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Kernel estimates for elliptic operators with second-order discontinuous coefficients

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Kernel estimates for elliptic operators with second-order discontinuous coefficients

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Contents

Introduction

The notion of heat kernel has long represented an essential milestone in the development of the theory of parabolic partial differential equations. It is widely known, for example, that the Laplace operator Δ on $\mathbb{R}^N$ has the explicit kernel

$$p_\Delta(t,x,y) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4t}}, \quad t > 0,\ x, y \in \mathbb{R}^N, \tag{1}$$

so that the heat equation $\partial_t u - \Delta u = 0$, $u(0) = f$ has, for suitable initial data f, the explicit solution

$$e^{t\Delta} f(x) = \int_{\mathbb{R}^N} p_\Delta(t,x,y) f(y)\, dy. \tag{2}$$

The properties of heat kernels play a fundamental role in approaching several important questions of different sections of analysis. For instance, based on (1) and (2), one can derive all the main inequalities of the theory of Sobolev spaces, in particular the Sobolev inequality

$$\|f\|_{\frac{2N}{N-2}} \le C \|\nabla f\|_2,$$

and the Nash inequality

$$\|f\|_2^{2+\frac{4}{N}} \le C \|\nabla f\|_2^2 \, \|f\|_1^{\frac{4}{N}},$$

which are valid for every $f \in C_c^\infty\left(\mathbb{R}^N\right)$.

It appears clear that one of the most significant problems becomes to determine whether the heat kernel of a general elliptic operator admits some Gaussian bounds. From an upper estimates one can infer, for example, $L^p - L^q$ estimates, the analyticity of the semigroups in L^p for any $1 \le p \le \infty$, the p-independence of the spectrum, a bounded functional calculus, whereas a lower Gaussian bound is intimately related to some Harnack inequality for the solutions. For a detailed survey on the topic we suggest the classical books by Davies [23], Grigor'yan [35], Ouhabaz [67], Saloff-Coste [70] and references therein.

Historically the starting point of the theory was a 1967 paper of Aronson [6], where the author, using Moser's parabolic Harnack inequality [62], proved that if the Laplacian is replaced by a real elliptic operator $A = div(a(x) \cdot \nabla)$ satisfying $\lambda I \leq a(x) \leq \Lambda I$, then its heat kernel p_A satisfies

$$C_1 t^{-\frac{N}{2}} e^{-c_1 \frac{|x-y|^2}{t}} \leq p_A(t,x,y) \leq C_2 t^{-\frac{N}{2}} e^{-c_2 \frac{|x-y|^2}{t}}.$$

Since then, different proofs of this result appeared in literature, each one revealing a different aspect of the theory, especially its deep connection with the ideas developed by De Giorgi [25, 1957] and Nash [64, 1958] for the regularity of elliptic and parabolic equations and by Moser [62, 1964] for the Harnack inequalities. We mention the proof by Davies [22] based on a perturbation method (the "Davies's trick") together with logarithmic Sobolev inequalities, the one by Fabes and Stroock [30] using earlier ideas of Nash and the one by Coulhon [19] based on Moser's iteration technique (see also [8], [72]).

Nowadays Gaussian estimates have a compelling formulation in several branches of Mathematics, for example in the context of weighted Riemannian manifolds. Let $(\mathcal{M}, g)$ be a complete Riemannian manifold equipped with a measure $\mu = \phi d\nu$, where $d\nu$ stands for the Riemannian measure and ϕ is a smooth positive weight. If ∇_g is the Riemannian gradient, the associated Laplacian $A = \phi^{-1}\mathrm{div}\,(\phi\nabla_g)$ is a self-adjoint operator on $L^2(\mathcal{M}, \mu)$ and its heat kernel, when it does exist, is the positive function $p_A(t,x,y)$ which is defined for $t > 0,\ x, y \in \mathcal{M}$ and satisfies

$$e^{tA} f(x) = \int_{\mathcal{M}} p_A(t,x,y) f(y)\, \mu(y), \quad f \in L^2(\mathcal{M}, \mu).$$

In many favourable cases, one tries to realize if p_A satisfies the two-side Gaussian estimate

$$p_A(t,x,y) \simeq \frac{C}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}} e^{-\frac{d_g(x,y)^2}{ct}}, \quad t > 0,\ x, y \in \mathcal{M}. \tag{3}$$

Here $V(x,\sqrt{t})$ is the measure of the geodesic ball $B(x,\sqrt{t})$, d_g is the Riemannian distance induced by g and $\simeq$ means that both upper and lower inequalities hold for possibly different constants $C, c > 0$.

For the sake of completeness we summarize below the inspiring ingredients which usually occur in the derivation of (3).

Regarding the upper inequality, the crucial key is the derivation of the on-diagonal upper bound

$$p_A(t,x,x) \leq \frac{C}{V(x,\sqrt{t})}, \quad t > 0,\ x \in \mathcal{M}. \tag{4}$$

The general theory, indeed, allows to automatically improve (4) into the upper bound of (3): some known methods are the Davies perturbation method [22, 23], the integrated maximum principle [34, 36], the finite propagation speed for the wave equation [74] and an approach based on suitable Phragmén-Lindelöf theorems [21].

Typically one requires a control over the growth of V, namely the doubling condition

$$V(x,2r) \le CV(x,r), \quad r>0,\ x\in\mathcal{M}. \tag{5}$$

In this manner (4) becomes equivalent to some generalized local Faber-Krahn inequality [35]. In the uniform case, the estimate $p(t,x,x)\le\theta(t)$ can also be characterized in terms of Log-Sobolev [22, 23] or Nash type inequalities [20]. In particular, in analogy with the euclidean case, when $V(x,\sqrt{t})$ has polynomial growth $V(x,\sqrt{t})\simeq ct^{\frac{d}{2}}$, with $d>0$, (4) is equivalent to each of the following properties (see for example [35, Corollary 14.23]):

- The ultracontractivity of e^{tA}: $\|e^{tA}\|_{1\to\infty}\le Ct^{-\frac{d}{2}}$.
- The Nash inequality: $\|f\|_2^{2(1+\frac{2}{d})}\le C\|\nabla_g f\|_2^2\,\|f\|_1^{\frac{4}{d}}$, for every $f\in C_c^\infty(\mathcal{M})$.
- The Sobolev inequality: (if $d>2$) $\|f\|^2_{\frac{2d}{d-2}}\le C\|\nabla_g f\|_2^2$, for every $f\in C_c^\infty(\mathcal{M})$.
- The Log-Sobolev inequality: $\int_M f^2\log\frac{f}{\|f\|_2}d\mu\le\epsilon\|\nabla_g f\|_2^2{+}\beta(\epsilon)\|f\|_2$, for every positive $f\in C_c^\infty(\mathcal{M})$, $\epsilon>0$, where $\beta(\epsilon)=C-\frac{d}{4}\log\epsilon$.
- The Faber-Krahn inequality: $\lambda(B)\ge c\mu(B)^{-\frac{2}{d}}$, for every open relatively compact subset B of $\mathcal{M}$, where $\lambda(B)$ is the lowest Dirichlet eigenvalue in B.

On the other hand the lower estimates of (3) is intimately related to the regularity of the solutions of the heat equation, namely to the validity of the Harnack inequality which, for every positive solution u of $\partial_t u - Au = 0$ in the cylinder $D=(0,r^2)\times B(x,r)$, reads as

$$\sup_{D^-} u \le C\inf_{D^+} u, \tag{6}$$

where $D^-=(1/4\ r^2,1/2\ r^2)\times B(x,1/2\ r)$ and $D^+=(3/4\ r^2,r^2)\times B(x,1/2\ r)$. Indeed the two-side estimate (3) is characterized by the following equivalent properties (see for example [71, Theorem 3.1]):

- The parabolic Harnack inequality (6).
- The Poincaré inequality $\|f-f_B\|_{L^2(B(x,r))}\le Cr\|\nabla_g f\|_{L^2(B(x,r))}$ (for every $r>0,\ x\in\mathcal{M}$, $f\in C^1(B(x,r))$, with f_B its mean over $B(x,r)$) and the volume doubling property (5).

Unfortunately there is no general way to recognize when some of these characterizing conditions are satisfied since their validity heavily depends on the geometry of the manifolds.

The first eminent result was obtained by Li and Yau in their pioneering work [45], where they proved that the parabolic Harnack inequality (6) and the two-side heat kernel bounds (3) hold for every complete Riemannian manifold with non-negative Ricci curvature.

On the other hand few results are known in the case of incomplete Riemannian manifolds and the same happens if the Laplace-Beltrami operator is replaced by an elliptic operator with singular lower order terms. In the euclidean setting, Gaussian estimates are known when

$$A = div(a(x)\cdot\nabla) + c(x)\cdot\nabla - V(x)$$

and $V,\ c$ belong respectively to the Kato classes K_N, K_{N+1} (see for example Simon, Liskevich, Semenov, Voigt, Escauriaza, etc.., [1, 29, 40, 44, 73]). For $N \geq 3$, K_N is defined as the space of function q such that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^N}\int_{B(x,r)}\frac{1}{|x-y|^{N-2}}|q(y)|\,dy = 0.$$

Roughly speaking a singularity of V, say at 0, can be at most like $|x|^{-2+\varepsilon}$ and that of c like $|x|^{-1+\varepsilon}$. Non-autonomous cases are also treated and V, b are assumed to belong to suitable non-autonomous Kato classes.

Our goal, in this dissertation, is to prove sharp upper and lower bounds for the heat kernels of the operators

$$L = \Delta + (a-1)\sum_{i,j=1}^{N}\frac{x_ix_j}{|x|^2}D_{ij} + c\frac{x}{|x|^2}\cdot\nabla - \frac{b}{|x|^2}, \tag{7}$$

where $a > 0$, $b,\ c \in \mathbb{R}$. The leading coefficients of L are uniformly elliptic but discontinuous at 0, if $a \neq 1$, and singularities in the lower order terms appear when b or c is different from 0 and, correspondingly, the potential $\frac{b}{|x|^2}$ or the drift term $c\frac{x}{|x|^2}$ does not belong to the Kato classes K_N, K_{N+1}. L is associated with the Riemannian manifold $\left(\mathbb{R}^N\setminus\{0\}, g\right)$ (see (2.5) in Proposition 2.1.1) where g is the Riemannian metric

$$g = \sum_{i,j=1}^{N}\left(\delta_{ij} + \left(\frac{1}{a}-1\right)\frac{x_ix_j}{|x|^2}\right)dx_i\otimes dx_j.$$

As a reflection of the singularity at 0 of the operator, the manifold is not complete for $a \neq 1$. Moreover it is flat if either $N = 2$ or $a = 1$. For $N \geq 3$, its Ricci tensor Ric satisfies

$$(1-a)\ \mathrm{Ric} \geq 0$$

and, if $a > 1$, Ric is not bounded from below (see Proposition 2.1.4).

In the special case b = c = 0, these operators have been introduced to provide counterexamples to the elliptic regularity (see for example [69] and [78]). Positive results have also been obtained by Manselli and Ragnedda, see [47], [48] and [49], who proved existence and uniqueness results in Sobolev spaces in a bounded domain containing the origin and spectral properties in the two-dimensional case. When $a = 1$, $c = 0$ the operator becomes the Schrödinger operator with inverse square potential

$$L = \Delta - \frac{b}{|x|^2} \tag{8}$$

for which we recover sharp heat kernel bounds even in the critical case $D := b + \left(\frac{N-2}{2}\right)^2 = 0$.

Concerning (8), Milman and Semenov prove in [58, Theorem 1] the same upper and lower bounds as in our Theorem 7.1.1 with (almost) precise constants in the Gaussian factor and including the critical case; we also mention Ishige, Kabeya and Ouhabaz [38], Barbatis, Filippas and Tertikas [11] and [43, 44]. We refer to [32] for sharp bounds in bounded domains when the potential in (8) degenerates as the inverse of square of the distance from the boundary.

Our methods work for the more general operators (7). This generalization is important to obtain precise bounds on the heat kernels of certain operators with unbounded coefficients, as shown in Chapter 7.

We point out that generation properties and domain description for our operator have been previously investigated in [56]. If $1 < p < \infty$, we define the maximal operator $L_{p,max}$ through the domain

$$D(L_{p,max}) = \{u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N \setminus \{0\}) : Lu \in L^p(\mathbb{R}^N)\}$$

and $L_{p,min} \subset L_{p,max}$ is defined as the closure, in $L^p(\mathbb{R}^N)$ of $(L, C_c^\infty(\mathbb{R}^N \setminus \{0\})$.

The equation $Lu = 0$ has radial solutions $|x|^{-s_1}$, $|x|^{-s_2}$ where s_1, s_2 are the roots of the indicial equation $f(s) = -as^2 + (N-1+c-a)s + b = 0$ given by

$$s_1 := \frac{N-1+c-a}{2a} - \sqrt{D}, \quad s_2 := \frac{N-1+c-a}{2a} + \sqrt{D}$$

where

$$D := \frac{b}{a} + \left(\frac{N-1+c-a}{2a}\right)^2. \tag{9}$$

The above numbers are real if and only if $D \geq 0$. When $D < 0$ the equation $u - Lu = f$ cannot have positive distributional solutions for certain positive f, see [56] and [55]. This fact constitutes an elliptic counterpart of a famous result due to Baras and Goldstein, see [10], in the case of the Schrödinger operator with inverse square potential where the above condition reads $b + (N-2)^2/4 \geq 0$.

Assuming $D \geq 0$ it has shown in [56] and [54] that there exists an intermediate operator $L_{p,min} \subset L_{p,int} \subset L_{p,max}$ which generates a semigroup in $L^p(\mathbb{R}^N)$ if and only if $\frac{N}{p} \in (s_1, s_2 + 2)$.

The main result of the dissertation consists in the following two-side estimates for the heat kernel p of L with respect to the measure $|y|^\gamma\,dy$, see Theorem 4.2.2,

$$p(t,x,y) \simeq c_1 t^{-\frac{N}{2}} |x|^{-\frac{\gamma}{2}} |y|^{-\frac{\gamma}{2}} \left[\left(\frac{|x|}{t^{\frac12}} \wedge 1\right)\left(\frac{|y|}{t^{\frac12}} \wedge 1\right)\right]^{-\frac{N}{2}+1+\sqrt{D}} \exp\left(-\frac{c_2|x-y|^2}{t}\right) \tag{10}$$

where $D \geq 0$ is defined in (9) and $\gamma = (N-1+c)/a - N + 1$. Here c_1, c_2 are positive constants which may be different in the lower and upper bounds. Our estimates involve a Gaussian factor, a power $|x|^\gamma$ which takes into account the asymmetry of the operator in L^2, with respect to the Lebesgue measure, and the term $\left(|x|/\sqrt{t}\ \wedge 1\right)^{-\frac{N}{2}+1+\sqrt{D}}$ which is related to the singularity at 0. A different but equivalent form of the above bounds, in terms of the eigenfunction $|x|^{-s_1}$, is shown in Corollary 4.2.4, see also Remark 4.2.3.

The upper bound can be improved and extended for complex time $z \in \mathbb{C}_+$:

$$\begin{aligned}|p(z,x,y)| \leq C\,(Re\, z)^{-\frac{N}{2}} \left(1 + Re\,\frac{d_g^2(x,y)}{4z}\right)^{\alpha} |x|^{-\frac{\gamma}{2}} |y|^{-\frac{\gamma}{2}} \\ \times \left[\left(\frac{|x|}{(Re\, z)^{\frac12}} \wedge 1\right)\left(\frac{|y|}{(Re\, z)^{\frac12}} \wedge 1\right)\right]^{-\frac{N}{2}+1+\sqrt{D}} \exp\left(-Re\,\frac{d_g(x,y)^2}{4z}\right),\end{aligned} \tag{11}$$

where $\alpha = \frac{N}{2}$ if $0 \leq D \leq \left(\frac{N-2}{2}\right)^2$, $\alpha = \frac{N+\gamma-2s_1}{2}$ if $D > \left(\frac{N-2}{2}\right)^2$ and d_g is the distance on $\mathbb{R}^N \setminus \{0\}$ associated to the operator L and which is expressed by

$$d_g(x,y) = \sqrt{\frac{1}{a}\left[|x|^2 + |y|^2 - 2|x||y| \cos\left(\pi \wedge \sqrt{a}\arccos\left(\langle \frac{x}{|x|}, \frac{y}{|y|}\rangle\right)\right)\right]}. \tag{12}$$

As a consequence we improve the result proved in [57] and we obtain sharp bounds for the Green function. For example, if $N > 2$ and $D > 0$, then (writing the kernel with respect to the measure $|y|^\gamma\,dy$)

$$(|x||y|)^{\frac{\gamma}{2}}\, G_\lambda(x,y) \simeq e^{-c\sqrt{\lambda}|x-y|}\, |x-y|^{2-N} \left(1 \wedge \frac{|x||y|}{|x-y|^2}\right)^{\sqrt{D}-\frac{N-2}{2}}.$$

The critical cases $N = 2$ and $D = 0$ are also considered.

Let us briefly describe the contents of the individual chapters.

In Chapter 1 we collect some preliminary properties about L and we write it in spherical coordinates. We observe that, on subspaces defined as tensor products of radial functions and spherical harmonics, it reduces to one-dimensional Bessel operators. This property allows, in Chapter 4, to decompose the kernel of L in terms of its one-dimensional counterparts. We construct the operator, via form methods, in the weighted space $L^2\left(\mathbb{R}^N, d\mu\right)$, where L is a self-adjoint operator and generates an analytic semigroup of angle $\pi/2$. In

Section 1.3 we treat the one-dimensional case, i.e. the Bessel operator, and we give an analytic proof of the explicit form of its heat kernel. We conclude with Section 1.4, where we present the main results concerning the generation in $L^p(\mathbb{R}^N)$ proved in [56].

Chapter 2 is devoted to the study of the Riemannian manifolds $(\mathbb{R}^N \setminus \{0\}, g)$ associated to L. The core of this chapter is Section 2.2, where we compute the explicit formula (12) of the geodesic distance induced by g. Basic knowledge of Riemannian Geometry is required and we refer the reader to Appendix A for a brief survey on the main notions needed.

The proof of the optimal upper bounds (11) is the subject of Chapter 3. As observed in the first part of the intoduction, there are different methods to prove Gaussian estimates. Here we follow the approach of [21] where the authors use the Phragmén-Lindelöf theorem to deduce them from the ultracontractivity of the semigroup and some L^2 Gaussian bounds, the so-called Davies-Gaffney estimates (3.17). In Section 3.1, to overcome the singularity at 0, we perform another change in the measure and use form methods to construct an equivalent operator in the space $L^2\left(\mathbb{R}^N, d\nu\right)$, where $d\nu$ is defined in (3.10). In Section 3.4 we show that the the analytic semigroup generated by L satisfies the Davies-Gaffney estimates. This property, combined with some ultracontractivity bounds, obtained using Gagliardo-Nirenberg type inequalities, and with [21, Theorems 4.1], ensures the validity of (3.2). The upper bound of (10) immediately follows from Corollary 3.5.5.

Chapter 4 deals with the lower estimate of (10). Section 4.1 is devoted to the decomposition of the heat kernel of L as the (infinite) sum of heat kernels of one-dimensional Bessel operators. In Section 4.2 we get the main result by combining kernel estimates near the origin, obtained thanks to the explicit formula of one-dimensional Bessel operators, with Gaussian estimates faraway from the origin already known for uniformly elliptic operators.

In Chapter 5 we prove that the spectrum of L coincides with $(-\infty, 0]$ and compute estimates of the Green functions G_λ, whereas Chapter 6 treats the derivation of some Gaussian bounds for the time derivative and for the space gradient of p.

We conclude with Chapter 7, where we present some application to some special cases, including Schödinger operators and homogeneous operators with unbounded coefficients introduced in [52]. In Section 7.4 we derive a new proof of Gegenbauers generalization of the Poisson integral representation of Bessel functions $I_{\nu+n}$, where $\nu = \left(\frac{N-2}{2}\right)$ and $n \in N_0$.

The last part of the dissertation contains some reference material. Appendix A and B collect, respectively, the main results about Riemannian geometry and spherical harmonics which are used throughout the exposition; in Appendix C we recall, following [67], the equivalence between ultracontractivity and Gagliardo-Nirenberg type inequalities and provide a brief survey on the approach of Coulhon and Sikora [21] for the derivation of Gaussian estimates.

Unless otherwise specified, all the original results contained in this dissertation are based on [15, 50, 51].

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Chapter 1

The elliptic operator

This Chapter is devoted to the introduction and the analysis of the elliptic operator L defined in (1.1). In section 1.1 we collect some preliminary properties and we write L in spherical coordinates. We observe that, on subspaces defined as tensor products of radial functions and spherical harmonics, it reduces to one-dimensional Bessel operators. This property allows, in Chapter 4, to obtain a decomposition of the kernel of L in terms of its one-dimensional counterparts. Section 1.2 is concerned with the construction of the operator, via form methods, in the weighted space $L^2\left(\mathbb{R}^N, d\mu\right)$, where L becomes a self-adjoint operator. In Section 1.3 we treat the one-dimensional case, i.e. the Bessel operator, and we give an analytic proof of the explicit form of its heat kernel. Finally, in Section 1.4, we present the main results concerning generation in $L^p(\mathbb{R}^N)$ proved in [56].

Unless otherwise specified, all the results of this Chapter are based on [50].

1.1 The operator on smooth functions: first properties

Let $a > 0$, b, $c \in \mathbb{R}$ and let L be the elliptic operator formally defined, on smooth functions, by

$$L = \Delta + (a-1)\sum_{i,j=1}^{N} \frac{x_i x_j}{|x|^2} D_{ij} + c\frac{x}{|x|^2} \cdot \nabla - \frac{b}{|x|^2}. \tag{1.1}$$

The leading coefficients are uniformly elliptic (see Proposition 1.1.1 below) but discontinuous at 0, if $a \neq 1$, and singularities in the lower order terms appear when b or c is different from 0. We note that:

- If $a = 1, c = 0$, L becomes the **Schrödinger** operator with inverse-square potential:

$$L = \Delta - \frac{b}{|x|^2}.$$

- If $N = 1$, L is the one-dimensional **Bessel Operator**:

$$L = aD_{rr} + \frac{c}{r}D_r - \frac{b}{r^2}.$$

- L is equivalent, after a suitable isometry in L^p spaces (see Lemma 7.3.1), to the operator with **unbounded coefficients**:

$$J_p \circ L \circ J_p^{-1} = |x|^\alpha \Delta + \tilde{c}|x|^{\alpha-1}\frac{x}{|x|} \cdot \nabla - \tilde{b}|x|^{\alpha-2}.$$

L is invariant by rescaling that is, if $\lambda \in \mathbb{R}$ and M_λ is the dilation defined by $M_\lambda u\,(x) := u(\lambda x)$, then

$$L(M_\lambda u)(x) = \lambda^2 Lu\,(\lambda x),$$

and if Q is an orthogonal matrix in $\mathbb{R}^N$ and $M_Q u(x) = u(Qx)$, then

$$L(M_Q u)(x) = Lu(Qx).$$

For every $x \in \mathbb{R}^N \setminus \{0\}$, let $\bar{a}(x)$ be the diffusion matrix of L at x. If I is the identity matrix and $x \otimes x = \big(x_i x_j\big)_{i,j=1,\dots,N}$, then

$$\bar{a}(x) = I + (a-1)\frac{x \otimes x}{|x|^2},$$

and L takes the compact form

$$Lu = \mathrm{tr}(\bar{a}D^2) + c\frac{x}{|x|^2} \cdot \nabla - \frac{b}{|x|^2} = \sum_{i,j=1}^{N} \bar{a}_{ij}D_{ij} + c\frac{x}{|x|^2} \cdot \nabla - \frac{b}{|x|^2}.$$

We list, in the following Proposition, the main properties about the matrix $\bar{a}(x)$ whose proofs are immediate to check.

Proposition 1.1.1 *Let $x \in \mathbb{R}^N \setminus \{0\}$ and $\bar{a}(x) := I + (a-1)\frac{x\otimes x}{|x|^2}$. The following properties hold.*

(i) $\bar{a}(x) \cdot \xi = \xi + (a-1)\frac{(x,\xi)}{|x|^2}x$, *for every* $\xi \in \mathbb{R}^N$.

