setminus \{\boldsymbol{v}\}) \cup \{\boldsymbol{g} + \boldsymbol{v} \mid \boldsymbol{g} \in G(S) \setminus \{\boldsymbol{v}\}\} \cup \{2\boldsymbol{v}, 3\boldsymbol{v}\}$.*

Remark 7.1.2. If S is a generalized numerical semigroup and $\mathbf{v}$ is an effective generator of S with respect to a fixed relaxed monomial order, by the preceding proposition we can produce a finite system of generators, $\mathcal{G}(G, \mathbf{v})$, of $S \setminus \{\mathbf{v}\}$. However $\mathcal{G}(G, \mathbf{v})$ is not, in general, a minimal set of generators for $S \setminus \{\mathbf{v}\}$. The next example shows this fact.

Example 7.1.3. Let $S = \mathbb{N}^2 \setminus \{(1,0)\}$, it is $G(S) = \{(2,0),(3,0),(1,1),(0,1)\}$. Let $\prec$ be the lexicographic order, so $\mathbf{F}_\prec = (1,0)$, and $(1,1)$ is an effective generator. Let $S' = S \setminus \{(1,1)\} = \mathbb{N}^2 \setminus \{(1,0),(1,1)\}$. From Proposition 7.1.1 $\mathcal{G}(G,(1,1)) = \{(2,0),(3,0),(0,1),(3,1),(4,1),(1,2),(2,2),(3,3)\}$ is a set generating S'. However $(3,0)+(0,1) = (3,1)$ so it is not minimal.

Now we want to consider the following membership problem: let $\mathbf{b} \in \mathbb{N}^d$ and A a finite subset of $\mathbb{N}^d$, we ask if $\mathbf{b} \in \langle A \rangle$ (linear combination with nonnegative integer coefficients). There are different ways to solve this problem and there are computer algebra systems in which such methods are implemented. Anyway we provide here a simple method, useful also to justify our results from theoretical point of view.

Definition 7.1.4. Let $A \subseteq \mathbb{N}^d$ be a finite set. Let F_A be the polynomial

$$F_A = \sum_{\mathbf{v} \in A} x^{\mathbf{v}}$$

where $x^{\mathbf{v}} = x_1^{v_1} x_2^{v2} \cdots x_d^{v_d}$, since $\mathbf{v} = (v_1, v_2, \dots, v_d)$. We call $x^{\mathbf{v}}$ the associated monomial to $\mathbf{v}$.
We also define the *power series expansion* of $1/(1-F_A)$ as the formal power series:

$$P(F_A) = \sum_{k=0}^{\infty} (F_A)^k$$

The following lemma (that is also a generalization of a property suggested by [33, Lemma 2.2] for $d = 1$) is obtained by applying *Leibnitz's rule*:

$$(a_1 + a_2 + \cdots + a_m)^n = \sum_{h_1+h_2+\cdots+h_m=n} \frac{n!}{h_1!h_2!\dots h_m!} a_1^{h_1} a_2^{h_2} \cdots a_m^{h_m}$$

Lemma 7.1.5. *Let $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n\} \subseteq \mathbb{N}^d$ and $\boldsymbol{b} \in \mathbb{N}^d$. Then $\boldsymbol{b}$ is a linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n$ with nonnegative integer coefficients if and only if the coefficient of $x^{\boldsymbol{b}}$ in $P(F_A)$ is nonzero.*

Proof. By Leibnitz's rule we obtain:

$$(F_A)^t = (x_1^{a_1^{(1)}} x_2^{a_1^{(2)}} \cdots x_d^{a_1^{(d)}} + x_1^{a_2^{(1)}} x_2^{a_2^{(2)}} \cdots x_d^{a_2^{(d)}} + \cdots + x_1^{a_n^{(1)}} x_2^{a_n^{(2)}} \cdots x_d^{a_n^{(d)}})^t =$$

$$= \sum K \cdot x_1^{a_1^{(1)}h_1 + a_2^{(1)}h_2 + \cdots a_n^{(1)}h_n} \cdot x_2^{a_1^{(2)}h_1 + a_2^{(2)}h_2 + \cdots a_n^{(2)}h_n} \cdot \cdots \cdot x_d^{a_1^{(d)}h_1 + a_2^{(d)}h_2 + \cdots a_n^{(d)}h_n}$$

where the sum is extended to $h_1, \dots, h_n \in \mathbb{N}$ with $h_1 + \cdots + h_n = t$ and K is a nonzero coefficient.

If $\mathbf{b} = \sum_{i=1}^n \lambda_i \mathbf{a}_i$, set $t = \sum_{i=1}^n \lambda_i$, then $x^{\mathbf{b}}$ is a monomial in $(F_A)^t$. Conversely, if $x^{\mathbf{b}}$ has nonzero coefficient in $P(F_A)$ then

$$x^{\mathbf{b}} = x_1^{a_1^{(1)}h_1 + a_2^{(1)}h_2 + \cdots a_n^{(1)}h_n} \cdot x_2^{a_1^{(2)}h_1 + a_2^{(2)}h_2 + \cdots a_n^{(2)}h_n} \cdot \cdots \cdot x_d^{a_1^{(d)}h_1 + a_2^{(d)}h_2 + \cdots a_n^{(d)}h_n}$$

with $h_i \in \mathbb{N}$ for $i = 1, \dots, n$ that is $\mathbf{b} = \sum_{i=1}^n h_i \mathbf{a}_i$.

□

The previous result permits to give a method to produce the minimal system of generators of a finitely generated monoid from any finite set of generators for it. However we need to improve that, because we can't do a sum with infinite values. Now we show that it is possible to truncate the power series expansion $P(F_{A\setminus\{\mathbf{v}\}})$, in order to obtain the same consequence but with a finite sum.

Definition 7.1.6. Let $A = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\} \subseteq \mathbb{N}^d$ with $\mathbf{a}_i = (a_i^{(1)}, a_i^{(2)}, \dots, a_i^{(d)})$ for $i = 1, 2, \dots, n$, and let $\mathbf{b} \in \mathbb{N}^d$.
Let $t = \min\{\sum_{j=1}^d a_i^{(j)} := |\mathbf{a}_i| \mid i = 1, 2, \dots, n\}$. We define the positive integer

$$N_{\mathbf{b}} := \left\lfloor \frac{|\mathbf{b}|}{t} \right\rfloor$$

where $\lfloor x \rfloor$ denotes the floor of x.

Proposition 7.1.7. *Let $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n\} \subseteq \mathbb{N}^d$ and $\boldsymbol{b} \in \mathbb{N}^d$. Then $\boldsymbol{b} \in \langle A \rangle$ if and only if the coefficient of $x^{\boldsymbol{b}}$ is nonzero in the polynomial:*

$$F(x_1, x_2, \dots, x_d) = \sum_{k=0}^{N_b} (F_A)^k$$

Proof. By lemma 7.1.5 it is enough to show that the coefficient of $x^{\mathbf{b}}$ is zero in $F(x_1, \ldots, x_d)$ if and only if it is zero also in $P(F_A)$, that is $\sum_{k=0}^{\infty}(F_A)^k$. We suppose that the coefficient of $x^{\mathbf{b}}$ is nonzero in $P(F_A)$. Then there exists $r \in \mathbb{N}$ such that $x^{\mathbf{b}}$ is a monomial in $(F_A)^r$. By Leibnitz's rule we obtain:

$$(F_A)^r = (x_1^{a_1^{(1)}} x_2^{a_1^{(2)}} \cdots x_d^{a_1^{(d)}} + x_1^{a_2^{(1)}} x_2^{a_2^{(2)}} \cdots x_d^{a_2^{(d)}} + \cdots + x_1^{a_n^{(1)}} x_2^{a_n^{(2)}} \cdots x_d^{a_n^{(d)}})^r$$

$$= \sum_{\mathbf{h}} K \cdot x_1^{a_1^{(1)}h_1 + a_2^{(1)}h_2 + \cdots a_n^{(1)}h_n} \cdot x_2^{a_1^{(2)}h_1 + a_2^{(2)}h_2 + \cdots a_n^{(2)}h_n} \cdot \cdots \cdot x_d^{a_1^{(d)}h_1 + a_2^{(d)}h_2 + \cdots a_n^{(d)}h_n}$$

where $\mathbf{h} = (h_1, \ldots, h_n)$ with $h_1 + h_2 + \cdots + h_n = r$ and K is the correspondent coefficient, but we do not need its exact value.
If in the sum $x_1^{b^{(1)}} x_2^{b^{(2)}} \ldots x_d^{b^{(d)}}$ appears, then there exist $h_1, h_2, \ldots, h_n$ with $h_1 + h_2 + \cdots + h_n = r$, such that the following equalities are satisfied:

$$\begin{aligned} a_1^{(1)}h_1 + a_2^{(1)}h_2 + \cdots a_n^{(1)}h_n &= b^{(1)} \\ a_1^{(2)}h_1 + a_2^{(2)}h_2 + \cdots a_n^{(2)}h_n &= b^{(2)} \\ &\vdots \\ a_1^{(d)}h_1 + a_2^{(d)}h_2 + \cdots a_n^{(d)}h_n &= b^{(d)} \end{aligned}$$

