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Graded algebras: theoretical and computational aspects

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Graded algebras: theoretical and computational aspects

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Declaration of Authorship

I, Luca AMATA, declare that this thesis titled "Graded algebras: theoretical and computational aspects" has been composed by myself and that the work presented in it has been generated as result of the original research by me jointly with my supervisor, Prof. Marilena CRUPI, with the same contributions. This work was done wholly or mainly while in candidature for a research degree at this University. Where I have consulted the published work of others, this is always clearly attributed.

It was like a drop of silver
in which one dipped and illumined
the darkness of the past.

Virginia Woolf (*To the lighthouse*)

Die Wahrheit ist eben kein Kristall,
den man in die Tasche stecken kann,
sondern eine unendliche Flüssigkeit,
in die man hineinfällt.

Robert Musil (*Der Mann ohne Eigenschaften*)

University of Catania

Abstract

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Department of Mathematics and Computer Sciences,
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Doctor of Philosophy

Graded algebras:
theoretical and computational aspects

by Luca Amata

In this dissertation we study by a computational approach Hilbert functions and minimal graded free resolutions of finitely generated graded modules over two significant graded K-algebras, K being a field.
More precisely, if E is the exterior algebra of a finite dimensional K-vector space and F is a finitely generated graded free E-module with a homogeneous basis, we characterize the Hilbert functions of graded E–modules of the type F/M, with M graded submodule of F, *via* the unique lexicographic submodule of F having the same Hilbert function as M. Furthermore, we study projective and injective resolutions over E. In particular, we give upper bounds for the graded Betti numbers and the graded Bass numbers of classes of E-modules.
Moreover, we give a criterion to determine the extremal Betti numbers of a special class of monomial ideals of a standard polynomial ring S known as the t-spread strongly stable ideals, where t is an integer ≥ 0. We are able to find a complete numerical characterization (positions as well as values) for the case $t = 0$ and $t = 1$. Instead, for the case $t = 2$ we determine the structure of the t-spread strongly stable ideals with the maximal number of extremal Betti numbers.
The approach to these topics is mainly computational because of the algorithmic nature of the topic themselves.
Finally, we present some packages in order to work and manipulate specific objects in both contexts.

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Introduction

Classical problems in commutative algebra include the study of the Hilbert functions and minimal graded free resolutions of finitely generated graded modules over graded algebras. These topics represent important tools in algebraic geometry and are becoming increasingly important both in combinatorics and computational algebra. Many authors have focused their attention on such notions (see for instance [Mac27, Sta75, Hul95, Gas97, BH96, HH11, Hoe11] and the references therein) both in the polynomial ring context and in the exterior algebra one. Indeed, it is well known that, even if the exterior algebra is not commutative, it behaves like a commutative local ring or *local ring ([BH96]) in many cases. Hence, many notions and results that hold in one context can be translated to the other one with some suitable modifications.
This dissertation aims to deepen the study of the above mentioned topics approaching some open problems in order to integrate the existing literature and developing some packages that can be useful in the framework of commutative algebra and algebraic geometry. All the algorithms presented in this thesis have been implemented and some of them are included in *Macaulay2* version 1.14.

Let K be a field. The graded K-algebras we consider in this thesis are the standard polynomial ring $S = K[x_1, \dots, x_n]$ and the exterior algebra $E = K\langle e_1, \dots, e_n \rangle$ of a K-vector space with basis $e_1, \dots, e_n$.
Let $R \in \{S, E\}$. Our work environment is $\mathcal{M}$, the category of finitely generated $\mathbb{Z}$-graded left and right R-modules M, and we will denote by $F \in \mathcal{M}$ a finitely generated graded free module with homogeneous basis $g_1, \dots, g_r$. In R one can introduce the notions of monomial and monomial ideal and therefore that of monomial submodule of F. More in details, a monomial submodule M of F is a submodule of the form $M = \oplus_{i=1}^r I_i g_i$, with I_i ($i = 1, \dots, r$) monomial ideals in R, *i.e.*, ideals generated by monomials of R. It is clear that a monomial submodule is in the category $\mathcal{M}$. Monomial modules over graded algebras are the *fulcrum* of our interest.

The Hilbert function of a graded K-algebra computes the vector space dimension of its graded components. It encodes important information on the graded K-algebra such as its Krull dimension or its multiplicity [HH11]. The Macaulay's key idea about the existence of highly structured monomial ideals, the lexicographic ideals, which attain all Hilbert functions of quotients of polynomial rings, has revealed crucial in the polynomial ring context. The pivotal property is that a lexicographic ideal grows as slowly as possible. Macaulay's theorem has been the inspiration for many similar classifications. Stanley wrote Macaulay's theorem

in its modern form in [Sta75] (see also [BH96]). Kruskal proved a theorem on bounding the f-vectors of simplicial complexes in a way similar to Macaulay's theorem [Kal01]. Katona independently proved an equivalent result phrased in terms of Sperner families [Kru63]. The Kruskal–Katona theorem is the `squarefree` analogue of Macaulay's theorem and may be also interpreted as a theorem on Hilbert functions of quotients of exterior algebras in [AHH97]. Finally, Macaulay's theorem was extended to modules by many authors, in particular by Hulett [Hul95] and Gasharov in [Gas97]. In the polynomial ring context the Hilbert functions are characterized both for ideals and modules, whereas in the exterior algebra context the main results hold for ideals; therefore, in this thesis we have focused our attention on graded modules over the exterior algebra.
In fact, we generalize the combinatorial Kruskal–Katona theorem [AHH97, Theorem 4.1] for finitely generated modules over exterior algebras. More precisely, we describe the possible Hilbert functions of graded E–modules of the form F/M, with M graded submodule of F. Our result bounds the growth of Hilbert function of such a kind of modules *via* the class of lexicographic submodules (Definition 1.2.16). The construction of such a submodule can be realized by using the classical way (which involves suitable sets of monomials of F). More in details, if M is a graded submodule of F, the construction of the lexicographic submodule M^{lex} with the same Hilbert function of M proceeds as follows: for each graded component M_j of M, let M_j^{lex} be the K–vector space spanned by the (unique) lexicographic segment L_j of F (Definition 1.2.14) with $|L_j| = \dim_K M_j$. Then one defines $M^{\text{lex}} = \oplus_j M_j^{\text{lex}}$.
Hereafter, we describe an alternative way for determining the lexicographic submodule we are looking for. Our approach (Theorem 2.3.2) manipulates sequences of nonnegative integers. More precisely, if M is a graded submodule of $F = \oplus_{i=1}^r E g_i$, we associate to F/M the sequence $Hs_{F/M} = (H_{F/M}(f_1), H_{F/M}(f_1+1), \dots, H_{F/M}(f_r+n)) \in \mathbb{N}_0^{f_r+n-f_1+1}$, where $f_i = \deg g_i$, $i = 1, \dots, r$. We call $Hs_{F/M}$ the *Hilbert sequence* of the graded E-module F/M. Using the Kruskal–Katona theorem (Theorem 2.2.4) and operating on the given Hilbert sequence by repeated subtractions, one obtains r suitable sequences which are the Hilbert sequences of r graded K–algebras E/I_i, with I_i $(i = 1, \dots r)$ lexicographic ideals of E, and $L = \oplus_{i=1}^r I_i g_i$ will be the unique lexicographic submodule of F with $H_{F/L} = H_{F/M}$. Consequently, we get a new criterion (Criterion 2.3.3) able, given a sequence H of nonnegative integers (of a certain length), to find out if H determines the Hilbert function of a quotient of the type F/M. We have also created a *Macaulay2* package, `ExteriorModules`, in order to manage monomial submodules of F, and in particular to compute the lex submodule M^{lex} associated to a submodule M.

As far as the minimal graded resolutions are concerned, many authors have been interested in the problem of giving upper bounds for the graded Betti numbers and the graded Bass numbers of graded submodules of a finitely generated graded free module with homogeneous basis, both in the polynomial and in the exterior algebra context (see for instance [Big93, Bee07, AHH97, CM13, CU07, CF15, Hul93, Hul95, AH00, CF12, CF13, Par94, Par96]).

If $R \in \{S, E\}$ and M is a graded R-module, then M has a unique minimal graded free resolution over R: $F_\bullet : \ldots \to F_2 \to F_1 \to F_0 \to M \to 0$, where $F_i = \oplus_j R(-j)^{\beta_{i,j}(M)}$. The integers $\beta_{i,j}(M) = \dim_K \operatorname{Tor}_i^R(M, K)_j$ are called the *graded Betti numbers* of M, whereas the numbers $\beta_i(M) = \sum_j \beta_{i,j}(M)$ are called the *Betti numbers* of M. The problem to bound the graded Betti numbers of a module M can be reformulated as follows: *is it possible to find a graded R-module L such that $\beta_{i,j}(L) \geq \beta_{i,j}(M)$ (for all i, j), for all graded R-modules M with the same Hilbert function of L?* In the case of modules over the polynomial ring and ideals in the exterior algebra, the answer is positive and a fundamental tool is the class of lexicographic submodules and the class of lexicographic ideals, respectively. More in details, Bigatti [Big93] and Hulett [Hul93] showed independently that among all graded ideals in a standard polynomial ring S with a given Hilbert function, the lexicographic ideal has the largest graded Betti numbers in characteristic zero. Then Hulett [Hul95] extended the result to graded submodules of a free module over S. The previous results were proved by Pardue [Par94, Par96] in any characteristic. The result of Bigatti [Big93], Hulett [Hul93] and Pardue [Par94, Par96] was generalized to the exterior algebra context by Aramova, Herzog and Hibi in [AHH97]. In this dissertation we will show how the class of lexicographic submodules reveals fundamental also in getting upper bounds for graded submodules of a free module over the exterior algebra.

It is known that the R-module M has also a unique minimal graded injective resolution: $I_\bullet : 0 \to M \to I^0 \to I^1 \to I^2 \to \ldots$, where $I^i = \oplus_j R(n-j)^{\mu_{i,j}(M)}$. The integers $\mu_{i,j}(M) = \dim_K \operatorname{Ext}_R^i(K, M)_j$ are called the *graded Bass numbers* of M [BH96, KÏ0]. Also for these invariants, the problem to find a bound (once the Hilbert function has been fixed) for modules over the polynomial ring and ideals of the exterior algebra has been solved ([AHH97, Bee07]). Hence, we will still focus on modules over the exterior algebra $E = K\langle e_1, \ldots, e_n \rangle$. Our aim is to give upper bounds for such invariants in the class of all graded submodules in F with the same Hilbert function, again *via* lexicographic modules.

An important role in the exterior algebra context is assumed by a special monomial submodule of F, as in the polynomial ring context. Precisely, if $M \in \mathcal{M}$ is a graded submodule of F, fixing a proper monomial order on F, there exists a monomial submodule of F called the generic initial module, denoted by $\operatorname{Gin}(M)$, which contains some information related to M. Indeed, if K is infinite then $\operatorname{Gin}(M)$ is strongly stable (see Definition 1.2.8) and the inequality $\beta_{i,j}(M) \leq \beta_{i,j}(\operatorname{Gin}(M))$ holds for all i, j (see [Gre98, AHH97]). So, for the solution of our problem we can take into account, without loss of generality, strongly stable submodules. In order to face the analogous problem on the graded Bass numbers, we need some further remarks. Let M^* be the right (left) E-module $\operatorname{Hom}_E(M, E)$. The duality between projective and injective resolutions implies the existence of a relation ([AHH97, Proposition 5.2]) between the graded Bass numbers of a module and the graded Betti numbers of its dual: $\beta_{i,j}(M) = \mu_{i,n-j}(M^*)$, for all i, j. An important observation is that $\mathrm{Hom}_E(E/I, E) \simeq 0 : I$ ([AHH97]), where $0 : I$ is the annihilator of I. Furthermore, if I is lex than $0 : I$ is lex too. In our study, the crucial point is to relate the submodules F/M^{lex} with $\mathrm{Hom}(F/M, E)^{\mathrm{lex}}$. We have obtained a partial result in the case $F = E^r$, as we will see later.

An important subset of the graded Betti numbers of a graded ideal of a polynomial ring is the one consisting of the *extremal Betti numbers*. These invariants were introduced by Bayer, Charalambous and Popescu in [BCP99], as a refinement of some important invariants of the graded ideal I. More precisely, let $S = K[x_1, \dots, x_n]$ be the standard polynomial ring in n variables over a field K and let I be a graded ideal of S. A graded Betti number $\beta_{k,k+\ell}(I) \neq 0$ is called *extremal* if $\beta_{i,\,i+j}(I) = 0$ for all $i \geq k$, $j \geq \ell$, $(i, j) \neq (k, \ell)$ [BCP99]. The pair (k, ℓ) is called a *corner* of I. If $\beta_{k_i,k_i+\ell_i}(I)$ $(i = 1, \dots, r)$ are extremal Betti numbers of a graded ideal I, then the set $\mathrm{Corn}(I) = \{(k_1, \ell_1), (k_2, \ell_2), \dots, (k_r, \ell_r)\}$ will be called the corner sequence of I. In the *Macaulay2* or CoCoA Betti diagram of I, the graded Betti number $\beta_{i,j}(I)$ is plotted in column i and row $j - i$. Using such a notation, a graded Betti number $\beta_{k,k+\ell}(I)$ is extremal if it is the only entry in the quadrant where it is the northwest corner (see Figure 1.2). Projective dimension measures the column index of the easternmost extremal Betti number, whereas regularity measures the row index of the southernmost one. Indeed, the extremal Betti numbers are a generalization of such meaningful algebraic invariants.
Recently, Ene, Herzog, and Qureshi have introduced the notion of t–spread monomial ideal [EHQ19] (see also [AEL19, AC19f]), where t is a nonnegative integer. More precisely, if $t \geq 0$ is an integer, a monomial $x_{i_1} x_{i_2} \cdots x_{i_d}$ with $1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq n$ is called t–spread, if $i_j - i_{j-1} \geq t$ for $2 \leq j \leq d$. A monomial ideal in S is called a t–spread monomial ideal, if it is generated by t–spread monomials. Such a notion generalizes the notion of (squarefree) monomial ideal. Indeed, it is clear that every monomial ideal of S is a 0–spread monomial ideal, whereas every squarefree monomial ideal of S is a 1–spread monomial ideal.
We recall that a squarefree monomial ideal of S is a monomial ideal generated by squarefree monomials. Such ideals are also known as Stanley–Reisner ideals, and quotients by them are called Stanley–Reisner rings. The combinatorial nature of these algebraic objects comes from their close connections to simplicial topology. Many authors have studied the class of squarefree monomial ideals from the viewpoint of commutative algebra and combinatorics (see, for example [AHH98, AHH00, CSW14, MS05, CU09], and the references therein). We analyze the following problem.

Problem 1 Given three nonnegative integers t, n, r ($n \geq 2$ and $1 \leq r \leq n - 1$), r pairs of positive integers (k_1, ℓ_1), …, (k_r, ℓ_r) such that $n - 1 \geq k_1 > k_2 > \cdots > k_r \geq 1$, $1 \leq \ell_1 < \ell_2 < \cdots < \ell_r$, and r positive integers $a_1, \dots, a_r$, under which conditions does there exist a t–spread ideal I of $S = K[x_1, \dots, x_n]$ such that $\beta_{k_1,k_1+\ell_1}(I) = a_1$, …, $\beta_{k_r,k_r+\ell_r}(I) = a_r$ are its extremal Betti numbers?

A positive answer to this problem has been given for K field of characteristic zero and $t = 0$ [CU00, CU03, HSV14]. Indeed, under these hypotheses, the generic initial ideal $\mathrm{Gin}(I)$, with respect to the graded reverse lexicographic order induced by $x_1 > \cdots > x_n$ ([Eis95, HH11]), of a graded ideal of S is a strongly stable ideal, and the extremal Betti numbers of I, as well as their positions, are preserved by passing from I to $\mathrm{Gin}(I)$ [BCP99, Corollary 1]. A similar result holds also in the squarefree case even if the fact that I is squarefree does not imply that $\mathrm{Gin}(I)$ is squarefree. Indeed, in [AHH00], the authors have introduced a certain operator σ which transforms $\mathrm{Gin}(I)$ to a squarefree monomial ideal of S. Such an ideal, denoted by $\mathrm{Gin}(I)^\sigma$, is squarefree strongly stable [AHH00, Lemma 1.2.]. On the other hand, [AHH00, Theorem 2.4.] assures that if I is a squarefree ideal, then the extremal Betti numbers are preserved when we pass from I to $\mathrm{Gin}(I)^\sigma$. Hence, Problem 1 can be reformulated in terms of strongly stable ideals for $t = 0$ and in terms of squarefree strongly stable ideals for $t = 1$.

In this thesis we analyze the behavior of the extremal Betti numbers of t–spread strongly stable ideals for $t \geq 0$ (Definition 4.1.1). This class of ideals is a natural generalization of the class of (squarefree) strongly stable ideals [EK90, AHH98]. Hence, one can use similar methods as in [AC19a, AC19e, Cru16, CU00, CU03, CU09, CF16] for establishing criteria to determine their extremal Betti numbers. The discussion of this topic has been gradual. Since for $t = 0$, *i.e.* in the case of monomial ideals of S, Problem 1 has been solved [CU03, Theorem 3.1], [Cru16, Propositions 3.4, 3.5, Theorem 3.7] and [HSV14, Theorem 6.7], we have created a package in CoCoA for "manipulating" the extremal Betti numbers in this case. By using computer algebra systems, CAS, (for instance, CoCoA [AB, ABL] or Macaulay2 [GS]), given a graded ideal I of the polynomial ring S, functions for determining the extremal Betti numbers of I are available. On the contrary, to the best of our knowledge, it seems that packages for the inverse problem, *i.e.*, Problem 1, have not been implemented yet. The key idea is to identify appropriate segments of monomials to determine the extremal Betti numbers for each degree. So, we have improved some known results and we have developed a CoCoA package (`ExtrBettiNumbers`) for computing the *smallest strongly stable ideal* of S solution of Problem 1 ([AC19a]). In particular, the package is able to determine all the possible r-tuples of positive integers $(a_1, \ldots, a_r)$ for which such an ideal does exist.
For the case $t = 1$, *i.e.* squarefree ideals of S, the first result on the behavior of the extremal Betti numbers of such a class of squarefree monomial ideals can be found in [CU03, Propostion 4.1]. More precisely, the authors in [CU03] gave the following criterion to determine whether a graded Betti number is extremal: *let I be a squarefree strongly stable ideal of S. $\beta_{k,\,k+\ell}(I)$ is an extremal Betti number if and only if $k+\ell = \max\{\mathrm{m}(u) : u \in G(I)_\ell\}$ and $\mathrm{m}(u) < k+j$, for all $j > \ell$ and for all $u \in G(I)_j$* (Theorem 4.1.12); $G(I)_\ell$ is the set of monomials u of $G(I)$ such that $\deg u = \ell$. They did not give any numerical characterization of the possible extremal Betti numbers of such a class of ideals. Later, such a criterion was generalized to the class of squarefree strongly stable submodules of a finitely generated graded

free S–module with a homogeneous basis in [CF16, Theorem 4.3]. Moreover, a criterion for determining their positions and their number was also given in [CF16, Section 5]. Differently from the non–squarefree case, not much is known about the numerical characterization of the possible extremal Betti numbers (values and positions) of the class of squarefree strongly stable ideals. We are able to give such a characterization (Theorem 4.3.29). Our techniques involve tools from enumerative combinatorics to detect some particular monomials that characterize the positions of the extremal Betti numbers and hence some combinatorial formulas to establish the bounds for their values.
For the case $t = 2$, *i.e.* the case of 2–spread ideals of S, we determine the maximal admissible number of extremal Betti numbers of a 2–spread strongly stable ideal. Many surprising situations occur in such a case ([AC19b]). At the present time, it seems to be a difficult combinatorial problem to determine such a number for all $t \geq 1$ and solve Problem 1.

This thesis is structured in four chapters. Chapter 1 contains a brief sketch of the notions which are intensively used along the thesis. Moreover, it fixes the notations and gives a short overview on computational methods in commutative algebra.

In Chapter 2 we discuss in details the Hilbert functions of quotients of graded free E-modules. The study of the behavior of these functions is crucial for the development of the main result. We state a new expression for such Hilbert functions (Proposition 2.2.1), and give their characterization (Theorem 2.2.4) *via* lexicographic submodules. Moreover, we describe a new procedure for the construction of the unique lexicographic submodule for a given Hilbert function (Theorem 2.3.2). A new criterion (Criterion 2.3.3) to verify if a sequence of nonnegative integers determines the Hilbert function of quotients of graded E–algebras is also given. After this, we show some examples illustrating our results and procedures. Finally, we present two *Macaulay2* packages, and show the use of their methods by means of suitable examples. This chapter is based on the papers [AC18b, AC19c, AC20].

Chapter 3 is dedicated to minimal graded free resolutions, and in particular to devise bounds for the graded Betti numbers and the graded Bass numbers in the exterior algebra context. We start by analyzing the generic initial module of a graded module $M \in \mathcal{M}$. Generic initial modules preserve much information of the original module and, furthermore, they are strongly stable (Proposition 1.2.7) if the base field K is infinite. Therefore, in many situations it is a successful strategy to pass on to the generic initial module and then exploit the nice properties of strongly stable submodules. After this, we discuss both the class of almost lexicographic submodules (Definition 3.2.2) and the class of lexicographic submodules of F. We prove that the almost lexicographic submodules provide a first upper bound for the Betti numbers of all graded submodules of F with the same Hilbert function (Proposition 3.2.3). Such a bound is not maximal in general. Finally, we give a characterization of the class of lexicographic submodules (Definition 1.2.16). Indeed, if $F = \oplus_{i=1}^{r} Eg_i$ is the free E-module with homogeneous basis $g_1, \ldots, g_r$, such that $\deg g_1 \leq \deg g_2 \leq \cdots \leq \deg g_r$, we show that the lexicographic submodules give upper bounds for the graded Betti numbers of the class of graded submodules of F with the same Hilbert function (Theorem 3.3.9). Our techniques generalize the ones discussed in [AHH97, AHH98]. Moreover, upper bounds for the graded Bass numbers of the class of graded submodules of $F \simeq E^r$ with a given Hilbert function, are stated. Indeed, in such a case, setting $M^{\text{lex}} = \oplus_{t=1}^{r} J_t g_t \subset E^r$, we have that $\text{Hom}(E^r/M, E)^{\text{lex}} = \oplus_{t=1}^{r}(0 : J_{r-t-1})g_t$ (these are lex submodules of E^r with the same Hilbert function). This allows us to state that $\mu_{i,j}(E^r/M) \leq \mu_{i,j}(E^r/M^{\text{lex}})$, for all i, j. Furthermore, some remarks on the annihilator of classes of monomial submodules in F are given (Theorem 3.4.5). Finally, we present other functionalities of the *Macaulay2* packages introduced in the previous chapter and we show some explicative examples. This chapter is based on the papers [AC18b, AC18a, AC19d].

In Chapter 4, we investigate the behavior of the extremal Betti numbers of t–spread strongly stable ideals (Theorem 4.1.12, Corollary 4.1.13). A fundamental tool is the Ene, Herzog, Qureshi formula [EHQ19] for computing the graded Betti numbers of such a class of monomial ideals. We have approached the problem step by step, for low values of t, *i.e.*, for $t \in \{0, 1\}$. First, we study the possible extremal Betti numbers of a strongly stable ideal of S (*i.e.*, $t = 0$) with initial degree ≥ 2. Since the characterization of the extremal Betti numbers is obtained by a detailed description of suitable sets of monomials (Proposition 4.2.5, Theorem 4.2.6), we exhibit two algorithms (Algorithm 4.1 and 4.2) for the computation of all the sets of monomials involved in the characterization. As a final result, our procedure returns the strongly stable ideal we are looking for (Algorithm 4.3). Moreover, we describe a further algorithm (Algorithm 4.4) able to compute all the admissible values for the r-tuple of positive integers $(a_1, \ldots, a_r)$ satisfying Problem 1. Hence, an example which illustrates the implemented functions is given. We also describe in detail the *CoCoA* package *ExtrBettiNumbers* (tested with CoCoA System, version 5.1.4) that has been built.
Furthermore, for $t = 1$, we identify the admissible corner sequences of a squarefree strongly stable ideal of S for $n = 2, 3, 4$. Then, we determine the maximal number of corners allowed for a squarefree strongly stable ideal I of S with a corner in its initial degree (Propositions 4.3.7, 4.3.9). Moreover, given $n - \ell_1$ ($n \geq 5$) pairs of positive integers (k_1, ℓ_1), (k_2, ℓ_2), $\ldots$, $(k_{n-\ell_1}, \ell_{n-\ell_1})$, with $1 \leq k_{n-\ell_1} < k_{n-\ell_1-1} < \cdots < k_1 \leq n - 3$ and $3 \leq \ell_1 < \ell_2 < \cdots < \ell_{n-\ell_1} \leq n - 1$, we determine the conditions under which there exists a squarefree lex ideal I of $K[x_1, \ldots, x_n]$ of initial degree ℓ_1 having $\beta_{k_i, k_i+\ell_i}(I)$, $i = 1, \ldots, r$, as extremal Betti numbers (Theorem 4.3.13). A complete description of the minimal system of monomial generators of I is given. Next, we face Problem 1, and solve it when $\text{char}(K) = 0$ (Theorem 4.3.29). In such a case, the question is equivalent to the characterization of the possible extremal Betti numbers of a squarefree strongly stable ideal of S as we have underlined before. The idea behind Theorem 4.3.29 is to establish the bounds for the integers a_i ($i = 1, \ldots, r$), starting with a_r and then arriving to a_1, by computing the cardinality of

suitable sets of monomials. The key result in this case is Theorem 4.3.17. Let (k,ℓ) be a pair of positive integers and let $A^s(k,\ell)$ be the set of all squarefree monomials u of S of degree ℓ and such that $\max(u) = k+\ell$, with $\max(u) = \max\{i : x_i \text{ divides } u\}$, ordered by the squarefree lex order $\geq_{\text{slex}}$. If $u \in A^s(k,\ell)$, Theorem 4.3.17 shows a method for determining the cardinality of the set of all squarefree monomials $w \in A^s(k,\ell)$ such that $w \geq_{\text{slex}} u$. We provide some examples illustrating the main obstructions to the issue.
For $t = 2$, we analyze the extremal Betti numbers of 2–spread strongly stable ideals in the polynomial ring $S = K[x_1,\dots,x_n]$. More precisely, we face the following problem: *given the set $\mathcal{S}_{2,n}$ of all 2–spread strongly stable ideals in S, what is the maximal number of extremal Betti numbers allowed for an ideal in $\mathcal{S}_{2,n}$?* The study of this problem has led us to distinguish the cases n odd and n even (Theorems 4.4.2, 4.4.9). Moreover, if $n \geq 11$ is an odd integer, given $r = \frac{n-3}{2}$ pairs of positive integers (k_1,ℓ_1), …, (k_r,ℓ_r) such that $n-3 \geq k_1 > k_2 > \cdots > k_r \geq 1$ and $2 = \ell_1 < \ell_2 < \cdots < \ell_r$, we determine the conditions under which there exists a 2–spread strongly stable ideal I of $S = K[x_1,\dots,x_n]$ of initial degree $\ell_1 = 2$ with $\beta_{k_1,k_1+\ell_1}(I)$, …, $\beta_{k_r,k_r+\ell_r}(I)$ as extremal Betti numbers (Theorem 4.4.6). A similar result is proved for $n \geq 12$ even integer (Theorem 4.4.12). We provide some examples illustrating the main results. Finally, we present a *Macaulay2* package, `SquarefreeIdeals`, to manipulate squarefree monomial ideals. This package provides some methods to solve the Problem 1 for $t = 1$; we are currently working to implement methods to face the general problem where $t \geq 1$. This chapter is based on the papers [AC19a, AC19e, AC19b].

All the examples in the dissertation are constructed by means of CoCoA or *Macaulay2* packages, some of which developed by the author of this work.

Chapter 1

Basic notions

This chapter summarizes the fundamental notions about algebraic structures, and the ideas to we will refer to in this dissertation. We illustrate classical definitions and properties about graded algebras, monomial modules, Hilbert functions and minimal resolutions. These objects and tools, here introduced theoretically, in the situations where it is possible, will be investigated through an algorithmic and computational point of view in the next chapters. The aspects here discussed can be found in any textbook about commutative algebra (see, for instance, [BH96], [HH11], [Eis95], [Pee11], [Eis05]).

1.1 Graded Algebras

In this section, we discuss about rings and algebras which admit a decomposition of their elements into homogeneous components. Throughout this thesis we assume that all rings are Noetherian, commutative or skew-commutative and with identity. All modules involved are finitely generated unless otherwise stated and we fix an infinite field K.

Definition 1.1.1 A graded ring is a ring R together with a decomposition $R = \bigoplus_{i\in\mathbb{Z}} R_i$ (as a $\mathbb{Z}$–module) such that $R_iR_j \subset R_{i+j}$, for all $i, j \in \mathbb{Z}$.
One calls R_i the i–th homogeneous (or graded) component of R. The elements $x \in R_i$ are called *homogeneous* of degree i.

According to this definition, the zero element is homogeneous of arbitrary degree. The degree of x is denoted by $\deg x$. An arbitrary element $x \in R$ has a unique presentation $x = \sum_i x_i$ as a sum of homogeneous elements $x_i \in R_i$. The elements x_i are called the *homogeneous components* of x. Note that every ring R has the trivial grading given by $R_0 = R$ and $R_i = 0$ for $i \neq 0$.

A (not necessarily commutative) R–algebra A is graded if, in addition to the definition, $A_iA_j \subset A_{i+j}$. Let K be a field, A is called standard graded if it is a finitely generated

K-algebra and all its generators are of degree 1. Any other standard graded K-algebra is isomorphic to the polynomial ring modulo a graded ideal.

Proposition 1.1.2 *Let R be a positively graded R_0–algebra, and $x_1, \ldots, x_n$ homogeneous elements of positive degree. Then the following are equivalent:*

(*i*) *$x_1, \ldots, x_n$ generate the ideal $\mathfrak{m} = \sum_{i=1}^{\infty} R_i$.*

(*ii*) *$x_1, \ldots, x_n$ generate R as an R_0–algebra.*

The last assertion of Proposition 1.1.2 holds for graded rings in general (see [BH96, Proposition 1.5.4]).

Definition 1.1.1 can be generalized for modules over a graded ring.

A graded R–module is an R–module M together with a decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ (as a $\mathbb{Z}$–module) such that $R_i M_j \subset M_{i+j}$, for all $i, j \in \mathbb{Z}$. One calls M_i the i–th homogeneous (or graded) component of M. The elements $x \in M_i$ are called *homogeneous* (of degree i).

According to this definition, the zero element is homogeneous of arbitrary degree. The degree of x is denoted by $\deg x$. An arbitrary element $x \in M$ has a unique presentation $x = \sum_i x_i$ as a sum of homogeneous elements $x_i \in M_i$. The elements x_i are called the *homogeneous components* of x.

Note that R_0 is a ring with $1 \in R_0$, all summands M_i are R_0–modules, and so $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a direct sum decomposition of M as an R_0–module.

Let R be a graded ring. The category of graded R-modules, denoted by $\mathcal{M}_0(R)$, has as objects the graded R–modules. A morphism $\varphi : M \to N$ in $\mathcal{M}_0(R)$ is an R–module homomorphism satisfying $\varphi(M_i) \subset N_i$ for all $i \in \mathbb{Z}$. An R–module homomorphism which is a morphism in $\mathcal{M}_0(R)$ will be called homogeneous.

Let M be a graded R–module and N a submodule of M. N is called a *graded submodule* if it is a graded module such that the inclusion map is a morphism in $\mathcal{M}_0(R)$. This is equivalent to the condition $N_i \subset N \cap M_i$ for all $i \in \mathbb{Z}$. In other words, N is a graded submodule of M if and only if N is generated by the homogeneous elements of M which belong to N. In particular, if $x \in N$, then all homogeneous components of x belong to N. Furthermore, M/N is graded in a natural way. If φ is a morphism in $\mathcal{M}_0(R)$, then $\operatorname{Ker} \varphi$ and $\mathrm{im}\varphi$ are graded.

The graded submodules of R are called *graded ideals*.

Remark 1.1.3 Let I be an arbitrary ideal of R. Then the graded ideal I^* is defined to be the ideal generated by all homogeneous elements $a \in I$. It is clear that I^* is the largest graded ideal contained in I, and that R/I^* inherits a natural structure as a graded ring.

In this thesis, due to their algorithmic structure, the focus of our arguments will be two graded algebras with reference to the polynomial ring and the exterior algebra.

Example 1.1.4 (*i*) Let K be a field, and $S = K[x_1, \dots, x_n]$ a polynomial ring over K. Then for every choice of integers $d_1, \dots, d_n$ there exists a unique grading on R such that $\deg x_i = d_i$ and $\deg a = 0$ for all $a \in K$. The m–th graded component is the S–module generated by all the products $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ such that $\sum \alpha_i d_i = m$. If one chooses $d_i = 1$ for all i, then one obtains the grading of the polynomial ring corresponding to the total degree of a product of indeterminates. Unless indicated otherwise, we will always consider S to be graded in this way.

(*ii*) Let R be a ring, and M an R–module. We consider R as a graded ring by giving it the trivial grading. Let $M^{\otimes i}$ denotes the i–th tensor power of M, *i.e.*, $M \bigotimes \cdots \bigotimes M$ of i factors of M for $i > 0$ and R for $i = 0$. The tensor power form a graded R–module $\bigotimes M = \bigoplus_{i=0}^{\infty} M^{\otimes i}$. Defining the product of $x_1 \otimes \cdots \otimes x_n$ and $y_1 \otimes \cdots \otimes y_m$ as $x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_m$, one gives $\bigotimes M$ the structure of a graded associative R–algebra (not commutative in general). Let us consider the two-sided graded ideal I generated by the homogeneous elements $x \otimes x$, $x \in M$. The exterior algebra $\bigwedge M$ is the graded R–algebra $\bigwedge M = \bigotimes M/I$ The product of $x, y \in \bigwedge M$ is denoted by $x \wedge y$. One has

$$\begin{array}{ll} x \wedge y = (-1)^{(\deg x)(\deg y)}\ y \wedge x & \text{for homogeneous } x, y \in \bigwedge M, \\ x \wedge x = 0 & \text{for homogeneous } x \in \bigwedge M,\ \deg x \text{ odd.} \end{array}$$

Given $x_1, \dots, x_n \in M$ and a permutation π of S_n, then $x_{\pi(1)} \wedge \ldots \wedge x_{\pi(n)} = \operatorname{sgn}(\pi) x_1 \wedge \ldots \wedge x_n$ ($\operatorname{sgn}:\ S_n \to \{-1, 1\}$, the sign of a permutation).

Remark 1.1.5 Throughout this thesis, if K is a field, we will denote by $E = K \langle e_1, \dots, e_n \rangle$ the exterior algebra of a K-vector space V with basis $e_1, \dots, e_n$.
For any subset $\sigma = \{i_1, \dots, i_d\}$ of $\{1, \dots, n\}$ with $i_1 < i_2 < \cdots < i_d$ we write $e_\sigma = e_{i_1} \wedge \ldots \wedge e_{i_d}$. We set $e_\sigma = 1$, if $\sigma = \emptyset$. One can easily verify that the set of all products of indeterminates in E forms a K-basis of E of cardinality 2^n.
In order to simplify the notation, we put $fg = f \wedge g$ for any two elements f and g in E. An element $f \in E$ is called *homogeneous* of degree j if $f \in E_j$, where $E_j = \bigwedge^j V$.

Now, let R be a graded algebra and let $\mathcal{M}$ be the category of finitely generated $\mathbb{Z}$–graded left and right R-modules M.

It is possible to introduce graded free modules over a graded algebra. From now on we will define and discuss graded free modules over a graded algebra R, when R is the polynomial ring $S = K[x_1, \dots, x_n]$ or the exterior algebra $E = K \langle e_1, \dots, e_n \rangle$.

Let $F \in \mathcal{M}$ be a graded free module with homogeneous basis $g_1, \dots, g_r$, where $\deg(g_i) = f_i$ for each $i = 1, \dots, r$, with $f_1 \leq f_2 \leq \cdots \leq f_r$. We write $F = \bigoplus_{i=1}^{r} Rg_i$.

The degree of an element of the form xg_i, where x is homogeneous in R, is $\deg(xg_i) = \deg(x) + \deg(g_i)$.

Definition 1.1.6 If M is a graded submodule of a finitely generated graded free module F, we denote by $\operatorname{indeg}(M)$ the *initial degree* of M, *i.e.*, the minimum i such that $M_i \neq 0$.

1.2 Monomials

Special elements called *monomials* can be introduced in both graded algebras S and E. In order to point out the differences between the polynomial ring and the exterior algebra, we will proceed with some "splitted" definitions.

Definition 1.2.1 Let K be a field.

(i) Let us consider the polynomial ring $S = K[x_1, \ldots, x_n]$ as a $\mathbb{Z}$-graded ring where $\deg x_i = 1$, $i = 1, \ldots, n$. Any product $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, with α_i non–negative integers, is called a *monomial* of S of degree $\sum \alpha_i$.

(ii) If $E = K\langle e_1, \ldots, e_n\rangle$ is the exterior algebra of a K-vector space V with basis $e_1, \ldots, e_n$, any product $e_{i_1} \ldots e_{i_d}$, with $1 \leq i_1 < i_2 < \cdots < i_d \leq n$, is called a *monomial* of E of degree d.

(iii) Let F be a graded free module with homogeneous basis $g_1, \ldots, g_r$ over $R \in \{S, E\}$. We write $F = \oplus_{i=1}^r Rg_i$. The elements of the form ug_i, where u is a monomial of R, are called *monomials* of F of degree $\deg(u) + \deg(g_i)$.

In the following we will refer to graded algebra $R \in \{S, E\}$. Indeed, in both these cases the definition above can be given.

For a monomial $1 \neq u \in R$, we set

$$\operatorname{supp}(u) = \{i : x_i \text{ divides } u\},$$

and we write

$$\max(u) = \max\{i : i \in \operatorname{supp}(u)\}, \qquad \min(u) = \min\{i : i \in \operatorname{supp}(u)\}.$$

Moreover, we set $\max(1) = \min(1) = 0$.

Definition 1.2.2 Let F be a finitely generated graded free R-module with homogeneous basis $g_1, \ldots, g_r$. A graded submodule M of F is a monomial submodule if M is a submodule generated by monomials of F, *i.e.*,

$$M = I_1 g_1 \oplus \cdots \oplus I_r g_r,$$

with I_i a monomial ideal of R, for each i.

One can easily observe that monomial modules are graded modules. Moreover, if $r = 1$ and $\deg g_1 = 0$ then every monomial submodule of F is a monomial ideal of R.

For a subset T of F, we denote by $\mathrm{Mon}(T)$ ($\mathrm{Mon}_d(T)$, respectively) the set of all monomials (monomials of degree d, respectively) of T, and by $|T|$ its cardinality. It is clear that $\mathrm{Mon}(F)$ is a K-basis of F. In particular, if I is an ideal of R, then $\mathrm{Mon}(I)$ is a K-basis of I.

There exist some special classes of monomial modules that have wide applications to algebraic geometry, commutative algebra, and combinatorics.

Indeed, in several cases it is convenient to consider particular monomial modules associated to a graded module in order to obtain information on it. To this aim, it is necessary to introduce the notion of monomial order.

Definition 1.2.3 Let $F \in \mathcal{M}$ be a free R-module with basis. A *monomial order* on F is a total order $>$ on the monomials of F such that if $u, v \in \mathrm{Mon}(F)$ and $w \in \mathrm{Mon}(R)$, then $u > v$ implies $wu > wv > v$.

Now, it is possible to introduce the definition of *initial monomial* of an element of F. Let $y \in F$ and let $>$ a monomial order on F. The initial monomial of y, written $\mathrm{in}_>(y)$, is the greatest monomial of y with respect to the monomial order $>$.
If M is a submodule of F, then the *initial module* of M indicated by $\mathrm{in}_>(M)$ is the monomial submodule of F generated by the monomials $\mathrm{in}_>(y)$ for all $y \in M$.
Fixed a monomial order, we will use to write $\mathrm{in}(y)$ and $\mathrm{in}(M)$, respectively.

Example 1.2.4 *(i)* Let $S = K[x_1, \ldots, x_n]$ and let $\mathrm{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\mathrm{x}^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ be monomials of S with α_i, β_i nonnegative integers. We denote by $>_{\mathrm{lex}}$ the *lexicographic order* (lex order, for short) on $\mathrm{Mon}(S)$, *i.e.* $\mathrm{x}^\alpha >_{\mathrm{lex}} \mathrm{x}^\beta$ (lexicographically greater than) if either $\sum_{i=1}^n \alpha_i > \sum_{i=1}^n \beta_i$ or $\alpha_1 = \beta_1, \ldots, \alpha_{s-1} = \beta_{s-1}$ and $\alpha_s > \beta_s$ for some $1 \leq s \leq n$.

(ii) Let $E = K\langle e_1, \ldots, e_n \rangle$ be the exterior algebra. Let $e_\sigma = e_{i_1} e_{i_2} \cdots e_{i_d}$ and $e_\tau = e_{j_1} e_{j_2} \cdots e_{j_d}$ be monomials belonging to $\mathrm{Mon}_d(E)$ with $1 \leq i_1 < i_2 < \cdots < i_d \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_d \leq n$. We denote by $>_{\mathrm{lex}}$ the *lexicographic order* (lex order, for short) on $\mathrm{Mon}_d(E)$, *i.e.* $e_\sigma >_{\mathrm{lex}} e_\tau$ (lexicographically greater than) if $i_1 = j_1, \ldots, i_{s-1} = j_{s-1}$ and $i_s < j_s$ for some $1 \leq s \leq d$.

(iii) Let $F = \oplus_{i=1}^r S g_i$ be a graded free module over S or E with a homogeneous basis. Denote by $\mathrm{Mon}(F)$ the set of all monomials of F. We order such a set by the ordering $>_{\mathrm{lex}_F}$ defined as follows: if ug_i and vg_j are monomials of F, then $ug_i >_{\mathrm{lex}_F} vg_j$ (F-lexicographically greater than) if $\deg(ug_i) > \deg(vg_j)$ or if $\deg(ug_i) = \deg(vg_j)$ and either $i < j$ or $i = j$ and $u >_{\mathrm{lex}} v$.

For instance, if $E = K\langle e_1, e_2, e_3\rangle$ and $F = Eg_1 \oplus Eg_2$, with $\deg g_1 = 2$ and $\deg g_2 = 3$, then

$\mathrm{Mon}_2(F)$	g_1
$\mathrm{Mon}_3(F)$	$e_1g_1 >_{\mathrm{lex}_F} e_2g_1 >_{\mathrm{lex}_F} e_3g_1 >_{\mathrm{lex}_F} g_2$
$\mathrm{Mon}_4(F)$	$e_1e_2g_1 >_{\mathrm{lex}_F} e_1e_3g_1 >_{\mathrm{lex}_F} e_2e_3g_1 >_{\mathrm{lex}_F} e_1g_2 >_{\mathrm{lex}_F} e_2g_2 >_{\mathrm{lex}_F} e_3g_2$
$\mathrm{Mon}_5(F)$	$e_1e_2e_3g_1 >_{\mathrm{lex}_F} e_1e_2g_2 >_{\mathrm{lex}_F} e_1e_3g_2 >_{\mathrm{lex}_F} e_2e_3g_2$
$\mathrm{Mon}_6(F)$	$e_1e_2e_3g_2$

Theorem 1.2.5 *(Macaulay)*
Let R be a graded algebra, F a free R-module with homogeneous basis, and M an arbitrary submodule of F. For any monomial order $>$ on F, the set B of all monomials not in $\mathrm{in}_>(M)$ forms a basis for F/M.

The initial module depends also on the choice of coordinates, but there is an object, the initial module in generic coordinates, which is coordinate-independent both for variables of S and for homogeneous free generators of F (see [Par96]). So, given a graded R-module M, in generic coordinates, and a monomial order through the initial module $\mathrm{in}(M)$, we can read off information about the module M, for example the depth of F/M or the regularity of M. One can observe that is more convenient to handle initial modules with respect to a given coordinates. So instead of making a generic transformation of them and considering the initial module, it is suitable to transform a module by a generic linear transformation and consider its initial module in the given coordinates.

The first result of Galligo about the *generic initial ideals* $\mathrm{Gin}(I)$ is referred to characteristic 0. Bayer, Stillman worked about this idea and achieved the definition of generic initial ideal in finite characteristic. Pardue ([Par94]) has extended the definition to the *generic initial modules.*

More in details, let K be a field. The linear group $\mathrm{GL}_n(K) = \mathrm{GL}(n)$ is a Zariski open subset of $M_n(K)$. Let $I \subset S$ be a graded ideal and $<$ a monomial order on S. Then there exists a nonempty open subset $U \subset \mathrm{GL}(n)$ such that $\mathrm{in}_<(\alpha I) = \mathrm{in}_<(\alpha' I)$ for all $\alpha, \alpha' \in U$. This result leads out to the definition of $\mathrm{Gin}(I) = \mathrm{in}_<(\alpha I)$ for $\alpha \in U$. Let us denote by $\mathcal{B}(n)$ the subgroup of $\mathrm{GL}(n)$ of all nonsingular upper triangular matrices. $\mathcal{B}(n)$ is called the *Borel subgroup* of $\mathrm{GL}(n)$. The following result holds.

Theorem 1.2.6 *(Galligo, Bayer and Stillman)[Gal74, BS87]*
If $I \subset S$ is a homogeneous ideal then $\mathrm{Gin}(I)$ is Borel-fixed in the sense that for all $g \in \mathcal{B}(n)$, then $g(\mathrm{Gin}(I)) = \mathrm{Gin}(I)$.

Let F be the graded free module over S with homogeneous basis. In order to generalize Theorem 1.2.6, one can consider $\mathrm{GL}(F)$ to be the group of S-linear graded automorphisms of F. Since conjugation is an action of $\mathrm{GL}(n)$ on $\mathrm{GL}(F)$, we can write $G = \mathrm{GL}(n) \rtimes GL(F)$.

This is an algebraic group that acts on F through K-linear graded automorphisms that take submodules to submodules. Also in this case there exists an open subset $U \subset G$ such that $\mathrm{in}_<(M)$ is invariant for all transformation in G. The subgroup $\mathcal{B}(F) \subset \mathrm{GL}(F)$ of all nonsingular upper triangular matrices is called the *Borel subgroup* of $\mathrm{GL}(F)$. Then $\mathcal{B}(n)$ acts on $\mathcal{B}(F)$ and $\mathcal{B} = \mathcal{B}(n) \rtimes \mathcal{B}(F) \subset G$ is called the Borel subgroup of G. The following result holds.

Proposition 1.2.7 *([Par96, Proposition 6])*
Let $F = \oplus_{i=1}^r Sg_i$. A submodule $M \subset F$ is fixed by the action of $\mathcal{B}$ on F if and only

(i) $M = I_1 g_1 \oplus \cdots \oplus I_r g_r$ *is a monomial submodule*

(ii) for every monomial $u \in M$, if $x_j^\ell \mid u$, $x_j^{\ell+1} \nmid u$ and $i < j$ then $(x_i/x_j)^d u \in M$ for every $d \leq \ell$

(iii) $\mathfrak{m}^{d_j - d_i} I_j \subseteq I_i$ *for every $i < j$.*

Bayer proved Proposition 1.2.7 in the case of $F = S$ ([Bay82]).
The previous result forced Pardue ([Par96]) to give the following definition.

Definition 1.2.8 A submodule $N \subseteq F$ is a *Borel-fixed* submodule if N is fixed by $\mathcal{B}(F)$. A submodule $N \subseteq F$ is a *standard Borel-fixed* submodule if N satisfies conditions (i) and (iii) of Proposition 1.2.7 and furthermore for every monomial $mg_i \in N$, if $x_j \mid m$ then $(x_i/x_j)mg_i \in N$ for every $i < j$.

A standard Borel-fixed submodule is Borel-fixed. If the characteristic of K is zero, then every Borel-fixed submodule is standard. A Borel-fixed submodule which is not standard is called nonstandard.

One can observe that for $r = 1$ and $\deg(g_1) = 0$, a standard Borel-fixed submodule M of F is a strongly stable ideal of S ([EK90]). We recall that a monomial ideal $I \subset S = K[x_1, \ldots, x_n]$ is called *strongly stable* if for each monomial $u \in I$ and each $x_j \mid u$ one has $(x_i/x_j)u \in I$, for all $i < j$. For this reason, throughout the paper, a standard Borel-fixed submodule of F will be called a strongly stable submodule. Hence, we give the following definition.

Definition 1.2.9 A monomial submodule $M = \oplus_{i=1}^r I_i g_i$ of F is a *strongly stable* submodule if I_i is a strongly stable ideal of S, for each i, and $(x_1, \ldots, x_n)^{f_{i+1} - f_i} I_{i+1} \subseteq I_i$, for $i = 1, \ldots, r-1$.

In [EK90], next class of monomial ideals was introduced.

Definition 1.2.10 Let I be a monomial ideal of $S = K[x_1, \ldots, x_n]$. I is called *stable* if for each monomial $u \in I$ and each $j < \mathrm{m}(u)$ one has $(x_j/x_{\mathrm{m}(u)})u \in I$.

Therefore, we give next definition.

Definition 1.2.11 Let $F = \oplus_{i=1}^r S g_i$. A monomial submodule $M = \oplus_{i=1}^r I_i g_i$ is called *stable* if I_i is a stable ideal of S, for each i, and $(x_1, \dots, x_n)^{f_{i+1}-f_i} I_{i+1} \subseteq I_i$, for $i = 1, \dots, r-1$.

The above definitions can be introduced also in the exterior algebra context with the suitable modifications (see Section 3.1 in Chapter 3).

Example 1.2.12 We give an example of a (strongly) stable ideal of R and of a (strongly) stable submodule of a graded free module F.

(*i*) Let $S = K[x_1, x_2, x_3]$. The ideal $I = (x_1^2, x_1x_2, x_2^2, x_2x_3)$ is an example of stable ideal which is not a strongly stable ideal. Indeed, $x_1x_3 \notin I$. The ideal $J = I \cup \{x_1x_3\} = (x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3)$ is the smallest strongly stable ideal containing I.

(*ii*) Let $E = K\langle e_1, e_2, e_3, e_4\rangle$ and $F = Eg_1 \oplus Eg_2 \oplus Eg_3$, with $\deg g_1 = -2$, $\deg g_2 = -1$ and $\deg g_3 = 3$. The submodule

$$M = (e_1e_3, e_2e_3, e_1e_2e_4)g_1 \oplus (e_1e_2, e_2e_3, e_2e_4, e_3e_4)g_2 \oplus (e_1e_2e_3, e_2e_3e_4)g_3$$

is not a stable submodule of F. The smallest stable submodule containing M is

$$Ms = (e_1e_2, e_1e_3, e_2e_3)g_1 \oplus (e_1e_2, e_1e_3, e_2e_3, e_2e_4, e_3e_4)g_2 \oplus (e_1e_2e_3, e_2e_3e_4)g_3.$$

Ms is not a strongly stable submodule of F, so we can compute the smallest one containing it:

$$\begin{aligned} Mss = (e_1e_2, e_1e_3, e_2e_3)g_1 \oplus (e_1e_2, e_1e_3, e_1e_4, e_2e_3, e_2e_4, e_3e_4)g_2 \\ \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4)g_3. \end{aligned}$$

Remark 1.2.13 Some of the results discussed about generic initial ideals in S and generic initial S-modules also hold in the context of the exterior algebra E and E-modules with the appropriate modifications. Generic initial modules are defined exactly as before and the analogues of Theorem 1.2.6 and Proposition 1.2.7 have been stated by Aramova, Herzog, Hibi in [AHH97] (see also Green [Gre98].

A fundamental subclass of strongly stable modules is that one of the lexsegment modules. Indeed, this modules have some properties useful to classify important geometrical invariants, as we will see in the sequel.

Let $R \in \{S, E\}$ and let us consider a graded free R-module F with homogeneous basis endowed with a lexicographic order $>_{\mathrm{lex}_F}$ (Example 1.2.4).

Definition 1.2.14 A nonempty subset N of $\mathrm{Mon}_d(F)$ is called a *lexicographic segment* of F (lex_F segment, for short) of degree d if for all $v \in N$ and all $u \in \mathrm{Mon}_d(F)$ such that $u >_{\mathrm{lex}_F} v$, then $u \in N$.

Example 1.2.15 Let $E = K\langle e_1, e_2, e_3\rangle$ and $F = E^2$. The subset $N = \{e_1e_2g_1,\ e_1e_3g_1,\ e_2e_3g_1,\ e_1e_2g_2\}$ is a lex_F segment of degree 2 of F; on the contrary, $N' = \{e_1e_2g_1,\ e_1e_3g_1,\ e_1e_2g_2\}$ is not a lex_F segment of degree 2. Indeed, the monomial $e_2e_3g_1 >_{\mathrm{lex}_F} e_1e_2g_2$ does not belong to N'.

Definition 1.2.16 Let L be a monomial submodule of F. L is a *lexicographic submodule* (lex submodule, for short) if for all $u, v \in \mathrm{Mon}_d(F)$ with $v \in L$ and $u >_{\mathrm{lex}_F} v$, one has $u \in L$, for every d, *i.e.*, $\mathrm{Mon}_d(L)$ is a lex_F segment of degree d, for each degree d.