(ii) $\bar{a}(x)$ *is positive definite and it has eigenvalues a with eigenvector x and 1 with eigenspace the orthogonal complement of x.*

(iii) $(1 \wedge a)|\xi|^2 \leq (\bar{a}(x)\xi, \xi) \leq (1 \vee a)|\xi|^2$, *for every* $\xi \in \mathbb{R}^N$.

(iv) $\det \bar{a}(x) = a > 0$, $\bar{a}(x)$ *is non-singular and its inverse is given by the matrix*

$$\bar{a}^{-1}(x) = I + \left(\frac{1}{a} - 1\right)\frac{x \otimes x}{|x|^2}.$$

Let us employ spherical coordinates on $\mathbb{R}^N \setminus \{0\}$ (see Section B.1 in Appendix B for further details). For every $x \in \mathbb{R}^N \setminus \{0\}$ we write

$$x = r\omega, \quad \text{where} \quad r := |x|, \quad \omega := \frac{x}{|x|} \in \mathbb{S}^{N-1}.$$

If $u \in C^2(\mathbb{R}^N)$, let $D_r u$ and $D_{rr} u$ be the radial derivatives of u and let $\nabla_\tau u$ be the tangential component of its gradient. They are defined through the formulas

$$D_r u = \sum_{i=1}^{N} D_i u \, \frac{x_i}{r}, \qquad D_{rr} u = \sum_{i,j=1}^{N} D_{ij} u \, \frac{x_i x_j}{r^2}, \qquad \nabla u = D_r u \frac{x}{|x|} + \frac{\nabla_\tau u}{r}. \tag{1.2}$$

Moreover let Δ_0 be the Laplace-Beltrami operator on the sphere $\mathbb{S}^{N-1}$ (see Definition A.3.4). Besides its geometric intrinsic definition, we can define Δ_0 as follows. Given a function $f \in C^2(\mathbb{S}^{N-1})$, we consider its extension $\tilde{f}$ to $\mathbb{R}^N \setminus \{0\}$ given by $\tilde{f}(x) := f(\frac{x}{|x|})$. The gradient ∇_τ and the Laplacian Δ_0 on $\mathbb{S}^{N-1}$ are then given by

$$\nabla_\tau f(x) = \nabla \tilde{f}(x), \qquad \Delta_0 f(x) = \Delta \tilde{f}(x), \qquad \text{for every } x \in \mathbb{S}^{N-1}.$$

In spherical coordinates, we have the following decomposition for the Laplacian Δ on $\mathbb{R}^N$ in terms of radial derivatives and of the Laplace-Beltrami operator Δ_0 on $\mathbb{S}^{N-1}$:

$$\Delta = D_{rr} + \frac{N-1}{r} D_r + \frac{\Delta_0}{r^2}. \tag{1.3}$$

We refer the reader to Sections A.3 and A.4.2 in Appendix A and to Appendix B for a geometric definition of the differential operators introduced so far and for further details. In particular a proof of formulas (1.2) and (1.3) can be found in Proposition B.1.4. We suggest, furthermore, [79, Section 5, Chapter IX] for finding the explicit expression of Δ_0 in spherical coordinates.

In analogy with formula (1.3), we highlight, in the following proposition, the expression of L in spherical coordinates. We recall, preliminary, that a spherical harmonic P of order $n \in \mathbb{N}_0$ is the restriction to $\mathbb{S}^{N-1}$ of a homogeneous harmonic polynomial of degree n (see Section B.2). P is an eigenfunction of the Laplace-Beltrami operator Δ_0 whose eigenvalue is $-\lambda_n = -(n^2 + (N-2)n)$; P, therefore satisfies:

$$\Delta_0 P = -(n^2 + (N-2)n)P.$$

Proposition 1.1.2 *Let Δ_0 be the Laplace-Beltrami operator of $\mathbb{S}^{N-1}$. Then, employing the spherical coordinates $x = r\omega$, the operator L defined in (1.1) admits the following decomposition:*

$$\begin{aligned} L &= \Delta + (a-1) \sum_{i,j=1}^{N} \frac{x_i x_j}{|x|^2} D_{ij} + c \frac{x}{|x|^2} \cdot \nabla - \frac{b}{|x|^2} \qquad (1.4) \\ &= a D_{rr} + \frac{N-1+c}{r} D_r - \frac{b - \Delta_0}{r^2}. \end{aligned}$$

Let P be a spherical harmonic of order n and $u \in C_c^\infty(\mathbb{R}^+)$. If uP is the function of $\mathbb{R}^N \setminus \{0\}$ defined by $(uP)(x) = u(r)P(\omega)$, then $uP \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ and

$$L(uP) = \left(au_{rr} + \frac{N-1+c}{r} u_r - \frac{b+\lambda_n}{r^2} u \right) P(\omega). \tag{1.5}$$

Proof. (1.4) follows by inserting formulas (1.2) and (1.3) into the expression (1.1). The second claim is, then, a consequence of the decomposition of L just proved and of the relation $\Delta_0 P = -\lambda_n P$.

□

According to equation (1.4), L splits into a sum of two operators, namely

$$L = L_0 + \frac{\Delta_0}{r^2},$$

where L_0 is the radial operator

$$L_0 := aD_{rr} + \frac{N-1+c}{r} D_r - \frac{b}{r^2}.$$

The latter equality defines a one-dimensional Bessel operator which is the one-dimensional counterpart of L. A detailed analysis of Bessel operators will be the object of section 1.3.

Defining L_n as the Bessel operator

$$L_n := aD_{rr} + \frac{N-1+c}{r} D_r - \frac{b+\lambda_n}{r^2}, \tag{1.6}$$

then equation (1.5) becomes

$$L(uP) = \left(au_{rr} + \frac{N-1+c}{r} u_r - \frac{b+\lambda_n}{r^2} u \right) P(\omega) = (L_n u)(r) P(\omega).$$

L_n is obtained from the radial part of L, substituting b with $b+\lambda_n$. The last relation shows that, on subspaces defined as tensor products of radial functions and spherical harmonics, the operator L reduces to one-dimensional Bessel operator. This property is the starting point which leads, in Section 4.1, to a complete decomposition of L in terms of its one-dimensional counterparts $(L_n)_{n \in \mathbb{N}_0}$.

On the other hand, formula (1.4) allows to find radial solutions of the homogeneous equation $Lu = 0$.

Indeed, if $s \in \mathbb{R}$, $r = |x|$ and $u(x) = r^s$, then $Lu = 0$ is equivalent to $L_0 r^s = 0$. A simple calculation shows, therefore, that $Lu = 0$ has radial solutions

$$|x|^{-s_1}, \quad |x|^{-s_2},$$

where s_1, s_2 are the roots of the indicial equation

$$f(s) = -as^2 + (N-1+c-a)s + b = 0.$$

s_1 and s_2 are consequently given by

$$s_1 := \frac{N-1+c-a}{2a} - \sqrt{D}, \quad s_2 := \frac{N-1+c-a}{2a} + \sqrt{D} \tag{1.7}$$

where D is the discriminant

$$D := \frac{b}{a} + \left(\frac{N-1+c-a}{2a}\right)^2. \tag{1.8}$$

The above numbers are real if and only if $D \geq 0$. In what follows we refer to D as the discriminant of the operator L and, unless otherwise specified, we will always assume $D \geq 0$.

Remark 1.1.3 *In [56] the authors show that the positivity of D, as well as the reality of s_1 and s_2, takes a fundamental role in the generation properties of L in $L^p(\mathbb{R}^N)$.*
Indeed assuming $D \geq 0$ and $1 < p < \infty$, there exists a realization of L which generates a positive semigroup in $L^p(\mathbb{R}^N)$ if and only if $\frac{N}{p} \in (s_1, s_2+2)$ (see section 1.4).

On the other hand, when $D < 0$, the equation $u - Lu = f$ cannot have positive distributional solutions for certain positive f, see [56], [55] and Proposition 1.3.1. This fact constitutes an elliptic counterpart of a famous result due to Baras and Goldstein (see [10]) in the case of the Schrödinger operator with inverse square potential where the above condition reads $D = b + (N-2)^2/4 \geq 0$.
We point out, however, that even when $b + (N-2)^2/4$ is negative there are realizations of the Schrödinger operator L in $L^2(\mathbb{R}^N)$ which generate analytic semigroups, see [53]. Such semigroups are not positive and these realizations are necessarily non self-adjoint.

1.2 The operator in $L^2(\mathbb{R}^N, d\mu)$

Let

$$L = \Delta + (a-1)\sum_{i,j=1}^{N} \frac{x_i x_j}{|x|^2} D_{ij} + c\frac{x}{|x|^2}\cdot\nabla - \frac{b}{|x|^2} = aD_{rr} + \frac{N-1+c}{r}D_r - \frac{b-\Delta_0}{r^2},$$

where D_r, D_{rr} denote radial derivatives and Δ_0 is the Laplace-Beltrami on $\mathbb{S}^{N-1}$. In what follows we consider also the one-dimensional situation, by agreeing that Δ_0 is not present and that $\mathbb{R}^N$ stands for $(0,\infty)$ when $N = 1$.

Let us introduce the parameter

$$\gamma := \frac{N-1+c}{a} - N + 1, \tag{1.9}$$

Recalling the definition of the diffusion matrix $\overline{a} = (\overline{a}_{ij})$,

$$\overline{a}_{ij}(x) = \delta_{ij} + (a-1)|x|^{-2}x_i x_j, \quad x \in \mathbb{R}^N \setminus \{0\},$$

the operator L can be written in the weighted divergence form

$$L = |x|^{-\gamma}\mathrm{div}(|x|^{\gamma}\overline{a}\cdot\nabla) - \frac{b}{|x|^2}. \tag{1.10}$$

We recall that the condition

$$D = \frac{b}{a} + \left(\frac{N-1+c-a}{2a}\right)^2 \geq 0$$

is necessary and sufficient to get positive solutions, see [56], and we shall always assume it. Note that L is formally self-adjoint with respect to the measure $d\mu := |x|^{\gamma}dx$ and, in particular, $\gamma = 0$ if and only if L is formally self-adjoint with respect to the Lebesgue measure.

Using (1.10), we define L through a symmetric form in a weighted space. To this aim we note that, by Proposition 1.1.1, the matrix $\overline{a}(x)$ has eigenvalues a with eigenvector x and 1 with eigenspace the orthogonal complement of x.

Definition 1.2.1 *Consider the sesquilinear form $\mathfrak{a}$ in $L^2_\mu = L^2(\mathbb{R}^N, d\mu)$ with $d\mu = |x|^{\gamma}dx$ defined by*

$$\mathfrak{a}(u,v) := \int_{\mathbb{R}^N}\left(\langle\overline{a}\nabla u, \nabla v\rangle + \frac{b}{|x|^2}u\overline{v}\right)d\mu,$$
$$D(\mathfrak{a}) := C_c^\infty(\mathbb{R}^N\setminus\{0\}).$$

Denoting by $\nabla_\tau u$ the tangential component of the gradient and recalling that $\nabla u = u_r\frac{x}{|x|} + \frac{\nabla_\tau u}{r}$, the form $\mathfrak{a}$ becomes

$$\begin{aligned}\mathfrak{a}(u,v) &:= \int_{\mathbb{S}^{N-1}}\int_0^\infty\left[\left\langle\overline{a}\left(u_r\frac{x}{|x|}+\frac{\nabla_\tau u}{r}\right), v_r\frac{x}{|x|}+\frac{\nabla_\tau v}{r}\right\rangle + \frac{b}{r^2}u\overline{v}\right] r^{\frac{N-1+c}{a}}\,dr\,d\sigma\\ &= \int_{\mathbb{S}^{N-1}}\int_0^\infty\left(au_r\overline{v}_r + \frac{\nabla_\tau u\,\nabla_\tau\overline{v}}{r^2} + \frac{b}{r^2}u\overline{v}\right) r^{\frac{N-1+c}{a}}\,dr\,d\sigma.\end{aligned}$$

We provide preliminary a simple proof of the Hardy inequality which can be found in [24, Lemma 5.3.1].

Lemma 1.2.2 *Let $b, s \in \mathbb{R}$. For every $u \in C_c^\infty((0,+\infty))$ setting $v = ur^{\frac{s-1}{2}}$ one has*

$$\int_0^\infty\left[|u'|^2 + \frac{b}{r^2}u^2\right]r^s\,dr = \int_0^\infty\left[|v'|^2 + \left(b + \frac{(s-1)^2}{4}\right)\frac{v^2}{r^2}\right]r\,dr.$$

Proof. We have

$$|u'|^2 + \frac{b}{r^2}u^2 = r^{1-s}\left[|v'|^2 + \frac{(s-1)^2}{4}\frac{v^2}{r^2} - \frac{s-1}{r}vv' + \frac{b}{r^2}v^2\right]$$

and observing that $\int_0^\infty vv'\,dr = 0$ we obtain

$$\int_0^\infty \left[|u'|^2 + \frac{b}{r^2}u^2\right] r^s\,dr = \int_0^\infty \left[|v'|^2 + \left(b + \frac{(s-1)^2}{4}\right)\frac{v^2}{r^2} - \frac{s-1}{r}vv'\right] r\,dr$$
$$= \int_0^\infty \left[|v'|^2 + \left(b + \frac{(s-1)^2}{4}\right)\frac{v^2}{r^2}\right] r\,dr.$$

□

We immediately deduce from the last lemma Hardy's inequality in $\mathbb{R}^N$. We recall that D is defined in (1.8).

Corollary 1.2.3 *Given $u, v \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ let $u = u_1|x|^{-\frac{N-1+c-a}{2a}}$ and $v = v_1|x|^{-\frac{N-1+c-a}{2a}}$. Then*

$$\mathfrak{a}(u,v) = \int_{\mathbb{R}^N} |x|^{2-N}\left[a(u_1)_r\overline{(v_1)}_r + \frac{\nabla_\tau u_1\,\nabla_\tau \overline{v_1}}{|x|^2} + \frac{aD}{|x|^2}u_1\overline{v_1}\right] dx$$
$$= \int_{\mathbb{S}^{N-1}}\int_0^\infty r\left[a(u_1)_r\overline{(v_1)}_r + \frac{\nabla_\tau u_1\,\nabla_\tau \overline{v_1}}{r^2} + \frac{aD}{r^2}u_1\overline{v_1}\right] dr\,d\sigma.$$

In particular

$$\mathfrak{a}(u,u) = \int_{\mathbb{S}^{N-1}}\int_0^\infty r\left[a|(u_1)_r|^2 + \frac{|\nabla_\tau u_1|^2}{r^2} + \frac{aD}{r^2}|u_1|^2\right] dr\,d\sigma \geq 0 \tag{1.11}$$

if $D \geq 0$.

Note that if $a = 1$, $c = 0$ then $\gamma = 0$ and $D = b + (N-2)^2/4$ and we recover the classical Hardy's inequality

$$\mathfrak{a}(u,u) = \int_{R^N}\left(|\nabla u|^2 - \frac{(N-2)^2}{4}\frac{|u|^2}{|x|^2}\right) dx \geq 0. \tag{1.12}$$

The following lemma corresponds to the Friedrichs extension of $\mathfrak{a}$. We give a direct proof which is elementary and allows to describe the domain of the closure.

Lemma 1.2.4 *If $D \geq 0$, $\mathfrak{a}$ is nonnegative, symmetric and closable in $L^2_\mu = L^2(\mathbb{R}^N, d\mu)$. Denoting by $\tilde{\mathfrak{a}}$ the closure of $\mathfrak{a}$, the following properties hold:*

(i) if $D = 0$, $D(\tilde{\mathfrak{a}}) = \left\{u \in L^2_\mu : u|x|^{\frac{N-1+c-a}{2a}} \in H^1_0\left(\mathbb{R}^N \setminus \{0\}, |x|^{2-N}dx\right)\right\}$, where $H^1_0\left(\mathbb{R}^N \setminus \{0\}, |x|^{2-N}dx\right)$ is the closure of $C_c^\infty(\mathbb{R}^N \setminus \{0\})$ with respect to the norm

$$\|v\|^2_{H^1_0(\mathbb{R}^N,|x|^{2-N}dx)} = \int_{\mathbb{R}^n}\left[|\nabla v|^2 + |v|^2\right]|x|^{2-N}dx;$$

(ii) if $D > 0$ then $D(\tilde{\mathfrak{a}}) = \left\{u \in L^2_\mu : u|x|^{\frac{\gamma}{2}} \in H^1_0\left(\mathbb{R}^N\right)\right\}$;

(iii) if M_s is the dilation defined by $M_s u(x) = u(sx)$ then for every $u, v \in D(\tilde{\mathfrak{a}})$ and $s > 0$ we have $M_s u, M_s v \in D(\tilde{\mathfrak{a}})$ and

$$\tilde{\mathfrak{a}}(M_s u, M_s v) = s^{2-\gamma-N}\, \tilde{\mathfrak{a}}(u, v);$$

(iv) if Q is an orthogonal matrix in $\mathbb{R}^N$ and $M_Q u(x) = u(Qx)$, then for every $u, v \in D(\tilde{\mathfrak{a}})$, $M_Q u, M_Q v \in D(\tilde{\mathfrak{a}})$ and

$$\tilde{\mathfrak{a}}(M_Q u, M_Q v) = \tilde{\mathfrak{a}}(u, v).$$

Proof. Clearly $\mathfrak{a}$ is a non-negative symmetric form in $L^2(\mathbb{R}^N, d\mu)$, from the previous corollary. To prove its closability, by [67, Proposition 1.13], it is sufficient to show that if $(u_n)_n \subset C_c^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$ satisfies $u_n \to 0$ in L^2_μ as $n \to \infty$ and $\mathfrak{a}(u_n - u_m, u_n - u_m) \to 0$ as $n, m \to \infty$, then $\mathfrak{a}(u_n, u_n) \to 0$ as $n \to \infty$. Since $\mathfrak{a}$ is locally uniformly elliptic then $u_m \to 0$ in $H^1_{\mathrm{loc}}\left(\mathbb{R}^N \setminus \{0\}\right)$. Passing to a subsequence we may assume that $u_m, \nabla u_m \to 0$ pointwise. From (1.11) setting $u_n = v_n |x|^{-\frac{N-1+c-a}{2a}}$ we have

$$\mathfrak{a}(u_n, u_n) = \int_{\mathbb{S}^{N-1}} \int_0^\infty r \left[a|(v_n)_r|^2 + \frac{|\nabla_\tau v_n|^2}{r^2} + \frac{aD}{r^2}|v_n|^2 \right] dr\, d\sigma.$$

and then Fatou's Lemma yields

$$\mathfrak{a}(u_n, u_n) \le \liminf_{m \to \infty} \mathfrak{a}(u_n - u_m, u_n - u_m).$$

This proves the closability of $\mathfrak{a}$.
Let now $\tilde{\mathfrak{a}}$ be the closure of $\mathfrak{a}$. By [67, Proposition 1.13] $D(\tilde{\mathfrak{a}})$ is the closure of $C_c^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$ with respect to the norm $||u||_{\mathfrak{a}} := \sqrt{\mathfrak{a}(u, u) + ||u||^2_{L^2_\mu}}$.
If $D = 0$, let $u \in C_c^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$ and set $u = v|x|^{-\frac{N-1+c-a}{2a}}$. From (1.11)

$$\mathfrak{a}(u, u) = \int_{\mathbb{R}^N} |x|^{2-N} \left[a|v_r|^2 + \frac{|\nabla_\tau v|^2}{|x|^2} \right] dx$$

and recalling that $a > 0$ and $\nabla v = v_r \frac{x}{|x|} + \frac{\nabla_\tau v}{r}$ we easily recognize that the last integral is equivalent to $\|\nabla v\|^2_{L^2(\mathbb{R}^N, |x|^{2-N} dx)}$. Since also the norms of u in L^2_μ and v in $L^2(|x|^{2-N}\, dx)$ coincide, we see that the norms $||u||_{\mathfrak{a}}$ and $||v||_{H^1_0(\mathbb{R}^N, |x|^{2-N} dx)}$ are equivalent on $C_c^\infty(\mathbb{R}^N \setminus \{0\})$ and (i) follows.
Suppose now that $D > 0$ and let $u \in C_c^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$. Setting $u = v|x|^{-\frac{\gamma}{2}}$ and proceeding as before we obtain

$$\mathfrak{a}(u, u) = \int_{\mathbb{R}^N} \left[a|v_r|^2 + \frac{|\nabla_\tau v|^2}{|x|^2} + a\left(D - \frac{(N-2)^2}{4} \right) \frac{v^2}{|x|^2} \right] dx$$

Then from the Hardy inequality (1.12), $||u||_{\mathfrak{a}}$ and $||v||_{H^1_0(\mathbb{R}^N)}$ are equivalent norms and so

$$D(\tilde{\mathfrak{a}}) = \left\{ u \in L^2(\mathbb{R}^N, d\mu) : u|x|^{\frac{\gamma}{2}} \in H^1_0\left(\mathbb{R}^N\right) \right\}$$

since $C_c^\infty(\mathbb{R}^N\setminus\{0\})$ is dense in $H_0^1(\mathbb{R}^N)$, $N\geq 2$, and clearly $C_c^\infty(0,\infty)$ is dense in $H_0^1(0,\infty)$ when $N=1$.

Consider now the third statement. Let $u\in D(\tilde{\mathfrak{a}})$ and $s>0$. From the definition of $\tilde{\mathfrak{a}}$, there exists a sequence $\{u_n\}_n \in C_c^\infty\left(\mathbb{R}^N\setminus\{0\}\right)$ such that $u_n\to u$ in L^2_μ and $\mathfrak{a}(u_n-u_m,u_n-u_m)\to 0$ as $n,m\to\infty$. Therefore $M_s u_n\to M_s u$ in L^2_μ,

$$
\begin{aligned}
&\mathfrak{a}(M_s u_n - M_s u_m, M_s u_n - M_s u_m)\\
&= \int_{\mathbb{R}^N}\left(s^2\langle \overline{a}\nabla(u_n-u_m)(sx),\nabla(u_n-u_m)(sx)\rangle + \frac{b}{|x|^2}|(u_n-u_m)(sx)|^2\right)|x|^\gamma\,dx\\
&= s^{2-\gamma-N}\int_{\mathbb{R}^N}\left(\langle \overline{a}\nabla(u_n-u_m)(y),\nabla(u_n-u_m)(y)\rangle + \frac{b}{|y|^2}|(u_n-u_m)(y)|^2\right)|y|^\gamma\,dy\\
&= s^{2-\gamma-N}\mathfrak{a}(u_n-u_m,u_n-u_m)\to 0,
\end{aligned}
$$

and hence $M_s u\in D(\tilde{\mathfrak{a}})$.

Finally let $u,\ v\in D(\tilde{\mathfrak{a}})$ and $s>0$. By approximating u and v with $\{u_n\}_n,\{v_n\}_n\in C_c^\infty\left(\mathbb{R}^N\setminus\{0\}\right)$ as above and using

$$\mathfrak{a}(M_s u_n, M_s v_n) = s^{2-\gamma-N}\,\mathfrak{a}(u_n,v_n)$$

we obtain (iii) letting n to infinity. The proof of (iv) is similar. □

Remark 1.2.5 *We point out that, by construction,*

$$W_c^{1,\infty}(\mathbb{R}^N\setminus\{0\}) := \left\{u\in W^{1,\infty}(\mathbb{R}^N)\ ;\ \textit{supp}\, u \textit{ is compact in } \mathbb{R}^N\setminus\{0\}\right\}\subseteq D(\tilde{\mathfrak{a}})$$

and, for every $u,v\in W_c^{1,\infty}(\mathbb{R}^N\setminus\{0\})$,

$$\tilde{\mathfrak{a}}(u,v) = \int_{\mathbb{R}^N}\left(\langle\overline{a}\nabla u,\nabla v\rangle + \frac{b}{|x|^2}u\overline{v}\right)d\mu.$$

If $D\geq 0$, let $-L$ be the operator associated to $\tilde{\mathfrak{a}}$, that is

$$
\begin{aligned}
&D(L) := \left\{u\in D(\tilde{\mathfrak{a}})\ ;\ \exists v\in L^2_\mu \text{ s.t. } \tilde{\mathfrak{a}}(u,w) = \int_{\mathbb{R}^N} v\overline{w}\,d\mu \quad \forall w\in D(\mathfrak{a})\right\},\\
&-Lu := v.
\end{aligned}
\tag{1.13}
$$

The basic properties of L are listed below, see also [50, 56].