We sum the righ-hand side and the left-hand side of all equalities, obtaining that:

$$\begin{aligned} r = h_1 + h_2 + \cdots + h_n &\leq |\mathbf{a}_1|h_1 + |\mathbf{a}_2|h_2 + \cdots + |\mathbf{a}_n|h_n = \\ &= b^{(1)} + b^{(2)} + \cdots + b^{(d)} \end{aligned}$$

Finally, if $t = \min\{|\mathbf{a}_i| \mid i = 1, 2, \ldots, n\}$ then $\frac{|\mathbf{a}_i|}{t} \geq 1$ for $i = 1, 2, \ldots, d$. So we can divide the right-hand side of inequality by t and we obtain:

$$\begin{aligned} r = h_1 + h_2 + \cdots + h_n &\leq \\ \leq \frac{|\mathbf{a}_1|}{t}h_1 + \frac{|\mathbf{a}_2|}{t}h_2 + \cdots + \frac{|\mathbf{a}_n|}{t}h_n &= \frac{|\mathbf{b}|}{t} \end{aligned}$$

It follows that $r \leq N_{\mathbf{b}}$. So, if the coefficient of $x^{\mathbf{b}}$ in $P(F_A)$ is nonzero then the greatest power in which it is obtained is at last $N_{\mathbf{b}}$, for greater powers we are sure that monomial does not appear.

□

Let $S \subseteq \mathbb{N}^d$ be a finitely generated monoid and $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ be a system of generators for S. We denote by M the $d \times n$ matrix whose i-th column is the vector $\mathbf{a}_i \in \mathbb{N}^d$ for $i = 1, \ldots, n$. It is easy to see that an element $\mathbf{b} \in S$ if and only if the system $M\mathbf{x} = \mathbf{b}$ admits solutions in $\mathbb{N}^n$. In fact this statement is equivalent to say that $\mathbf{b}$ is a linear combination of $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} \subseteq \mathbb{N}^d$ with nonnegative integer coefficients.
An application of the previous proposition is the following criterion for the existence of $\mathbb{N}$-solutions in a linear system with nonnegative integer coefficients.

Corollary 7.1.8. *Let M be a $d \times n$ matrix with entries in $\mathbb{N}$ whose columns are the vectors of the set $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$ and let $\boldsymbol{b} \in \mathbb{N}^d$. Then the linear system $M\boldsymbol{x} = \boldsymbol{b}$ admits solutions $\boldsymbol{x} \in \mathbb{N}^n$ if and only if the coefficient of $x^{\boldsymbol{b}}$ is nonzero in the polynomial:*

$$F(x_1, x_2, \ldots, x_d) = \sum_{k=0}^{N_{\boldsymbol{b}}} (F_A)^k$$

Corollary 7.1.9. *Let $S \subseteq \mathbb{N}^d$ be a finitely generated monoid, $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$ be a finite system of generators for S and $\boldsymbol{v} \in \mathbb{N}^d$. Then $\boldsymbol{v} \in S$ if and only if the coefficient of $x^{\boldsymbol{v}}$ is nonzero in the polynomial:*

$$F(x_1, x_2, \ldots, x_d) = \sum_{k=0}^{N_{\boldsymbol{v}}} (F_A)^k$$

Another straigthforward application, useful for our aim, is the following:

Corollary 7.1.10. *Let $S \subseteq \mathbb{N}^d$ be a finitely generated monoid generated by the finite set A and $\boldsymbol{v} \in A$. Then $\boldsymbol{v}$ is a minimal generator if and only if the coefficient of $x^{\boldsymbol{v}}$ is zero in the polynomial:*

$$F(x_1, x_2, \ldots, x_d) = \sum_{k=0}^{N_{\boldsymbol{v}}} (F_{A \setminus \{\boldsymbol{v}\}})^k$$

If S is a generalized numerical semigroup and a finite system of generators for S is known, then Corollary 7.1.9 and Corollary 7.1.10 provide a way to establish whether an element $\mathbf{v} \in S$ or if it is a minimal generator. It can be done with a finite computation, that is the building of a polynomial. These results are useful for a generalized numerical semigroup for which we do not know its set of holes. If the set of holes is known then it is simpler to verify the previous facts, as stated by the following:

Proposition 7.1.11. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup generated by the finite set A. Let $\boldsymbol{v} \in \mathbb{N}^d$. Then:*

1. *$\boldsymbol{v} \in S$ if and only if $\boldsymbol{v} \notin \mathrm{H}(S)$.*

2. *$\boldsymbol{v}$ is a minimal generators of S if and only if $\boldsymbol{v} - \boldsymbol{a} \in \mathrm{H}(S)$ for all $\boldsymbol{a} \in A$ with $\boldsymbol{a} \leq \boldsymbol{v}$ with respect to the natural partial order in $\mathbb{N}^d$.*

Proof. The first statement is trivial. The second one follows from the fact that $\mathbf{v}$ is not a minimal generator of S if and only if $\mathbf{v} = \mathbf{a} + \mathbf{b}$ with $\mathbf{a}, \mathbf{b} \in S$. □

An interesting combinatoric property involving the number of generalized numerical semigroups in $\mathbb{N}^d$ of a given genus is provided in [22]. Recall that if $A \subseteq \mathbb{N}^d$ we denote by $\mathrm{Span}_{\mathbb{R}}(A)$ the $\mathbb{R}$-vector space spanned by the elements of A. We have provided the following proposition in a previous chapter:

Proposition 7.1.12 ([22], Proposition 5.2)**.** *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and $\mathrm{H}(S)$ be the set of its holes. Then $\mathrm{Span}_{\mathbb{R}}(\mathrm{H}(S))$ is a coordinate linear space.*

We will use the Notation 4.3.2.

Theorem 7.1.13. [22, Proposition 5.3] *Let $g \in \mathbb{N}$ and consider the following polynomial:*

$$F_g(d) = \sum_{i=1}^{g} N_{g,i}^{(i)} \binom{d}{i}$$

Then $F_g(d)$ is a degree g polynomial in $\mathbb{Q}[d]$ and $F_g(d) = N_{g,d}$.

7.2 An algorithm generating all generalized numerical semigroups of a given genus

Now we write a complete algorithm generating all generalized numerical semigroups in $\mathbb{N}^d$ of a given genus g. Recall the min ideas of the algorithm: starting from the trivial generalized numerical semigroup N^d of genus 0, whose generators are $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$, the standard basis vectors of the vector space $\mathbb{R}^d$, we produce all generalized numerical semigroups of genus k from all ones of genus $k-1$, for k to 1 up to g. This is the procedure: fixed a relaxed monomial order $\prec$, from all generalized numerical semigroups S of genus $k-1$, if $\mathbf{g}_1, \ldots, \mathbf{g}_m$ are the effective generators of S with respect to

$\prec$ we produce the generalized numerical semigroups $S \setminus \{\mathbf{g}_1\}, \ldots, S \setminus \{\mathbf{g}_m\}$ whose genus is k. In this way all generalized numerical semigroups of genus k are produced without redundancy.

Algorithm 1: Algorithm for computing $N_{g,d}$

Data: Two integers $g, d \in \mathbb{N}$ and a relaxed monomial order $\prec$.
Result: $N_{g,d}$

1 $G = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d\}$, $S_{0,d} = \{(\mathbb{N}^d, G)\}$, $N_{0,d} = 1$, $\mathbf{F}_\prec(\mathbb{N}^d) = \mathbf{0}$, $\mathrm{H}(\mathbb{N}^d) = \emptyset$.
for $i = 0, g$ **do**
2     $S_{i,d} = \{(S^{(j)}, A^{(j)}) \mid j = 1, \ldots, N_{i,d}\}$, where $\langle A^{(j)} \rangle = S^{(j)}$, $|A^{(j)}| < \infty$.
    $N_{i+1,d} = 0$.
    for $j = 1, N_{i,d}$ **do**
3   From $(S^{(j)}, A^{(j)})$ find out $G^{(j)} = \mathbf{G}(S^{(j)})$ and $E^{(j)} = \mathbf{E}_\prec(S^{(j)})$.
        $N_{i+1,d} = N_{i+1,d} + |E^{(j)}|$.
4   **if** $i + 1 = g$ **then**
        return $N_{g,d}$.
        STOP
    $S_{i,d} = \{(S^{(j)}, G^{(j)}, E^{(j)}) \mid j = 1, \ldots, N_{i,d}\}$.
    $S_{i+1,d} = \emptyset$.
    for $j = 1, N_{i,d}$ **do**
        $\{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{|E^{(j)}|}\} = E^{(j)}$.
        for $k = 1, |E^{(j)}|$ **do**
            $S^{(j,k)} = S^{(j)} \setminus \{\mathbf{g}_k\}$.
            $\mathbf{F}_\prec(S^{(j,k)}) = \mathbf{g}_k$.
5   Build $A^{(j,k)}$, with $|A^{(j,k)}| < \infty$, such that $\langle A^{(j,k)} \rangle = S^{(j,k)}$.
            $\mathrm{H}(S^{(j,k)}) = \mathrm{H}(S^{(j)}) \cup \{\mathbf{g}_k\}$.
        $S_{i+1,d} = S_{i+1,d} \cup \{(S^{(j,k)}, A^{(j,k)})\}$.