Remark 1.2.17 A monomial submodule $\mathcal{L}$ of F is a lexicographic submodule if $\mathrm{Mon}_d(\mathcal{L})$ is a lex_F segment of degree d, for each degree d; $\mathrm{Mon}_d(\mathcal{L})$ is the set of all monomials of degree d of $\mathcal{L}$.

Example 1.2.18 Let $E = K\langle e_1, e_2, e_3, e_4\rangle$ and $F = Eg_1 \oplus Eg_2 \oplus Eg_3$, with $\deg g_1 = -2$, $\deg g_2 = -1$ and $\deg g_3 = 3$. The submodule

$$L = (e_1e_2, e_1e_3, e_2e_3e_4)g_1 \oplus (e_1e_2, e_1e_3, e_1e_4, e_2e_3)g_2 \oplus (e_1e_2e_3, e_1e_2e_4)g_3$$

is a lex submodule of F.

It is easy to show the relations between the classes of stable, strongly stable and lexicographic modules:

$$\{\text{lex modules}\} \subsetneq \{\text{strongly stable modules}\} \subsetneq \{\text{stable modules}\}$$

For instance, the submodule Ms in Example 1.2.12[(*ii*)] is stable but not strongly stable. Moreover, the submodule Mss is strongly stable but not lexicographic. So, we can compute the lexicographic submodule L of F that contains Mss.

$$\begin{aligned} L = (e_1e_2, e_1e_3, e_1e_4, e_2e_3)g_1 \oplus (e_1e_2, e_1e_3, &e_1e_4, e_2e_3, e_2e_4, e_3e_4)g_2 \\ &\oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4)g_3. \end{aligned}$$

1.3 Hilbert Functions

Hilbert functions represent numerical invariants of projective algebraic sets. Invariant theory has been of great importance after the second half of the nineteenth century and it originated to obtain properties of geometric objects defined by equations, that were invariant under some geometrically defined set of transformations.

Definition 1.3.1 Let $M = \bigoplus_{d\in\mathbb{Z}} M_d$ be a finitely generated graded module over R, with grading by degree. The numerical function $H_M :\ \mathbb{Z} \to \mathbb{Z}$ defined by

$$H_M(d) = \dim_K M_d$$

is called the *Hilbert function* of M.

Note that the dimensions in the definition are all finite. Indeed, if M_d were not finite dimensional, then the submodule $\bigoplus_{d\in\mathbb{Z}} M_d$ would not be finitely generated, contradicting the fact that because R is Noetherian and M finitely generated then M is Noetherian.

If M is a finitely generated graded module over a standard graded algebra R (for instance as S or E with n indeterminates) then $H_M(d)$ agrees, for large d, with a polynomial of degree $\leq n-1$.
This polynomial, denoted $P_M(d)$ is called the *Hilbert polynomial* of M.

Note that, if all variables of R have degree 1, then the Hilbert function agrees with a polynomial function of d for large d. This is not true when the variables have different degrees. In this case $H_M(d)$ still agrees with a periodic polynomial, but it is often more convenient to use the Hilbert series instead.
The *Hilbert series* of M is defined to be the formal Laurent series in one variable t given by

$$h_M(t) = \sum_{d\in\mathbb{Z}} H_M(d)t^d.$$

The Hilbert series of a graded module, for large d, agrees with a rational function.

Now, let R be a graded algebras as well as $S = K[x_1, \ldots, x_n]$ or $E = K\langle e_1, \ldots, e_n\rangle$, with K a field. Some fundamental properties about Hilbert functions can be stated using Macaulay's results. This approach take into account the theory of monomial orders.

Theorem 1.3.2 *Let P be a finitely generated, graded R-module, given by generators and relations as $P = F/M$, where F is a free module with a homogeneous basis and M is a submodule generated by homogeneous elements. The Hilbert function of P is the same as the Hilbert function of $F/\operatorname{in}(M)$.*

Since $\operatorname{in}(M)$ is a monomial submodule of F with the same Hilbert function as M, Theorem 1.3.2 allows us to assume M itself to be a monomial submodule without changing the Hilbert function. Indeed, as an immediate consequence of the theorem we obtain that $H_{F/M}(d) = H_{F/\operatorname{in}(M)}(d)$ for all $d \in \mathbb{Z}$.

In order to quote other important results, we have to assume that K is a field of characteristic 0. Assume $S = K[x_1, \ldots, x_n]$ endowed with the lexicographic monomial order $>_{\text{lex}}$ induced by the order $x_1 > \cdots > x_n$. The next well known properties concern graded ideals of S. Some generalizations will be described later in this dissertation.

Theorem 1.3.3 *[HH11, Theorem 6.3.1]*
Let $I \subseteq S$ be a graded ideal. Then there exists a unique lexsegment ideal, denoted by I^{lex}, such that S/I and S/I^{lex} have the same Hilbert function.

Theorem 1.3.3 is a fundamental step to characterize the Hilbert functions of standard graded K-algebras. An important tool in this context is the so-called Macaulay expansion of a positive integer.

Let a and i be two positive integers. Then a has the unique i-th Macaulay expansion [HH11, Lemma 6.3.4]

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

with $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$.

We define

$$a^{<i>} = \binom{a_i+1}{i+1} + \binom{a_{i-1}+1}{i} + \cdots + \binom{a_j+1}{j+1} \text{ and}$$

$$a^{(i)} = \binom{a_i}{i+1} + \binom{a_{i-1}}{i} + \cdots + \binom{a_j}{j+1}.$$

We also set $0^{(i)} = 0^{<i>} = 0$ for all $i \geq 1$.

Example 1.3.4 Let be $a = 17$ and $i = 3$. The integer 17 has the unique 3-rd Macaulay expansion

$$17 = \binom{5}{3} + \binom{4}{2} + \binom{1}{1}.$$

Hence:

$$17^{<3>} = \binom{5+1}{3+1} + \binom{4+1}{2+1} + \binom{1+1}{1+1} = \binom{6}{4} + \binom{5}{3} + \binom{2}{2} = 26,$$

$$17^{(3)} = \binom{5}{3+1} + \binom{4}{2+1} + \binom{1}{1+1} = \binom{5}{4} + \binom{4}{3} + \binom{1}{2} = 9$$

Theorem 1.3.5 ([HH11, Theorem 6.3.8])
Let $h:\ \mathbb{Z}_+ \to \mathbb{Z}_+$ be a numerical function. The following conditions are equivalent:

- (i) *h is the Hilbert function of a standard graded K-algebra;*
- (ii) *there exists an integer $n \geq 0$ and a lexsegment ideal $I \subset S = K[x_1, \ldots, x_n]$ such that $h_d = H_{S/I}(d)$ for all $d \geq 0$;*
- (iii) *$h(0) = 1$ and $h_{i+1} \leq h_i^{<i>}$ for all $i > 0$.*

Example 1.3.6 Let us consider the function $h:\ \mathbb{Z}_+ \to \mathbb{Z}_+$ defined as follows:

$$h(0) = 1,\ h(1) = 3,\ h(2) = 5,\ h(3) = 7,\ h(4) = 8$$

and $h(i) = 0$ for all integers $i < 0$ and $i > 4$. Then we have

$$\begin{aligned} h(1) &= \binom{3}{1} \\ h(2) &= \binom{3}{2} + \binom{2}{1} \\ h(3) &= \binom{4}{3} + \binom{3}{2} \\ h(4) &= \binom{5}{4} + \binom{3}{3} + \binom{2}{2} + \binom{1}{1} \end{aligned}$$

We note that $h(0) = 1$ and the following inequalities hold

$$h(2) = 5 \leq 6 = \binom{4}{2} = h(1)^{<1>}$$
$$h(3) = 7 \leq 7 = \binom{4}{3} + \binom{3}{2} = h(2)^{<2>}$$
$$h(4) = 8 \leq 9 = \binom{5}{4} + \binom{4}{3} = h(3)^{<3>}$$

Hence condition [(iii)] of Theorem 1.3.5 holds. Let $n = 4$ and $S = K[x_1, x_2, x_3, x_4]$, then

$$I = (x_1, x_2^2, x_2x_3^3, x_4^5, x_3x_4^4, x_2x_4^4, x_3^2x_4^3, x_2x_3x_4^3, x_3^3x_4^2, x_2x_3^2x_4^2, x_3^4x_4, x_3^5)$$

is a lex ideal such that the Hilbert function of S/I is equal to h.

Theorem 1.3.5 holds also in the context of the exterior algebra and it is known as the Kruskal–Katona theorem.

Theorem 1.3.7 ([AHH97, Theorem 4.1])
Let $(h_1, \ldots, h_n)$ be a sequence of non-negative integers. Then the following conditions are equivalent:

(i) $1 + \sum_{i=1}^{n} h_i t^i$ *is the Hilbert series of a graded K-algebra E/I;*

(ii) $0 < h_{i+1} \leq h_i^{(i)}$, $0 < i \leq n - 1$.

The proof of this theorem use the following idea. If I is a graded ideal of E, then there exists a unique lex segment ideal of E, usually denoted by I^{lex}, such that $H_{E/I} = H_{E/I^{\mathrm{lex}}}$ (see also Theorem 1.3.3).

Example 1.3.8 Consider the sequence $(h_1, h_2, h_3, h_4, h_5) = (5, 9, 5, 1, 0)$. Then we have

$$h_1 = \binom{5}{1}$$
$$h_2 = \binom{4}{2} + \binom{3}{1}$$
$$h_3 = \binom{4}{3} + \binom{2}{2}$$
$$h_4 = \binom{4}{4}$$
$$h_5 = 0$$

We note that $h_1 \leq 5$. The following inequalities hold

$$
\begin{aligned}
h_2 &= 9 \leq 10 = \binom{5}{2} = h_1^{(1)} \\
h_3 &= 5 \leq 7 = \binom{4}{3} + \binom{3}{2} = h_2^{(2)} \\
h_4 &= 1 \leq 1 = \binom{4}{4} + \binom{2}{3} = h_3^{(3)} \\
h_5 &= 0 \leq 0 = \binom{4}{5} = h_4^{(4)}
\end{aligned}
$$

The conditions in Theorem 1.3.7 are verified. Let $n = 5$ and $E = K\langle e_1, e_2, e_3, e_4, e_5\rangle$, then the lex ideal

$$I = (e_1e_2, e_1e_3e_4, e_1e_3e_5)$$

is such that the Hilbert series of E/I is $1 + \sum_{i=1}^{5} h_i t^i$.

1.4 Minimal Resolutions

The minimal resolution of a module M is a good tool for extracting information about M. Hilbert originally has studied free resolutions because their discrete invariants. The degrees of the generators of its free modules, not only yield the Hilbert function (as would be true for any resolution) but form a finer invariant.

If R is a graded ring, then a graded free R-module F is a direct sum of modules of the form $R(d)$, for various d. This notation indicates that the graded module R *shift* its grading by d steps (d-th *twist* of R). Note that $R(d)$ is isomorphic to R. So, if F is finitely generated, then $R(d) = Rg_i$ for some i.
Note that $R(d)_e = R_{d+e}$, then $R(d)_{-d} = R_0$. So $R(d)$ has its generator in degree $-d$, not d.

Definition 1.4.1 A *complex* of R-modules is a sequence of modules F_i and maps $F_i \to F_{i-1}$ such that the compositions $F_{i+1} \to F_i \to F_{i-1}$ are all zero.
The *homology* of this complex at F_i is the module

$$H_i(F_\bullet) = \ker(F_i \to F_{i-1})/\mathrm{im}(F_{i+1} \to F_i).$$

A *free resolution* of an R-module M is a complex

$$F_\bullet : \cdots \to F_s \xrightarrow{\varphi_s} \cdots \to F_1 \xrightarrow{\varphi_1} F_0,$$

of free R-modules such that $\mathrm{coker}\varphi_1 = M$ and is exact.

In this thesis we will sometimes abuse this notation and use

$$F_\bullet : \cdots \to F_s \xrightarrow{\varphi_s} \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \to M \to 0,$$

to say that an exact sequence $F_\bullet$ is a resolution of M. The image of the map φ_i is called the i-th syzygy module of M. Note that, $F_\bullet$ is a free resolution if and only if $H_i(F_\bullet) = 0$ for $i \neq 0$ and $H_0(F_\bullet) = M$.
A resolution $F_\bullet$ is a *graded free resolution* if R is a graded ring, the F_i are graded free modules, and the maps are homogeneous maps of degree 0. If there exists a natural s such that $F_{s+1} = 0$ and $F_i \neq 0$ for $0 < i < s$, then we shall say that $F_\bullet$ is a finite resolution of length s.

Recall also that given a complex $F_\bullet$ of finitely generated free modules and a R-module M, then $F_\bullet \otimes_R M$, $M \otimes_R F_\bullet$, $\mathrm{Hom}_R(F_\bullet, M)$ and $\mathrm{Hom}_R(M, F_\bullet)$ are still complexes with complex maps induced by $\varphi \otimes_R M$, $M \otimes_R \varphi$, $\mathrm{Hom}_R(\varphi, M)$ and $\mathrm{Hom}_R(M, \varphi)$.

Remark 1.4.2 Note that, to construct a free resolution for a module M one can begin by taking a set of generators for M and map a free module onto M sending the free generators of the free module to the given generators of M. Let M_1 be the kernel of this map. After that, one can repeat the procedure starting with M_i.
Hence, it is clear that every module has a free resolution and every graded module has a graded free resolution (observe that only graded modules can have graded free resolutions).

Moreover, an important and useful result on commutative algebra has been given by Hilbert.

Theorem 1.4.3 *(Hilbert syzygy theorem)*
If $S = K[x_1, \ldots, x_n]$, then every finitely generated graded S-module has a finite graded free resolution of length $< n$, by finitely generated free modules.

Let M be a finitely generated graded S-module and let

$$F_\bullet : 0 \to F_s \xrightarrow{\varphi_s} \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \to M \to 0,$$

be a finite graded free resolution of M. Since the φ_i preserve degrees, we get an exact sequence of finite dimensional vector spaces by taking the degree d part of each module in this sequence. Thus the Hilbert function of M

$$H_M(d) = \sum_{i=0}^{s} (-1)^i H_{F_i}(d)$$

is a linear combination of the Hilbert functions of the free modules F_i.

A free resolution of M depends strongly on the choice of generators for M, as well as the subsequent choices of generators of M_1, and so on. If M is a finitely generated graded module, the operations described in Remark 1.4.2 can be replicated choosing a minimal set of homogeneous generators m_i for a finitely generated grade module M. So, map a graded free module F_0 onto M by sending a basis for F_0 to the set of m_i. Let M' be the kernel of the map $F_0 \to M$, and repeat the procedure, starting with a minimal system of homogeneous generators of M'. In such a case, one obtains a *minimal graded free resolution.*

Formally, let us $S = K[x_1, \dots, x_n]$. We will use standard notation $\mathfrak{m}$ to denote the homogeneous maximal ideal $(x_1, \dots, x_n)$.

Definition 1.4.4 A graded free resolution of S-module

$$F_\bullet : \cdots \to F_s \xrightarrow{\varphi_s} \cdots \to F_1 \xrightarrow{\varphi_1} F_0,$$

is called *minimal* if the image of φ_i is contained in $\mathfrak{m}F_{i-1}$ for all $i \in \mathbb{N}$.

This means, that no invertible elements (non-zero constants) appear in the matrices representing the maps φ_i. This definition are related to the constructive method by a graded version of the Nakayama's Lemma. Moreover, some fundamental results following the Lemma allow us to discuss some properties of minimal graded free resolutions.

Lemma 1.4.5 *(Nakayama)[Eis05, Lemma 1.4] If M is a finitely generated graded S-module and $m_1, \dots, m_n \in M$ generate $M/\mathfrak{m}M$ then $m_1, \dots, m_n$ generate M.*

Corollary 1.4.6 *If*

$$F_\bullet : \cdots \to F_s \xrightarrow{\varphi_s} \cdots \to F_1 \xrightarrow{\varphi_1} F_0,$$

is a graded free resolution, then $F_\bullet$ is minimal if and only if for the maps φ_i takes a basis of F_i to a minimal set of generators of the image of φ_i, for all $i \in \mathbb{N}$.

Theorem 1.4.7 *Let M be a finitely generated graded S-module. If $F_\bullet$ and $G_\bullet$ are minimal graded free resolutions of M, then there is a graded isomorphism of complexes $F_\bullet \to G_\bullet$ inducing the identity map on M. Any free resolution of M contains the minimal free resolution as a direct summand.*

Example 1.4.8 Let $S = K[x_1, x_2, x_3]$ and let $I = (x_1^2 x_2, x_1 x_2 x_3^3, x_2 x_3^2, x_1 x_2^2)$ be a monomial ideal of S. A minimal set of generators of I is $G_0 = \{x_2 x_3^2, x_1 x_2^2, x_1^2 x_2\}$. We now compute a set of generators for the syzygy module of G_0. It can be represented as column vectors a_{ij} such that $a_{ij1}(x_2 x_3^2) + a_{ij2}(x_1 x_2^2) + a_{ij3}(x_1^2 x_2) = 0$. So, we obtain the following 3×3 matrix representing the map $\varphi_1 : \ F_1 \to F_0$

$$\varphi_1 = \begin{bmatrix} x_1 x_2 & x_1^2 & 0 \\ -x_3^2 & 0 & x_1 \\ 0 & -x_2^2 & -x_2 \end{bmatrix}$$

Now, let G_1 a minimal set of generators for F_1. We compute a set of generators for the syzygy module of G_1, hence a matrix representing the map $\varphi_2 : \ F_2 \to F_1$

$$\varphi_2 = \begin{bmatrix} -x_1 \\ x_2 \\ -x_3^2 \end{bmatrix}$$

The next step is trivial. All this information is represented in the following free resolution

$$F_\bullet : 0 \to F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \to I \to 0.$$

By the construction, it is clearly a minimal free resolution. In order to verify that it is a minimal graded free resolution we can write it more in detail

$$F_\bullet : 0 \to S(-6) \xrightarrow{\begin{bmatrix} -x_1 \\ x_2 \\ -x_3^2 \end{bmatrix}} S(-5)^2 \oplus S(-4) \xrightarrow{\begin{bmatrix} x_1x_2 & x_1^2 & 0 \\ -x_3^2 & 0 & x_1 \\ 0 & -x_2^2 & -x_2 \end{bmatrix}} S(-3)^3 \to S(0) \to 0.$$

An important consequence of the uniqueness of minimal free resolutions is the fact that, if $F_\bullet$ is the minimal graded free resolution of a finitely generated graded S-module M, then the number of generators of each degree required for the free modules F_i depends only on M. The easiest way to state a precise result is to use the functor Tor. If R is a graded algebra over a field K and M, N are graded R-modules, then ([Rö1])

$$\mathrm{Tor}_i^R(N, M) \cong H_i(N \otimes F_\bullet) \quad \text{and} \quad \mathrm{Ext}_R^i(M, N) \cong H^i(\mathrm{Hom}(F_\bullet, N)).$$

Proposition 1.4.9 *[Eis05, Proposition 1.7] If $F_\bullet : \cdots \to F_s \xrightarrow{\varphi_s} \cdots \to F_1 \xrightarrow{\varphi_1} F_0$, is the minimal free resolution of a finitely generated graded S-module M, and $\mathcal{K}$ denotes the residue field $S/\mathfrak{m}$ then any minimal set of homogeneous generators of F_i contains precisely* $\dim_K \mathrm{Tor}_i^S(\mathcal{K}, M)_j$ *generators of degree j.*

The previous results allow to define some important invariants.

Definition 1.4.10 Let $F_\bullet$ be the minimal graded free S-resolution of a graded finitely generated S-module M:

$$F_\bullet : 0 \to F_s \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where $F_i = \oplus_{j\in\mathbb{Z}} S(-j)^{\beta_{i,j}}$. The integers $\beta_{i,j} = \beta_{i,j}(M) = \dim_K \mathrm{Tor}_i(\mathcal{K}, M)_j$ are called the *graded Betti numbers* of M. Moreover, $\beta_i = \sum_{j\in\mathbb{Z}} \beta_{i,j}$ is said to be the ith-total Betti number of M.

The computer program *Macaulay2* (see also CoCoA) displays the (graded) Betti numbers in a table called the Betti diagram.

		0	1	$\cdots$	i	$\cdots$
$\cdots$	:	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
0	:	$\beta_{0,0}$	$\beta_{1,0}$	$\cdots$	$\beta_{i,i+0}$	$\cdots$
1	:	$\beta_{0,1}$	$\beta_{1,2}$	$\cdots$	$\beta_{i,i+1}$	$\cdots$
$\cdots$	:	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
j	:	$\beta_{0,j}$	$\beta_{1,1+j}$	$\cdots$	$\beta_{i,i+j}$	$\cdots$
$\cdots$	:	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Table 1.1: Betti diagram

For instance, if we consider the minimal graded free resolution of $I = (x_1^2x_2, x_1x_2x_3^3, x_2x_3^2, x_1x_2^2) \subset S = K[x_1, x_2, x_3]$ in Example 1.4.8 the Betti diagram of S/I as S-module is:

		0	1	2	3
0	:	1	$-$	$-$	$-$
1	:	$-$	$-$	$-$	$-$
2	:	$-$	3	1	$-$
3	:	$-$	$-$	2	1

Next fundamental definition can be introduced.

Definition 1.4.11 [RÖ1] Let M be a graded submodule of a free module F over $S = K[x_1, \dots, x_n]$. Then

$$\mathrm{pd}(M) = \sup\{i \in \mathbb{Z} \ : \ \beta_{i,j}(M) \neq 0 \text{ for some } j \in \mathbb{Z}\}$$

is said to be the *projective dimension* of M and

$$\mathrm{reg}(M) = \sup\{j \in \mathbb{Z} \ : \ \beta_{i,i+j}(M) \neq 0 \text{ for some } i \in \mathbb{Z}\}$$

is called the *Castelnuovo-Mumford regularity* of M.

Theorem 1.4.3 assures that $\mathrm{pd}(M) \leq n$ and $\mathrm{reg}(M) \leq \infty$. Bayer, Charalambous and Popescu introduced in [BCP99] the following refinement of the projective dimension and the regularity.

Definition 1.4.12 Let M a graded submodule of the free module F over $S = K[x_1, \dots, x_n]$. A graded Betti number $\beta_{i,i+j}(M) \neq 0$ is called *extremal* if $\beta_{l,\,l+r}(M) = 0$ for all $l \geq i$, $r \geq j$, $(i, j) \neq (l, r)$.

From the Betti diagram of the minimal graded free resolution of M one can observe that the outside corners of the dashed lines give the positions of the extremal Betti numbers.

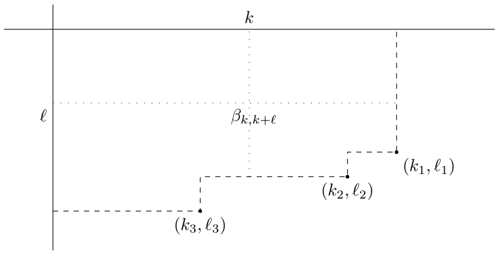

Table 1.2: Extremal Betti diagram

Note that, if β_{i_1,i_1+j_1}, β_{i_2,i_2+j_2}, $\ldots$, β_{i_t,i_t+j_t} $(i_1 < \cdots < i_t)$ are all the extremal Betti numbers of a graded module M, then $\operatorname{reg}(M) = j_1$ and $\operatorname{pd}(M) = i_t$.

Example 1.4.13 Let $S = K[x_1, x_2, x_3, x_4, x_5]$ and let

$$I = (x_2^2x_3, x_2x_3^3, x_2x_3^2x_4, x_2x_3^2x_5, x_3^5)$$

be a graded ideal of S. The extremal Betti numbers of I are $\beta_{3,3+4} = 1$, $\beta_{1,1+5} = 1$, as the Betti diagram of I shows:

		0	1	2	3
3	:	1	−	−	−
4	:	3	6	4	1
5	:	1	1	−	−

In particular, $\operatorname{reg}(M) = 5$ and $\operatorname{pd}(M) = 3$.

In this dissertation we analyze graded Betti numbers in some particular cases.

Remark 1.4.14 Note that, if we consider the exterior algebra $E = K\langle e_1, \ldots, e_n\rangle$ then it can be proved an analogous version of Nakayama's Lemma over E. Moreover, many results for graded modules over a commutative local or $\star$local ring can be proved to modules over an exterior algebra. For example, every projective E-module is free over E.

Every graded E-module M have a unique minimal graded free resolution and it can be obtained as well as for graded S-modules. Moreover, it also preserves the property of graded Betti numbers of M. These arguments can be found in [AAH00].

However, an important result does not hold: the Hilbert syzygies theorem. Indeed, the minimal graded projective resolution of a non free E-module has always infinite length. Therefore the projective dimension is not significant. Even though the projective dimension of M is infinite (unless M is free) the regularity can be defined as is remarked in [AHH97].

In the case of the exterior algebra E, an important role to compute the graded Betti numbers is played by the *Cartan complex* $C_\bullet(e_1, \ldots, e_n; E)$. It is defined by the complex whose i-chains $C_i(e_1, \ldots, e_n; E)$ are the elements of degree i of the free divided power algebra $E < x_1, \ldots, x_n >$. If M is a graded E-module, then $C_\bullet(e_1, \ldots, e_n; M) = M \otimes_E C_\bullet(e_1, \ldots, e_n; E)$ and $H_i(e_1, \ldots, e_n; M) = H_i(C_\bullet(e_1, \ldots, e_n; M))$ is the i-th *Cartan homology module*. An important result relates graded Betti numbers and Cartan homology complex, indeed $\mathrm{Tor}_i^E(M, K) \cong H_i(e_1, \ldots, e_n; M)$, for each $i \geq 0$. More details on this subject can be found in [AHH97].

1.5 On algorithms

Computer Algebra is a subject of science devoted to methods for solving mathematically formulated problems by symbolic algorithms, and to implementation of these algorithms. It is based on the exact finite representation of mathematical objects and structures, and allows for symbolic and abstract manipulation by a computer.

The interplay between computation and many areas of algebra is a natural phenomenon in view of the algorithmic character of the latter. The existence of inexpensive but powerful computational resources has enhanced these links by the opening up of many new areas of investigation in algebra.

A frequent task in computational algebra is to certify that a given object has a certain property, also providing rather elaborate examples. Moreover, they have contributed to a new view of algorithmic methods not only as tools, but as new objects worthy of mathematical study. This concerns both the design, verification, and complexity analysis of computer algebra algorithms, as well as non-algorithmic structural mathematics. In fact, an algorithmic approach to a classical problem may lead to a significant refinement of classical structure theory irrespective of algorithmic considerations.

In the last years, the theory of *Gröbner basis* has become a major research area in computational algebra and computer science because of its usefulness in providing computational tools which are applicable to a wide range of problems. Gröbner basis were introduced in 1965 by Buchberger. The basic idea behind the theory can be described as a generalization of the theory of polynomials in one variable, indeed a Gröbner bases is the analogue of greatest common divisors in the multivariate case. But the true significance of Gröbner bases is the fact that they can be computed algorithmically.

Let I be an ideal of $S = K[x_1, \ldots, x_n]$, a Gröbner basis for I is a set of generators with an additional property. Buchberger's algorithm yields a simple and effective method for computing Gröbner bases and syzygies. Through use of Gröbner bases, many questions about ideals in polynomial rings can be reduced to questions about monomial ideals, which are far easier.

Definition 1.5.1 A *Gröbner basis* with respect to a monomial order $>$ on a free module F with basis is a set of elements $h_1, \dots, h_t \in F$ such that if M is the submodule of F generated by $h_1, \dots, h_t$, then $\text{in}_>(h_1), \dots, \text{in}_>(h_t)$ generate $\text{in}_>(M)$.

Important well–known results from theory assure that every nonzero ideal I of $S = K[x_1, \dots, x_n]$ has a Gröbner basis. Such a basis can be obtained using Buchberger's algorithm as long as a system of generators of I is given. Moreover, if one imposes simple conditions on $\text{in}_>(g_1), \dots, \text{in}_>(g_t)$ then a minimal Gröbner basis for the ideal I can be computed.

Many *Computer Algebra Systems* have a Gröbner basis package, for example CoCoa, Macaulay2, Singular, Maple, and Mathematica. In particular, all the computations in the examples in this dissertation were performed using *Macaulay2* and *CoCoA*. These systems allow the user to build customized packages in order to extend some features not yet included. The algorithms presented in this thesis have been implemented (some of them is also included with *Macaulay2* version 1.14).

Here we collect some applications of Gröbner basis. The type of problems that can be solved with Gröbner bases can be divided into two groups: constructive module theory and *elimination theory.*

- *Constructive module theory*

  Let F be a graded free S-module with homogeneous basis and let M be a graded submodule of F endowed with a monomial order $>$. Let $G = \{h_1, \dots, h_t\}$ be a Gröbner basis for the module M.

  – *Module membership*

    Let $f \in F$ be an element of the free module then a characterization for G is that the remainder of the division of f by the elements in G is unique (with an appropriate definition of division). So, f is in M if and only if the remainder of the division of f by the elements in G is zero. Many other problems about operations with ideals are related to this one.

  – *Compute Syzygies*

    It is possible to compute syzygies of M on a fixed set of its generators (using Schreyer's [Eis95, Theorem 15.10]). Indeed, the Buchberger's algorithm is used to obtain a Gröbner basis for M but also the syzygies on the Gröbner basis elements. This process usually will not return a minimal set of syzygies: to replace it with a minimal set (obtaining a minimal resolution of M) one can analyse the nonminimal syzygies and using them to eliminate superfluous relations. Hence, some other problems related to the minimal free resolution of M can be solved.

  – *Compute module of Homomorphisms*

    Let M, N be finitely generated graded submodules of F. It is possible to compute a presentation of $\mathrm{Hom}(M, N)$ given two explicitly presentations of M and N. As direct consequence, all the previous observations can be used to give results on free resolutions and the computation of $\mathrm{Tor}_i^S(M, N)$ and $\mathrm{Ext}_S^i(M, N)$.

  - *Compute the Hilbert function*
    By the Definition 1.5.1 of G, it is immediate to compute a system of generators of initial ideal $\mathrm{in}(M)$. Theorem 1.3.2 assures that to compute the Hilbert functions of an arbitrary graded module M it is enough to compute the Hilbert function of the initial module $\mathrm{in}(M)$. The Hilbert functions of monomial modules are characterized in [Hul95, Corollary 6].

- Elimination theory

  - *Elimination*
    Compute the intersection between an ideal $I \subset K[x_1, \ldots, x_n]$ with a subring $K[x_1, \ldots, x_r]$. From a geometric point of view, this represent a projection of a variety of A^n defined by the vanishing polynomial in I to A^r. One of the main use of elimination is finding solutions for a system of polynomial equations, *i.e.* finding points of a variety.
  - *Closure*
    Compute the equations satisfied by given elements of an affine ring, *i.e.* compute the closure of the image of an affine or projective variety under a morphism.

Remark 1.5.2 Gröbner basis theory for the exterior algebra $E = K\langle e_1, \ldots, e_n\rangle$ is very similar to that for the polynomial ring $S = K[x_1, \ldots, x_n]$. The fact that the exterior algebra has zero divisors is responsible for modifications of some technical tools. All the results and strategies on for S hold for E, too. Some of them can be found in [HH11] or are argument of this dissertation.

Chapter 2

Generalizations of Kruskal–Katona's Theorem

Let K be a field, E the exterior algebra of a finite dimensional K-vector space, and F a finitely generated graded free E-module with homogeneous basis $g_1, \ldots, g_r$ such that $\deg g_1 \leq \deg g_2 \leq \cdots \leq \deg g_r$. We characterize the Hilbert functions of graded E–modules of the type F/M, with M graded submodule of F. The existence of a unique lexicographic submodule of F with the same Hilbert function as M plays a crucial role. This result is obtained both through a classical theoretical approach and through a new algorithmic approach. Such an approach allows us to establish a criterion for determining if a sequence of nonnegative integers defines the Hilbert function of a quotient of a free E–module only *via* the combinatorial Kruskal–Katona's theorem.

2.1 The Hilbert function of graded E-modules

In this Section, we discuss the Hilbert functions of quotients of free modules over the exterior algebra.

Let K be a field and let $E = K\langle e_1, \ldots, e_n\rangle$ be the exterior algebra of a K-vector space V with basis $e_1, \ldots, e_n$ (Remark 1.1.5). Let $\mathcal{M}$ be the category of finitely generated $\mathbb{Z}$-graded left and right E-modules M satisfying $am = (-1)^{\deg a \deg m} ma$ for all homogeneous elements $a \in E$, $m \in M$. Note that if I is a graded ideal of E, then $I \in \mathcal{M}$ and $E/I \in \mathcal{M}$.

Let $F \in \mathcal{M}$ be a free module with homogeneous basis $g_1, \ldots, g_r$, where $\deg(g_i) = f_i$ for each $i = 1, \ldots, r$, with $f_1 \leq f_2 \leq \cdots \leq f_r$. We write $F = \oplus_{i=1}^r E g_i$ and when we write $F = E^r$, we mean that F is the free E-module $F = \oplus_{i=1}^r E g_i$ with homogeneous basis $g_1, \ldots, g_r$, where g_i $(i = 1, \ldots, r)$ is the r-tuple where the unique non zero–entry is 1 in the i–th position, and such that $\deg(g_i) = 0$, for all i.

If $a = (a_1, \dots, a_p)$ and $b = (b_1, \dots, b_p)$ are two sequences of nonnegative integers, we say that $a > b$ if $(a_1, \dots, a_p) > (b_1, \dots, b_p)$ in the lexicographic ordering, *i.e.*, the difference $a_s - b_s$ is positive for the first index $1 \le s \le p$ where it is not zero.

We make the following conventions:

$$\binom{m}{k} = 0 \quad \text{if} \quad m < k \quad \text{or} \quad k < 0.$$

One can observe that if $E = K\langle e_1, \dots, e_n\rangle$, then $H_E(d) = \mathrm{Mon}(E_d) = \binom{n}{d}$, where $\binom{n}{d}$ is the number of monomials of degree d in E. Hence, if I is a graded ideal of E, it follows that

$$H_{E/I}(d) + H_I(d) = \binom{n}{d}.$$

Furthermore, if $F = \oplus_{i=1}^r Eg_i$, we have that

$$H_F(d) = \sum_{i=1}^r H_{Eg_i}(d) = \sum_{i=1}^r \binom{n}{d - f_i},$$

and consequently, if M is a graded submodule of F, one has

$$H_{F/M}(d) + H_M(d) = \sum_{i=1}^r \binom{n}{d - f_i},$$

where $\binom{n}{d-f_i}$ is the number of monomials of degree $d - f_i$ in F.

Important tools to proceed with the investigation are the definition of the Macaulay expansion (Section 1.4 of Chapter 1) and the Kruskal–Katona theorem (Theorem 1.3.7) which classifies Hilbert functions of quotients of exterior algebras

From now on, if $1 + \sum_{i=1}^n h_i t^i$ is the Hilbert series of a graded K-algebra E/I, $I \subsetneq E$, the sequence $(1, h_1, \dots, h_n)$ is called the *Hilbert sequence* of E/I. We will denote it by $Hs_{E/I}$.

From the Kruskal–Katona theorem, one can deduce that a sequence of nonnegative integers $(h_0, h_1, \dots, h_n)$ is the Hilbert sequence of a graded K–algebra E/I, with $I \subsetneq E$ graded ideal of initial degree ≥ 1, if $h_0 = 1$, $h_1 \le n$ and condition (b) in Theorem 1.3.7 holds. Note that if $I = 0$, then $Hs_{E/I} = Hs_E = (1, n, \binom{n}{2}, \cdots, \binom{n}{n})$.

Finally, we set $Hs_{E/I} = (\underbrace{0, \dots, 0}_{n+1})$, if $I = E$.

Let us consider the graded E-module $F = \oplus_{i=1}^r Eg_i$. One can quickly verify that

$$H_F(d) = \dim_K F_d = 0, \quad \text{for} \quad d < f_1 \quad \text{and} \quad d > f_r + n. \tag{2.1.1}$$

Now, we discuss the Hilbert function of a graded E–algebra F/M, with M submodule of F.

Discussion 2.1.1 Assume M is a monomial submodule of F. From (2.1.1), it follows that

$$H_{F/M}(t) = \sum_{i=f_1}^{f_r+n} H_{F/M}(i)t^i,$$

and we can associate to F/M the following sequence

$$(H_{F/M}(f_1), H_{F/M}(f_1+1), \dots, H_{F/M}(f_r+n)) \in \mathbb{N}_0^{f_r+n-f_1+1}. \tag{2.1.2}$$

Such a sequence is called *the Hilbert sequence* of F/M and it is denoted by $Hs_{F/M}$. The integers $f_1, f_1+1, \dots, f_r+n$ are called the $Hs_{F/M}$-degrees. It is clear that $Hs_{F/M} \le Hs_F$ component-wise.

Moreover, we define

$$\text{indeg} Hs_{F/M} = \min\{d : H_{F/M}(d) \neq 0\}, \quad \text{for } d = f_1, \dots, f_r+n.$$

We use the standard notation $[p]$ for the set $\{1, 2, \dots, p\}$.

Consider the sequence $Hs_{F/M}$ defined in (2.1.2). The entries $H_{F/M}(f_i)$ $(i = 1, \dots, r)$ are called the *critical values* of $Hs_{F/M}$. Moreover, we define

$$\mu_{f_i} = |\{s \in [r] : f_s = f_i\}|, \quad \text{for } \ i = 1, 2, \dots, r,$$

and we call μ_{f_i} the *multiplicity* of $H_{F/M}(f_i)$.

Now, let us consider the case $H_{F/M}(f_1) = 0$. In such a situation, one has:

$$M = Eg_1 \oplus T_2,$$

where T_2 is a submodule of $Eg_2 \oplus \cdots \oplus Eg_r$. Indeed, if $H_{F/M}(f_1) = 0$, then $M_{f_1} = F_{f_1}$ and so $M_j = F_j$, for $j = f_1, \dots, f_2 - 1$ (it is clear because $1_K g_1 \in M$). Hence, $H_{F/M}(j) = 0$, for $j = f_1, \dots, f_2 - 1$.

Now, let us consider the critical value $H_{F/M}(f_2)$.

If $H_{F/M}(f_2) = 0$, we can repeat the same reasoning done for $H_{F/M}(f_1) = 0$, *i.e.*, $H_{F/M}(j) = 0$, for $j = f_2, \dots, f_3 - 1$, and $M = Eg_1 \oplus Eg_2 \oplus T_3$, where T_3 is a submodule of $Eg_3 \oplus \cdots \oplus Eg_r$. And so on.

Now, let k be the minimum integer such that $H_{F/M}(f_k) \neq 0$, *i.e.*, $\text{indeg} Hs_{F/M} = f_k$. Note that $M = Eg_1 \oplus \cdots \oplus Eg_{k-1} \oplus T_k$, where T_k is a submodule of $Eg_k \oplus \cdots \oplus Eg_r$.We have:

$$H_{F/M}(f_k) \le \mu_{f_k},$$

and

$$H_{F/M}(f_k + 1) \le n\mu_{f_k} + \mu_{f_k+1}.$$

The integer $H_{F/M}(f_k)$ is called the *initial critical value* (of F/M) and f_k the *initial critical degree* (of F/M).

2.2 A generalization of Kruskal–Katona theorem

In this Section, we state a generalization of the Kruskal–Katona theorem. We characterize the Hilbert functions of quotients of the fixed free E-module $F = \oplus_{i=1}^r Eg_i$.

Our first result gives a new expression for the Hilbert functions of graded E-modules.

Proposition 2.2.1 *Let M be a graded submodule of $F = \oplus_{i=1}^{r} E g_i$ and let $H_{F/M}$ the Hilbert function of F/M. There exists an integer $N \le r$ such that we have the unique expression*

$$H_{F/M}(d) = \sum_{i=N+1}^{r} \binom{n}{d-f_i} + \binom{a_0}{d-f_N} + \binom{a_1}{d-f_N-1} + \cdots + \binom{a_s}{d-f_N-s},$$

where

$$\binom{a_0}{d-f_N} + \binom{a_1}{d-f_N-1} + \cdots + \binom{a_s}{d-f_N-s} < \binom{n}{d-f_N}$$

and $a_0 > a_1 > \cdots > a_s$ and $a_i \ge d - f_N - i$, for all $0 \le i \le s$.

Then,

$$H_{F/M}(d+1) \le \sum_{i=N+1}^{r} \binom{n}{d-f_i+1} + \binom{a_0}{d-f_N+1} + \binom{a_1}{d-f_N} + \cdots + \binom{a_s}{d-f_N-s+1},$$

for $d \ge \operatorname{indeg} Hs_{F/M} + 1$.

Proof. Since $\dim_K F_d = \sum_{i=1}^{r} \binom{n}{d-f_i}$, one has that

$$H_{F/M}(d) \le \sum_{i=1}^{r} \binom{n}{d-f_i}.$$

Let N be the greatest positive integer less than or equal to r such that

$$H_{F/M}(d) = \sum_{i=N+1}^{r} \binom{n}{d-f_i} + a = \sum_{i=N+1}^{r} H_E(d-f_i) + a, \qquad a < \binom{n}{d-f_N}.$$

We may assume there exists a graded ideal I of E generated in degree $d - f_N$ such that $H_{E/I}(d - f_N) = a$. If

$$a = \binom{a_0}{d-f_N} + \binom{a_1}{d-f_N-1} + \cdots + \binom{a_s}{d-f_N-s}$$

is the $(d - f_N)$–th Macaulay representation of a, one has:

$$H_{F/M}(d) = \sum_{i=N+1}^{r} \binom{n}{d-f_i} + \binom{a_0}{d-f_N} + \binom{a_1}{d-f_N-1} + \cdots + \binom{a_s}{d-f_N-s},$$

for $d \ge \operatorname{indeg} Hs_{F/M} + 1$. Therefore, from Theorem 1.3.7, it follows that:

$$\begin{aligned} H_{F/M}(d+1) &= \sum_{i=N+1}^{r} H_E(d+1-f_i) + H_{E/I}(d+1-f_N) \\ &\le \sum_{i=N+1}^{r} \binom{n}{d+1-f_i} + H_{E/I}(d-f_N)^{(d-f_N)} = \sum_{i=N+1}^{r} \binom{n}{d+1-f_i} + a^{(d-f_N)} \\ &= \sum_{i=N+1}^{r} \binom{n}{d+1-f_i} + \binom{a_0}{d-f_N+1} + \binom{a_1}{d-f_N} + \cdots + \binom{a_s}{d-f_N-s+1}. \end{aligned}$$

$\square$

If T is a set of monomials of degree $d < f_r + n$ of F, we denote by $\mathrm{Shad}(T)$ the following set of monomials of degree $d+1$ of F:

$$\mathrm{Shad}(T) = \{(-1)^{\alpha(\sigma,j)} e_j e_\sigma g_i : \, e_\sigma g_i \in T, \; j \notin \mathrm{supp}(e_\sigma), \, j = 1, \ldots, n, \, i = 1, \ldots r\},$$

$\alpha(\sigma, j) = |\{r \in \sigma : r < j\}|$. Such a set is called the *shadow* of T (see [CF15], for the $r = 1$ case). Moreover, let us define the i-th *shadow* recursively by $\mathrm{Shad}^i(T) = \mathrm{Shad}(\mathrm{Shad}^{i-1}(T))$, $\mathrm{Shad}^0(T) = T$.

Remark 2.2.2 Usually, the shadow of a set T of monomials of degree d of E, $d < n$, is defined as follows:

$$\mathrm{Shad}(T) = \{e_j e_\sigma : \, e_\sigma \in T, \; j \notin \mathrm{supp}(e_\sigma), \, j = 1, \ldots, n\}.$$

We observe that this definition is a little bit imprecise. In fact, if $j < \min(e_\sigma)$, then $e_j e_\sigma \in \mathrm{Mon}_{d+1}(E)$. Suppose $j > \min(e_\sigma)$ and $e_\sigma = e_{i_1} e_{i_2} \cdots e_{i_d}$. Since $e_h e_i = -e_i e_h$, $i, h \in \{1, \ldots, n\}$, then $e_j e_\sigma = (-1)^t e_{i_1} e_{i_2} \cdots e_{i_t} e_{i_j} e_{i_{t+1}} \cdots e_{i_d}$, where t is the largest integer such that $i_t < j$, that is, $t = \alpha(\sigma, j)$. Note that if t is odd, then $e_j e_\sigma \notin Mon_{d+1}(E)$.

Furthermore, if M is a monomial submodule of F, and M_d ($d \geq f_1$) is the K-vector space generated by all monomials of degree d belonging to M, we set $\mathrm{Shad}(M_d) = \mathrm{Shad}(\mathrm{Mon}(M_d))$ and by $E_1 M_d$ the K-vector space spanned by $\mathrm{Shad}(M_d)$.

For $p, q \in \mathbb{Z}$ with $p < q$, let us define the following set:

$$[p, q] = \{j \in \mathbb{Z} : p \leq j \leq q\}.$$

Remark 2.2.3 An important role in the next theorem is played by the class of lex submodules of F (Definition 1.2.16) and some results about lex ideals (Theorem 1.3.7). For our purpose, we state that the trivial ideals of E are monomial lex ideals.

Theorem 2.2.4 *Let $(f_1, f_2, \ldots, f_r) \in \mathbb{Z}^r$ be an r–tuple such that $f_1 \leq f_2 \leq \cdots \leq f_r$ and let $(h_{f_1}, h_{f_1+1}, \ldots, h_{f_r+n})$ be a sequence of nonnegative integers. Set*

$$s = \min\{k \in [f_1, f_r + n] : h_k \neq 0\},$$

and

$$\tilde{r}_j = |\{p \in [r] : f_p = s + j\}|, \quad \textit{for} \quad j = 0, 1.$$

Then the following conditions are equivalent:

(a) *$\sum_{i=s}^{f_r+n} h_i t^i$ is the Hilbert series of a graded E-module F/M, with $F = \oplus_{i=1}^r E g_i$ finitely generated graded free E-module with the basis elements g_i of degrees f_i;*

(b) $h_s \leq \tilde{r}_0$, $h_{s+1} \leq n\tilde{r}_0 + \tilde{r}_1$, $h_i = \sum_{j=N+1}^{r} \binom{n}{i-f_j} + a$, *where a is a positive integer less than* $\binom{n}{i-f_N}$, $0 < N \leq r$, *and* $h_{i+1} \leq \sum_{j=N+1}^{r} \binom{n}{i-f_j+1} + a^{(i-f_N)}$, $i = s+1, \ldots, f_r + n$;

(c) *there exists a unique lexicographic submodule L of a finitely generated graded free E-module $F = \oplus_{i=1}^{r} Eg_i$ with the basis elements g_i of degrees f_i and such that $\sum_{i=s}^{f_r+n} h_i t^i$ is the Hilbert series of F/L.*

Proof. (a)$\Rightarrow$ (b). It follows from Proposition 2.2.1 and Discussion 2.1.1. Note that s is the initial critical degree, $\widetilde{r}_0 = \mu_s$ and $\widetilde{r}_1 = \mu_{s+1}$.

(b)$\Rightarrow$ (c). We construct a lexicographic submodule L of F such that $H_{F/L}(t) = \sum_{i=s}^{f_r+n} h_i t^i$.

Setting $L_p = \langle \mathrm{Mon}(F_p) \rangle$ ($p = f_1, \ldots, s-1$), let L_{s+j} be the K-vector space generated by the lex_F segment of length $\dim_K F_{s+j} - h_{s+j}$, $j = 0, 1$, where $h_s \leq \tilde{r}_0$ and $h_{s+1} \leq n\tilde{r}_0 + \tilde{r}_1$.

Now, suppose L_k, $s \leq k \leq i$, has already been constructed.

By hypothesis, $\dim_K F_i/L_i = h_i = \sum_{j=N+1}^{r} \binom{n}{i-f_j} + a$, where a is a positive integer less than $a < \binom{n}{i-f_N}$. Hence,

$$\dim_K F_{i+1}/E_1 L_i = \sum_{j=N+1}^{r} \binom{n}{i - f_j + 1} + a^{(i-f_N)}$$

and

$$h_{i+1} \leq \dim_K F_{i+1}/E_1 L_i. \tag{2.2.1}$$

Let L_{i+1} be the K-vector space spanned by the lex_F segment of length $\dim_K F_{i+1} - h_{i+1}$. From (2.2.1), one has

$$\dim_F L_{i+1} = \dim_K F_{i+1} - h_{i+1} \geq \dim_K F_{i+1} - \dim_K F_{i+1}/E_1 L_i = \dim_K E_1 L_i.$$

Hence $E_1 L_i \subseteq L_{i+1}$. It follows that $L = \oplus_d L_d$ is a submodule of F. The uniqueness of L is clear from the definition of lex submodules.

(c)$\Rightarrow$ (a). It follows immediately. □

If M is a monomial submodule of a finitely generated graded free E-module $F = \oplus_{i=1}^{r} Eg_i$, we will denote by M^{lex} the unique lexicographic submodule of F with the same Hilbert function of M. Such a monomial submodule will be called the *lex submodule associated* to M.

Remark 2.2.5 We have obtained a generalization of Kruskal–Katona's theorem (Theorem 2.2.4) *via* results on ideals in an exterior algebra (Proposition 2.2.1). We believe that such a characterization could also be obtained using the same techniques as in [AHH97], *i.e.*, extending [AHH97, Theorem 4.2] to graded E–modules.

2.3 The Lex–Algorithm

In this Section, fixed a graded submodule M of F, we give a new procedure for the construction of M^{lex}. The algorithmic construction of the lex submodule is based on the additive property of Hilbert functions and on Kruskal–Katona's theorem. The idea dates back to the computation of all admissible Hilbert sequences of quotients of exterior algebras in [AC18b] and from the realisation that, given a Hilbert sequence $Hs_{F/M}$, there exist only $r = \operatorname{rank} F$ Hilbert sequences of the type E/I (I graded ideal in E), which determine M^{lex}. The choice of such r sequence is forced by some restrictions, as next theorem will point out.

Let $p, q \in \mathbb{Z}$ such that $p < q$. A finite sequence H of nonnegative integers is called $[p, q]$–sequence if it is indexed by the set $[p, q]$:

$$H = (h_i)_{i \in [p,q]} = (h_p, h_{p+1}, \dots, h_q).$$

We set

$$H(j) = h_j, \quad \text{for} \quad j \in [p, q];$$

the integers j are called H–degrees.

One can observe that the sequence $Hs_{F/M}$ is a $[f_1, f_r + n]$–sequence, and the integers $j \in [f_1, f_r + n]$ are the $Hs_{F/M}$–degrees.

Moreover, if $p = 0$, then H is the $(q + 1)$–tuple $(h_0, h_1, \dots, h_q)$.

Example 2.3.1 Let $p = -2$ and $q = 1$. Then $[-2, 1] = \{-2, -1, 0, 1\}$. If $H = (0, 2, 7, 3)$ is a $[-2, 1]$–sequence, one has $H(-2) = 0$, $H(-1) = 2$, $H(0) = 7$, and $H(1) = 3$.

Theorem 2.3.2 *(The Lex–Algorithm) Let $(h_{f_1}, \dots, h_{f_r+n})$ be the Hilbert sequence of a graded E– module F/M. Then, there exists a unique lex submodule L of F such that $H_{F/L} = H_{F/M}$.*

Proof. Set $Hs_{F/M} = (h_{f_1}, \dots, h_{f_r+n})$. We want to construct a lex submodule $L = \oplus_{i=1}^r I_i g_i$ of F such that $H_{F/L} = H_{F/M}$. Let us define

$$0_p = (0, \dots, 0) \in \mathbb{N}^p, \text{ for } p \geq 1.$$

Step 1. Construction of I_r.

Let us consider the following subsequence of $Hs_{F/M}$:

$$(h_{f_r}, \dots, h_{f_r+n}) = (H_{F/M}(f_r), \dots, H_{F/M}(f_r + n)). \tag{2.3.1}$$

Define

- $H_r(0) := \min\{1, H_{F/M}(f_r)\}$,

- $H_r(1) := \begin{cases} \min\{n, H_{F/M}(f_r + 1)\} & \text{if } H_r(0) = 1 \\ 0 & \text{if } H_r(0) = 0, \end{cases}$

- $H_r(2) := \min\{H_r(1)^{(1)}, H_{F/M}(f_r+2)\}$,

- $H_r(i) := \min\{H_r(i-1)^{(i-1)}, H_{F/M}(f_r+i)\}$, for $3 \leq i \leq n$.

Setting $H_r = (H_r(0), \ldots, H_r(n))$, if $H_r(0) = 1$, then Kruskal–Katona's theorem (Theorem 1.3.7) assures that such a sequence is the largest extractable Hilbert sequence from (2.3.1) for which there exists a lex ideal $I_r \subsetneq E$ such that

$$Hs_{E/I_r} = H_r;$$

on the contrary, if $H_r(0) = 0$, then the only admissible Hilbert sequence is the null sequence. In such a case, the corresponding lex ideal is $I_r = E$.

Step 2. Construction of I_{r-1}.