Proposition 1.2.6 *If $D\geq 0$, the operator $-L$ defined in (1.13) is nonnegative and self-adjoint. Moreover,* *(i)* $C_c^\infty(\mathbb{R}^N \setminus \{0\}) \hookrightarrow D(L)$ *and for every* $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$

$$Lu = \sum_{i,j=1}^N \overline{a}_{ij} D_{ij} u + c\frac{x}{|x|^2} \cdot \nabla u - \frac{b}{|x|^2} u.$$

(ii) $D(L) \hookrightarrow \{u \in L^2_\mu \cap W^{2,p}_{\rm loc}(\mathbb{R}^N \setminus \{0\})\ ;\ Lu \in L^2_\mu\}$.

(iii) $C_c^\infty\left(\mathbb{R}^N \setminus \{0\}\right) \hookrightarrow D(L^n) \hookrightarrow H^{2n}_{loc}\left(\mathbb{R}^N \setminus \{0\}\right)$.

(iv) L *is invariant under orthogonal transformation and its scale homogeneity is* 2*, which means*

$$s^2 L = M_s^{-1} L M_s, \qquad L = M_{Q^*} L M_Q, \tag{1.14}$$

where $s > 0$ *and* Q *is an orthogonal matrix.*

(v) L *generates a contractive analytic semigroup* $\left\{e^{zL} :\ z \in \mathbb{C}_+\right\}$ *in* L^2_μ *which satisfies*

$$e^{s^2 zL} = M_s^{-1} e^{zL} M_s, \qquad e^{zL} = M_{Q^*} e^{zL} M_Q \qquad z \in \mathbb{C}_+,\ s > 0. \tag{1.15}$$

For $t \geq 0$ *the semigroup* e^{tL} *is positive and irreducible, that is* $e^{tL} f > 0$ *a.e. if* $f \geq 0$*,* $f \neq 0$.

(vi) The semigroup is represented by a kernel p *with respect to the measure* $d\mu = |y|^\gamma\, dy$ *which satisfies*

$$p(s^2 z, x, y) = s^{-N-\gamma} p\left(z, \frac{x}{s}, \frac{y}{s}\right), \qquad p(z, x, y) = p(z, Qx, Qy) \tag{1.16}$$

for $z \in \mathbb{C}_+$*,* $s > 0$*,* $x, y \in \mathbb{R}^N \setminus \{0\}$.

Proof. (i) is clear by construction and (ii) follows from interior elliptic regularity. (iii) follows from (i) and (ii), by induction. Concerning (iv), let $u \in D(L)$ and $v \in D(\tilde{\mathfrak{a}})$. Then

$$\tilde{\mathfrak{a}}(M_s u, v) = s^{2-\gamma-N} \tilde{\mathfrak{a}}(u, M_{s^{-1}} v) = -s^{2-\gamma-N} \int_{\mathbb{R}^N} (Lu) \overline{M_{s^{-1}} v}\, d\mu = -s^2 \int_{\mathbb{R}^N} (M_s L u) \overline{v}\, d\mu$$

hence $M_s u \in D(L)$ and $L M_s u = s^2 M_s L u$. Similarly for M_Q. The generation property of L follows by standard results on non-negative and self-adjoint operators (see also Section C.1); (1.15) reflects (1.14). Concerning the positivity of e^{tL}, it follows from [67, Theorem 2.6], since the form is real, $u^+, u^- \in D(\mathfrak{a})$ when $u \in D(\mathfrak{a})$ and $\mathfrak{a}(u^+, u^-) = 0$. Irreducibility follows from [67, Theorem 4.5]. The existence of the Heat kernel p will be proved in Theorem 3.5.3 in Chapter 3. Finally, properties (vi) are simple consequences of (v), taking into account that the reference measure is $|y|^\gamma\, dy$.

$\square$

In Chapter 4 we will prove that the resolvent of L and the generated semigroup are the direct sum of the corresponding resolvents and semigroups generated by the one-dimensional Bessel operators L_n, $n \in \mathbb{N}_0$, defined in (1.6). In the following section, we analyse this operators in details.

1.3 The one dimensional case: Bessel operators

In this section we find an explicit formula for the heat kernel of the operator

$$Lu = u_{rr} + \frac{c}{r}u_r - \frac{b}{r^2}u$$

considered in $L^2((0,\infty), r^c dr)$. L is the one dimensional version of the operator defined in (1.13) choosing $a = 1$ and restricted to the positive half-line $(0,\infty)$. As before we assume the condition $D = b + \frac{(c-1)^2}{4} \geq 0$, which is necessary and sufficient for the existence of a positive resolvent. In fact, when $D < 0$, the next proposition shows that, through the change of variable $r = e^s$ and the Sturm Comparison Theorem, that every solution of the homogeneuous equation $\lambda u - Lu = 0$, $\lambda > 0$, oscillates near zero and therefore there is no way to construct a positive resolvent. We refer the reader also to [52] for the proof and to [53] and [55] for an investigation of the generation properties of L when $D < 0$ and for uniqueness problems.

Proposition 1.3.1 [52] *Let $D = b + \left(\frac{c-1}{2}\right)^2 < 0$. Then for every $\lambda > 0$ there exists a nonnegative function $0 \leq \phi \in C_c^\infty((0,\infty))$, $\phi \not\equiv 0$, such that the problem*

$$\lambda v - Lv = \phi \tag{1.17}$$

does not admit any positive solution in $(0,\infty)$.

Proof. By scaling we may assume that $\lambda = 1$. Suppose that there exists a distributional solution $v \geq 0$ of (1.17) in $(0,\infty)$. By local elliptic regularity, $v \in C^\infty(0,+\infty)$. Setting $w(s) = e^{(\frac{c-1}{2})s}v(e^s)$ we get

$$w''(s) = (D + e^{2s})w(s) - e^{(\frac{c+3}{2})s}\phi(e^s), \quad s \in \mathbb{R},$$

where, by hypothesis,

$$D = b + \left(\frac{c-1}{2}\right)^2 < 0.$$

We choose $m \in \mathbb{R}$ such that $(D + e^{2s}) \leq D/2 < 0$ for $s \leq m$. By the Sturm comparison theorem all non-zero solutions of the homogeneous equation

$$\zeta''(s) = (k + e^{2s})\zeta(s) \tag{1.18}$$

are oscillating for $s \le m$. In particular any solution of the homogeneous equation (1.17), with $\phi = 0$, is oscillating near 0.

By variation of parameters we write

$$w(s) = u_2(s) \int_{-\infty}^{s} u_1(t)g(t)dt + u_1(s) \int_{s}^{\infty} u_2(t)g(t)dt + c_1u_1(s) + c_2u_2(s),$$

where $c_1, c_2 \in \mathbb{C}$, $g(s) = e^{(\frac{c+3}{2})s}\phi(e^s)$ and u_i, $i = 1, 2$ are linearly independent solutions of (1.18) with Wronskian equal to 1. Since g is compactly supported we have for s near $-\infty$

$$w(s) = u_1(s) \int_{supp\, g} u_2(t)g(t)dt + c_1u_1(s) + c_2u_2(s).$$

However w is non-negative, because $v \ge 0$, and also oscillating near $-\infty$ since solves (1.18). Hence $w = 0$ near $-\infty$ and therefore

$$c_1 = -\int_{supp\, g} u_2(t)g(t)dt, \quad c_2 = 0.$$

This gives

$$\begin{aligned} w(s) &= u_2(s) \int_{-\infty}^{s} u_1(t)g(t)dt + u_1(s) \int_{s}^{\infty} u_2(t)g(t)dt - u_1(s) \int_{supp\, g} u_2(t)g(t)dt \\ &= u_2(s) \int_{-\infty}^{s} u_1(t)g(t)dt - u_1(s) \int_{-\infty}^{s} u_2(t)g(t)dt = \\ &\int_{-\infty}^{s} (u_1(t)u_2(s) - u_1(s)u_2(t))g(t)dt. \end{aligned}$$

For fixed s the function $t \mapsto G(s,t) = u_1(t)u_2(s) - u_1(s)u_2(t)$ is also oscillating near $t = -\infty$. Therefore, if we choose $g \ne 0$ such that $G(s,t) < 0$ on $supp\, g$, we get $w(s) < 0$ and this contradicts $v \ge 0$.

□

The heat kernel of the Bessel operator is usually deduced by probabilistic tools. We refer, however, to [41] where the author uses the Weyl-Kodaira theory for Sturm Liouville problems. We give a purely analytic proof of the heat kernel formula which has also the advantage to appear consistent with the construction of the operator, see also Remark 1.3.9.

1.3.1 Definition of the operator

Let us consider now $b, c \in \mathbb{R}$ such that $D = b + \left(\frac{c-1}{2}\right)^2 \ge 0$ and

$$Lu = u_{rr} + \frac{c}{r}u_r - \frac{b}{r^2}u = r^{-c}\frac{d}{dr}\left(r^c u_r\right) - \frac{b}{r^2}.$$

Note that the parameter γ in (1.9) coincides with c. If $u = r^{-\frac{c}{2}}v$ and setting $\nu^2 = b + \left(\frac{c-1}{2}\right)^2$ then

$$Lu = r^{-\frac{c}{2}}\left[v_{rr} - \left(b + \left(\frac{c-1}{2}\right)^2 - \frac{1}{4}\right)\frac{v}{r^2}\right] = r^{-\frac{c}{2}}L_\nu v.$$

As in the N-dimensional case L can be defined through a symmetric form in the weighted space $L^2((0,\infty), r^c dr)$. Let

$$\mathfrak{b}(u,v) := \int_0^\infty \left(u_r \overline{v_r} + b\frac{u\overline{v}}{r^2}\right) r^c dr, \qquad D(\mathfrak{b}) := C_c^\infty((0,\infty))$$

The following Lemma is the one-dimensional version of Lemma 1.2.4.

Lemma 1.3.2 *$\mathfrak{b}$ is a nonnegative, symmetric and closable form in $L^2((0,\infty), r^c dr)$. Denoting by $\tilde{\mathfrak{b}}$ the closure of $\mathfrak{b}$, the following proprieties hold:*

(i) if $b + \left(\frac{c-1}{2}\right)^2 = 0$, $D(\tilde{\mathfrak{b}}) = \left\{u \in L^2((0,\infty), r^c dr) : r^{\frac{c-1}{2}} u \in H_0^1\big((0,\infty), r dr\big)\right\}$;

(ii) if $b + \left(\frac{c-1}{2}\right)^2 > 0$, $D(\tilde{\mathfrak{b}}) = \left\{u \in L^2((0,\infty), r^c dr) : r^{\frac{c}{2}} u \in H_0^1\big((0,\infty), dr\big)\right\}$;

(iii) if M_s is the dilation defined by $M_s u(r) = u(sr)$ then for every $u, v \in D(\tilde{\mathfrak{b}})$ and $s > 0$ we have $M_s u \in D(\tilde{\mathfrak{b}})$ and

$$\tilde{\mathfrak{b}}(M_s u, v) = s^{1-c}\,\tilde{\mathfrak{b}}(u, M_{s^{-1}} v), \qquad \tilde{\mathfrak{b}}(M_s u, M_s v) = s^{1-c}\,\tilde{\mathfrak{b}}(u,v).$$

Let $-L$ be the operator associated to $\tilde{\mathfrak{b}}$, that is

$$D(L) := \left\{u \in D(\tilde{\mathfrak{b}})\ ;\ \exists v \in L^2((0,\infty), r^c dr) \text{ s.t. } \tilde{\mathfrak{b}}(u,w) = \int_0^\infty v\overline{w}\, r^c dr \quad \forall w \in D(\mathfrak{b})\right\},$$

$$-Lu := v.$$

The next proposition shows the basic properties of L.

Proposition 1.3.3 *$-L$ is nonnegative and self-adjoint. Moreover,*

(i) $Lu = u_{rr} + \frac{c}{r} u_r - \frac{b}{r^2} u$ for every $u \in C_c^\infty((0,\infty))$.

(ii) $s^2 L = M_s^{-1} L M_s$, $\quad s > 0$, where M_s is the dilation defined by $M_s u(r) = u(sr)$.

If $c = 0$ The operator L becomes a Schrödinger operator with inverse square potential. Since we have assumed $b \geq -\frac{1}{4}$ in order to get positive solutions, we may write our operator in the form

$$L_\nu u = u_{rr} - \left(\nu^2 - \frac{1}{4}\right)\frac{u}{r^2}.$$

It follows from the previous results that L_ν is the operator associated to the the Friedrichs extension $\tilde{\mathfrak{a}_\nu}$ of the sesquilinear form $\mathfrak{a}_\nu$ defined in $L^2(0,\infty)$ by

$$\mathfrak{a}_\nu(u,v) := \int_0^\infty \left(u_r\overline{v_r} + \left(\nu^2 - \frac{1}{4}\right)\frac{u\overline{v}}{r^2}\right) dr,$$

$$D(\mathfrak{a}_\nu) := C_c^\infty((0,\infty)).$$

We investigate now the relation between the operators L and L_ν. Let us set $\nu^2 = b + \left(\frac{c-1}{2}\right)^2$ and consider, for this reason, the map

$$\Phi : u \in L^2\left((0,\infty), r^c dr\right) \mapsto r^{\frac{c}{2}} u \in L^2\left(0,\infty\right).$$

Obviously Φ is an isometry which preserves $C_c^\infty\left(0,\infty\right)$. Moreover integrating by parts we easily obtain $a_\nu(\Phi u, \Phi v) = \mathfrak{b}(u,v)$, for $u, v \in C_c^\infty\left(0,\infty\right)$. This relation between the two forms translates into a similar relation between the associated operator which we point out in the next proposition.

Proposition 1.3.4 *Let L and L_ν be the operators defined above. Then*

$$L = \Phi^{-1} L_\nu \Phi.$$

Proof. We have already observed that $a_\nu(\Phi u, \Phi v) = \mathfrak{b}(u,v)$, for $u, v \in C_c^\infty\left((0,\infty)\right)$. The same relation is inherited by the extensions and so $\tilde{b}(u,v) = \tilde{a_\nu}(\Phi u, \Phi v)$. For every $u \in D(\mathfrak{b})$ and $w \in C_c^\infty\left(0,+\infty\right)$

$$\begin{aligned} -\int_0^\infty \left(\Phi^{-1} L_\nu \Phi u\right) \overline{w}\, r^c dr &= -\int_0^\infty \left(L_\nu \Phi u\right) \overline{w}\, r^{\frac{c}{2}} dr = \\ &= \tilde{\mathfrak{a}_\nu}(\Phi u, \Phi w) = \tilde{\mathfrak{b}}(u,w) \end{aligned}$$

This proves $L = \Phi^{-1} L_\nu \Phi$. □

1.3.2 The resolvent and the heat kernel of L

We first consider the case $c = 0$ corresponding to the operator L_ν and then apply Proposition 1.3.4.

We recall some well-known facts about the modified Bessel functions I_ν and K_ν which constitute a basis of solutions of the modified Bessel equation

$$r^2 \frac{d^2 v}{dr^2} + r \frac{dv}{dr} - (r^2 + \nu^2) v = 0, \quad r > 0.$$

We recall that

$$I_\nu(r) = \left(\frac{r}{2}\right)^\nu \sum_{m=0}^\infty \frac{1}{m!\,\Gamma(\nu + 1 + m)} \left(\frac{r}{2}\right)^{2m}, \quad K_\nu(r) = \frac{\pi}{2} \frac{I_{-\nu}(r) - I_\nu(r)}{\sin \pi\nu},$$

where limiting values are taken for the definition of K_ν when ν is an integer. The basic properties of these functions we need are collected in the following lemma, see e.g., [2, 9.6 and 9.7].

Lemma 1.3.5 *For every $\nu \geq 0$, I_ν is increasing, K_ν is decreasing and satisfy the following asymptotic behaviour. If $r \to \infty$,*

$$|I_\nu(r)| \approx r^{-\frac{1}{2}} e^r, \quad |I'_\nu(r)| \approx r^{-\frac{1}{2}} e^r, \quad |K_\nu(r)| \approx r^{-\frac{1}{2}} e^{-r}, \quad |K'_\nu(r)| \approx r^{-\frac{1}{2}} e^{-r}.$$

Moreover, if $\nu > 0$, then as $r \to 0$,

$$|I_\nu(r)| \approx r^\nu, \quad |I'_\nu(r)| \approx r^{\nu-1}, \quad |K_\nu(r)| \approx r^{-\nu}, \quad |K'_\nu(r)| \approx r^{-\nu-1},$$

and

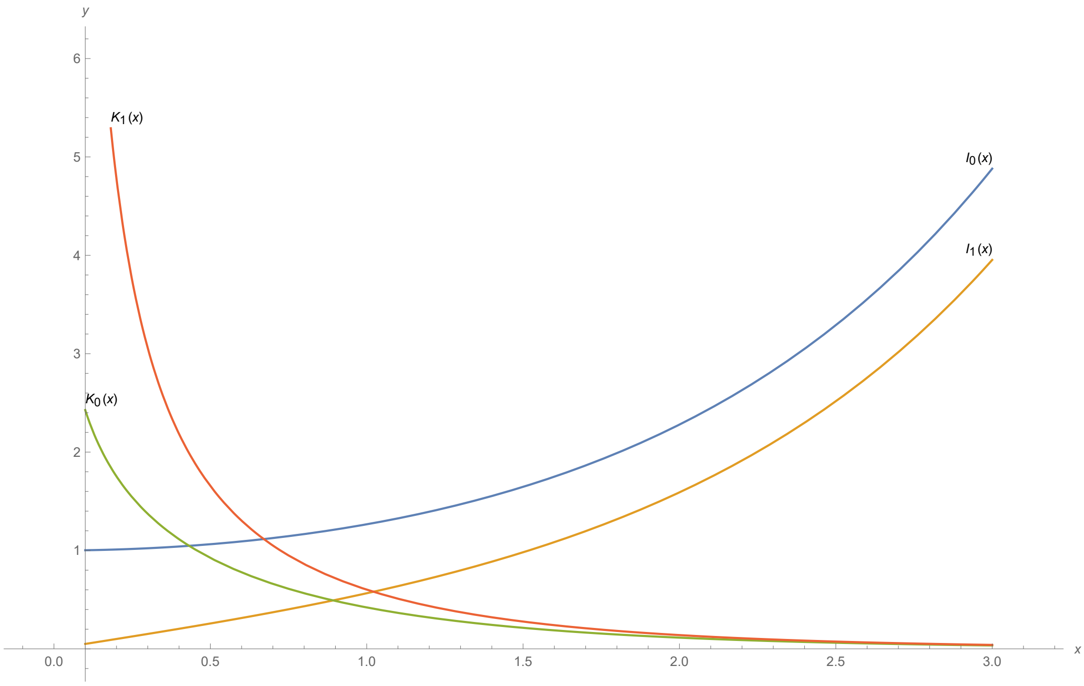
$$|I_0(r)| \approx 1, \quad |I'_0(r)| \to 0, \quad |K_0(r)| \approx |\log r|, \quad |K'_0(r)| \approx r^{-1}.$$

Note that

$$C_1 \, (1 \wedge r)^{\nu+\frac{1}{2}} \, \frac{e^r}{\sqrt{r}} \leq I_\nu(r) \leq C_2 \, (1 \wedge r)^{\nu+\frac{1}{2}} \, \frac{e^r}{\sqrt{r}} \tag{1.19}$$

for suitable $C_1, C_2 > 0$.

Figure 1.1: I_0, K_0, I_1 and K_1.



Let us compute the resolvent operator of L_ν.

Proposition 1.3.6 *Let $\lambda > 0$. Then, for every $f \in L^2((0,\infty))$,*

$$(\lambda - L_\nu)^{-1} f = \int_0^\infty G(\lambda, r, s) f(s) ds$$

with

$$G(\lambda, r, s) = \begin{cases} \sqrt{rs}\, I_\nu(\sqrt{\lambda}\, r) K_\nu(\sqrt{\lambda}\, s) & r \le s \\ \sqrt{rs}\, I_\nu(\sqrt{\lambda}\, s) K_\nu(\sqrt{\lambda}\, r) & r \ge s. \end{cases} \tag{1.20}$$

PROOF. Let us first consider the case $\lambda = 1$. The homogeneous equation

$$u_{rr} - \left(\nu^2 - \frac{1}{4}\right)\frac{u}{r^2} - u = 0$$

transforms into

$$v_{rr} + \frac{v_r}{r} - \left(\frac{\nu^2}{r^2} + 1\right) v = 0$$

by setting $v(r) = \frac{u(r)}{\sqrt{r}}$. Therefore $u_1(r) = \sqrt{r} I_\nu$ and $u_2(r) = \sqrt{r} K_\nu$ constitute a basis of solutions. Since the Wronskian of K_ν, I_ν is $1/r$, , see [2, 9.6 and 9.7] that of $\sqrt{r}K_\nu$,$\sqrt{r}I_\nu$ is 1. It follows that every solution of

$$u_{rr} - \left(\nu^2 - \frac{1}{4}\right)\frac{u}{r^2} - u = f$$

is given by

$$u(r) = \int_0^\infty G(1, r, s) f(s)\, ds + c_1 \sqrt{r} I_\nu(r) + c_2 \sqrt{r} K_\nu(r), \tag{1.21}$$

with $c_1,\ c_2 \in \mathbb{R}$ and

$$G(1, r, s) = \begin{cases} \sqrt{rs}\, I_\nu(r) K_\nu(s) & r \le s \\ \sqrt{rs}\, I_\nu(s) K_\nu(r) & r \ge s \end{cases}$$

Elementary computations using Lemma 1.3.5 show that

$$\sup_{r \in (0, +\infty)} \int_0^\infty G(1, r, s) ds < +\infty.$$

By the symmetry of the kernel and Young's inequality the integral operator T defined by G is therefore bounded in $L^2((0,\infty))$.

Let $f \in C_c^\infty((0,\infty))$ with support in (a, b) and $u = (1 - L_\nu)^{-1} f \in D(L_\nu)$. Then u is given by (1.21) with $c_1 = 0$, since T is bounded in $L^2((0,\infty))$, K_ν is exponentially decreasing and I_ν is exponentially increasing near ∞. Since

$$u(r) = \int_0^r \sqrt{rs} K_\nu(r) I_\nu(s) f(s)\, ds + \int_r^b \sqrt{rs} K_\nu(s) I_\nu(r) f(s)\, ds + c_2 \sqrt{r} K_\nu(r).$$

we have for $r < a$

$$u(r) = \int_a^b \sqrt{rs} K_\nu(s) I_\nu(r) f(s)\, ds + c_2 \sqrt{r} K_\nu = c\sqrt{r} I_\nu(r) + c_2 \sqrt{r} K_\nu(r)$$

for some $c \in \mathbb{R}$. If $c_2 \neq 0$, by Lemma 1.3.5 $u'(r) \approx c_2(\frac{1}{2} - \nu) r^{-\frac{1}{2} - \nu}$ when $\nu \neq 0$ and $\frac{u}{\sqrt{r}} = cI_\nu(r) + c_2 K_\nu(r)$ for $r < a$ when $\nu = 0$. In both cases, by Lemma 1.3.2 (i), (ii), with

$c = 0$, $u \notin D(\tilde{\mathfrak{a}}_\nu)$. Therefore $c_2 = 0$ and, by density, $(I - L_\nu)^{-1} = T$, since both operators are bounded and coincide on compactly supported functions.