Algorithm 1 reproduced in the table above is the pseudocode for the algorithm to compute $N_{g,d}$. We give a description of it:

- Line 1 contains the initial step of the algorithm, that is the root of $\mathcal{T}_\prec$: it starts from the trivial generalized numerical semigroup $\mathbb{N}^d$, that is finitely generated by the standard basis vectors and whose hole set is empty. Moreover it is the unique generalized numerical semigroup of genus 0, so $N_{0,d} = 1$. Even if $\mathbb{N}^d$ has not a Frobenius element it is convenient for the algorithm to fix $\mathbf{F}_{\prec}(S) = \mathbf{0}$.

- Line 2: $S_{i,d}$ is the set of all generalized numerical semigroups in $\mathbb{N}^d$ of genus i, we represent each one of these semigroups $S^{(j)}$ by a finite system of generators $A^{(j)}$.

- Line 3: $A^{(j)}$ is not in general the minimal system of generators for $S^{(j)}$. It is needed to find the minimal generators of $S^{(j)}$ and its effective generators with respect to $\prec$ in order to continue the process. For the first request a possible choice is to make use of Proposition 7.1.10 or Proposition 7.1.11, for the latter it suffices to compare each given generator with the Frobenius element, with respect to the order $\prec$.

- Lines 4 and successive: if $i+1 = g$ then we have computed $N_{g,d}$ and this stops the algorithm; if not, we have to produce each generalized numerical semigroup of genus $i+1$ from all generalized numerical semigroups of genus i applying Lemma 2.2.6.

- Lines 5: a finite system of generators $A^{(j,k)}$ for $S^{(j,k)}$ can be obtained by Proposition 7.1.1.

7.3 An algorithm to compute minimal generators of a generalized numerical semigroup from its set of holes

Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and let $\prec$ a relaxed monomial order. We suppose that the hole set $\mathrm{H}(S)$ is known and we want to find the set of minimal generators of S. Obviously if $S = \mathbb{N}^d$ then we know that S is generated by $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d\}$, the standard basis vectors of $\mathbb{R}^d$. If S is a proper subset of $\mathbb{N}^d$ we suggest an algorithm, based on the following result that can be found in a more general setting also in [27, Corollary 9].

Corollary 7.3.1. *Let $S \subset \mathbb{N}^d$ be a generalized numerical semigroup of genus g and let $\prec$ be a relaxed monomial order in $\mathbb{N}^d$. Suppose that $\mathrm{H}(S) = \{\boldsymbol{h}_1 \prec \boldsymbol{h}_2 \prec \ldots \prec \boldsymbol{h}_g\}$. Then*

- *$\boldsymbol{h}_1$ is a minimal generator of $\mathbb{N}^d$,*

- *for every $i \in \{2, \dots, g\}$ $\boldsymbol{h}_i$ is an effective generators, with respect to $\prec$, of the generalized numerical semigroup $S \setminus \{\boldsymbol{h}_1, \boldsymbol{h}_2, \dots, \boldsymbol{h}_{i-1}\}$.*

Proof. It follows easily from Lemma 2.2.5, since $\mathbf{h}_i$ is the Frobenius element with respect to $\prec$ of $S \setminus \{\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_i\}$. □

Algorithm 2: Algorithm to compute $\mathbf{G}(S)$ from the set $\mathrm{H}(S)$

Data: A set $H = \{\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_g\} \subseteq \mathbb{N}^d$, a relaxed monomial order $\prec$

Result: If $S = \mathbb{N}^d \setminus H$ is a generalized numerical semigroup, $\mathbf{G}(S)$ is computed

1 $H_{\preceq} = \{\mathbf{h}_{j_1} \preceq \mathbf{h}_{j_2} \prec \dots \prec \mathbf{h}_{j_g}\}$, $lH = \emptyset$, $G = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d\}$.

for $i \in \{1, \dots, g\}$ **do**

  if $\mathbf{h}_{j_i} \notin G$ **then**

    $\mathbb{N}^d \setminus H$ is not a generalized numerical semigroup.

    STOP

  $lH = lH \cup \{\mathbf{h}_{j_i}\}$.

  $G = G \setminus \{\mathbf{h}_{j_i}\}$

2   $A = \{\mathbf{g} + \mathbf{h}_{j_i} \mid \mathbf{g} \in G\} \cup \{2\mathbf{h}_{j_i}, 3\mathbf{h}_{j_i}\}$

  for $\mathbf{v} \in A$ **do**

3   **if** $\mathbf{v}$ is a minimal generator of $\mathbb{N}^d \setminus lH$ **then**

      $G = G \cup \{\mathbf{v}\}$.

return G

- Line 1. We arrange the elements of H with respect to $\prec$.
- Line 2. At this point $G \cup A$ is a finite set of generators of $\mathbb{N}^d \setminus lH$, by Proposition 7.1.1. From that, we have to produce the minimal set of generators of $\mathbb{N}^d \setminus lH$.
- Line 3: The elements in G are already minimal generators of $\mathbb{N}^d \setminus lH$, we have to check for the elements in A and this can be done by Proposition 7.1.11.

Observe that Algorithm 1 is similar to Algorithm 2: the first covers all the branches of the tree $\mathcal{T}_{\prec}$ with respect to $\prec$ (up to depth g) while the second covers the unique branch linking $\mathbb{N}^d$ with $S = \mathbb{N}^d \setminus H$, in the case S is actually a generalized numerical semigroup such that $\mathrm{H}(S) = H$. Moreover, Algorithm 2 can verify if a given finite set $H \subseteq \mathbb{N}^d$ has the property that $\mathbb{N}^d \setminus H$ is a generalized numerical semigroup. Another way to test if a given set is the set of holes of a submonoid in $\mathbb{N}^d$ is provided by the next result.

Proposition 7.3.2. *Let $H \subset \mathbb{N}^d$. Then $\mathbb{N}^d \setminus H$ is a semigroup if and only if for every $\mathbf{h} \in H$, $\mathbf{h} - \mathbf{x} \in H$ for all $\mathbf{x} \in \pi(\mathbf{h}) \setminus H$.*

Proof. ($\Rightarrow$) Suppose there exists $\mathbf{h} \in H$ such that $\mathbf{y} = \mathbf{h} - \mathbf{x} \notin H$ for some $\mathbf{x} \in \pi(\mathbf{h}) \setminus H$. Then $\mathbf{h} = \mathbf{x} + \mathbf{y}$ with $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d \setminus H$; a contradiction.
($\Leftarrow$) Suppose that $\mathbb{N}^d \setminus H$ is not a semigroup. Then there exists $\mathbf{h} \in H$ such that $\mathbf{h} = \mathbf{x} + \mathbf{y}$ with $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d \setminus H$. In particular, $\mathbf{x} \in \pi(\mathbf{h}) \setminus H$ and $\mathbf{h} - \mathbf{x} \notin H$, contradicting the hypothesis. □

Observe that $\pi(\mathbf{h})$ is a finite set for each $\mathbf{h} \in \mathbb{N}^d$. So if H is a finite set, the condition in the previous proposition is easy to test. In particular, it is also possible to obtain a procedure to test if a given finite set H is the set of holes of a generalized numerical semigroup, without using Algorithm 2.

7.4 An algorithm to compute the set of holes of a generalized numerical semigroup from a finite set of generators

Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and suppose that a finite set of generators is known (even if it is not minimal). A possible way to compute $\mathrm{H}(S)$ consists in considering the characterization of a set of generators for a generalized numerical semigroup and its consequences described in Theorem 2.1.10. In particular we remind that the set A of generators of a generalized numerical semigroup in $\mathbb{N}^d$ must fulfil the following two conditions:

1. For all $j = 1, 2, \ldots, d$ there exist $a_1^{(j)}\mathbf{e}_j, a_2^{(j)}\mathbf{e}_j, \ldots, a_{r_j}^{(j)}\mathbf{e}_j \in A$, $r_j \in \mathbb{N}\setminus\{0\}$, such that $\gcd(a_1^{(j)}, a_2^{(j)}, \ldots, a_{r_j}^{(j)}) = 1$ (that is, the elements $a_i^{(j)}$, $1 \leq i \leq r_j$, generate a numerical semigroup).
2. For every i, k, $1 \leq i < k \leq d$ there exist $\mathbf{x}_{ik}, \mathbf{x}_{ki} \in A$ such that $\mathbf{x}_{ik} = \mathbf{e}_i + n_i^{(k)}\mathbf{e}_k$ and $\mathbf{x}_{ki} = \mathbf{e}_k + n_k^{(i)}\mathbf{e}_i$ with $n_i^{(k)}, n_k^{(i)} \in \mathbb{N}$.