Let us define

$$\widetilde{H}_r = 0_{f_r - f_1} \uplus Hs_{E/I_r} = (\underbrace{0, \ldots, 0}_{f_r - f_1}, H_r(0), \ldots, H_r(n)),$$

and consider the $[f_1, f_r + n]$–sequence

$$\begin{aligned} Hs_{F/M} - \widetilde{H}_r &= (h_{f_1}, \ldots, h_{f_r - 1}, h_{f_r} - H_r(0), \ldots, h_{f_r + n} - H_r(n)) = \\ &= (h_{f_1}, \ldots, h_{f_r - 1}, h_{f_r} - H_r(0), \ldots, h_{f_{r-1}+n} - H_r(n - f_r + f_{r-1}), \underbrace{0, \ldots, 0}_{f_r - f_{r-1}}). \end{aligned}$$

Note that if $f_{r-1} < f_r$, then the last $f_r - f_{r-1}$ entries of $Hs_{F/M}$ concern only the ideal I_r. Furthermore, if $f_1 = f_2 = \ldots = f_r$, then $\widetilde{H}_r = H_r$.

Set

$$\overline{H}_r = Hs_{F/M} - \widetilde{H}_r.$$

Starting from the $(r-1)$-th critical degree, we can repeat on $\overline{H}_r$ the same reasoning done for $Hs_{F/M}$. More precisely, define

- $H_{r-1}(0) := \min\{1, \overline{H}_r(f_{r-1})\}$,

- $H_{r-1}(1) := \begin{cases} \min\{n, \overline{H}_r(f_{r-1}+1)\} & \text{if } H_{r-1}(0) = 1 \\ 0 & \text{if } H_{r-1}(0) = 0, \end{cases}$

- $H_{r-1}(2) := \min\{H_{r-1}(1)^{(1)}, \overline{H}_r(f_{r-1}+2)\}$,

- $H_{r-1}(i) := \min\{H_{r-1}(i-1)^{(i-1)}, \overline{H}_r(f_{r-1}+i)\}$, for $3 \leq i \leq n$,

and let I_{r-1} be the unique lex ideal of E such that

$$Hs_{E/I_{r-1}} = H_{r-1} = (H_{r-1}(0), \ldots, H_{r-1}(n)).$$

Setting

$$\widetilde{H}_{r-1} = 0_{f_{r-1}-f_1} \uplus Hs_{E/I_{r-1}} \uplus 0_{f_r-f_{r-1}} = (\underbrace{0,\dots,0}_{f_{r-1}-f_1}, H_{r-1}(0),\dots,H_{r-1}(n), \underbrace{0,\dots,0}_{f_r-f_{r-1}}),$$

let us consider the $[f_1, f_r+n]$–sequence

$$\begin{aligned}\overline{H}_{r-1} = \overline{H}_r - \widetilde{H}_{r-1} = \big(&\overline{H}_r(f_1),\dots,\overline{H}_r(f_{r-1}-1), \overline{H}_r(f_{r-1}) - H_{r-1}(0),\dots \\ &\dots, \overline{H}_r(f_{r-1}+n) - H_{r-1}(n), \underbrace{0,\dots,0}_{f_r-f_{r-1}}\big) = \\ = (h_{f_1},\dots,h_{f_{r-1}-1}, &\overline{H}_r(f_{r-1}) - H_{r-1}(0),\dots,\overline{H}_r(f_{r-2}+n) - H_{r-1}(n-f_{r-1}+f_{r-2}), \\ &\underbrace{0,\dots,0}_{f_r-f_{r-2}}).\end{aligned}$$

Proceeding as before, we will get a Hilbert sequence H_{r-2} and a lex ideal I_{r-2} such that $H_{r-2} = Hs_{E/I_{r-2}}$. Finally, iterating the previous procedure, after r steps, we will obtain r lex ideals $I_r,\dots,I_1$. The monomial submodule $L = \oplus_{i=1}^r I_i g_i$ is the lex submodule we are looking for. Indeed, the suitable choice of the r Hilbert sequences $H_r, H_{r-1},\dots,H_1$ assures that L_d is generated (as a K-vector space) by a lex_F segment of monomials of degree d of F.

Note that the r subtractions will return the (f_r+n-f_1+1)–tuple $0_{f_r+n-f_1+1}$, and consequently $H_{F/M} = H_{F/L}$, *i.e.*, $Hs_{F/M} = Hs_{F/L} = \sum_{i=1}^r \widetilde{H}_i$. $\square$

In order to outline the basic idea behind the Theorem 2.3.2, we present a sketch of the algorithm as pseudocode in Algorithm 2.1.

Algorithm 2.1: Lexicographic submodule computation

Input: Sequence hs, free E-module F
Output: Lexicographic submodule with Hilbert sequence hs

```
begin
    n ← number of indeterminates of the exterior algebra E;
    r ← rank of F;
    degs ← list of degrees of a basis of F;
    length ← max{degs} − min{degs} + n + 1;
    foreach j ∈ {1..r} do
        ind ← degs(r − j) − min{degs};
        seq(0) ← min{hs(ind), 1};
        if seq(0) = 1 then
            seq(1) ← min{hs(ind + 1), n};
        else
            seq(1) ← 0;
        end
        foreach k ∈ {2..n} do
            seq(k) ← min{hs(ind + k), seq(k − 1)^(k−1)};
        end
        hs ← hs − (0_ind ⊎ seq ⊎ 0_(length−1−n−ind));
        I_(r−j) ← lex ideal with Hilbert sequence seq;
    end
    if hs = 0_length then
        return M = ⊕_{i=1}^r I_i g_i;
    else
        Error: "expected a Hilbert sequence";
    end
end
```

Algorithm in Theorem 2.3.2

The procedure in Theorem 2.3.2, allows us to give a criterion for determining when a sequence of nonnegative integers is the Hilbert function of a graded E–algebra of the type F/M, with M graded submodule of F.

Criterion 2.3.3 Let $F = \oplus_{i=1}^{r} Eg_i$ be a finitely generated graded free E-module and the generators g_i of degrees f_i are ordered such that $f_1 \leq f_2 \leq \cdots \leq f_r$.

A sequence of nonnegative integers

$$H = (h_{f_1}, \dots, h_{f_r+n})$$

is the Hilbert sequence of graded E–module F/M, if applying the algorithm in Theorem 2.3.2, after r steps, the repeated subtractions from H of the largest Hilbert sequences (in the sense of the aforementioned theorem) of graded K–algebras of the type E/I, return the null sequence $0_{f_r+n-f_1+1}$.

2.4 Examples

In this Section, we collect some examples in order to illustrate our results. In particular the strategy used in Theorem 2.3.2.

Example 2.4.1 Let $E = K\langle e_1, e_2, e_3, e_4\rangle$, $F = E^3$, and consider the $[0, 4]$–sequence

$$H = (3, 11, 13, 3, 0) = (h_0, h_1, \dots, h_4).$$

Using the procedure described in Theorem 2.2.4, we can guess if H is a Hilbert sequence of a quotient F/M (M graded submodule of F), and we can also construct the lex submodule L of F such that $H_{F/L} = H$.

With the same notations as in Theorem 2.2.4. We have $s = f_1 = 0$, $\tilde{r}_0 = 3$ and $\tilde{r}_1 = 0$. In fact, the initial critical value is the first element of the sequence and has multiplicity equal to 3, and there do not exist critical degrees different from it. Therefore, the first two conditions in Theorem 2.2.4 (b) are realized:

$$h_0 = 3 \le 3 = \tilde{r}_0,$$

$$h_1 = 11 \le 12 = n\tilde{r}_0 + \tilde{r}_1.$$

By Proposition 2.2.1, we have to verify the following inequalities

$$\begin{aligned}
h_1 = 11 = \binom{4}{1} + \binom{4}{1} + \underbrace{\binom{3}{1}}_{a} \quad &\Rightarrow \quad h_2 = 13 \quad \le \quad 15 = \binom{4}{2} + \binom{4}{2} + \underbrace{\binom{3}{2}}_{a^{(1)}},\\
h_2 = 13 = \binom{4}{2} + \binom{4}{2} + \underbrace{\binom{2}{2}}_{a} \quad &\Rightarrow \quad h_3 = 3 \quad \le \quad 8 = \binom{4}{3} + \binom{4}{3} + \underbrace{\binom{2}{3}}_{a^{(2)}},\\
h_3 = 3 = \underbrace{\binom{3}{3} + \binom{2}{2} + \binom{1}{1}}_{a} \quad &\Rightarrow \quad h_4 = 0 \quad \le \quad 0 = \underbrace{\binom{3}{4} + \binom{2}{3} + \binom{1}{2}}_{a^{(3)}}.
\end{aligned}$$

a is the integer defined in Proposition 2.2.1 (see its proof). Hence H is the Hilbert sequence of a quotient of F.

In order to assure this, we construct the lex submodule $L = \oplus_{d=0}^{4} L_d$ of F such that $H_{F/L} = H$; L_d is the K–vector space generated by a lex segment of length $\dim_K F_d - h_d$, for $d = 0, \ldots, 4$.

Firstly, one can observe that $\dim_K L_0 = \dim_K F_0 - h_0 = 0$. Hence $L_0 = 0$.
Furthermore, $\dim_K L_1 = \dim_K F_1 - h_1 = 12 - 11 = 1$, and so

$$L_1 = \langle e_1 g_1 \rangle.$$

In degree 2, we have $\dim_K L_2 = \dim_K F_2 - h_2 = 3\binom{4}{2} - 13 = 5$. Since, $\mathrm{Shad}(L_1) = \{e_1e_2g_1,\ e_1e_3g_1,\ e_1e_4g_1\}$,

$$L_2 = \langle u \in \mathrm{Shad}(L_1), e_2e_3g_1,\ e_2e_4g_1 \rangle.$$

In degree 3, we have $\dim_K L_3 = \dim_K F_3 - h_3 = 3\binom{4}{3} - 3 = 9$. Since $|\mathrm{Shad}(L_2)| = 4$ ($e_\sigma g_1 \in \mathrm{Shad}(L_2)$, for all $e_\sigma \in E_3$), one has

$$L_3 = \langle u \in \mathrm{Shad}(L_2), e_1e_2e_3g_2,\ e_1e_2e_4g_2,\ e_1e_3e_4g_2,\ e_2e_3e_4g_2,\ e_1e_2e_3g_3 \rangle.$$

Finally, we have $\dim_K L_4 = \dim_K F_4 - h_4 = 3\binom{4}{4} = 3$ and all the monomials we need are in $\mathrm{Shad}(L_3)$, *i.e.*, $L_4 = \langle u \in \mathrm{Shad}(L_3) \rangle$.

Hence, we have constructed the unique lex submodule $L = \oplus_{i=1}^{r} I_i g_i$ with $H_{F/L} = (3, 11, 13, 3, 0)$. More in details:

$$L = (e_1, e_2e_3, e_2e_4)g_1 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4)g_2 \oplus (e_1e_2e_3)g_3.$$

A more general example can be given if one considers a free-module F with a basis in different degrees.

Example 2.4.2 Let $E = K\langle e_1, e_2, e_3, e_4 \rangle$, $F = \oplus_{i=1}^{3} Eg_i$ with $f_1 = -2, f_2 = 0, f_3 = 3$, and let us consider the $[-2, 7]$–sequence

$$H = (1, 4, 5, 4, 5, 2, 4, 3, 1, 0) = (h_{-2}, h_{-1}, \ldots, h_7).$$

As in Example 2.4.1, we will verify that H is a Hilbert sequence, and then we will construct the lex submodule L of F such that $H_{F/L} = H$.

Since $s = f_1 = -2$, $\tilde{r}_{-2} = 1$ and $\tilde{r}_{-1} = 0$, we have:

$$h_{-2} = 1 \leq 1 = \tilde{r}_{-2}, \qquad h_{-1} = 4 \leq 4 = n\tilde{r}_{-2} + \tilde{r}_{-1}.$$

Moreover, next inequalities hold (Proposition 2.2.1):

$$h_{-1} = 4 = \binom{4}{-4} + \binom{4}{-1} + \binom{4}{1} \qquad \Rightarrow \quad h_0 = 5 \quad \leq \quad 7 = \binom{4}{-3} + \binom{4}{0} + \binom{4}{2}$$

$$h_0 = 5 = \binom{4}{-3} + \binom{4}{0} + \underbrace{\binom{3}{2} + \binom{1}{1}}_{a} \quad \Rightarrow \quad h_1 = 4 \quad \leq \quad 5 = \binom{4}{-2} + \binom{4}{1} + \underbrace{\binom{3}{3} + \binom{1}{2}}_{a^{(2)}}$$

$$
\begin{array}{llll}
h_1 = 4 = \binom{4}{-2} + \binom{4}{1} & \Rightarrow & h_2 = 5 & \le \; 6 = \binom{4}{-1} + \binom{4}{2} \\
h_2 = 5 = \binom{4}{-1} + \underbrace{\binom{3}{2} + \binom{2}{1}}_{a} & \Rightarrow & h_3 = 2 & \le \; 3 = \binom{4}{0} + \underbrace{\binom{3}{3} + \binom{2}{2}}_{a^{(2)}} \\
h_3 = 2 = \binom{4}{0} + \underbrace{\binom{3}{3}}_{a} & \Rightarrow & h_4 = 4 & \le \; 4 = \binom{4}{1} + \underbrace{\binom{3}{4}}_{a^{(3)}} \\
h_4 = 4 = \binom{4}{1} & \Rightarrow & h_5 = 3 & \le \; 6 = \binom{4}{2} \\
h_5 = 3 = \underbrace{\binom{3}{2}}_{a} & \Rightarrow & h_6 = 1 & \le \; 1 = \underbrace{\binom{3}{3}}_{a^{(2)}} \\
h_6 = 1 = \underbrace{\binom{3}{3}}_{a} & \Rightarrow & h_7 = 0 & \le \; 0 = \underbrace{\binom{3}{4}}_{a^{(3)}}
\end{array}
$$

It is worthy of being stressed that in order to get the right expression for the h_i's ($i = -1, \dots, 6$), we firstly compute the binomial coefficient $\binom{4}{i-f_3}$, then the other admissible ones.

For instance, $h_{-1} = 4 = \binom{4}{-1-3} + \binom{4}{-1-0} + \binom{4}{-1+2}$.

Now, we can construct the lex submodule $L = \oplus_{d=-2}^{7} L_d$ of F such that $H_{F/L} = H$, where L_d ($d = -2, \dots, 7$) is the K–vector space generated by a lex segment of length $\dim_K F_d - h_d$, for $d = -2, \dots, 7$.

At first, we observe that $\dim_K L_{-2} = \dim_K F_{-2} - h_{-2} = 0$. Moreover, $\dim_K L_{-1} = \dim_K F_{-1} - h_{-1} = 0$. Hence $L_{-2} = L_{-1} = 0$.
In degree 0, $\dim_K L_0 = \dim_K F_0 - h_0 = \binom{4}{-3} + \binom{4}{0} + \binom{4}{2} - 5 = 2$ and so

$$L_0 = \langle e_1e_2g_1, e_1e_3g_1 \rangle.$$

In degree 1, $\dim_K L_1 = \dim_K F_1 - h_1 = \binom{4}{-2} + \binom{4}{1} + \binom{4}{3} - 4 = 4$. Since $\mathrm{Shad}(L_0) = \{e_1e_2e_3g_1, e_1e_2e_4g_1, \ e_1e_3e_4g_1\}$, we choose

$$L_1 = \langle u \in \mathrm{Shad}(L_0), e_2e_3e_4g_1 \rangle.$$

In degree 2, $\dim_K L_2 = \dim_K F_2 - h_2 = \binom{4}{-1} + \binom{4}{2} + \binom{4}{4} - 5 = 2$. Since $\mathrm{Shad}(L_1) = \{e_1e_2e_3e_4g_1\}$, we set

$$L_2 = \langle u \in \mathrm{Shad}(L_1), e_1e_2g_2 \rangle.$$

In degree 3, $\dim_K L_3 = \dim_K F_3 - h_3 = \binom{4}{0} + \binom{4}{3} + \binom{4}{5} - 2 = 3$. Since $\mathrm{Shad}(L_2) = \mathrm{Shad}^2(L_1) \cup \{e_1e_2e_3g_2, e_1e_2e_4g_2\} = \{e_1e_2e_3g_2, e_1e_2e_4g_2\}$, we get

$$L_3 = \langle u \in \mathrm{Shad}(L_2), e_1e_3e_4g_2 \rangle.$$

In degree 4, $\dim_K L_4 = \dim_K F_4 - h_4 = \binom{4}{1} + \binom{4}{4} + \binom{4}{6} - 4 = 1$. Since $\mathrm{Shad}(L_3) = \{e_1e_2e_3e_4g_2\}$, we have that $L_4 = \langle u \in \mathrm{Shad}(L_3)\rangle$.
In degree 5, $\dim_K L_5 = \dim_K F_5 - h_5 = \binom{4}{2} + \binom{4}{5} + \binom{4}{7} - 3 = 3$. Since $\mathrm{Shad}(L_4)$ is empty, we have

$$L_5 = \langle e_1e_2g_3, e_1e_3g_3, e_1e_4g_3\rangle.$$

In degree 6, $\dim_K L_6 = \dim_K F_6 - h_6 = \binom{4}{3} + \binom{4}{6} + \binom{4}{8} - 1 = 3$. Since $\mathrm{Shad}(L_5) = \{e_1e_2e_3g_3, e_1e_2e_4g_3,\ e_1e_3e_4g_3\}$, we set $L_6 = \langle u \in \mathrm{Shad}(L_5)\rangle$.
Finally, in degree 7, $\dim_K L_7 = \dim_K F_7 - h_7 = \binom{4}{4} + \binom{4}{7} + \binom{4}{9} - 0 = 1$. Since $\mathrm{Shad}(L_6) = \{e_1e_2e_3e_4g_3\}$, we have $L_7 = \langle u \in \mathrm{Shad}(L_6)\rangle$.

In so doing, we have determined the lex submodule $L = \oplus_i^r I_i g_i$ with $H_{F/L} = (1, 4, 5, 4, 5, 2, 4, 3, 1, 0)$. More in details:

$$L = (e_1e_2, e_1e_3, e_2e_3e_4)g_1 \oplus (e_1e_2, e_1e_3e_4)g_2 \oplus (e_1e_2, e_1e_3, e_1e_4)g_3.$$

Now, we show an example of a sequence of nonnegative integers H that is not a Hilbert sequence of a quotient of a free E–module.

Example 2.4.3 Let $E = K\langle e_1, e_2, e_3, e_4\rangle$, $F = \oplus_{i=1}^3 Eg_i$ with $f_1 = -3, f_2 = -2, f_3 = 1$ and let us consider the [-2, 5]–sequence

$$H = (1, 3, 3, 4, 2, 4, 5, 1, 0) = (h_{-2}, h_{-1}, \dots, h_5).$$

We proceed as in the previuos examples.

It is $s = f_1 = -3$, $\tilde{r}_{-3} = 1$ and $\tilde{r}_{-2} = 1$, and consequently

$$h_{-3} = 1 \le 1 = \tilde{r}_{-3}, \qquad h_{-2} = 3 \le 5 = n\tilde{r}_{-3} + \tilde{r}_{-2}.$$

By Proposition 2.2.1, we can test the required bounds:

$$\begin{aligned}
h_{-2} = 3 = \binom{4}{-3} + \binom{4}{0} + \underbrace{\binom{2}{1}}_{a} \quad &\Rightarrow \quad h_{-1} = 3 \quad \le \quad 5 = \binom{4}{-2} + \binom{4}{1} + \underbrace{\binom{2}{2}}_{a^{(1)}},\\
h_{-1} = 3 = \binom{4}{-2} + \underbrace{\binom{3}{1}}_{a} \quad &\Rightarrow \quad h_0 = 4 \quad \not\le \quad 3 = \binom{4}{-1} + \underbrace{\binom{3}{2}}_{a^{(1)}}
\end{aligned}$$

The integer h_0 does not satisfy the required inequality. We will see that there does not exist the lex submodule $L = \oplus_{d=-2}^5 L_d$ of F such that $H_{F/L} = H$.

Indeed, $\dim_K L_{-3} = \dim_K F_{-3} - h_{-3} = 0$. Hence, $L_{-3} = 0$.
Moreover, in degree -2, $\dim_K L_{-2} = \dim_K F_{-2} - h_{-2} = \binom{4}{-3} + \binom{4}{0} + \binom{4}{1} - 3 = 2$. Hence,

$$L_{-2} = \langle e_1g_1, e_2g_1\rangle.$$

In degree -1, we have $\dim_K L_{-1} = \dim_K F_{-1} - h_{-1} = \binom{4}{-2} + \binom{4}{1} + \binom{4}{2} - 3 = 7$. On the other hand, $\mathrm{Shad}(L_{-2}) = \{e_1e_2g_1, e_1e_3g_1, e_1e_4g_1, e_2e_3g_1, e_2e_4g_1\}$, then

$$L_{-1} = \langle u \in \mathrm{Shad}(L_{-2}), e_3e_4g_1, e_1g_2\rangle.$$

In degree 0, we have $\dim_K L_0 = \dim_K F_0 - h_0 = \binom{4}{-1} + \binom{4}{2} + \binom{4}{3} - 4 = 6$. Since, $\mathrm{Shad}(L_{-1}) = \{e_1e_2e_3g_1, e_1e_2e_4g_1, e_1e_3e_4g_1, e_2e_3e_4g_1, e_1e_2g_2, e_1e_3g_2, e_1e_4g_2\}$, then $|\mathrm{Shad}(L_{-1})| > 6$. This situation implies that it is not possible the construction of the lex submodule L with $H_{F/L} = (1, 3, 3, 4, 2, 4, 5, 1, 0)$.

On the contrary, one can verify that for $h_0 = 3$, there exists the lex submodule $L = \oplus_i^r I_i g_i$ of F with $H_{F/L} = (1, 3, 3, 3, 2, 4, 5, 1, 0)$.

In order to simplify the notation, once we fix a sequence of nonnegative integers H, when we say that a graded ideal I of E has H as Hilbert sequence, or that H is the Hilbert sequence of a graded ideal I, we mean that $Hs_{E/I} = H$. Moreover, in what follows, we refer to Hilbert sequences of quotients of the type E/I (I graded ideal of E), whenever it is not specified.

Example 2.4.4 Let $E = K\langle e_1, e_2, e_3, e_4\rangle$, $F = E^3$ and

$$M = (e_1e_2, e_3e_4)g_1 \oplus (e_1e_2, e_2e_3e_4)g_2 \oplus (e_2e_3e_4)g_3$$

a submodule of F. M is not a lex submodule of F. The Hilbert sequence of F/M is

$$Hs_{F/M} = (3, 12, 15, 4, 0).$$

Setting $J_1 = (e_1e_2,\ e_3e_4)$, $J_2 = (e_1e_2,\ e_2e_3e_4)$ and $J_3 = (e_2e_3e_4)$, one has

$$Hs_{E/J_1} = (1, 4, 4, 0, 0),\ Hs_{E/J_2} = (1, 4, 5, 1, 0),\ Hs_{E/J_3} = (1, 4, 6, 3, 0),$$

and $H_{F/M}(d) = \sum_{i=1}^{3} H_{E/J_i}(d)$, $d \geq 0$, as the next table shows

Hs–degrees	**0**	1	2	3	4	
Hs_{E/J_1}	(**1**,	**4**,	**4**,	**0**,	**0**)	+
Hs_{E/J_2}	(**1**,	**4**,	**5**,	**1**,	**0**)	+
Hs_{E/J_3}	(**1**,	**4**,	**6**,	**3**,	**0**)	=
$Hs_{F/M}$	(3,	12,	15,	4,	0)	

Now, we want to describe our new point of view.

Let us consider the sequence $H = (3, 12, 15, 4, 0)$. The largest Hilbert sequence of a graded ideal that can be extracted from H is $H_3 = (1, 4, 6, 4, 0)$. Indeed, there exists the lex ideal I_3 of E such that $Hs_{E/I_3} = H_3$. It is $I_3 = (e_1e_2e_3e_4)$.

Using the same notations as in Theorem 2.3.2, let $\overline{H}_3 = H - \widetilde{H}_3 = (3, 12, 15, 4, 0) - (1, 4, 6, 4, 0) = (2, 8, 9, 0, 0)$. The largest Hilbert sequence that can be extracted from $\overline{H}_3$ is

$H_2 = (1, 4, 6, 0, 0)$. In fact, $H_2 = Hs_{E/I_2}$, with $I_2 = (e_1e_2e_3, e_1e_3e_4, e_1e_3e_4, e_2e_3e_4)$ lex ideal of E.

Next, consider the sequence $\overline{H}_2 = \overline{H}_3 - \widetilde{H}_2 = (2, 8, 9, 0, 0) - (1, 4, 6, 0, 0) = (1, 4, 3, 0, 0)$. The largest Hilbert sequence that can be extracted from $\overline{H}_2$ is $H_1 = \overline{H}_2$. The lex ideal whose Hilbert sequence is H_1 is $I_1 = (e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4)$.

We can observe that in such a case the sequence $\overline{H}_1 = \overline{H}_2 - \widetilde{H}_1 = (1, 4, 3, 0, 0) - (1, 4, 3, 0, 0) = 0_5$.

Next table describes our procedure:

H–degrees	**0**	1	2	3	4	
H	(3,	12,	15,	4,	0)	$-$
Hs_{E/I_3}	(**1**,	**4**,	**6**,	**4**,	**0**)	$-$
Hs_{E/I_2}	(**1**,	**4**,	**6**,	**0**,	**0**)	$-$
Hs_{E/I_1}	(**1**,	**4**,	**3**,	**0**,	**0**)	$=$
0_5	(0,	0,	0,	0,	0)	

Observe that in our situation $f_1 = f_2 = f_3 = 0$, and so $\widetilde{H}_i = Hs_{E/I_i}$ $(i = 1, 2, 3)$.
Finally, $M^{\text{lex}} = \oplus_i^r I_i g_i$ is the unique lex submodule with Hilbert sequence $H = (3, 12, 15, 4, 0)$. More in details:

$$M^{\text{lex}} = (e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4)g_1 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4)g_2 \oplus (e_1e_2e_3e_4)g_3.$$

Remark 2.4.5 Note that, given a Hilbert sequence H of a quotient of a free E-module F, $\operatorname{rank} F = r$, if one applies to H r repeated subtractions by the *non–largest admissible Hilbert sequences* of K-algebras E/T_i, with T_i lex ideals of E, for $i = 1, \ldots, r$ (in the sense of Theorem 2.3.2 and according to Kruskal–Katona Theorem), then the submodule $N = \oplus_i^r T_i g_i$ is not a lex submodule.

Indeed, let us consider Example 2.4.4. We can subtract from H the Hilbert sequences $(1, 4, 6, 3, 0)$, $(1, 4, 5, 1, 0)$, $(1, 4, 4, 0, 0)$, and, consequently, we can get the corresponding lex ideals $T_3 = (e_1e_2e_3)$, $T_2 = (e_1e_2, e_1e_3e_4)$, $T_1 = (e_1e_2, e_1e_3, e_2e_3e_4)$. But, $(e_1, e_2, e_3, e_4)^{\operatorname{indeg} T_2}$ $= (e_1, e_2, e_3, e_4)^2 \nsubseteq T_1$ and $N = \oplus_i^r T_i g_i$ is not a lex submodule.

Example 2.4.6 Let $E = K\langle e_1, e_2, e_3, e_4\rangle$ and $F = \oplus_{i=1}^3 E g_i$ with $f_1 = -2, f_2 = 0, f_3 = 2$. Consider the monomial submodule

$$M = (e_1e_2, e_3e_4)g_1 \oplus (e_1e_2, e_2e_3e_4)g_2 \oplus (e_2e_3e_4)g_3$$

of F. The Hilbert sequence of F/M is

$$Hs_{F/M} = (1, 4, 5, 4, 6, 5, 6, 3, 0).$$

Setting $J_1 = (e_1e_2,\ e_3e_4)$, $J_2 = (e_1e_2,\ e_2e_3e_4)$ and $J_3 = (e_2e_3e_4)$, one has

$$Hs_{E/J_1} = (1, 4, 4, 0, 0),\ Hs_{E/J_2} = (1, 4, 5, 1, 0),\ Hs_{E/J_3} = (1, 4, 6, 3, 0).$$

and $H_{F/M}(d) = \sum_{i=1}^{3} H_{E/J_i}(d - f_i)$, $d \geq -2$, as shown in next table,

Hs–degrees	$\mathbf{-2}$	-1	$\mathbf{0}$	1	$\mathbf{2}$	3	4	5	6	
$\widetilde{H}_1$	(**1**,	**4**,	**4**,	**0**,	**0**,	0,	0,	0,	0)	+
$\widetilde{H}_2$	(0,	0,	**1**,	**4**,	**5**,	**1**,	**0**,	0,	0)	+
$\widetilde{H}_3$	(0,	0,	0,	0,	**1**,	**4**,	**6**,	**3**,	**0**)	=
$Hs_{F/M}$	(1,	4,	5,	4,	6,	5,	6,	3,	0)	

where $\widetilde{H}_i = 0_{f_i - f_1} \uplus Hs_{E/J_i} \uplus 0_{f_r - f_i}$ $(i = 1, 2, 3)$. We have indicated the Hilbert sequences Hs_{E/J_i} $(i = 1, 2, 3)$ in bold.

Let us consider the $[-2, 6]$–sequence $H = (1, 4, 5, 4, 6, 5, 6, 3, 0)$. The largest Hilbert sequence of a graded ideal I that can be extracted from the subsequence $(H(2), \dots, H(6)) = (6, 5, 6, 3, 0)$ is $H_3 = (1, 4, 6, 3, 0)$. Indeed, there exists the lex ideal $I_3 = (e_1e_2e_3)$ of E such that $Hs_{E/I_3} = H_3$.

With the same notations as in Theorem 2.3.2. Set $\overline{H}_3 = H - \widetilde{H}_3 = (1, 4, 5, 4, 6, 5, 6, 3, 0) - (0, 0, 0, 0, 1, 4, 6, 3, 0) = (1, 4, 5, 4, 5, 1, 0, 0, 0)$, the largest extractable Hilbert sequence from the subsequence $(\overline{H}_3(0), \dots, \overline{H}_3(4)) = (5,\ 4,\ 5,\ 1,\ 0)$ is $H_2 = (1, 4, 5, 1, 0) = Hs_{E/I_2}$, with $I_2 = (e_1e_2, e_1e_3e_4)$.

Next, consider the sequence $\overline{H}_2 = \overline{H}_3 - \widetilde{H}_2 = (1, 4, 5, 4, 5, 1, 0, 0, 0) - (0, 0, 1, 4, 5, 1, 0, 0, 0) = (1, 4, 4, 0, 0, 0, 0, 0, 0)$. The largest Hilbert sequence that can be extracted from the subsequence $(\overline{H}_2(-2), \dots, \overline{H}_2(2)) = (1, 4, 4, 0, 0)$ is $H_1 = (1, 4, 4, 0, 0) = Hs_{E/I_1}$, with $I_1 = (e_1e_2, e_1e_3, e_2e_3e_4)$.

We can observe that in such a case the sequence $\overline{H}_1 = \overline{H}_2 - \widetilde{H}_1 = (1, 4, 4, 0, 0, 0, 0, 0, 0) - (1, 4, 4, 0, 0, 0, 0, 0, 0) = 0_9$.

H–degrees	$\mathbf{-2}$	-1	$\mathbf{0}$	1	$\mathbf{2}$	3	4	5	6	
H	(1,	4,	5,	4,	6,	5,	6,	3,	0)	−
$\widetilde{H}_3$	(0,	0,	0,	0,	**1**,	**4**,	**6**,	**3**,	**0**)	−
$\widetilde{H}_2$	(0,	0,	**1**,	**4**,	**5**,	**1**,	**0**,	0,	0)	−
$\widetilde{H}_1$	(**1**,	**4**,	**4**,	**0**,	**0**,	0,	0,	0,	0)	=
0_9	(0,	0,	0,	0,	0,	0,	0,	0,	0)	

$\widetilde{H}_i = 0_{f_i - f_1} \uplus Hs_{E/I_i} \uplus 0_{f_r - f_i}$ $(i = 1, 2, 3)$, and we have indicated the Hilbert sequences Hs_{E/I_i} $(i = 1, 2, 3)$ in bold. Finally,

$$M^{\mathrm{lex}} = \oplus_i^r I_i g_i = (e_1e_2, e_1e_3, e_2e_3e_4)g_1 \oplus (e_1e_2, e_1e_3e_4)g_2 \oplus (e_1e_2e_3)g_3.$$

The Hilbert sequence of a quotient of the form F/M can have zeros as initial values. The number of such zeros is $f_k - f_1$, where f_k is the initial critical degree of the Hilbert sequence, as next example will show. Moreover, we can note that the existence of initial zeros implies the presence of some improper ideals as initial components in the direct decomposition. The converse is not true.

Example 2.4.7 Let $E = K\langle e_1, e_2, e_3, e_4\rangle$ and $F = \oplus_{i=1}^{8} Eg_i$ with $f_1 = -3, f_2 = f_3 = -1, f_4 = f_5 = f_6 = 2, f_7 = f_8 = 7$. Let us consider the $[-3,7]$–sequence

$$H = (0,0,1,4,6,7,13,7,1,0,1,4,5,2,0).$$

By applying the algorithm in Theorem 2.3.2, as in the previous examples, we obtain by repeated subtractions from H, the following Hilbert sequences $H_i = Hs_{E/I_i}$, with I_i lex ideal of E $(i = 1,\dots,8)$:

$$H_8 = (1,4,5,2,0),\ H_7 = (0,0,0,0,0),\ H_6 = (1,4,6,1,0),\ H_5 = (1,4,1,0,0),$$
$$H_4 = (1,4,0,0,0),\ H_3 = (1,4,6,4,1),\ H_2 = (0,0,0,0,0),\ H_1 = (0,0,0,0,0).$$

and

$$I_8 = (e_1e_2),\ I_7 = E,\ I_6 = (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4),\ I_5 = (e_1e_2, e_1e_3, e_1e_4, e_2e_3, e_2e_4),$$
$$I_4 = (e_1e_2, e_1e_3, e_1e_4, e_2e_3, e_2e_4, e_3e_4),\ I_3 = (0),\ I_2 = E,\ I_1 = E.$$

H–degrees	**−3**	−2	**−1**	0	1	**2**	3	4	5	6	**7**	8	9	10	11	
H	(0,	0,	1,	4,	6,	7,	13,	7,	1,	0,	1,	4,	5,	2,	0)	−
$\widetilde{H}_8$	(0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	**1**,	**4**,	**5**,	**2**,	**0**)	−
$\widetilde{H}_7$	(0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	**0**,	**0**,	**0**,	**0**,	**0**)	−
$\widetilde{H}_6$	(0,	0,	0,	0,	0,	**1**,	**4**,	**6**,	**1**,	**0**,	0,	0,	0,	0,	0)	−
$\widetilde{H}_5$	(0,	0,	0,	0,	0,	**1**,	**4**,	**1**,	**0**,	**0**,	0,	0,	0,	0,	0)	−
$\widetilde{H}_4$	(0,	0,	0,	0,	0,	**1**,	**4**,	**0**,	**0**,	**0**,	0,	0,	0,	0,	0)	−
$\widetilde{H}_3$	(0,	0,	**1**,	**4**,	**6**,	**4**,	**1**,	0,	0,	0,	0,	0,	0,	0,	0)	−
$\widetilde{H}_2$	(0,	0,	**0**,	**0**,	**0**,	**0**,	**0**,	0,	0,	0,	0,	0,	0,	0,	0)	−
$\widetilde{H}_1$	(**0**,	**0**,	**0**,	**0**,	**0**,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0)	=
0_{15}	(0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,	0)	

($\widetilde{H}_i = 0_{f_i - f_1} \uplus Hs_{E/I_i} \uplus 0_{f_r - f_i}$ $(i = 1,\dots,8)$, and the Hilbert sequences Hs_{E/I_i} $(i = 1,\dots,8)$ are indicated in bold.)

By means of repeated subtractions, we obtain the null sequence 0_{15}. Hence, the sequence H is the Hilbert sequence of a quotient of a free E–module. Indeed, there exists the lex submodule

$$L = \oplus_i^r I_i g_i = Eg_1 \oplus Eg_2 \oplus (0)g_3 \oplus (e_1e_2, e_1e_3, e_1e_4, e_2e_3, e_2e_4, e_3e_4)g_4 \oplus$$
$$(e_1e_2, e_1e_3, e_1e_4, e_2e_3, e_2e_4)g_5 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4)g_6 \oplus Eg_7 \oplus (e_1e_2)g_8.$$

such that $H = Hs_{F/L} = (0,0,1,4,6,7,13,7,1,0,1,4,5,2,0) = \sum_{i=1}^{8} \widetilde{H}_i$.

We close this Section with an example of a sequence of nonnegative integers H that is not a Hilbert sequence of a quotient of a free E–modules.

Example 2.4.8 Let $E = K\langle e_1, e_2, e_3, e_4\rangle$ and $F = \oplus_{i=1}^3 Eg_i$ with $f_1 = -3, f_2 = -1, f_3 = 2$. Let us consider the $[-3, 2]$–sequence

$$H = (1, 2, 2, 4, 3, 3, 4, 5, 2, 0).$$

By using the Lex–Algorithm and by repeated subtractions from H, we obtain the Hilbert sequences $H_i = Hs_{E/I_i}$, with I_i lex ideal of E $(i = 1, 2, 3)$:

$$H_3 = (1, 4, 5, 2, 0),\ H_2 = (1, 4, 3, 1, 0),\ H_1 = (1, 2, 1, 0, 0),$$

$$I_3 = (e_1e_2),\ I_2 = (e_1e_2, e_1e_3, e_1e_4),\ I_1 = (e_1, e_2).$$

Next table describes the construction.

H–degrees	$\mathbf{-3}$	-2	$\mathbf{-1}$	0	1	$\mathbf{2}$	3	4	5	6	
H	(1,	2,	2,	4,	3,	3,	4,	5,	2,	0)	$-$
$\widetilde{H}_3$	(0,	0,	0,	0,	0,	**1**,	**4**,	**5**,	**2**,	**0**)	$-$
$\widetilde{H}_2$	(0,	0,	**1**,	**4**,	**3**,	**1**,	**0**,	0,	0,	0)	$-$
$\widetilde{H}_1$	(**1**,	**2**,	**1**,	**0**,	**0**,	0,	0,	0,	0,	0)	$=$
	(0,	0,	0,	0,	0,	**1**,	0,	0,	0,	0)	

At the end, we do not obtain the null sequence 0_{10}, and so H is not a Hilbert sequence of a quotient of the given free E-module F according to Criterion 2.3.3.

On the other hand, it is relevant to analyze the second difference that comes into play (according to the Lex–Algorithm):

$$\overline{H}_2 = \overline{H}_3 - \widetilde{H}_2 = (1, 2, 2, 4, 3, 2, 0, 0, 0, 0) - (0, 0, 1, 4, 3, 1, 0, 0, 0, 0) = (1, 2, 1, 0, 0, 1, 0, 0, 0, 0).$$

In this case, the largest Hilbert sequence of a graded K-algebra E/I is $(1, 4, 3, 1, 0)$. In fact, for the sequence $(1, 4, 3, 2, 0)$ no ideal I of E with $Hs_{E/I} = (1, 4, 3, 2, 0)$ does exist (see Kruskal–Katona's theorem). Finally, the submodule

$$N = \oplus_{i=1}^3 I_i g_i = (e_1, e_2)g_1 \oplus (e_1e_2, e_1e_3, e_1e_4)g_2 \oplus (e_1e_2)g_3$$

has Hilbert sequence $Hs_{F/N} = (1, 2, 2, 4, 3, 2, 4, 5, 2, 0) < H$. $Hs_{F/N}$ is the largest extractable Hilbert sequence from H attaining a submodule of F.

All the examples in this chapter have been constructed by such packages.

2.5 *Macaulay2* packages

The procedures described in this work are part of two *Macaulay2* packages "ExteriorIdeals.m2" [AC18b], "ExteriorModules.m2", and tested with Macaulay 1.10. We believe that these packages may reveal useful for further applications. Indeed, functions for computing monomial ideals in a polynomial ring are available in many computer algebra systems, CAS, (for instance, CoCoA [ABL], Macaulay2 [GS] and Singular [DGPS16]); on the contrary, to the best of our knowledge, specific packages for manipulating classes of monomial ideals (or monomial submodules) in an exterior algebra have not been implemented yet.

More precisely, in the package "ExteriorIdeals" we implement some algorithms in order to easily check whether an $(n+1)$-tuple $(1, h_1, \dots, h_n)$ ($h_1 \le n = \dim_K V$) of non–negative integers is the Hilbert function of a graded K–algebra of the form E/I, with I graded ideal of E, is given. In particular, if $H_{E/I}$ is the Hilbert function of a graded K-algebra E/I, the package is able to construct the unique lexsegment ideal I^{lex} such that $H_{E/I} = H_{E/I^{\text{lex}}}$. Whereas in the "ExteriorModules" we extend these functions to modules.

In this Section, we collect some examples in order to describe the algorithms.

Example 2.5.1 Let E be an exterior algebra with n generators over a field K and $h = (h_0, h_1, \dots, h_n)$ a sequence of nonnegative integers, we describe how one can verify if h is a Hilbert sequence. The key tools in our package are the methods `isHilbertSequence(list,exterior algebra)` and `lexIdeal(list,exterior algebra)`. The first function verifies if a list of nonnegative integers of length $n+1$ is a Hilbert function; the second one returns a lex ideal of E if and only if the *list* is a Hilbert sequence. In more detail, if $(h_0,\ h_1,\ \dots,\ h_n)$ is a Hilbert sequence, the lex ideal of E produced by the function `lexIdeal(`$\{h_0, \dots, h_n\}$`, E)` is the unique lex ideal I of E with $H_{E/I}(d) = h_d$ $(d = 0, \dots, n)$. The procedure for the computation of the required lex ideal is based on the constructive proof of Theorem 1.3.7 (see [AHH97, Theorem 4.1, (b) $\Rightarrow$ (a)]).

We start with some examples of sequences which are not Hilbert sequences. The property is verified by using either `isHilbertSequence(list,exterior algebra)` or `lexIdeal(list,exterior algebra)`:

```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : loadPackage "ExteriorIdeals"
i2 : E=QQ[e_1..e_5,SkewCommutative=>true]
i3 : isHilbertSequence({2,4,3,0,0,0},E)
o3 = false
```

```
i4 : isHilbertSequence({0,4,3,0,0,0},E)
o4 : false
i5 : lexIdeal({1,6,3,0,0,0,0},E)
stdio:24:1:(3): error: expected a Hilbert sequence
i6 : lexIdeal({1,5,10,10,5,1,0},E)
stdio:26:1:(3): error: expected a Hilbert sequence
```

Moreover, the next statements provide some examples of the lex ideal *produced* by a Hilbert sequence. The length of the sequence can be at most $n+1$; if the length is less than $n+1$, then the sequence will be completed by adding zeros on the right.

```
i6 : lexIdeal({1,4,3,0,0,0},E)
o6 = ideal (e_1, e_2e_3, e_2e_4, e_2e_5, e_3e_4e_5)
o6 : Ideal of E
i7 : lexIdeal({1,4,4},E)
o7 = ideal (e_1, e_2e_3, e_2e_4, e_3e_4e_5)
o7 : Ideal of E
i8 : lexIdeal({1,5,7,4,0,0},E)
o9 = ideal (e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4e_5)
o9 : Ideal of E
```

The function `lexIdeal(list,exterior algebra)`, above defined, plays a relevant role also in the next algorithm.

Example 2.5.2 Given an exterior algebra E and a graded ideal I in E, we illustrate how to obtain the unique lex ideal I^{lex} with the same Hilbert function as I. In more detail, we describe two different methods for computing such lex ideal.

```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : loadPackage "ExteriorIdeals";
i2 : E=QQ[e_1..e_5,SkewCommutative=>true]
i3 : I=ideal {e_1*e_2*e_3+e_3*e_4*e_5,e_1*e_3+e_4*e_5,e_2*e_3*e_4}
o3 = ideal (e_1e_2e_3+e_3e_4e_5, e_1e_3+e_4e_5, e_2e_3e_4)
o3 : Ideal of E
i4 : hilbSeq=hilbertSequence(I)
o4 = {1, 5, 9, 3, 0, 0}
o4 : List
```

A first way for computing the lex ideal we are looking for is to use the function `lexIdeal(list,exterior algebra)`:

```
i5 : Ilex1=lexIdeal(hilbSeq,E)
o5 = ideal (e_1e_2, e_1e_3e_4, e_1e_3e_5, e_1e_4e_5, e_2e_3e_4)
o5 : Ideal of E
i6 : isLexIdeal Ilex1
o6 = true
i7 : hilbertSequence(Ilex1)
o7 = {1, 5, 9, 3, 0, 0}
o7 : List
```

and a second one is *via* the new function `lexIdeal(ideal)`, which returns directly the required lex ideal:

```
i8 : Ilex2=lexIdeal(I)
o8 = ideal (e_1e_2, e_1e_3e_4, e_1e_3e_5, e_1e_4e_5, e_2e_3e_4)
o8 : Ideal of E
i9 : hilbertSequence(Ilex2)
o9 = {1, 5, 9, 3, 0, 0}
o9 : List
```

Finally, last example is related to the algorithm for the computation of Hilbert sequences.

Example 2.5.3 Given an exterior algebra E, we illustrate how to get all the Hilbert sequences of quotients of E.

```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : loadPackage "ExteriorIdeals";
i2 : E=QQ[e_1..e_4,SkewCommutative=>true]
i3 : hilbSeqs=allHilbertSequences(E)
o3 = {{1, 4, 6, 4, 1}, {1, 4, 6, 4, 0}, {1, 4, 6, 3, 0}, {1, 4, 6, 2, 0},
     ------------------------------------------------------------------------
     {1, 4, 6, 1, 0}, {1, 4, 6,0, 0}, {1, 4, 5, 2, 0}, {1, 4, 5, 1, 0},
     ------------------------------------------------------------------------
     {1, 4, 5, 0, 0}, {1, 4, 4, 1, 0}, {1, 4, 4, 0, 0}, {1,4, 3, 1, 0},
     ------------------------------------------------------------------------
     {1, 4, 3, 0, 0}, {1, 4, 2, 0, 0}, {1, 4, 1, 0, 0}, {1, 4, 0, 0, 0},
```

```
     -----------------------------------------------------------------
     {1, 3, 3, 1,0}, {1, 3, 3, 0, 0}, {1, 3, 2, 0, 0}, {1, 3, 1, 0, 0},
     -----------------------------------------------------------------
     {1, 3, 0, 0, 0}, {1, 2, 1, 0, 0}, {1, 2,0, 0, 0}, {1, 1, 0, 0, 0},
     -----------------------------------------------------------------
     {1, 0, 0, 0, 0}}
o3 : List
i4 : transpose matrix hilbSeqs
o4 = | 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 |
     | 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 3 3 3 3 3 2 2 1 0 |
     | 6 6 6 6 6 6 5 5 5 4 4 3 3 2 1 0 3 3 2 1 0 1 0 0 0 |
     | 4 4 3 2 1 0 2 1 0 1 0 1 0 0 0 0 1 0 0 0 0 0 0 0 0 |
     | 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 |
o4 : Matrix ZZ^5  <--- ZZ^25
```

Note that the method `allHilbertSequences` returns an object of type `List`; for a more compact view it could be displayed as a matrix.

Now, we extend to modules all can we have done for ideals.

Example 2.5.4 Given a graded submodule M of F, we illustrate how to obtain the unique lex submodules M^{lex} with the same Hilbert function as M. More in detail, we describe two different methods for computing such lex submodule.

```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : loadPackage "ExteriorModules";
i2 : E=QQ[e_1..e_4,SkewCommutative=>true];
i3 : F=E^3;
i4 : I_1=ideal {e_1, e_2*e_3*e_4};
i5 : I_2=ideal {e_1*e_2, e_1*e_3*e_4};
i6 : I_3=ideal {e_1*e_2*e_3};

i7 : M=createModule({I_1, I_2, I_3},F)
o7 = image|e_1 e_2e_3e_4 0      0          0        |
          |0   0         e_1e_2 e_1e_3e_4 0        |
          |0   0         0      0          e_1e_2e_3|
o7 : E-module, submodule of E^3
```

```
i8 : isAlmostLexModule M
o8 = true
i9 : isLexModule M
o9 = false
```

Given a submodule M, a first way for computing the lex submodule we are looking for is to use the function `lexModule(module)`:

```
i10 : L=lexModule M
o10 = image|e_1 e_2e_3 0         0         0         0         0           |
           |0   0      e_1e_2e_3 e_1e_2e_4 e_1e_3e_4 e_2e_3e_4 0           |
           |0   0      0         0         0         0         e_1e_2e_3e_4|
o10 : E-module, submodule of E^3
i11 : hilbertSequence M
o11 = {3, 11, 14, 4, 0}
i12 : hilbertSequence M==hilbertSequence L
o12 = true
```

It is interesting get another similar computation in a more general situation:

```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : loadPackage "ExteriorModules";
i2 : E=QQ[e_1..e_4,SkewCommutative=>true];
i3 : F=E^{2,0,-1};
i4 : I_1=ideal {e_1*e_2, e_3*e_4};
i5 : I_2=ideal {e_1*e_2, e_2*e_3*e_4};
i6 : I_3=ideal {e_2*e_3*e_4};

i7 : M=createModule({I_1, I_2, I_3},F)
o7 = image {-2}|e_1e_2 e_3e_4 0      0         0        |
           {0} |0      0      e_1e_2 e_2e_3e_4 0        |
           {1} |0      0      0      0         e_2e_3e_4|
o7 : E-module, submodule of E^3

i8 : L=lexModule M
o8 = image {-2}|e_1e_2 e_1e_3 e_2e_3e_4 0      0         0        |
           {0} |0      0      0         e_1e_2 e_1e_3e_4 0        |
           {1} |0      0      0         0      0         e_1e_2e_3|
```

```
o8 : E-module, submodule of E^3
i9 : hilbertSequence M
o9 = {1, 4, 5, 5, 9, 7, 3, 0}
09 : List
i10 : hilbertSequence M==hilbertSequence L
o10 = true
```

Example 2.5.5 Similarly to what has been done for ideals, there exists a second way to obtain lex submodules given a sequence of nonnegative integers, if such a sequence is a Hilbert sequence.

```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : loadPackage "ExteriorModules";
i2 : E=QQ[e_1..e_4,SkewCommutative=>true];
i3 : F=E^3;
i4 : hs={3, 12, 16, 6, 0};
i5 : lexModule(hs,F)
o5 = image|e_1e_2 e_1e_3 e_2e_3e_4 0         0         0            |
          |0      0      0         e_1e_2e_3 e_1e_2e_4 0            |
          |0      0      0         0         0         e_1e_2e_3e_4|
o5 : E-module, submodule of E^3

i6 : F=E^{2,0,-2};
i7 : hs={1, 4, 5, 4, 6, 5, 6, 3, 0};
i8 : lexModuleBySequences(hs,F)
o8 = image {-2}|e_1e_3 e_1e_2 e_2e_3e_4 0      0         0        |
           {0} |0      0      0         e_1e_2 e_1e_3e_4 0        |
           {2} |0      0      0         0      0         e_1e_2e_3|
o8 : E-module, submodule of E^3

i9 : F=E^{3,1,-2};
i10 : hs={1, 2, 2, 4, 3, 3, 4, 5, 2, 0};
i11 : isHilbertSequence(hs,F)
o11 = false
```

Chapter 3

Bounds for Betti numbers

Let K be a field, V a finite n–dimensional K-vector space, $E = K\langle e_1, \dots, e_n \rangle$ the exterior algebra of V, and F a finitely generated graded free E-module with a homogeneous basis. Let $\mathcal{M}$ the category of finitely generated $\mathbb{Z}$-graded left and right E-modules M, satisfying $am = (-1)^{\deg a \deg m} ma$ for all homogeneous elements $a \in E$, $m \in M$.
We study projective and injective resolutions over E. More precisely, we give upper bounds for the graded Betti numbers and the graded Bass numbers of classes of modules in $\mathcal{M}$. In order to do this, we firstly describe some classes of graded submodules of F, and finally we state that the lexicographic submodules of F have the maximal Betti numbers among all the graded submodules of F with the same Hilbert function. A similar result holds for the Bass numbers. In this chapter Theorem 2.2.4 in Chapter 2 plays a crucial role.

3.1 The generic initial module

In this Section, we study the generic initial module of a graded module $M \in \mathcal{M}$. Such a module can be defined as in the polynomial case [AH00, Par94, Par96].

In order to point out some peculiarities of the exterior algebra, we rewrite some definitions introduced in Section 1.2 of Chapter 1 and in Section 2.1 of Chapter 2.

Let K be a field, $E = K\langle e_1, \dots, e_n \rangle$ the exterior algebra of a n–dimensional K-vector space V, and Let $F \in \mathcal{M}$ be a free module with homogeneous basis $g_1, \dots, g_r$, where $\deg(g_i) = f_i$ for each $i = 1, \dots, r$, with $f_1 \leq f_2 \leq \cdots \leq f_r$. We write $F = \oplus_{i=1}^r E g_i$.

Definition 3.1.1 Let I be a monomial ideal of E. I is called *stable* if for each monomial $e_\sigma \in I$ and each $j < \mathrm{m}(e_\sigma)$ one has $e_j e_{\sigma \setminus \{\mathrm{m}(e_\sigma)\}} \in I$. I is called *strongly stable* if for each monomial $e_\sigma \in I$ and each $j \in \sigma$ one has $e_i e_{\sigma \setminus \{j\}} \in I$, for all $i < j$.