Finally let us compute the resolvent for a general $\lambda > 0$. Recalling that $M_{\sqrt{\lambda}} L_\nu M_{\sqrt{\lambda}^{-1}} = \lambda L_\nu$,

$$(\lambda - L_\nu)^{-1} f = \lambda^{-1} M_{\sqrt{\lambda}} (I - L_\nu)^{-1} M_{\sqrt{\lambda}^{-1}} f = \frac{1}{\lambda} \int_0^\infty G(1, r\sqrt{\lambda}, s) f\left(\frac{s}{\sqrt{\lambda}}\right) ds$$
$$= \frac{1}{\sqrt{\lambda}} \int_0^\infty G(1, r\sqrt{\lambda}, s\sqrt{\lambda}) f(s)\, ds$$

which gives (1.20). □

Remark 1.3.7 *Observe that the above proof shows also the boundedness of $(\lambda - L_\nu)^{-1}$ in $L^p((0, \infty))$ for every $1 \le p \le \infty$, $\lambda > 0$.*

Remark 1.3.8 *We point out that the function $\sqrt{r} K_\nu$ does not belong to the domain of the form, but belongs to $L^2(0, \infty)$ if and only if $\nu < 1$ or $b < 3/4$. In this range other boundary conditions at $r = 0$ are possible and our choice, consistent with the rest of the paper, corresponds to a minimal resolvent, in the sense of positivity, see [55].*

Remark 1.3.9 *When $b \ge 3/4$, L_ν is essentially self-adjoint on $C_c^\infty((0, \infty))$. Moreover, when $b > 3/4$, the domain of L_ν is given by $D(L_\nu) = \{u \in H^2(0, \infty) : \frac{u}{r^2}, \frac{u'}{r} \in L^2(0, \infty)\}$, see [52, Example 7.1]. This last coincides with $H^2_0((0, \infty)) = \{u \in H^2((0, \infty)) : u(0) = u'(0) = 0\}$. In fact, the inclusion $\{u \in H^2((0, \infty)) : \frac{u}{r^2}, \frac{u'}{r} \in L^2((0, \infty))\} \subset H^2_0((0, \infty))$ is immediate since a function $u \in H^2((0, \infty))$ has finite values $u(0), u'(0)$ and these vanish when $u/r^2, u'/r$ are integrable near zero. Conversely, if $u \in H^2_0((0, \infty))$, then $|u(r)| = |\int_0^r (r - s) u''(s)\, ds| \le r \int_0^r |u''(s)|\, ds$ and $u/r^2 \in L^2((0, \infty))$, by Hardy inequality. A similar argument holds for u'/r.*

We denote now by $p_\nu(t, r, s)$ the heat kernel of the operator L_ν. Its existence is well-known, due to the local regularity of the coefficients. We show below a simple way to compute it, even without assuming its existence. We look for a smooth function $p(t, r, s)$ such that, for every $f \in L^2((0, \infty))$

$$e^{tL_\nu} f(r) = \int_0^\infty p(t, r, s) f(s)\, ds.$$

The function p should then satisfy

$$\begin{cases} p_t(t, r, s) = p_{rr}(t, r, s) - \frac{1}{r^2}\left(\nu^2 - \frac{1}{4}\right) p(t, r, s) \\ p(0, r, s) = \delta_s. \end{cases} \tag{1.22}$$

Since $\lambda^2 L_\nu = M_\lambda^{-1} L_\nu M_\lambda$ we obtain $e^{t\lambda^2 L_\nu} = M_\lambda^{-1} e^{tL_\nu} M_\lambda$. Rewriting this identity using the kernel p and setting $\lambda^2 t = 1$ we obtain

$$p(t,r,s) = \frac{1}{\sqrt{t}} p\left(1, \frac{r}{\sqrt{t}}, \frac{s}{\sqrt{t}}\right) := \frac{1}{\sqrt{t}} F\left(\frac{r}{\sqrt{t}}, \frac{s}{\sqrt{t}}\right).$$

Then (1.22) becomes

$$F_{rr}\left(\frac{r}{\sqrt{t}}, \frac{s}{\sqrt{t}}\right) - \frac{1}{r^2}\left(\nu^2 - \frac{1}{4}\right) tF\left(\frac{r}{\sqrt{t}}, \frac{s}{\sqrt{t}}\right) + \\ + \frac{1}{2}F\left(\frac{r}{\sqrt{t}}, \frac{s}{\sqrt{t}}\right) + \frac{1}{2}\frac{r}{\sqrt{t}}F_r\left(\frac{r}{\sqrt{t}}, \frac{s}{\sqrt{t}}\right) + \frac{1}{2}\frac{s}{\sqrt{t}}F_s\left(\frac{r}{\sqrt{t}}, \frac{s}{\sqrt{t}}\right) = 0$$

that is

$$F_{rr}(r,s) - \frac{1}{r^2}\left(\nu^2 - \frac{1}{4}\right) F(r,s) + \frac{1}{2}F(r,s) + \frac{1}{2} rF_r(r,s) + \frac{1}{2} sF_s\left(\frac{r}{\sqrt{t}}, \frac{s}{\sqrt{t}}\right) = 0.$$

Since for large r the operator L_ν behaves like D^2, having in mind the Gaussian kernel, we look for a solution of the form

$$F(r,s) = \frac{1}{\sqrt{4\pi}} \exp\left\{-\frac{(r-s)^2}{4}\right\} H(rs)$$

with H depending only on the product of the variables. By straightforward computations, we deduce

$$s^2 H_{rr}(rs) + s^2 H_r(rs) - \frac{1}{r^2}\left(\nu^2 - \frac{1}{4}\right) H(rs) = 0$$

or

$$H_{xx}(x) + H_x(x) - \frac{1}{x^2}\left(\nu^2 - \frac{1}{4}\right) H(x) = 0.$$

Setting $H(x) = u(x)e^{-\frac{x}{2}}$, u solves

$$u_{xx} - \frac{1}{4}u(x) - \frac{1}{x^2}\left(\nu^2 - \frac{1}{4}\right) u(x) = 0$$

and $v(x) = u(2x)$ satisfies

$$v_{xx} - v(x) - \frac{1}{x^2}\left(\nu^2 - \frac{1}{4}\right) u(x) = 0.$$

It follows that $v(x) = c_1\sqrt{x}I_\nu(x) + c_2\sqrt{x}K_\nu(x)$. Since the function H captures the behaviour of the heat kernel near the origin (the behaviour at infinity is governed by the Gaussian factor) and since the resolvent of L_ν is constructed with I_ν near the origin, we choose $c_2 = 0$ and write c instead of c_1. Therefore $u(x) = v\left(\frac{x}{2}\right) = c\sqrt{\frac{x}{2}} I_\nu\left(\frac{x}{2}\right)$, $H(rs) = u(rs)e^{-\frac{rs}{2}} = c\sqrt{\frac{rs}{2}} I_\nu\left(\frac{rs}{2}\right) e^{-\frac{rs}{2}}$,

$$F(r,s) = \frac{c}{\sqrt{4\pi}} \exp\left\{-\frac{(r-s)^2}{4}\right\} \sqrt{\frac{rs}{2}} I_\nu\left(\frac{rs}{2}\right) e^{-\frac{rs}{2}} = \frac{c}{\sqrt{4\pi}}\sqrt{\frac{rs}{2}} \exp\left\{-\frac{r^2+s^2}{4}\right\} I_\nu\left(\frac{rs}{2}\right)$$

and

$$p(t,r,s) = \frac{1}{\sqrt{4\pi t}} H\left(\frac{rs}{t}\right) \exp\left\{-\frac{(r-s)^2}{4t}\right\} = \frac{c}{t\sqrt{4\pi}}\sqrt{\frac{rs}{2}} \exp\left\{-\frac{r^2+s^2}{4t}\right\} I_\nu\left(\frac{rs}{2t}\right). \quad (1.23)$$

Theorem 1.3.10 *Let $p_\nu(t,r,s)$ be the heat kernel of L_ν. Then*

$$p_\nu(t,r,s) = \frac{1}{2t}\sqrt{rs}I_\nu\left(\frac{rs}{2t}\right)\exp\left\{-\frac{r^2+s^2}{4t}\right\}.$$

Proof. The Laplace transform of the right hand side of (1.23) is given by, see [28, p.200],

$$\begin{cases} \frac{2c}{\sqrt{4\pi}}\sqrt{\frac{rs}{2}}I_\nu(r\sqrt{\lambda})K_\nu(s\sqrt{\lambda}) & r \le s \\ \frac{2c}{\sqrt{4\pi}}\sqrt{\frac{rs}{2}}I_\nu(s\sqrt{\lambda})K_\nu(r\sqrt{\lambda}) & r \ge s. \end{cases}$$

For $c = \sqrt{2\pi}$ it coincides with the kernel $G(\lambda,r,s)$ of the resolvent operator $(\lambda - L_\nu)^{-1}$, see Proposition 1.3.6 (note that $\sqrt{2\pi}$ appears in the asymptotic expansion at infinity of I_ν). Let $S(t)$ be the operator defined through the kernel p_ν, that is p with $c = \sqrt{2\pi}$ and let $G(t)$ be the Gauss-Weierstrass semigroup in $\mathbb{R}$. By (1.23) and since $H(r) = c\sqrt{\frac{r}{2}}I_\nu\left(\frac{r}{2}\right)e^{-\frac{r}{2}}$ is bounded by Lemma 1.3.5, then $|S(t)f| \le CG(t)|f|$, pointwise and $\|S(t)\| \le C$ in $L^2((0,\infty))$. Given $f \in C_c^\infty((0,\infty))$, let $u(t,r) = S(t)f(r)$. By the construction of the kernel p we have $u_t = L_\nu u$ pointwise. Finally, for $\lambda > 0$,

$$\begin{aligned}\int_0^\infty e^{-\lambda t}u(t,r)\,dt &= \int_0^\infty e^{-\lambda t}\,dt\int_0^\infty p(t,r,s)f(s)\,ds = \int_0^\infty f(s)\,ds\int_0^\infty e^{-\lambda t}p(t,r,s)\,dt \\ &= \int_0^\infty G(\lambda,r,s)f(s)\,ds.\end{aligned}$$

It follows that the Laplace transform of $S(t)f$ coincides with the resolvent of L_ν, hence, by uniqueness, $S(t)$ is the generated semigroup and $p = p_\nu$ its kernel, as in the statement. □

Remark 1.3.11 *Observe that in the proof we have avoided the verification of the semigroup law and of the strong continuity, which hold a-posteriori.*

The general case, now, immediately follows.

Proposition 1.3.12 *Let $\lambda > 0$. Then, for every $f \in L^2((0,+\infty), r^c dr)$,*

$$(\lambda - L)^{-1}f = r^{-\frac{c}{2}}\int_0^\infty G(\lambda,r,s)s^{-\frac{c}{2}}f(s)\,s^c ds$$

with $G(\lambda,r,s)$ defined in (1.20).

Proof. The result immediately follows by Proposition 1.3.6 by observing that

$$(\lambda - L)^{-1} = \Phi^{-1}(\lambda - L_\nu)^{-1}\Phi.$$

□

The same argument gives the heat kernel of L.

Theorem 1.3.13 *Let p be the heat kernel kernel of L with respect to the measure $s^c\,ds$. Then*

$$p(t,r,s)=r^{-\frac{c}{2}}\frac{1}{2t}\sqrt{rs}\,I_\nu\left(\frac{rs}{2t}\right)\exp\left\{-\frac{r^2+s^2}{4t}\right\}s^{-\frac{c}{2}}$$

that is for every $f\in L^2((0,+\infty),r^c dr)$

$$e^{tL}f(r)=r^{-\frac{c}{2}}\frac{1}{2t}\int_0^\infty \sqrt{rs}\,I_\nu\left(\frac{rs}{2t}\right)\exp\left\{-\frac{r^2+s^2}{4t}\right\}s^{-\frac{c}{2}}f(s)s^c\,ds.$$

The asymptotic behaviour of Bessel functions allows to deduce explicit bounds for the heat kernel p. We need first the following elementary lemma.

Lemma 1.3.14 *With $C(\epsilon):=\frac{1+\sqrt{1+\frac{2}{\epsilon}}}{2}$ we have for every $r,s>0$*

$$1\le\frac{1\wedge rs}{(1\wedge r)(1\wedge s)}\le C(\epsilon)\,e^{\epsilon|r-s|^2}.$$

Proof. We observe preliminarily that

$$\frac{1\wedge rs}{(1\wedge r)(1\wedge s)}=\begin{cases}1, & \text{if}\quad r,s\le 1\quad\text{or}\quad r,s\ge 1;\\ s, & \text{if}\quad rs\le 1\quad\text{and}\quad r\le 1\le s;\\ \frac{1}{r}, & \text{if}\quad rs\ge 1\quad\text{and}\quad r\le 1\le s;\\ r, & \text{if}\quad rs\le 1\quad\text{and}\quad s\le 1\le r;\\ \frac{1}{s}, & \text{if}\quad rs\ge 1\quad\text{and}\quad s\le 1\le r.\end{cases}\tag{1.24}$$

It is easily seen from (1.24) that $\frac{1\wedge rs}{(1\wedge r)(1\wedge s)}\ge 1$ in every case. To prove the other inequality we fix $\epsilon>0$ and consider the function $g(r,s):=\frac{1\wedge rs}{(1\wedge r)(1\wedge s)}e^{-\epsilon|r-s|^2}$. For fixed $0<r_0\le 1$ the function $0<s\mapsto f(s):=se^{-\epsilon|r_0-s|^2}$ has maximum in $s_0=\frac{r_0+\sqrt{r_0^2+\frac{2}{\epsilon}}}{2}$ which gives

$$se^{-\epsilon|r_0-s|^2}\le f(s_0)=\frac{r_0+\sqrt{r_0^2+\frac{2}{\epsilon}}}{2}e^{-\epsilon|r_0-s_0|^2}\le\frac{1+\sqrt{1+\frac{2}{\epsilon}}}{2}.$$

Now, using (1.24), we distinguish three cases:

(i) if $r,s\le 1$ or $r,s\ge 1$ we have $g(r,s)=e^{-\epsilon|r-s|^2}\le 1\le\frac{1+\sqrt{1+\frac{2}{\epsilon}}}{2}$;

(ii) if $rs\le 1$ and $r\le 1\le s$ we get, recalling (1.3.2), $g(r,s)=se^{-\epsilon|r-s|^2}\le\frac{1+\sqrt{1+\frac{2}{\epsilon}}}{2}$;

(iii) similarly, if $rs\ge 1$ and $r\le 1\le s$, $g(r,s)=\frac{1}{r}e^{-\epsilon|r-s|^2}\le se^{-\epsilon|r-s|^2}\le\frac{1+\sqrt{1+\frac{2}{\epsilon}}}{2}$.

The other cases follow by symmetry interchanging the role of r and s. □

Proposition 1.3.15 *The heat kernel p of L, with respect to the measure $d\mu = s^c ds$, satisfies*

$$p(t,r,s) \leq C(\epsilon)\frac{1}{\sqrt{t}}(rs)^{-\frac{c}{2}}\left[\left(1\wedge\frac{r}{\sqrt{t}}\right)\left(1\wedge\frac{s}{\sqrt{t}}\right)\right]^{\nu+\frac{1}{2}}\exp\left(-(1-\epsilon)\frac{|r-s|^2}{4t}\right)$$

$$p(t,r,s) \geq C\frac{1}{\sqrt{t}}(rs)^{-\frac{c}{2}}\left[\left(1\wedge\frac{r}{\sqrt{t}}\right)\left(1\wedge\frac{s}{\sqrt{t}}\right)\right]^{\nu+\frac{1}{2}}\exp\left(-\frac{|r-s|^2}{4t}\right).$$

PROOF. From Theorem 1.3.13 and using (1.19) we have

$$\begin{aligned} p(t,r,s) &= \frac{1}{2t}(rs)^{-\frac{c}{2}}\sqrt{rs}\,I_\nu\left(\frac{rs}{2t}\right)\exp\left\{-\frac{r^2+s^2}{4t}\right\} \\ &\simeq \frac{1}{\sqrt{t}}(rs)^{-\frac{c}{2}}\left(1\wedge\frac{rs}{t}\right)^{\nu+\frac{1}{2}}\exp\left(-\frac{|r-s|^2}{4t}\right). \end{aligned}$$

Applying lemma 1.3.14 we conclude the proof.

□

1.4 The elliptic operator in $L^p(\mathbb{R}^N)$

In this section we analyse elliptic and parabolic problems in $L^p(\mathbb{R}^N)$ associated to the operator

$$L = \Delta + (a-1)\sum_{i,j=1}^N \frac{x_i x_j}{|x|^2}D_{ij} + c\frac{x}{|x|^2}\cdot\nabla - \frac{b}{|x|^2}$$

where $a > 0$ and $b,\ c$ real coefficients. We collect, without proofs, the main results concerning generation and domain characterization proved in [56]. We recall that the condition $D \geq 0$ is necessary to get positive solutions.

If $1 < p < \infty$, we define the maximal operator $L_{p,\max}$ through the domain

$$D(L_{p,\max}) = \{u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N\setminus\{0\})\ ;\ Lu \in L^p(\mathbb{R}^N)\}$$

and note that, by local elliptic regularity, $L_{p,\max}$ is closed and

$$D(L_{p,\max}) = \{u \in L^p(\mathbb{R}^N)\ ;\ Lu \in L^p(\mathbb{R}^N) \text{ as a distribution in } \mathbb{R}^N\setminus\{0\}\}.$$

The operator $L_{p,\min}$ is defined as the closure, in $L^p(\mathbb{R}^N)$ of $(L, C_c^\infty(\mathbb{R}^N\setminus\{0\})$ (the closure exists since this operator is contained in the closed operator $L_{p,\max}$) and it is clear that $L_{p,\min} \subset L_{p,\max}$.

The formal adjoint of L is given by

$$L^* = \Delta + (a-1)\sum_{i,j=1}^N \frac{x_i x_j}{|x|^2}D_{ij} + c^*\frac{x}{|x|^2}\cdot\nabla - b^*|x|^{-2} \tag{1.25}$$

where $c^* = 2(N-1)(a-1) - c$ and $b^* = b + (N-2)(c - (N-1)(a-1))$.

From Proposition 1.1.2 we have, in spherical coordinates $x = r\omega$,

$$L = aD_{rr} + \frac{N-1+c}{r} D_r - \frac{b - \Delta_0}{r^2},$$

where Δ_0 is the Laplace-Beltrami on $\mathbb{S}^{N-1}$ and, if P is a spherical harmonic of degree n, then $\Delta_0 P = -\lambda_n P$, where $\lambda_n = n^2 + (N-2)n, n \in \mathbb{N}_0$.

We recall, from Section 1.1, that the equation $Lu = 0$ has radial solutions $|x|^{-s_1}$, $|x|^{-s_2}$ where s_1, s_2 are the roots of the indicial equation $f(s) = -as^2 + (N-1+c-a)s + b = 0$ given by

$$s_1 := \frac{N-1+c-a}{2a} - \sqrt{D}, \quad s_2 := \frac{N-1+c-a}{2a} + \sqrt{D} \tag{1.26}$$

where

$$D := \frac{b}{a} + \left(\frac{N-1+c-a}{2a}\right)^2. \tag{1.27}$$

The above numbers are real if and only if $D \geq 0$ and when $D < 0$ the equation $u - Lu = f$ cannot have positive distributional solutions for certain positive f, see Proposition 1.3.1.

The main result consists in showing that, assuming $D \geq 0$, there exists an intermediate operator $L_{p,\min} \subset L_{p,\text{int}} \subset L_{p,\max}$ which generates a semigroup in $L^p(\mathbb{R}^N)$ if and only if $\frac{N}{p} \in (s_1, s_2 + 2)$. Before stating the relative theorems, we give a short explanation of this result referring to [56, Section 3] for the proper proofs and further details.

For a fixed $1 < p < \infty$, let us decompose $L^p(\mathbb{R}^N)$ into the direct sum

$$L^p(\mathbb{R}^N) = L^p_{<n} \oplus L^p_{\geq n},$$

where the spaces $L^p_{<n}$, $L^p_{\geq n}$ are the closure of the linear span of functions of the form $\sum_j f_j(r) P_j(\omega)$, where the sums are finite and the spherical harmonics P_j have degree less than n or greater or equal than n, respectively.

Using improved Hardy and Poincaré inequalities it can be proved that, if n is large enough, $L_{p,\min} = L_{p,\max}$ is complex dissipative on $L^p_{\geq n}(\mathbb{R}^N)$ and so generates an analytic semigroup of contraction in $L^p_{\geq n}$; the domain can be characterized using Rellich inequalities proved in [54].

On the other hand

$$L^p_{<n} = \bigoplus_{i \in J} \left(L^p_{\text{rad}} \otimes P_j \right)$$

where $L^p_{\text{rad}} = L^p((0,\infty), r^{N-1}\, dr)$, J is finite and $\{P_j, j \in J\}$ is an orthogonal basis of spherical harmonics of degree less than n. If $v(x) = u(r)P_j(\omega) \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, using Proposition 1.1.2 and recalling (1.6), it follows that

$$Lv = \left(au_{rr} + \frac{N-1+c}{r} u_r - \frac{b + \lambda_k}{r^2} u \right) P_j = L_k u\, P_j,$$

where $k = \deg \mathrm{P_j} < n$ and the Bessel operator

$$L_k u = au'' + \frac{N-1+c}{r}u' - \frac{b+\lambda_k}{r^2}u$$

is obtained from the radial part of L, substituting b with $b+\lambda_k$. The equation $L_k u = 0$ has solutions $r^{-s_1^{(k)}}$, $r^{-s_2^{(k)}}$ where the numbers $s_1^{(k)}, s_2^{(k)}$ are defined as in (1.26), (1.27) with $b+\lambda_k$ instead of b; $s_1^{(k)}$ decrease to $-\infty$, whereas $s_2^{(k)}$ increase to $+\infty$.

The operator in $L^p_{<n}$ is, therefore, reduced to a finite number of ordinary differential operators of Bessel type L_k. As k increases the potentials $-(b+\lambda_k)r^{-2}$ become more and more negative and, correspondingly, L_k have more regularizing effect. Thus the most critical equation appears for $k=0$ and corresponds to radial functions.
The conditions $D \geq 0$ and $\frac{N}{p} \in (s_1, s_2+2)$ come from the 1-d analysis and guarantees the existence of a positive resolvent in $L^p\left((0,\infty), r^{N-1}dr\right)$ which can be written explicitly, using the same methods as in Section 1.3.

Finally, putting together the results in $L^p_{\geq n}$ and in $L^p_{<n}$, we obtain the necessary and sufficient conditions for the generation in $L^p(\mathbb{R}^N)$. When these conditions are satisfied, the semigroup turns out to be analytic and positive.

We list, below, the main theorems about the generation results presented so far and proved in [56]. We refer the reader, also, to section 4.1 where the decomposition of L is proved in $L^2(\mathbb{R}^N, d\mu)$.

1.4.1 Generation results and domain characterization

The following lemma follows from elliptic regularity, see [52, Proposition 2.2].

Lemma 1.4.1 *Let $1 < p < \infty$. Then the adjoint of $L_{p,\min}$, $L_{p,\max}$ are $L^*_{p',\max}$, $L^*_{p',\min}$, respectively.*

Let us compute the numbers s_1^*, s_2^*, D^* defined as in (1.7), (1.8) and relative to L^*. We have

$$D^* := \frac{b^*}{a} + \left(\frac{N-1+c^*-a}{2a}\right)^2 = D, \tag{1.28}$$

$$s^*_{1,2} := \frac{N-1+c^*-a}{2a} \mp \sqrt{D^*} = s_{1,2} + \frac{(a-1)(N-1)-c}{a} = N-2-s_{2,1}. \tag{1.29}$$

Observe that $\frac{N}{p} > s_1$ is equivalent to $\frac{N}{p'} < s_2^*+2$ and $\frac{N}{p} < s_2$ is equivalent to $\frac{N}{p'} > s_1^*+2$. Similarly, $\frac{N}{p} > s_1+2$ is equivalent to $\frac{N}{p'} < s_2^*$ and $\frac{N}{p} < s_2+2$ is equivalent to $\frac{N}{p'} > s_1^*$.

L is formally self-adjoint, that is $L = L^*$, if and only if $c = (a-1)(N-1)$. In this case

$$s_{1,2} = \frac{N-2}{2} \mp \sqrt{\frac{b}{a} + \frac{(N-2)^2}{4}}.$$

When $D = 0$ we write s_0 for $s_1 = s_2$.

Theorem 1.4.2 [56] *Assume that $D > 0$. If $\frac{N}{p} \in (s_1, s_2 + 2)$ that is $s_1 < \frac{N}{p} - 2\theta < s_2$ for some $\theta \in (0,1]$, then L endowed with domain*

$$D(L_{p,int}) = \{u \in D(L_{p,max}) \; ; \; |x|^{-2\theta} u \in L^p\}$$

generates a bounded positive analytic semigroup on L^p. Moreover,

$$D(L_{p,int}) = D(L_{p,reg}) := \{u \in D(L_{p,max}) \; ; \; (1 \wedge |x|)^{2-2\theta} D^2 u, (1 \wedge |x|)^{1-2\theta} \nabla u, |x|^{-2\theta} u \in L^p\}$$

for all/one θ as above. In particular, if $s_1 + 2 < \frac{N}{p} < s_2 + 2$, then $\theta = 1$ and

$$D(L_{p,int}) = \{u \in W^{2,p}(\mathbb{R}^N) \; ; \; |x|^{-1} \nabla u, |x|^{-2} u \in L^p\}.$$

When $\frac{N}{p} \notin (s_1, s_2 + 2)$, then $\sigma(L) = \mathbb{C}$ for every $L_{p,min} \subset L \subset L_{p,max}$.