Furthermore, let S_j be the numerical semigroup generated by $\{a_1^{(j)}, a_2^{(j)}, \ldots, a_{r_j}^{(j)}\}$ and $F^{(j)} = \max\{0, F(S_j)\}$, for $j = 1, \ldots, d$. Let $\mathbf{v} = (v^{(1)}, v^{(2)}, \ldots, v^{(d)}) \in \mathbb{N}^d$ defined by:

$$v^{(j)} = \sum_{i \neq j}^{d} F^{(i)} n_i^{(j)} + F^{(j)}$$

Then $\mathrm{H}(S) \subseteq \pi(\mathbf{v})$.

Algorithm 3: Algorithm to compute $\mathrm{H}(S)$ from a finite set A such that $\langle A \rangle = S$

Data: A finite set $A \subseteq \mathbb{N}^d$

Result: If $S := \langle A \rangle$ is a generalized numerical semigroup, $\mathrm{H}(S)$ is computed

```
for j=1,d do
    Gather in a set A_j the element a ∈ ℕ such a e_j ∈ A .
    if S_j = ⟨A_j⟩ is a numerical semigroup then
        Compute the Frobenius numbers F^(j) of S_j
    else
        ⟨A⟩ is not a generalized numerical semigroup
        STOP
for i = 1,d do
    for k = 1,d, k ≠ i do
        Found n_i^(k) ∈ ℕ such that e_i + n_i^(k) e_k ∈ A
        if n_i^(k) does not exist then
            ⟨A⟩ is not a generalized numerical semigroup
            STOP
for j=1,d do
    v^(j) = Σ_{i≠j}^d F^(i) n_i^(j) + F^(j)
v = (v^(1), v^(2), ..., v^(d))
π(v) = {n ∈ ℕ^d | n ≤ v}
H(S) = ∅
for x ∈ π(v) do
    if x ∉ ⟨A⟩ then
        H(S) = H(S) ∪ {x}
```

A brief description of some points:

- Line 1: We check if the first condition of Theorem 2.1.10 is satisfied, otherwise A does not generate a generalized numerical semigroup.
- Line 2: We check if the second condition of Theorem 2.1.10 is satisfied, otherwise A does not generate a generalized numerical semigroup.
- Line 3: We compute all coordinates of the vector $\mathbf{v}$ such that $\mathrm{H}(S) \subseteq \pi(\mathbf{v}) = \{\mathbf{n} \in \mathbb{N}^d \mid \mathbf{n} \leq \mathbf{v}\}$.

- Line 4: Observe that $\pi(\mathbf{v})$ is a finite set, so we have to look for the holes of the semigroup among a finite number of elements. Proposition 7.1.7 gives one of the possible method to verify if $\mathbf{x} \notin \langle A \rangle$.

For the sake of completeness we cite that when $S \subseteq \mathbb{N}^d$ is a finitely generated monoid it is possible to give a finite presentation of the set $\mathbb{N}^d \setminus S$ even if this set is not finite, and there exist algorithms to obtain such a presentation (see [29] and [31]).

7.5 The tree of all generalized numerical semigroups of a given genus: an alternative procedure to compute all generalized numerical semigroups of a given genus

The algorithm described in section 2 allows to obtain all generalized numerical semigroups in $\mathbb{N}^d$ up to genus g, in particular to compute the cardinality of $\mathcal{S}_{g,d}$. The peculiarity of this procedure is that it is needed to compute not only all generalized numerical semigroups of genus g, but also all generalized numerical semigroup of genus $g' < g$. In fact it produces the semigroup tree $T_{\prec}$ up to genus g, and such a tree contains generalized numerical semigroups of different genus. In the following section we define a rooted tree that contains all generalized numerical semigroups in $\mathbb{N}^d$ of fixed genus g, in order to compute the set $S_{g,d}$ without considering all generalized numerical semigroups of genus $g' < g$. Such a rooted tree is inspired by the *ordinarization tranform* for numerical semigroups, as stated in [7].

Proposition 7.5.1. *Let $\preceq$ be a relaxed monomial order in $\mathbb{N}^d$ and $\mathbf{s} \in \mathbb{N}^d$. Suppose that the set $\{\mathbf{t} \in \mathbb{N}^d \mid \mathbf{t} \preceq \mathbf{s}\}$ is finite. Then the set $S = \{\mathbf{x} \in \mathbb{N}^d \mid \mathbf{s} \prec \mathbf{x}\} \cup \{\mathbf{0}\}$ is a generalized numerical semigroup.*

Proof. By hypothesis $\mathbb{N}^d \setminus S$ is finite. Moreover if $\mathbf{x}, \mathbf{y} \in S$ then $\mathbf{s} \preceq \mathbf{x}$ and $\mathbf{s} \preceq \mathbf{y}$, so $\mathbf{s} \preceq \mathbf{x} + \mathbf{y}$ since $\preceq$ is a relaxed monomial order. □

Definition 7.5.2. Let $\preceq$ be a relaxed monomial order and let $S \subseteq \mathbb{N}^d$ be a monoid satisfying the hypothesis of the above Proposition. We call S an *ordinary* generalized numerical semigroup with respect to $\preceq$.
Let $\{\mathbf{0} = \mathbf{s}_0 \preceq \mathbf{s}_1 \preceq \cdots \preceq \mathbf{s}_g\}$ the list of the first $g+1$ elements in $\mathbb{N}^d$, ordered by $\preceq$. We define $R_{g,d}(\preceq) = \{\mathbf{x} \in \mathbb{N}^d \mid \mathbf{s}_g \prec \mathbf{x}\} \cup \{\mathbf{0}\}$, that is the ordinary generalized numerical semigroup in $\mathbb{N}^d$ of genus g, with respect to $\preceq$.

Observe that the previous definition of ordinary generalized numerical semigroup depends strongly on the relaxed monomial order defined: different relaxed monomial orders define different ordinary generalized numerical semigroup of a given genus.

Proposition 7.5.3. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and $\prec$ be a relaxed monomial order in $\mathbb{N}^d$. Then $S \setminus \{\mathbf{m}_\prec(S)\}$ is a generalized numerical semigroup.*

Proof. Obviously $S \setminus \{\mathbf{m}_\prec(S)\}$ has finite complement in $\mathbb{N}^d$. Let $\mathbf{s}_1, \mathbf{s}_2 \in S \setminus \{\mathbf{m}_\prec(S)\}$, then $\mathbf{s}_1 + \mathbf{s}_2 \in S$ and $\mathbf{s}_1 + \mathbf{s}_2 \neq \mathbf{m}_\prec(S)$, since $\mathbf{m}_\prec(S) \prec \mathbf{s}_1, \mathbf{s}_2$, hence $\mathbf{m}_\prec(S) \prec \mathbf{s}_1 + \mathbf{s}_2$. □

Proposition 7.5.4. *Let $S \subseteq \mathbb{N}^d$ be a generalized numerical semigroup, $\prec$ a relaxed monomial order in $\mathbb{N}^d$ and suppose that S is not ordinary with respect to $\prec$. Then $T = S \cup \{\mathbf{F}_\prec(S)\} \setminus \{\mathbf{m}_\prec(S)\}$ is a generalized numerical semigroup with $\mathbf{m}_\prec(S) \prec \mathbf{m}_\prec(T)$ and $\mathbf{F}_\prec(S) \succ \mathbf{F}_\prec(T)$.*

Proof. It follows from Proposition 7.5.3 considering that $\mathbf{m}_\prec(S) \prec \mathbf{F}_\prec(S)$, so $\mathbf{m}_\prec(S \cup \{\mathbf{F}_\prec(S)\}) = \mathbf{m}_\prec(S)$. The second statement is trivial. □

Let $\mathcal{S}_d$ be the set of all generalized numerical semigroups in $\mathbb{N}^d$. As for numerical semigroup we define the *ordinarization transform* with respect to $\prec$:

$$\mathcal{A}_\prec : \mathcal{S}_d \to \mathcal{S}_d \quad \text{defined by} \quad \mathcal{A}_\prec(S) = S \cup \{\mathbf{F}_\prec(S)\} \setminus \{\mathbf{m}_\prec(S)\}$$

If $S \in \mathcal{S}_d$ and $S \neq R_{g,d}(\prec)$ then, by Proposition 7.5.4, there exists $n \in \mathbb{N}$ such that $\mathcal{A}^n_\prec(S) = R_{g,d}(\prec)$. Moreover if $S \in \mathcal{S}_{g,d}$ then $\mathcal{A}_\prec(S) \in \mathcal{S}_{g,d}$.

If $S \subseteq \mathbb{N}^d$ is a generalized numerical semigroup recall that $\mathrm{SG}(S) = \{\mathbf{h} \in \mathrm{H}(S) \mid 2\mathbf{h} \in S, \mathbf{h} + \mathbf{s} \in S \text{ for all } \mathbf{s} \in S\}$ and its elements are called *special gaps* of S. Observe that $\mathrm{H}(S)$ and $\mathrm{SG}(S)$ do not depend on fixed relaxed monomial order. Moreover we have seen that $S \cup \{\mathbf{h}\}$ is a generalized numerical semigroup if and only if $\mathbf{h} \in \mathrm{SG}(S)$.