Definition 3.1.2 A monomial submodule $M = \oplus_{i=1}^r I_i g_i$ of F is an *almost (strongly) stable* submodule if I_i is a (strongly) stable ideal of E, for each i.

Definition 3.1.3 A monomial submodule $M = \oplus_{i=1}^r I_i g_i$ of F is a *(strongly) stable* submodule if I_i is a (strongly) stable ideal of E, for each i, and $(e_1, \ldots, e_n)^{f_{i+1}-f_i} I_{i+1} \subseteq I_i$, for $i = 1, \ldots, r-1$.

Example 3.1.4 Let $E = K\langle e_1, e_2, e_3, e_4\rangle$ and $F = E^2$. The submodule

$$M = (e_1e_2)g_1 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4)g_2$$

of F is an almost strongly stable submodule; whereas

$$N = (e_1e_2, e_1e_3)g_1 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4)g_2$$

is a strongly stable submodule.
If we consider the free E-module $F' = Eg_1 \oplus Eg_2$ with $f_1 = \deg g_1 = -2$, $f_2 = \deg g_2 = -1$, the submodule $M = (e_1e_2)g_1 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4)g_2 \subset F'$ is strongly stable, in fact $(e_1, \ldots, e_4)^{-1+2} I_2 = (e_1e_2e_3e_4) \subseteq I_1$.

Now, order the monomials of F in the *degree reverse lexicographic order*, $>_{\mathrm{degrevlex}_F}$, as follows: let $e_\sigma g_i$ and $e_\tau g_j$ be monomials of F, then $e_\sigma g_i >_{\mathrm{degrevlex}_F} e_\tau g_j$ if

- $\deg(e_\sigma g_i) > \deg(e_\tau g_j)$, or
- $\deg(e_\sigma g_i) = \deg(e_\tau g_j)$, and either $e_\sigma >_{\mathrm{revlex}} e_\tau$, or $e_\sigma = e_\tau$ and $i < j$;

$>_{\mathrm{revlex}}$ is the usual reverse lexicographic order on E with $e_1 >_{\mathrm{revlex}} \cdots >_{\mathrm{revlex}} e_n$ (see [AHH97], for instance).

Now, let $\mathrm{GL}(n)$ be the group of $n \times n$ invertible matrices with entries in the field K, or equivalently, the group of K-linear graded automorphisms of E.

If $\varphi = (a_{i,j}) \in \mathrm{GL}(n)$, one can define the action of φ on E_1 as follows:

$$\varphi(e_j) = \sum_{i=1}^n a_{i,j} e_i, \qquad a_{i,j} \in K$$

and

$$\varphi(\sum_{i=1}^n a_i e_i) = \sum_{i=1}^n a_i \varphi(e_i), \qquad a_i \in K.$$

Furthermore, such an action can be extended to E_d as follow:

$$\varphi(e_\sigma) = \varphi(e_{i_1}) \cdots \varphi(e_{i_d}), \quad \text{for } e_\sigma = e_{i_1} \cdots e_{i_d} \in \mathrm{Mon}_d(E).$$

The automorphism φ induces a natural compatible action on $F = \oplus_{i=1}^r Eg_i$ by

$$\varphi(\sum_{i=1}^r f_i g_i) = \sum_{i=1}^r \varphi(f_i) g_i, \qquad f_i \in E.$$

Now, let $\mathrm{GL}(F)$ be the group of E-linear graded automorphism of F. An element of $\mathrm{GL}(F)$ sends g_i to $\sum_{j=1}^r f_{ij}g_j$, where $f_{ij} \in E_{d_i-d_j}$. If $\varphi_1 \in \mathrm{GL}(n)$ and $\varphi_2 \in \mathrm{GL}(F)$, then $\varphi_1\varphi_2\varphi_1^{-1}$ is an E-linear graded automorphism of F and so we have an action of $\mathrm{GL}(n)$ on $\mathrm{GL}(F)$. Therefore, we can consider the semidirect product $G = \mathrm{GL}(n) \rtimes \mathrm{GL}(F)$. G acts on F through graded K-vector space automorphisms; this action takes submodules to submodules.

Let B be the subgroup of G consisting of all automorphisms taking g_i to a E-linear combination of $g_1, \dots, g_i$ and e_i to a K-linear combination of $e_1, \dots, e_i$. B is the Borel group of G and it is naturally realized by upper triangular matrices.

Definition 3.1.5 A submodule M of F is Borel-fixed if $\varphi(M) = M$, for every $\varphi \in B$.

The following result is the analogue of a general result of Galligo's theorem [Eis95] on generic initial ideals proved in [Par94]. Since its proof is quite similar to the one on submodules of a finitely generated graded free module on a polynomial ring, we omit its proof (see also [AHH97, Theorem 1.6] for the rank one case).

Proposition 3.1.6 *Assume the base field K is infinite and let G and B as above. Then for each graded submodule M of F there exists a nonempty open subset $U \subseteq G$ such that*

(1) *there is a monomial submodule N of F such that $N = \mathrm{in}(\varphi(M))$ for all $\varphi \in U$;*

(2) *N is a Borel-fixed submodule of F, that is $\varphi(N) = N$ for all $\varphi \in B$.*

The monomial submodule $N = \mathrm{in}(\varphi(M))$ of F is denoted by $\mathrm{Gin}(M)$ and called the *generic initial module* of M.

Proposition 3.1.7 *Let K be infinite and let M be a graded submodule of F. Then* $\mathrm{Gin}(M)$ *is a strongly stable submodule of F with the same Hilbert function as M.*

Proof. Since E is noetherian (see for instance [KÏ0]) using the same arguments as in [Hul95, Lemmas 14, 15], we may assume that $M = I_1g_1 \oplus \cdots \oplus I_rg_r$, is a monomial submodule of F such that $(e_1, \dots, e_n)^{f_{i+1}-f_i}I_{i+1} \subseteq I_i$ $(i = 1, \dots, r-1)$, where $f_i = \deg(g_i)$, for all i, without changing the Hilbert function. Moreover, since $\mathrm{in}(P)\,\mathrm{in}(Q) \subset \mathrm{in}(PQ)$, with P, Q graded ideals of E, one has that $(e_1, \dots, e_n)^{f_{i+1}-f_i}\,\mathrm{in}(\varphi(I_{i+1})) \subseteq \mathrm{in}(\varphi(I_i))$, for all $\varphi \in B$.

Hence, $\mathrm{Gin}(M) = \oplus_{i=1}^r J_ig_i$, with J_i monomial ideal of E, for all i, and such that $(e_1, \dots, e_n)^{f_{i+1}-f_i}J_{i+1} \subseteq J_i$, for $i = 1, \dots, r-1$.

Now, we prove that every J_i $(i = 1, \dots, r)$ is a strongly stable ideal of E.

Assume there exists an integer $i \in \{1, \dots, r\}$ such that J_i is not a strongly stable ideal of E. Hence, there exist a monomial $e_\sigma \in J_i$ and a pair (h, j) of positive integers with $h < j$, $j \in \mathrm{supp}(e_\sigma)$, such that $e_he_{\sigma\setminus\{j\}} \notin J_i$. Let $\varphi \in \mathrm{GL}(n)$ with $\varphi(e_j) = e_j + e_h$ and $\varphi(e_k) = e_k$ for $k \neq j$. Then, $\varphi(e_\sigma) = e_\sigma + e_he_{\sigma\setminus\{j\}}$ and consequently $\varphi(J_i) \nsubseteq J_i$. Therefore, $\varphi(e_\sigma)g_i$ does not belong to $\mathrm{Gin}(M)$. A contradiction.

Finally, from (3.3.1), $\mathrm{Gin}(M)$ is a strongly stable submodule of F with the same Hilbert function as M. $\square$

From now on, we will assume that the base field K is infinite.

3.2 (Almost) Lexicographic submodules

In this Section, we analyze two special classes of monomial submodules of F that will play a fundamental role for the development of the work: the *almost lexicographic submodules* and the *lexicographic submodules*.

Definition 1.2.16 in Chapter 1 is equivalent to the following one [AC18a, Proposition 3.12] (see also [CF16, Proposition 3.8]).

Definition 3.2.1 Let L be a graded submodule of F. L is a lex submodule of F if $L = \oplus_{i=1}^{r} I_i g_i$, with I_i lex ideals of E $(i = 1, \dots, r)$, and $(e_1, \dots, e_n)^{\rho_i + f_i - f_{i-1}} \subseteq I_{i-1}$, for $i = 2, \dots, r$, with $\rho_i = \mathrm{indeg} I_i$.

Now, we give the definition of a particular class of monomial submodules that includes all lex submodules as subclass.

Definition 3.2.2 A monomial submodule $M = \oplus_{i=1}^{r} I_i g_i$ of F is an almost lexicographic submodule (almost lex submodule, for short) if I_i is a lex ideal of E, for each i.

Next result associates to a graded submodule M of F an almost lex submodule of F which preserves the Hilbert function and provides an upper bound for the Betti numbers of the class of all graded submodules of F with given Hilbert function.

For a monomial submodule $M = \oplus_{i=1}^{r} I_i g_i$ of F, let us denote by $\mathcal{D}(M)$ the set of all the monomial ideals I_i which appear in the direct decomposition of M.

Proposition 3.2.3 *Let M be a graded submodule of F. Then there exists an almost lex submodule $\mathcal{L}$ of F such that*

(a) $H_{F/M} = H_{F/\mathcal{L}}$;

(b) $\beta_{p,q}(F/M) \leq \beta_{p,q}(F/\mathcal{L})$, *for all* p, q.

Proof. First of all, from (3.3.1), (3.3.2), we may assume that M is a monomial submodule of F.
Set $M = \oplus_{j=1}^{r} I_j g_j$, with I_j monomial ideal of E, for all j. From Theorem 1.3.7 and [AHH97, Theorem 4.1], for every $I_j \in \mathcal{D}(M)$ $(j = 1, \dots, r)$ there exists a unique lex ideal I_j^{lex} of E such that $H_{E/I_j} = H_{E/I_j^{\mathrm{lex}}}$ and $\beta_{p,q}(E/I_j) \leq \beta_{p,q}(E/I_j^{\mathrm{lex}})$, for all p, q.

Hence, setting $\mathcal{L} = \oplus_{j=1}^{r} I_j^{\mathrm{lex}} g_j$, $\mathcal{L}$ is an almost lex submodule of F such that

$$\begin{aligned} H_{F/M}(d) &= \sum_{j=1}^{r} H_{Eg_j/I_jg_j}(d) = \sum_{j=1}^{r} H_{E/I_j}(d-f_i) = \sum_{j=1}^{r} H_{E/I_j^{\mathrm{lex}}}(d-f_i) = \\ &= \sum_{j=1}^{r} H_{Eg_j/I_j^{\mathrm{lex}}g_j}(d) = H_{F/\mathcal{L}}(d), \quad \text{for all} \ \ d, \end{aligned}$$

and

$$\begin{aligned} \beta_{p,q}(F/M) &= \sum_{j=1}^{r} \beta_{p,q}(Eg_j/I_jg_j) = \sum_{j=1}^{r} \beta_{p,q-f_j}(E/I_j) \leq \\ &\leq \sum_{j=1}^{r} \beta_{p,q-f_j}(E/I_j^{\mathrm{lex}}) = \sum_{j=1}^{r} \beta_{p,q}(Eg_j/I_j^{\mathrm{lex}}g_j) = \beta_{p,q}(F/\mathcal{L}), \quad \text{for all} \ \ p,q. \end{aligned}$$

The assertions (a), (b) follow. $\square$

If $M = \oplus_{j=1}^{r} I_j g_j$ is a monomial submodule of F, we will denote by M^{alex} the almost lex submodule of F defined in Proposition 3.2.3, *i.e.*, $M^{\mathrm{alex}} = \oplus_{j=1}^{r} I_j^{\mathrm{lex}} g_j$. Such a monomial submodule will be called the *almost lex submodule associated* to M.

Note that Proposition 3.2.3 implies that if M is a graded submodule of F, we may assume M itself to be an almost lex submodule (hence, an almost strongly stable submodule) without changing the Hilbert function.

Example 3.2.4 Let $E = K\langle e_1, e_2, e_3, e_4\rangle$ and $F = E^3$. Consider the monomial submodule

$$M = (e_1e_2, e_3e_4)g_1 \oplus (e_1e_2, e_2e_3e_4)g_2 \oplus (e_2e_3e_4)g_3$$

of F. Set $I_1 = (e_1e_2,\ e_3e_4)$, $I_2 = (e_1e_2,\ e_2e_3e_4)$ and $I_3 = (e_2e_3e_4)$.
If one considers the monomial ideal I_1, one has $H_{E/I_1} = (1,4,4,0,0)$ and consequently $I_1^{\mathrm{lex}} = (e_1e_2, e_1e_3, e_2e_3e_4)$. Furthermore, $H_{E/I_2} = (1,4,5,1,0)$ and $I_2^{\mathrm{lex}} = (e_1e_2, e_1e_3e_4)$; whereas, $H_{E/I_3} = (1,4,6,3,0)$ and $I_3^{\mathrm{lex}} = (e_1e_2e_3)$. Therefore,

$$M^{\mathrm{alex}} = (e_1e_2, e_1e_3, e_2e_3e_4)g_1 \oplus (e_1e_2, e_1e_3e_4)g_2 \oplus (e_1e_2e_3)g_3,$$

and $H_{F/M^{\mathrm{alex}}} = (3,12,15,4,0) = H_{F/M}$. Finally, if we compare the Betti diagrams of M and M^{alex}

total	5	14	29	52	85	130
2	3	6	9	12	15	18
3	2	8	20	40	70	112

Betti diagram for M

total	6	18	38	68	110	166
2	3	7	12	18	25	33
3	3	11	26	50	85	133

Betti diagram for M^{alex}

the inequalities on the Betti numbers of Proposition 3.2.3 (b) are verified.

Remark 3.2.5 It is worthy to be highlighted that if M is a graded submodule of F, then almost lex submodules which are not equal to M^{alex} but with the same Hilbert function as M could exist. Indeed, let $E = K\langle e_1, e_2, e_3, e_4\rangle$, $F = E^3$ and

$$M = (e_1e_2, e_3e_4)g_1 \oplus (e_1e_2, e_2e_3e_4)g_2 \oplus (e_2e_3e_4)g_3.$$

The almost lex submodule of F associated to M is

$$M^{\text{alex}} = (e_1e_2, e_1e_3, e_2e_3e_4)g_1 \oplus (e_1e_2, e_1e_3e_4)g_2 \oplus (e_1e_2e_3)g_3,$$

and $H_{F/M^{\text{alex}}} = (3, 12, 15, 4, 0) = H_{F/M}$. The following submodule

$$N = (e_1e_2, e_1e_3, e_1e_4)g_1 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4)g_2 \oplus (e_1e_2e_3, e_1e_2e_4)g_3,$$

of F is an almost lex submodule (different from M^{alex}) such that $H_{F/N} = (3, 12, 15, 4, 0) = H_{F/M}$.

We can observe that every lex submodule of F is a strongly stable submodule (see [CF16, Proposition 3.9]). Moreover, it is clear that a lex submodule is an almost lex submodule. The converse does not hold, as next example illustrates.

Example 3.2.6 *(1)* Let $E = K\langle e_1, e_2, e_3, e_4, e_5\rangle$ and $F = E^3$. The submodule

$$\begin{aligned} M =&(e_1e_2, e_1e_3, e_1e_4e_5, e_2e_3e_4e_5)g_1 \oplus (e_1e_2, e_1e_3e_4, e_1e_3e_5, e_2e_3e_4e_5)g_2 \oplus \\ &(e_1e_2e_3, e_1e_2e_4, e_1e_3e_4e_5)g_3 \end{aligned}$$

of F is not a lex submodule of F even if the ideals $(e_1e_2, e_1e_3, e_1e_4e_5, e_2e_3e_4e_5)$, $(e_1e_2, e_1e_3e_4, e_1e_3e_5, e_2e_3e_4e_5)$, $(e_1e_2e_3, e_1e_2e_4, e_1e_3e_4e_5)$ are lex ideals of E. In fact, $e_1e_2g_2 \in M_2$ but $e_2e_3g_1 >_{\text{lex}_F} e_1e_2g_2$ and $e_2e_3g_1 \notin M_2$. Observe that $(e_1, e_2, e_3, e_4, e_5)^2 \nsubseteq (e_1e_2, e_1e_3, e_1e_4e_5, e_2e_3e_4e_5)$. M is an almost lex submodule of F.

(2) Let $E = K\langle e_1, e_2, e_3, e_4, e_5\rangle$ and $F = \oplus_{i=1}^3 Eg_i$ with $\deg g_1 = -1$, $\deg g_2 = \deg g_3 = 1$. Let us consider the M of *(1)* as submodule $M' \subset F$. It is an almost lex submodule, indeed this property remains true when the degrees of g_i changes. In such a case, we observe that $e_1e_2g_2 \in M'_3$ and $e_1e_2e_3e_4g_1, \ldots, e_2e_3e_4e_5g_1 \in M'_3$. In fact, $(e_1, e_2, e_3, e_4, e_5)^{2+1+1} \subseteq (e_1e_2, e_1e_3, e_1e_4e_5, e_2e_3e_4e_5)$. But also in this case M' is not a lex submodule due to second relation: $(e_1, e_2, e_3, e_4, e_5)^{3+1-1} \nsubseteq (e_1e_2, e_1e_3e_4, e_1e_3e_5, e_2e_3e_4e_5)$.

(3) Let $E = K\langle e_1, e_2, e_3, e_4, e_5\rangle$ and $F = E^3$. The submodule

$$\begin{aligned} \mathcal{L} =&(e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4, e_2e_3e_5, e_2e_4e_5, e_3e_4e_5)g_1 \oplus \\ &(e_1e_2e_3, e_1e_2e_4, e_1e_2e_5, e_1e_3e_4e_5, e_2e_3e_4e_5)g_2 \oplus (e_1e_2e_3e_4, e_1e_2e_3e_5, e_1e_2e_4e_5, e_1e_3e_4e_5)g_3 \end{aligned}$$

is a lex submodule of F.

We have already shown in Chapter 2 that lexicographic submodules play a fundamental role in the classification of the Hilbert functions of quotient of finitely generated graded free E–modules. Theorem 2.2.4 points out that if M is a graded submodule of F, then there exists a unique lex submodule of F with the same Hilbert function as M, *i.e.* $H_{F/M} = H_{F/M^{\mathrm{lex}}}$, among all almost lex submodules sharing this properties.

Example 3.2.7 Let $E = K\langle e_1, \ldots, e_4\rangle$. Consider the following submodule of E^3:

$$M = (e_1e_2, e_3e_4)g_1 \oplus (e_1e_2, e_2e_3e_4)g_2 \oplus (e_2e_3e_4)g_3.$$

M is a monomial submodule with $H_{E^3/M} = (3, 12, 15, 4, 0)$. One has that

$$M^{\mathrm{lex}} = (e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4)g_1 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4)g_2 \oplus (e_1e_2e_3e_4)g_3.$$

We can notice that $H_{E^3/M} = H_{E^3/M^{\mathrm{lex}}}$. Finally, one can observe that $M^{\mathrm{lex}} \neq M^{\mathrm{alex}}$ (Example 3.2.4).

3.3 Graded Betti numbers

In this Section we generalize the "higher" Kruskal–Katona Theorem [AHH97, Theorem 4.4]. We show that if $\mathcal{H}$ is a class of graded submodules of the free E –module $F = \oplus_{i=1}^{r} Eg_i$ with a given Hilbert function H, then the unique lex submodule belonging to $\mathcal{H}$ (Theorem 2.2.4) gives upper bounds for the graded Betti numbers of any graded submodule in $\mathcal{H}$.

Using the same arguments as in the polynomial case ([Eis95, Ch. 15], [MS05, Ch. 8.3], [CR09, Her02]; see also [AHH97] for the rank one case), one has that

$$H_{F/M} = H_{F/\operatorname{in}(M)} \tag{3.3.1}$$

and

$$\beta_{i,j}(F/M) \leq \beta_{i,j}(F/\operatorname{in}(M)), \text{ for all } i, j. \tag{3.3.2}$$

Since $\operatorname{in}(M)$ is a monomial submodule of F with the same Hilbert function as M, we may assume M itself is a monomial submodule without changing the Hilbert function.

Example 3.3.1 Let $E = K\langle e_1, e_2, e_3, e_4, e_5\rangle$ and $F = E^2$. Consider the graded submodule

$$M = (e_1e_2e_3 + e_3e_4e_5, e_1e_3 + e_4e_5, e_2e_3e_4)g_1 \oplus (e_1e_2 + e_1e_3, e_4e_5)g_2$$

of F. M is not a monomial submodule and the initial module of M is

$$\operatorname{in}(M) = (e_1e_3, e_1e_4e_5, e_2e_3e_4, e_2e_4e_5, e_3e_4e_5)g_1 \oplus (e_1e_2, e_4e_5)g_2.$$

Note that $H_{F/M} = (2, 10, 17, 7, 0, 0) = H_{F/\operatorname{in}(M)}$. Finally, by comparing the Betti diagrams (as displayed by the computer program Macaulay2 [GS]) of M and $\operatorname{in}(M)$

total	5	20	56	123	234	404	650
2	3	4	6	8	10	12	14
3	2	16	50	115	224	392	636

Betti diagram for M

total	7	25	63	132	245	417	665
2	3	6	9	12	15	18	21
3	4	19	54	120	230	399	644

Betti diagram for $\mathrm{in}(M)$

one can verify that the inequality in (3.3.2) is satisfied.

If M is a monomial submodule of F, we denote by $G(M)$ the unique minimal set of monomial generators of M, and by $G(M)_d$ the set of all monomials $u \in G(M)$ such that $\deg(u) = d$, and by $G(M)_{\geq d}$ the set of monomials $u \in G(M)$ such that $\deg u \geq d$. For every monomial submodule $M = \oplus_{i=1}^r I_i g_i$ of F, we have

1. $G(M) = \{ug_i \,:\, u \in G(I_i), i = 1, \dots, r\}$,
2. $G(M)_d = \{ug_i \,:\, u \in G(I_i)_{d-f_i},\ i = 1, \dots, r\}$.
3. $G(M)_{\geq d} = \{ug_i \,:\, u \in G(I_i)_{\geq d-f_i},\ i = 1, \dots, r\}$.

For a monomial $e_\sigma g_i$ of $F = \oplus_{i=1}^r E g_i$, setting

$$\mathrm{m}_F(e_\sigma g_i) = \mathrm{m}(e_\sigma), \qquad 1 \leq i \leq r,$$

define

$$G(M : j) = \{e_\sigma g_i \in G(M) : \mathrm{m}_F(e_\sigma g_i) = j\},$$

and

$$\mathrm{m}_j^F(M) = |G(M : j)|,\ 1 \leq j \leq n, \qquad \mathrm{m}_{\leq t}^F(M) = \sum_{j=1}^t \mathrm{m}_j^F(M),\ 1 \leq t \leq n.$$

One can observe that $\mathrm{m}_{\leq n}^F(M) = |G(M)|$.

If $M = \oplus_{i=1}^r I_i g_i$ is an (almost) stable submodule of F, then we can use the *Aramova-Herzog-Hibi formula* [AHH97, Corollary 3.3] for computing the graded Betti numbers of M:

$$\beta_{k,k+\ell}(M) = \sum_{i=1}^r \beta_{k,\,k+\ell}(I_i g_i) = \sum_{u \in G(M)_\ell} \binom{\mathrm{m}_F(u) + k - 1}{\mathrm{m}_F(u) - 1}, \qquad \text{for all } k. \tag{3.3.3}$$

Indeed, one can easily observe that

$$\sum_{u \in G(M)_\ell} \binom{\mathrm{m}_F(u) + k - 1}{\mathrm{m}_F(u) - 1} = \sum_{i=1}^r \left[\sum_{u \in G(I_i)_{\ell - f_\ell}} \binom{\mathrm{m}(u) + k - 1}{\mathrm{m}(u) - 1} \right]. \tag{3.3.4}$$

As in the case when ideals of a polynomial ring are considered [AHH98, Lemma 3.6], next characterization of an almost strongly stable submodule of F easily follows.

Lemma 3.3.2 *Let M be a monomial submodule of F. Assume $M = M' + M''$, with $M' = \oplus_{i=1}^r I_i' g_i$, $M'' = \oplus_{i=1}^r I_i'' e_n g_i$ and I_i', I_i'' ideals of the exterior algebra $\widetilde{E} = K\langle e_1, \ldots, e_{n-1}\rangle$ $i = 1, \ldots, r$. Set $\widetilde{M''} = \oplus_{i=1}^r I_i'' g_i$. Then the following conditions are equivalent:*

(i) *M is an almost strongly stable submodule;*

(ii) *M', $\widetilde{M''}$ are almost strongly stable submodules, and $I_i''(e_1, \ldots, e_{n-1}) \subset I_i'$, for all i.*

Remark 3.3.3 One can quickly verify that if M is a strongly stable submodule of F, then M admits a decomposition of the type defined in Lemma 3.3.2 with M' strongly stable submodule of M, too; whereas $\widetilde{M''}$ could not be a strongly stable submodule.

Example 3.3.4 Let $E = K\langle e_1, \ldots, e_4\rangle$ and $F = Eg_1 \oplus Eg_2$ with $\deg g_1 = -2$, $\deg g_2 = 1$.

$$M = (e_1e_2, e_1e_3, e_1e_4, e_2e_3)g_1 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4)g_2$$

is a strongly stable submodule of F. We can write M as follows

$$M = M' + M'',$$

where $M' = (e_1e_2, e_1e_3, e_2e_3)g_1 \oplus (e_1e_2e_3)g_2$ and $M'' = (e_1)e_4g_1 \oplus (e_1e_2, e_1e_3, e_2e_3)e_4g_2$.

One can observe that M' is a strongly stable submodule, whereas $\widetilde{M''} = (e_1)g_1 \oplus (e_1e_2, e_1e_3, e_2e_3)g_2$ is an almost strongly stable submodule, which is not strongly stable.

Remark 3.3.5 One can observe that if I is a strongly stable ideal of the exterior algebra $\widetilde{E} = K\langle e_1, \ldots, e_{n-1}\rangle$, then I is a strongly stable ideal of the exterior algebra $E = K\langle e_1, \ldots, e_n\rangle$; whereas, one can easily find a monomial ideal I which is lex in the exterior algebra $\widetilde{E} = K\langle e_1, \ldots, e_{n-1}\rangle$, but not in E.

Following [AHH98], the following map can be defined

$$\alpha : \mathrm{Mon}_d(E) \to \mathrm{Mon}_d(E),$$

with

- $\alpha(e_\sigma) = e_\sigma$, if $n \notin \mathrm{supp}(e_\sigma)$;
- $\alpha(e_\sigma) = (-1)^{\alpha(\sigma,j)} e_j e_{\sigma\setminus\{n\}}$, if $n \in \mathrm{supp}(e_\sigma)$ and j is the largest integer $< n$ which does not belong to $\mathrm{supp}(e_\sigma)$, $\alpha(\sigma, j) = |\{t \in \sigma : t < j\}|$.

Such a map is order preserving [AHH98], *i.e.*, if $e_\sigma, e_\tau \in \mathrm{Mon}_d(E)$ and $e_\sigma \geq_{\mathrm{lex}} e_\tau$, then $\alpha(e_\sigma) \geq_{\mathrm{lex}} \alpha(e_\tau)$. The map α can be extended to $\mathrm{Mon}_d(F)$ as follows:

$$\alpha_F : \mathrm{Mon}_d(F) \to \mathrm{Mon}_d(F), \quad \alpha_F(e_\sigma g_i) = \alpha(e_\sigma)g_i,\ 1 \leq i \leq r.$$

The map α_F is order preserving too.

Let $e_\sigma g_i, e_\tau g_j \in \mathrm{Mon}_d(F)$ with $e_\sigma g_i \geq_{\mathrm{lex}_F} e_\tau g_j$. We distinguish two cases: $i = j$, $i \neq j$.

Let $i = j$. If $e_\sigma g_i \geq_{\mathrm{lex}_F} e_\tau g_i$, then $e_\sigma \geq_{\mathrm{lex}} e_\tau$. Since $\alpha_F(e_\sigma g_i) = \alpha(e_\sigma)g_i$, $\alpha_F(e_\tau g_i) = \alpha(e_\tau)g_i$ and α is order preserving, then $\alpha_F(e_\sigma g_i) \geq_{\mathrm{lex}_F} \alpha_F(e_\tau g_i)$.

Let $i \neq j$. If $e_\sigma g_i \geq_{\mathrm{lex}_F} e_\tau g_j$, then $i < j$. Hence, $\alpha_F(e_\sigma g_i) = \alpha(e_\sigma)g_i \geq_{\mathrm{lex}_F} \alpha(e_\tau)g_j = \alpha_F(e_\tau g_j)$.

For a non empty subset M of $\mathrm{Mon}(F)$, let us denote by $\min(M)$ the smallest monomial of M with respect to $\leq_{\mathrm{lex}_F}$.

Lemma 3.3.6 *Let $M = \oplus_{i=1}^r I_i g_i = M' + M''$ be an almost strongly stable submodule of F, with $M' = \oplus_{i=1}^r I'_i g_i$, $M'' = \oplus_{i=1}^r I''_i e_n g_i$ and I'_i, I''_i $(i = 1, \ldots, r)$ ideals of $\widetilde{E} = K\langle e_1, \ldots, e_{n-1}\rangle$. Then $\alpha_F(\min(G(M)) = \alpha_F(\min(G(M'))$.*

Proof. Since $\min(G(M')) \geq_{\mathrm{lex}} \min(G(M))$, then $\min(G(M')) = \alpha_F(\min(G(M')) \geq_{\mathrm{lex}_F} \alpha_F(\min(G(M))$. On the other hand, since M is almost strongly stable, then $\alpha_F(\min(G(M)) \in G(M')$ and $\min(G(M')) \leq_{\mathrm{lex}_F} \alpha_F(\min(G(M))$. $\square$

Theorem 3.3.7 *Let M and L be monomial submodules of F generated in degree s. Assume*

(1) *M is an almost strongly stable submodule,*

(2) *L is a lex submodule, and*

(3) $\dim_K L_s \leq \dim_K M_s$.

Then

$$\mathrm{m}^F_{\leq i}(L) \leq \mathrm{m}^F_{\leq i}(M) \tag{3.3.5}$$

for all i.

Proof. Set $\widetilde{E} = K\langle e_1, \ldots, e_{n-1}\rangle$. We proceed by induction on $n = \dim_K E_1$. By hypotheses, $\mathrm{m}^F_{\leq n}(L) = \dim_K L_s \leq \dim_K M_s = \mathrm{m}^F_{\leq n}(M)$. In order to prove the inequality in (3.3.5) for $i < n$, we write M and L as follows:

$$M = \oplus_{i=1}^r I_i g_i = M' + M'',$$

with $M' = \oplus_{i=1}^r I'_i g_i$, $M'' = \oplus_{i=1}^r I''_i e_n g_i$, and I'_i, I''_i $(i = 1, \ldots, r)$ ideals of E generated by monomials in $e_1, \ldots, e_{n-1}$, *i.e.*, monomial ideals of $\widetilde{E}$, and

$$L = \oplus_{i=1}^r J_i g_i = L' + L'',$$

with $L' = \oplus_{i=1}^r J'_i g_i$, $L'' = \oplus_{i=1}^r J''_i e_n g_i$ and J'_i, J''_i monomial ideals of $\widetilde{E}$.

It is clear that M' is an almost strongly stable submodule and that L' is a lex submodule.

Hence, if we prove that $\dim_K L'_s \leq \dim_K M'_s$, from the inductive hypothesis the assertion will follows.

Set $\widetilde{M}'' = \oplus_{i=1}^r I_i'' g_i$. We can assume that M' and $\widetilde{M}''$ are lex submodules.

Indeed, let $\widetilde{M} = \oplus_{i=1}^r \widetilde{I}_i g_i$ ($\widetilde{L} = \oplus_{i=1}^r \widetilde{J}_i g_i$, respectively) be the lex submodules of F generated by those monomials ug_i with u monomial of $\widetilde{E}$ and such that $\dim_K \widetilde{M}_s = \dim_K M'_s$ ($\dim_K \widetilde{L}_{s-1} = \dim_K \widetilde{M}''_{s-1}$, respectively).

Let $N = \widetilde{M} + \widetilde{L} = \oplus_{i=1}^r \widetilde{I}_i g_i + \oplus_{i=1}^r \widetilde{J}_i g_i$. We prove that N is an almost strongly stable submodule.

First of all note that $\widetilde{I}_i$, $\widetilde{J}_i$ are lex ideals and so strongly stable ideals, for all i. On the other hand, by [AHH98, Lemma 3.7, Theorem 3.9], one can verify that $\widetilde{J}_i(e_1, \ldots, e_{n-1}) \subset \widetilde{I}_i$, for all i. Hence, N is an almost strongly stable submodule.

Now, we are in the following situation:

$$M = \oplus_{i=1}^r I'_i g_i + \oplus_{i=1}^r I''_i e_n g_i, \quad L = \oplus_{i=1}^r J'_i g_i + \oplus_{i=1}^r J''_i e_n g_i$$

where M is an almost strongly stable submodule and L is a lex submodule, and in addition $M' = \oplus_{i=1}^r I'_i g_i$, $\widetilde{M}'' = \oplus_{i=1}^r I''_i g_i$ are lex submodules. Assuming that $\dim_K L_s \leq \dim_K M_s$ we want to prove that

$$\dim_K L'_s \leq \dim_K M'_s. \tag{3.3.6}$$

Thanks to Lemma 3.3.6 we have

$$\min(G(M')) = \alpha_F(\min(G(M)) \leq_{\mathrm{lex}_F} \min(G(L')) = \alpha_F(\min(G(L')).$$

Since the submodules L' and M' are lex, the inequality (3.3.6) holds. Hence, by the inductive hypothesis, the required inequality (3.3.5) follows. $\square$

By using combinatorial arguments one can quickly verify the following lemma.

Lemma 3.3.8 *Let M be an almost strongly stable submodule of F generated in degree d. If $M_{\langle d+1\rangle}$ is the submodule of F generated by the elements of M_{d+1}, then*

$$\mathrm{m}_i(M_{\langle d+1\rangle}) = \mathrm{m}_{\leq i-1}(M)$$

for all i.

If M is a set of monomials of degree $d < n$ of F, we denote by $M\{e_1, \ldots, e_n\}$ the following set of monomials of degree $d+1$ of F [AC18b, CF15]:

$$M\{e_1, \ldots, e_n\} = \{(-1)^{\alpha(\sigma,j)} e_j e_\sigma g_i : e_\sigma g_i \in M,\ j \notin \mathrm{supp}(e_\sigma),\ j = 1, \ldots, n,\ i = 1, \ldots r\},$$

$\alpha(\sigma, j) = |\{r \in \sigma : r < j\}|$. Such a set is usually called the *shadow* of M.

Theorem 3.3.7 and Lemma 3.3.8 yield the following result.

Theorem 3.3.9 *Let M be a graded submodule of F. Then*

$$\beta_{i,j}(M) \leq \beta_{i,j}(M^{\mathrm{lex}}),$$

for all i, j.

Proof. The proof is quite similar to [AC18a, Theorem 4] (see also [AHH97]). Due to (3.3.1) and (3.3.2), from Proposition 3.1.7, we may assume that M is a strongly stable submodule.

From (3.3.3) we have:

$$\beta_{i,i+j}(M) = \sum_{u \in G(M)_j} \binom{\mathrm{m}_F(u)+i-1}{\mathrm{m}_F(u)-1}, \quad \text{for } i \geq 1. \tag{3.3.7}$$

Since $G(M)_j = G(M_{\langle j\rangle}) - G(M_{\langle j-1\rangle})\{e_1, \ldots, e_n\}$, the above sum can be written as a difference $\beta_{i,i+j}(M) = C - D$, with

$$\begin{aligned}
C &= \sum_{u \in G(M_{\langle j\rangle})} \binom{\mathrm{m}_F(u)+i-1}{\mathrm{m}_F(u)-1} \\
&= \sum_{t=1}^{n} \sum_{u \in G(M_{\langle j\rangle};t)} \binom{t+i-1}{t-1} = \sum_{t=1}^{n} \mathrm{m}_t(M_{\langle j\rangle}) \binom{t+i-1}{t-1} \\
&= \sum_{t=1}^{n} (\mathrm{m}_{\leq t}(M_{\langle j\rangle}) - \mathrm{m}_{\leq t-1}(M_{\langle j\rangle})) \binom{t+i-1}{t-1} \\
&= \mathrm{m}_{\leq n}(M_{\langle j\rangle}) \binom{n+i-1}{n-1} \\
&+ \sum_{t=1}^{n-1} \mathrm{m}_{\leq t}(M_{\langle j\rangle}) \left[\binom{t+i-1}{t-1} - \binom{t+1+i-1}{t} \right] \\
&= \mathrm{m}_{\leq n}(M_{\langle j\rangle}) \binom{n+i-1}{n-1} - \sum_{t=j}^{n-1} \mathrm{m}_{\leq t}(M_{\langle j\rangle}) \binom{t+i-1}{t}
\end{aligned}$$

and

$$\begin{aligned}
D &= \sum_{u \in G(M_{\langle j-1\rangle})\{e_1,\ldots,e_n\}} \binom{\mathrm{m}_F(u)+i-1}{\mathrm{m}_F(u)-1} \\
&= \sum_{t=2}^{n} m_{\leq t-1}(M_{\langle j-1\rangle}) \binom{t+i-1}{t-1},
\end{aligned}$$

from Lemma 3.3.8. On the other hand, since the number of generators of $M_{\langle d\rangle}$ and $M^{\mathrm{lex}}_{\langle d\rangle}$ are equal for all d, we have $\mathrm{m}_{\leq n}(M_{\langle d\rangle}) = \mathrm{m}_{\leq n}(M^{\mathrm{lex}}_{\langle d\rangle})$. Hence, from Theorem 3.3.7, $\mathrm{m}_{\leq i}(M^{\mathrm{lex}}_{\langle d\rangle})$

$\leq \mathrm{m}_{\leq i}(M_{\langle d\rangle})$ for $1 \leq i \leq n$, and consequently:

$$\begin{aligned}
\beta_{i,i+j}(M) = {} & \mathrm{m}_{\leq n}(M_{\langle j\rangle})\binom{n+i-1}{n-1} - \sum_{t=j}^{n-1} \mathrm{m}_{\leq t}(M_{\langle j\rangle})\binom{t+i-1}{t} \\
& - \sum_{t=2}^{n} m_{\leq t-1}(M_{\langle j-1\rangle})\binom{t+i-1}{t-1} \leq \\
& \leq \mathrm{m}_{\leq n}(M^{\mathrm{lex}}_{\langle j\rangle})\binom{n+i-1}{n-1} - \sum_{t=j}^{n-1} \mathrm{m}_{\leq t}(M^{\mathrm{lex}}_{\langle j\rangle})\binom{t+i-1}{t} \\
& - \sum_{t=2}^{n} m_{\leq t-1}(M^{\mathrm{lex}}_{\langle j-1\rangle})\binom{t+i-1}{t-1} = \beta_{i,i+j}(M^{\mathrm{lex}}).
\end{aligned}$$

□

Finally, from Proposition 3.2.3 and Theorem 3.3.9, next result follows.

Corollary 3.3.10 *Let M be a graded submodule of F. Then*

$$\beta_{i,j}(M) \leq \beta_{i,j}(M^{\mathrm{alex}}) \leq \beta_{i,j}(M^{\mathrm{lex}}), \quad \textit{for all } i, j.$$

Example 3.3.11 *(1)* Let $E = K\langle e_1, e_2, e_3, e_4, e_5\rangle$ and $F = E^3$. The submodule

$$M = (e_1e_2, e_1e_4, e_3e_4e_5)g_1 \oplus (e_1e_3, e_1e_4e_5, e_2e_3e_4)g_2 \oplus (e_1e_2e_4, e_1e_3e_5)g_3$$

of F is not an almost lex submodule of F. It is sufficient to observe that the ideal $(e_1e_2, e_1e_4, e_3e_4e_5)$ is not a lex ideals of E. Consider the almost lex submodule

$$\begin{aligned}
M^{\mathrm{alex}} = {} & (e_1e_2, e_1e_3, e_1e_4e_5, e_2e_3e_4e_5)g_1 \oplus (e_1e_2, e_1e_3e_4, e_1e_3e_5, e_2e_3e_4e_5)g_2 \oplus \\
& (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4e_5)g_3,
\end{aligned}$$

which is not a lex submodule of F (see Example 3.2.6), and the lex submodule

$$\begin{aligned}
M^{\mathrm{lex}} = {} & (e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4, e_2e_3e_5, e_2e_4e_5, e_3e_4e_5)g_1 \oplus \\
& (e_1e_2e_3, e_1e_2e_4, e_1e_2e_5, e_1e_3e_4e_5, e_2e_3e_4e_5)g_2 \oplus \\
& (e_1e_2e_3e_4, e_1e_2e_3e_5, e_1e_2e_4e_5, e_1e_3e_4e_5)g_3
\end{aligned}$$

One can quickly verify that $H_{F/M} = (3, 15, 27, 17, 1, 0) = H_{F/M^{\mathrm{alex}}} = H_{F/M^{\mathrm{lex}}}$.

Moreover, using the computer program *Macaulay2*, if one compares the Betti diagrams of the submodules above considered, one has the Corollary 3.3.10:

total	8	26	59	113	195	313	476
2	3	7	12	18	25	33	42
3	5	18	42	80	135	210	308
4	–	1	5	15	35	70	126

Betti diagram for M

total	11	43	113	243	460	796	1288
2	3	7	12	18	25	33	42
3	5	21	56	120	225	385	616
4	3	15	45	105	210	378	630

Betti diagram for M^{alex}

total	16	69	190	419	805	1406	2289
2	3	9	19	34	55	83	119
3	7	31	86	190	365	637	1036
4	6	29	85	195	385	686	1134

Betti diagram for M^{lex}

(2) Now, let $F' = \oplus_{i=1}^{3} Eg_i$ with $\deg g_1 = -1$, $\deg g_2 = 0$ and $\deg g_3 = 1$. Now, let us consider M of *(1)* as submodule $M' \subset F'$. For the same reason as before, it is not an almost lex submodule of F'. Consider the almost lex submodule

$$\begin{aligned} M'^{\text{alex}} =& (e_1e_2, e_1e_3, e_1e_4e_5, e_2e_3e_4e_5)g_1 \oplus (e_1e_2, e_1e_3e_4, e_1e_3e_5, e_2e_3e_4e_5)g_2 \oplus \\ & (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4e_5)g_3, \end{aligned}$$

which is not a lex submodule of F' (see Example 3.2.6), and the lex submodule

$$\begin{aligned} M'^{\text{lex}} =& (e_1e_2, e_1e_3, e_1e_4e_5, e_2e_3e_4)g_1 \oplus \\ & (e_1e_2e_3, e_1e_2e_4, e_1e_2e_5, e_1e_3e_4, e_1e_3e_5, e_2e_3e_4e_5)g_2 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4e_5)g_3 \end{aligned}$$

One can quickly verify that $H_{F/M'} = (1, 6, 14, 18, 15, 8, 1, 0) = H_{F/M'^{\text{alex}}} = H_{F/M'^{\text{lex}}}$. Moreover, using the computer program *Macaulay2*, if one compares the Betti diagrams of the submodules above considered, one has the Corollary 3.3.10:

total	8	26	59	113	195	313	476
1	2	5	9	14	20	27	35
2	2	6	13	24	40	62	91
3	2	8	20	40	70	112	168
4	2	6	12	20	30	42	56
5	–	1	5	15	35	70	126

Betti diagram for M'

total	11	43	113	243	460	796	1288
1	2	5	9	14	20	27	35
2	2	7	18	39	75	132	217
3	3	14	40	90	175	308	504
4	3	12	31	65	120	203	322
5	1	5	15	35	70	126	210

Betti diagram for M'^{alex}

total	13	52	136	289	540	923	1477
1	2	5	9	14	20	27	35
2	2	9	25	55	105	182	294
3	5	21	56	120	225	385	616
4	3	12	31	65	120	203	322
5	1	5	15	35	70	126	210

Betti diagram for M'^{lex}

3.4 Graded Bass numbers

In this section we analyze the graded Bass numbers of graded submodules of F. We are interested in determining upper bounds for such invariants.

If $M \in \mathcal{M}$, we recall that M has a unique minimal graded injective resolution:

$$I_\bullet : 0 \to M \to I^0 \to I^1 \to I^2 \to \dots,$$

where $I^i = \oplus_j E(n-j)^{\mu_{i,j}(M)}$. The integers $\mu_{i,j}(M) = \dim_K \operatorname{Ext}^i_E(K, M)_j$ are called the *graded Bass numbers* of M [BH96, KÏ0].

Let M^* be the right (left) E-module $\operatorname{Hom}_E(M, E)$. The duality between projective and injective resolutions implies the following relation ([AHH97, Proposition 5.2]) between the graded Bass numbers of a module and the graded Betti numbers of its dual.

Proposition 3.4.1 *Let $M \in \mathcal{M}$. Then*

$$\beta_{i,j}(M) = \mu_{i,n-j}(M^*), \qquad \textit{for all } i, j.$$

For the reader's convenience, we recall some notions and results from [AHH97, KÏ0]. Let $M \in \mathcal{M}$ and let M^* be the right (left) E-module $\operatorname{Hom}_E(M, E)$.

We quote next result from [AHH97, Proposition 5.1].

Lemma 3.4.2 *Let $M \in \mathcal{M}$. Then*

$$\dim_K M_i^* = \dim_K M_{n-i}, \quad \textit{for all } i.$$

Let us consider the dual module $\mathrm{Hom}_E(F/M, E)$, where M is a graded submodule of F. If $\operatorname{rank} F = 1$ with $f_1 = 0$, *i.e.*, $F = E$ and $M = I$ is a graded ideal of E, then

$$\mathrm{Hom}_E(E/I, E) \simeq 0 : I, \tag{3.4.1}$$

where $0 : I$ is the annihilator of I, *i.e.*, the set of all elements $b \in E$ such that $ba = 0$, for all $a \in I$. Moreover, from Lemma 3.4.2 (see also [AHH97, Corollary 5.3])

$$\dim_K(E/I)_i = \dim_k(0 : I)_{n-i} \quad \text{for all } i. \tag{3.4.2}$$

Remark 3.4.3 The ideal $0 : I$ is spanned as K-vector space by all monomials $e_{\bar{\sigma}}$ such that $e_\sigma \notin I$, where $\bar{\sigma}$ is the complement of σ in the set $\{1, \ldots, n\}$ (see [AHH97, Proposition 5.7], proof). Furthermore, if I is a lex ideal in E, then $0 : I$ is a lex ideal in E, too. Note that $0 : I$ is the exterior version of the Alexander dual of a squarefree monomial ideal in a polynomial ring.

The next example will be useful for describing our strategy in Theorem 3.4.5.

Example 3.4.4 Let $E = K\langle e_1, e_2, e_3, e_4\rangle$ and $F = E^3$. Let us consider the lex submodule of F:

$$L = (e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4)g_1 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4)g_2 \oplus (e_1e_2e_3e_4)g_3.$$

Setting $I_1 = (e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4)$, $I_2 = (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4)$ and $I_3 = (e_1e_2e_3e_4)$, one has

$$\begin{aligned} 0 : I_1 &= (e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4), \\ 0 : I_2 &= (e_1e_2, e_1e_3, e_1e_4, e_2e_3, e_2e_4, e_3e_4), \\ 0 : I_3 &= (e_1, e_2, e_3, e_4). \end{aligned}$$

Even though the annihilators above are lex ideals, the submodule $N = \oplus_{t=1}^r (0 : I_t)g_t$ is not a lex submodule of F (see for instance Proposition 3.2.1). Indeed, the monomial $e_2e_4 \notin 0 : I_1$. Equivalently, $e_2e_4g_1 >_{\mathrm{lex}_F} e_2e_4g_2$, but $e_2e_4g_2 \in N$, whereas $e_2e_4g_1 \in F \setminus N$. Conversely,

$$\widetilde{N} = (0 : I_3)g_1 \oplus (0 : I_2)g_2 \oplus (0 : I_1)g_3$$

is a lex submodule in F. Note that $(F/L)^* \simeq N \simeq \widetilde{N}$ as E–graded modules (see (3.4.1)) and $H_{F/N} = (3, 8, 3, 0, 0) = H_{F/\widetilde{N}}$.

Theorem 3.4.5 *Let M be a graded submodule of E^r, $r \geq 1$. Then*

$$\mu_{i,j}(E^r/M) \leq \mu_{i,j}(E^r/M^{\mathrm{lex}}), \quad \textit{for all } i, j.$$

Proof. Set $F = E^r$. The case $r = 1$ has been proved in [AHH97, Corollary 5.8]. Assume $r > 1$.

From Proposition 3.4.1 and Theorem 3.3.9, one has

$$\mu_{i,j}(F/M) = \beta_{i,n-j}(\mathrm{Hom}_E(F/M, E)) \leq \beta_{i,n-j}((\mathrm{Hom}_E(F/M, E))^{\mathrm{lex}}). \tag{3.4.3}$$

Let us consider the lex submodule M^{lex}. It is $M^{\mathrm{lex}} = \oplus_{t=1}^r J_t g_t$, with each J_t lex ideal in E and $(e_1, \ldots, e_n)^{\mathrm{indeg} J_t} \subseteq J_{t-1}$, for $t = 2, \ldots, r$. Moreover, from (3.4.1),

$$\mu_{i,j}(F/M^{\mathrm{lex}}) = \beta_{i,n-j}(\oplus_{t=1}^r (0 : J_t) g_t).$$

Now, consider the submodule $\oplus_{t=1}^r (0 : J_t) g_t$ of F. It is not a lex submodule in general (see for instance Example 3.4.4), nevertheless the behavior of the ideals J_t, together with the fact that $\deg g_t = 0$ for all t, assures that $\oplus_{t=1}^r (0 : J_{r-t-1}) g_t$ is a lex submodule of F (see also Remark 3.4.3).

Moreover, it is clear that $\oplus_{t=1}^r (0 : J_t) g_t \simeq \oplus_{t=1}^r (0 : J_{r-t-1}) g_t$. Hence

$$\mu_{i,j}(F/M^{\mathrm{lex}}) = \beta_{i,n-j}(\oplus_{t=1}^r (0 : J_{r-t-1}) g_t). \tag{3.4.4}$$

Claim. The graded E–modules $(\mathrm{Hom}_E(F/M, E))^{\mathrm{lex}}$ and $\oplus_{t=1}^r (0 : J_{r-t-1}) g_t$ have the same Hilbert function.

Set $P = (\mathrm{Hom}_E(F/M, E))^{\mathrm{lex}}$ and $Q = \oplus_{t=1}^r (0 : J_{r-t-1}) g_t$. From Lemma 3.4.2 and (3.4.2), we have

$$\begin{aligned}
\dim_K P_i &= \dim_K ((\mathrm{Hom}_E(F/M, E))^{\mathrm{lex}})_i = \dim_K (\mathrm{Hom}_E(F/M, E))_i \\
&= \dim_K (F/M)_{n-i} = \dim_K (F/M^{\mathrm{lex}})_{n-i} \\
&= \sum_{t=1}^r \dim_K (0 : J_t)_i = \sum_{t=1}^r \dim_K (0 : J_{r-t-1})_i \\
&= \dim_K Q_i.
\end{aligned}$$

The claim follows.

Therefore, since P and Q are lex submodules of F with the same Hilbert function, then they coincide. Finally, from (3.4.3) and (3.4.4),

$$\mu_{i,j}(F/M) \leq \beta_{i,n-j}(P) = \beta_{i,n-j}(Q) = \mu_{i,j}(F/M^{\mathrm{lex}}),$$

for all i, j. $\square$

We close this Section discussing the annihilator of a submodule of F. The next proposition generalizes some results in [AHH97] (Remark 3.4.3).

Proposition 3.4.6 *Let M be a graded submodule of F.*

(1) *If M is a (strongly) stable submodule, then $0 : M$ is a strongly stable ideal in E.*

(2) *If M is a lex submodule, then $0 : M$ is a lex ideal in E.*

Proof. (1). Since $M = \oplus_{i=1}^{r} I_i g_i$ is a monomial submodule of F, then

$$0 : M = \cap_{i=1}^{r}(0 : I_i g_i) = \cap_{i=1}^{r}(0 : I_i),$$

and each ideal $0 : I_i$ is strongly stable [CF13, Lemma 4.1]. The definition of a strongly stable submodule assures us that the ideal $0 : M$ is not null and strongly stable.

Similarly, one can verify that (2) holds. □

If I is a graded ideal of E, then $0 : I^{\mathrm{lex}} = (0 : I)^{\mathrm{lex}}$ [AHH97, Proposition 5.7]. The next example shows that such a property does not hold if I is a graded submodule of F.