Theorem 1.4.3 [56] *Assume that $D = 0$. If $\frac{N}{p} \in (s_0, s_0 + 2)$, then L endowed with domain*

$$D(L_{p,int}) = \left\{u \in D(L_{p,max}) \; ; \; |x|^{-2\theta_0} |\log|x||^{-\frac{2}{p}} u \in L^p(B_{\frac{1}{2}})\right\}$$

with $\theta_0 = \frac{1}{2}(s_0 - \frac{N}{p}) \in (0,1)$ generates a bounded positive analytic semigroup on L^p. Moreover,

$$D(L_{p,int}) = D(L_{p,reg}) := \left\{ u \in D(L_{p,max}) \; ; \; \begin{array}{l} u \in W^{2,p}(\mathbb{R}^N \setminus B_{\frac{1}{2}}), \\ |x|^{2-2\theta_0} |\log|x||^{-\frac{2}{p}} D^2 u \in L^p(B_{\frac{1}{2}}), \\ |x|^{1-2\theta_0} |\log|x||^{-\frac{2}{p}} \nabla u \in L^p(B_{\frac{1}{2}}), \\ |x|^{-2\theta_0} |\log|x||^{-\frac{2}{p}} u \in L^p(B_{\frac{1}{2}}) \end{array} \right\}.$$

When $\frac{N}{p} \notin (s_0, s_0 + 2)$, then $\sigma(L) = \mathbb{C}$ for every $L_{p,min} \subset L \subset L_{p,max}$.

As a consequence, L generates a semigroup in some $L^p(\mathbb{R}^N)$, $1 < p < \infty$, if and only if $(s_1, s_2 + 2) \cap (0, N) \neq \varnothing$.

Corollary 1.4.4 [56] *If p, q satisfy the hypotheses of Theorems 1.4.2 or 1.4.3, then the generated semigroups coincide in $L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$. Moreover $(L_{p,int})^* = L^*_{p',int}$.*

Next we state when $L_{p,int}$ coincides with $L_{p,min}$ or $L_{p,max}$.

Proposition 1.4.5 [56] *Assume that $D > 0$. Then $L_{p,int} = L_{p,max}$ if and only if $\frac{N}{p} \in (s_1, s_2]$ and $L_{p,int} = L_{p,min}$ if and only if $\frac{N}{p} \in [s_1 + 2, s_2 + 2)$. Therefore, if $s_1 + 2 \leq s_2$ and if $\frac{N}{p} \in [s_1 + 2, s_2]$, then $L_{p,int} = L_{p,min} = L_{p,max}$.*

However, if $D = 0$ and $\frac{N}{p} \in (s_0, s_0 + 2)$, then $L_{p,min} \subsetneq L_{p,int} \subsetneq L_{p,max}$.

We refer the reader to [56, Theorem 3.29] for a detailed discussion of the inclusion $D(L_{p,int}) \subset W^{2,p}(\mathbb{R}^N)$.

We conclude the section with the following result concerning the action of the operator in the space of continuous functions, referring to [57] for its proof.

Theorem 1.4.6 [57] *If $s_1 < 0 < s_2 + 2$, then L generates a positive bounded analytic semigroup $T(z)$ of angle $\frac{\pi}{2}$ in $C_0(\Omega)$. If $s_1 = 0$, then $T(z)$ is positive bounded analytic semigroup of angle $\frac{\pi}{2}$ in $C_0(\mathbb{R}^N)$.*

Chapter 2

The Riemannian manifold associated with the elliptic operator

In this Chapter we study the Riemannian manifolds $(\mathbb{R}^N \setminus \{0\}, g)$ associated with the self-adjoint elliptic operator

$$A = \Delta + (a-1)\sum_{i,j=1}^{N} \frac{x_i x_j}{|x|^2} D_{ij} + (a-1)(N-1)\frac{x}{|x|^2}\cdot\nabla = \text{div}\,(\bar{a}(x)\nabla)\,,$$

where $N \geq 2$, $a > 0$ and $\bar{a}(x) = I + (a-1)\frac{x\otimes x}{|x|^2}$. The core of this chapter is Section 2.2, where we compute the explicit formula of the geodesic distance induced by g: such formula plays, in Chapter 3, an essential role in the derivation of sharp upper estimates for the heat kernel of the general operator L defined in (1.1).
Basic knowledge of Riemannian Geometry is required throughout the chapter and we refer the reader to Appendix A for a brief survey on the main notions and results needed.

This Chapter is mainly based on [15].

2.1 The Riemannian manifold

Let $a > 0$ and let us consider the Riemannian manifold $(\mathbb{R}^N \setminus \{0\}, g)$, where $N \geq 2$ and g is the Riemannian metric defined in Cartesian coordinates by

$$g = \sum_{i,j=1}^{N} \left(\delta_{ij} + \left(\frac{1}{a} - 1\right)\frac{x_i x_j}{|x|^2}\right) dx_i \otimes dx_j. \tag{2.1}$$

Let us set

$$\bar{a}(x) := I + (a-1)\frac{x\otimes x}{|x|^2}$$

where I is the identity matrix and $x \otimes x = \left(x_i x_j\right)_{i,j=1,\dots,N}$. The matrix $\bar{a}(x)$ has eigenvalues a with eigenvector x and 1 with eigenspace the orthogonal complement of x. In particular $\det \bar{a} = a > 0$ and $\bar{a}$ is non-singular with inverse given by the matrix

$$\bar{a}^{-1}(x) = I + \left(\frac{1}{a} - 1\right) \frac{x \otimes x}{|x|^2}$$

that defines the metric tensor.

Throughout the chapter we always assume the canonical identification of the tangent space $T_p(\mathbb{R}^N) \equiv \{p\} \times \mathbb{R}^N \equiv \mathbb{R}^N$ given by (A.8). With this assumption, denoting by $\langle \cdot, \cdot \rangle$ the euclidean inner product of $\mathbb{R}^N$, we have for every vector fields X, Y,

$$g(X,Y) = \langle \bar{a}^{-1} X, Y \rangle, \quad |X|_g = \langle \bar{a}^{-1} X, X \rangle^{\frac{1}{2}} = |\bar{a}^{-\frac{1}{2}} X|. \tag{2.2}$$

In particular

$$\left(1 \wedge \frac{1}{\sqrt{a}}\right) |X| \le |X|_g \le \left(1 \vee \frac{1}{\sqrt{a}}\right) |X|. \tag{2.3}$$

The manifold $(\mathbb{R}^N \setminus \{0\}, g)$ is naturally associated with its Laplace-Beltrami operator (see Section A.3) defined as the second order elliptic operator in divergence form

$$A = \operatorname{div}\, (\bar{a}(x) \nabla) = \Delta + (a-1) \sum_{i,j=1}^{N} \frac{x_i x_j}{|x|^2} D_{ij} + (a-1)(N-1) \frac{x}{|x|^2} \cdot \nabla.$$

For $a = 1$, the metric tensor g reduces to the Euclidean metric of $\mathbb{R}^N \setminus \{0\}$ and A becomes the classical Laplace operator Δ of $\mathbb{R}^N \setminus \{0\}$.

Let us employ spherical coordinates on $\mathbb{R}^N \setminus \{0\}$ (see Section B.1 in Appendix B for further details). For every $x \in \mathbb{R}^N \setminus \{0\}$ we write $x = r\omega$, where $r := |x|$, $\omega := \frac{x}{|x|} \in \mathbb{S}^{N-1}$. If $u \in C^2(\mathbb{R}^n)$, $\partial_r u$, $\partial_{rr} u$ are the radial derivatives of u and $\nabla_\tau u$ is the tangential component of its gradient. Recalling Proposition B.1.4, they can be defined through the formulas

$$\partial_r u = \sum_{i=1}^{N} \partial_i u \, \frac{x_i}{r}, \qquad \partial_{rr} u = \sum_{i,j=1}^{N} \partial_{ij} u \, \frac{x_i x_j}{r^2}, \qquad \nabla u = \partial_r u \frac{x}{|x|} + \frac{\nabla_\tau u}{r}.$$

Moreover, in spherical coordinates, we have the following relation between the Laplacian Δ and the Laplace-Beltrami operator Δ_0 on the sphere $\mathbb{S}^{N-1}$:

$$\Delta = D_{rr} + \frac{N-1}{r} D_r + \frac{\Delta_0}{r^2}.$$

The following Proposition gives the expression, in Cartesian coordinates, of the g-gradient of functions of $\mathbb{R}^N \setminus \{0\}$.

Proposition 2.1.1 *For every $u \in C^1\left(\mathbb{R}^N \setminus \{0\}\right)$ the gradient $\nabla_g u$, taken with respect the metric g, is given, in Cartesian coordinates, by*

$$\nabla_g u = \bar{a}\,\nabla u = \nabla u + (a-1)\partial_r u \frac{x}{|x|}, \tag{2.4}$$

where $\partial_r u$ is the radial derivative of u. In particular, for every $u, v \in C^1\left(\mathbb{R}^N \setminus \{0\}\right)$, one has

(i) $g(\nabla_g u, \nabla_g v) = \langle \bar{a}\nabla u, \nabla v\rangle$;

(ii) $|\nabla_g u|_g^2 = \langle \bar{a}\nabla u, \nabla u\rangle = |\bar{a}^{\frac{1}{2}}\nabla u|^2$.

Furthermore,

$$L = |x|^{-\gamma}\mathrm{div}(|x|^{\gamma}\nabla_g) - \frac{b}{|x|^2}, \tag{2.5}$$

where $\gamma = \frac{N-1+c}{a} - N + 1$ and $|x|^{-\gamma}\mathrm{div}(|x|^{\gamma}\nabla_g)$ is the weighted Laplace-Beltrami operator on $(\mathbb{R}^N \setminus \{0\}, g)$.

Proof. Let $u \in C^1\left(\mathbb{R}^N \setminus \{0\}\right)$. Using (A.2), one has

$$\nabla_g u = \bar{a}\,\nabla u = \nabla u + (a-1)\left\langle \frac{x}{|x|}, \nabla u\right\rangle \frac{x}{|x|} = \nabla u + (a-1)\partial_r u\,\frac{x}{|x|}.$$

(i) and (ii) are then trivial consequences of (2.2). (2.5) follows from (2.4) and the relation

$$L = |x|^{-\gamma}\mathrm{div}(|x|^{\gamma}\overline{a}\,\nabla) - \frac{b}{|x|^2},$$

which can be checked directly.

□

Remark 2.1.2 *Let d_g be the distance function on $\mathbb{R}^N \setminus \{0\}$ induced by the metric tensor g and let f be a function on $\mathbb{R}^N \setminus \{0\}$. Then, by standard result on Lipschitz functions on Riemannian manifold (see for example [35, Theorem 11.3]), $f \in \mathrm{Lip}(\mathbb{R}^N \setminus \{0\}, d_g)$ if and only if $f \in W^{1,\infty}_{loc}(\mathbb{R}^N \setminus \{0\})$ and*

$$\mathrm{Lip}(f, d_g) = \sup_{x \in \mathbb{R}^N \setminus \{0\}} |\nabla_g f(x)|_g = \|\langle \bar{a}\nabla f, \nabla f\rangle\|_{\infty}^{\frac{1}{2}} < \infty. \tag{2.6}$$

We compute, now, the metric tensor g in spherical coordinates.

Proposition 2.1.3 *Let $(\mathbb{R}^N \setminus \{0\}, g)$ be the Riemannian manifold defined by (2.1). The metric tensor g has the following representation in spherical coordinates $x = r\omega$:*

$$g = \frac{1}{a}dr^2 + r^2 h.$$

Here h is the canonical spherical metric on $\mathbb{S}^{N-1}$ and $dr^2 := dr \otimes dr$ is the canonical euclidean metric on R^+ which satisfies $dr^2 = \sum_{i,j=1}^N \frac{x_i x_j}{|x|^2} dx_i \otimes dx_j$.

PROOF. The claim follows immediately by applying Proposition B.1.1.

□

The previous Proposition shows that the metric tensor g is homothetic by a factor $\frac{1}{a}$ to

$$\tilde{g} = dr^2 + ar^2 h. \tag{2.7}$$

This gives us a useful interpretation of $(\mathbb{R}^N \setminus \{0\}, g)$ in terms of warped products (see Section A.4.3 in Appendix A). More specifically we have the following result.

Proposition 2.1.4 *The Riemannian manifold $(\mathbb{R}^N \setminus \{0\}, \tilde{g})$ is isometric to the warped product $\mathbb{R}^+ \times_{\sqrt{a}r} \mathbb{S}^{N-1}$. As a consequence the metric tensor g can be extended smoothly at 0 if and only if $a = 1$ and, in such a case, it coincides with the Euclidean metric of $\mathbb{R}^N$. $(\mathbb{R}^N \setminus \{0\}, g)$ is flat if either $N = 2$ or $a = 1$ and, for $N \geq 3$, its Ricci tensor Ric satisfies*

$$(1 - a)\mathrm{Ric} \geq 0$$

i.e. $(\mathbb{R}^N \setminus \{0\}, g)$ has non-negative or non-positive Ricci curvature accordingly to whether $a < 1$ or $a > 1$. Furthermore, in the last case, the Ricci tensor is not bounded from below.

PROOF. Recalling Definition A.4.12, the first claim is an immediate consequence of equation (2.7). The required isometry is, in particular, given by the map

$$Q : \mathbb{R}^N \setminus \{0\} \to]0, \infty[\times \mathbb{S}^{N-1}, \;\; x \mapsto Q(x) = \left(|x|, \frac{x}{|x|}\right).$$

The second assertion follows from a result on rotationally symmetric metrics (see for example [68, page 12-13]) or directly observing that the term $\frac{x_i x_j}{|x|^2}$ in (2.1) cannot be extended smoothly to 0.

To prove the last two claims we apply Proposition A.4.14 of the Appendix A with $(B, g_B) = (\mathbb{R}^+, dr)$, $(F, g_F) = (\mathbb{S}^{N-1}, h)$ and $f(r) = \sqrt{a}r$. We follow the same notation adopted there to denote vector fields on B and F and their respective lifts on $B \times F$.
Let R be the Riemannian curvature tensor of $\mathbb{R}^+ \times_{\sqrt{a}r} \mathbb{S}^{N-1}$. Since $(\mathbb{R}^+, dr)$ is flat and $f'(r) = \sqrt{a}$, $f''(r) = 0$, all the curvatures in (i)-(iv) of Proposition A.4.14 vanish identically and we have only to manage the spherical terms in (v).

Let $R^{\mathbb{S}^{N-1}}$, $\mathrm{Ric}^{\mathbb{S}^{N-1}}$ be respectively the Riemannian curvature and the Ricci Tensor of $\left(\mathbb{S}^{N-1}, h\right)$; let X, Y, Z be vector fields on $\mathbb{S}^{N-1}$. Example A.4.11 shows that

$$R^{\mathbb{S}^{N-1}}(X, Y)Z = \Big[h(Y, Z)X - h(X, Z)Y\Big]$$

$$\mathrm{Ric}^{\mathbb{S}^{N-1}}(X) = (N - 2)X.$$

Inserting the first of the last equalities in the expression (v) of Proposition A.4.14, we get

$$\begin{aligned}R(X,Y)Z &= R^{\mathbb{S}^{N-1}}(X,Y)Z - \frac{1}{r^2}\left[ar^2 h(Y,Z)X - ar^2 h(X,Z)Y\right] \\ &= (1-a)R^{\mathbb{S}^{N-1}}(X,Y)Z.\end{aligned}$$

This implies $R = 0$ if either $R^{\mathbb{S}^{N-1}} = 0$ or $a = 1$. Since $R^{\mathbb{S}^{N-1}} = 0$ only for the one dimensional sphere, the previous conditions are equivalent to $a = 1$ or $N = 2$.

To prove the last claim let $(E_1, \dots, E_{N-1})$ be an orthonormal frame on $\left(\mathbb{S}^{N-1}, h\right)$; then $(F_1, \dots, F_N) := \left(\frac{1}{\sqrt{a}r}E_1, \dots, \frac{1}{\sqrt{a}r}E_{N-1}, \partial_r\right)$ is an orthonormal frame field on $\mathbb{R}^+ \times_{\sqrt{a}r} \mathbb{S}^{N-1}$. An analogous computation shows that

$$\begin{aligned}&\mathrm{Ric}(F_i) = (N-2)\frac{1-a}{ar^2}F_i, \quad i < N; \\ &\mathrm{Ric}(\partial_r) = 0;\end{aligned} \tag{2.8}$$

(see also [68, Section 3.2.3, page 70]). This implies, recalling Definition A.2.14, that the warped product $\mathbb{R}^+ \times_{\sqrt{a}r} \mathbb{S}^{N-1}$ has non-negative Ricci curvature for $a < 1$ and non-positive Ricci curvature for $a > 1$. In the last case, Ric is not bounded from below since, from (2.8), one has

$$\begin{aligned}\mathrm{Ric}(F_i, F_i) &= (N-2)\frac{1-a}{ar^2}g(F_i, F_i) \\ &= (N-2)\frac{1-a}{ar^2} \longrightarrow -\infty, \quad \text{as } r \to 0.\end{aligned}$$

Being $g = \frac{1}{a}\,\tilde{g}$, the same result holds for $(\mathbb{R}^N \setminus \{0\}, g)$.

□

Remark 2.1.5 *The non-completeness of the manifold and, for $N \geq 3,\ a > 1$, the unboundedness from below of its Ricci tensor Ric are the geometric reflection of the singularity which appears, for $a \neq 1$, in the leading term of the operator L defined in* (1.1) .

The next Proposition points out the spherical symmetry of $(\mathbb{R}^N \setminus \{0\}, g)$. For every $k < N$, we consider the Riemannian Manifolds $(\mathbb{R}^N \setminus \{0\}, g)$ and $(\mathbb{R}^k \setminus \{0\}, g)$, where we keep the same notation g to denote both metric tensors on $\mathbb{R}^N \setminus \{0\}$ and $\mathbb{R}^k \setminus \{0\}$.

Proposition 2.1.6 *Let $O(N)$ be the group of the orthogonal transformations of $\mathbb{R}^N$. Then every $T \in O(N)$ induces an isometry of $(\mathbb{R}^N \setminus \{0\}, g)$. Moreover every Hyperplane containing the origin is a totally geodesic Riemannian submanifold of $(\mathbb{R}^N \setminus \{0\}, g)$. In particular, for every $k < N$, $(\mathbb{R}^k \setminus \{0\}, g)$ is a Riemannian submanifold of $(\mathbb{R}^N \setminus \{0\}, g)$.*

PROOF. Let $T \in O(N)$. Since $T^tT = I$ and $|Tx| = |x|$, we have, for every $x \in \mathbb{R}^N \setminus \{0\}$,

$$\begin{aligned} T^t\,\bar{a}^{-1}(Tx)\,T &= T^t\,T + \left(\frac{1}{a} - 1\right) T^t\,\frac{Tx \otimes Tx}{|Tx|^2}\,T \\ &= I + \left(\frac{1}{a} - 1\right) \frac{(T^tT)x \otimes (T^tT)x}{|Tx|^2} \\ &= I + \left(\frac{1}{a} - 1\right) \frac{x \otimes x}{|x|^2} = \bar{a}^{-1}(x). \end{aligned} \tag{2.9}$$

Recalling (A.6) in Appendix A, the last equation implies that T is an isometry of $(\mathbb{R}^N \setminus \{0\}, g)$.
To prove the second claim we observe that every Hyperplane containing the origin coincides, for some subset $\Theta \subseteq o(N)$, with the set $\mathrm{Fix}(\Theta) = \{x \in \mathbb{R}^N : T(x) = x \ \forall T \in \Theta\}$. The required property, then, follows from Proposition A.4.10.

Finally, arguing as in (2.9), we can easily prove that the immersion i of $\mathbb{R}^k \setminus \{0\}$ onto $\mathbb{R}^N \setminus \{0\}$ is, actually, an isometric immersion of $(\mathbb{R}^k \setminus \{0\}, g)$ onto $(\mathbb{R}^N \setminus \{0\}, g)$.

□

2.2 The distance function

In this section we denote by d_g the distance function on $\mathbb{R}^N \setminus \{0\}$ induced by the metric tensor g defined in (2.1). By definition, $d_g(p, q)$ is given by the infimum of the lengths $\mathcal{L}(\gamma)$ of all piecewise C^1 curves γ from P to Q. Analogously we write $d_{\tilde{g}}$ to denote the distance function on $\mathbb{R}^N \setminus \{0\}$ induced by the metric tensor $\tilde{g} = dr^2 + ar^2h$.

Proposition 2.1.3 shows that the metric g is homothetic by a factor $\frac{1}{a}$ to $\tilde{g}$. Therefore, by the definition of d_g, we have , for any $p, q \in \mathbb{R}^N \setminus \{0\}$,

$$d_g(p, q) = \frac{1}{\sqrt{a}}\, d_{\tilde{g}}(p, q). \tag{2.10}$$

Moreover, recalling (2.3), d_g is equivalent to the euclidean distance on $\mathbb{R}^N \setminus \{0\}$:

$$\left(1 \wedge \frac{1}{\sqrt{a}}\right) |p - q| \le d_g(p, q) \le \left(1 \vee \frac{1}{\sqrt{a}}\right) |p - q|. \tag{2.11}$$

The main result of this chapter consists in proving the following explicit expression for the distance function d_g.

Theorem 2.2.1 *For any $p, q \in \mathbb{R}^N \setminus \{0\}$, given in spherical coordinates by $p = r_p\,\omega_p$, $q = r_q\,\omega_q$, the distance d_g between p and q is given by*

$$d_g(p, q) = \sqrt{\frac{1}{a}\left[r_p^2 + r_q^2 - 2r_pr_q \cos\left(\pi \wedge \sqrt{a}\arccos\left(\langle \omega_p, \omega_q \rangle\right)\right)\right]}. \tag{2.12}$$

Before proving Theorem 2.2.1, we give an interpretation of $d_{\tilde{g}}$ in terms of cone metrics. We recall the following.

Definition 2.2.2 [14] A *cone* $\mathrm{con}(X)$ over a topological space X is the quotient of the product $X \times [0, \infty)$, obtained by gluing together (identifying) all points in the fiber $X \times \{0\}$. This point is called the *origin* (or apex) of the cone, and denoted by O.

A point $p \in \mathrm{con}(X)$ can be described as (x, r), where $x \in X$ and $r = |Op|$. If (X, d_X) is a metric space, then d_X induces a metric d_C on $\mathrm{con}(X)$ (called *cone metric*), explicitly given, for any $p = (x, r), q = (y, s) \in \mathrm{con}(X)$, by

$$d_C(p,q) = \begin{cases} \sqrt{r^2 + s^2 - 2rs\cos(d_X(x,y))} & \text{if } d_X(x,y) \le \pi, \\ r + s & \text{if } d_X(x,y) > \pi \end{cases}$$

(see [14, pp. 90-93]). Observe that $r + s = \sqrt{r^2 + s^2 - 2rs\cos(\pi)}$, so that we can rewrite d_C as $d_C(p,q) = \sqrt{r^2 + s^2 - 2rs\cos(\pi \wedge d_X(x,y))}$.

Clearly, the above extremely general construction applies to $(\mathbb{R}^N \setminus \{0\}, d_{\tilde{g}})$ (the case when $a = 1$ was explicitly described in [14]). In fact, $(\mathbb{R}^N \setminus \{0\}, d_{\tilde{g}})$ can be seen as the cone over $(\mathbb{S}^{N-1}, d_{\tilde{h}})$, where $\tilde{h} = ah$ and $d_{\tilde{h}}$ is the corresponding distance.

Consider now two points $p = r_p\omega_p, q = r_q\omega_q \in \mathbb{R}^n \setminus \{0\}$, where (r_p, ω_p) are the spherical coordinates of p , so that $r_p = |p|, \omega_p = p/|p|$ (and correspondingly for q). Then, we have

$$d_{\tilde{g}}(p,q) = \sqrt{r_p^2 + r_q^2 - 2r_pr_q\cos(\pi \wedge d_{\tilde{h}}(\omega_p, \omega_q))}.$$

On the other hand, it is well known that

$$d_h(\omega_p, \omega_q) = \arccos(\langle \omega_p, \omega_q \rangle).$$

Since $\tilde{h} = ah$, we have $d_{\tilde{h}} = \sqrt{a}\, d_h$, so that the above equation permits to calculate $d_{\tilde{h}}(p,q)$. Substituting into the above formula for $d_{\tilde{g}}(p,q)$ and using (2.10), we obtain (2.12).