Lemma 7.5.5. *Let $S \subseteq \mathbb{N}^d$, $\prec$ be a relaxed monomial order and suppose S not ordinary with respect to $\prec$. Let $T = S \cup \{\mathbf{F}_\prec(S)\} \setminus \{\mathbf{m}_\prec(S)\}$. Then $m_\prec(S) \in \mathrm{SG}(T)$ and $\mathbf{F}_\prec(S)$ is an effective generator of $T \cup \{\mathbf{m}_\prec(S)\}$, with respect to $\prec$.*

Proof. Observe that $T \cup \{\mathbf{m}_\prec(S)\} = S \cup \{\mathbf{F}_\prec(S)\}$ is a generalized numerical semigroup, so $\mathbf{m}_\prec(S) \in \mathrm{SG}(T)$. Moreover $\mathbf{F}_\prec(S)$ is a minimal generator of $T \cup \{\mathbf{m}_\prec(S)\}$ and $\mathbf{F}_\prec(S) \succ \mathbf{F}_\prec(T)$. □

Lemma 7.5.6. *Let $T \subseteq \mathbb{N}^d$ be a generalized numerical semigroup and $\prec$ a relaxed monomial order in $\mathbb{N}^d$. Suppose that there exists $\boldsymbol{h} \in \mathrm{SG}(T)$ with $\boldsymbol{h} \prec \mathbf{m}_{\prec}(T)$ and let $\boldsymbol{x}$ be an effective generator of $T \cup \{\boldsymbol{h}\}$, with respect to $\prec$. Consider the semigroup $S = (T \cup \{\boldsymbol{h}\}) \setminus \{\boldsymbol{x}\}$, then $\mathcal{A}_{\prec}(S) = T$.*

Proof. Let S be the semigroup as below. We have $T = (S \cup \{\mathbf{x}\}) \setminus \{\mathbf{h}\}$, so we have to prove that $\mathbf{h} = \mathbf{m}_{\prec}(S)$ and $\mathbf{x} = \mathbf{F}_{\prec}(S)$. Since $\mathbf{h} \prec \mathbf{m}_{\prec}(T)$, then $\mathbf{h} \prec \mathbf{t}$ for every $\mathbf{t} \in T$, hence $\mathbf{h} = \mathbf{m}_{\prec}(S)$. Furthermore, since $\mathbf{x}$ is an effective generator of $T \cup \{\mathbf{m}_{\prec}(S)\}$, then $\mathbf{x} \succ \mathbf{F}_{\prec}(T \cup \{\mathbf{h}\}) = \mathbf{F}_{\prec}(S \cup \{\mathbf{x}\})$, so $\mathbf{x} = \mathbf{F}_{\prec}(S)$. □

Definition 7.5.7. Let $g \in \mathbb{N}$ and let $\mathcal{S}_{g,d}$ be the set of all generalized numerical semigroups in $\mathbb{N}^d$ of genus g. Let us fix a relaxed monomial order $\prec$ in $\mathbb{N}^d$. We define the oriented graph $\mathcal{T}^d_{g,\prec} = (\mathcal{S}_{g,d}, \mathcal{V})$, where $\mathcal{V}$ is the set of all couples $(S, \mathcal{A}_{\prec}(S))$.

Theorem 7.5.8. *Let $g \in \mathbb{N}$. The graph $\mathcal{T}^d_{g,\prec}$ is a rooted tree where the root is the semigroup $R_{g,d}(\prec)$. Moreover if $T \in \mathcal{S}_{g,d}$ then the sons of T are the semigroups $T_{\mathbf{h},\boldsymbol{x}} = (T \cup \{\boldsymbol{h}\}) \setminus \{\boldsymbol{x}\}$ for all $\boldsymbol{h} \in \mathrm{SG}(T)$ and for all $\boldsymbol{x} \in \mathbf{E}_{\prec}(T \cup \{\boldsymbol{h}\})$ with $\boldsymbol{x} \neq \mathbf{m}_{\prec}(T \cup \{\boldsymbol{h}\})$.*

Proof. Let $T \in \mathcal{S}_{g,d}$, we define the following sequence:

- $T_0 = T$.
- $T_{i+1} = \begin{cases} \mathcal{A}_{\prec}(T_i) & \text{if } T_i \neq R_{g,d}(\prec) \\ R_{g,d}(\prec) & \text{otherwise} \end{cases}$

in particular $T_i = \mathcal{A}^i_{\prec}(T)$ for all i. We know that there exists a minimum nonnegative integer k such that $T_k = \mathcal{A}^k_{\prec}(T) = R_{g,d}(\prec)$. So the edges $(T_0, T_1), (T_1, T_2), \ldots, (T_{k-1}, T_k)$ provide a path from T to the ordinary numerical semigroup of genus g.
Let $T_{\mathbf{h},\mathbf{x}}$ be the semigroup as described above. By Lemma 7.5.6 every pair $(T_{\mathbf{h},\mathbf{x}}, T)$ is an edge of $\mathcal{T}^d_{g,\prec}$ for every possible choice of $\mathbf{h}$ and $\mathbf{x}$, so every semigroup $T_{\mathbf{h},\mathbf{x}}$ is a son of T and these semigroups are exactly all the sons of T by Lemma 7.5.5. □

Corollary 7.5.9. *Let $T \in \mathcal{S}_{g,d}$. T is a leaf in $\mathcal{T}^d_{g,\prec}$ if and only if for all $\boldsymbol{h} \in \mathrm{SG}(T)$ either $\boldsymbol{h} \succ \mathbf{m}_{\prec}(T)$ or $\mathbf{E}_{\prec}(T \cup \{\boldsymbol{h}\}) \subseteq \{\boldsymbol{h}\}$.*

Theorem 7.5.8 allows us to write another algorithm to produce all generalized numerical semigroups in $\mathbb{N}^d$ of genus g and the previous corollary gives us the condition to stop computation.

Algorithm 4: Algorithm for computing the set $\mathcal{S}_{g,d}$

Data: Two integers $g, d \in \mathbb{N}$ and a relaxed monomial order $\prec$.
Result: $\mathcal{S}_{g,d}$
Compute $R_{g,d}(\prec)$.
$\mathcal{S}_{g,d} = \{R_{g,d}(\prec)\}, \mathcal{L} = \{R_{g,d}(\prec)\}$.
1 **while** $\exists S \in L$ such that $\mathrm{SG}(S) \cap \{\boldsymbol{h} \in \mathrm{H}(S) \mid \boldsymbol{h} \prec \mathbf{m}_{\prec}(S)\} \neq \emptyset$ **do**
  $\mathcal{I} = \emptyset$
  for $S \in \mathcal{L}$ **do**
    if $\mathrm{SG}(S) \cap \{\boldsymbol{h} \in \mathrm{H}(S) \mid \boldsymbol{h} \prec \mathbf{m}_{\prec}(S)\} \neq \emptyset$ **then**
      $\mathcal{R} = \emptyset$
2   **for** $\boldsymbol{h} \in \mathrm{SG}(S) \cap \{\boldsymbol{h} \in \mathrm{H}(S) \mid \boldsymbol{h} \prec \mathbf{m}_{\prec}(S)\}$ **do**
        $\mathcal{R} = \mathcal{R} \cup \{S \cup \{\mathbf{h}\}\}$
3   **for** $T \in \mathcal{R}$ **do**
        for $\boldsymbol{x} \in \mathbf{E}_{\prec}(T)$ *with* $\boldsymbol{x} \neq \mathbf{m}_{\prec}$ **do**
          $\mathcal{I} = \mathcal{I} \cup \{T \setminus \{x\}\}$
  $\mathcal{S}_{g,d} = \mathcal{S}_{g,d} \cup \mathcal{I}$
4   $\mathcal{L} = \mathcal{I}$
return $\mathcal{S}_{g,d}$

- Line 1: The algorithm stops when all computed semigroups are leaves of $\mathcal{T}^d_{g,\prec}$. We gather in $\mathcal{L}$ the semigroups for which we have to do computations at each step.
- Line 2: For each semigroup S of the current step we have to compute its sons, and for this aim we need all semigroups $S \cup \{\mathbf{h}\}$ with $\mathbf{h} \in \mathrm{SG}(S)$ and smaller than $\mathbf{m}_{\prec}(S)$ with respect to $\prec$.
- Line 3: For each semigroup T computed in the previous line we compute $T \setminus \{\mathbf{x}\}$ for all $\mathbf{x} \in \mathbf{E}_{\prec}(T)$. These are the sons of all semigroups in $\mathcal{L}$, that we gather in $\mathcal{I}$.
- Line 4: In the next step we have to repeat the same procedure considering the semigroups in $\mathcal{I}$.

Example 7.5.10. Let $\prec$ be the lexicographic order in $\mathbb{N}^2$. Consider the semigroup $R_{3,2}(\prec) = \mathbb{N}^2 \setminus \{(0,1),(0,2),(0,3)\}$, we compute its sons in $\mathcal{T}^2_{3,\prec}$.