Example 3.4.7 Let $E = K\langle e_1, e_2, e_3, e_4, e_5\rangle$ and $F = E^2$. Consider the following submodules of F:

$$M = (e_1e_2, e_1e_3, e_1e_4e_5, e_2e_3e_4, e_2e_4e_5, e_3e_4e_5)g_1 \oplus (e_1e_2, e_2e_3e_4)g_2,$$

$$M^{\mathrm{lex}} = (e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4, e_2e_3e_5, e_2e_4e_5, e_3e_4e_5)g_1 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_2e_5, e_1e_3e_4e_5)g_2.$$

One has

$$\begin{aligned}
0 : M \quad &= (e_1e_4, e_1e_2e_3, e_1e_2e_5, e_1e_3e_5, e_2e_3e_4, e_2e_3e_5),\\
(0 : M)^{\mathrm{lex}} &= (e_1e_2, e_1e_3e_4, e_1e_3e_5, e_1e_4e_5, e_2e_3e_4, e_2e_3e_5),\\
0 : M^{\mathrm{lex}} &= (e_1e_2e_3, e_1e_2e_4, e_1e_2e_5, e_1e_3e_4, e_1e_3e_5, e_1e_4e_5, e_2e_3e_4).
\end{aligned}$$

Hence, $0 : M^{\mathrm{lex}} \neq (0 : M)^{\mathrm{lex}}$.

3.5 *Macaulay2* packages

In this Section, we describes other procedures of the packages "ExteriorIdeals.m2" and "ExteriorModules.m2" introduced in the Section 2.5. We collect some examples in order to describe the algorithms to easily compute stable, strongly stable and lexsegment ideals in E and the (almost) stable, (almost) strongly stable and (almost) lex submodules of F.

Example 3.5.1 Given a monomial ideal I in an exterior algebra E, we illustrate how some functions from our package allow one to check whether I is (strongly) stable or lex and to produce (strongly) stable ideals containing I. The core of the algorithms is based on the fact that the minimal monomial generators of a (strongly) stable ideal must satisfy the criterion in Definition 3.1.1 (it is sufficient to apply it only on a set of generators of the ideal) and on the fact that the shadow of a lexsegment of monomials is again a lexsegment [HH11].

```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : loadPackage "ExteriorIdeals"
i2 : E=QQ[e_1..e_5,SkewCommutative=>true]
i3 : I=ideal {e_2*e_3,e_3*e_4*e_5}
o3 = ideal (e_2e_3, e_3e_4e_5)
o3 : Ideal of E
i4 : isStableIdeal I
o4 = false
```

The ideal I is not stable. Indeed, the monomial e_1e_2 is not in I even though e_2e_3 is. Hence, by the function `StableIdeal(ideal)`, we compute the smallest stable ideal Is containing I:

```
i5 : Is=stableIdeal I
o5 = ideal (e_1e_2, e_1e_3e_4, e_2e_3, e_3e_4e_5)
o5 : Ideal of E
i6 : isStableIdeal Is
o6 = true
i7 : isStronglyStableIdeal Is
o7 = false
```

The ideal Is is stable but not strongly stable in E. Note that the monomial e_1e_3 is not in Is even though e_2e_3 is. Using the function `stronglyStableIdeal(ideal)`, we compute the smallest strongly stable ideal (Iss) containing Is, and consequently I:

```
i8 : Iss=stronglyStableIdeal Is
o8 = ideal (e_1e_2, e_1e_3, e_1e_4e_5, e_2e_3, e_2e_4e_5, e_3e_4e_5)
o8 : Ideal of E
i9 : isStronglyStableIdeal Iss
o9 = true
i10 : Iss2=stronglyStableIdeal I
o10 = ideal (e_1e_2, e_1e_3, e_1e_4e_5, e_2e_3, e_2e_4e_5, e_3e_4e_5)
o10 : Ideal of E
i11 : Iss==Iss2
o11 = true
```

Now, we extend to submodules of F all can we have done for ideals.

Example 3.5.2 Let M be a monomial submodule of a graded free module F, we illustrate functions from "ExteriorModules" package analogously to those for ideals: to check whether M is (strongly) stable or lex and to produce (strongly) stable modules containing M.

```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : loadPackage "ExteriorModules";
i2 : E=QQ[e_1..e_5,SkewCommutative=>true];
i3 : F=E^2;
i4 : I_1=ideal {e_1*e_2, e_1*e_3, e_1*e_4*e_5};
i5 : I_2=ideal {e_1*e_2, e_2*e_3*e_4};
i6 : M=createModule({I_1, I_2},F)
o6 = image|e_1e_3 e_1e_2 e_1e_4e_5 0      0        |
          |0      0      0         e_1e_2 e_2e_3e_4|
o6 : E-module, submodule of E^2
i7 : isStableModule M
o7 = false
```

The module M is almost stable but not stable: the monomial $e_2e_3e_4$ is not in I_1 (Definition 1.2.9). We can compute the smallest stable module containing M by the function `StableModule(module)`.

```
i8 : Ms=stableModule M
o8 = image|e_1e_2 e_1e_3 e_1e_4e_5 e_2e_3e_4 0      0        |
          |0      0      0         0         e_1e_2 e_2e_3e_4|
o8 : E-module, submodule of E^2
i9 : isStronglyStableModule Ms
o9 = false
```

The ideal Ms is stable and but not either almost strongly stable and strongly stable. In fact, the ideals $(e_1e_2, e_2e_3e_4)$ is not strongly stable. We compute the smallest strongly stable module containing Ms by using the function `stronglyStableModule(module)`:

```
i10 : Mss=stronglyStableModule Ms
o10 = image|e_1e_2 e_1e_3 e_1e_4e_5 e_2e_3e_4 0      0         0        |
           |0      0      0         0         e_1e_2 e_1e_3e_4 e_2e_3e_4|
o10 : E-module, submodule of E^2
i11 : isStronglyStableModule Mss
o11 = true
i12 : Mss==stronglyStableModule M
o12 = true
```

The module Mss is not an almost lex submodule of F: the ideal $(e_1e_3e_4, e_2e_3e_4)$ is not lex. One can verify it and compute the almost lex submodule associated to Mss.

```
i13 : isLexIdeal (getIdeals Mss)_1
o13 = false
i14 : isAlmostLexModule Mss
o14 = false
i15 : Al=almostLexModule Mss
o15 = image|e_1e_2 e_1e_3 e_1e_4e_5 e_2e_3e_4
           |0      0      0         0
                          0      0         0         0           |
                          e_1e_2 e_1e_3e_4 e_1e_3e_5 e_2e_3e_4e_5|
o15 : E-module, submodule of E^2
```

Finally, we compute the Betti tables of the almost lex submodule and the lex submodules associated to a monomial submodule M.

Example 3.5.3 Let M be a monomial submodule of a graded free module F, we illustrate some functions from "ExteriorModules" package to compare the Betti diagrams of M, M^{alex} and M^{lex}.

```
Macaulay2, version 1.10
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : loadPackage "ExteriorModules";
i2 : E=QQ[e_1..e_5,SkewCommutative=>true];
i3 : F=E^{2,1,-1};
i4 : I_1=ideal({e_1*e_4,e_3*e_4*e_5});
i5 : I_2=ideal({e_1*e_4*e_5,e_2*e_3*e_4});
i6 : I_3=ideal({e_1*e_3*e_5});
i7 : M=createModule({I_1, I_2, I_3},F)
o7 = image {-2}|e_1e_4 e_3e_4e_5 0         0         0        |
           {-1}|0      0         e_1e_4e_5 e_2e_3e_4 0        |
           {1} |0      0         0         0         e_1e_3e_5|
o7 : E-module, submodule of E^3
i8 : isAlmostLexModule M
o8 = false
i9 : Malex=almostLexModule M
o9 = image {-2}|e_1e_2 e_1e_3e_4 0         0         0            0        |
           {-1}|0      0         e_1e_2e_3 e_1e_2e_4 e_1e_3e_4e_5 0        |
           {1} |0      0         0         0         0            e_1e_2e_3|
o9 : E-module, submodule of E^3
```

```
i10 : isLexModule Malex
o10 = false
i11 : Mlex=lexModule M
o11 = image {-2}|e_1e_2 e_1e_3e_4 e_2e_3e_4e_5
            {-1}|0      0         0
            {1} |0      0         0
                        {-2} 0         0            0            0       |
                        {-1} e_1e_2e_3 e_1e_2e_4e_5 e_1e_3e_4e_5 0       |
                        {1}  0         0            0            e_1e_2e_3|
o11 : E-module, submodule of E^3
```

Now, we can compare the Betti diagrams of the submodules we have found.

```
i12 : minimalBettiNumbers M
             0  1  2  3   4   5   6
o12 = total: 5 16 36 69 120 195 301
          0: 1  2  3  4   5   6   7
          1: 1  4 10 20  35  56  84
          2: 2  6 12 20  30  42  56
          3: .  1  5 15  35  70 126
          4: 1  3  6 10  15  21  28
o12 = BettiTally
i13 : minimalBettiNumbers Malex
             0  1  2  3   4   5   6
o13 = total: 6 21 50 99 175 286 441
          0: 1  2  3  4   5   6   7
          1: 1  4 10 20  35  56  84
          2: 2  7 16 30  50  77 112
          3: 1  5 15 35  70 126 210
          4: 1  3  6 10  15  21  28
o13 = BettiTally
i14 : minimalBettiNumbers Mlex
             0  1  2   3   4   5   6
o14 = total: 7 27 70 149 280 482 777
          0: 1  2  3   4   5   6   7
          1: 1  4 10  20  35  56  84
          2: 2  8 21  45  85 147 238
          3: 2 10 30  70 140 252 420
          4: 1  3  6  10  15  21  28
o14 = BettiTally
```

Chapter 4

Extremal Betti numbers

Let K be a field and let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K of characteristic 0. We analyze the behavior of the extremal Betti numbers of special classes of monomial ideals of S known as the t–spread strongly stable ideals, where t is an integer ≥ 0.
We focus our attention on the cases $t = 0, 1, 2$.

4.1 A hierarchy of monomial ideals

Let us consider the polynomial ring $S = K[x_1, \ldots, x_n]$ as an $\mathbb{N}$-graded ring where $\deg x_i = 1$, $i = 1, \ldots, n$. Some definitions we will use in this chapter have been given in Chapter 1 (Section 1.2).

A monomial $x_{i_1}x_{i_2}\cdots x_{i_d} \in S$ is *squarefree* if $1 \leq i_1 < i_2 < \cdots < i_d \leq n$. A graded ideal I of S is a *squarefree monomial ideal* if I is generated by squarefree monomials.

Definition 4.1.1 Let I be a squarefree monomial ideal of S. I is called a *squarefree stable ideal* if for all $u \in G(I)$ one has $(x_j u)/x_{\max(u)} \in I$ for all $j < \max(u), j \notin \operatorname{supp}(u)$.
I is called a *squarefree strongly stable ideal* if for all $u \in G(I)$ one has $(x_j u)/x_i \in I$ for all $i \in \operatorname{supp}(u)$ and all $j < i$, $j \notin \operatorname{supp}(u)$.

In [EHQ19], the notion of a t–spread monomial ideal has been introduced.

Let $t \geq 0$ be an integer. A monomial $x_{i_1}x_{i_2}\cdots x_{i_d}$ with $1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq n$ is called t–spread, if $i_j - i_{j-1} \geq t$ for $2 \leq j \leq d$. Note that, any monomial is 0–spread, while the squarefree monomials are 1–spread.

For example, the monomial $x_2x_5x_8 \in K[x_1, \ldots, x_8]$ is 3–spread.

Definition 4.1.2 A monomial ideal in S is called a t–spread monomial ideal, if it is generated by t–spread monomials.

It is clear that if $t \geq 1$, then every t–spread monomial is a squarefree monomial ideal.

Definition 4.1.3 A t–spread monomial ideal I of S is called t–spread stable, if for all t–spread monomials $u \in I$ and for all $i < \max(u)$ such that $x_i(u/x_{\max(u)})$ is a t–spread monomial, it follows that $x_i(u/x_{max(u)}) \in I$.

The ideal I is called t–spread strongly stable, if for all t–spread monomials $u \in I$, all $j \in \operatorname{supp}(u)$ and all $i < j$ such that $x_i(u/x_j)$ is t–spread, it follows that $x_i(u/x_j) \in I$.

Every t–spread strongly stable ideal is also t–spread stable.

One can notice that the notion of t–spread (strongly) stable ideal generalizes the notion of (strongly) stable ideal and of squarefree (strongly) stable ideal.

Remark 4.1.4 The defining property of a t–spread strongly stable ideal needs to be checked only for the set of monomial generators. Indeed, if I is a t–spread monomial ideal of S, then I is t–spread strongly stable if and only if the ideal I satisfies the following condition: for $u \in G(I)$ and $j \in \operatorname{supp}(u)$, if $i < j$ and $x_i(u/x_j)$ is a t–spread monomial, then $x_i(u/x_j) \in I$ [EHQ19, Lemma 1.2].

Let $u_1, \dots, u_m$ be t–spread monomials in S. The unique t–spread strongly stable ideal containing $u_1, \dots, u_m$ will be denoted by $B_t(u_1, \dots, u_m)$ [EHQ19]. The monomials $u_1, \dots, u_m$ are called t– spread Borel generators.

In the sequel, we refer to $B_t(u_1, \dots, u_m)$ as the finitely generated t–spread Borel ideal. If $t = 0$, we set $B_0(u_1, \dots, u_m) = \langle u_1, \dots, u_m \rangle$. We will call such ideals *finitely generated Borel ideals* (FGBI, for short). The ideal $B_0(u_1) = \langle u_1 \rangle$ is called a *principal Borel ideal* (PBI, for short).

Example 4.1.5 Let $S = K[x_1, \dots, x_8]$ and let us consider the set $P = \{x_1x_8, x_2x_6x_8\}$. We want to compute some finitely generated t-Borel ideals with the monomials in P as Borel generators.

$$\begin{aligned} B_0(x_1x_8, x_2x_6x_8) = (&x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_1x_8, x_2^3, x_2^2x_3, x_2^2x_4, x_2^2x_5,\\ &x_2^2x_6, x_2^2x_7, x_2^2x_8, x_2x_3^2, x_2x_3x_4, x_2x_3x_5, x_2x_3x_6, x_2x_3x_7, x_2x_3x_8,\\ &x_2x_4^2, x_2x_4x_5, x_2x_4x_6, x_2x_4x_7, x_2x_4x_8, x_2x_5^2, x_2x_5x_6, x_2x_5x_7,\\ &x_2x_5x_8, x_2x_6^2, x_2x_6x_7, x_2x_6x_8); \end{aligned}$$

$$\begin{aligned} B_1(x_1x_8, x_2x_6x_8) = (&x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_1x_8, x_2x_3x_4, x_2x_3x_5, x_2x_3x_6,\\ &x_2x_3x_7, x_2x_3x_8, x_2x_4x_5, x_2x_4x_6, x_2x_4x_7, x_2x_4x_8, x_2x_5x_6, x_2x_5x_7,\\ &x_2x_5x_8, x_2x_6x_7, x_2x_6x_8); \end{aligned}$$

$$\begin{aligned} B_2(x_1x_8, x_2x_6x_8) = (&x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_1x_8, x_2x_4x_6, x_2x_4x_7, x_2x_4x_8, x_2x_5x_7,\\ &x_2x_5x_8, x_2x_6x_8). \end{aligned}$$

For an arbitrary monomial ideal I, we denote by I_j, the j–th graded component of I and call the set of t–spread monomials in I_j, the t–spread part of I_j and denote it by $[I_j]_t$.

Now, let $M_{n,d,t}$ be the set of all t–spread monomials of degree d in S and let N be a non–empty subset of $M_{n,d,t}$. If T is a subset of S, we denote by $\text{Mon}_d(T) = M_{n,d,0} \cap T$ the set of all monomials in T and by $\text{Mon}_d^s(T) = M_{n,d,1} \cap T$ the set of all squarefree monomials of degree d in T.

Let us define the following set:

$$\text{Shad}_t(N) = \{x_i w : w \in N, i = 1, \ldots, n\} \cap M_{n,d+1,t}.$$

It is clear that $\text{Shad}_t(N)$ could be empty. The set $\text{Shad}_t(N) \neq \emptyset$ will be called the t–shadow of N.

A special class of t–spread strongly stable ideals consists of t-spread lex ideals, which are defined as follows [AC19f].

Definition 4.1.6 (a) A subset L of $M_{n,d,t}$ is called a t–spread lex set, if for all $u \in L$ and for all $v \in M_{n,d,t}$ with $v >_{\text{lex}} u$, it follows that $v \in L$.
(b) Let I be a t–spread monomial ideal. Then I is called a t–spread lex ideal, if $[I_j]_t$ is a t–spread lex set for all j.

Example 4.1.7 Let $S = K[x_1, x_2, x_3, x_4, x_5]$. The ideal $I = (x_1x_2x_3,\ x_1x_2x_4,\ x_1x_2x_5,\ x_1x_3x_4,\ x_2x_3x_4x_5)$ is a squarefree lexsegment ideal of S.

We recall the next definition from [AC19e].

Definition 4.1.8 Let $u = x_{i_1} \cdots x_{i_q}$ be a squarefree monomial of S of degree $q < n$. We say that u has a j-gap if $i_{j+1} - i_j > 1$ for some $1 \leq j < q$. The positive integer $i_{j+1} - i_j - 1$ will be called the width of the j-gap.

The j-gap of a squarefree monomial $u = x_{i_1} \cdots x_{i_q} \in S$ will be denoted by $j\text{-gap}(u)$, whereas its width will be denoted by $\text{wd}(j\text{-gap}(u))$. Moreover, we define

$$\text{Gap}(u) := \{j \in [q] : \text{there exists a } j\text{-gap}(u)\}.$$

One can observe that for $t \geq 1$, $\text{Shad}_t(N) \neq \emptyset$ if there exists a squarefree monomial $u = x_{i_1} \cdots x_{i_d} \in N$ satisfying at least one of the following conditions:

(i) $i_1 > t$;

(ii) $\text{wd}(j\text{-gap}(u)) \geq 2t$, for $1 \leq j < d$;

(iii) $i_d \leq n - t$.

For instance, if $x_3x_5x_9 \in M_{9,3,2}$, then $\mathrm{Shad}_2(\{x_3x_5x_9\}) = \{x_1x_3x_5x_9, x_3x_5x_7x_9\} \subset M_{9,4,2}$; if $x_4x_9 \in M_{12,2,3}$, then $\mathrm{Shad}_3(\{x_4x_9\}) = \{x_1x_4x_9, x_4x_9x_{12}\} \subset M_{12,3,3}$; whereas, if $x_3x_6x_9 \in M_{10,3,3}$, one has $\mathrm{Shad}_3(\{x_3x_6x_9\}) = \emptyset$.

Now, we study the extremal Betti numbers of t–spread strongly stable ideals.

For any graded ideal I of S, there is a minimal graded free S-resolution [BH96]

$$F_\bullet : 0 \to F_s \to \cdots \to F_1 \to F_0 \to I \to 0,$$

where $F_i = \oplus_{j\in\mathbb{Z}} S(-j)^{\beta_{i,j}}$. The integers $\beta_{i,j} = \beta_{i,j}(I) = \dim_K \mathrm{Tor}_i(K, I)_j$ are called the *graded Betti numbers* of I.

If I is a t–spread strongly stable ideal, there exists a formula to compute the graded Betti numbers of I [EHQ19, Corollary 1.12]:

$$\beta_{k,\,k+\ell}(I) = \sum_{u\in G(I)_\ell} \binom{\max(u) - t(\ell-1) - 1}{k}. \tag{4.1.1}$$

Note that the formula in (4.1.1) becomes the well–known formula of Eliahou and Kervaire [EK90] for (strongly) stable ideals for $t = 0$; whereas, if $t = 1$ then it coincides with the formula stated by Aramova, Herzog and Hibi for the class of squarefree (strongly) stable ideals [AHH98].

Definition 4.1.9 *[BCP99]* A graded Betti number $\beta_{k,k+\ell}(I) \neq 0$ is called *extremal* if $\beta_{i,\,i+j}(I) = 0$ for all $i \geq k$, $j \geq \ell$, $(i,j) \neq (k,\ell)$.

The pair (k,ℓ) is called a *corner* of I.

Next results, that lead to a characterization of the extremal Betti numbers of a t–spread strongly stable ideal, are quite similar to the ones in [CU00, CU03]. We include them in this section for completeness of information.

Lemma 4.1.10 *Let I be a t–spread strongly stable ideal of S. If $\beta_{i,i+j}(I) \neq 0$, then $\beta_{k,k+j}(I) \neq 0$ for $k = 0,\dots,i$.*

Proof. If $\beta_{i,i+j}(I) \neq 0$, by (4.1.1) there exists $u \in G(I)_j$ such that $\max(u) - t(j-1) - 1 \geq i$, *i.e.*, $\max(u) \geq i + t(j-1) + 1$. It follows that $\max(u) \geq k + t(j-1) + 1$, for $k = 0,\dots,i$, and again from (4.1.1), the assertion follows. □

From Definition 4.1.9, it follows:

Corollary 4.1.11 *Let I be a t–spread strongly stable ideal. The following conditions are equivalent:*

(a) $\beta_{k,k+\ell}(I)$ *is extremal;*

(b) (b.1) $\beta_{k,k+\ell}(I) \neq 0$*;*

(b.2) $\beta_{k,k+j}(I) = 0$, *for* $j > \ell$;

(b.3) $\beta_{i,i+\ell}(I) = 0$, *for* $i > k$.

Lemma 4.1.10 and Corollary 4.1.11 yield the following characterization.

Theorem 4.1.12 *Let I be a t–spread strongly stable ideal of S.*

The following conditions are equivalent:

(1) $\beta_{k,\,k+\ell}(I)$ *is extremal;*

(2) $k + t(\ell - 1) + 1 = \max\{\max(u) : u \in G(I)_\ell\}$ *and* $\max(u) < k + t(j - 1) + 1$, *for all* $j > \ell$ *and for all* $u \in G(I)_j$.

Proof. (1)⇒ (2). By (4.1.1) $\beta_{k,\,k+\ell}(I) \neq 0$ if and only if there exists a monomial $u \in G(I)_\ell$ such that $\max(u) \geq k + t(\ell - 1) + 1$. Hence $\max\{\max(u) : u \in G(I)_\ell\} \geq k + t(\ell - 1) + 1$.

Suppose $j + t(\ell - 1) + 1 := \max\{\max(u) : u \in G(I)_\ell\} > k + t(\ell - 1) + 1$. Hence $\beta_{j,j+\ell}(I) \neq 0$, for $j > k$. This is a contradiction from Corollary 4.1.11, (b.3). Hence

$$k + t(\ell - 1) + 1 = \max\{\max(u) : u \in G(I)_\ell\}.$$

Suppose there exist an integer $j > \ell$ and a monomial $u \in G(I)_j$ such that $\max(u) \geq k+t(j-1)+1$. From (4.1.1), then $\beta_{k,\,k+j}(I) \neq 0$. Again a contradiction from Corollary 4.1.11, (b.2).

(2)⇒ (1). Since $k + t(\ell - 1) + 1 = \max\{\max(u) : u \in G(I)_\ell\}$, then $\beta_{k,\,k+\ell}(I) \neq 0$ and $\beta_{i,\,i+\ell}(I) = 0$, for all $i > k$. On the other hand $\max(u) < k + t(j - 1) + 1$, for all $j > \ell$ and for all $u \in G(I)_j$, implies $\beta_{k,\,k+j}(I) = 0$. Hence from Corollary 4.1.11, we get the assertion. □

As a consequence we obtain the following:

Corollary 4.1.13 *Let I be a t–spread strongly stable ideal of S and let $\beta_{k,\,k+\ell}(I)$ an extremal Betti number of I. Then*

$$\beta_{k,\,k+\ell}(I) = |\{u \in G(I)_\ell : \max(u) = k + t(\ell - 1) + 1\}|.$$

Now, let $t \geq 1$ and let $M_{n,\ell,t}$ be the set of all t–spread monomials of degree ℓ in S. From [AC19f] (see also [EHQ19, Theorem 2.3]), one has

$$|M_{n,\ell,t}| = \binom{n - (\ell - 1)(t - 1)}{\ell}. \tag{4.1.2}$$

Hence, if (k, ℓ) is a pair of positive integers such that $k + t(\ell - 1) + 1 \leq n$, one has

$$|\{u \in M_{n,\ell,t} : \max(u) = k + t(\ell - 1) + 1\}| = \binom{k + t(\ell - 1) + 1 - (\ell - 1)(t - 1) - 1}{\ell - 1} \tag{4.1.3}$$

$$= \binom{k + \ell - 1}{\ell - 1}. \tag{4.1.4}$$

As a consequence, if I is a t–spread strongly stable ideal of S and $\beta_{k,\,k+\ell}(I)$ is an extremal Betti number of I, then from Theorem 4.1.12, we have the following bound:

$$1 \leq \beta_{k,k+\ell}(I) \leq \binom{k+\ell-1}{\ell-1}. \tag{4.1.5}$$

4.2 Algorithms for a FGBI

In this Section, we examine the following problem.

Problem 4.2.1 Given two positive integers n, r, $1 \leq r \leq n-1$, r pairs of positive integers (k_1, ℓ_1), …, (k_r, ℓ_r) such that $n-1 \geq k_1 > k_2 > \cdots > k_r \geq 1$, $1 \leq \ell_1 < \ell_2 < \cdots < \ell_r$, and r positive integers $a_1, \ldots, a_r$, under which conditions does there exist a graded ideal I of S such that $\beta_{k_1,k_1+\ell_1}(I) = a_1$, …, $\beta_{k_r,k_r+\ell_r}(I) = a_r$ are its extremal Betti numbers?

An answer to Problem 4.2.1 can be found in [CU00, CU03, Cru16, Cru17, Cru19, HSV14]. In [HSV14] a crucial role is played by the class of piecewise lexsegment ideals. The notion of piecewise lexsegment ideal has been introduced by Shakin in [Sha03] (see also [Mur07]). A monomial ideal $I \subseteq S$ is said to be *piecewise lexsegment* if for any monomial $u \in G(I)$ of degree d and for any monomial $v \in S$ of degree d such that $v > u$ and $\mathrm{m}(v) \leq \mathrm{m}(u)$ we have that $v \in I$, or equivalently, if there exists lexsegment ideals L_i of $K[x_1, \ldots, x_i]$ $(i = 1, \ldots, n)$ such that $L = L_1S + \ldots + L_nS$ [Sha03, Proposition 2.4].

We realize some procedures to construct FGBI's of initial degree ≥ 2 with given extremal Betti numbers (positions as well as values).

From now on, we assume $S = K[x_1, \ldots, x_n]$ endowed with the lexicographic order $>_{\mathrm{lex}}$ induced by the ordering $x_1 > x_2 > \cdots > x_n$.

For the reader's convenience, we recall some notations from [Cru16].

For $u, v \in \mathrm{Mon}_d(S)$, $u \geq_{\mathrm{lex}} v$, define the set

$$\mathcal{L}(u, v) = \{z \in \mathrm{Mon}_d(S) : u \geq_{\mathrm{lex}} z \geq_{\mathrm{lex}} v\}.$$

Let $\mathcal{M}$ be a set of monomials of degree d of S. The set of monomials of degree $d+1$ of S

$$\mathrm{Shad}(\mathcal{M}) = \{x_i u : u \in \mathcal{M}, \quad i = 1, \ldots, n\}$$

is called the *shadow* of $\mathcal{M}$. We define the i-th *shadow* recursively by $\mathrm{Shad}^0(\mathcal{M}) = \mathcal{M}$, $\mathrm{Shad}^i(\mathcal{M}) = \mathrm{Shad}(\mathrm{Shad}^{i-1}(\mathcal{M}))$.

Moreover, we denote by $\min(\mathcal{M})$ ($\max(\mathcal{M})$, respectively) the smallest (the greatest, respectively) monomial of $\mathcal{M}$ with respect to $\geq_{\mathrm{lex}}$. Setting $w = \min(\mathcal{M})$, if $\ell > d$ is an integer, we define the following set of monomials of degree ℓ in S:

$$\mathrm{LexShad}^{\ell-d}(\mathcal{M}) = \mathcal{L}(x_1^{\ell}, w x_n^{\ell-d}).$$

We call such a set the *lexicographic shadow* of $\mathcal{M}$.

Finally, given two positive integers k, d, with $1 \leq k < n$ and $d \geq 2$, we consider the following sets of monomials:

$$A(k, d) = \{u \in \mathrm{Mon}_d(S) : \mathrm{m}(u) = k + 1\}, \tag{4.2.1}$$

and

$$A(\leq k, d) = \{u \in \mathrm{Mon}_d(S) : \mathrm{m}(u) \leq k + 1\}. \tag{4.2.2}$$

Setting $A(k, d) = \{u_1, \ldots, u_q\}$, we can suppose, after a permutation of the indices, that

$$u_1 >_{\mathrm{lex}} u_2 >_{\mathrm{lex}} \cdots >_{\mathrm{lex}} u_q. \tag{4.2.3}$$

For the i-th monomial u of degree d with $\mathrm{m}(u) = k+1$, we mean the monomial of $A(k, d)$ that appears in the i-th position of (4.2.3), for $1 \leq i \leq q$. Note that $u_1 = x_1^{d-1} x_{k+1}$, $u_q = x_{k+1}^d$, and $q = |A(k, d)| = \binom{k+d-1}{d-1}$.

Furthermore, if u_i, u_j, $i < j$, are two monomials in (4.2.3), we will denote by $[u_i, u_j]$ the subset of $A(k, d)$ defined as follows:

$$[u_i, u_j] = \{w \in A(k, d) : u_i \geq_{\mathrm{lex}} w \geq_{\mathrm{lex}} u_j\};$$

$[u_i, u_j]$ will be called a *segment of $A(k, d)$ of initial element u_i and final element u_j*, and its cardinality will be called the length of $[u_i, u_j]$. Note that $A(k, d) = [x_1^{d-1} x_{k+1}, x_{k+1}^d]$. If $i = j$, we set $[u_i, u_j] = \{u_i\}$.

The sets in (4.2.1) and (4.2.2) are the first objects involved in the determination of the ideal we are looking for.

In Algorithm 4.1 we give the pseudocode of the procedure for computing such sets of monomials.

Algorithm 4.1: Computation of $A(k,d)$ or $A(\leq k,d)$

```
Input: Ring R, string s, index k and degree l
Output: Set of monomials: A(k,l)
begin
    n ← number of indeterminates of the ring R;
    if k + 1 ≤ n then
        M ← monomials of degree l ;
        foreach i ∈ {k + 2 . . n} do                    // A(≤ k,d) computation
            M ← list of monomials of M not divisible by x_i ;
        end
        if s = " = " then                                 // A(k,d) computation
            M ← list of monomials of M divisible by x_{k+1} ;
        end
    end
    return M;
end
```

Now, for our purpose, we give a reformulation of the numerical characterization of all possible extremal Betti numbers of any graded strongly stable ideal of S [Cru16, Proposition 3.4, Theorem 3.7].

For every subset $\mathcal{M} \subseteq \mathrm{Mon}_d(S)$, $d \geq 1$, and for every monomial $u \in \mathcal{M}$, we introduce the following set of monomials:

$$\mathcal{M}_{,u} = \{v \in \mathcal{M} : v \geq_{\mathrm{lex}} u\};$$

if $\mathcal{M} = \{u\}$, then $\mathcal{M}_{,u} = \mathcal{M}$.

Moreover, if $(k_1, \ell_1), \ldots, (k_r, \ell_r)$, with $n-1 \geq k_1 > k_2 > \cdots > k_r \geq 1$ and $1 \leq \ell_1 < \ell_2 < \cdots < \ell_r$, are the corners of a graded ideal I, according to [Cru16], the following notions can be introduced:

$$\mathrm{Corn}(I) = \{(k_1, \ell_1), \ldots, (k_r, \ell_r)\}, \ \ a(I) = (\beta_{k_1,k_1+\ell_1}(I), \ldots, \beta_{k_r,k_r+\ell_r}(I)).$$

$\mathrm{Corn}(I)$ is called the *corner sequence* of I, and $a(I)$ the *corner values sequence* of I.

The next definition, introduced in [Cru16], was motivated by the fact that, for every graded ideal I of S, $\mathrm{Corn}(I)$ defines a corner sequence.

Definition 4.2.2 *[Cru16, Definition 4.1]* Let $(k_1, \ldots, k_r)$ and $(\ell_1, \ldots, \ell_r)$ be two sequences of positive integers such that $n-1 \geq k_1 > k_2 > \cdots > k_r \geq 1$ and $1 \leq \ell_1 < \ell_2 < \cdots < \ell_r$. The set $\mathcal{C} = \{(k_1, \ell_1), \ldots, (k_r, \ell_r)\}$ is called a corner sequence and $\ell_1, \ldots, \ell_r$ are called the *corner degrees* of $\mathcal{C}$.

Definition 4.2.3 *[Cru16, Definition 4.3]* A totally ordered corner sequence $\mathcal{C} = \{(k_1, \ell_1), \ldots, (k_r, \ell_r)\}$ is a corner sequence such that $(k_1, \ell_1) \succ (k_2, \ell_2) \succ \cdots \succ (k_r, \ell_r)$ where

$$(k_i, \ell_i) \succ (k_j, \ell_j) \quad \text{if and only if} \quad k_i \geq k_j \quad \text{and} \quad \ell_i \leq \ell_j;$$

we refer to (k_i, ℓ_i) as the i-th element of the ordered corner sequence.

From now on, when we refer to a corner sequence $\mathcal{C}$, we assume that $\mathcal{C}$ is totally ordered with respect to $\succ$.

Example 4.2.4 Let $S = K[x_1, x_2, x_3, x_4, x_5, x_6]$ and let

$$I = (x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2x_3x_4, x_2x_3x_5, x_2x_3x_6, x_2x_4x_5, x_2x_4x_6, x_3x_4x_5x_6)$$

be a squarefree strongly stable ideal of S. The extremal Betti numbers of I are $\beta_{3,6}(I) = 2$, $\beta_{2,6}(I) = 1$, as the Betti table of I shows:

		0	1	2	3
2	:	4	6	4	1
3	:	5	11	8	2
4	:	1	2	1	−

Hence, the corner sequence and the corner values sequence of I are

$$\mathrm{Corn}(I) = \{(3, 3), (2, 4)\}, \text{ and } a(I) = (2, 1).$$

Proposition 4.2.5 *Let $n \geq 4$ be an integer. Let $(k_1, \ell_1), (k_2, \ell_2)$ be two pairs of positive integers such that $n - 1 \geq k_1 > k_2 \geq 2$ and $2 = \ell_1 < \ell_2$, and let a_1, a_2 be two positive integers. If K is a field of characteristic 0, then the following conditions are equivalent:*

(1) *there exists a graded ideal $J \subsetneq S$, with extremal Betti numbers $\beta_{k_1,k_1+\ell_1}(J) = a_1$ and $\beta_{k_2,k_2+\ell_2}(J) = a_2$;*

(2) *there exists a strongly stable ideal $I \subsetneq S$, with extremal Betti numbers $\beta_{k_1,k_1+\ell_1}(I) = a_1$ and $\beta_{k_2,k_2+\ell_2}(I) = a_2$;*

(3) *the integers a_i satisfy the conditions:*

$$1 \leq a_i \leq |A_i \setminus \mathrm{LexShad}^{\ell_i - \ell_{i-1}}(A_{i-1})|, \quad \textit{for} \;\; i = 1, 2, \tag{4.2.4}$$

where $A_0 = \emptyset$,

(i) $A_1 = \{u \in A(k_1, \ell_1 = 2) : u \geq_{\mathrm{lex}} x_{k_2}x_{k_1+1}\}$;

(ii) $A_2 = \{u \in A(k_2, \ell_2) : u \geq_{\mathrm{lex}} x_{k_2+1}^{\ell_2}\}$.

and if $a_1 = |[u,v]|$, *with* $u, v \in A_1$, *then* $1 \leq a_2 \leq |A_2 \setminus \mathrm{LexShad}^{\ell_1-2}([u,v])|$.

Proof. (1) $\Leftrightarrow$ (2). [Cru16, Proposition 3.4].
(2) $\Rightarrow$ (3). The inequalities in (4.2.4) satisfied by the integers a_1, a_2 are proved in [Cru16, Proposition 3.4]. Note that $A_1 = [x_1x_{k_1+1}, x_{k_2}x_{k_1+1}]$ and $A_2 = [x_1^{\ell_2-1}x_{k_2+1}, x_{k_2+1}^{\ell_2}]$.
From Characterization 4.1.12, a_1 is the number of all monomials $z \in A(k_1, \ell_1)$ which determine the corner (k_1, ℓ_1). Furthermore, by (4.1.13), such monomials form a segment $[u, v]$ of $A(k_1, \ell_1)$ of length a_1; $u = x_1x_{k_1+1}$, and v is the a_1-th monomial of A_1 with respect to $\geq_{\mathrm{lex}}$.

On the other hand, the existence of the extremal Betti number $\beta_{k_2,k_2+\ell_2}(I) = a_2$, ensures that there exist a_2 monomials of A_2 not belonging to $\mathrm{LexShad}^{\ell_2-2}([u,v])|$. Hence, $1 \leq a_2 \leq |A_2 \setminus \mathrm{LexShad}^{\ell_2-2}([u,v])|$.
(3) $\Rightarrow$ (2). A natural construction of a FGBI I of S with $\mathrm{Corn}(I) = \{(k_1\ell_1), (k_2, \ell_2)\}$ and $a(I) = (a_1, a_2)$ proceeds as follows.
Let $m(a_1)$ be the a_1-th monomial of A_1, and let $m(a_2)$ be the a_2-th monomial of $A_2 \setminus \mathrm{LexShad}^{\ell_2-2}(A_{1,m(a_1)})$. Note that $A_{1,m(a_1)} = [x_1x_{k_1+1}, m(a_1)]$ and $a_1 = |A_{1,m(a_1)}|$.

We construct a strongly stable ideal I of S in degrees $\ell_1 = 2 < \ell_2$, using the following *criterion*:

- $G(I)_2 = \{v \in A(\leq k_1, 2) : x_1^2 \geq_{\mathrm{lex}} v \geq_{\mathrm{lex}} m(a_1)\}$,
- $G(I)_{\ell_2} = \{z \in A(\leq k_2, \ell_2) : m(a_1)x_n^{\ell_2-2} >_{\mathrm{lex}} z \geq_{\mathrm{lex}} m(a_2)\}$, where $m(a_1)x_n^{\ell_2-2}$ is the smallest monomial belonging to $\mathrm{Shad}^{\ell_2-2}(G(I)_2)$.

I is the FGBI we are looking for. $\square$

Theorem 4.2.6 *Given two positive integers* n, r *such that* $1 \leq r \leq n-1$, *let* $(k_1, \ell_1), (k_2, \ell_2), \ldots, (k_r, \ell_r)$ *be* r *pairs of positive integers with* $n-1 \geq k_1 > k_2 > \cdots > k_r \geq 1$ *and* $2 \leq \ell_1 < \ell_2 < \cdots < \ell_r$, *and let* $a_1, a_2, \ldots, a_r$ *be* r *positive integers.*

If K *is a field of characteristic* 0, *then the following conditions are equivalent:*

(1) *there exists a graded ideal* $J \subsetneq S$, *with extremal Betti numbers* $\beta_{k_i,k_i+\ell_i}(J) = a_i$, *for* $i = 1, \ldots, r$;

(2) *there exists a strongly stable ideal* $I \subsetneq S$, *with extremal Betti numbers* $\beta_{k_i,k_i+\ell_i}(I) = a_i$, *for* $i = 1, \ldots, r$;

(3) *set* $t = \max\{i : \ell_i \leq r - i\}$. *The integers* a_i *satisfy the conditions:*

$$1 \leq a_i \leq |A_i \setminus \mathrm{LexShad}^{\ell_i-\ell_{i-1}}(A_{i-1})|, \quad \textit{for} \;\; i = 1, \ldots, r, \tag{4.2.5}$$

where $A_0 = \emptyset$,

(i) $A_1 = \{u \in A(k_1, \ell_1) : u \geq_{\mathrm{lex}} x_{k_r-1}x_{k_1+1}\}$, *whenever* $\ell_1 = 2$;

(ii) $A_i = \{u \in A(k_i, \ell_i) : u \geq_{\text{lex}} x_{k_r} x_{k_{r-1}} \cdots x_{k_{r-\ell_i+3}} x_{k_{r-\ell_i+2}-1} x_{k_i+1}\}$, *for* $i = 1, \ldots, t$, *whenever* $\ell_1 \geq 3$, *and for* $i = 2, \ldots, t$, *whenever* $\ell_1 = 2$;

(iii) $A_i = \{u \in A(k_i, \ell_i) : u \geq_{\text{lex}} x_{k_r} x_{k_{r-1}} \cdots x_{k_{i+1}} x_{k_i+1}^{\ell_i-(r-i)}\}$, *for* $i = t+1, \ldots, r-1$;

(iv) $A_r = \{u \in A(k_r, \ell_r) : u \geq_{\text{lex}} x_{k_r+1}^{\ell_r}\}$,

and if $a_i = |[u, v]|$, *with* $u, v \in A_i$, *then* $1 \leq a_{i+1} \leq |A_{i+1} \setminus \text{LexShad}^{\ell_{i+1}-\ell_i}([u, v])|$, *for all* $i = 1, \ldots r-1$,

with $2 < r \leq n-2$ *(it has to be* $n \geq 5$*),* $k_r \geq 2$, *whenever* $\ell_1 = 2$, *and* $1 \leq r \leq n-1$, $k_r \geq 1$, *whenever* $\ell_1 \geq 3$.

Proof. (1) $\Leftrightarrow$ (2). [Cru16, Proposition 3.5], [CU03, Theorem 3.1].
(2) $\Rightarrow$ (3). The proof is quite similar to Proposition 4.2.5. The inequalities in (4.2.5) satisfied by the integers a_i $(1 \leq i \leq r)$ are proved in [Cru16, Proposition 3.5] and [CU03, Theorem 3.1] (see also, [Cru16, Theorem 3.7]).
From Characterization 4.1.12, a_i is the number of all monomials $z \in A(k_i, \ell_i)$ which determine the corner (k_i, ℓ_i) $(1 \leq i \leq r)$. Such monomials form a segment $[u, v]$ of $A(k_i, \ell_i)$ of length a_i, with $u, v \in A_i$. More precisely, u is the greatest monomial of A_i not belonging to $\text{Shad}^{\ell_i-\ell_{i-1}}(\text{Mon}(I_{\ell_{i-1}})$ $(i = 1, \ldots, r)$, where $\text{Mon}(I_{\ell_{i-1}})$ is the set of the monomials of degree ℓ_{i-1} belonging to $I_{\ell_{i-1}}$, whereas v is the a_i-th monomial of the segment $[u, \max(A_i)]$, with respect to $\geq_{\text{lex}}$. On the other hand, the existence of the extremal Betti number $\beta_{k_{i+1}, k_{i+1}+\ell_{i+1}}(I) = a_{i+1}$ implies that there exist a_{i+1} monomials of A_{i+1} not belonging to $\text{LexShad}^{\ell_{i+1}-\ell_i}([u, v])|$. Hence, $1 \leq a_{i+1} \leq |A_{i+1} \setminus \text{LexShad}^{\ell_{i+1}-\ell_i}([u, v])|$.
(3) $\Rightarrow$ (2). Let $m(a_1)$ be the a_1-th monomial of A_1. Setting

- $\widetilde{A}_1 = A_1$, and
- $\widetilde{A}_2 = A_2 \setminus \text{LexShad}^{\ell_2-\ell_1}(\widetilde{A}_{1,m(a_1)}) = A_2 \setminus \text{LexShad}^{\ell_2-\ell_1}(A_{1,m(a_1)})$,

let $m(a_2)$ be the a_2-th monomial of $\widetilde{A}_2$. Note that $\widetilde{A}_{1,m(a_1)} = A_{1,m(a_1)} = [x_1^{\ell_1-1} x_{k_1+1}, m(a_1)]$. For $i \geq 3$, let us denote by $m(a_i)$ the a_i-th monomial of $\widetilde{A}_i = A_i \setminus \text{LexShad}^{\ell_i-\ell_{i-1}}(\widetilde{A}_{i-1,m(a_{i-1})})$.

We construct a FGBI I of S in degrees $2 \leq \ell_1 < \ell_2 < \cdots < \ell_r$ as follows:

- $G(I)_{\ell_1} = \{v \in A(\leq k_1, \ell_1) : x_1^{\ell_1} \geq_{\text{lex}} v \geq_{\text{lex}} m(a_1)\}$;
- $G(I)_{\ell_i} = \{z \in A(\leq k_i, \ell_i) : m(a_{i-1}) x_n^{\ell_i-\ell_{i-1}} >_{\text{lex}} z \geq_{\text{lex}} m(a_i)\}$, with $m(a_{i-1}) x_n^{\ell_i-\ell_{i-1}}$ the smallest monomial belonging to $\text{Shad}^{\ell_i-\ell_{i-1}}(\text{Mon}(I_{\ell_{i-1}}))$, for $i = 2, \ldots, r$, where $\text{Mon}(I_{\ell_{i-1}})$ is the set of the monomials of degree ℓ_{i-1} belonging to $I_{\ell_{i-1}}$.

One has that $\text{Corn}(I) = \{(k_i, \ell_i)\}_{i=1,\ldots,r}$, and $a(I) = (a_1, \ldots, a_r)$. $\square$

As we have underlined, a numerical characterization (different in nature from Proposition 4.2.5 and Theorem 4.2.6) of the possible extremal Betti numbers of a graded ideal in a polynomial ring over a field of characteristic zero has been given by Herzog, Sharifan and Varbaro in [HSV14]. In their paper, the authors have introduced the following notion.

Definition 4.2.7 *[HSV14, Definition 3.6]* Let $\mathbf{i} = (i_1, \ldots, i_k)$ and $\mathbf{j} = (j_1, \ldots, j_k)$ be such that $0 < i_1 < i_2 < \cdots < i_k < n$, $j_1 > j_2 > \cdots j_k > 0$. We say that a graded ideal I of S is a $(\mathbf{i}, \mathbf{j})$-lex ideal if $I = \sum_{p=1}^{k} L_p S$, where L_p is a lexsegment ideal generated in degree j_p in $K[x_1, \ldots, x_{i_p+1}]$.

Such an ideal is related to the FGBI defined in Theorem 4.2.6 (see also Proposition 4.2.5), as the following result shows.

Corollary 4.2.8 *Given two positive integers n, r such that $1 \leq r \leq n-1$, let $(k_1, \ell_1), (k_2, \ell_2), \ldots, (k_r, \ell_r)$ be r pairs of positive integers with $n-1 \geq k_1 > k_2 > \cdots > k_r \geq 1$ and $2 \leq \ell_1 < \ell_2 < \cdots < \ell_r$, and let $a_1, a_2, \ldots, a_r$ be r positive integers.*

Let I be a graded ideal of S, then the following conditions are equivalent:

(1) *I is the FGBI with extremal Betti numbers $\beta_{k_1, k_1+\ell_1}(I) = a_1$, ..., $\beta_{k_r, k_1+\ell_r}(I) = a_r$;*

(2) *I is the smallest $(\mathbf{k}, \boldsymbol{\ell})$-lex ideal with extremal Betti numbers $\beta_{k_1, k_1+\ell_1}(I) = a_1$, ..., $\beta_{k_r, k_1+\ell_r}(I) = a_r$, for $\mathbf{k} = (k_r, \ldots, k_1)$ and $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_r)$.*

Proof. (1) $\Rightarrow$ (2). Let I be the FGBI with extremal Betti numbers $\beta_{k_i, k_i+\ell_i}(I) = a_i$ $(i = 1, \ldots, r)$ constructed by the criterion in Theorem 4.2.6. One can note that I equals the $(\mathbf{k}, \boldsymbol{\ell})$-lex ideal $L_1S + L_2S + \cdots + L_rS$ defined as follows:

- $G(L_1) = G(I)_{\ell_1} = \mathcal{L}(x_1^{\ell_1}, m(a_1)) \cap K[x_1, \ldots, x_{k_1+1}]$;
- $G(L_i) = G(I)_{\ell_i} \cup \Big(\mathrm{Shad}^{\ell_i - \ell_{i-1}}(\mathrm{Mon}(I_{\ell_{i-1}})) \cap K[x_1, \ldots, x_{k_i+1}]\Big)$
  $= \mathcal{L}(x_1^{\ell_i}, m(a_i)) \cap K[x_1, \ldots, x_{k_i+1}]$, for $i = 2, \ldots, r$.

Its structure assures that Condition (2) is satisfied.
(2) $\Rightarrow$ (1). Let $I = L_1S + L_2S + \cdots + L_rS$ be the smallest $(\mathbf{k}, \boldsymbol{\ell})$-lex ideal with extremal Betti numbers $\beta_{k_i, k_i+\ell_i}(I) = a_i$ $(i = 1, \ldots, r)$, for $\mathbf{k} = (k_r, \ldots, k_1)$ and $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_r)$. With the same notations as in Theorem 4.2.6, since a piecewise lex ideal is strongly stable [Sha03, Proposition 3.5], one has that $\min L_i = m(a_i)$ $(i = 1, \ldots, r)$. Hence, if one minimises the set of generators of I, the assertion follows. $\square$

Algorithm 4.2 shows the pseudocode of the procedure returning the sets $\widetilde{A}_i$ $(1 \leq i \leq r)$ which arise in the construction of the strongly stable ideal we are interested to.

To simplify the notation, in the Algorithms 4.2 and 4.3, we set $LS_i = \mathrm{LexShad}^{\ell_i - \ell_{i-1}}(A_{i-1})$, and denote by $\widetilde{A}_i[a_i]$ the a_i-th monomial of $\widetilde{A}_i$ with respect to $\geq_{\mathrm{lex}}$, for $i = 1, \ldots, r$.

Algorithm 4.2: $\widetilde{A}_i$ Computation

```
Input: Ring R, Corner sequence {(k_i, ℓ_i)}_{i=1,...,r}, r–tuple (a_i)_{i=1,...,r}
Output: Sets of monomials Ã_i
begin
    n ← number of indeterminates of the ring R;
    t ← max ({i : ℓ_i ≤ r − i} ∪ {0});
    foreach i ∈ {1..r} do
        Ak_i l_i ← A(k_i, ℓ_i) ;                          // calling Algorithm 4.1
        if (i = 1) ∧ (ℓ_1 = 2) then
            m ← x_{k_r −1} x_{k_1 +1} ;
            if r = 2 then                                  // Proposition 4.2.5
                m ← m · x_{k_r}
            end
        end
        if ((i ∈ 1..t) ∧ (ℓ_1 ≥ 3)) ∨ ((i ∈ 2..t) ∧ (ℓ_1 = 2)) then
            m ← x_{k_r} ··· x_{k_{r−ℓ_i+3}} x_{k_{r−ℓ_i+2} −1} x_{k_i +1};
        end
        if (i ∈ t + 1..r − 1) then
            m ← x_{k_r} ··· x_{k_{i+1}} x_{k_{i+1}}^{ℓ_i −(r−i)};
        end
        if (i = r) then
            m ← x_{k_r +1}^{ℓ_r};
        end
        Al_i ← {u ∈ Ak_i l_i : u ≥ m} ;                   // A_i computation
        if (i = 1) then
            LS_i ← ∅ ;
        else
            mi ← Ã_{i−1}[a_{i−1}] · x_n^{ℓ_i −ℓ_{i−1}};
            LS_i ← {u ∈ Ak_i l_i : m ≤ u ≤ x_1^{ℓ_i}};
        end
        Ã_i ← Al_i \ LS_i ;                               // Ã_i computation
    end
    return list of Ã;
end
```

Finally, Algorithm 4.3 presents the pseudocode of the procedure giving the required FGBI (or, equivalently the $(\mathbf{k}, \boldsymbol{\ell})$-lex ideal, $\mathbf{k} = (k_r, \ldots, k_1)$, $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_r)$), solution of Problem 4.2.1.

Algorithm 4.3: FGBI Computation

Input: Ring R, Corner sequence $\{(k_i, \ell_i)\}_{i=1,\dots,r}$, r–tuple $(a_i)_{i=1,\dots,r}$

Output: FGBI I

begin

  $n \leftarrow$ number of indeterminates of the ring R;

  $\widetilde{A} \leftarrow$ list of $\widetilde{A}_i$; `// calling Algorithm 4.2`

  foreach $i \in \{1\,..\,r\}$ **do**

    $Cond \leftarrow Cond \vee (a_i < 1) \vee (a_i > |\widetilde{A}_i|)$;

  end

  $I \leftarrow (0)$;

  if $\overline{Cond}$ **then**

    $Alk_1l_1 \leftarrow A(\leq k_1, \ell_1)$; `// calling Algorithm 4.1`

    $G_1 \leftarrow \{u \in Alk_1l_1 : \widetilde{A}_1[a_1] \leq u \leq x_1^{\ell_1}\}$; `//` $G(I)_{\ell_1}$ `computation`

    foreach $i \in \{2\,..\,r\}$ **do**

      $Alk_il_i \leftarrow A(\leq k_i, \ell_i)$; `// calling Algorithm 4.1`

      $max \leftarrow \widetilde{A}_{i-1}[a_{i-1}] \cdot x_n^{\ell_i - \ell_{i-1}}$;

      $G_i \leftarrow \{u \in Alk_il_i : \widetilde{A}_i[a_i] \leq u < max\}$; `//` $G(I)_{\ell_i}$ `computation`

    end

    foreach $i \in \{1\,..\,r\}$ **do**

      $L_i \leftarrow \{u \in Alk_il_i : \widetilde{A}_i[a_i] \leq u\}$; `//` L_i `computation`

    end

    $I \leftarrow$ ideal spanned by monomials in the lists G_i;

  end

  return I;

end

4.2.1 Admissible corner values sequences

In this Section, we provide a procedure to determine the admissible corner values sequences of a strongly stable ideal with a given sequence of corners.