Remark 2.2.3 *Let (X, g_X) be a complete Riemannian manifold. The latter construction fails to be applied in the context of Riemannian manifolds since, in general, con(X) is not smooth at its apex O. In our case, indeed, the metric tensor $\tilde{g}$ cannot be extended smoothly at* 0 *if $a \neq 1$ (see Proposition 2.1.4). On the other hand, con(X) is a complete metric space provided that (X, d_X) is complete.*

In order to prove Theorem 2.2.1, we need some preparation. We start by finding a lower bound for $d_{\tilde{g}}$ which will be a crucial key to prove (2.12).

Proposition 2.2.4 *Let $\overline{h}$ be a metric tensor on the sphere $\mathbb{S}^{N-1}$ and let us consider the warped product $\mathbb{R}^+ \times_r \mathbb{S}^{N-1}$. Then, for every C^1 curve $\gamma : [a,b] \to \mathbb{R}^+ \times \mathbb{S}^{N-1}$, $t \mapsto \gamma(t) := (r(t), \omega(t))$, one has*

$$\mathcal{L}(\gamma) \geq \sqrt{r(a)^2 + r(b)^2 - 2r(a)r(b)\cos\Big(\pi \wedge \mathcal{L}(\omega)\Big)}.$$

Here $\mathcal{L}(\gamma)$ is the length of the curve γ in $\mathbb{R}^+ \times_r \mathbb{S}^{N-1}$ and $\mathcal{L}(\omega)$ is the length of ω in $\big(\mathbb{S}^{N-1}, \overline{h}\big)$. In particular, if d_C and $d_{\overline{h}}$ are the Riemannian distances on $\mathbb{R}^+ \times_r \mathbb{S}^{N-1}$ and $\big(\mathbb{S}^{N-1}, \overline{h}\big)$ respectively, then, for every $p = (r_p, \omega_p)$, $q = (r_q, \omega_q) \in \mathbb{R}^+ \times \mathbb{S}^{N-1}$, one has

$$d_C(p,q) \geq \sqrt{r_p^2 + r_q^2 - 2r_p r_q \cos\Big(\pi \wedge d_{\overline{h}}(\omega_p, \omega_q)\Big)}. \tag{2.13}$$

Proof. Let $\gamma : [a,b] \to \mathbb{R}^+ \times \mathbb{S}^{N-1}$, $t \mapsto \gamma(t) := (r(t), \omega(t))$ be a C^1 curve of $\mathbb{R}^+ \times \mathbb{S}^{N-1}$ and, for simplicity, let us assume $[a,b] = [0,1]$. Let $\theta(t) := \mathcal{L}\left(\omega_{|_{[0,t]}}\right)$ be the length of $\omega_{|_{[0,t]}}$ in $\big(\mathbb{S}^{N-1}, \overline{h}\big)$. Since

$$\mathcal{L}\left(\omega_{|_{[0,t]}}\right) = \int_0^t \sqrt{\overline{h}\left(\dot{\omega}(s), \dot{\omega}(s)\right)}\, ds,$$

a simple application of the fundamental theorem of calculus yields $\dot{\theta}(t) = \sqrt{\overline{h}\left(\dot{\omega}(t), \dot{\omega}(t)\right)}$.

Let us define, now, the curve α of $\mathbb{R}^2$ given in polar coordinates by $\alpha(t) \equiv (r(t), \theta(t))$ and let $p = \alpha(0)$, $q = \alpha(1)$. Then it follows, by construction,

$$\mathcal{L}(\gamma) = \int_0^1 \sqrt{\dot{r}(t)^2 + r(t)^2\, \overline{h}\left(\dot{\omega}(t), \dot{\omega}(t)\right)}\, dt = \int_0^1 \sqrt{\dot{r}(t)^2 + r(t)^2\, \dot{\theta}(t)^2}\, dt = \mathcal{L}(\alpha).$$

If $\mathcal{L}(\omega) = \theta(1) \leq \pi$ (see Figure 2.1), then it follows immediately by the properties of the Euclidean distance of $\mathbb{R}^2$ that

$$\sqrt{\Big[r(0)^2 + r(1)^2 - 2r(0)r(1)\cos(\theta(1))\Big]} = |\overline{pq}| \leq \mathcal{L}(\alpha) = \mathcal{L}(\gamma).$$

Analogously, if $\mathcal{L}(\omega) = \theta(1) > \pi$ (see Figure 2.2), then

$$r(0) + r(1) = |p| + |q| \leq \mathcal{L}(\alpha) = \mathcal{L}(\gamma).$$

Putting together the previous inequalities, we get

$$\mathcal{L}(\gamma) \geq \sqrt{\Big[r(0)^2 + r(1)^2 - 2r(0)r(1)\cos\Big(\pi \wedge \mathcal{L}(\omega)\Big)\Big]}.$$

To prove the last claim, let $p = (r_p, \omega_p)$, $q = (r_q, \omega_q) \in \mathbb{R}^+ \times \mathbb{S}^{N-1}$ and let $\gamma(t) := (r(t), \omega(t))$ be a C^1 curve of $\mathbb{R}^+ \times \mathbb{S}^{N-1}$ connecting p to q. Then the previous estimate and $\mathcal{L}(\omega) \geq d_{\overline{h}}(\omega_p, \omega_q)$ yield

$$\mathcal{L}(\gamma) \geq \sqrt{\Big[r_p^2 + r_q^2 - 2r_p r_q \cos\Big(\pi \wedge \mathcal{L}(\omega)\Big)\Big]} \geq \sqrt{\Big[r_p^2 + r_q^2 - 2r_p r_q \cos\Big(\pi \wedge d_{\overline{h}}(\omega_p, \omega_q)\Big)\Big]}.$$

Optimizing over γ we get the desired claim.

$\square$

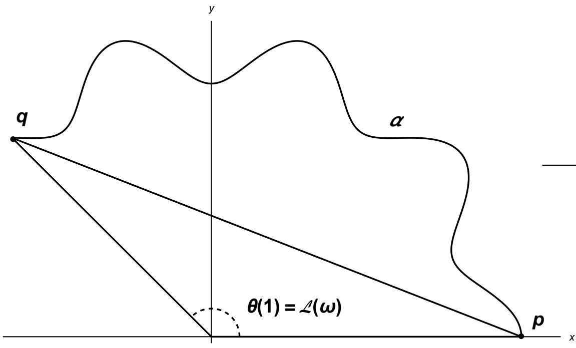


Figure 2.1: $\mathcal{L}(\omega) \leq \pi$

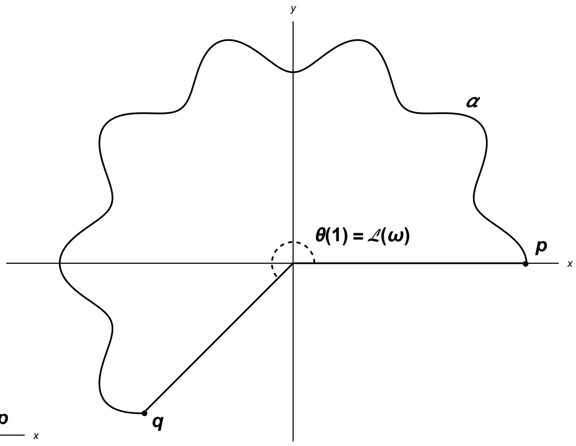


Figure 2.2: $\mathcal{L}(\omega) > \pi$

The Riemannian manifold $\left(\mathbb{R}^2 \setminus \{0\}, \tilde{g}\right)$ is, by Proposition 2.1.4, a flat manifold and, consequently, it is locally isometric to the Euclidean plane. This property allows to solve, at a first stage, the two-dimensional problem.

Proposition 2.2.5 *For any $p, q \in \mathbb{R}^2 \setminus \{0\}$, defined in spherical coordinates by $p = r_p\, \omega_p$, $q = r_q\, \omega_q$, the distance $d_{\tilde{g}}$ between p and q is given by*

$$d_{\tilde{g}}(p,q) = \sqrt{r_p^2 + r_q^2 - 2r_p r_q \cos\Big(\pi \wedge \sqrt{a}\arccos\left(\langle \omega_p, \omega_q\rangle\right)\Big)}. \tag{2.14}$$

Proof. Let us employ polar coordinates on $\mathbb{R}^2 \setminus \{0\}$:

$$\mathbb{R}^+ \times [0, 2\pi[\ni (r,\theta) \mapsto (x,y) = (r\cos\theta, r\sin\theta) \equiv re^{i\theta} \in \mathbb{R}^2 \setminus \{0\}.$$

The metric tensor takes the form

$$\tilde{g} = dr^2 + ar^2 d\theta^2.$$

On the other hand we have, for every $e^{i\theta_p}\, e^{i\theta_q} \in \mathbb{S}^1$,

$$d_h(e^{i\theta_p}, e^{i\theta_q}) = \arccos\Big(\langle e^{i\theta_p}, e^{i\theta_q}\rangle\Big) = \min\left\{|\theta_p - \theta_q|, 2\pi - |\theta_p - \theta_q|\right\}.$$

Let $p = r_p e^{i\theta_p}$, $q = r_q e^{i\theta_q} \in \mathbb{R}^2 \setminus \{0\}$ and let us define

$$d_1(p,q) := \sqrt{r_p^2 + r_q^2 - 2r_p r_q \cos\Big(\pi \wedge \sqrt{a} d_h(e^{i\theta_p}, e^{i\theta_q})\Big)}.$$

Applying Proposition 2.1.4 and Proposition 2.2.4, with $\overline{h} = ah$, we immediately get

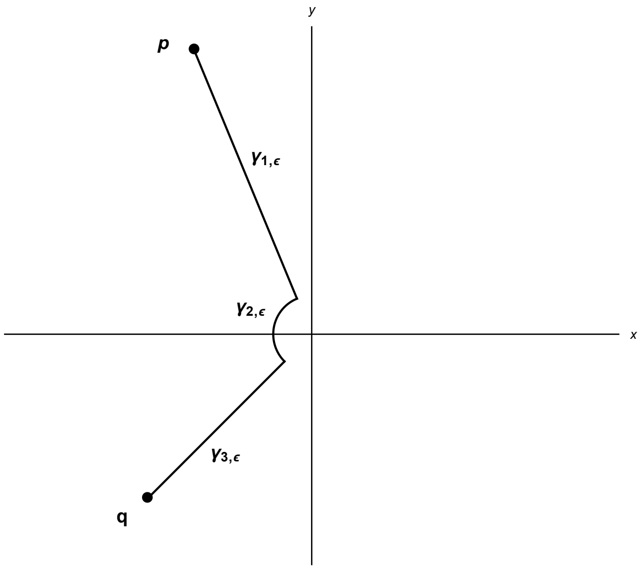
$$d_{\tilde{g}}(p,q) \geq d_1(p,q). \tag{2.15}$$

It is, therefore, sufficient to prove $d_{\tilde{g}}(p,q) \leq d_1(p,q)$.

Since, by Proposition 2.1.6, $\left(\mathbb{R}^2 \setminus \{0\}, \tilde{g}\right)$ is invariant under rotations, we can suppose, without loss of generality, $\frac{\pi}{2} \leq \theta_p \leq \theta_q \leq \frac{3}{2}\pi$. In particular $d_h(e^{i\theta_p}, e^{i\theta_q}) = (\theta_q - \theta_p)$.

Let us treat, firstly, the case $\sqrt{a}\,(\theta_q - \theta_p) \geq \pi$.
For every $0 < \epsilon < \min\{r_p, r_q\}$, let us consider the curve of $\mathbb{R}^2 \setminus \{0\}$ $\gamma_\epsilon := \gamma_{1,\epsilon} \cup \gamma_{2,\epsilon} \cup \gamma_{3,\epsilon}$ defined as the union of the three curves $\gamma_{1,\epsilon} : [\epsilon, r_p] \ni r \mapsto re^{i\theta_p}$, $\gamma_{2,\epsilon} : [\theta_p, \theta_q] \ni \theta \mapsto \epsilon e^{i\theta}$, $\gamma_{3,\epsilon} : [\epsilon, r_q] \ni r \mapsto re^{i\theta_q}$ (see Figure 2.3). Obviously γ_ϵ connects p to q and a direct

Figure 2.3: the support of the curve $\gamma_\epsilon = \gamma_{1,\epsilon} \cup \gamma_{2,\epsilon} \cup \gamma_{3,\epsilon}$.



computation easily gives $\mathcal{L}(\gamma_{1,\epsilon}) = r_p - \epsilon$, $\mathcal{L}(\gamma_{2,\epsilon}) = \epsilon\sqrt{a}(\theta_q - \theta_p)$, $\mathcal{L}(\gamma_{3,\epsilon}) = r_q - \epsilon$. This yields

$$d_{\tilde{g}}(p, q) \leq \mathcal{L}(\gamma_\epsilon) = r_p + r_q - 2\epsilon + \epsilon\sqrt{a}(\theta_q - \theta_p).$$

Taking the limit, in the last inequality, as $\epsilon \to 0$, we get $d_{\tilde{g}}(p, q) \leq r_p + r_q = d_1(p, q)$. This implies, recalling (2.15), $d_{\tilde{g}}(p, q) = d_1(p, q)$ which is the required claim for $\sqrt{a}\,(\theta_q - \theta_p) \geq \pi$.

Let us suppose, now, $\sqrt{a}\,(\theta_q - \theta_p) < \pi$.
$\left(\mathbb{R}^2 \setminus \{0\}, \tilde{g}\right)$ is, by Proposition 2.1.4, a flat Riemannian manifold and therefore it can be locally developed on the euclidean plane. Let us fix, to this aim, a sufficiently small $\epsilon > 0$ such that $I :=]\theta_p - \epsilon, \theta_q + \epsilon[\subseteq [0, 2\pi[$ and $\sqrt{a}\,(\theta_q - \theta_p) + 2\epsilon\sqrt{a} < \pi$. Then, setting $\theta_0 := \theta_p - \epsilon$ and $J := \sqrt{a}\, I - \sqrt{a}\theta_0$, we have

$$J =]0, \sqrt{a}\,(\theta_q - \theta_p) + 2\epsilon\sqrt{a}[\subseteq]0, \pi[.$$

Let U, V be the open cones of $\mathbb{R}^2 \setminus \{0\}$ defined by $U = \{re^{i\theta} : r \in \mathbb{R}^+, \theta \in I\}$ and $V = \{re^{i\theta} : r \in \mathbb{R}^+, \theta \in J\}$ (see Figure 2.4). Clearly $p, q \in U$ and V, having opening angle

$\kappa := \sqrt{a}\,(\theta_q - \theta_p) + 2\epsilon\sqrt{a} < \pi$, is convex. Let $f : U \to V$ be the application given, in polar coordinates, by

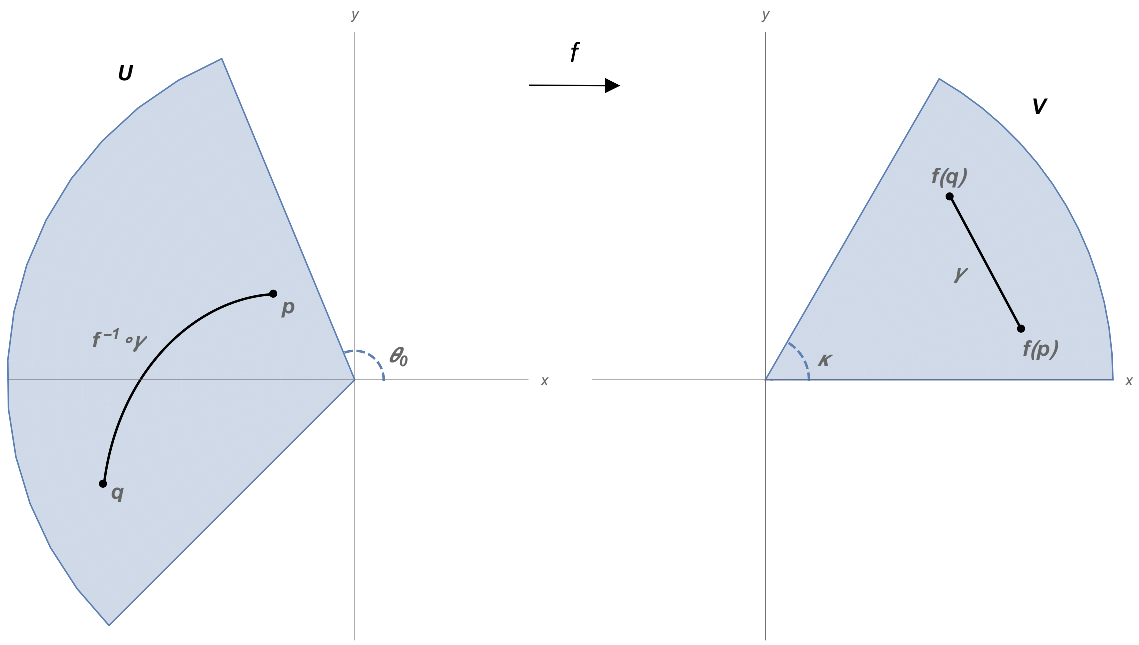
$$f : \mathbb{R}^+ \times I \ni (r, \theta) \mapsto \big(r, \sqrt{a}\,(\theta - \theta_0)\big) \in \mathbb{R}^+ \times J. \tag{2.16}$$

f is, clearly, a diffeomorphism whose Jacobian, at any point, satisfies $Jf = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \sqrt{a} \end{smallmatrix}\right)$. Let $g_0 = dr^2 + r^2 d\theta^2$ be the euclidean metric of $\mathbb{R}^2 \setminus \{0\}$. Since

$$\begin{pmatrix} 1 & 0 \\ 0 & ar^2 \end{pmatrix} = Jf^t \circ \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \circ Jf,$$

equation (A.6) in Appendix A proves that f is a Riemannian isometry between $(U, \tilde{g})$ and the euclidean cone (V, g_0).

Figure 2.4: the action of $f(r, \theta) = (r, \sqrt{a}\,(\theta - \theta_0))$.



Since V is convex, the segment γ that joins $f(p) = r_p e^{i\sqrt{a}(\theta_p - \theta_0)}$ to $f(q) = r_q e^{i\sqrt{a}(\theta_q - \theta_0)}$ lies entirely in V and has euclidean length equals to

$$|\overline{f(p)f(q)}| = \sqrt{r_p^2 + r_q^2 - 2r_p r_q \cos\big(\sqrt{a}(\theta_q - \theta_p)\big)} = d_1(p, q).$$

Since f is length preserving, the curve $\alpha := f^{-1} \circ \gamma$ of U joins p to q and satisfies

$$d_{\tilde{g}}(p, q) \le \mathcal{L}(\alpha) = |\overline{f(p)f(q)}| = d_1(p, q).$$

The last inequality, recalling (2.15), proves $d_{\tilde{g}}(p, q) = d_1(p, q)$ i.e the required claim in the remaining case.

$\square$

Remark 2.2.6 *If $a \leq 1$, the same proof of Proposition 2.2.5 shows that $\left(\mathbb{R}^2 \setminus \{0\}, \tilde{g}\right)$ can be globally developed on the quotient Riemannian manifold $V/\sim$ (see [13, Chapter 3, Definition 2.1]), where*

$$V = \{re^{i\theta} : r \in \mathbb{R}^+, 0 \leq \theta \leq \sqrt{a}\, 2\pi\}$$

is the cone equipped with the euclidean metric g_0 induced by the immersion in R^2 and $\sim$ is the equivalence relation that identifies the boundary rays given by $\theta = 0$, $\theta = \sqrt{a}2\pi$ (see Figure 2.5). The isometry is described, in polar coordinates, by

$$F : \mathbb{R}^+ \times [0, 2\pi[\ni (r, \theta) \mapsto \left(r, \sqrt{a}\,\theta\right) \in \mathbb{R}^+ \times [0, \sqrt{a}\, 2\pi[.$$

In this case, denoting by d_{g_0} the distance induced on $(V/\sim, g_0)$, one can recognize that

$$d_{\tilde{g}}(p, q) = d_{g_0}(F(p), F(q)) = \sqrt{r_p^2 + r_q^2 - 2r_p r_q \cos\left(\sqrt{a} d_h(e^{i\theta_p}, e^{i\theta_q})\right)}.$$

If $a > 1$, F is well defined only for $\sqrt{a}\,\theta < 2\pi$. To overcome this limit one can consider the helicoid of $\mathbb{R}^3$,

$$\mathcal{M} = \{(r\cos\theta, r\sin\theta, \theta) \in \mathbb{R}^3 : r > 0,\ \theta \in \mathbb{R}\}$$

provided with the metric tensor $dr^2 + r^2 d\theta^2$. We remark that the projection

$$p : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}, \quad p(x, y, z) = (x, y)$$

is a Riemannian covering map from $\left(\mathcal{M},\, dr^2 + r^2 d\theta^2\right)$ to $\left(\mathbb{R}^2 \setminus \{0\}, g_0\right)$ (see [66, Chapter 7, Definition 11]. With this setting, $\left(\mathbb{R}^2 \setminus \{0\}, \tilde{g}\right)$ is isometric to $\mathcal{M}'/\sim$ where

$$\mathcal{M}' = \{(r\cos\theta, r\sin\theta, \theta) \in \mathbb{R}^3 : r > 0,\ 0 \leq \theta \leq \sqrt{a}\, 2\pi\}$$

is a Riemannian submanifold of $\mathcal{M}$ and $\sim$ is the equivalence relation that identifies the boundary rays given by $\theta = 0$, $\theta = \sqrt{a}2\pi$ (see Figure 2.6). The isometry is given by

$$F : \mathbb{R}^2 \setminus \{0\} \to \mathcal{M}'/\sim,\ re^{i\theta} \mapsto \left(r\cos(\sqrt{a}\theta), r\sin(\sqrt{a}\theta), \sqrt{a}\theta\right).$$

In this case, if d' is the distance induced on $\left(\mathcal{M}'/\sim,\, dr^2 + r^2 d\theta^2\right)$, one can recognize that

$$d_{\tilde{g}}(p, q) = d'(F(p), F(q)) = \sqrt{r_p^2 + r_q^2 - 2r_p r_q \cos\left(\pi \wedge \sqrt{a} d_h(e^{i\theta_p}, e^{i\theta_q})\right)}.$$

Figure 2.5: the action of $F(r, \theta) = (r, \sqrt{a}\,\theta)$ for $a < 1$.