The set of special gaps less than $\mathbf{m}_{\prec}(R_{3,2}(\prec)) = (0,4)$ is $\{(0,2),(0,3)\}$. So we consider:

- $T_{(0,2)} = R_{3,2}(\prec) \cup \{(0,2)\} = \mathbb{N}^2 \setminus \{(0,1),(0,3)\}$, with $\mathbf{E}_{\prec}(S) = \{(1,0),(1,1),(0,5)\}$.
- $T_{(0,3)} = R_{3,2}(\prec) \cup \{(0,3)\} = \mathbb{N}^2 \setminus \{(0,1),(0,2)\}$, with $\mathbf{E}_{\prec}(S) = \{(1,0),(1,1),(0,3),(1,2),(0,4),(0,5)\}$

So the sons of $R_{3,2}(\prec)$ are the following:

- $S_1 = T_{(0,2)} \setminus \{(1,0)\} = \mathbb{N}^2 \setminus \{(0,1),(0,3),(1,0)\}$.
- $S_2 = T_{(0,2)} \setminus \{(1,1)\} = \mathbb{N}^2 \setminus \{(0,1),(0,3),(1,1)\}$.
- $S_3 = T_{(0,2)} \setminus \{(0,5)\} = \mathbb{N}^2 \setminus \{(0,1),(0,3),(0,5)\}$.
- $S_4 = T_{(0,3)} \setminus \{(1,0)\} = \mathbb{N}^2 \setminus \{(0,1),(0,2),(1,0)\}$.
- $S_5 = T_{(0,3)} \setminus \{(1,1)\} = \mathbb{N}^2 \setminus \{(0,1),(0,2),(1,1)\}$.
- $S_6 = T_{(0,3)} \setminus \{(1,2)\} = \mathbb{N}^2 \setminus \{(0,1),(0,2),(1,2)\}$.
- $S_7 = T_{(0,3)} \setminus \{(0,4)\} = \mathbb{N}^2 \setminus \{(0,1),(0,2),(0,4)\}$.
- $S_8 = T_{(0,2)} \setminus \{(0,5)\} = \mathbb{N}^2 \setminus \{(0,1),(0,2),(0,5)\}$.

Observe that $S_1, S_2, S_3, S_6, S_7, S_8$ are leaves in $\mathcal{T}^2_{3,\prec}$. If we continue the procedure we obtain all generalized numerical semigroups of genus g.
The sons of S_4 are:

- $S_9 = S_4 \cup \{(0,2)\} \setminus \{(1,1)\} = \mathbb{N}^2 \setminus \{(0,1),(1,0),(1,1)\}$.
- $S_{10} = S_4 \cup \{(0,2)\} \setminus \{(2,1)\} = \mathbb{N}^2 \setminus \{(0,1),(1,0),(2,1)\}$.
- $S_{11} = S_4 \cup \{(0,2)\} \setminus \{(1,2)\} = \mathbb{N}^2 \setminus \{(0,1),(1,0),(1,2)\}$.
- $S_{12} = S_4 \cup \{(0,2)\} \setminus \{(2,0)\} = \mathbb{N}^2 \setminus \{(0,1),(1,0),(2,0)\}$
- $S_{13} = S_4 \cup \{(0,2)\} \setminus \{(1,1)\} = \mathbb{N}^2 \setminus \{(0,1),(1,0),(3,0)\}$

The sons of S_5 are:

- $S_{14} = S_5 \cup \{(0,2)\} \setminus \{(2,1)\} = \mathbb{N}^2 \setminus \{(0,1),(1,1),(2,1)\}$.

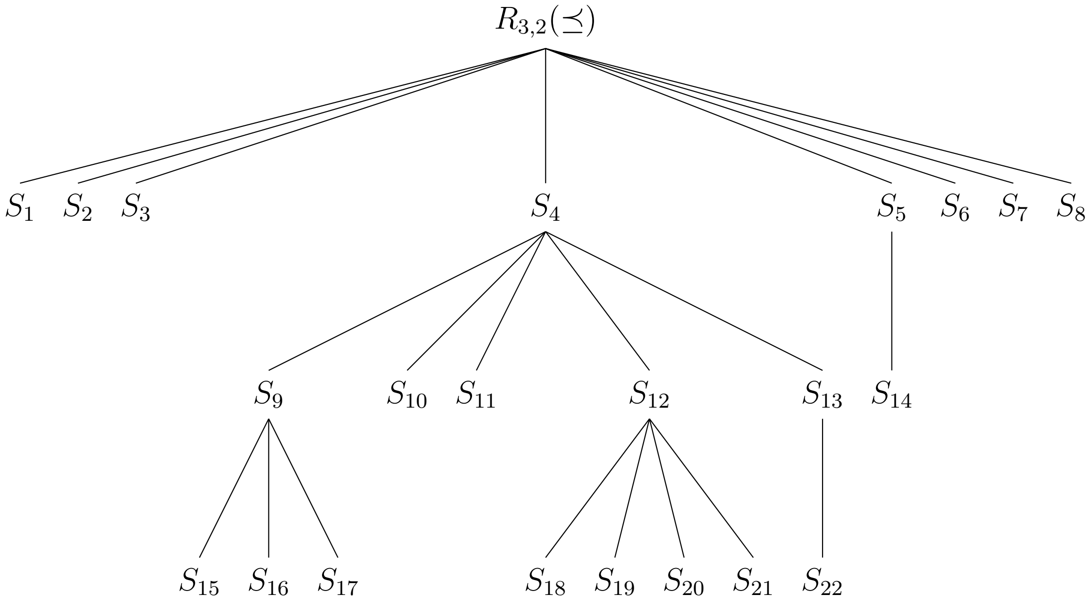


Figure 7.1: The tree $\mathcal{T}^2_{3,\preceq}$, with $\preceq$ the lexicographic order.

The semigroups S_{10}, S_{11}, S_{14} are leaves in $\mathcal{T}^2_{3,\prec}$.
The sons of S_9 are:

- $S_{15} = S_9 \cup \{(0,1)\} \setminus \{(2,0)\} = \mathbb{N}^2 \setminus \{(1,0),(1,1),(2,0)\}$.
- $S_{16} = S_9 \cup \{(0,1)\} \setminus \{(3,0)\} = \mathbb{N}^2 \setminus \{(1,0),(1,1),(3,0)\}$.
- $S_{17} = S_9 \cup \{(0,1)\} \setminus \{(1,2)\} = \mathbb{N}^2 \setminus \{(1,0),(1,1),(1,2)\}$.

The sons of S_{12} are:

- $S_{18} = S_{12} \cup \{(0,1)\} \setminus \{(2,1)\} = \mathbb{N}^2 \setminus \{(1,0),(2,0),(2,1)\}$.
- $S_{19} = S_{12} \cup \{(0,1)\} \setminus \{(3,0)\} = \mathbb{N}^2 \setminus \{(1,0),(2,0),(3,0)\}$.
- $S_{20} = S_{12} \cup \{(0,1)\} \setminus \{(4,0)\} = \mathbb{N}^2 \setminus \{(1,0),(2,0),(4,0)\}$.
- $S_{21} = S_{12} \cup \{(0,1)\} \setminus \{(5,0)\} = \mathbb{N}^2 \setminus \{(1,0),(2,0),(5,0)\}$.

The sons of S_{13} are:

- $S_{22} = S_{13} \cup \{(0,1)\} \setminus \{(5,0)\} = \mathbb{N}^2 \setminus \{(1,0),(3,0),(5,0)\}$.

By Theorem 7.5.8 it is possible to produce all generalized numerical semigroups of genus g starting from the ordinary generalized numerical semigroup of genus g with respect to a fixed relaxed monomial order. This procedure works as in the previous example and it can avoid to consider all generalized numerical semigroups of genus less than g, like in the standard algorithm described for instance in [22].

7.6 Some computational results

Now we provide some numerical data obtained by implementation of the previous algorithms. In this section we are interested in counting the number of generalized numerical semigroups in $\mathbb{N}^d$ of given genus. A first implementation has been made in my master degree thesis using the computer algebra system REDUCE [36] and considering Algorithm 1. Successively I have utilized a more appropriate software, that is GAP [24], using Algorithm 4. In fact, using GAP, one can take advantage of some tools and functions of the package `Numericalsgps` (see [17]), that allows to deal with numerical semigroups and affine semigroups in a simpler and more efficient way. See the Appendix for other details. Let, as above, $N_{g,d}$ be the number of all generalized numerical semigroups in $\mathbb{N}^d$ of genus g. If $d = 1$, we know that the sequence $\{N_{g,1}\}$ has a Fibonacci-like behaviour (see [47]), as conjectured by M. Bras-Amóros in [5]. That conjecture was justified by the computation of the values $N_{g,1}$ for $g = 1$ up to $g = 50$. We question if also the sequence $\{N_{g,d}\}$ with $d > 1$ has a particular behaviour. For $d = 2$ the values computed are contained contained Table 7.1.