Let r be a positive integer, and let $B = [b_1, \dots, b_r]$ a list of positive integers. Setting $N = \max\{b_1, \dots, b_r\}$, let us consider the subset of the set $\mathcal{D}'(N, r)$ of all ordered selections with repetition of r items from a set of size N [Ros00], say

$$\mathcal{D}'(B, r) = \{(y_1, \dots, y_r) : y_i \leq b_i, i = 1, \dots, r\}.$$

We will call $\mathcal{D}'(B, r)$ the set of *list ordered selections*.

Algorithm 4.4 gives the pseudocode of the procedure returning the admissible corner values sequences.

	Admissible Values for a_i																												
$(5,3)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	a_1
$(4,4)$	1	1	1	1	1	1	1	2	2	2	2	2	3	3	3	3	4	4	4	4	5	5	6	7	1	1	2	3	a_2
$(3,5)$	1	2	3	4	5	6	7	1	2	3	4	5	1	2	3	4	1	2	3	4	1	2	1	1	1	2	1	1	a_3
$(1,7)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	a_4

Table 4.1: List of admissible value for a_i of I

Algorithm 4.4: Admissible Corner Values Sequences Computation

Input: Ring R, Corner sequence $\{(k_i, \ell_i)\}_{i=1,\ldots,r}$

Output: Table of admissible values

begin

  $Ub \leftarrow$ list of upper bounds ; `// from (4.1.5)`

  repeat

    $a \leftarrow$ next list ordered selections $\mathcal{D}'(Ub, r)$;

    $I \leftarrow FGBI(Corners, a)$; `// calling Algorithm 4.3`

    if I *is not NULL* **then**

      $AV \leftarrow a$; `// update list of admissible values`

    else

      $Ub \leftarrow$ max consistent value for a ; `// dynamic optimization`

    end

  until $a_1 > Ub_1$ // a_1 is the 1-st component of a;

  return AV;

end

Next example illustrates how our procedures work.

Example 4.2.9 Let $S = \mathbb{Q}[x_1, \ldots, x_7]$. Consider the corner sequence

$$\mathcal{C} = \{(5,3), (4,4), (3,5), (1,7)\}.$$

All the admissible choices for the sequence of positive integers $a = (a_1, a_2, a_3, a_4)$ for which there exists a strongly stable ideal I of S such that $\mathrm{Corn}(I) = \mathcal{C}$ and $a(I) = a$ are (Algorithm 4.4):

Let us consider the case where $a = (1, 2, 5, 1)$. Use of Algorithm 4.2 yields:

A_1	$x_1^2x_6,\quad x_1x_2x_6$
LS_1	$\emptyset$
$\widetilde{A_1}$	$x_1^2x_6,\quad x_1x_2x_6$
A_2	$x_1^3x_5,\quad x_1^2x_2x_5,\quad x_1^2x_3x_5,\quad x_1^2x_4x_5,\quad x_1^2x_5^2,\quad x_1x_2^2x_5,\quad x_1x_2x_3x_5,\quad x_1x_2x_4x_5,$ $x_1x_2x_5^2,\quad x_1x_3^2x_5,\quad x_1x_3x_4x_5,\quad x_1x_3x_5^2$
LS_2	$x_1^3x_5,\quad x_1^2x_2x_5,\quad x_1^2x_3x_5,\quad x_1^2x_4x_5,\quad x_1^2x_5^2$
$\widetilde{A_2}$	$x_1x_2^2x_5,\quad x_1x_2x_3x_5,\quad x_1x_2x_4x_5,\quad x_1x_2x_5^2,\quad x_1x_3^2x_5,\quad x_1x_3x_4x_5,\quad x_1x_3x_5^2$
A_3	$x_1^4x_4,\quad x_1^3x_2x_4,\quad x_1^3x_3x_4,\quad x_1^3x_4^2,\quad x_1^2x_2^2x_4,\quad x_1^2x_2x_3x_4,\quad x_1^2x_2x_4^2,\quad x_1^2x_3^2x_4,$ $x_1^2x_3x_4^2,\quad x_1^2x_4^3,\quad x_1x_2^3x_4,\quad x_1x_2^2x_3x_4,\quad x_1x_2^2x_4^2,\quad x_1x_2x_3^2x_4,\quad x_1x_2x_3x_4^2,\quad x_1x_2x_4^3,$ $x_1x_3^3x_4,\quad x_1x_3^2x_4^2,\quad x_1x_3x_4^3,\quad x_1x_4^4$
LS_3	$x_1^4x_4,\quad x_1^3x_2x_4,\quad x_1^3x_3x_4,\quad x_1^3x_4^2,\quad x_1^2x_2^2x_4,\quad x_1^2x_2x_3x_4,\quad x_1^2x_2x_4^2,\quad x_1^2x_3^2x_4,$ $x_1^2x_3x_4^2,\quad x_1^2x_4^3,\quad x_1x_2^3x_4,\quad x_1x_2^2x_3x_4,\quad x_1x_2^2x_4^2,\quad x_1x_2x_3^2x_4,\quad x_1x_2x_3x_4^2$
$\widetilde{A_3}$	$x_1x_2x_4^3,\quad x_1x_3^3x_4,\quad x_1x_3^2x_4^2,\quad x_1x_3x_4^3,\quad x_1x_4^4$
A_4	$x_1^6x_2,\quad x_1^5x_2^2,\quad x_1^4x_2^3,\quad x_1^3x_2^4,\quad x_1^2x_2^5,\quad x_1x_2^6,\quad x_2^7$
LS_4	$x_1^6x_2,\quad x_1^5x_2^2,\quad x_1^4x_2^3,\quad x_1^3x_2^4,\quad x_1^2x_2^5,\quad x_1x_2^6$
$\widetilde{A_4}$	x_2^7

and, by using Algorithm 4.3, the minimal system of monomial generators of the desired FGBI is obtained:

$x_1^3,$	$x_1^2x_2,$	$x_1^2x_3,$	$x_1^2x_4,$	$x_1^2x_5,$	$\underline{x_1^2x_6},$	$x_1x_2^3,$
$x_1x_2^2x_3,$	$x_1x_2^2x_4,$	$x_1x_2^2x_5,$	$x_1x_2x_3^2,$	$x_1x_2x_3x_4,$	$\underline{x_1x_2x_3x_5},$	$x_1x_2x_4^3,$
$x_1x_3^4,$	$x_1x_3^3x_4,$	$x_1x_3^2x_4^2,$	$x_1x_3x_4^3,$	$\underline{x_1x_4^4},$	$\underline{x_2^7},$	

where the underlined monomials are the Borel generators. Furthermore, it is easy to compute $I = L_1S + L_2S + L_3S + L_4S$ as a piecewise lex ideal (according to Definition 3.3):

L_1	$x_1^3,\quad x_1^2x_2,\quad x_1^2x_3,\quad x_1^2x_4,\quad x_1^2x_5,\quad x_1^2x_6$
L_2	$x_1^4,\quad x_1^3x_2,\quad x_1^3x_3,\quad x_1^3x_4,\quad x_1^3x_5,\quad x_1^2x_2^2,\quad x_1^2x_2x_3,\quad x_1^2x_2x_4,$ $x_1^2x_2x_5,\quad x_1^2x_3^2,\quad x_1^2x_3x_4,\quad x_1^2x_3x_5,\quad x_1^2x_4^2,\quad x_1^2x_4x_5,\quad x_1^2x_5^2,\quad x_1x_2^3,$ $x_1x_2^2x_3,\quad x_1x_2^2x_4,\quad x_1x_2^2x_5,\quad x_1x_2x_3^2,\quad x_1x_2x_3x_4,\quad x_1x_2x_3x_5$
L_3	$x_1^5,\quad x_1^4x_2,\quad x_1^4x_3,\quad x_1^4x_4,\quad x_1^3x_2^2,\quad x_1^3x_2x_3,\quad x_1^3x_2x_4,\quad x_1^3x_3^2,$ $x_1^3x_3x_4,\quad x_1^3x_4^2,\quad x_1^2x_2^3,\quad x_1^2x_2^2x_3,\quad x_1^2x_2^2x_4,\quad x_1^2x_2x_3^2,\quad x_1^2x_2x_3x_4,\quad x_1^2x_2x_4^2,$ $x_1^2x_3^3,\quad x_1^2x_3^2x_4,\quad x_1^2x_3x_4^2,\quad x_1^2x_4^3,\quad x_1x_2^4,\quad x_1x_2^3x_3,\quad x_1x_2^3x_4,\quad x_1x_2^2x_3^2,$ $x_1x_2^2x_3x_4,\quad x_1x_2^2x_4^2,\quad x_1x_2x_3^3,\quad x_1x_2x_3^2x_4,\quad x_1x_2x_3x_4^2,\quad x_1x_2x_4^3,\quad x_1x_3^4,\quad x_1x_3^3x_4,$ $x_1x_3^2x_4^2,\quad x_1x_3x_4^3,\quad x_1x_4^4$
L_4	$x_1^7,\quad x_1^6x_2,\quad x_1^5x_2^2,\quad x_1^4x_2^3,\quad x_1^3x_2^4,\quad x_1^2x_2^5,\quad x_1x_2^6,\quad x_2^7$

Finally, as a check of our procedures, we report the Betti table of the FGBI above defined, computed by native functions of CoCoA:

		0	1	2	3	4	5
3	:	6	15	20	15	6	1
4	:	7	19	20	10	2	−
5	:	6	17	16	5	−	−
6	:	−	−	−	−	−	−
7	:	1	1	−	−	−	−

Remark 4.2.10 It is worth of being recalled that if I is a strongly stable ideal with given extremal Betti numbers, then there does not always exist a unique finitely generated Borel ideal with the same extremal Betti numbers (positions and values).

Example 4.2.11 Let $S = K[x_1, \dots, x_6]$. Consider the strongly stable ideal $I = (x_1^2,\ x_1x_2,\ x_1x_3,\ x_1x_4,\ x_1x_5,\ x_1x_6,\ x_2^3,\ x_2^2x_3,\ x_2x_3^3,\ x_2x_3^2x_4,\ x_2x_3^2x_5,\ x_3^5)$. The extremal Betti numbers of I are

$$\beta_{5,5+2}(I) = \beta_{4,4+4}(I) = \beta_{2,2+5}(I) = 1, \tag{4.2.6}$$

as the Betti table of I shows:

		0	1	2	3	4	5
2	:	6	15	20	15	6	1
3	:	2	3	1	−	−	−
4	:	3	9	10	5	1	−
5	:	1	2	1	−	−	−

Note that I is not the finitely generated Borel ideal with the extremal Betti numbers described in (4.2.6). Indeed, the finitely generated Borel ideal J such that $\mathrm{Corn}(J) = \mathrm{Corn}(I)$ and $a(J) = a(I)$ is (Theorem 4.2.6):

$$J = (x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_2^4, x_2^3x_3, x_2^3x_4, x_2^3x_5, x_2^2x_3^3),$$

with the following Betti table

		0	1	2	3	4	5
2	:	6	15	20	15	6	1
3	:	−	−	−	−	−	−
4	:	4	10	10	5	1	−
5	:	1	2	1	−	−	−

4.2.2 CoCoa package

Algorithms described in this section are part of a *CoCoA package* "ExtrBettiNumbers.cpkg5", and tested with CoCoA System 5.1.4. We are confident that this package may reveal useful for further applications. The source code of the package can be requested directly to the authors. The package contains two public functions:

- "StronglyStableIdealEB(R, Corners, a, print)",
- "AdmissibleValues(R, Corners)".

The function "StronglyStableIdealEB(R, Corners, a, print)" requires as input parameters a polynomial ring (R), a corner sequence (Corners), a corner values sequence (a), and a boolean value to control the printing. The ring (R) may be defined by the user, and the functions redefine the ring, forcing $\mathbb{Q}$ as base field, and the lex ordering as monomial ordering. The sets A_i, $\widetilde{A}_i$, computed in Example 4.2.9, are not displayed, and only the minimal set of monomial generators of the FGBI are printed. The function returns a FGBI, and so it may be used for further computations. All results are written and returned in the variables chosen by users.
The function "AdmissibleValues(R, Corners)" requires as input parameters a polynomial ring (R), and a corner sequence (Corners). Let $\mathcal{C} = \{(k_i, \ell_i)\}_{i=1,\dots,r}$ be a corner sequence. The function computes the set $\mathcal{D}'(Ub, r)$, where Ub is the list whose i-th component is the binomial coefficient $\binom{k_i+\ell_i-1}{\ell_i-1}$, and $r = |\mathcal{C}|$. Ub is the so called *list of upper bounds*. The structure of the FGBI we are looking for allows us to dynamically update the list Ub by checking the existence of the ideal by the function FGBI(Corners, a). Hence, setting $a = (a_1, \dots, a_r)$, it is possible to reach an optimal list of upper bounds for each a_i. As a result, the procedure returns a table whose columns are the admissible r-tuple $a = (a_1, \dots, a_r)$.

As far as Example 4.2.9 is concerned, within CoCoA (with the "ExtrBettiNumbers.cpkg5" package installed), the following statements provide the table of all admissible values:

```
use R::=QQ[x[1..7]];
Corners:=[[5,3],[4,4],[3,5],[1,7]];
av:=AdmissibleValues(R,Corners);
av;
```

furthermore, the following statements

```
use R::=QQ[x[1..7]];
Corners:=[[5,3],[4,4],[3,5],[1,7]];
a:=[1,2,5,1];
I:=StronglyStableIdealEB(R,Corners,a,1);
I;
IsStronglyStable(I);
PrintRes(I);
PrintBettiDiagram(I);
```

allow to compute the FGBI I with $\mathrm{Corn}(I) = \{(5,3),(4,4),(3,5),(1,7)\}$ and $a(I) = a = (1,2,5,1)$.

4.3 Squarefree strongly stable ideals

In this Section, we examine the possible extremal Betti numbers of squarefree strongly stable ideals in $S = K[x_1, \dots, x_n]$. More precisely, we identify the admissible corner sequence of a squarefree strongly stable ideal in S.

From now on, we assume $\mathrm{Mon}_{\ell}{}^{s}(S)$ to be endowed with the squarefree lex order $>_{\mathrm{slex}}$ induced by $x_1 > x_2 > \cdots > x_n$.

At first, we analyze the simple cases occurring when $n = 2, 3$.

Case 1. Let $n = 2$ and $S = K[x_1, x_2]$. A squarefree strongly stable ideal I of S can have at most one corner. More precisely, $\mathrm{Corn}(I) = \{(1, 1)\}$ with $a(I) = (1)$, *i.e.*, $I = (x_1, x_2)$.

Case 2. Let $n = 3$ and $S = K[x_1, x_2, x_3]$. Also in such a case, a squarefree strongly stable ideal I of S can have at most one corner (k, ℓ), $k + \ell \leq 3$. Indeed, the only situations that may occur are listed in Table 4.2.

Corner sequence	**Corner values**	**Squarefree strongly stable ideal**
$\mathrm{Corn}(I) = \{(2, 1)\}$	$a(I) = (1)$	$I = (x_1, x_2, x_3)$
$\mathrm{Corn}(I) = \{(1, 1)\}$	$a(I) = (1)$	$I = (x_1, x_2)$
$\mathrm{Corn}(I) = \{(1, 2)\}$	$a(I) = (1)$	$I = (x_1x_2, x_1x_3)$
$\mathrm{Corn}(I) = \{(1, 2)\}$	$a(I) = (2)$	$I = (x_1x_2, x_1x_3, x_2x_3)$

Table 4.2: Squarefree corner sequences for $n = 3$.

Such easy cases allow us to yield the next result.

Proposition 4.3.1 *Let $S = K[x_1, \dots, x_n]$, $n \geq 2$. If I is a squarefree strongly stable ideal of S with $(k, 1) \in \mathrm{Corn}(I)$, then $|\mathrm{Corn}(I)| = 1$. More precisely, $I = (x_1, x_2, \dots, x_{k+1})$.*

Proof. First of all one can observe that $G(I)_1 = \{x_1, \dots, x_{k+1}\}$. If $G(I)_{\geq 2} \neq \emptyset$, then there exists a monomial $u \in G(I)$ of degree $\ell \geq 2$ such that $\max(u) \geq k + 2$. A contradiction, since $(k, 1)$ is a corner of I. □

Now, let us consider the case $n = 4$.

Case 3. Let $n = 4$ and $S = K[x_1, x_2, x_3, x_4]$. Assume I to be a squarefree strongly stable ideal S of initial degree ≥ 2 (Proposition 4.3.1). Since a pair $(k, \ell) \in \mathrm{Corn}(I)$ must satisfy the inequality $k + \ell \leq 4$, the situations that can occur in such a case are described in Table 4.3.

Corner sequence	**Corner values**	**Squarefree strongly stable ideal**
$\mathrm{Corn}(I) = \{(2,2),(1,3)\}$	$a(I)$ =(1,1)	$I = (x_1x_2, x_1x_3, x_1x_4, x_2x_3x_4)$
$\mathrm{Corn}(I) = \{(1,2)\}$	$a(I) = (1)$	$I = (x_1x_2, x_1x_3)$
$\mathrm{Corn}(I) = \{(1,2)\}$	$a(I) = (2)$	$I = (x_1x_2, x_1x_3, x_2x_3)$
$\mathrm{Corn}(I) = \{(2,2)\}$	$a(I) = (1)$	$I = (x_1x_2, x_1x_3, x_1x_4)$
$\mathrm{Corn}(I) = \{(2,2)\}$	$a(I) = (2)$	$I = (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4)$
$\mathrm{Corn}(I) = \{(2,2)\}$	$a(I) = (3)$	$I = (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4)$
$\mathrm{Corn}(I) = \{(1,3)\}$	$a(I) = (1)$	$I = (x_1x_2x_3, x_1x_2x_4)$
$\mathrm{Corn}(I) = \{(1,3)\}$	$a(I) = (2)$	$I = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_4)$
$\mathrm{Corn}(I) = \{(1,3)\}$	$a(I) = (3)$	$I = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4)$

Table 4.3: Squarefree corner sequences for $n = 4$.

Remark 4.3.2 All the squarefree strongly stable ideals described in Tables 4.2 and 4.3 are the smallest strongly stable ideals with the given data.

Let T be a subset of $\mathrm{Mon}_d^s(S)$, $d < n$. The set of squarefree monomials of degree $d+1$ of S

$$\mathrm{Shad}(T) = \{x_i u : u \in T,\, i \notin \mathrm{supp}(u),\, i = 1, \dots, n\}$$

is called the *squarefree shadow* of T. Moreover, we define the i-th *squarefree shadow* recursively by $\mathrm{Shad}^i(T) = \mathrm{Shad}(\mathrm{Shad}^{i-1}(T))$, $i \geq 1$, with $\mathrm{Shad}^0(T) = T$.

Next notion will be crucial for the further developments in this chapter.

Definition 4.3.3 Let $u = x_{i_1} \cdots x_{i_q}$ be a squarefree monomial of S of degree $q < n$. We say that u has a j-gap if $i_{j+1} - i_j > 1$ for some $1 \leq j < q$. The positive integer $i_{j+1} - i_j - 1$ will be called the width of the j-gap.

The j-gap of a squarefree monomial $u = x_{i_1} \cdots x_{i_q} \in S$ will be denoted by j-gap(u), whereas its width will be denoted by wd(j-gap(u)). Moreover, we define

$$\mathrm{Gap}(u) := \{j \in [q] : \text{there exists a } j\text{-gap}(u)\}.$$

Definition 4.3.4 A squarefree monomial $u = x_{i_1} \cdots x_{i_q}$ of S will be said gap–free if $\mathrm{Gap}(u) = \emptyset$.

Example 4.3.5 Let $S = K[x_1, \dots, x_{11}]$. The monomial $u = x_1x_3x_4x_6x_{10} \in S$ has three gaps. Indeed, $\mathrm{Gap}(u) = \{1, 3, 4\}$, 1-gap(u), 3-gap(u) have both width equal to 1 and 4-gap(u) has width equal to 3; on the contrary, the monomial $v = x_2x_3x_4x_5x_6 \in S$ is gap–free.

Lemma 4.3.6 *Let $u = x_{i_1} \cdots x_{i_q}$ be a squarefree monomial of degree $q < n-1$ of S. Assume u has a gap whose width is ≥ 2, or u has at least two gaps.*

Then there exist at least two squarefree monomials $v, w \in S$ of degree $q+1$ with $v >_{\mathrm{slex}} w$, $\max(v) = \max(w) = n$ and such that

(i) *v is a multiple of u;*

(ii) *w is not a multiple of u.*

Proof. If $\max(u) < n$, we can choose $v = ux_n = x_{i_1} \cdots x_{i_q} x_n$. Setting $t = \max \mathrm{Gap}(v)$, the greatest squarefree monomial following v in the squarefree lex order is

$$\tilde{v} = x_{i_1} \cdots x_{i_{t-1}} x_{i_t+1} \cdots x_{i_t+q-t+2}.$$

If $i_t + q - t + 2 = n$, we choose $w = \tilde{v}$, otherwise, if $i_t + q - t + 2 < n$, we choose $w = x_{i_1} \cdots x_{i_{t-1}} x_{i_t+1} \cdots x_{i_t+q-t+1} x_n$. Finally, $v >_{\mathrm{slex}} w$, $u \mid v$ and $u \nmid w$. Note that $t \leq q$.

Now, assume $\max(u) = n$. If $t = \max \mathrm{Gap}(u)$, let

$$v = x_{i_1} \cdots x_{i_t} x_{i_{t+1}-1} x_{i_{t+1}} \cdots x_{i_{q-1}} x_{i_q} = x_{i_1} \cdots x_{i_t} x_{i_{t+1}-1} x_{i_{t+1}} \cdots x_{i_{q-1}} x_n.$$

Furthermore, if $p = \max \mathrm{Gap}(v)$, then the greatest squarefree monomial following v in the squarefree lex order is

$$\tilde{v} = x_{i_1} \cdots x_{i_{p-1}} x_{i_p+1} \cdots x_{i_p+q-p+2}.$$

Hence, if $i_p + q - p + 2 = n$, we choose $w = \tilde{v}$, otherwise, if $i_p + q - p + 2 < n$, we choose $w = x_{i_1} \cdots x_{i_{p-1}} x_{i_p+1} \cdots x_{i_p+q-p+1} x_n$.

Note that the assumption on the gaps of the squarefree monomial u assures us that we can construct both the monomials v and w. $\square$

Now, we recall some notations from [CF16] that will be useful in the sequel.

Let I be a squarefree stable ideal of S. If I is generated in one degree ℓ, then I has a unique extremal Betti number $\beta_{m-\ell,\, m}(I)$, where $m = \max\{\max(u) : u \in G(I)\}$.

Assume I to be generated in degrees $1 \leq \ell_1 < \ell_2 < \cdots < \ell_r \leq n$, and denote by $[r]$ the set $\{1, \ldots, r\}$.

Setting

$$m_{\ell_j} = \max\{\max(u) : u \in G(I)_{\ell_j}\}, \quad \text{for } j = 1, \ldots, r,$$

let us consider the following sequence of nonnegative integers associated to I:

$$\mathbf{ds}(I) = (m_{\ell_1} - \ell_1, m_{\ell_2} - \ell_2, \ldots, m_{\ell_r} - \ell_r). \tag{4.3.1}$$

Such a sequence is called the *degree-sequence* of I.

One can observe that, if

$$m_{\ell_1} - \ell_1 > m_{\ell_2} - \ell_2 > \cdots > m_{\ell_r} - \ell_r, \tag{4.3.2}$$

then, from Characterization 4.1.12, $\beta_{m_{\ell_i}-\ell_i, m_{\ell_i}}(I)$ is an extremal Betti number of I, for $i = 1, \dots, r$. If (4.3.2) does not hold, one can construct a suitable subsequence of the *degree-sequence* $\mathbf{ds}(I)$, say

$$\widehat{\mathbf{ds}(I)} = (m_{\ell_{i_1}} - \ell_{i_1}, m_{\ell_{i_2}} - \ell_{i_2}, \dots, m_{\ell_{i_q}} - \ell_{i_q}), \tag{4.3.3}$$

with $\ell_1 \leq \ell_{i_1} < \ell_{i_2} < \cdots < \ell_{i_q} = \ell_r$, and such that, for $j = 1, \dots, q$, $\beta_{m_{\ell_{i_j}}-\ell_{i_j}, m_{\ell_{i_j}}}(I)$ is an extremal Betti number of I.
The integer $q \leq r$, denoted by $\mathbf{dl}(I)$, and called the *degree-length* of I, gives the number of the extremal Betti numbers of the squarefree stable ideal I.
For more details on this subject see [CF16].

Next results easily follow.

Proposition 4.3.7 *Let I be a squarefree strongly stable ideal of $S = K[x_1, \dots, x_n]$, $n \geq 4$, with initial degree 2 and with a corner in degree 2. Then*

(1) *I has at most $n-2$ corners for $n = 4$;*

(2) *I has at most $n-3$ corners for $n \geq 5$.*

Proof. (1). It follows from Case 3.
(2). Let $n \geq 5$. An admissible degree–sequence of I is the following one

$$\mathbf{ds}(I) = (n-2, n-3, \cdots, n-(n-2) = 2).$$

Indeed, setting $w_1 = x_1x_n$, since 1-gap(w_1) has width $n-2$, then Lemma 4.3.6 assures that there exist at least $n-4$ squarefree monomials $w_2, \dots, w_{n-3}$ in S of degrees $3, \dots, n-2$, respectively, with $\max(w_i) = n$, and $n-4$ squarefree monomials $v_2, \dots, v_{n-3}$ of degrees $3, \dots, n-2$, respectively, with $\max(v_i) = n$ and such that $v_i >_{\text{slex}} w_i$, $w_{i-1} \mid v_i$, $v_i \nmid w_i$, for $i = 2, \dots, n-3$. Using the same techniques as in Lemma 4.3.6, one can easily verify that $w_i \nmid w_{i+1}$ $(i = 1, \dots, n-4)$.

The monomials w_i $(i = 1, \dots, n-3)$ will be called *basic monomials*.

$$\begin{tikzcd}
w_1 \arrow[r, dotted] \arrow[dr] & v_2 & w_3 \arrow[r, dotted] \arrow[dr] & v_4 & w_5 \arrow[r, dotted] \arrow[dr] & v_6 & \cdots \\
 & w_2 \arrow[r, dotted] \arrow[ur] & v_3 & w_4 \arrow[r, dotted] \arrow[ur] & v_5 & w_6 \arrow[r, dotted] \arrow[ur] & v_7
\end{tikzcd}$$

Next tables list the basic monomials for $n = 5, \dots, 9$. For $n \geq 10$, the construction of such elements proceeds smoothly.

v_i	w_i
	$\mathbf{x_1x_5}$
$x_1x_4x_5$	$\mathbf{x_2x_3x_5}$

(a) $\mathbf{n=5}$

v_i	w_i
	$\mathbf{x_1x_6}$
$x_1x_5x_6$	$\mathbf{x_2x_3x_6}$
$x_2x_3x_5x_6$	$\mathbf{x_2x_4x_5x_6}$

(b) $\mathbf{n=6}$

v_i	w_i
	$\mathbf{x_1x_7}$
$x_1x_6x_7$	$\mathbf{x_2x_3x_7}$
$x_2x_3x_6x_7$	$\mathbf{x_2x_4x_5x_7}$
$x_2x_4x_5x_6x_7$	$\mathbf{x_3x_4x_5x_6x_7}$

(c) $\mathbf{n=7}$

v_i	w_i
	$\mathbf{x_1x_8}$
$x_1x_7x_8$	$\mathbf{x_2x_3x_8}$
$x_2x_3x_7x_8$	$\mathbf{x_2x_4x_5x_8}$
$x_2x_4x_5x_7x_8$	$\mathbf{x_2x_4x_6x_7x_8}$
$x_2x_4x_5x_6x_7x_8$	$\mathbf{x_3x_4x_5x_6x_7x_8}$

(d) $\mathbf{n=8}$

v_i	w_i
	$\mathbf{x_1x_9}$
$x_1x_8x_9$	$\mathbf{x_2x_3x_9}$
$x_2x_3x_8x_9$	$\mathbf{x_2x_4x_5x_9}$
$x_2x_4x_5x_8x_9$	$\mathbf{x_2x_4x_6x_7x_9}$
$x_2x_4x_6x_7x_8x_9$	$\mathbf{x_2x_5x_6x_7x_8x_9}$
$x_2x_4x_5x_6x_7x_8x_9$	$\mathbf{x_3x_4x_5x_6x_7x_8x_9}$

(e) $\mathbf{n=9}$

Table 4.4: Tables of fundamental squarefree monomials for initial degree 2

Note that the construction of the basic elements ends up as soon as one gets a gap–free monomial. $\square$

Example 4.3.8 Let $S = K[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$, and let

$$I = (x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_1x_8, x_2x_3x_4, x_2x_3x_5, x_2x_3x_6, x_2x_3x_7, x_2x_3x_8,\\ x_2x_4x_5x_6, x_2x_4x_5x_7, x_2x_4x_5x_8, x_2x_4x_6x_7x_8, x_3x_4x_5x_6x_7x_8)$$

be a squarefree strongly stable ideal of S. The *degree-sequence* of I is

$$\mathbf{ds}(I) = (m_2 - 2, m_3 - 3, m_4 - 4, m_5 - 5, m_6 - 6) = (6, 5, 4, 3, 2).$$

I has initial degree 2 and $\mathbf{dl}(I) = 5$. The extremal Betti numbers of I are $\beta_{8-2,8}(I) = \beta_{8-3,8}(I) = \beta_{8-4,8}(I) = \beta_{8-5,8}(I) = \beta_{8-6,8}(I) = 1$, as the Betti table of I shows:

		0	1	2	3	4	5	6
2	:	7	21	35	35	21	7	1
3	:	5	15	20	15	6	1	–
4	:	3	9	10	5	1	–	–
5	:	1	3	3	1	–	–	–
6	:	1	2	1	–	–	–	–

Proposition 4.3.9 *Let $n \geq 5$ and let I be a squarefree strongly stable ideal of $S = K[x_1, \ldots, x_n]$ with initial degree $\ell \geq 3$ and with a corner in degree ℓ. Then I has at most $n - \ell$ corners.*

Proof. Using the same reasoning as in Proposition 4.3.7, an admissible degree–sequence of I is the following one:

$$\mathbf{ds}(I) = (n - \ell, n - (\ell + 1), \cdots, n - (n - 1) = 1),$$

with $\mathbf{dl}(I) = n - \ell$.

Next tables show the basic monomials for $n = 5, \ldots, 8$ and $\ell = 3$. For $n \geq 8$ ($\ell = 3$), the construction of such elements proceeds smoothly.

v_i	w_i
	$\mathbf{x_1x_2x_5}$
$x_1x_2x_4x_5$	$\mathbf{x_1x_3x_4x_5}$

(a) $\mathbf{n = 5}$

v_i	w_i
	$\mathbf{x_1x_2x_6}$
$x_1x_2x_5x_6$	$\mathbf{x_1x_3x_4x_6}$
$x_1x_3x_4x_5x_6$	$\mathbf{x_2x_3x_4x_5x_6}$

(b) $\mathbf{n = 6}$

v_i	w_i
	$\mathbf{x_1x_2x_7}$
$x_1x_2x_6x_7$	$\mathbf{x_1x_3x_4x_7}$
$x_1x_3x_4x_6x_7$	$\mathbf{x_1x_3x_5x_6x_7}$
$x_1x_3x_4x_5x_6x_7$	$\mathbf{x_2x_3x_4x_5x_6x_7}$

(c) $\mathbf{n = 7}$

v_i	w_i
	$\mathbf{x_1x_2x_8}$
$x_1x_2x_7x_8$	$\mathbf{x_1x_3x_4x_8}$
$x_1x_3x_4x_7x_8$	$\mathbf{x_1x_3x_5x_6x_8}$
$x_1x_3x_5x_6x_7x_8$	$\mathbf{x_1x_4x_5x_6x_7x_8}$
$x_1x_3x_4x_5x_6x_7x_8$	$\mathbf{x_2x_3x_4x_5x_6x_7x_8}$

(d) $\mathbf{n = 8}$

Table 4.5: Tables of fundamental squarefree monomials for initial degree 3

Also in this case, the construction of the basic elements ends up as soon as one gets a gap–free monomial. $\square$

Example 4.3.10 Let $S = K[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$ and let

$$I = (x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_2x_6, x_1x_2x_7, x_1x_2x_8, x_1x_3x_4x_5, x_1x_3x_4x_6, x_1x_3x_4x_7,$$
$$x_1x_3x_4x_8, x_1x_3x_5x_6x_7, x_1x_3x_5x_6x_8, x_1x_4x_5x_6x_7x_8, x_2x_3x_4x_5x_6x_7x_8)$$

be a squarefree strongly stable ideal of S initial degree 3. The *degree-sequence* of I is

$$\mathbf{ds}(I) = (m_2 - 3, m_3 - 4, m_4 - 5, m_5 - 6, m_6 - 7) = (5, 4, 3, 2, 1).$$

The extremal Betti numbers of I are $\beta_{8-3,8}(I) = \beta_{8-4,8}(I) = \beta_{8-5,8}(I) = \beta_{8-6,8}(I) = \beta_{8-7,8}(I) = 1$, as the Betti table of I shows

		0	1	2	3	4	5
3	:	6	15	20	15	6	1
4	:	4	10	10	5	1	–
5	:	2	5	4	1	–	–
6	:	1	2	1	–	–	–
7	:	1	1	–	–	–	–

Betti Table of I

The next example considers a squarefree monomial ideal I of S without a corner in its initial degree, and shows the construction of a squarefree monomial ideal J of S with a corner in its initial degree and with the same extremal Betti numbers (positions and values) of I.

Example 4.3.11 Consider the following monomial ideal I of $S = K[x_1, \dots, x_5]$:

$$I = (x_1x_2, x_1x_3x_4, x_1x_3x_5, x_2x_3x_4x_5).$$

I is squarefree strongly stable of initial degree 2 and with $\mathrm{Corn}(I) = \{(2,3), (1,4)\}$. From the Betti table of I, one can note that there is no corner in its initial degree:

		0	1	2
2	:	1	–	–
3	:	2	3	1
4	:	1	1	–

Betti Table of I

Furthermore, we can construct a squarefree strongly stable ideal J in S with initial degree 3 and $\mathrm{Corn}(J) = \{(2,3), (1,4)\}$. It is

$$J = (x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_3x_4x_5).$$

Note that J is the smallest squarefree strongly stable ideal of S with corner sequence $\{(2,3), (1,4)\}$:

		0	1	2
3	:	3	3	1
4	:	1	1	–

Betti Table of J

Remark 4.3.12 It is worthy to point out that a squarefree strongly stable ideal I of $S = K[x_1, \dots, x_n]$ $(n \geq 5)$ of initial degree $\ell \geq 2$ with a corner in degree ℓ and such that

$$\mathbf{ds}(I) = (n-2, n-3, \dots, 2), \text{ for } \ell = 2,$$

$$\mathbf{ds}(I) = (n-\ell, n-\ell-1, \dots, 1), \text{ for } \ell \geq 3$$

is a squarefree lex ideal of S.

Hence, one can observe that a squarefree lex ideal of the polynomial ring S of initial degree $\ell \geq 2$ and with a corner in degree ℓ can have at most $n - \ell$ corners unlike the non–squarefree case. Indeed, a non–squarefree lex ideal I of a polynomial ring can have at most 2 corners [CU00].

For $u, v \in \mathrm{Mon}^s_d(S)$, $u \geq_{\mathrm{slex}} v$, let us define the following set of squarefree monomials:

$$\mathcal{L}(u, v) = \{z \in \mathrm{Mon}^s_d(S) : u \geq_{\mathrm{slex}} z \geq_{\mathrm{slex}} v\}.$$

Theorem 4.3.13 *Let $n \geq 5$ and $\ell_1 \geq 3$ two integers. Given $n - \ell_1$ pairs of positive integers*

$$(k_1, \ell_1), (k_2, \ell_2), \dots, (k_{n-\ell_1}, \ell_{n-\ell_1}), \tag{4.3.4}$$

with $1 \leq k_{n-\ell_1} < k_{n-\ell_1-1} < \cdots < k_1 \leq n-3$ and $3 \leq \ell_1 < \ell_2 < \cdots < \ell_{n-\ell_1} \leq n-1$, then there exists a squarefree lex ideal I of S of initial degree ℓ_1 and with the pairs in (4.3.4) as corners if and only if $k_i + \ell_i = n$, for $i = 1, \dots, n - \ell_1$.

Proof. Set $S = K[x_1, \dots, x_n]$. If there exists a squarefree lex ideal I of $K[x_1, \dots, x_n]$ of initial degree ℓ_1 and with the pairs in (4.3.4) as corners, then Proposition 4.3.9 forces that $k_i + \ell_i = n$, for $i = 1, \dots, n - \ell_1$.

Conversely, assume there exist $n - \ell_1$ pairs of positive integers

$$(k_1, \ell_1), (k_2, \ell_2), \dots, (k_{n-\ell_1}, \ell_{n-\ell_1}), \tag{4.3.5}$$

with $1 \leq k_{n-\ell_1} < k_{n-\ell_1-1} < \cdots < k_1 \leq n-3$, $3 \leq \ell_1 < \ell_2 < \cdots < \ell_{n-\ell_1} \leq n-1$ and $k_i + \ell_i = n$, for $i = 1, \dots, n - \ell_1$.
We prove that there exists a squarefree lex ideal I of S generated in degrees $\ell_1, \ell_2, \dots, \ell_{n-\ell_1}$ with $\mathrm{Corn}(I) = \{(k_1, \ell_1), \dots, (k_{n-\ell_1}, \ell_{n-\ell_1})\}$.

Setting $s = \max\{i : \ell_1 + 2i - 3 \leq n - 2\}$, the required monomial ideal I can be constructed as follows.

Step 1. For $i = 1, \dots, s$, let

- $G(I)_{\ell_1} = \mathcal{L}(u_1, v_1)$, with $u_1 = x_1 x_2 \cdots x_{\ell_1}$ and $v_1 = x_1 x_2 \cdots x_{\ell_1 - 1} x_n$;

- $G(I)_{\ell_i} = G(I)_{\ell_1+i-1} = \mathcal{L}\,(u_i, v_i)$, with

$$\begin{aligned} u_i &= x_1x_2\cdots x_{\ell_1-2}\prod_{j=0}^{i-2} x_{\ell_1+2j}x_{\ell_1+2(i-2)+1}x_{\ell_1+2(i-2)+2} \\ &= x_1x_2\cdots x_{\ell_1-2}\prod_{j=0}^{i-2} x_{\ell_1+2j}x_{\ell_1+2i-3}x_{\ell_1+2i-2} \end{aligned}$$

and

$$v_i = x_1\cdots x_{\ell_1-2}\prod_{j=0}^{i-2} x_{\ell_1+2j}x_{\ell_1+2(i-2)+1}x_n = x_1\cdots x_{\ell_1-2}\prod_{j=0}^{i-2} x_{\ell_1+2j}x_{\ell_1+2i-3}x_n.$$

Step 2. Let us consider the squarefree monomial

$$v_s = x_1x_2\cdots x_{\ell_1-2}\prod_{j=0}^{s-2} x_{\ell_1+2j}x_{\ell_1+2s-3}x_n.$$

Since, $\ell_1+2s-3 \le n-2$, the smallest monomial belonging to the $\mathrm{Shad}(G(I)_{\ell_s})$ is

$$w_{s+1} = x_1x_2\cdots x_{\ell_1-2}\prod_{j=0}^{s-2} x_{\ell_1+2j}x_{\ell_1+2s-3}x_{n-1}x_n.$$

We distinguish two cases: $\ell_1+2s-3 = n-2$, and $\ell_1+2s-3 < n-2$.

Claim 1. If $\ell_1+2s-3 < n-2$, then $\ell_1+2s-3 = n-3$.

Indeed, by the meaning of s, $\ell_1+2(s+1)-3 \ge n-1$. Hence, $\ell_1+2s-3 \ge n-3$ and

$$n-3 \le \ell_1+2s-3 < n-2$$

and consequently $\ell_1+2s-3 = n-3$. The claim follows.

Let us consider $\ell_1+2s-3 = \ell_1+2(s-2)+1 = n-2$. In such a case,

$$w_{s+1} = x_1\cdots x_{\ell_1-2}\prod_{j=0}^{s-2} x_{\ell_1+2j}x_{\ell_1+2(s-2)+1}x_{n-1}x_n = x_1\cdots x_{\ell_1-2}\prod_{j=0}^{s-3} x_{\ell_1+2j}x_{n-3}x_{n-2}x_{n-1}x_n.$$

Hence, the greatest squarefree monomial of S following w_{s+1} is

$$u_{s+1} = x_1x_2\cdots x_{\ell_1-2}\prod_{j=0}^{s-4} x_{\ell_1+2j}x_{\ell_1+2(s-3)+1}x_{\ell_1+2(s-3)+2}\cdots x_{\ell_1+2(s-3)+5}.$$

Note that $\max(u_{s+1}) = \ell_1+2(s-3)+5 = \ell_1+2s-3+2 = n-2+2 = n$, whereupon we choose

$$G(I)_{\ell_{s+1}} = \{u_{s+1}\}.$$

The smallest squarefree monomial belonging to $\mathrm{Shad}(G(I)_{\ell_{s+1}})$ is

$$\begin{aligned} w_{s+2} &= x_1x_2\cdots x_{\ell_1-2}\prod_{j=0}^{s-4}x_{\ell_1+2j}x_{\ell_1+2(s-3)}x_{\ell_1+2(s-3)+1}x_{\ell_1+2(s-3)+2}\cdots x_{\ell_1+2(s-3)+5}\\ &= x_1x_2\cdots x_{\ell_1-2}\prod_{j=0}^{s-4}x_{\ell_1+2j}x_{n-5}x_{n-4}x_{n-3}x_{n-2}x_{n-1}x_n. \end{aligned}$$

Therefore, the greatest squarefree monomial of S following w_{s+2} is

$$u_{s+2} = x_1x_2\cdots x_{\ell_1-2}\prod_{j=0}^{s-5}x_{\ell_1+2j}x_{\ell_1+2(s-4)+1}x_{\ell_1+2(s-4)+2}\cdots x_{\ell_1+2(s-4)+7}.$$

Note that $\max(u_{s+2}) = \ell_1+2(s-4)+7 = \ell_1+2s-3+2 = n-2+2 = n$. Thus, we choose

$$G(I)_{\ell_{s+2}} = \{u_{s+2}\},$$

and so on. In general,

$$G(I)_{\ell_{s+q}} = \{u_{s+q}\},$$

with

$$u_{s+q} = x_1x_2\cdots x_{\ell_1-2}\prod_{j=0}^{s-2-(q+1)}x_{\ell_1+2j}x_{\ell_1+2(s-2-q)+1}x_{\ell_1+2(s-2-q)+2}\cdots x_{\ell_1+2(s-2-q)+2q+3},$$

for $q=1,\dots,t$, where t is the positive integer such that $s-2-(t+1)=0$. It is easy to verify that $\max(u_{s+q}) = n$.

Claim 2. $s+t = n-\ell_1-2$.
Since, $\max(u_{s+t}) = n$, and $t+1 = s-2$ ($t = s-3$), then

$$n = \ell_1+2(s-2-t)+2t+3 = \ell_1+2(t+1-t)+2t+3 = \ell_1+2t+5.$$

Hence,

$$n-\ell_1-2 = \ell_1+2t+5-\ell_1-2 = 2t+3 = 2s-3 = s+t.$$

The claim follows.

Finally, we choose

$$\begin{aligned} G(I)_{\ell_{n-\ell_1-1}} &= G(I)_{s+t+1} = \{u_{s+t+1}\} = \{x_1x_2\cdots x_{\ell_1-2}x_{\ell_1+1}\cdots x_n\},\\ G(I)_{\ell_{n-\ell_1}} &= G(I)_{s+t+2} = \{u_{s+t+2}\} = \{x_1x_2\cdots x_{\ell_1-3}x_{\ell_1-1}x_{\ell_1}\cdots x_n\}. \end{aligned}$$

Now, let us consider the case $\ell_1+2s-3 = n-3$. In such a case, the smallest monomial belonging to $\mathrm{Shad}(G(I)_{\ell_s})$ is

$$w_{s+1} = x_1x_2\cdots x_{\ell_1-2}\prod_{j=0}^{s-2}x_{\ell_1+2j}x_{\ell_1+2(s-2)+1}x_{n-1}x_n = x_1x_2\cdots x_{\ell_1-2}\prod_{j=0}^{s-2}x_{\ell_1+2j}x_{n-3}x_{n-1}x_n.$$

Therefore, the greatest squarefree monomial of S following w_{s+1} is

$$u_{s+1} = x_1x_2\cdots x_{\ell_1-2}\prod_{j=0}^{s-2} x_{\ell_1+2j}x_{n-2}x_{n-1}x_n.$$

Since $\max(u_{s+1}) = n$, we choose

$$G(I)_{\ell_{s+1}} = \{u_{s+1}\}.$$

By hypothesis, $\ell_1 + 2(s-2) = n-4$, then the smallest squarefree monomial belonging to $\mathrm{Shad}(G(I)_{\ell_{s+1}})$ is

$$w_{s+2} = x_1\cdots x_{\ell_1-2}\prod_{j=0}^{s-2} x_{\ell_1+2j}x_{n-3}x_{n-2}x_{n-1}x_n = x_1\cdots x_{\ell_1-2}\prod_{j=0}^{s-3} x_{\ell_1+2j}x_{n-4}x_{n-3}x_{n-2}x_{n-1}x_n$$

and, consequently, the greatest squarefree monomial of S following w_{s+2} is

$$u_{s+2} = x_1x_2\cdots x_{\ell_1-2}\prod_{j=0}^{s-4} x_{\ell_1+2j}x_{\ell_1+2(s-3)+1}x_{\ell_1+2(s-3)+2}\cdots x_{\ell_1+2(s-3)+6}.$$

Note that $\max(u_{s+2}) = \ell_1 + 2(s-3) + 6 = \ell_1 + 2s = n$, whence we choose

$$G(I)_{\ell_{s+2}} = \{u_{s+2}\}.$$

In general,

$$G(I)_{\ell_{s+q}} = \{u_{s+q}\},$$

with

$$u_{s+q} = x_1x_2\cdots x_{\ell_1-2}\prod_{j=0}^{s-2-q} x_{\ell_1+2j}x_{\ell_1+2(s-2-(q-1))+1}\cdots x_{\ell_1+2(s-2-(q-1))+2q+2},$$

for $q = 1,\dots,t$, where t is the positive integer such that $s-2-t=0$ $(t = s-2)$. It is easy to verify that $\max(u_{s+q}) = n$.

Also in such a case we can verify that $s+t = n-\ell_1-2$. Indeed, since $\max(u_{s+t}) = n$, and $t = s-2$, then

$$n = \ell_1 + 2(s-2-(t-1)) + 2t + 2 = \ell_1 + 2t + 4,$$

and

$$n - \ell_1 - 2 = \ell_1 + 2t + 4 - \ell_1 - 2 = 2t + 2 = 2(s-2) + 2 = s + t.$$

Finally, as in the previous case, we can choose

$$G(I)_{\ell_{n-\ell_1-1}} = G(I)_{s+t+1} = \{x_1x_2\cdots x_{\ell_1-2}x_{\ell_1+1}\cdots x_n\},$$

and

$$G(I)_{\ell_{n-\ell_1}} = G(I)_{s+t+2} = \{x_1x_2\cdots x_{\ell_1-3}x_{\ell_1-1}x_{\ell_1}\cdots x_n\}.$$

It is worthy observing that I is the smallest squarefree lex ideal of S with $\mathrm{Corn}(I) = \{(k_1,\ell_1),(k_2,\ell_2),\dots,(k_r,\ell_r)\}$ and such that $\beta_{k_i,\,k_i+\ell_i}(I) = 1$, for all i, *i.e.*, $a(I) = (1,\dots,1)$. $\square$

4.3.1 A numerical characterization

In this Subection, we face the following problem.

Problem 4.3.14 Given three positive integers $n \geq 4$, $\ell_1 \geq 2$ and $1 \leq r \leq n-\ell_1$, r pairs of positive integers (k_1,ℓ_1), …, (k_r,ℓ_r) such that $n-3 \geq k_1 > k_2 > \cdots > k_r \geq 2$ and $2 \leq \ell_1 < \ell_2 < \cdots < \ell_r$, $k_i+\ell_i \leq n$ $(i = 1,\dots,r)$, and r positive integers $a_1,\dots,a_r$, under which conditions does there exist a squarefree monomial ideal I of $S = K[x_1,\dots,x_n]$ such that $\beta_{k_1,k_1+\ell_1}(I) = a_1$, …, $\beta_{k_r,k_r+\ell_r}(I) = a_r$ are its extremal Betti numbers?

For a pair of positive integers (k,ℓ) such that $k+\ell \leq n$, we define the following set:

$$A^s(k,\ell) = \{u \in \mathrm{Mon}^s_\ell(S) : \max(u) = k+\ell\}.$$

Setting $A^s(k,\ell) = \{u_1,\dots,u_q\}$, we can suppose, possibly after a permutation of the indices, that

$$u_1 >_{\text{slex}} u_2 >_{\text{slex}} \cdots >_{\text{slex}} u_q. \tag{4.3.6}$$

For the i-th monomial u of degree ℓ with $\max(u) = k+\ell$, we mean the monomial of $A^s(k,\ell)$ that appears in the i-th position of (4.3.6), for $1 \leq i \leq q$. Note that $u_1 = x_1x_2\cdots x_{\ell-1}x_{k+\ell}$, $u_q = x_{k+1}\cdots x_{k+\ell}$, and $q = |A^s(k,\ell)| = \binom{k+\ell-1}{\ell-1}$.

Furthermore, if u_i, u_j, $i < j$, are two monomials in (4.3.6), we define the following subsets of $A^s(k,\ell)$:

$$[u_i,u_j] = \{w \in A^s(k,\ell) : u_i \geq_{\text{slex}} w \geq_{\text{slex}} u_j\},$$

$$[u_i,u_j) = \{w \in A^s(k,\ell) : u_i \geq_{\text{slex}} w >_{\text{slex}} u_j\};$$

$[u_i,u_j]$ will be called the *segment* of $A^s(k,\ell)$ of initial element u_i and final element u_j, whereas $[u_i,u_j)$ will be called the *left segment* of $A^s(k,\ell)$ of initial element u_i and final element u_j. If $i = j$, we set $[u_i,u_j] = \{u_i\}$.

Remark 4.3.15 From (4.1.1), if (k,ℓ) is a corner of a squarefree stable ideal I and $\beta_{k,k+\ell}(I) = a$, then there exists a segment $[v_1,v_a]$ of $A^s(k,\ell)$ such that $a = |[v_1,v_a]|$.

Next lemma will be crucial in the sequel.

Lemma 4.3.16 *Let n and $q \geq 1$ be two positive integers such that $n \geq q$. Then*

$$\binom{n}{q} = \binom{n-1}{q-1} + \binom{n-2}{q-1} + \cdots + \binom{q-1}{q-1}.$$

Proof. Let $\mathrm{Mon}_q^s(S)$ be the set of all squarefree monomials of degree q of $S = K[x_1, \ldots, x_n]$. It is well–known that

$$\binom{n}{q} = |\mathrm{Mon}_q^s(S)|.$$

Setting,

$$b_i = |\{u \in \mathrm{Mon}_q^s(S) : \min(u) = i\}|,$$

one has

$$\binom{n}{q} = \sum_{i=1}^{n-q+1} b_i.$$

On the other hand,

$$b_i = \binom{n-i}{q-1}, \quad i = 1, \ldots, n-q+1.$$

The assertion follows. $\square$

Given a monomial $u \in A^s(k,\ell)$, the next proposition shows a method, involving Lemma 4.3.16, to count the number of monomials $v \in A^s(k,\ell)$ such that $v \geq_{slex} u$.