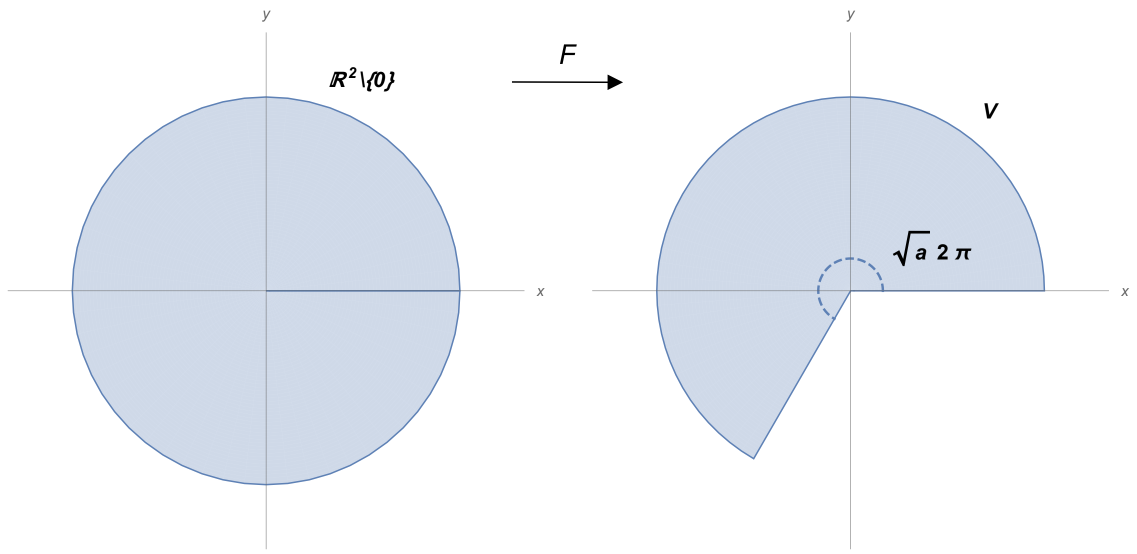
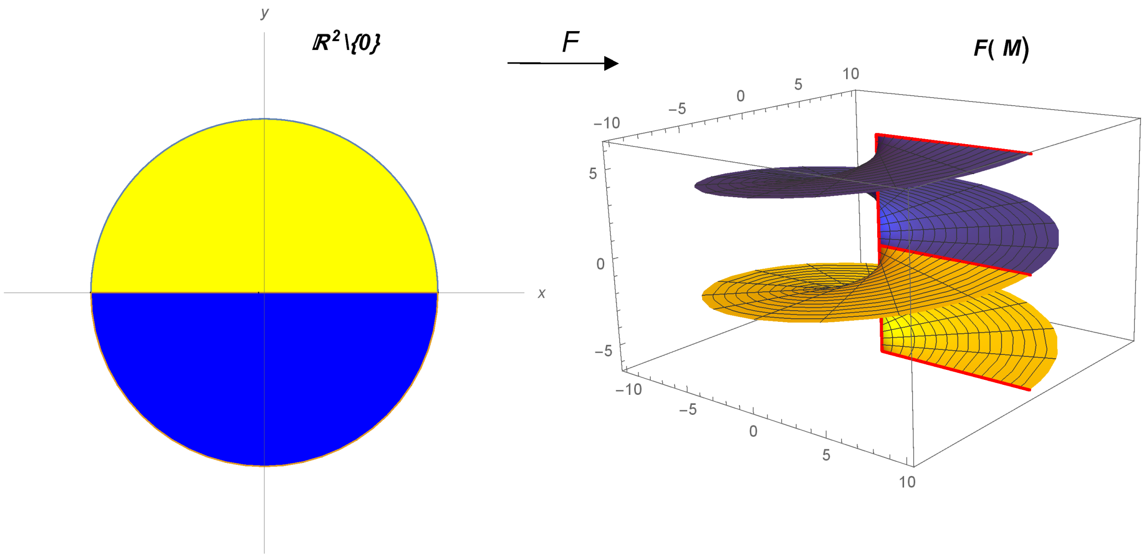


Figure 2.6: the action of $F(re^{i\theta}) = \left(r\cos(\sqrt{a}\theta), r\sin(\sqrt{a}\theta), \sqrt{a}\theta\right)$ for $a = 2$.



Finally, for $N > 2$, the spherical symmetry of $(\mathbb{R}^N \setminus \{0\}, g)$, proved in Proposition 2.1.6, combined with the lower bound (2.13), allows to deduce the N-dimensional distance from the two-dimensional one.

Proof of Theorem 2.2.1. Let $p, q \in \mathbb{R}^N \setminus \{0\}$, given in spherical coordinates by $p = r_p\, \omega_p,\ q = r_q\, \omega_q$. Recalling (2.10), we can equivalently prove

$$d_{\tilde{g}}(p, q) = \sqrt{r_p^2 + r_q^2 - 2 r_p r_q \cos\Big(\pi \wedge \sqrt{a}\arccos\left(\langle \omega_p, \omega_q\rangle\right)\Big)}.$$

By Proposition 2.1.6, $\big(\mathbb{R}^N \setminus \{0\}, \tilde{g}\big)$ is invariant under orthogonal transformations. We can, therefore, suppose $p, q \in \mathbb{R}^2 \setminus \{0\}$ since, otherwise, we can find an orthogonal transformation T that maps the plane π containing $\{p, q, 0\}$ into $\mathbb{R}^2$ and for which $d_{\tilde{g}}(T(p), T(q)) = d_{\tilde{g}}(p, q)$.

$(\mathbb{R}^2 \setminus \{0\}, \tilde{g})$ is, by Proposition 2.1.6, a Riemannian submanifold of $(\mathbb{R}^N \setminus \{0\}, \tilde{g})$: in particular, if d_2 is the induced distance on $(\mathbb{R}^2 \setminus \{0\}, \tilde{g})$ given by Formula (2.14), we apply Propositions A.4.3 and deduce

$$d_{\tilde{g}}(p, q) \le d_2(p, q) = \sqrt{r_p^2 + r_q^2 - 2 r_p r_q \cos\Big(\pi \wedge \sqrt{a}\arccos\left(\langle \omega_p, \omega_q\rangle\right)\Big)}.$$

On the other hand, the result of Proposition 2.2.4, with $\overline{h} = ah$, yields

$$d_{\tilde{g}}(p, q) \ge \sqrt{r_p^2 + r_q^2 - 2 r_p r_q \cos\Big(\pi \wedge \sqrt{a}\arccos\left(\langle \omega_p, \omega_q\rangle\right)\Big)}.$$

Combining the last two inequalities we prove the Theorem.

□

2.3 A characterization of the Riemannian distance

Let $(\mathcal{M}, g)$ denote a Riemannian manifold of dimension N and d_g the distance function on $\mathcal{M}$ induced by the metric tensor g. By definition, d_g is given by the infimum of the lengths $L(\gamma)$ of all C^1 curves γ from P to Q.

We denote by $\mathrm{Lip}(\mathcal{M}, d_g)$ the class of real functions defined on $\mathcal{M}$ which are Lipschitz-continuous with respect to the distance d_g. For $f \in \mathrm{Lip}(\mathcal{M}, d_g)$, we write $\mathrm{Lip}(f, d_g)$ to indicate the best constant L such that $|f(p) - f(q)| \le L\, d_g(p, q)$ for every $p,\, q \in \mathcal{M}$.

We start by recalling a result which allows the regularization of Lipschitz functions. For the proof, we may refer to [9].

Lemma 2.3.1 [9, Theorem 1] *Let $(\mathcal{M}, g)$ be a Riemannian manifold and let $\psi \in \mathrm{Lip}(\mathcal{M}, d_g)$. For every $\epsilon > 0$ there exists $\psi_\epsilon \in C^\infty(\mathcal{M})$, such that* *(i)* $\sup_{\mathcal{M}} |\psi_\epsilon - \psi| \le \epsilon$*;*

(ii) $\mathrm{Lip}(\psi_\epsilon, d_g) \le \mathrm{Lip}(\psi, d_g) + \epsilon$*.*

In particular, $\psi_\epsilon \to \psi$ *uniformly on* $\mathcal{M}$ *as* $\epsilon \to 0$*, and* $\limsup_{\epsilon \to 0} \mathrm{Lip}(\psi_\epsilon, d_g) \le \mathrm{Lip}(\psi, d_g)$*.*

Remark 2.3.2 *The same result with* $\mathrm{Lip}(\psi_\epsilon, d_g) = \mathrm{Lip}(\psi, d_g) + \epsilon$ *holds if* $(\mathcal{M}, g)$ *is a complete Riemannian manifold (see [33, Theorem 2]).*

The following proposition provides a useful characterization of the Riemannian distance function in terms of Lipschitz functions.

Proposition 2.3.3 *Let* $(\mathcal{M}, g)$ *be a Riemannian manifold. Then, for every* $p, q \in \mathcal{M}$*,*

$$d_g(p,q) = \sup\left\{\psi(p) - \psi(q) : \psi \in \mathrm{Lip}(\mathcal{M}, d_g),\ \mathrm{Lip}(\psi, d_g) \le 1\right\}. \tag{2.17}$$

Proof. Given two points $p, q \in \mathcal{M}$, we first consider the function f defined for every $z \in \mathcal{M}$ by $f(z) = d_g(z, q)$. The triangle inequality then yields at once that $f \in \mathrm{Lip}(\mathcal{M}, d_g)$ and $\mathrm{Lip}(f) \le 1$. Since $d_g(p,q) = f(p) - f(q)$, we immediately deduce that

$$d_g(p,q) \le \sup\left\{\psi(p) - \psi(q) : \psi \in \mathrm{Lip}(\mathcal{M}, d_g),\ \mathrm{Lip}(f) \le 1\right\}.$$

For the converse inequality, let $\gamma : [0,1] \to \mathcal{M}$ be a C^1 curve with $\gamma(0) = p$, $\gamma(1) = q$ and let $\psi \in \mathrm{Lip}(\mathcal{M}, d_g)$ with $\mathrm{Lip}(\psi, d_g) \le 1$. For every $\epsilon > 0$ let $\psi_\epsilon \in C^\infty(\mathcal{M})$ as in the previous Lemma. We observe that by definition we have $\frac{d}{dt}(\psi_\epsilon \circ \gamma)(t) = g\left(\nabla_g \psi_\epsilon(\gamma(t)), \dot\gamma(t)\right)$, where $\nabla_g \psi_\epsilon(p)$ is the $(\mathcal{M}, g)$ gradient of ψ_ϵ at p. The Cauchy-Schwarz inequality then yields

$$\begin{aligned}\psi_\epsilon(p) - \psi_\epsilon(q) &= \int_0^1 \frac{d}{dt}(\psi_\epsilon \circ \gamma)(t)\, dt \\ &= \int_0^1 g\left(\nabla_g \psi_\epsilon(\gamma(t)), \dot\gamma(t)\right)\, dt \le \int_0^1 \|\nabla_g \psi_\epsilon(\gamma(t))\|_g\, \|\dot\gamma(t)\|_g\, dt.\end{aligned}$$

Since $\sup_{z \in \mathcal{M}} \|\nabla_g \psi_\epsilon(z)\|_g = \mathrm{Lip}(\psi_\epsilon, d_g) \le \mathrm{Lip}(\psi, d_g) + \epsilon$, we get

$$\psi_\epsilon(p) - \psi_\epsilon(q) \le \int_0^1 (1+\epsilon)\, \|\dot\gamma(t)\|_g\, dt = (1+\epsilon)\, \mathcal{L}(\gamma).$$

Minimizing over γ and taking the limit for $\epsilon \to 0$, we conclude that $\psi(p) - \psi(q) \le d_g(p\,, q)$. This proves the desired inequality. $\square$

Remark 2.3.4 *Since for every* $\psi \in \mathrm{Lip}(\mathcal{M}, d_g)$*,* $-\psi \in \mathrm{Lip}(\mathcal{M}, d_g)$ *with* $\mathrm{Lip}(-\psi, d_g) = \mathrm{Lip}(\psi, d_g)$*, one has, equivalently,*

$$d_g(p,q) = \sup\left\{|\psi(p) - \psi(q)| : \psi \in \mathrm{Lip}(\mathcal{M}, d_g),\ \mathrm{Lip}(\psi, d_g) \le 1\right\}.$$

Moreover, every $\psi \in \mathrm{Lip}(\mathcal{M}, d_g)$ *with* $\mathrm{Lip}(\psi, d_g) \leq 1$ *can be approximated in* $L^\infty_{loc}(\mathcal{M})$ *(in* $L^\infty(\mathcal{M})$ *if* ψ *is bounded) with a sequence of function* $\phi_\epsilon \in C^\infty(\mathcal{M})$ *such that* $\mathrm{Lip}(\phi_\epsilon, d_g) = \sup_{z\in\mathcal{M}} \|\nabla_g \phi_\epsilon(z)\|_g \leq 1$*. Indeed, if* ψ_ϵ *is the approximating sequence of Proposition 2.3.1, then the sequence* $\phi_\epsilon = \frac{1}{1+\epsilon}\,\psi_\epsilon$ *satisfies the required property. This implies, also,*

$$\begin{aligned} d_g(p,q) &= \sup\left\{\psi(p) - \psi(q) : \psi \in C^\infty(\mathcal{M}),\ \sup_{z\in\mathcal{M}} \|\nabla_g \psi(z)\|_g \leq 1\right\} \\ &= \sup\left\{|\psi(p) - \psi(q)| : \psi \in C^\infty(\mathcal{M}),\ \sup_{z\in\mathcal{M}} \|\nabla_g \psi(z)\|_g \leq 1\right\}. \end{aligned}$$

We now come back to the Riemannian manifold $(\mathbb{R}^N \setminus \{0\}, g)$, where

$$g = \sum_{i,j=1}^{N} \left(\delta_{ij} + \left(\frac{1}{a} - 1\right)\frac{x_i x_j}{|x|^2}\right) dx_i dx_j,$$

and let us define the set

$$\Psi := \left\{\psi \in C^\infty(\mathbb{R}^N \setminus \{0\})\ :\ \langle \overline{a}\nabla\psi, \nabla\psi\rangle \leq 1\right\}. \tag{2.18}$$

Recalling (2.6), Ψ is the set of the functions $\psi \in C^\infty(\mathbb{R}^N \setminus \{0\}) \cap \mathrm{Lip}(\mathbb{R}^N \setminus \{0\}, d_g)$ which satisfy $\mathrm{Lip}(f, d_g) \leq 1$.

Proposition 2.3.5 *The distance* d_g *on* $\mathbb{R}^N \setminus \{0\}$ *induced by the metric* (2.1) *can be equivalently defined as*

$$d_g(x,y) := \sup\{\psi(x) - \psi(y)\ ;\ \psi \in \Psi\}, \quad x, y \in \mathbb{R}^N \setminus \{0\}. \tag{2.19}$$

Proof. The claim is an immediate consequence of Remark 2.3.4.

□

Remark 2.3.6 *For a generic metric space* $\mathcal{M}$*, formula* (2.17) *defines a distance on* $\mathcal{M}$ *which is called the functional distance. Functional distances play a preferred role in the discussion of second-order operators: for example in [22] Davies uses them to implement his famous perturbation method to prove Gaussian upper bounds for symmetric purely second order operators with* L^∞*-coefficients.*

We also remark that equality (2.19) *shows, in particular, that* d_g *is consistent with the metric used in [57, Lemma 4.11] to deduce upper estimate for the heat kernel of* L *with unspecified constants.*

Chapter 3

Gaussian upper bound for the Heat Kernel

In this Chapter we prove optimal upper complex bounds for the heat kernel of the operator

$$L = \Delta + (a-1)\sum_{i,j=1}^{N} \frac{x_i x_j}{|x|^2} D_{ij} + c\frac{x}{|x|^2}\cdot\nabla - \frac{b}{|x|^2}, \tag{3.1}$$

$a > 0$, $b,\ c \in \mathbb{R}$.

The main result is Theorem 3.5.3 where we prove that the heat kernel p of L, with respect to the measure $\mu = |x|^\gamma dy$, satisfies

$$\begin{aligned} |p(z,x,y)| \leq{}& C\,(Re\, z)^{-\frac{N}{2}}\left(1 + \mathrm{Re}\,\frac{d_g^2(x,y)}{4z}\right)^{\alpha} |x|^{-\frac{\gamma}{2}}|y|^{-\frac{\gamma}{2}} \\ &\times \left[\left(\frac{|x|}{(Re\, z)^{\frac{1}{2}}}\wedge 1\right)\left(\frac{|y|}{(Re\, z)^{\frac{1}{2}}}\wedge 1\right)\right]^{-\frac{N}{2}+1+\sqrt{D}} \exp\left(-Re\,\frac{d_g(x,y)^2}{4z}\right). \end{aligned} \tag{3.2}$$

where $D \geq 0$ is defined in (3.5), $\gamma = (N-1+c)/a - N + 1$ and $\alpha = \frac{N}{2}$ if $0 \leq D \leq \left(\frac{N-2}{2}\right)^2$, $\alpha = \frac{N+\gamma-2s_1}{2}$ if $D > \left(\frac{N-2}{2}\right)^2$. d_g is the distance on $\mathbb{R}^N \setminus \{0\}$ associated to the operator L which, as proved in Chapter 2, is expressed by

$$d_g(x,y) = \sqrt{\frac{1}{a}\left[|x|^2 + |y|^2 - 2|x||y|\cos\left(\pi \wedge \sqrt{a}\arccos\left(\langle \frac{x}{|x|}, \frac{y}{|y|}\rangle\right)\right)\right]}.$$

As a consequence we improve the result proved in [57].

There are different methods to prove Gaussian estimates. Here we follow the approach of [21] where the authors use the Phragmén-Lindelöf theorem to deduce them from some L^2 Gaussian bounds, the so-called Davies-Gaffney estimates (3.17), and the ultracontractivity of the semigroup.

Such technique is known to apply, for example, to the Laplace-Beltrami operator on a complete Riemannian manifold $\mathcal{M}$ (see [21, Theorems 1.1]): if $q(z,x,y)$ is the corresponding heat kernel, the Davies-Gaffney estimates and the on-diagonal upper bounds

$$q(t,x,x) \le Ct^{-\frac{D}{2}}, \quad \forall\, t>0,\, x\in M,$$

imply

$$|q(z,x,y)| \le C(Re\, z)^{-D/2}\left(1+Re\,\frac{d^2(x,y)}{4z}\right)^{D/2}\exp\left(-Re\,\frac{d^2(x,y)}{4z}\right),$$

where $z\in\mathbb{C}_+$, $x,y\in\mathcal{M}$ and d is the Riemannian distance on $\mathcal{M}$.

In this paper we apply the method to the operator defined by (3.1) whose associated Riemannian metric g, as a reflection of the singularity of L, is not complete for $a\neq 1$ (see Chapter 2).

In Section 3.1, we analyse the operator in the metric measure space $\left(\mathbb{R}^N\setminus\{0\}, d_g, \mu\right)$, where it is nonnegative and self-adjoint. To overcome the singularity at 0 we perform another change in the measure and use form methods to construct an equivalent operator in the space $L^2\left(\mathbb{R}^n, d\nu\right)$, where $d\nu$ is defined in (3.10).

In Section 3.4 we show that the the analytic semigroup $\{e^{zL}:\ z\in\mathbb{C}+\}$ generated by L satisfies the Davies-Gaffney estimates (3.17). This property, combined in Section 3.5 with some ultracontractivity bounds and with [21, Theorems 4.1], ensures the validity of (3.2).

We refer the reader to Appendix C for a brief survey on the main notions and results needed.

Unless otherwise specified, all the results presented in this Chapter are collected in [15].

3.1 Symmetric forms associated to $-L$

Let

$$L=\Delta+(a-1)\sum_{i,j=1}^N \frac{x_ix_j}{|x|^2}D_{ij}+c\frac{x}{|x|^2}\cdot\nabla-\frac{b}{|x|^2}=aD_{rr}+\frac{N-1+c}{r}D_r-\frac{b-\Delta_0}{r^2},$$

where D_r, D_{rr} denote radial derivatives and Δ_0 is the Laplace-Beltrami on S^{N-1}.

Setting $\overline{a}=(\overline{a}_{ij})$ with

$$\overline{a}_{ij}(x)=\delta_{ij}+(a-1)|x|^{-2}x_ix_j, \quad x\in\mathbb{R}^N\setminus\{0\}, \quad \gamma=\frac{N-1+c}{a}-N+1, \tag{3.3}$$

L becomes

$$L=|x|^{-\gamma}\mathrm{div}(|x|^{\gamma}\overline{a}\,\nabla)-\frac{b}{|x|^2}. \tag{3.4}$$

Note that $\gamma=0$ if and only if L is formally self-adjoint with respect to the Lebesgue measure.

In Chapter 1 we showed that $-L$ is associated with the sesquilinear form $\mathfrak{a}$ defined in $L^2_\mu = L^2(\mathbb{R}^N, d\mu)$, where $d\mu = |x|^\gamma dx$ and

$$\mathfrak{a}(u,v) := \int_{\mathbb{R}^N} \left(\langle \overline{a}\nabla u, \nabla v\rangle + \frac{b}{|x|^2} u\overline{v} \right) d\mu,$$
$$D(\mathfrak{a}) := C_c^\infty(\mathbb{R}^N \setminus \{0\}).$$

We recall that the condition

$$D = \frac{b}{a} + \left(\frac{N-1+c-a}{2a} \right)^2 \geq 0 \tag{3.5}$$

is necessary and sufficient to get positive solutions (see Proposition 1.3.1) and we shall always assume it. For the reader's convenience we summarize below the main properties about L proved in Section 1.2. See also [50, Lemma 3.4, Proposition 3.5] and [57, Lemma 4.2].

Proposition 3.1.1 *If $D \geq 0$, then $\mathfrak{a}$ is a non-negative, symmetric and closable form in L^2_μ. Denoting by $\tilde{\mathfrak{a}}$ the closure of $\mathfrak{a}$, the Friedrichs extension operator*

$$D(L) := \left\{ u \in D(\tilde{\mathfrak{a}}) \; ; \; \exists v \in L^2_\mu \text{ s.t. } \tilde{\mathfrak{a}}(u,w) = \int_{\mathbb{R}^N} v\overline{w}\, d\mu \quad \forall w \in D(\mathfrak{a}) \right\},$$
$$-Lu := v$$

is well-defined, non-negative and self-adjoint. Moreover,

(i) $C_c^\infty(\mathbb{R}^N \setminus \{0\}) \hookrightarrow D(L)$ *and for every* $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$

$$Lu = \sum_{i,j=1}^N \overline{a}_{ij} D_{ij} u + c\frac{x}{|x|^2} \cdot \nabla u - \frac{b}{|x|^2} u.$$

(ii) $D(L) \hookrightarrow \{u \in L^2_\mu \cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\}) \; ; \; Lu \in L^2_\mu\}$.

(iii) The scale homogeneity of L is 2:

$$s^2 L = M_s^{-1} L M_s, \quad M_s u(x) = u(sx), \quad s > 0. \tag{3.6}$$

(iv) L generates a contractive analytic semigroup $\left\{ e^{zL} : \; z \in \mathbb{C}_+ \right\}$ in L^2_μ which satisfies

$$e^{s^2 zL} = M_s^{-1} e^{zL} M_s, \quad M_s u(x) = u(sx), \quad z \in \overline{\mathbb{C}}_+, s > 0. \tag{3.7}$$

(v) For $t \geq 0$ the semigroup e^{tL} is positive and irreducible, that is $e^{tL} f > 0$ a.e. if $f \geq 0$, $f \neq 0$.

To derive kernel estimates, we introduce the following functions which allow to control the singularity of L at 0.

Definition 3.1.2 We fix $\eta \in C^\infty((0,\infty))$ satisfying $\eta = 1$ on $(0, 1/2]$ and $\eta = 0$ on $[2,\infty)$ and define a radial function $\phi \in C^\infty(\mathbb{R}^N \setminus \{0\})$ by

$$\phi(r) := \eta(r)r^{-s_1} + (1-\eta(r))r^{-\frac{\gamma}{2}}, \quad r = |x| > 0, \tag{3.8}$$

where s_1, γ are defined above (recall that $L|x|^{-s_1} = 0$).

We also define for $\alpha, \beta \in \mathbb{R}$,

$$\varrho_{\alpha,\beta}(x) := \begin{cases} |x|^{-\alpha} & \text{if } 0 < |x| < 1, \\ |x|^{-\beta} & \text{if } 1 \le |x| < \infty. \end{cases} = |x|^{-\beta}(1 \wedge |x|)^{-\alpha+\beta}. \tag{3.9}$$

The following lemma describes some properties of ϕ.

Lemma 3.1.3 *ϕ satisfies the following properties:*

(i) There exists a constant $0 < c_0 \le 1$ *such that*

$$c_0 \varrho_{s_1,\frac{\gamma}{2}}(x) \le \phi(x) \le c_0^{-1} \varrho_{s_1,\frac{\gamma}{2}}(x) \quad x \in \mathbb{R}^N \setminus \{0\}$$

where, since $s_1 = N/2 - 1 - \sqrt{D} + \gamma/2$,

$$\varrho_{s_1,\frac{\gamma}{2}}(x) = |x|^{-\frac{\gamma}{2}}(1 \wedge |x|)^{-\frac{N}{2}+1+\sqrt{D}}.$$

(ii) There exists a constant $c_0' \ge 0$ *such that*

$$\left|\nabla(\phi^2|x|^\gamma)\right| \le c_0'|x|^{\gamma-2s_1-1}\chi_{\{|x|<2\}}(x), \quad x \in \mathbb{R}^N \setminus \{0\}.$$

(iii) ϕ is a Lyapunov function for L, that is there exists $C_0 \ge 0$ *such that*

$$L\phi(x) \le C_0\phi(x), \quad x \in \mathbb{R}^N \setminus \{0\}.$$

Proof. **(i)** If $r \in (0,\infty) \setminus [\frac{1}{2}, 2]$, then $\phi(x) = \varrho_{s_1,\frac{\gamma}{2}}(x)$ for $|x| = r$. If $r \in [\frac{1}{2}, 2]$, then noting that ϕ and $\varrho_{s_1,\frac{\gamma}{2}}$ are continuous and positive on $[\frac{1}{2}, 2]$, we see that there exists $0 < c < 1$ such that $c \le \phi(x) \le c^{-1}$ and $c \le \varrho_{s_1,\frac{\gamma}{2}} \le c^{-1}$. Choosing $c_0 = c^2$, we obtain the first inequality. **(ii)** is proved similarly.
(iii) For $r \in (0, \frac{1}{2})$, $L\phi = 0$. For $r \in (2,\infty)$, we have

$$L\phi = -\left[b + \frac{\gamma}{2}\left(N - 1 + c - a - \frac{a\gamma}{2}\right)\right] r^{-\frac{\gamma}{2}-2} \le C_1 r^{-\frac{\gamma}{2}} = C_1\phi.$$

For $r \in [\frac{1}{2}, 2]$, from the continuity of ϕ, ϕ' and ϕ'' we deduce $L\phi \le C_2\phi$. Therefore ϕ is a Lyapunov function for L.