We computed also some values of $N_{g,d}^{(r)}$ in order to build the polynomial $F_g(d)$ of Theorem 7.1.13. The polynomials

- $F_1(d) = d$
- $F_2(d) = \frac{3}{2}d^2 + \frac{1}{2}d$
- $F_3(d) = \frac{5}{3}d^3 + \frac{5}{2}d^2 - \frac{1}{6}d$

are given in [22].
Other values of $N_{g,d}^{(r)}$ have been computed, in particular:

Table 7.1: Computational results for $N_{g,2}$

g	$N_{g,2}$	$N_{g-1,2} + N_{g-2,2}$	$\frac{N_{g-1,2}+N_{g-2,2}}{N_{g,2}}$	$\frac{N_{g,2}}{N_{g-1,2}}$
0	1			
1	2			2
2	7	3	0,4285714286	3,5
3	23	9	0,3913043478	3,2857142857
4	71	30	0,4225352113	3,0869565217
5	210	94	0,4476190476	2,9577464789
6	638	281	0,4404388715	3,0380952381
7	1894	848	0,4477296727	2,9686520376
8	5570	2532	0,4545780969	2,9408658923
9	16220	7464	0,4601726264	2,9120287253
10	46898	21790	0,4646253572	2,8913686806
11	134856	63118	0,4680399834	2,8755170796
12	386354	181754	0,4704338508	2,8649374147
13	1102980	521210	0,4725470997	2,8548429678
14	3137592	1489334	0,4746742088	2,8446499483
15	8892740	4240572	0,4768577514	2,8342563342
16	25114649	12030332	0,4790165294	2,8241744389
17	70686370	34007389	0,4811024954	2,8145473982
18	198319427	95801019	0,4830642184	2,8056247194
19	554813870	269005797	0,4848577362	2,797577012
20	1548231268	753133297	0,4864475434	2,7905417505
21	4310814033	2103045138	0,4878533664	2,7843476114

$N_{4,2}^{(2)} = 57$	$N_{5,2}^{(2)} = 186$	$N_{6,2}^{(2)} = 592$	$N_{7,2}^{(2)} = 1816$	$N_{8,2}^{(2)} = 5436$	$N_{9,2}^{(2)} = 15984$
$N_{4,3}^{(3)} = 100$	$N_{5,3}^{(3)} = 621$	$N_{6,3}^{(3)} = 3230$	$N_{7,3}^{(3)} = 15371$	$N_{8,3}^{(3)} = 69333$	$N_{9,3}^{(3)} = 301425$
$N_{4,4}^{(4)} = 41$	$N_{5,4}^{(4)} = 672$	$N_{6,4}^{(4)} = 6321$	$N_{7,4}^{(4)} = 47432$	$N_{8,4}^{(4)} = 315393$	$N_{9,4}^{(4)} = 1945238$
	$N_{5,5}^{(5)} = 196$	$N_{6,5}^{(5)} = 4745$	$N_{7,5}^{(5)} = 63205$	$N_{8,5}^{(5)} = 648115$	$N_{9,5}^{(5)} = 5742670$
		$N_{6,6}^{(6)} = 1057$	$N_{7,6}^{(6)} = 35480$	$N_{8,6}^{(6)} = 637312$	$N_{9,6}^{(6)} = 8584915$
			$N_{7,7}^{(7)} = 6322$	$N_{8,7}^{(7)} = 281099$	$N_{9,7}^{(7)} = 6563802$
				$N_{8,8}^{(8)} = 41393$	$N_{9,8}^{(8)} = 2355792$
					$N_{9,9}^{(9)} = 293608$

In [5] the following known values are shown:

$$N_{4,1}^{(1)} = 7 \quad N_{5,1}^{(1)} = 12 \quad N_{6,1}^{(1)} = 23 \quad N_{7,1}^{(1)} = 39 \quad N_{8,1}^{(1)} = 67 \quad N_{9,1}^{(1)} = 118$$

so the following polynomials can be expressed :

- $F_4(d) = \frac{41}{24}d^4 + \frac{77}{12}d^3 - \frac{65}{24}d^2 + \frac{19}{12}d$
- $F_5(d) = \frac{49}{30}d^5 + \frac{35}{3}d^4 - \frac{22}{3}d^3 + \frac{53}{6}d^2 - \frac{14}{5}d$
- $F_6(d) = \frac{1057}{720}d^6 + \frac{841}{48}d^5 - \frac{1045}{144}d^4 + \frac{563}{48}d^3 + \frac{148}{45}d^2 - \frac{15}{4}d$
- $F_7(d) = \frac{3161}{2520}d^7 + \frac{8257}{360}d^6 + \frac{127}{18}d^5 - \frac{1735}{72}d^4 + \frac{31757}{360}d^3 - \frac{15091}{180}d^2 + \frac{577}{21}d$
- $F_8(d) = \frac{41393}{40320}d^8 + \frac{38921}{1440}d^7 + \frac{128099}{2880}d^6 - \frac{9227}{62}d^5 + \frac{1875151}{5760}d^4 - \frac{467041}{1440}d^3 + \frac{1234271}{10080}d^2 - \frac{25}{24}d$
- $F_9(d) = \frac{5243}{6480}d^9 + \frac{14767}{504}d^8 + \frac{58399}{540}d^7 - \frac{203159}{720}d^6 + \frac{301811}{540}d^5 - \frac{24961}{144}d^4 - \frac{3909443}{6480}d^3 + \frac{536093}{840}d^2 - \frac{1427}{9}d$

Table 7.2: Some values of $N_{g,d}$ for $d > 2$

g	$N_{g,3}$	$N_{g,4}$	$N_{g,5}$	$N_{g,6}$	$N_{g,7}$	$N_{g,8}$	$N_{g,9}$	$N_{g,10}$
1	3	4	5	6	7	8	9	10
2	15	26	40	57	77	100	126	155
3	67	146	270	449	693	1012	1416	1915
4	292	811	1810	3512	6181	10122	15681	23245
5	1215	4320	11686	26538	53361	98096	168336	273522
6	5075	22885	74685	197960	453922	935426	1775943	3159590
7	20936	119968	472430	1461084	3818501	8815672	18505065	36024450
8	85842	625609	2973105	10725499	31932733	82542263	191448588	407552845
9	349731	3247314	18643540	78488473	266223972	770328304	1973498062	4591979390
10	1418323	16800886						
11	5731710	86739337						
12	23100916	447283982						
13	92882954	2304942650						
14	372648740							

Remark 7.6.1. The values of Table 7.1 and the values of Table 7.2 up to dimension 9 have been effectively computed, using Algorithm 4. Of course, they coincide with the values given by the polynomials $F_g(d)$ mentioned above, in the cases this value exists. As a consequence of our computations we observe that we are able to compute the number of generalized numerical semigroups with genus up to 9 for any dimension (using the polynomial given by Theorem 7.1.13). The last column of the table contains the values for dimension 10.

The values for dimension 3, up to genus 13 already appear in [26, Table 3], and these computations confirm them. Also the values for dimension 2 up to genus 18 appear in [26, Table 3] and are the same, but for genus 18 the value is different; the values for genus 19, 20 and 21 are new.

The numerical data collected is for the moment not sufficient to let us state any conjecture with some confidence, as happened with Bras-Amorós when she realized that the sequence of the number of numerical semigroups counted by genus had a Fibonacci-like behaviour. Since the algorithms and the implementations have still space to be improved, obtaining more numerical data may be seen as an active goal.

All the above computations were made by the system in alhambra.ugr.es, thanks to the the Centro de Servicios de Informática y Redes de Comunicaciones (CSIRC), Universidad de Granada, that provided the machine and the computing time. Big thanks go to Pedro García-Sánchez for arranging these computations in that machine.

We conclude with some tests on generalized Wilf's conjecture. Using these compuational tools we have seen that conjecture 4.1.8 is satisfied by all generalized numerical semigroups in $\mathbb{N}^2$ up to genus $g = 13$, and in $\mathbb{N}^3$ up to genus $g = 10$. Moreover the function `RandomAffineSemigroupWithGenusAndDimension` allows to produce a random generalized numerical semigroup in $\mathbb{N}^d$ of genus g, so it is possible to make a random test of the conjecture. Considering for each value of the genus g a random generalized numerical semigroup of genus g, from genus $g = 1$ up to $g = 500$ we have checked that different random tests give a positive answer for conjecture 4.1.8 in $\mathbb{N}^d$ from $d = 2$ up to $d = 5$. We summarize the computational positive answers of the conjecture in the following table:

	genus	Test
$\mathbb{N}^2$	1 to 13 1 to 500	All semigroups Random test
$\mathbb{N}^3$	1 to 10 1 to 500	All semigroups Random test
$\mathbb{N}^4$	1 to 500	Random test
$\mathbb{N}^5$	1 to 500	Random test

Considering the number of such semigroups the previous test confirm a positive answer to conjecture 4.1.8 for a wide number of generalized numerical semigroups.

Appendix

This appendix is thought for providing the implementation of some algorithms described in this work. For implementation we use the computer algebra software GAP [24]. Furthermore the GAP package `numericalsgps` [17] offers many useful tools and functions to deal with numerical semigroups and affine semigroups. The documentation produced for these functions can be consulted in the GAP help system or in the manual of the package. We took advantage of such tools, that are both standard functions of GAP or some specific of `numericalsgps`. For instance, for what concerns computation for affine semigroups, the function `MinimalGenerators` of the package `numericalsps` allows to compute a minimal set of generators of an affine semigroup if a finite set of generators is known (using a different idea to that described in Proposition 7.1.10). The implementations of algorithms 2 and 3 of the last chapter are contained in the package `numericalsgps`, in particular in the files `affine.*` and `afine-def.*`. We give here the GAP code of some algorithms not contained in the package. Almost all the examples and tests of this work have been produced and tested by all these implementations.