Theorem 4.3.17 *Let (k,ℓ) be a pair of positive integers, $\ell \geq 2$, and let $u = x_{i_1}x_{i_2}\cdots x_{i_{\ell-1}}x_{i_\ell}$ be a monomial of $A^s(k,\ell)$. Setting $\tilde{u} = x_{i_1}x_{i_2}\cdots x_{i_{\ell-1}}$, then $|[x_1x_2\cdots x_{\ell-1}x_{k+\ell}, u]|$ is a sum of t suitable binomial coefficients, where*

$$t = \begin{cases} i_1, & \text{if} \quad \mathrm{Gap}(\tilde{u}) = \emptyset, \\ \\ i_1 + \sum_{s=1}^{p} \mathrm{wd}(g_s\text{-gap}(\tilde{u})), & \text{if} \quad \mathrm{Gap}(\tilde{u}) = \{g_1, \ldots, g_p\} \neq \emptyset. \end{cases}$$

Proof. Set $m = |[x_1x_2\cdots x_{\ell-1}x_{k+\ell}, u]|$. m is the number of all monomials $w \in A^s(k,\ell)$ such that $w \geq_{slex} u$. By Lemma 4.3.16, the binomial coefficient $\binom{k+\ell-1}{\ell-1} = |A^s(k,\ell)|$ can be decomposed as a sum of $k+1$ binomial coefficients, as follows:

$$\binom{k+\ell-1}{\ell-1} = \sum_{j=1}^{k+1} \binom{k+\ell-1-j}{\ell-2} = \binom{k+\ell-2}{\ell-2} + \binom{k+\ell-3}{\ell-2} + \cdots + \binom{\ell-2}{\ell-2}. \quad (4.3.7)$$

One can observe that $\binom{k+\ell-2}{\ell-2}$ counts the monomials $w \in A^s(k,\ell)$ such that $\min(w) = 1$, the binomial coefficient $\binom{k+\ell-3}{\ell-2}$ counts the monomials $w \in A^s(k,\ell)$ such that $\min(w) = 2$. In general, the binomial coefficient $\binom{k+\ell-i}{\ell-2}$ counts the monomials $w \in A^s(k,\ell)$ such that $\min(w) = i-1$, for $i = 4, \ldots, k+2$. Note that $\binom{\ell-2}{\ell-2} = \binom{k+\ell-(k+2)}{\ell-2}$ counts the monomials $w \in A^s(k,\ell)$ with $\min(w) = k+1$. Indeed, there exists only a monomial w of such a type. It is $w = x_{k+1}x_{k+2}\cdots x_{k+\ell} = \min A^s(k,\ell)$. It is clear that all monomials $w \in A^s(k,\ell)$

with $\min(w) < i_1 = \min(\tilde{u}) = \min(u)$ are greater than u. Hence, the first $i_1 - 1$ binomial coefficients in (4.3.7) give a contribute for the computation of m.
We need to distinguish two cases: $\mathrm{Gap}(\tilde{u}) = \emptyset$, $\mathrm{Gap}(\tilde{u}) \neq \emptyset$.
Note that $\mathrm{Gap}(\tilde{u}) = \mathrm{Gap}(u)$, or $\mathrm{Gap}(\tilde{u}) = \mathrm{Gap}(u) - 1$.
Case 1. Let $\mathrm{Gap}(\tilde{u}) = \emptyset$. In such a case, u is the greatest monomial of $A^s(k,\ell)$ with $\min(u) = i_1$. More precisely, the following sum of binomial coefficients

$$\sum_{j=1}^{i_1-1} \binom{k+\ell-1-j}{\ell-2} \tag{4.3.8}$$

gives the number of all monomials $w \in A^s(k,\ell)$ greater than u. Since $i_1, i_2, \ldots, i_\ell$ are consecutive integers, then other monomials greater than u which are different from the w's counted by (4.3.8) do not exist. Hence,

$$m = |[x_1x_2\cdots x_{\ell-1}x_{k+\ell}, u]| = \sum_{j=1}^{i_1-1} \binom{k+\ell-1-j}{\ell-2} + 1.$$

On the other hand, $1 = \binom{0}{0}$, and consequently m is the sum of $t = i_1 - 1 + 1 = i_1 = \min(\tilde{u}) = \min(u)$ binomial coefficients.
Case 2. Let $\mathrm{Gap}(\tilde{u}) = \{g_1, \ldots, g_p\}$, $p \geq 1$. It is worthy to point out that the existence of the gaps g_j $(j = 1, \ldots, p)$ implies that $i_{g_j+1} - i_{g_j} - 1 > 0$, *i.e.*, $\mathrm{supp}(\tilde{u}) \cap \{q : i_{g_j} < q < i_{g_j+1}\} = \emptyset$, for all $j \in [p]$. Thus, all monomials $w \in A^s(k,\ell)$ of the type $x_{i_1}x_{i_2}\cdots x_{i_{g_j}}z$, where z is a monomial of degree $\ell - g_j$ and $\max(z) = k+\ell$ such that $\mathrm{supp}(z) \cap \{q : i_{g_j} < q < i_{g_j+1}\} \neq \emptyset$, are greater than u.
It is clear that all these monomials make up the left segment $[x_1x_2\cdots x_{\ell-1}x_{k+\ell}, u)$.
Let us consider the i_1–th binomial in (4.3.7):

$$\binom{k+\ell-1-i_1}{\ell-2} = \sum_{j=1}^{k+1} \binom{k+\ell-1-i_1-j}{\ell-3}. \tag{4.3.9}$$

In order to compute all monomials w of the type $x_{i_1}x_{i_2}\cdots x_{i_{g_1}}z$, we need to evaluate g_1 successive binomial decompositions until the next one:

$$\binom{k+\ell-i_{g_1}-1}{\ell+i_1-i_{g_1}-2} = \sum_{j=1}^{k-i_1+1} \binom{k+\ell-i_{g_1}-1-j}{\ell+i_1-i_{g_1}-3}. \tag{4.3.10}$$

The sum of the first $\mathrm{wd}(g_1\text{-gap})(\tilde{u}) = i_{g_1+1} - i_{g_1} - 1$ binomial coefficients in (4.3.10) gives the number of all monomials $w \in A^s(k,\ell)$ we are looking for.
In order to compute all monomials $w \in A^s(k,\ell)$ of the type $x_{i_1}x_{i_2}\cdots x_{i_{g_2}}z$, we consider the $(\mathrm{wd}(g_1\text{-gap})(\tilde{u}) - 1)$–th binomial in (4.3.10):

$$\begin{aligned}\binom{k+\ell-i_{g_1}-1-\mathrm{wd}(g_1\text{-gap})(\tilde{u})-1}{\ell+i_1-i_{g_1}-3} &= \binom{k+\ell-i_{g_1+1}-1}{\ell+i_1-i_{g_1}-3} = \\ &= \sum_{j=1}^{k-i_1+i_{g_1}-i_{g_1+1}+2} \binom{k+\ell-i_{g_1+1}-1-j}{\ell+i_1-i_{g_1}-4}.\end{aligned}$$

Hence, evaluating the $i_{g_2} - i_{g_1+1}$ successive binomial decompositions until

$$\binom{k+\ell-i_{g_2}-1}{\ell+i_1-i_{g_1}-i_{g_2}+i_{g_1+1}-3} = \sum_{j=1}^{k-i_1+i_{g_1}-i_{g_1+1}+2} \binom{k+\ell-i_{g_2}-1-j}{\ell+i_1-i_{g_1}-i_{g_2}+i_{g_1+1}-4}, \tag{4.3.11}$$

the number of all required monomials $w \in A^s(k,\ell)$ will be given by the sum of the first $\mathrm{wd}(g_2\text{-gap}(\tilde{u})) = i_{g_2+1} - i_{g_2} - 1$ binomial coefficients in (4.3.11).
The procedure can be iterated for all $g_j \in \mathrm{Gap}(\tilde{u})$, $j \geq 3$.

Finally, $|[x_1x_2\cdots x_{\ell-1}x_{k+\ell}, u)| = i_1 - 1 + \sum_{s=1}^{p} \mathrm{wd}(g_s\text{-gap})(\tilde{u})$. Hence, in order to get $|[x_1x_2\cdots x_{\ell-1}x_{k+\ell}, u]|$, we must take into account the binomial $\binom{0}{0}$ which counts the monomial u:

$$t = i_1 - 1 + \sum_{s=1}^{p} \mathrm{wd}(g_s\text{-gap})(\tilde{u}) + 1 = i_1 + \sum_{s=1}^{p} \mathrm{wd}(g_s\text{-gap})(\tilde{u}).$$

The assertion follows. □

Remark 4.3.18 Our choice to focus on the monomial $\tilde{u} = x_{i_1}x_{i_2}\cdots x_{i_{\ell-1}}$, instead of u, in Theorem 4.3.17 is due to the fact that if $i_{\ell-1} < k+\ell-1$, *i.e.*, $\mathrm{Gap}(\tilde{u}) = \mathrm{Gap}(u) - 1$, then all monomials $z \in A^s(k,\ell)$ such that $k+\ell-1 \in \mathrm{supp}(z)$ are smallest than u, with respect to $\geq_{slex}$.

Next example illustrates Theorem 4.3.17.

Example 4.3.19 Let $S = K[x_1,\ldots,x_9]$ and consider the monomial $u = x_2x_5x_7x_8$. Set $\tilde{u} = x_2x_5x_7$. From Equation 4.1.3, $|A^s(4,4)| = \binom{7}{3} = 35$.
In order to compute $m = |[x_1x_2x_3x_8, u]|$, we consider the following binomial decomposition:

$$\binom{7}{3} = \binom{6}{2} + \binom{5}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2}.$$

Since, $\min(u) = 2$, then all monomials $w \in A^s(4,4)$ with $\min(w) = 1$ are greater than u, so we must take into account the binomial coefficient $\boxed{\binom{6}{2} = 15}$ for the computation of m.
Now, let us consider the following binomial decomposition:

$$\binom{5}{2} = \binom{4}{1} + \binom{3}{1} + \binom{2}{1} + \binom{1}{1}.$$

Since $\mathrm{Gap}(\tilde{u}) = \{1,2\}$ and $\mathrm{wd}(1\text{-gap}(\tilde{u})) = 2$, the sum $\boxed{\binom{4}{1} + \binom{3}{1} = 7}$ gives the number of all monomials of the type $x_2z \in A^s(4,4)$, with z squarefree monomial of degree 3 and $\max(z) = 8$ such that $\mathrm{supp}(z) \cap \{q : 2 < q < 5\} \neq \emptyset$.
At this stage, we have $\boxed{15 + 7 = 22}$ monomials.
The next decomposition we need to consider is

$$\binom{2}{1} = \binom{1}{0} + \binom{0}{0}.$$

Since $2 \in \mathrm{Gap}(\tilde{u})$, and $\mathrm{wd}(2\text{-gap}(\tilde{u})) = 1$, we must take into account $\boxed{\binom{1}{0} = 1}$.
Finally, we have obtained $\boxed{22 + 1 = 23}$ monomials of $A^s(4,4)$ greater than u, and so $m = |[x_1x_2x_3x_8, u]| = 23 + 1 = 24$.
The following scheme summarizes the previous calculations.

$$\binom{7}{3} = \boxed{\binom{\mathbf{6}}{\mathbf{2}}} + \binom{5}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2}$$
$$\binom{5}{2} = \boxed{\binom{\mathbf{4}}{\mathbf{1}} + \binom{\mathbf{3}}{\mathbf{1}}} + \binom{2}{1} + \binom{1}{1}$$
$$\binom{2}{1} = \boxed{\binom{\mathbf{1}}{\mathbf{0}}} + \binom{0}{0}.$$

Now, consider the monomial $v = x_3x_4x_7x_8$. Let $\tilde{v} = x_3x_4x_7$.
Proceeding as before, since $\mathrm{Gap}(\tilde{v}) = \{1\}$, then $|[x_1x_2x_3x_8, u]| = 27 + 1$, where 27 is given by the sum of the highlighted binomial coefficients in the next scheme:

$$\binom{7}{3} = \boxed{\binom{\mathbf{6}}{\mathbf{2}} + \binom{\mathbf{5}}{\mathbf{2}}} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2}$$
$$\binom{4}{2} = \binom{3}{1} + \binom{2}{1} + \binom{1}{1}$$
$$\binom{3}{1} = \boxed{\binom{\mathbf{2}}{\mathbf{0}} + \binom{\mathbf{1}}{\mathbf{0}}} + \binom{0}{0}.$$

Here is the list of all monomials which come into play for u and v:

$$x_1x_2x_3x_8, x_1x_2x_4x_8, x_1x_2x_5x_8, x_1x_2x_6x_8, x_1x_2x_7x_8,$$
$$x_1x_3x_4x_8, x_1x_3x_5x_8, x_1x_3x_6x_8, x_1x_3x_7x_8,$$
$$x_1x_4x_5x_8, x_1x_4x_6x_8, x_1x_4x_7x_8,$$
$$x_1x_5x_6x_8, x_1x_5x_7x_8,$$
$$x_1x_6x_7x_8,$$
$$x_2x_3x_4x_8, x_2x_3x_5x_8, x_2x_3x_6x_8, x_2x_3x_7x_8,$$
$$x_2x_4x_5x_8, x_2x_4x_6x_8, x_2x_4x_7x_8,$$
$$x_2x_5x_6x_8, \mathbf{x_2x_5x_7x_8},$$
$$x_2x_6x_7x_8,$$
$$x_3x_4x_5x_8, x_3x_4x_6x_8, \mathbf{x_3x_4x_7x_8},$$
$$x_3x_5x_6x_8, x_3x_5x_7x_8,$$
$$x_3x_6x_7x_8,$$
$$x_4x_5x_6x_8, x_4x_5x_7x_8,$$
$$x_4x_6x_7x_8,$$
$$x_5x_6x_7x_8$$

Now, let $u_1, \dots u_r$ be squarefree monomials of degree q of S. We denote by $B(u_1, \dots, u_r)$ the smallest squarefree strongly stable set of $\mathrm{Mon}_q^s(S)$ containing the monomials $u_1, \dots, u_r$.

It is well known that if $q < n$, $\mathrm{Shad}(B(u_1, \ldots, u_r))$ is a squarefree strongly stable set of monomials of degree $q+1$ of S, and consequently $\mathrm{Shad}^i(B(u_1, \ldots, u_r))$ is a squarefree strongly stable set of degree $q+i$, for $1 \leq i \leq n-q$.

Now, let (k_1, ℓ_1) and (k_2, ℓ_2) be two pairs of positive integers such that $k_1 > k_2$, $\ell_1 < \ell_2$, $k_i + \ell_i \leq n$ $(i = 1, 2)$. If $u_1, \ldots, u_r \in \mathrm{Mon}^s_{\ell_1}(S)$ are squarefree monomials of S such that $\max(u_j) = k_1 + \ell_1$, $j = 1, \ldots, r$, we define the following set:

$$\mathrm{BShad}(u_1, \ldots, u_r)_{(k_2,\ell_2)} = \{v \in \mathrm{Shad}^{\ell_2 - \ell_1}(B(u_1, \ldots, u_r)) : \max(v) \leq k_2 + \ell_2\}.$$

One can quickly observe that $\mathrm{BShad}(u_1, \ldots, u_r)_{(k_2,\ell_2)}$ is a squarefree strongly stable set of degree ℓ_2 of S.

Remark 4.3.20 It is worthy to underline that if one wants to compute the minimum of $\mathrm{BShad}(u_1, \ldots, u_r)_{(k_2,\ell_2)}$, it is sufficient to determine $\min \mathrm{BShad}(u_r)_{(k_2,\ell_2)}$. Furthermore, in order to obtain such a monomial, one can suitably manage the integers in $\mathrm{supp}(u_r)$, as we will see in a while.

Definition 4.3.21 Let u be a squarefree monomial of degree q of S, $q < n$. Let $p \leq n$ a positive integer such that $[p] \setminus \mathrm{supp}(u) \neq \emptyset$ and $\{j_1, \ldots, j_t\}$ a subset of $[p] \setminus \mathrm{supp}(u)$, with $j_1 < j_2 < \cdots < j_t$, $q + t \leq n$. The monomial $x_{j_1} \cdots x_{j_t} u \in \mathrm{Mon}^s_{q+t}(S)$ is called the joint of u with the variables $x_{j_1}, \ldots, x_{j_t}$.

Example 4.3.22 Let $u = x_1x_3x_6x_8 \in K[x_1, \ldots, x_9]$. Let $p = 7$ and consider the set $\{2, 4, 7\} \subset [7] \setminus \{1, 3, 6, 8\}$. The joint of u with x_2, x_4, x_7 is the squarefree monomial $x_1x_2x_3x_4x_6x_7x_8 \in \mathrm{Mon}^s_7(S)$.

With the same notations as before, we give the construction of $\min \mathrm{BShad}(u)_{(k_2,\ell_2)}$ for a given squarefree monomial $u \in A^s(k_1, \ell_1)$.

Construction 4.3.23 Let (k_1, ℓ_1) and (k_2, ℓ_2) be two pairs of positive integers such that $k_1 > k_2$, $2 \leq \ell_1 < \ell_2$ and $k_i + \ell_i \leq n$, for $i = 1, 2$. Let $u = x_{i_1} \cdots x_{i_{\ell_1}}$ be a squarefree monomial of $A^s(k_1, \ell_1)$. Assume i_t to be the greatest integer belonging to $\mathrm{supp}(u)$ such that $i_t < k_2 + \ell_2$, and write

$$u = x_{i_1} \cdots x_{i_t} \cdots x_{i_{\ell_1}}.$$

Let us consider the monomial $\overline{u} = x_{i_1} \cdots x_{i_t}$ and let $j_1, \ldots, j_{\ell_2 - t}$ be the greatest integers belonging to $[k_2 + \ell_2] \setminus \mathrm{supp}(\overline{u})$. Then,

$$\min \mathrm{BShad}(u)_{(k_2,\ell_2)} = x_{j_1} \cdots x_{j_{\ell_2 - t}} \overline{u} \in A^s(k_2, \ell_2).$$

Example 4.3.24 Let $S = K[x_1, \ldots, x_{10}]$, $(k_1, \ell_1) = (5, 4)$ and $(k_2, \ell_2) = (3, 5)$. All the conditions of Construction 4.3.23 hold. Now, let $u = x_1x_4x_8x_9 \in A^s(5, 4)$.

Hence, $i_t = 4$ and we can write $\overline{u} = x_1x_4$. We have $\ell_2 - t = 3$, so we can take the three greatest variables of the set $[8] \setminus \mathrm{supp}(x_1x_4) = \{2,3,5,6,7,8\}$: $j_1 = 6, j_2 = 7, j_3 = 8$. Then

$$\min \mathrm{BShad}(x_1x_4x_8x_9)_{(3,5)} = x_6x_7x_8\overline{u} = x_1x_4x_6x_7x_8 \in A^s(k_2,\ell_2).$$

Construction 4.3.23 assures the correctness of the next algorithm.

Algorithm 4.5: Computation of $\min \mathrm{BShad}(u)_{(k,\ell)}$

Input: Polynomial ring S, monomial u, positive integer k, positive integer ℓ

Output: monomial v

```
begin
    j ← k + ℓ;
    t ←| {i ∈ supp(u) : i < j} |;
    v ← the first t variables of u;
    q ← ℓ − t;
    while q > 0 do
        if j ∉ supp(v) then
            if j > 0 then
                v ← v ∗ S_j;
            else
                error no monomial;
            end
            q ← q − 1;
        end
        j ← j − 1;
    end
    return v;
end
```

Lemma 4.3.25 *Take two pairs of positive integers (k_1,ℓ_1) and (k_2,ℓ_2) such that $k_1 > k_2$, $2 \le \ell_1 < \ell_2$ with $k_i + \ell_i \le n$, for $i = 1,2$. Let u be a squarefree monomial of degree ℓ_1 with $\max(u) = k_1 + \ell_1$ and let $v = \min \mathrm{BShad}(u)_{(k_2,\ell_2)}$. If $\mathrm{Gap}(v) \neq \emptyset$, then there exists a monomial $w \in A^s(k_2,\ell_2) \setminus \mathrm{BShad}(u)_{(k_2,\ell_2)}$.*

Proof. Let $v = \min \mathrm{BShad}(u)_{(k_2,\ell_2)} = x_{r_1} \cdots x_{r_{\ell_2}}$. One has $\max(v) = k_2 + \ell_2$.
Assume $p = \max \mathrm{Gap}(v)$, then the greatest squarefree monomial following v in the squarefree lex order is $x_{r_1} \cdots x_{r_{p-1}} x_{r_p+1} \cdots x_{r_p+\ell_2-p+1}$, with $r_p + \ell_2 - p + 1 \le k_2 + \ell_2$. Hence, if $r_p + \ell_2 - p + 1 = k_2 + \ell_2$, we choose

$$w = x_{r_1} \cdots x_{r_{p-1}} x_{r_p+1} \cdots x_{r_p+\ell_2-p+1}.$$

Otherwise, if $r_p + \ell_2 - p + 1 < k_2 + \ell_2$, let

$$w = x_{r_1} \cdots x_{r_{p-1}} x_{r_p+1} \cdots x_{r_p+\ell_2-p} x_{k_2+\ell_2}.$$

□

Example 4.3.26 As in the Example 4.3.24, let $S = K[x_1, \dots, x_{10}]$, $(k_1, \ell_1) = (5, 4)$, $(k_2, \ell_2) = (3, 5)$, $u = x_1x_4x_8x_9 \in A^s(5, 4)$, $v = \min \mathrm{BShad}(x_1x_4x_8x_9)_{(3,5)} = x_1x_4x_6x_7x_8 \in A^s(k_2, \ell_2)$. We have $p = \max \mathrm{Gap}(x_1x_4x_6x_7x_8) = \max\{1, 2\} = 2$, hence we consider the monomial $x_1x_5x_6x_7x_8$. Since $\max(x_1x_5x_6x_7x_8) = k_2 + \ell_2 = 8$ then the monomial desired is $w = x_1x_5x_6x_7x_8 \in A^s(3, 5) \setminus \mathrm{BShad}(x_1x_4x_8x_9)_{(3,5)}$.

Next pseudocode describes the procedure in Lemma 4.3.25.

Algorithm 4.6: Computation of the next monomial smaller than u in $A^s(k, \ell)$

```
Input: Polynomial ring S, monomial u
Output: monomial w
begin
    m ← max supp(u);
    ℓ ← deg(u);
    if Gap(u) ≠ ∅ then
        t ← max Gap(u);
        w ← the first t − 1 variables of u;
        j ← index of variable of u at position t;
        foreach i ∈ {1..ℓ − t} do
            j ← j + 1 ;
            w ← w * S_j;
        end
        w ← w * S_m;
    else
        error no monomial;
    end
    return w;
end
```

The discussion below is significant for solving Problem 4.3.14.

Discussion 4.3.27 Let (k_1, ℓ_1) and (k_2, ℓ_2) be two pairs of positive integers such that $k_1 > k_2$, $2 \le \ell_1 < \ell_2$ with $k_i + \ell_i \le n$ $(i = 1, 2)$ and let a_1, a_2 be two positive integers.

Let T be a segment of $A^s(k_2, \ell_2)$ of cardinality $a_2 < \binom{k_2+\ell_2-1}{\ell_2-1}$. We want to determine the admissible values for $a_1 \le \binom{k_1+\ell_1-1}{\ell_1-1}$ so that there exists a segment $[u_1, u_{a_1}]$ of $A^s(k_1, \ell_1)$

of cardinality a_1 and such that $\mathrm{BShad}([u_1, u_{a_1}])_{(k_2,\ell_2)} \not\supseteq T$. It is clear that it should be $a_1 < \binom{k_1+\ell_1-1}{\ell_1-1}$.

Now, set $T = [z_1, z_{a_2}]$, and assume $T \not\subseteq \mathrm{BShad}([u_1, u_{a_1}])_{(k_2,\ell_2)}$. Let $v_1 \in A^s(k_1, \ell_1)$ be the smallest monomial such that $z_1 \notin \mathrm{BShad}(v_1)_{(k_2,\ell_2)}$. Such a monomial allows us to determine the bound on a_1 for which there exists the segment T.

Indeed, we can compute the following cardinalities (Theorem 4.3.17):

$$\begin{aligned} n_1 &= |\{u \in A^s(k_1, \ell_1) : u \geq v_1\}| = |[x_1x_2 \cdots x_{\ell_1-1}x_{k_1+\ell_1}, v_1]|, \\ p_1 &= |\{v \in A^s(k_1, \ell_1) : v > u_1\}| = |[x_1x_2 \cdots x_{\ell_1-1}x_{k_1+\ell_1}, u_1)|. \end{aligned}$$

Hence, since $[u_1, u_{a_1}] \subseteq [x_1x_2 \cdots x_{\ell_1-1}x_{k_1+\ell_1}, v_1]$, we get the following coarse bound for a_1:

$$a_1 \leq n_1;$$

then, we can refine such a bound *via* p_1 as follows:

$$a_1 \leq n_1 - p_1.$$

One can notice, that if $u_1 = \max A^s(k_1, \ell_1)$, then $p_1 = 0$.

Example 4.3.28 Given $S = K[x_1, \dots, x_{10}]$, let us consider the pairs of positive integers $(5, 4)$ and $(2, 6)$, the positive integers $a_1 = 8$ and $a_2 = 6$, and the following segment of $A^s(5, 4)$ of cardinality $a_1 = 8$:

$$\begin{aligned} [x_1x_3x_4x_9, x_1x_4x_7x_9] = \{&x_1x_3x_4x_9, x_1x_3x_5x_9, x_1x_3x_6x_9, x_1x_3x_7x_9, x_1x_3x_8x_9, x_1x_4x_5x_9, \\ &x_1x_4x_6x_9, x_1x_4x_7x_9\}. \end{aligned}$$

We want to verify if there exists a segment of $A^s(2, 6)$ of cardinality $a_2 = 6$ not contained in $\mathrm{BShad}([x_1x_3x_4x_9, x_1x_4x_7x_9])_{(2,6)}$.

First, from Equation 4.1.3, we know that $a_1 \leq \binom{8}{3} = 56$ and $a_2 \leq \binom{7}{5} = 21$.

In order to determine $p_1 = |\{v \in A^s(5, 4) : v > x_1x_3x_4x_9\}| = |[x_1x_2x_3x_9, x_1x_3x_4x_9)|$, we need to consider a suitable sequence of binomial decompositions. The first binomial decomposition that we have to examine is

$$\binom{8}{3} = \binom{7}{2} + \binom{6}{2} + \binom{5}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2}.$$

Then, applying the procedure described in Theorem 4.3.17 (see also Example 4.3.19), we obtain the following sequence of binomial decompositions,

$$\begin{aligned} \tbinom{8}{3} &= \tbinom{7}{2} + \tbinom{6}{2} + \tbinom{5}{2} + \tbinom{4}{2} + \tbinom{3}{2} + \tbinom{2}{2} \\ \tbinom{7}{2} &= \boxed{\tbinom{\mathbf{6}}{\mathbf{1}}} + \tbinom{5}{1} + \tbinom{4}{1} + \tbinom{3}{1} + \tbinom{2}{1} + \tbinom{1}{1}, \end{aligned}$$

whereupon $p_1 = 6$.
In order to compute n_1, we consider the set A_2 consisting of the smallest $a_2 = 6$ monomials of $A^s(2,6)$:

$$A_2 = \{x_2x_3x_4x_5x_6x_8, x_2x_3x_4x_5x_7x_8, x_2x_3x_4x_6x_7x_8, x_2x_3x_5x_6x_7x_8, x_2x_4x_5x_6x_7x_8, x_3x_4x_5x_6x_7x_8\}.$$

These monomials can be found using the "reversal" of Algorithm 4.6.
The smallest monomial z of $A^s(5,4)$ such that $\max A_2 = x_2x_3x_4x_5x_6x_8 \notin \mathrm{BShad}(z)_{(2,6)}$ is $z = x_1x_7x_8x_9$. The number of all monomials $w \in A^s(5,4)$ greater than or equal to z is determined by the following binomial sequences:

$$\begin{aligned} \binom{8}{3} &= \binom{7}{2} + \binom{6}{2} + \binom{5}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2} \\ \binom{7}{2} &= \boxed{\binom{\mathbf{6}}{\mathbf{1}} + \binom{\mathbf{5}}{\mathbf{1}} + \binom{\mathbf{4}}{\mathbf{1}} + \binom{\mathbf{3}}{\mathbf{1}} + \binom{\mathbf{2}}{\mathbf{1}}} + \binom{1}{1} \end{aligned}$$

Hence, we have $n_1 = (6+5+4+3+2)+1 = 21$ monomials. Finally, we have $a_1 \leq n_1 - p_1 = 21 - 6 = 15$. For $a_1 = 15$, then a segment of $A^s(2,6)$ of length $a_2 = 6$ is

$$A_2 = [x_2x_3x_4x_5x_6x_8, x_3x_4x_5x_6x_7x_8].$$

Discussion 4.3.27 yields the following result.

Theorem 4.3.29 *Consider three positive integers $n \geq 5$, $\ell_1 \geq 3$ and $1 \leq r \leq n - \ell_1$, r pairs of positive integers (k_1, ℓ_1), …, (k_r, ℓ_r) such that $n - 3 \geq k_1 > k_2 > \cdots > k_r \geq 2$ and $2 \leq \ell_1 < \ell_2 < \cdots < \ell_r$, $k_i + \ell_i \leq n$ $(i = 1, \dots, r)$, and r positive integers $a_1, \dots, a_r$. Let K be a field of characteristic zero. The following conditions are equivalent:*

(1) *There exists a squarefree graded ideal J of $S = K[x_1, \dots, x_n]$ with $\beta_{k_1,k_1+\ell_1}(J) = a_1$, …, $\beta_{k_r,k_r+\ell_r}(J) = a_r$ as extremal Betti numbers.*

(2) *There exists a squarefree strongly stable ideal I of $S = K[x_1, \dots, x_n]$ with $\beta_{k_1,k_1+\ell_1}(I) = a_1$, …, $\beta_{k_r,k_r+\ell_r}(I) = a_r$ as extremal Betti numbers.*

(3) *Setting*

(i) $v_r = x_{k_r+1} \cdots x_{k_r+\ell_r}$,

$A_r = [w_r, v_r]$, *with* $w_r \in A^s(k_r, \ell_r)$ *and such that* $|A_r| = a_r$*;*

(ii) *for* $i = 1, \dots, r-1$,

$v_{r-i} = \min\{u \in A^s(k_{r-i}, \ell_{r-i}) : \max A_{r-i+1} \notin \mathrm{BShad}(u)_{(k_{r-i+1},\ell_{r-i+1})}\}$,

$A_{r-i} = [w_{r-i}, v_{r-i}]$, *with* $w_{r-i} \in A^s(k_{r-i}, \ell_{r-i})$ *and such that* $|A_{r-i}| = a_{r-i}$*;*

(iii) *for* $i = 1, \dots, r$, $n_i = |\{u \in A^s(k_i, \ell_i) : u \geq v_i\}|$, *then the integers* a_i *satisfy the following conditions:*

$$a_i \leq n_i.$$

If $a_i = |[u_{i,1}, u_{i,a_i}]|$, $u_{i,j} \in A^s(k_i, \ell_i)$ *(*$j = 1, \dots, a_i$*) and* $p_i = |\{v \in A^s(k_i, \ell_i) : v > u_{i,1}\}|$, *then* $a_i \leq n_i - p_i$, *for* $i = 1, \dots, r$.

Proof. (1) $\Leftrightarrow$ (2). See [AHH00] and the introduction in this dissertation.
(2) $\Rightarrow$ (3). It follows applying iteratively Discussion 4.3.27, for $i = 1, \dots, r$. Note that $v_r = \min A^s(k_r, \ell_r)$, and consequently $n_r = \binom{k_r + \ell_r - 1}{\ell_r - 1}$; whereas $p_1 = 0$.
(3) $\Rightarrow$ (2). We construct a squarefree strongly stable ideal I of S generated in degrees $\ell_1, \dots, \ell_r$ as follows:

- $G(I)_{\ell_1} = B(u_{1,1}, \dots, u_{1,a_1})$;
- $G(I)_{\ell_2} = B(u_{2,1}, \dots, u_{2,a_2}) \setminus \mathrm{BShad}^{\ell_2 - \ell_1}(G(I)_{\ell_1})_{(k_2, \ell_2)}$;
- $G(I)_{\ell_i} = B(u_{i,1}, \dots, u_{i,a_i}) \setminus \mathrm{BShad}^{\ell_i - \ell_{i-1}}(\mathrm{Mon}^s(I_{\ell_{i-1}}))_{(k_i, \ell_i)}$, for $i = 3, \dots, r$, where $\mathrm{Mon}^s(I_{\ell_{i-1}})$ is the set of all squarefree monomials of degree ℓ_{i-1} belonging to $I_{\ell_{i-1}}$.

The monomials $u_{i,1}, \dots, u_{i,a_i}$, for $i = 1, \dots, r$, are the *basic monomials* of I. $\square$

Remark 4.3.30 A similar statement can be formulated in the case $\ell_1 = 2$ and $n \geq 5$.

Theorem 4.3.29 assures the correctness of the next algorithm.

Algorithm 4.7: Computation of the basic monomials for the given data

```
Input: Polynomial ring S, list of corners {(k_i, ℓ_i)}, list of values (a_i)
Output: list of monomials mons
begin
    hyp ← logical conditions required as hypotheses of the Theorem 4.3.29;
    if hyp then
        m ← k_0 + ℓ_0;
        w ← S_1 * ... * S_{ℓ_0 − 1} * S_m;                     // first corner
        mons ← {w};
        foreach j ∈ {2 .. a_0} do
            w ←next monomial of w;                              // calling Algorithm 4.6
            if no monomial then
                error no ideal;
            else
                mons ← mons ∪ {w};
            end
        end
        r ← number of corners;                                  // successive corners
        foreach i ∈ {2 .. r} do
            w ← min BShad(mons)_{(k_{i−1}, ℓ_{i−1})} ;          // calling Algorithm 4.5
            foreach j ∈ {1 .. a_i} do
                w ← next monomial of w;                         // calling Algorithm 4.6
                if no monomial then
                    error no ideal;
                else
                    mons ← mons ∪ {w};
                end
            end
        end
    end
    return mons;
end
```

4.3.2 Some relevant examples

This subsection collects some nice examples on the theory developed in this Section.
Next example illustrates Theorem 4.3.29.

Example 4.3.31 Let $n = 11$, $r = 4$, $\mathcal{C} = \{(8,3),(4,5),(3,6),(2,9)\}$ and $a = (a_1, a_2, a_3, a_4) = (7,5,2,2)$. We want to construct a squarefree strongly stable ideal I of $S = K[x_1, \ldots, x_{11}]$ generated in degrees 3,5,6,9 and such that $\mathrm{Corn}(I) = \mathcal{C}$, $a(I) = a$.

Before starting the construction of the ideal, we verify if the coarse bounds are satisfied for each a_i, $i = 1, \dots, 4$.

First of all, $v_4 = x_3x_4x_5x_6x_7x_8x_9x_{10}x_{11}$ and $n_4 = |[x_1x_2x_3x_4x_5x_6x_7x_8x_{11}, v_4]| = \binom{10}{8} = 45$. Hence, $a_4 = 2 \le n_4$.

Moreover, $A_4 = \{x_2x_4x_5x_6x_7x_8x_9x_{10}x_{11}, x_3x_4x_5x_6x_7x_8x_9x_{10}x_{11}\}$, $v_3 = x_2x_3x_6x_7x_8x_9$, and from the binomial decompositions

$$\binom{8}{5} = \boxed{\binom{\mathbf{7}}{\mathbf{4}}} + \binom{6}{4} + \binom{5}{4} + \binom{4}{4}$$
$$\binom{6}{4} = \binom{5}{3} + \binom{4}{3} + \binom{3}{3}$$
$$\binom{5}{3} = \boxed{\binom{\mathbf{4}}{\mathbf{2}} + \binom{\mathbf{3}}{\mathbf{2}}} + \binom{2}{2}$$

we obtain $a_3 = 2 \le n_3 = |[x_1x_2x_3x_4x_5x_9, v_3]| = 35 + (6+3) + 1 = 45$.

Furthermore, $A_3 = \{x_2x_3x_5x_7x_8x_9, x_2x_3x_6x_7x_8x_9\}$ and $v_2 = x_2x_3x_5x_6x_9$. From the binomial decompositions

$$\binom{8}{4} = \boxed{\binom{\mathbf{7}}{\mathbf{3}}} + \binom{6}{3} + \binom{5}{3} + \binom{4}{3} + \binom{3}{3}$$
$$\binom{6}{3} = \binom{5}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2}$$
$$\binom{5}{2} = \boxed{\binom{\mathbf{4}}{\mathbf{1}}} + \binom{3}{1} + \binom{2}{1} + \binom{1}{1}$$

one has $a_2 = 5 \le n_2 = |[x_1x_2x_3x_4x_9, v_2]| = (35+4)+1 = 40$.

Finally, we have $A_2 = [x_2x_3x_4x_5x_9, x_2x_3x_5x_6x_9] = \{x_2x_3x_4x_5x_9,\ x_2x_3x_4x_6x_9,\ x_2x_3x_4x_7x_9,\ x_2x_3x_4x_8x_9,\ x_2x_3x_5x_6x_9\}$ and $v_1 = x_1x_{10}x_{11}$. The binomial decompositions

$$\binom{10}{2} = \binom{9}{1} + \binom{8}{1} + \binom{7}{1} + \binom{6}{1} + \binom{5}{1} + \binom{4}{1} + \binom{3}{1} + \binom{2}{1} + \binom{1}{1}$$
$$\binom{9}{1} = \boxed{\binom{\mathbf{8}}{\mathbf{0}} + \binom{\mathbf{7}}{\mathbf{0}} + \binom{\mathbf{6}}{\mathbf{0}} + \binom{\mathbf{5}}{\mathbf{0}} + \binom{\mathbf{4}}{\mathbf{0}} + \binom{\mathbf{3}}{\mathbf{0}} + \binom{\mathbf{2}}{\mathbf{0}} + \binom{\mathbf{1}}{\mathbf{1}}} + \binom{0}{0}$$

imply $a_1 = 7 \le n_1 = |[x_1x_2x_{11}, v_1]| = 8 + 1 = 9$.

Now, we proceed with the construction of the ideal I we are looking for, and so doing we refine the previous bounds for the a_i's.

- The greatest monomial of $A^s(8,3)$ is $x_1x_2x_{11}$. Since p_1 must be equal to 0 and $a_1 = 7 \le n_1 - p_1 = 9$, one can consider the greatest $a_1 = 7$ monomials of $A^s(8,3)$. Such monomials can be obtained by Algorithm 4.6. Hence, we set

$$G(I)_3 = B(x_1x_2x_{11}, x_1x_3x_{11}, x_1x_4x_{11}, x_1x_5x_{11}, x_1x_6x_{11}, x_1x_7x_{11}, x_1x_8x_{11}).$$

- Let us consider the corner $(4,5)$. By Algorithm 4.5, we compute the smallest monomial of $\mathrm{BShad}^2(G(I)_3)_{(4,5)}$, *i.e.*, the monomial $x_1x_6x_7x_8x_9$; whereas, by Algorithm 4.6, we determine the greatest monomial of $A^s(4,5) \setminus \mathrm{BShad}^2(G(I)_3)_{(4,5)}$, *i.e.*, $x_2x_3x_4x_5x_9$. Finally, from the binomial decomposition

$$\binom{8}{4} = \boxed{\binom{\mathbf{7}}{\mathbf{3}}} + \binom{6}{3} + \binom{5}{3} + \binom{4}{3} + \binom{3}{3}$$

it follows that $p_2 = |[x_1x_2x_3x_4x_9, x_2x_3x_4x_5x_9)| = 35$. Hence, $n_2 - p_2 = 40 - 35 = 5$ monomials are available. Therefore, since $a_2 = 5$, we set

$$G(I)_5 = B(x_2x_3x_4x_5x_9, x_2x_3x_4x_6x_9, x_2x_3x_4x_7x_9, x_2x_3x_4x_8x_9, x_2x_3x_5x_6x_9).$$

- Let us consider the corner $(3,6)$. One has $\min \mathrm{BShad}(G(I)_5)_{(3,6)} = x_2x_3x_5x_6x_8x_9$ and $\max(A^s(3,6) \setminus \mathrm{BShad}(G(I)_5)_{(3,6)} = x_2x_3x_5x_7x_8x_9$, and from

$$\binom{8}{5} = \boxed{\binom{\mathbf{7}}{\mathbf{4}}} + \binom{6}{4} + \binom{5}{4} + \binom{4}{4}$$
$$\binom{6}{4} = \binom{5}{3} + \binom{4}{3} + \binom{3}{3}$$
$$\binom{5}{3} = \boxed{\binom{\mathbf{4}}{\mathbf{2}}} + \binom{3}{2} + \binom{2}{2}$$
$$\binom{3}{2} = \boxed{\binom{\mathbf{2}}{\mathbf{1}}} + \binom{1}{1}$$

we have $p_3 = |[x_1x_2x_3x_4x_5x_9, x_2x_3x_5x_7x_8x_9)| = 43$. Hence, $n_3 - p_3 = 45 - 43 = 2$. Since $a_3 = 2$, we set

$$G(I)_6 = B(x_2x_3x_5x_7x_8x_9, x_2x_3x_6x_7x_8x_9).$$

- If one considers the corner $(2,9)$, since $\min \mathrm{BShad}^3(G(I)_6)_{(2,9)} = x_2x_3x_5x_6x_7x_8x_9x_{10}x_{11}$, $\max(A^s(2,9) \setminus \mathrm{BShad}^3(G(I)_6))_{(2,9)} = x_2x_4x_5x_6x_7x_8x_9x_{10}x_{11}$, from

$$\binom{10}{8} = \boxed{\binom{\mathbf{9}}{\mathbf{7}}} + \binom{8}{7} + \binom{7}{7}$$
$$\binom{8}{7} = \boxed{\binom{\mathbf{7}}{\mathbf{6}}} + \binom{6}{6}$$

it follows $p_4 = |[x_1x_2x_3x_4x_5x_6x_7x_8x_{11}, x_2x_4x_5x_6x_7x_8x_9x_{10}x_{11})| = 43$.
So $n_4 - p_4 = \binom{10}{8} - p_4 = 45 - 43 = 2$. Hence, since $a_4 = 2$, we can set

$$G(I)_9 = B(x_2x_4x_5x_6x_7x_8x_9x_{10}x_{11}, x_3x_4x_5x_6x_7x_8x_9x_{10}x_{11}).$$

The Betti table of the squarefree strongly stable I just constructed is the following one:

		0	1	2	3	4	5	6	7	8
3	:	42	217	553	861	875	587	252	63	7
4	:	–	–	–	–	–	–	–	–	–
5	:	13	39	45	24	5	–	–	–	–
6	:	2	6	6	2	–	–	–	–	–
7	:	–	–	–	–	–	–	–	–	–
8	:	–	–	–	–	–	–	–	–	–
9	:	2	4	2	–	–	–	–	–	–

We close the Section with an example that illustrates a *situation* where the construction of a squarefree strongly stable ideal is not possible.

Example 4.3.32 Let $n = 10$, $r = 3$, $\mathcal{C} = \{(6, 2), (5, 4), (3, 7)\}$ and $a = (a_1, a_2, a_3) = (2, 1, 4)$.

We have $|A^s(3, 7)| = \binom{9}{6} = 84$, so it is possible to manage $a_3 = 4 \leq 84$ monomials.

Let us consider the smallest four monomials of $A^s(3, 7)$:

$$A_2 = \{x_3x_4x_5x_7x_8x_9x_{10}, x_3x_4x_6x_7x_8x_9x_{10}, x_3x_5x_6x_7x_8x_9x_{10}, x_4x_5x_6x_7x_8x_9x_{10}\}.$$

Let us try to get the smallest monomial $z \in A^s(5, 4)$ such that we have $x_3x_4x_5x_6x_8x_9x_{10} \notin \mathrm{BShad}(z)_{(3,7)}$. It is $z = x_2x_7x_8x_9$. Now, we compute $|[x_1x_2x_3x_9, z]|$ as bound for a_2:

$$\binom{8}{3} = \boxed{\binom{\mathbf{7}}{\mathbf{2}}} + \binom{6}{2} + \binom{5}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2}$$
$$\binom{6}{2} = \boxed{\binom{\mathbf{5}}{\mathbf{1}} + \binom{\mathbf{4}}{\mathbf{1}} + \binom{\mathbf{3}}{\mathbf{1}} + \binom{\mathbf{2}}{\mathbf{1}}} + \binom{1}{1}$$

So we have $n_2 = 21 + (5 + 4 + 3 + 2) + 1 = 36$ monomials greater than z. Hence, $a_2 = 1 \leq 36$. Note that if z does not exist, then it is clear that we can not go on.

Now, we try to verify the bound for a_1 taking into account the previous results. Consider the monomial $z \in A^s(5, 4)$, and take the greatest monomial w of $A^s(6, 2)$ such that $z \notin \mathrm{BShad}(w)_{(3,7)}$. It is $w = x_1x_8$. We can note that w is the smallest monomial of $A^s(6, 2)$, *i.e.*, $|[x_1x_8, w]| = 1$.

Hence, we have that $a_1 \leq 1$. For this reason the requested value for $a_1 = 2$ is not admissible and there does not exist any squarefree monomial ideal I of $K[x_1, \ldots, x_{10}]$ such that $\mathrm{Corn}(I) = \mathcal{C}$ and $a(I) = a$.

Nevertheless, there exists a squarefree monomial ideal J of S such that $\mathrm{Corn}(J) = \mathcal{C}$ and $a(J) = (1, 1, 4)$.

4.4 Corners of 2–spread strongly stable ideals

In this Section, we analyze the extremal Betti numbers of 2–spread strongly stable ideals in the polynomial ring $S = K[x_1, \ldots, x_n]$. If $\mathcal{S}_{2,n}$ is the set of all 2–spread strongly stable ideals in S, we determine the largest number of corners allowed for an ideal $I \in \mathcal{S}_{2,n}$. It is worth to point out that if $t \geq 2$, then a t–spread strongly stable ideal has initial degree ≥ 2.

For our purpose, we focus our attention on the ideals $I \in \mathcal{S}_{2,n}$ such that all the entries of their corner values sequence $a(I)$ are equal to 1, *i.e.*, every extremal Betti numbers of I equals 1. The subset of $\mathcal{S}_{2,n}$ consisting of such 2–spread strongly stable ideals will be denoted by $\mathcal{S}_{2,n,\mathbf{1}}$.

The study of this problem has shown that one has to consider two cases:

$$n \text{ odd}, \ n \text{ even}.$$

Firstly, we analyze the *odd case.*

Discussion 4.4.1 Let us consider $S = K[x_1, \dots, x_n]$, with $n \geq 3$ odd integer.

For $n = 3$, the only 2–spread strongly stable ideal $I \in \mathcal{S}_{2,n,\mathbf{1}}$ is $I = (x_1x_3)$ with $\mathrm{Corn}(I) = \{(0,2)\}$.

For $n = 5$, the only 2–spread strongly stable ideal $I \in \mathcal{S}_{2,n,\mathbf{1}}$ is $I = B_2(x_1x_5)$ with $\mathrm{Corn}(I) = \{(2,2)\}$.

For $n = 7, 9, 11$, the monomials which determine the largest number of admissible corners of a 2–spread strongly stable ideal in $\mathcal{S}_{2,n,\mathbf{1}}$ with a corner in degree 2 are the bold highlighted ones in Table 4.6 :

	$\mathbf{x_1x_7}$
$x_1x_5x_7$	$\mathbf{x_2x_4x_7}$

(a) $\mathbf{n = 7}$

	$\mathbf{x_1x_9}$
$x_1x_7x_9$	$\mathbf{x_2x_4x_9}$
$x_2x_4x_7x_9$	$\mathbf{x_2x_5x_7x_9}$

(b) $\mathbf{n = 9}$

	$\mathbf{x_1x_{11}}$
$x_1x_9x_{11}$	$\mathbf{x_2x_4x_{11}}$
$x_2x_4x_9x_{11}$	$\mathbf{x_2x_5x_7x_{11}}$
$x_2x_5x_7x_9x_{11}$	$\mathbf{x_3x_5x_7x_9x_{11}}$

(c) $\mathbf{n = 11}$

Table 4.6: 2–spread monomial generators for $n = 7, 9, 11$

In each of these cases, the finitely generated 2–spread Borel ideal with the bold highlighted monomials as generators is the ideal we are looking for.

For every $1 \leq d \leq n$, let us denote by $\mathrm{Mon}_d^s(S)$ the set of all squarefree monomials of degree d of S. We can order $\mathrm{Mon}_d^s(S)$ with the *squarefree lexicographic order* $\geq_{\mathrm{slex}}$ [AHH98]. More precisely, let

$$u = x_{i_1}x_{i_2}\cdots x_{i_d}, \qquad v = x_{j_1}x_{j_2}\cdots x_{j_d},$$

with $1 \leq i_1 < i_2 < \cdots < i_d \leq n$, $1 \leq j_1 < j_2 < \cdots < j_d \leq n$, be squarefree monomials of degree d in S, then

$$u >_{\mathrm{slex}} v \qquad \text{if} \qquad i_1 = j_1, \dots, i_{s-1} = j_{s-1} \qquad \text{and} \qquad i_s < j_s,$$

for some $1 \leq s \leq d$.

From now on, we assume that the sets $\mathrm{Mon}_d^s(S)$ $(1 \leq d \leq n)$ are endowed with the ordering $\geq_{\mathrm{slex}}$.

Theorem 4.4.2 *Let $n \geq 11$ be odd. A 2–spread strongly stable ideal $S = K[x_1, \dots, x_n]$ of initial degree 2 and with a corner in degree 2 can have at most $\frac{n-3}{2}$ corners.*

Proof. We will prove the existence of a 2–spread strongly stable ideal I of S generated in degrees $2, 3, \dots, \frac{n-1}{2}$ such that $|\mathrm{Corn}(I)| = \frac{n-3}{2}$ and $a(I) = (1, 1, \dots, 1)$.

Set $G(I)_2 = B_2(x_1x_n)$. One can observe that the monomial

$$x_3x_5\cdots x_{n-2}x_n$$

is a 2–spread monomial of the largest degree which is not multiple of x_1x_n. Moreover it is also the smallest 2–spread monomial of $M_{n,\frac{n-1}{2},2}$.

Claim 1. There exist 2–spread monomials $w_i \in S$, $i = 1, \dots, \frac{n-7}{2} = \frac{n-3}{2} - 2$, such that

$$I = B_2(x_1x_n, w_1, \dots, w_{\frac{n-7}{2}}, x_3x_5\cdots x_{n-2}x_n).$$

Proof of the Claim. We will verify that $\mathrm{wd}(1\text{-gap}(x_1x_n)) = n-2 \geq 9$ allows us to prove the existence of the desired w_i's.

The greatest 2–spread monomial not belonging to $\mathrm{Shad}_2(B_2(x_1x_n))$ is $x_2x_4x_n$. Hence, we choose $w_1 = x_2x_4x_n$. Therefore, since $\mathrm{wd}(2\text{-gap}(w_1)) = n-5 \geq 6$, we set $w_2 = x_2x_5x_7x_n$. It is the greatest 2–spread monomial not belonging to $\mathrm{Shad}_2(B_2(x_2x_4x_n))$.

Now, $\mathrm{wd}(3\text{-gap}(w_2)) = n-8 \geq 3$.

Let us distinguish the following cases:

$$n = 11, 13, 15 \text{ and } n \geq 17.$$

If $n = 11$, then $x_3x_5x_7x_9x_{11}$ is the greatest 2–spread monomial of degree $5 = \frac{11-1}{2}$ not belonging to $\mathrm{Shad}_2(B_2(x_2x_5x_7x_{11}))$ and the smallest 2–spread monomial in $M_{11,5,2}$. Hence, $I = B_2(x_1x_{11}, w_1, w_2, x_3x_5x_7x_9x_{11})$ is the 2–spread strongly stable ideal we are looking for.

If $n = 13$, then $w_3 = x_2x_5x_8x_{10}x_{13}$ is the greatest 2–spread monomial of degree 5 not belonging to $\mathrm{Shad}_2(B_2(x_2x_5x_7x_{13}))$. On the other hand, the greatest 2–spread monomial of degree $6 = \frac{13-1}{2}$ not belonging to $\mathrm{Shad}_2(B_2(w_3))$ is $x_3x_5x_7x_9x_{11}x_{13}$. Hence, $I = B_2(x_1x_{13}, w_1, w_2, w_3, x_3x_5x_7x_9x_{11}x_{13})$ is the wished 2–spread strongly stable ideal.

Similarly, if $n = 15$, then $w_3 = x_2x_5x_8x_{10}x_{15}$ is the greatest 2–spread monomial of degree 5 not belonging to $\mathrm{Shad}_2(B_2(x_2x_5x_7x_{15}))$. Moreover, the greatest 2–spread monomial of degree 6 not belonging to $\mathrm{Shad}_2(B_2(w_3))$ is $w_4 = x_2x_5x_8x_{11}x_{13}x_{15}$. Finally, the greatest 2–spread monomial of degree $7 = \frac{15-1}{2}$ not belonging to $\mathrm{Shad}_2(B_2(w_4))$ is $x_3x_5x_7x_9x_{11}x_{13}x_{15}$. Therefore, $I = B_2(x_1x_{15}, w_1, w_2, w_3, w_4, x_3x_5x_7x_9x_{11}x_{13}x_{15})$ is the 2–spread strongly stable ideal we are looking for.

Now, it is worth to point out that in the case when $n = 15$ a monomial generator with $x_2x_5x_8x_{11}$ as divisor appears for the first time. Such a monomial will play a crucial role for the proof of the claim.

Let $n \geq 17$. First of all, we set $w_3 = x_2x_5x_8x_{10}x_n$ and $w_4 = x_2x_5x_8x_{11}x_{13}x_n$. Then, one can observe that the number q of all 2–spread monomials z with $\max(z) = n$ and $x_2x_5x_8x_{11}$ as a divisor depends on the integer $n-11$. Indeed, one can quickly verify that q is bounded by the integer $m = \lfloor \frac{n-11-2-1}{2} \rfloor$. We will prove that $q = m$.

Since $n-14 \geq 3$ is odd, there exists a $5\text{-gap}(w_4)$ which allows us to get the smallest monomial of $\mathrm{Shad}_2(B_2(w_4))$, *i.e.*, $x_2x_5x_8x_{11}x_{13}x_{n-2}x_n$.

Let us distinguish two cases: $n = 17$, $n > 17$.

Let $n = 17$. Then $n - 14 = 3$ and $\lfloor \frac{n-14}{2} \rfloor = 1$. Indeed the greatest monomial not belonging to $\mathrm{Shad}_2(B_2(w_4))$ is $w_5 = x_2x_5x_9x_{11}x_{13}x_{15}x_{17}$. There exists only w_4 which is divisible by u.