The values in Table 7.1 and Table 7.2, together with the values of $N_{g,d}^{(r)}$ provided in the last chapter, were computed using the GAP code below. It is a recursive implementation of Algorithm 4 in order to explore the tree $\mathcal{T}_{g,\prec}^{d}$ in a depth first manner. This allows to compute only the number $N_{g,d}$ and not to store all the semigroups of a given genus. To produce and store all generalized numerical semigroups of a given genus it is needed an exploration in a breadth first manner which soon causes memory problems. The relaxed monomial order GAP used by default is the lexicographic order, our current implementations are not yet prepared to give the user the possibility of choosing another order. The only input data are the genus g and the dimension d. The output data are the number $N_{g,d}$ and a list L in such a way that L[i] (that is the i-th element of L) will contain the number of generalized numerical semigroups whose set of holes generates a vector space of dimension i.

```
numberSemigroups:= function(g,d)
  local  H, s, N, L, recursiveAffineSons, minGeneratorsMLex,
         addSpecialGapToAffineSemigroup,
         removeMinimalGeneratorFromAffineSemigroup, affineSons;

  H:=List([1..g],i->i*IdentityMat(d)[d]);
  s:=AffineSemigroupByGaps(H);
  N:=0;
  L:=List([1..d],i->0);

  ########################################
  ## recursive local function
  recursiveAffineSons:= function(s,N,L)
    local  H, gens, ml, smallgaps, smallPFs, SmSG, sons, t, E, i;

    H:=Gaps(s);
    gens:=Generators(s);
    #SmSG:=smallerSG(s);
    ml:=Minimum(gens); #multiplicity
    smallgaps := Filtered(H,j->j < ml);
    smallPFs := Filtered(smallgaps, g->not ForAny(gens, n -> n+g in H));
    SmSG := Filtered(smallPFs, g -> not(2*g in H));
    #special gaps smaller than the multiplicity
    #if the semigroup s has not special gaps smaller than the multiplicity
    #then it has not sons. If it has sons we apply the function itself on
    # the sons.
    if not(IsEmpty(SmSG)) then
      sons:=affineSons(s,SmSG);
      for t in sons do
        E:=recursiveAffineSons(t,N,L);
        N:=E[1];
        L:=E[2];
      od;
    fi;
    N:=N+1;
    i:=Rank(H);
    L[i]:=L[i]+1;
    return [N,L];
  end;;
  ## end of recursive local function
  ########################################
  ########################################
  ## other local functions
  #######
  #Computes minimal generators # similar to the function in the package
    minGeneratorsMLex := function(s)
    local  gens, len, non_minimal, y, x, mingens;

    gens:=Set(Generators(s));
    len := Length(gens);
    # compute the minimal generators
    non_minimal := [];
    for y in [2..len] do
      for x in [1..y-1] do
        if gens[y] - gens[x] in s then
          Add(non_minimal,gens[y]);
          break;
        fi;
```

```
    od;
  od;
  mingens := Difference(gens,non_minimal);
  #I set here the minimal generators, maybe it is better for affineSons.
  SetMinimalGenerators(s,mingens);
  return mingens;
end;;
#########
#Let S an affine semigroup with gaps and x be a Special gaps of S.
#We compute the unitary extension of S with x.
#In this form is useful for the successive function affineSons

addSpecialGapToAffineSemigroup:= function( x, a )
  local  H, gens, s;

  H:=Set(Gaps(a));
  RemoveSet(H,x);
  gens:=Union(Generators(a),[x]);
  s:=AffineSemigroup(gens);
  SetGaps(s,H);
  return s;
end;;
#########
#If S is an affine semigroup with gaps and x is a minimal generator of T, it computes
#the semigroup S\{x}. This form is useful for the succesive function affineSons.

removeMinimalGeneratorFromAffineSemigroup:=function(x,s)
  local  gens, H, t;

  gens:=MinimalGenerators(s);
  H:=Gaps(s);
  H:=Union(H,[x]);
  gens:=Difference(gens,[x]);
  gens:=Union(gens,gens+x);
  gens:=Union(gens,[2*x,3*x]);
  t:=AffineSemigroup(gens);
  SetGaps(t,H);
  return t;
end;;
########
#Computes the sons of an affine semigroup with gaps, in the ordinarization transform with
#respect to lexicographical order. That is: from S affine semigroup with gaps,
#it computes all semigroups (S U {h})\{x}, with h special gap smaller than multiplicity
#and x an effective generator. M is the list of special gaps smaller than ml
affineSons := function(s,M)
  local  L1, out, i, mingens, ml, Fb, biggens, j;

  #M:=smallerSG(s);
  #For each j in M we compute the unitary exstension of s with j.
  L1:=List(M,j->addSpecialGapToAffineSemigroup(j,s));
  out:=[];
  #For each semigroup S in L1 we have to compute the semigroup obtained removing by S
  #an effective generator, this have to be done for each effective generator of S.
  for i in L1 do
    mingens := minGeneratorsMLex(i);
    ml:=Minimum(Generators(i)); # multiplicity
    Fb:=Maximum(Gaps(i)); # Frobenius vector
    biggens:=Filtered(mingens,j->j>ml and j>Fb);
    for j in biggens do
```

```
      out:=Concatenation(out,[removeMinimalGeneratorFromAffineSemigroup(j,i)]);
    od;
  od;
  return out;
 end;;
 return recursiveAffineSons(s,N,L);
end;
```

The following code computes the decomposition of a generalized numerical semigroup as intersection of irreducible ones. It is an implementation of Algorithm 3.4.7. In particular it computes a minimal (non refinable) irreducible decomposition of it. The input is a generalized numerical semigroup a, otherwise an error is raised. The output is the list of generalized numerical semigroups that appear in a minimal irreducible decomposition of the semigroup represented by a.

```
AffineIrreducibleDecomposition := function( a )
local s,y,j,k,tag,Ga,SG,I,II,C,B,CS,sg,subAffineSemigroup;

sg:=function(gaps)
    return Filtered(gaps,g->not(2*g in gaps) and not(ForAny(Difference(gaps,[g]),
    h->not(h-g in gaps) and Minimum(h-g)>=0)));
end;

subAffineSemigroup:= function(s,t)
if ForAll(t,i->i in s) then
  return true;
else
  return false;
fi;
end;;


Ga:=Gaps(a);
SG:=sg(Ga);
if Length(SG)=1 then
  return a;
fi;
C:=[Ga];
I:=[];
while not(IsEmpty(C)) do
  B:=[];
  #From all semigroup in C, we gather all their unitary extension in B.
  for s in C do
    y:=sg(s);
    B:=Union(B,List(y,k->Difference(s,[k])));
  od;
   #In the following, test if all special gaps of "a" belongs to a semigroup S in B
  B:=Filtered(B,j->not(ForAll(SG,k->not(k in j))));

  #In the following we have to test if a semigroup S of I is contained in a semigroup S'
  #of B, this is equivalent to Gaps(S') is contained in Gaps(S) (step 5).
  B:=Filtered(B,i->not(ForAny(I,j->subAffineSemigroup(j,i))));

  #We split from B the irreducibles and not irreducibles   (Step 6 and 7)
  C:=Filtered(B,j->not(Length(sg(j))=1));
```

```
  I:=Union(I,Difference(B,C));
od;
CS:=[];
II:=[];
#CS[j] is the set of Special Gaps of "a" (the input) not belonging to the semigroup I[j],
#we gather both in a list [[C[j]],[I[j]]]. All these lists are gathered in II.
for j in [1..Length(I)] do
  CS[j]:=Filtered(SG,k-> k in I[j]);
  II:=Concatenation(II,[Concatenation([CS[j]],[I[j]])]);
  II:=Set(II);
od;
#We want to find a non refinable set of semigroups I[j] such that the union of the
#correspondent CS[j] is SG.
while true do
  tag:=[];
  for j in [1..Length(II)] do
    if Union(List(Difference(II,[II[j]]),k->k[1]))=SG then
      tag:=j;
      break;
    fi;
  od;
  if tag=[] then
    #return List(II,k->k[2]);
    return List(II,k->AffineSemigroupByGaps(k[2]));
  fi;
  II:=Difference(II,[II[j]]);
od;
end;;
```

The last code allows to test the generalized Wilf's conjecture 4.1.8. It requires a generalized numerical semigroup as input and gives the number $W = e(S)n(S) - dc(S)$ as output .

```
wilfgen:= function(s)
  local e,n,c,g,H,maxH,W,le;
  le:=function(x,y)
      return ForAll([1..Length(x)], i->x[i]<=y[i]);
  end;
  H:=Gaps(s);
  g:=Length(H);
  e:=Length(MinimalGenerators(s));
  maxH:=Filtered(H, a->not(ForAny(H, aa-> le(a,aa) and a<>aa)));
  c:=Length(Union(List(maxH,a->Cartesian(List(a,i->[0..i])))));
  n:=c-g;
  W:=e*n-Length(H[1])*c;
  return W;
end;;
```

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