Now, let $n > 17$. In such a case, the greatest monomial not belonging to $\mathrm{Shad}_2(B_2(w_4))$ is $w_5 = x_2x_5x_8x_{11}x_{14}x_{16}x_n$. Since $\mathrm{wd}(6\text{-gap}(w_5)) = n-17 = n-14-3 = \mathrm{wd}(5\text{-gap}(w_4))-3$ and $\mathrm{wd}(4\text{-gap}(w_5)) = 14 - 11 - 1 = 2 = \mathrm{wd}(4\text{-gap}(w_4)) + 1$, then $\lfloor \frac{n-14-3+1}{2} \rfloor = m - 1$. Hence, if $m - 1 > 1$ one obtains $w_6 = x_2x_5x_8x_{11}x_{14}x_{17}x_{19}x_n$.

After $m-1$ iterations, we have $\lfloor \frac{n-14-2m+2}{2} \rfloor = \lfloor \frac{n-12-(n-15)}{2} \rfloor = \lfloor \frac{3}{2} \rfloor = 1$. This assures that we can construct the last 2–spread monomial not belonging to $\mathrm{Shad}_2(B_2(w_{3+m-1}))$. It is $w_{3+m} = x_2x_5x_9x_{11}x_{13}x_{15}\cdots x_n$. It is the greatest 2–spread monomial not belonging to $\mathrm{Shad}_2(B_2(w_{3+m-1}))$.
Finally, we can observe that the monomial $x_3x_5x_7x_9\cdots x_n$ is the greatest 2–spread monomial not belonging to the $\mathrm{Shad}_2(B_2(x_2x_5x_9x_{11}x_{13}x_{15}\cdots x_n))$ and the smallest 2–spread monomial in $M_{n,\frac{n-1}{2},2}$.

Proceeding in this way at the end we get

$$1 + 3 + m + 2 = 6 + \lfloor \frac{n-14}{2} \rfloor = 6 + \frac{n-15}{2} = \frac{n-3}{2}$$

suitable monomials. The claim follows.

The construction of these monomials together with Theorem 4.1.12 guarantees that there exists an ideal $I \in \mathcal{S}_{2,n,\mathbf{1}}$ with a corner in degree 2 in S with $|\mathrm{Corn}(I)| = \deg(x_3x_5x_7x_9\cdots x_n) - 2 + 1 = \frac{n-1}{2} - 2 + 1 = \frac{n-3}{2}$.

More precisely,

$$\mathrm{Corn}(I) = \left\{ (k_i, \ell_i) : \quad k_i = n - 2(\ell_i - 1) - 1, \quad \ell_i = 2 + (i-1), \quad i = 1, \ldots, \frac{n-3}{2} \right\} =$$
$$= \left\{ (n-3, 2), (n-5, 3), \ldots, (2, \frac{n-1}{2}) \right\}.$$

The proof points out that there exist $\frac{n-3}{2}$ monomials of S of degrees $2, 3, \ldots, \frac{n-1}{2}$ each of which determines a corner. Moreover, the structure of $x_3x_5\cdots x_n$ assures that there does not exist a 2–spread monomial of degree $\deg(x_3x_5\cdots x_n) + 1$ that gives a contribution for the computation of a corner. Hence, $\frac{n-3}{2}$ is the maximal admissible number of corners of a 2–spread strongly stable ideal of S of initial degree 2. $\square$

The monomial generators x_1x_n, w_1, $\ldots$, $w_{\frac{n-7}{2}}$, $x_3x_5\cdots x_{n-2}x_n$ will be called 2–*spread basic monomials.*

For later use, we need to define a partial order $\succeq$ on the set $\mathrm{Mon}^s(S)$ of all squarefree monomials of S. More precisely, let u, v be two squarefree monomials of S, we say that

$$u \succeq v$$

- if $\deg u = \deg v$ and $u \geq_{\text{slex}} v$, or
- if $\deg u < \deg v$ and $u = x_i v/w$, with $i \notin \text{supp}(v)$, w divides v and $i < r$, for some $r \in \text{supp}(w)$.

We set $u \succ v$ if $u \succeq v$ and $u \neq v$.

For instance, if one considers the 2–spread monomials $x_1x_8, x_2x_4x_6, x_2x_6x_8 \in K[x_1, \ldots, x_8]$, then $x_1x_8 \succ x_2x_6x_8$. Indeed, $\deg x_1x_8 < \deg x_2x_6x_8$ and $x_1x_8 = x_1(x_2x_6x_8)/x_2x_6$.

On the contrary, $x_1x_8 \not\succeq x_2x_4x_6$ and $x_2x_4x_6 \not\succeq x_1x_8$; whereas $x_2x_4x_6 \succ x_2x_6x_8$.

With the same notation as in Theorem 4.4.2, setting

$$A = \{x_1x_n, w_1, \ldots, w_{\frac{n-7}{2}}, x_3x_5 \cdots x_{n-2}x_n\},$$

then

$$x_1x_n \succ w_1 \succ \ldots \succ w_{\frac{n-7}{2}} \succ x_3x_5 \cdots x_{n-2}x_n.$$

Now, we give a nice explicit description of the finitely generated Borel 2–spread ideal of $\mathcal{S}_{2,n,\mathbf{1}}$ of initial degree 2 with the maximal number of corners, for all odd $n \geq 5$. We will denote it by $B_{2,n,\mathbf{1}}$.

Discussion 4.4.3 Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring, with $n \geq 5$ odd integer. In what follows both Discussion 4.4.1 and Theorem 4.4.2 (proof) will be crucial.

Firstly, let $n = 5, 7, 9$. In such cases, the finitely generated 2–spread Borel ideals $B_{2,n,\mathbf{1}}$ of $\mathcal{S}_{2,n,\mathbf{1}}$ are described in Table 4.7.

n	**Corner sequence**	**2–spread strongly stable ideal**
5	$\{(2,2)\}$	$B_{2,5,\mathbf{1}} = B_2(x_1x_5) = (x_1x_3, x_1x_4, x_1x_5)$
7	$\{(4,2),(2,3)\}$	$B_{2,7,\mathbf{1}} = B_2(x_1x_7, x_2x_4x_7)$
9	$\{(6,2),(4,3),(2,4)\}$	$B_{2,9,\mathbf{1}} = B_2(x_1x_9, x_2x_4x_9, x_2x_5x_7x_9)$

Table 4.7: 2–spread corner sequences for $n = 5, 7, 9$.

One can observe that the monomial of the type x_1x_n ($n = 5, 7, 9$) appears as 2–spread Borel generator in all three ideals $B_{2,5,\mathbf{1}}$, $B_{2,7,\mathbf{1}}$ and $B_{2,9,\mathbf{1}}$; the monomial $x_2x_4x_n$ ($n = 7, 9$) appears as 2–spread Borel generator in the ideals $B_{2,7,\mathbf{1}}$ and $B_{2,9,\mathbf{1}}$; whereas the monomial $x_2x_5x_7x_9$ appears only in the ideal $B_{2,9,\mathbf{1}}$ as 2–spread Borel generator.

For $n \geq 11$, the monomials in the following set

$$\{x_1x_n, x_2x_4x_n, x_2x_5x_7x_n\} \tag{4.4.1}$$

will be always 2–spread Borel generators for $B_{2,n,\mathbf{1}}$.

Let us consider the case $n = 11$. From Theorem 4.4.2 (proof) we have to introduce the monomial $x_3x_5x_7x_9x_{11}$ to complete the minimal system of monomial generators of $B_{2,11,\mathbf{1}}$.

Moreover, if $n \equiv 11 \pmod 6$, we need to add $r_1 = \frac{n-11}{6} + 1$ monomials of the type

$$\prod_{i=0}^{2k-1} x_{2+3i}x_{j+2}x_{j+4}\cdots x_{n-2}x_n, \quad j = 6k+1, \quad k = 0,\ldots,r_1-1 \tag{4.4.2}$$

to the set in (4.4.1) to get the minimal system of monomial generators of $B_{2,n,\mathbf{1}}$. We refer to them as the *right-form* basic monomials.

Note that, setting $\prod_{i=0}^{2k-1} x_{2+3i} = 1$ for $k = 0$, then $\prod_{i=0}^{2k-1} x_{2+3i}x_{j+2}x_{j+4}\cdots x_{n-2}x_n = x_3x_5x_7x_9\cdots x_{n-2}x_n$.

Hence, the monomials in

$$\{x_1x_n, x_2x_4x_n, x_2x_5x_7x_n, \prod_{i=0}^{2k-1} x_{2+3i}x_{j+2}\cdots x_{n-2}x_n, j = 6k+1, k = 0,\ldots,r_1-1\} \tag{4.4.3}$$

will belong to the minimal set of monomial generators of $B_{2,n,\mathbf{1}}$, for all odd integer $n \geq 11$.

Let us consider $n = 13$. In such a case one has the monomial $x_2x_5x_8x_{10}x_{13}$. Such a monomial is smaller than all monomials in (4.4.1) and greater than the *right–form* ones, with respect to $\succeq$.

In general, if $n \equiv 13 \pmod 6$, we need to add $r_2 = \frac{n-13}{6} + 1$ monomials of the type

$$x_2x_5x_8 \prod_{i=0}^{2k-1} x_{11+3i}x_jx_n, \quad j = 6k+10, \quad k = 0,\ldots,r_2-1 \tag{4.4.4}$$

to the set in (4.4.3) to get the minimal system of monomial generators of $B_{2,n,\mathbf{1}}$. We refer to them as the *first-left-form* basic monomials.

Note that, setting $\prod_{i=0}^{2k-1} x_{11+3i} = 1$ for $k = 0$, then $x_2x_5x_8\prod_{i=0}^{2k-1} x_{11+3i}x_jx_n = x_2x_5x_8x_{10}x_n$. Hence, the monomials in (4.4.4) together with the ones in (4.4.3) will belong to the minimal set of monomial generators of $B_{2,n,\mathbf{1}}$, for all odd $n \geq 13$.

Now, let us consider $n = 15$. In such a case one has the monomial $x_2x_5x_8x_{11}x_{13}x_{15}$. Such a monomial is greater than the *right-form* ones.

In general, if $n \equiv 15 \pmod 6$ then $r_3 = \frac{n-15}{6} + 1$ monomials of the type

$$x_2x_5x_8x_{11} \prod_{i=0}^{2k-1} x_{14+3i}x_jx_n, \quad j = 6k+13, \quad k = 0,\ldots,r_3-1 \tag{4.4.5}$$

will belong to the minimal set of monomial generators of $B_{2,n,\mathbf{1}}$. We refer to them as the *second-left-form* basic monomials.

Note that, setting $\prod_{i=0}^{2k-1} x_{14+3i} = 1$ for $k = 0$, then $x_2x_5x_8x_{11}\prod_{i=0}^{2k-1} x_{14+3i}x_jx_n = x_2x_5x_8x_{11}x_{13}x_n$.

Finally, the monomials in (4.4.5) together with the ones in (4.4.4) and the ones in (4.4.3) will determine the minimal set of monomial generators of $B_{2,n,\mathbf{1}}$, for all odd $n \geq 15$.

Remark 4.4.4 One can notice, that given an odd integer $n \geq 15$ in order to determine the set A of the 2–spread basic monomials, one can firstly consider the values n, $n-2$ and $n-4$. Then, if one writes down all the monomials (divisible by x_n) described in (4.4.5), (4.4.4), (4.4.2) *via* the integers n, $n-2$ and $n-4$ respectively, together with the monomials in (4.4.1), one gets:

$$|A| = (\frac{n-15}{6}+1)+(\frac{n-2-13}{6}+1)+(\frac{n-4-11}{6}+1)+3 = 6+\frac{n-15}{2} = \frac{n-3}{2}$$

which is the number of the desired generators.

The next example will illustrate Remark 4.4.4.

Example 4.4.5 Let us consider the polynomial ring $S = K[x_1, \dots, x_{21}]$. We want to construct the finitely generated 2–spread strongly stable ideal $I \in \mathcal{S}_{2,21,\mathbf{1}}$ with the greatest number of corners and such that $\operatorname{indeg}(I) = 2$, *i.e.*, $I = B_{2,21,\mathbf{1}}$.

One has $|\operatorname{Corn}(I)| = \frac{n-3}{2} = 9$ and

$$\operatorname{Corn}(I) = \{(18,2),(16,3),(14,4),(12,5),(10,6),(8,7),(6,8),(4,9),(2,10)\}.$$

In order to determine the 2–spread basic monomials that determine the minimal system of monomial generators $G(I)$ we proceed as follows.

Step 1. At first, we consider the 2–spread basic monomials x_1x_{21}, $x_2x_4x_{21}$ and $x_2x_5x_7x_{21}$.

Step 2. Since $n = 21 \equiv 15 \pmod 6$, we have $\frac{21-15}{6}+1 = 2$ second-left-form basic monomials of the type

$$x_2x_5x_8x_{11}\prod_{i=0}^{2k-1} x_{14+3i}x_jx_n,$$

with $j = 6k+13$ for $k = 0,1$. They are $x_2x_5x_8x_{11}x_{13}x_{21}$ ($k = 0$, $j = 13$) and $x_2x_5x_8x_{11}x_{14}x_{17}x_{19}x_{21}$ ($k = 1$, $j = 19$).

Step 3. Let us consider $n-2 = 19$. Since $n-2 = 19 \equiv 13 \pmod 6$, we have $\frac{19-13}{6}+1 = 2$ first-left-form basic monomials of the type

$$x_2x_5x_8\prod_{i=0}^{2k-1} x_{11+3i}x_jx_n,$$

with $j = 6k+10$ for $k = 0,1$. They are $x_2x_5x_8x_{10}x_{21}$ ($k = 0$, $j = 10$) and $x_2x_5x_8x_{11}x_{14}x_{16}x_{21}$ ($k = 1$, $j = 16$).

Step 4. Let us consider $n-4 = 17$. Since $n-4 = 17 \equiv 11 \pmod 6$, then we have $\frac{17-11}{6}+1 = 2$ right-form basic monomials of the type

$$\prod_{i=0}^{2k-1} x_{2+3i}x_{j+2}x_{j+4}\cdots x_{n-2}x_n,$$

with $j = 6k+1$ for $k = 0, 1$. They are $x_3x_5x_7x_9x_{11}x_{13}x_{15}x_{17}x_{19}x_{21}$ ($k = 0$, $j = 1$) and $x_2x_5x_9x_{11}x_{13}x_{15}x_{17}x_{19}x_{21}$ ($k = 1$, $j = 7$).

Finally, ordering the monomials in Steps 1–4 with respect to $\succeq$, we have the ideal

$$\begin{aligned} I = B_2(&x_1x_{21}, x_2x_4x_{21}, x_2x_5x_7x_{21}, x_2x_5x_8x_{10}x_{21}, x_2x_5x_8x_{11}x_{13}x_{21}, x_2x_5x_8x_{11}x_{14}x_{16}x_{21}, \\ &x_2x_5x_8x_{11}x_{14}x_{17}x_{19}x_{21}, x_2x_5x_9x_{11}x_{13}x_{15}x_{17}x_{19}x_{21}, x_3x_5x_7x_9x_{11}x_{13}x_{15}x_{17}x_{19}x_{21}). \end{aligned}$$

From Theorem 4.4.2 and Discussion 4.4.3, the next result follows.

Theorem 4.4.6 *Let $n \geq 11$ an odd integer and $\ell_1 = 2$. Given $\frac{n-3}{2}$ pairs of positive integers*

$$(k_1, \ell_1), (k_2, \ell_2), \dots, (k_{\frac{n-3}{2}}, \ell_{\frac{n-3}{2}}), \tag{4.4.6}$$

with $1 \leq k_{\frac{n-3}{2}} < k_{\frac{n-3}{2}-1} < \cdots < k_1 \leq n-3$ and $2 = \ell_1 < \ell_2 < \cdots < \ell_{\frac{n-3}{2}} \leq \frac{n-1}{2}$, then there exists a 2–spread strongly stable ideal I of S of initial degree ℓ_1 and with the pairs in (4.4.6) as corners if and only if $k_i + 2(\ell_i - 1) + 1 = n$, for $i = 1, \dots, \frac{n-3}{2}$.

Remark 4.4.7 For an arbitrary monomial ideal I, let I_j be the j–th graded component of I. Following [AC19f], we call the set of t–spread monomials in I_j, the t–spread part of I_j and denote it by $[I_j]_t$. A special class of t–spread strongly stable ideals consists of t-spread lex ideals, which are defined as follows [AC19f].

A subset L of $M_{n,d,t}$ is called a t–spread lex set, if for all $u \in L$ and for all $v \in M_{n,d,t}$ with $v >_{\text{lex}} u$, it follows that $v \in L$. A t–spread monomial ideal I is called a t–spread lex ideal, if $[I_j]_t$ is a t–spread lex set for all j.

It is clear that the 2–spread strongly stable ideal in Theorem 4.4.2 is a 2–spread lex ideal.

Now, we analyze the *even case*. The development will be very similar to the odd case. We include it for completeness and for highlighting the differences with the odd case.

Discussion 4.4.8 Let us consider $S = K[x_1, \dots, x_n]$, with $n \geq 4$ even.

For $n = 4$, the only 2–spread strongly stable ideal $I \in \mathcal{S}_{2,n}$ in S is $I = B_2(x_1x_4)$ with $\mathrm{Corn}(I) = \{(1, 2)\}$. For $n = 6, 8, 10, 12, 14$, the monomials which determine the maximal number of admissible corners of a 2–spread strongly stable ideal in $\mathcal{S}_{2,n,\mathbf{1}}$ with a corner in degree 2 are the bold highlighted ones in Table 4.8:

	$\mathbf{x_1x_6}$
$x_1x_4x_6$	$\mathbf{x_2x_4x_6}$

(a) $\mathbf{n = 6}$

	$\mathbf{x_1x_8}$
$x_1x_4x_8$	$\mathbf{x_2x_4x_8}$

(b) $\mathbf{n = 8}$

	$\mathbf{x_1x_{10}}$
$x_1x_8x_{10}$	$\mathbf{x_2x_4x_{10}}$
$x_2x_4x_8x_{10}$	$\mathbf{x_2x_5x_7x_{10}}$

(c) $\mathbf{n = 10}$

	$\mathbf{x_1x_{12}}$
$x_1x_{10}x_{12}$	$\mathbf{x_2x_4x_{12}}$
$x_2x_4x_{10}x_{12}$	$\mathbf{x_2x_5x_7x_{12}}$
$x_2x_5x_7x_{10}x_{12}$	$\mathbf{x_2x_5x_8x_{10}x_{12}}$

(d) $\mathbf{n = 12}$

	$\mathbf{x_1x_{14}}$
$x_1x_{12}x_{14}$	$\mathbf{x_2x_4x_{14}}$
$x_2x_4x_{12}x_{14}$	$\mathbf{x_2x_5x_7x_{14}}$
$x_2x_5x_7x_{12}x_{14}$	$\mathbf{x_2x_5x_8x_{10}x_{14}}$
$x_2x_5x_8x_{10}x_{12}x_{14}$	$\mathbf{x_2x_6x_8x_{10}x_{12}x_{14}}$

(e) $\mathbf{n = 14}$

Table 4.8: 2–spread monomial generators for $n = 6, 8, 10, 12, 14$

In each of the cases described in Table 4.8, the finitely generated 2-Borel ideal with the bold highlighted monomials as generators is the wished ideal.

Theorem 4.4.9 *Let $n \geq 14$ be even. A 2–spread strongly stable ideal $S = K[x_1, \ldots, x_n]$ of initial degree 2 and with a corner in degree 2 can have at most $\frac{n-4}{2}$ corners.*

Proof. The proof is verbatim the same of Theorem 4.4.2.
We prove the existence of a 2–spread strongly stable ideal I of S generated in degrees $2, 3, \ldots, \frac{n-2}{2}$ such that $|\mathrm{Corn}(I)| = \frac{n-4}{2}$ and $a(I) = (1, 1, \ldots, 1)$.

Set $G(I)_2 = B_2(x_1x_n)$. One can observe that $\mathrm{wd}(1\text{-gap}(x_1x_n)) = n - 2 \geq 12$. The greatest 2–spread monomial not belonging to $\mathrm{Shad}_2(B_2(x_1x_n))$ is $x_2x_4x_n$. Hence, we set $w_1 = x_2x_4x_n$.

Note that the monomial

$$x_2x_6x_8 \cdots x_{n-2}x_n,$$

of degree $\frac{n-2}{2}$ is a 2–spread monomial of $\mathrm{Mon}^s(S)$ of the largest degree which is not multiple both of x_1x_n and of w_1.

Claim 2. We prove the existence of certain 2–spread monomials $w_i \in S$, for $i = 2, \ldots, \frac{n-8}{2} = \frac{n-4}{2} - 2$, such that

$$I = B_2(x_1x_n, w_1, \ldots, w_{\frac{n-8}{2}}, x_2x_6x_8 \cdots x_{n-2}x_n).$$

Proof of the Claim. Firstly, since $\mathrm{wd}(2\text{-gap}(w_1)) = n - 5 \geq 9$, we set $w_2 = x_2x_5x_7x_n$. On the other hand, $\mathrm{wd}(3\text{-gap}(w_2)) = n - 8 \geq 6$. Then we set $w_3 = x_2x_5x_8x_{10}x_n$ and $\mathrm{wd}(4\text{-gap}(w_3)) = n - 11 \geq 3$. Let us distinguish the following cases:

$$n = 14, 16, 18 \text{ and } n \geq 20.$$

If $n = 14$, then $x_2x_6x_8x_{10}x_{12}x_{14}$ is the greatest 2–spread monomial of degree $6 = \frac{14-2}{2}$ not belonging to $\mathrm{Shad}_2(B_2(x_2x_5x_8x_{10}x_{14}))$. Hence, $I = B_2(x_1x_{14}, w_1, w_2, w_3, x_2x_6x_8x_{10}x_{12}x_{14})$ is the 2–spread strongly stable ideal we are looking for.

If $n = 16$, then $w_4 = x_2x_5x_8x_{11}x_{13}x_{16}$ is the greatest 2–spread monomial of degree 6 not belonging to $\mathrm{Shad}_2(B_2(x_2x_5x_8x_{10}x_{16}))$. Finally, we can construct the greatest 2–spread monomial of degree $7 = \frac{16-2}{2}$ not belonging to $\mathrm{Shad}_2(B_2(w_4))$. It is $x_2x_6x_8x_{10}x_{12}x_{14}x_{16}$. Hence, $I = B_2(x_1x_{16}, w_1, w_2, w_3, w_4, x_2x_6x_8x_{10}x_{12}x_{14}x_{16})$ is the 2–spread strongly stable ideal we are looking for.

If $n = 18$, then $w_4 = x_2x_5x_8x_{11}x_{13}x_{18}$ is the greatest 2–spread monomial of degree 6 not belonging to $\mathrm{Shad}_2(B_2(x_2x_5x_8x_{10}x_{18}))$. Moreover, the greatest 2–spread monomial of degree 7 not belonging to $\mathrm{Shad}_2(B_2(w_4))$ is $w_5 = x_2x_5x_8x_{11}x_{14}x_{16}x_{18}$. Finally, the greatest 2–spread monomial of degree $8 = \frac{18-2}{2}$ not into $\mathrm{Shad}_2(B_2(w_5))$ is $x_2x_6x_8x_{10}x_{12}x_{14}x_{16}x_{18}$. Hence, $I = B_2(x_1x_{18}, w_1, w_2, w_3, w_4, w_5, x_2x_6x_8x_{10}x_{12}x_{14}x_{16}x_{18})$ is the desired 2–spread strongly stable ideal.

Also in this case, the monomial generators with $u = x_2x_5x_8x_{11}$ as divisor will play a crucial role in the proof. We note that when $n = 18$ u does not divide any monomial generators.

Let $n \geq 20$. We set $w_4 = x_2x_5x_8x_{11}x_{13}x_n$ and $w_5 = x_2x_5x_8x_{11}x_{14}x_{16}x_n$. We observe that the number q of all 2–spread monomials z with $\max(z) = n$ and $x_2x_5x_8x_{11}$ as a divisor depends on the integer $n - 11$. Indeed, in this case q is bounded by the integer $m = \lfloor\frac{n-11-2-2}{2}\rfloor$. We will prove that $q = m$. We can observe that $n - 15 \geq 5$ is odd. This assures the existence of a 6-gap(w_5) which allows us to obtain the smallest monomial of $\mathrm{Shad}_2(B_2(w_5))$, *i.e.*, $x_2x_5x_8x_{11}x_{14}x_{16}x_{n-2}x_n$.

Let $n = 20$. Then $n-15 = 5$ and $\lfloor\frac{n-15}{2}\rfloor = 2$. Indeed the greatest monomial not belonging to $\mathrm{Shad}_2(B_2(w_5))$ is $w_6 = x_2x_5x_8x_{12}x_{14}x_{16}x_{18}x_{20}$. Hence there exist two monomials, w_4 and w_5, that are divisible by u.

Now, let us consider $n > 20$. Hence the greatest monomial not belonging to $\mathrm{Shad}_2(B_2(w_5))$ is $w_6 = x_2x_5x_8x_{11}x_{14}x_{17}x_{19}x_n$. One can observe that $\mathrm{wd}(7\text{-gap}(w_6)) = n-20 = n-17-3 = \mathrm{wd}(6\text{-gap}(w_5)) - 3$ and $\mathrm{wd}(5\text{-gap}(w_5)) = 17 - 14 - 1 = 2 = \mathrm{wd}(5\text{-gap}(w_4)) + 1$. This leads that $\lfloor\frac{n-15-3+1}{2}\rfloor = m - 1$.

Hence, if $m - 1 > 1$ we obtain the 2–spread monomial $w_7 = x_2x_5x_8x_{11}x_{14}x_{17}x_{20}x_{22}x_n$.

After $m - 1$ iterations we have $\lfloor\frac{n-15-2m+2}{2}\rfloor = \lfloor\frac{n-13-(n-16)}{2}\rfloor = \lfloor\frac{3}{2}\rfloor = 1$. This assures that we can construct the last 2–spread monomial not belonging to $\mathrm{Shad}_2(B_2(w_{3+m-1}))$. It is $w_{3+m} = x_2x_5x_8x_{12}x_{14}x_{16}\cdots x_n$ which is the greatest 2–spread monomial not belonging to this shadow.

Finally, we can observe that the monomial $x_2x_6x_8x_{10}\cdots x_{n-2}x_n$ is the greatest 2–spread monomial not belonging to the $\mathrm{Shad}_2(B_2(x_2x_5x_8x_{12}x_{14}x_{16}\cdots x_n))$.

Proceedings in this way we are able to identify

$$1 + 3 + m + 2 = 6 + \lfloor \frac{n-15}{2} \rfloor = 6 + \frac{n-16}{2} = \frac{n-4}{2}$$

monomials which are the ones we are looking for.

The construction of these monomials together with Theorem 4.1.12 leads to the existence of an ideal $I \in \mathcal{S}_{2,n,\mathbf{1}}$ of initial degree 2 in S with $|\mathrm{Corn}(I)| = \frac{n-2}{2} - 2 + 1 = \frac{n-4}{2}$. More in details,

$$\mathrm{Corn}(I) = \left\{ (k_i, \ell_i) : \quad k_i = n - 2(\ell_i - 1) - 1, \quad \ell_i = 2 + (i-1), \quad i = 1, \ldots, \frac{n-4}{2} \right\} =$$
$$= \left\{ (n-3, 2), (n-5, 3), \ldots, (3, \frac{n-2}{2}) \right\}$$

□

Now, we give an explicit description of the finitely generated Borel 2–spread ideal of $\mathcal{S}_{2,n,\mathbf{1}}$ of initial degree 2 with the maximal number of corners, for all even integer $n \geq 4$. We will denote it by $B_{2,n,\mathbf{1}}$ as in the case when n is odd.

Discussion 4.4.10 Let $S = K[x_1, \ldots, x_n]$ a polynomial ring, with $n \geq 4$ even integer .

Firstly, let us consider $n = 4, 6, 8, 10$. In such cases, the ideals $B_{2,n,\mathbf{1}}$ are described in Table 4.9.

n	**Corner sequence**	**2–spread strongly stable ideal**
4	$\{(1,2)\}$	$B_{2,4,\mathbf{1}} = B_2(x_1x_4) = (x_1x_3, x_1x_4)$
6	$\{(3,2),(1,3)\}$	$B_{2,6,\mathbf{1}} = B_2(x_1x_6, x_2x_4x_6)$
8	$\{(5,2),(3,3)\}$	$B_{2,8,\mathbf{1}} = B_2(x_1x_8, x_2x_4x_8)$
10	$\{(7,2),(5,3),(3,4)\}$	$B_{2,10,\mathbf{1}} = B_2(x_1x_{10}, x_2x_4x_{10}, x_2x_5x_7x_{10})$

Table 4.9: 2–spread corner sequences for $n = 4, 6, 8, 10$.

One can observe that the monomial of the type x_1x_n $(n = 4, 6, 8, 10)$ appears as 2–spread Borel generators in all four ideals $B_{2,4,\mathbf{1}}$, $B_{2,6,\mathbf{1}}$, $B_{2,8,\mathbf{1}}$ and $B_{2,10,\mathbf{1}}$; the monomial $x_2x_4x_n$ $(n = 6, 8, 10)$ appears as 2–spread Borel generators in the ideals $B_{2,6,\mathbf{1}}$, $B_{2,8,\mathbf{1}}$ and $B_{2,10,\mathbf{1}}$; whereas the monomial $x_2x_5x_7x_{10}$ appears only in the ideal $B_{2,10,\mathbf{1}}$, as 2–spread Borel generator.

It is worth to underline a difference from the n odd case. Indeed for two consecutive even values of n $(n = 6, 8)$, one has the same type of Borel generators.

For $n \geq 12$, the monomials in the following set

$$\{x_1x_n, x_2x_4x_n, x_2x_5x_7x_n\} \tag{4.4.7}$$

will be always 2–spread Borel generators for $B_{2,n,\mathbf{1}}$.

Let us consider $n = 12$. From the Theorem 4.4.9 (proof) we have to introduce the monomial $x_2x_5x_8x_{10}x_{12}$ to complete the minimal system of monomial generators of $B_{2,12,\mathbf{1}}$. Such a monomial is smaller than the monomials in (4.4.7), with respect to $\succeq$.

Furthermore, if $n \equiv 12 \pmod 6$, we need to add $r_1 = \frac{n-12}{6} + 1$ monomials of the type

$$x_2x_5x_8 \prod_{i=0}^{2k-1} x_{11+3i}x_jx_n, \quad j = 6k+10, \quad k = 0,\dots,r_1-1$$

to the set in (4.4.7) to get the minimal system of monomial generators of $B_{2,n,\mathbf{1}}$, for all even integer $n \geq 12$. We refer to them as the *first-left-form* basic monomials.
Note that, setting $\prod_{i=0}^{2k-1} x_{11+3i} = 1$ for $k = 0$, then $x_2x_5x_8 \prod_{i=0}^{2k-1} x_{11+3i}x_jx_n = x_2x_5x_8x_{10}x_n$.

Therefore, the monomials in

$$\{x_1x_n, x_2x_4x_n, x_2x_5x_7x_n, x_2x_5x_8 \prod_{i=0}^{2k-1} x_{11+3i}x_jx_n,\ j = 6k+10,\ k = 0,\dots,r_1-1\} \quad (4.4.8)$$

will belong to the minimal set of monomial generators of $B_{2,n,\mathbf{1}}$, for all even integer $n \geq 12$.

Let us consider $n = 14$. In such a case we introduce the monomial $x_2x_6x_8x_{10}x_{12}x_{14}$ as Borel generator. For $n \geq 14$, the monomial of the type $x_2x_6x_8x_{10}\cdots x_{n-2}x_n$ is the smallest generator of the ideal, with respect to $\succeq$.

In general, if $n \equiv 14 \pmod 6$ then we need to add $r_2 = \frac{n-14}{6} + 1$ monomials of the type

$$x_2 \prod_{i=0}^{2k-1} x_{5+3i}x_{j+2}x_{j+4}\cdots x_{n-2}x_n, \quad j = 6k+4, \quad k = 0,\dots,r_2-1 \quad (4.4.9)$$

to the set in (4.4.8) to get the minimal system of monomial generators of $B_{2,n,\mathbf{1}}$, for all even integer $n \geq 14$. We refer to them as the *right-form* basic monomials.

Note that, setting $\prod_{i=0}^{2k-1} x_{5+3i} = 1$ for $k = 0$, then $x_2 \prod_{i=0}^{2k-1} x_{5+3i}x_{j+2}x_{j+4}\cdots x_{n-2}x_n = x_2x_6x_8x_{10}\cdots x_{n-2}x_n$.

Hence, the monomials in (4.4.9) together with the ones in (4.4.8) will belong to the minimal set of monomial generators of $B_{2,n,\mathbf{1}}$, for all even $n \geq 14$.

Now, let us consider $n = 16$. In such a case we need the monomial $x_2x_5x_8x_{11}x_{13}x_{16}$. Such a monomial is greater than the *right-form* ones. In general, if $n \equiv 16 \pmod 6$ then $r_3 = \frac{n-16}{6} + 1$ monomials of the type

$$x_2x_5x_8x_{11} \prod_{i=0}^{2k-1} x_{14+3i}x_jx_n, \quad j = 6k+13, \quad k = 0,\dots,r_3-1 \quad (4.4.10)$$

will belong to the minimal set of monomial generators of $B_{2,n,\mathbf{1}}$, for all even $n \geq 16$. We refer to them as the *second-left-form* basic monomials.

Note that, setting $\prod_{i=0}^{2k-1} x_{14+3i} = 1$ for $k = 0$, then $x_2x_5x_8x_{11} \prod_{i=0}^{2k-1} x_{14+3i}x_jx_n = x_2x_5x_8x_{11}x_{13}x_n$.

Finally, the monomials in (4.4.10) together with the ones in (4.4.9) and the ones in (4.4.8) will determine the minimal set of monomial generators of $B_{2,n,\mathbf{1}}$, for all even $n \geq 16$.

The next example illustrates how given an even integer $n \geq 16$ in order to get the set of the generators of $B_{2,n,\mathbf{1}}$, one has to fix the the integers n, $n-2$, $n-4$. Reasoning as in Remark 4.4.4, the number of the monomials we need is given by

$$3+\frac{n-16}{6}+1+\frac{n-14-2}{6}+1+\frac{n-12-4}{6}+1=6+\frac{n-16}{2}=\frac{n-4}{2}.$$

and the 2–spread basic monomials can be obtained by (4.4.10), (4.4.9) and (4.4.8) *via* n, $n-2$, $n-4$, respectively.

Example 4.4.11 Let us consider the polynomial ring $S = K[x_1, \ldots, x_{20}]$. We want to construct the 2–spread strongly stable ideal $B_{2,20,\mathbf{1}}$ of S. Setting $I = B_{2,20,\mathbf{1}}$, one has $|\mathrm{Corn}(I)| = \frac{n-4}{2} = 8$ and

$$\mathrm{Corn}(I) = \{(17,2),(15,3),(13,4),(11,5),(9,6),(7,7),(5,8),(3,9)\}.$$

In order to get the 2–spread basic monomials that determine the minimal system of monomial generators $G(I)$ we proceed as follows.

Step 1. Consider the three 2–spread basic monomials x_1x_{20}, $x_2x_4x_{20}$ and $x_2x_5x_7x_{20}$.

Step 2. Since $n = 20 \equiv 14 \pmod 6$, we have $\frac{20-14}{6}+1 = 2$ right-form basic monomials of the type

$$x_2\prod_{i=0}^{2k-1} x_{5+3i}x_{j+2}x_{j+4}\cdots x_{n-2}x_n,$$

with $j = 6k+4$ for $k = 0,1$. They are $x_2x_6x_8x_{10}x_{12}x_{14}x_{16}x_{18}x_{20}$ ($k=0$, $j=4$) and $x_2x_5x_8x_{12}x_{14}x_{16}x_{18}x_{20}$ ($k=1$, $j=10$).

Step 3. Let us consider $n-2 = 18$. Since $n-2 = 18 \equiv 12 \pmod 6$, we have $\frac{18-12}{6}+1 = 2$ first-left-form basic monomials of the type

$$x_2x_5x_8\prod_{i=0}^{2k-1} x_{11+3i}x_jx_n,$$

with $j = 6k+10$ for $k = 0,1$. They are $x_2x_5x_8x_{10}x_{20}$ ($k=0$, $j=10$) and $x_2x_5x_8x_{11}x_{14}x_{16}x_{20}$ ($k=1$, $j=16$).

Step 4. Let us consider $n-4 = 16$. Since $n-4 = 16 \equiv 16 \pmod 6$, we have $\frac{16-16}{6}+1 = 1$ second-left-form basic monomial of the type

$$x_2x_5x_8x_{11}\prod_{i=0}^{2k-1} x_{14+3i}x_jx_n,$$

with $j = 13$ and $k = 0$. It is $x_2x_5x_8x_{11}x_{13}x_{20}$.

Finally, ordering the monomials in Steps 1–4 with respect to $\succeq$, we have the ideal

$$I = B_2(x_1x_{20}, x_2x_4x_{20}, x_2x_5x_7x_{20}, x_2x_5x_8x_{10}x_{20}, x_2x_5x_8x_{11}x_{13}x_{20}, x_2x_5x_8x_{11}x_{14}x_{16}x_{20},$$
$$x_2x_5x_8x_{12}x_{14}x_{16}x_{18}x_{20}, x_2x_6x_8x_{10}x_{12}x_{14}x_{16}x_{18}x_{20}).$$

From the Theorem 4.4.9 and Discussion 4.4.10, the next result follows.

Theorem 4.4.12 *Let $n \geq 12$ an even integer and $\ell_1 = 2$. Given $\frac{n-4}{2}$ pairs of positive integers*

$$(k_1, \ell_1), (k_2, \ell_2), \ldots, (k_{\frac{n-4}{2}}, \ell_{\frac{n-4}{2}}), \tag{4.4.11}$$

with $1 \leq k_{\frac{n-4}{2}} < k_{\frac{n-4}{2}-1} < \cdots < k_1 \leq n-3$ and $2 = \ell_1 < \ell_2 < \cdots < \ell_{\frac{n-4}{2}} \leq \frac{n-2}{2}$, then there exists a 2–spread strongly stable ideal I of S of initial degree ℓ_1 and with the pairs in (4.4.6) as corners if and only if $k_i + 2(\ell_i - 1) + 1 = n$, for $i = 1, \ldots, \frac{n-4}{2}$.

Also in such a case the 2–spread strongly stable ideal in Theorem 4.4.12 is a 2–spread lex ideal.

4.5 *Macaulay2* package

The algorithms described in this chapter have been implemented in a *Macaulay2* package: "SquarefreeIdeals.m2" (tested with Macaulay 1.13). This package contains procedures to handle squarefree (strongly) stable ideals and squarefree lex ideals. Moreover, the strongly combinatorial nature of squarefree monomial ideals has allowed to create customized algorithms to obtain the minimal generators for an ideal with given extremal Betti numbers.

More precisely, we implement some algorithms in order to compute, when possible, the smallest squarefree strongly stable ideal with given extremal Betti numbers (values as well as positions).

In this Section, we collect some examples in order to describe the principal algorithms in this package.

Example 4.5.1 Let n and $r < n$ be two positive integers. Let (k_1, ℓ_1), ..., (k_r, ℓ_r), r pairs of positive integers such that $n - 3 \geq k_1 > k_2 > \cdots > k_r \geq 2$ and $2 \leq \ell_1 < \ell_2 < \cdots < \ell_r$. Moreover, let $a_1, \ldots, a_r$, r positive integers.
We want to check if there exists a squarefree strongly stable ideal I of $S = K[x_1, \ldots, x_n]$ such that $\beta_{k_1,k_1+\ell_1}(I) = a_1$, ..., $\beta_{k_r,k_r+\ell_r}(I) = a_r$ are its extremal Betti numbers. In positive case we want to compute it.

```
Macaulay2, version 1.13
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
```

```
i1 : loadPackage "SquarefreeIdeals";
i2 : n=10;
i3 : S=QQ[x_1..x_n];
i4 : g={x_2x_8, x_3x_4x_5, x_3x_4x_8x_9, x_3x_5x_7x_9,
                                            x_4x_5x_6x_7x_8x_9x_10};

i5 : I=squarefreeStronglyStableIdeal ideal g
o5 = ideal (x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_1x_8, x_2x_3,
            x_2x_4, x_2x_5, x_2x_6, x_2x_7, x_2x_8, x_3x_4x_5, x_3x_4x_6x_7,
            x_3x_4x_6x_8, x_3x_4x_6x_9, x_3x_4x_7x_8, x_3x_4x_7x_9,
            x_3x_4x_8x_9, x_3x_5x_6x_7, x_3x_5x_6x_8, x_3x_5x_6x_9,
            x_3x_5x_7x_8, x_3x_5x_7x_9, x_4x_5x_6x_7x_8x_9x_10)
o5 : Ideal of S

i6 : minimalBettiNumbersIdeal I
            0  1   2   3  4  5 6
o6 = total: 26 94 154 139 71 19 2
         2: 13 42  70  70 42 14 2
         3:  1  2   1   .  .  . .
         4: 11 47  80  68 29  5 .
         5:  .  .   .   .  .  . .
         6:  .  .   .   .  .  . .
         7:  1  3   3   1  .  . .
o6 : BettiTally
```

The functions `extremalBettiCorners(ideal)` compute the corner sequence of an ideal using the definition of degree-sequence given in Equation 4.3.3. In this case, we want to compute the smallest squarefree strongly stable ideal with this corners. The main function is `extremalBettiMonomials(ring,integer,list,list)` that returns (if possible) the generators of the desired ideal. This procedure is based on the constructive proof of Theorem 4.3.29 and uses two specific algorithms (see Algorithm 4.5 and 4.6) to compute two kind of significant monomials.

```
i7 : corners=extremalBettiCorners I
o7 = {(6, 2), (5, 4), (3, 7)}
o7 : List
i8 : r=#corners;
i9 : a={2,5,1};
```

```
i10 : Bg=extremalBettiMonomials(S,r,corners,a)
o10 = {x_1x_8, x_2x_8, x_3x_4x_5x_9, x_3x_4x_6x_9, x_3x_4x_7x_9,
       x_3x_4x_8x_9, x_3x_5x_6x_9, x_4x_5x_6x_7x_8x_9x_10}
o10 : List
```

Now, we can compute the smallest squarefree strongly stable ideal containing the monomials in the list. The existence of the fundamental Borel generators with given corners assures that this ideal has the desired extremal Betti numbers (positions and values).

```
i11 : J=squarefreeStronglyStableIdeal ideal Bg
o11 = ideal(x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_1x_8, x_2x_3,
            x_2x_4, x_2x_5, x_2x_6, x_2x_7, x_2x_8, x_3x_4x_5x_6,
            x_3x_4x_5x_7, x_3x_4x_5x_8, x_3x_4x_5x_9, x_3x_4x_6x_7,
            x_3x_4x_6x_8, x_3x_4x_6x_9, x_3x_4x_7x_8, x_3x_4x_7x_9,
            x_3x_4x_8x_9, x_3x_5x_6x_7, x_3x_5x_6x_8, x_3x_5x_6x_9,
            x_4x_5x_6x_7x_8x_9x_10)
o11 : Ideal of S
i12 : minimalBettiNumbersIdeal J
                0  1   2   3  4  5 6
o12 = total: 27 97 157 140 71 19 2
          2: 13 42  70  70 42 14 2
          3:  .  .   .   .  .  . .
          4: 13 52  84  69 29  5 .
          5:  .  .   .   .  .  . .
          6:  .  .   .   .  .  . .
          7:  1  3   3   1  .  . .
o12 : BettiTally
```

Furthermore, we are trying to generalize some algorithms in order to manipulate t–spread strongly stable ideals and to solve the problem related with given extremal Betti numbers (values as well as positions).

Example 4.5.2 Let $t = 2$ and let two positive integers n and $r < n$. Let (k_1, ℓ_1), …, (k_r, ℓ_r), r pairs of positive integers such that $n - 3 \geq k_1 > k_2 > \cdots > k_r \geq 2$ and $2 \leq \ell_1 < \ell_2 < \cdots < \ell_r$. We want to check if there exists a 2–spread strongly stable ideal I of $S = K[x_1, \dots, x_n]$ such that $\beta_{k_1,k_1+\ell_1}(I) = 1$, …, $\beta_{k_r,k_r+\ell_r}(I) = 1$ are its extremal Betti numbers and in the positive case we want to compute it.

```
Macaulay2, version 1.13
with packages: ConwayPolynomials, Elimination, IntegralClosure,
InverseSystems, LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
```

```
i1 : loadPackage "SquarefreeIdeals";
i2 : n=13;
i3 : S=QQ[x_1..x_n];
i4 : t=2;
i5 : indeg=2;
i6 : k=n-t*(indeg-1)-1
o6 : 10
i7 : tot=(k-k%t)//t
o7 : 5
i8 : corners=for i to tot-1 list (k-t*i,indeg+i)
o8 : {(10, 2), (8, 3), (6, 4), (4, 5), (2, 6)}
o8 : List
i9 : a=toList(#corners:1);
i10 : Bg=tspreadExtremalBettiMonomials(S,corners,a,t)
o10 = {x_1x_13, x_2x_4x_13, x_2x_5x_7x_13, x_2x_5x_8x_10x_13,
                                          x_3x_5x_7x_9x_11x_13}
o10 : List
```

Now, we can compute the smallest squarefree strongly stable ideal containing the monomials in the list. The existence of the fundamental Borel generators with given corners assures that this ideal has the desired extremal Betti numbers (positions and values).

```
i14 : I=tspreadStronglyStableIdeal(t,ideal Bg)
o14 = ideal (x_1x_3, x_1x_4, x_1x_5, x_1x_6, x_1x_7, x_1x_8, x_1x_9,
             x_1x_10, x_1x_11, x_1x_12, x_1x_13, x_2x_4x_6, x_2x_4x_7,
             x_2x_4x_8, x_2x_4x_9, x_2x_4x_10, x_2x_4x_11, x_2x_4x_12,
             x_2x_4x_13, x_2x_5x_7x_9, x_2x_5x_7x_10, x_2x_5x_7x_11,
             x_2x_5x_7x_12, x_2x_5x_7x_13, x_2x_5x_8x_10x_12,
             x_2x_5x_8x_10x_13, x_3x_5x_7x_9x_11x_13)
o14 : Ideal of S
i15 : minimalBettiNumbersIdeal I
              0   1   2   3   4   5   6   7  8  9 10
o15 = total: 27 120 294 496 610 553 367 174 56 11  1
          2: 11  55 165 330 462 462 330 165 55 11  1
          3:  8  36  84 126 126  84  36   9  1  .  .
          4:  5  20  35  35  21   7   1   .  .  .  .
          5:  2   7   9   5   1   .   .   .  .  .  .
          6:  1   2   1   .   .   .   .   .  .  .  .
o15 : BettiTally
```

Final considerations

In this dissertation we have faced some open problems related to Hilbert functions and minimal free resolutions of graded submodules in the polynomial ring and in the exterior algebra. As we have already observed, the relation between this two algebras is very strong and most of the tools from commutative algebra can be used in both contexts. Our approach has been above all computational: we have took advantage of the combinatorial features of this structures to construct algorithms to manipulate ideals and submodules.

Our initial intent was to address some open problems and solve them in order to level knowledge in both contexts, where possible.

In Chapter 2 we have characterized the Hilbert functions of graded E–modules of the type F/M, with M graded submodule of F. A fundamental step was the construction of the unique lexicographic submodule of F with the same Hilbert function as M. This result is equivalent to what has already been done for the polynomial ring [Hul95].

In Chapter 3 we have proved the validity of the "higher" Kruskal–Katona's Theorem for E-submodules of a finitely generated graded free module F. Moreover, we have given upper bounds for the graded Bass numbers of E–modules of the type E^r/M, $r \geq 1$ (see [AHH97] for the rank one case).

Open 4.5.3 It would be nice to verify the inequality in Theorem 3.4.5 for quotients of the type F/M, with $F = \oplus_{i=1}^r Eg_i$, when the basis elements $g_1, \ldots, g_r$ have different degrees. We believe that such a statement should be proved by using a different approach, as next example illustrates.

Example 4.5.4 Let $E = K\langle e_1, e_2, e_3, e_4\rangle$ and $F = \oplus_{i=1}^r Eg_i$ with $\deg g_1 = \deg g_2 = -2$, $\deg g_3 = -1$. Let us consider the lex submodule of F

$$L = (e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4)g_1 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4)g_2 \oplus (e_1e_2e_3)g_3,$$

Setting $I_1 = (e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4)$, $I_2 = (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4)$, $I_3 = (e_1e_2e_3)$, one has

$$\begin{aligned} 0 : I_1 &= (e_1e_4, e_1e_3, e_1e_2, e_2e_3e_4), \\ 0 : I_2 &= (e_1, e_2e_3, e_2e_4, e_3e_4), \\ 0 : I_3 &= (e_1, e_2, e_3), \end{aligned}$$

and $N = \oplus_{t=1}^r (0 : I_t)g_t$ is not a lex submodule of F. Proceeding as in Example 3.4.4, let us consider the module

$$\widetilde{N} = (0 : I_3)g_1 \oplus (0 : I_2)g_2 \oplus (0 : I_1)g_3.$$

It is not a lex submodule of F ($e_4 \notin 0 : I_3$). Consider $F^* = \mathrm{Hom}_E(F, E)$. By using *Macaulay2*, it is $F^* = \oplus_{i=1}^{r} E\widetilde{g}_i$, with $\deg \widetilde{g}_1 = 1$, $\deg \widetilde{g}_2 = \deg \widetilde{g}_3 = 2$ and one can quickly verify that $\overline{N} = (0 : I_3)\widetilde{g}_1 \oplus (0 : I_2)\widetilde{g}_2 \oplus (0 : I_1)\widetilde{g}_3$ is a lex submodule of F^*. Note that, $F \simeq F^*$ as E–modules, but not as graded E–modules. Indeed, $H_F \neq H_{F^*}$. Hence, the arguments given in Theorem 3.4.5 do not work in the case of quotients of a free module with basis elements with different degrees.

In Chapter 4 we are in charge of analyzing the extremal Betti numbers of t–spread strongly stable ideals of S, where t is an integer ≥ 0, in order to characterize the possible extremal Betti numbers (values as well as positions). Also in this case, the approach has been mainly algorithmic and constructive. Indeed, in the managed cases we have computed the "fundamental" monomials to obtain the desired ideal, where possible, given an exhaustive configuration of extremal Betti numbers.

For the case $t = 0$, *i.e.* monomial ideals of S, the Corollary 4.2.8 points out that the main algorithm for FGBI is equivalent to the following:

Open 4.5.5 Given two positive integers n, r, $1 \leq r \leq n-1$, r pairs of positive integers (k_1, ℓ_1), ..., (k_r, ℓ_r) such that $n-1 \geq k_1 > k_2 > \cdots > k_r \geq 1$, $1 \leq \ell_1 < \ell_2 < \cdots < \ell_r$, and r positive integers $a_1, \ldots, a_r$. Under which conditions does there exist a piecewise lexsegment ideal L of S such that $\beta_{k_1,k_1+\ell_1}(L) = a_1$, ..., $\beta_{k_r,k_r+\ell_r}(L) = a_r$ are its extremal Betti numbers?

As we have just underlined, the class of piecewise lexsegment ideals has played a relevant role in [HSV14] in the numerical characterization of the possible extremal Betti numbers (values as well as positions) of any homogeneous ideal in a polynomial ring over a field of characteristic 0. Furthermore, [HSV14, Theorem 6.7 (iii)] states some conditions about the Macaulay representation of the positive integers $a_1, \ldots, a_r$ which guarantee the existence of a piecewise lexsegment ideal solution of Problem 4.5.5. Hence, we believe that it would be nice to implement new algorithms which determine the associated smallest possible piecewise lexsegment ideal *via* such conditions.

For the case $t = 1$, *i.e.* squarefree graded ideals of S, we have solved the problem of the extremal Betti numbers positions as well as values giving numerical bounds for the number of the corners and of the values. Moreover, for admissible corners sequences and values some constructive algorithms returns the smallest squarefree strongly stable ideal with given extremal Betti numbers.

For the case $t = 2$, *i.e.* 2–spread graded ideals of S, we have discussed the extremal Betti numbers of t–spread strongly stable ideals and we have determined the maximal number of the admissible corners of 2–spread strongly stable ideals. It would be nice to generalize the results in Section 4.3.2 to t–spread strongly stable ideals for all $t \geq 2$.

The following questions are currently under investigation and the generalization of the principal algorithms is in progress.

Open 4.5.6 Given an integer $t \geq 2$, let $\mathcal{S}_{t,n}$ be the set of all t–spread strongly stable ideals in S. What is the largest number of corners allowed for an ideal of $\mathcal{S}_{t,n}$?

Open 4.5.7 Given three positive integers $t \geq 2$, n and $r < n$, r pairs of positive integers (k_1, ℓ_1), …, (k_r, ℓ_r) such that $n-3 \geq k_1 > k_2 > \cdots > k_r \geq 2$ and $2 \leq \ell_1 < \ell_2 < \cdots < \ell_r$, and r positive integers $a_1, \ldots, a_r$, under which conditions does there exist a t–spread strongly stable ideal I of $S = K[x_1, \ldots, x_n]$ such that $\beta_{k_1,k_1+\ell_1}(I) = a_1$, …, $\beta_{k_r,k_r+\ell_r}(I) = a_r$ are its extremal Betti numbers?

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