$\square$

Following the argument used in [23, Section 4.2], we perform another change in the measure to get rid of the potential term $b|x|^{-2}$. We introduce the Hilbert space $L^2_\nu = L^2(\mathbb{R}^N, d\nu)$, with $d\nu = \phi^2\, d\mu$ and the non-negative and symmetric form

$$\mathfrak{b}(u,v) := \mathfrak{a}(\phi u, \phi v),$$
$$D(\mathfrak{b}) := C_c^\infty(\mathbb{R}^N \setminus \{0\}).$$

Observe that

$$d\nu = \begin{cases} |x|^{\gamma - 2s_1}\, dx = |x|^{-N+2+2\sqrt{D}}\, dx, & \text{if } |x| \le 1/2, \\ \\ dx, & \text{if } |x| \ge 2. \end{cases} \tag{3.10}$$

If $u, v \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ and $\tilde{L}u = \phi^{-1}L(\phi u)$, then

$$-\int_{\mathbb{R}^N} (\tilde{L}u)\overline{v}\, d\nu = -\int_{\mathbb{R}^N} L(\phi u)\phi \overline{v}\, d\mu = \mathfrak{a}(\phi u, \phi v) = \mathfrak{b}(u,v)$$

so that $\mathfrak{b}$ is formally associated to $-\tilde{L}$. Moreover, if $u, v \in D(\mathfrak{b})$ are real, then by (3.4) and integration by parts,

$$\begin{aligned} \mathfrak{b}(u,v) &= \int_{\mathbb{R}^N} \Big((\overline{a}\nabla u \cdot \nabla v)\phi^2 + (\overline{a}\nabla \phi \cdot \nabla u) v\phi + (\overline{a}\nabla\phi \cdot \nabla v)u\phi \\ &\qquad + (\overline{a}\nabla\phi \cdot \nabla\phi)uv + b|x|^{-2}\phi^2 uv \Big)\, d\mu \\ &= \int_{\mathbb{R}^N} \Big((\overline{a}\nabla u \cdot \nabla v)\phi^2 + (\overline{a}\nabla\phi \cdot \nabla(uv\phi)) + b|x|^{-2}\phi^2 uv \Big)\, d\mu \qquad (3.11) \\ &= \int_{\mathbb{R}^N} \Big((\overline{a}\nabla u \cdot \nabla v)\phi^2 - (L\phi)\phi uv \Big)\, d\mu. \end{aligned}$$

In particular $\mathfrak{b}$ is quasi-accretive:

$$\mathfrak{b}(u,u) \ge \int_{\mathbb{R}^N} (\overline{a}\nabla u \cdot \nabla u)\, d\nu - C_0 \int_{\mathbb{R}^N} u^2\, d\nu. \tag{3.12}$$

Remark 3.1.4 *We point out that, as in Remark 1.2.5, if $\tilde{\mathfrak{b}}$ is the closure of the form $\mathfrak{b}$ in L^2_ν, then*

$$W_c^{1,\infty}(\mathbb{R}^N \setminus \{0\}) := \big\{ u \in W^{1,\infty}(\mathbb{R}^N)\ ;\ \textit{supp}\, u \textit{ is compact in } \mathbb{R}^N \setminus \{0\} \big\} \subseteq D(\tilde{\mathfrak{b}})$$

and, for every $u, v \in W_c^{1,\infty}(\mathbb{R}^N \setminus \{0\})$,

$$\tilde{\mathfrak{b}}(u,v) = \tilde{\mathfrak{a}}(\phi u, \phi v) = \int_{\mathbb{R}^N} \Big((\overline{a}\nabla u \cdot \nabla v)\phi^2 - (L\phi)\phi uv \Big)\, d\mu. \tag{3.13}$$

In particular,

$$\tilde{\mathfrak{b}}(u,u) \ge \int_{\mathbb{R}^N} (\overline{a}\nabla u \cdot \nabla u)\, d\nu - C_0 \int_{\mathbb{R}^N} u^2\, d\nu. \tag{3.14}$$

Since $u \in L^2_\nu \to \phi u \in L^2_\mu$ is an isometry which maps $D(\mathfrak{b})$ onto $D(\mathfrak{a})$, we can define $\tilde{L}$ as the operator

$$D(\tilde{L}) = \phi^{-1}\, D(L),$$
$$\tilde{L}u = \phi^{-1} L(\phi u).$$

The following result is easily verified.

Proposition 3.1.5 *$-\tilde{L}$ is the non-negative and self-adjoint operator associated to the closure of the form $\mathfrak{b}$ in L^2_ν. $\tilde{L}$ generates a contractive analytic semigroup $\left\{e^{z\tilde{L}} :\ z \in \mathbb{C}_+\right\}$ in L^2_ν which satisfies*

$$e^{z\tilde{L}} f = \phi^{-1} e^{zL}(\phi f), \quad f \in L^2_\nu.$$

Furthermore, for every $t > 0$, the operator $e^{t\tilde{L}}$ is positive and

$$\|e^{t\tilde{L}} f\|_\infty \le e^{C_0 t}\|f\|_\infty, \quad f \in L^2_\nu \cap L^\infty(\mathbb{R}^N),\ t > 0,$$

where $C_0 \ge 0$ is the constant which appears in Lemma 3.1.3.

Proof. The first claim follows immediately by construction. The positivity of $e^{t\tilde{L}}$ is a consequence of [67, Theorem 2.6] since, for every $u \in D(\mathfrak{b})$, one has $\operatorname{Re} u, (\operatorname{Re} u)_+ \in D(\mathfrak{b})$ and

$$\mathfrak{b}(\operatorname{Re} u, \operatorname{Im} u) \in \mathbb{R},$$
$$\mathfrak{b}((\operatorname{Re} u)_+, (\operatorname{Re} u)_-) = 0.$$

To show the L^∞ bound, by [67, Theorem 2.13] it suffices to show that for every non-negative function $u \in D(\mathfrak{b})$, $(1 \wedge u) \in D(\mathfrak{b})$ and

$$\mathfrak{b}((1 \wedge u), (u-1)^+) \ge -C_0 \int_{\mathbb{R}^N} (u-1)^+\, d\nu, \tag{3.15}$$

(note that $u - (1 \wedge u) = (u-1)^+$). Indeed, the above inequality implies

$$\|e^{t\tilde{L}} f\|_\infty \le e^{C_0 t}\|f\|_\infty, \quad f \in L^2_\nu \cap L^\infty(\mathbb{R}^N), \quad f \ge 0,$$

which, combined with the positivity of $e^{t\tilde{L}}$, gives

$$\|e^{t\tilde{L}} f\|_\infty \le \|e^{t\tilde{L}}|f|\|_\infty \le e^{C_0 t}\|f\|_\infty, \quad f \in L^2_\nu \cap L^\infty(\mathbb{R}^N).$$

To prove (3.15) we note that $\nabla(1 \wedge u) = \chi_{\{u<1\}}\nabla u$ and $\nabla(u-1)^+ = \chi_{\{u>1\}}\nabla u$. This implies, recalling (3.11),

$$\mathfrak{b}((1 \wedge u), (u-1)^+) = \int_{\mathbb{R}^N} -\frac{L\phi}{\phi}(u-1)^+\, d\nu \ge -C_0 \int_{\mathbb{R}^N} (u-1)^+\, d\nu.$$

□

3.2 Caffarelli-Kohn-Nirenberg type inequalities

Some integral inequalities due to Caffarelli-Kohn-Nirenberg are crucial in deducing kernel estimates. We state those we need, providing a simple self-contained proof (see [57] for further details). The following lemma follows from Sobolev inequality, by scaling.

Lemma 3.2.1 *Let $N \in \mathbb{N}$. Then for every $q \in (2, \infty)$ satisfying $\frac{1}{q} \geq \frac{1}{2} - \frac{1}{N}$, there exists $\widetilde{C}_q > 0$ such that for every $v \in W^{1,2}(\mathbb{R}^N)$,*

$$\|v\|_q \leq \widetilde{C}_q \|\nabla v\|_2^{N\left(\frac{1}{2}-\frac{1}{q}\right)} \|v\|_2^{1-N\left(\frac{1}{2}-\frac{1}{q}\right)}.$$

The following lemma is a special case of Caffarelli-Kohn-Nirenberg inequalities. We provide a short proof for completeness.

Lemma 3.2.2 [57] *Let $\sigma \in \mathbb{R} \setminus \{-N\}$. Then for every $q \in (2, \infty)$ satisfying $\frac{1}{q} \geq \frac{1}{2} - \frac{1}{N}$, there exists $C_q > 0$ such that for every $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$,*

$$\left(\int_{\mathbb{R}^N} |u(x)|^q |x|^\sigma \, dx\right)^{\frac{1}{q}} \leq C_q \left(\int_{\mathbb{R}^N} |\nabla u(x)|^2 |x|^{(1-\frac{2}{N})\sigma} \, dx\right)^{\frac{N}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} \times \left(\int_{\mathbb{R}^N} |u(x)|^2 |x|^\sigma \, dx\right)^{\frac{1}{2}-\frac{N}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}.$$

Proof. Set $y = \Phi(x) := |x|^{\frac{\sigma}{N}} x$. Since $\sigma \neq -N$, Φ is bijective, $\Phi^{-1}(y) = |y|^{-\frac{\sigma}{N+\sigma}} y$ and $dy = \frac{N+\sigma}{N} |x|^\sigma \, dx$. Setting $v(y) := u(\Phi^{-1}(y))$ in Lemma 3.2.1 we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^N} |u(x)|^q |x|^\sigma \, dx\right)^{\frac{1}{q}} \\
&\quad = \left(\frac{N}{|N+\sigma|} \int_{\mathbb{R}^N} |v(y)|^q \, dy\right)^{\frac{1}{q}} \\
&\quad \leq N^{\frac{1}{q}} |N+\sigma|^{-\frac{1}{q}} \widetilde{C}_q \left(\int_{\mathbb{R}^N} |\nabla v(y)|^2 \, dy\right)^{\frac{N}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} \left(\int_{\mathbb{R}^N} |v(y)|^2 \, dy\right)^{\frac{1}{2}-\frac{N}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} \\
&\quad \leq N^{-(\frac{1}{2}-\frac{1}{q})} |N+\sigma|^{\frac{1}{2}-\frac{1}{q}} \widetilde{C}_q \left(\int_{\mathbb{R}^N} |\nabla v(\Phi(x))|^2 |x|^\sigma \, dx\right)^{\frac{N}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} \\
&\qquad \times \left(\int_{\mathbb{R}^N} |u(x)|^2 |x|^\sigma \, dx\right)^{\frac{1}{2}-\frac{N}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}.
\end{aligned}$$

Noting that

$$\nabla v(y) = \nabla\Big[u\big(\Phi^{-1}(y)\big)\Big] = (D\Phi^{-1})^*(y) \nabla u\big(\Phi^{-1}(y)\big),$$

we see

$$|\nabla v(\Phi(x))| \leq \max\left\{1, \frac{N}{|N+\sigma|}\right\} |\Phi(x)|^{-\frac{\sigma}{N+\sigma}} |\nabla u(x)| \leq \max\left\{1, \frac{N}{|N+\sigma|}\right\} |x|^{-\frac{\sigma}{N}} |\nabla u(x)|.$$

Therefore

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |u(x)|^q |x|^\sigma \, dx\right)^{\frac{1}{q}} \leq{}& N^{-(\frac{1}{2}-\frac{1}{q})} |N+\sigma|^{\frac{1}{2}-\frac{1}{q}} \max\left\{1, \frac{N}{|N+\sigma|}\right\}^{N\left(\frac{1}{2}-\frac{1}{q}\right)} \widetilde{C}_q \\ &\times \left(\int_{\mathbb{R}^N} |\nabla u(x)|^2 |x|^{(1-\frac{2}{N})\sigma} \, dx\right)^{\frac{N}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} \\ &\times \left(\int_{\mathbb{R}^N} |u(x)|^2 |x|^\sigma \, dx\right)^{\frac{1}{2}-\frac{N}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}.\end{aligned}$$

□

In the next lemma we use Hardy inequality to have the same weights in front of u and ∇u.

Lemma 3.2.3 [57] *Let $\sigma > 0$. Then for every $r \in (2, \infty)$ satisfying $\frac{1}{r} \geq \frac{1}{2} - \frac{1}{N+\sigma}$ when $N \geq 3$ and $\frac{1}{r} > \frac{1}{2} - \frac{1}{2+\sigma}$ when $N = 2$, there exists $\tilde{C}_r > 0$ such that for every $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$,*

$$\begin{aligned}&\left(\int_{\mathbb{R}^N} |u(x)|^r |x|^\sigma \, dx\right)^{\frac{1}{r}} \\ &\leq C_q \left(\int_{\mathbb{R}^N} |\nabla u(x)|^2 |x|^\sigma \, dx\right)^{\frac{N+\sigma}{2}\left(\frac{1}{2}-\frac{1}{r}\right)} \left(\int_{\mathbb{R}^N} |u(x)|^2 |x|^\sigma \, dx\right)^{\frac{1}{2}-\frac{N+\sigma}{2}\left(\frac{1}{2}-\frac{1}{r}\right)}.\end{aligned}$$

PROOF. First we observe that $\frac{\sigma(r-2)}{4} < 1$ and

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |u|^r |x|^\sigma \, dx\right)^{\frac{1}{r}} &= \left(\int_{\mathbb{R}^N} \left(|u|^2 |x|^{\sigma-2}\right)^{\frac{\sigma(r-2)}{4}} \left(|u|^2 |x|^\sigma\right)^{\frac{q}{2}(1-\frac{\sigma(r-2)}{4})} dx\right)^{\frac{1}{r}}, \\ &\leq \left(\int_{\mathbb{R}^N} |u|^2 |x|^{\sigma-2} \, dx\right)^{\frac{\sigma(r-2)}{4r}} \left(\int_{\mathbb{R}^N} |u|^q |x|^{\frac{q}{2}\sigma} \, dx\right)^{\frac{1}{r}-\frac{\sigma(r-2)}{4r}},\end{aligned}$$

where we set $q = \frac{4r-2\sigma(r-2)}{4-\sigma(r-2)} \in (r, \infty)$. By Hardy inequality, we have

$$\int_{\mathbb{R}^N} |u|^2 |x|^{\sigma-2} \, dx \leq \left(\frac{2}{N-2+\sigma}\right)^2 \int_{\mathbb{R}^N} |\nabla u|^2 |x|^\sigma \, dx.$$

Moreover, by the assumption $\frac{1}{r} \geq \frac{1}{2} - \frac{1}{N+\sigma}$ we have $\frac{1}{q} - \frac{1}{2} = -\frac{r-2}{2r-\sigma(r-2)} \geq -\frac{1}{N}$ and therefore applying Lemma 3.2.1 to $|x|^{\frac{\sigma}{2}}u$, and using Hardy inequality again, we deduce

$$\begin{aligned}
\int_{\mathbb{R}^N} |u|^q |x|^{\frac{q}{2}\sigma}\,dx &\le \widetilde{C}_q^q \|\nabla(|x|^{\frac{\sigma}{2}}u)\|_{L^2}^{\frac{N(q-2)}{2}} \left(\int_{\mathbb{R}^N} |u|^2|x|^\sigma\,dx\right)^{q-\frac{N(q-2)}{4}} \\
&\le \widetilde{C}_q^q \left(\||x|^{\frac{\sigma}{2}}\nabla u\|_{L^2} + \frac{\sigma}{2}\||x|^{\frac{\sigma}{2}-1}u\|_{L^2}\right)^{\frac{N(q-2)}{2}} \left(\int_{\mathbb{R}^N} |u|^2|x|^\sigma\,dx\right)^{q-\frac{N(q-2)}{4}} \\
&\le \widetilde{C}_q^q \left(\frac{N-2+2\sigma}{N-2+\sigma}\right)^{\frac{N(q-2)}{2}} \left(\int_{\mathbb{R}^N} |\nabla u|^2|x|^\sigma\,dx\right)^{\frac{N(q-2)}{4}} \\
&\qquad\times \left(\int_{\mathbb{R}^N} |u|^2|x|^\sigma\,dx\right)^{q-\frac{N(q-2)}{4}}.
\end{aligned}$$

Combining the above inequalities and noting that $\left(\frac{1}{r} - \frac{\sigma(r-2)}{4r}\right)\frac{N(q-2)}{4} = \frac{N(r-2)}{4r}$, we obtain the desired inequality. □

3.3 Ultracontractivity estimates

Using the Caffarelli-Kohn-Nirenberg type inequalities of Section 3.2, it is possible to prove the following two lemmas that provide some Gagliardo-Nirenberg type inequalities which are the main technical tool to prove upper bounds for the heat kernel of $\tilde{L}$. We refer the reader to [15, 57] for their proof.

For technical reasons we distinguish between the cases $0 \le D \le (\frac{N-2}{2})^2$ and $D > (\frac{N-2}{2})^2$ which, recalling (3.10), correspond respectively to the negativity and positivity of the exponent of the weight defining the measure $d\nu$ in a neighbourhood of 0.

Lemma 3.3.1 *Assume that $N \ge 2$ and that $0 \le D \le (\frac{N-2}{2})^2$. Then for every $q \in (2,\infty)$ such that $\frac{1}{q} \ge \frac{1}{2} - \frac{1}{N}$ and for every $u \in D(\mathfrak{b})$,*

$$\|u\|_{L^q_{d\nu}} \le C'_q \left(C_0\|u\|^2_{L^2_{d\nu}} + \mathfrak{b}(u,u)\right)^{\frac{\theta}{2}} \|u\|^{(1-\theta)}_{L^2_{d\nu}}$$

where $\theta = N\left(\frac{1}{2} - \frac{1}{q}\right) \in (0,1]$ and C'_q is a positive constant independent of u.

Proof. Since $0 \le D \le (\frac{N-2}{2})^2$, then $\sigma = \gamma - 2s_1 = 2 - N + 2\sqrt{D} \le 0$ and hence

$$c_0^2 \max\{1, |x|^{\gamma-2s_1}\} = c_0^2 \varrho_{-\sigma,0}(x) \le \phi^2(x)|x|^\gamma \le c_0^{-2}\varrho_{-\sigma,0}(x) = c_0^{-2}\max\{1,|x|^{\gamma-2s_1}\}.$$

Therefore, with $\theta = N\left(\frac{1}{2} - \frac{1}{q}\right) \in (0,1]$ as in the statement, we obtain from Lemma 3.2.1

$$\begin{aligned}
\left(\int_{\mathbb{R}^N} |u(x)|^q\,dx\right)^{\frac{1}{q}} &\le \widetilde{C}_q \left(\int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx\right)^{\frac{\theta}{2}} \left(\int_{\mathbb{R}^N} |u(x)|^2\,dx\right)^{\frac{1-\theta}{2}} \\
&\le c_0^{-1}\widetilde{C}_q \left(\int_{\mathbb{R}^N} |\nabla u(x)|^2\,d\nu\right)^{\frac{\theta}{2}} \left(\int_{\mathbb{R}^N} |u(x)|^2\,d\nu\right)^{\frac{1-\theta}{2}}.
\end{aligned}$$

Using Lemma 3.2.2 with $\sigma = \gamma - 2s_1$ and the same θ as above we have also

$$\begin{aligned}
\left(\int_{\mathbb{R}^N} |u(x)|^q |x|^{\gamma-2s_1}\,dx\right)^{\frac{1}{q}} &\le C_q \left(\int_{\mathbb{R}^N} |\nabla u(x)|^2 |x|^{(1-\frac{2}{N})(\gamma-2s_1)}\,dx\right)^{\frac{\theta}{2}} \\
&\quad \times \left(\int_{\mathbb{R}^N} |u(x)|^2 |x|^{\gamma-2s_1}\,dx\right)^{\frac{1-\theta}{2}} \\
&\le c_0^{-(1-\theta)} C_q \left(\int_{\mathbb{R}^N} |\nabla u(x)|^2 \max\{1, |x|^{\gamma-2s_1}\}\,dx\right)^{\frac{\theta}{2}} \\
&\quad \times \left(\int_{\mathbb{R}^N} |u(x)|^2\,d\nu\right)^{\frac{1-\theta}{2}} \\
&\le c_0^{-1} C_q \left(\int_{\mathbb{R}^N} |\nabla u(x)|^2\,d\nu\right)^{\frac{\theta}{2}} \left(\int_{\mathbb{R}^N} |u(x)|^2\,d\nu\right)^{\frac{1-\theta}{2}}.
\end{aligned}$$

Then

$$\begin{aligned}
\left(\int_{\mathbb{R}^N} |u(x)|^q\,d\nu\right)^{\frac{1}{q}} &\le c_0^{-\frac{2}{q}} \left(\int_{\mathbb{R}^N} |u(x)|^q \max\{1, |x|^{\gamma-2s_1}\}\,dx\right)^{\frac{1}{q}} \\
&\le c_0^{-1-\frac{2}{q}} (C_q^q + \widetilde{C}_q^q)^{\frac{1}{q}} \left(\int_{\mathbb{R}^N} |\nabla u(x)|^2\,d\nu\right)^{\frac{\theta}{2}} \left(\int_{\mathbb{R}^N} |u(x)|^2\,d\nu\right)^{\frac{1-\theta}{2}}.
\end{aligned}$$

Since (3.12) yields

$$(1\wedge a)\int_{\mathbb{R}^N} |\nabla u|^2\,d\nu \le \int_{\mathbb{R}^N} \langle \overline{a}\nabla u, \nabla u\rangle\,d\nu \le C_0 \|u\|_{L^2_\nu}^2 + \mathfrak{b}(u,u)$$

the proof is complete. □

Lemma 3.3.2 *Assume that $N \ge 2$ and that $D > (\frac{N-2}{2})^2$. Then for every $q \in (2,\infty)$ such that $\frac{1}{q} > \frac{1}{2} - \frac{1}{N+\gamma-2s_1}$ and $u \in D(\mathfrak{b})$,*

$$\|u\|_{L^q_{d\nu}} \le C_q'' \Big((1+C_0)\|u\|_{L^2_{d\nu}}^2 + \mathfrak{b}(u,u)\Big)^{\frac{\theta}{2}} \|u\|_{L^2_{d\nu}}^{1-\theta}$$

where $\theta = (N+\gamma-2s_1)\left(\frac{1}{2}-\frac{1}{q}\right) \in (0,1)$ and C_q'' is a positive constant independent of u.

Proof. Put $\sigma = \gamma - 2s_1 = 2 - N + \sqrt{D} > 0$ so that $\theta = (N+\sigma)\left(\frac{1}{2}-\frac{1}{q}\right) \in (0,1)$, by the assumption on q. Let η be defined in (3.8). Then noting that $\frac{1}{q} > \frac{1}{2} - \frac{1}{N+\sigma} > \frac{1}{2} - \frac{1}{N}$ and

$$c_1|x|^\sigma \le \phi(x)^2|x|^\gamma \le c_1^{-1}|x|^\sigma, \qquad x \in \operatorname{supp}\eta,$$

$$c_2 \le \phi(x)^2|x|^\gamma \le c_2, \qquad x \in \operatorname{supp}(1-\eta),$$

we see from Lemmas 3.2.3 and 3.2.1, respectively, that

$$\begin{aligned}
\|\eta u\|_{L^q_{d\nu}} &\le c_1^{-\frac{1}{q}} \||x|^{\frac{\sigma}{q}}\eta u\|_q \\
&\le c_1^{-\frac{1}{q}} C_q \||x|^{\frac{\sigma}{2}}\nabla(\eta u)\|_2^{\frac{(N+\sigma)(q-2)}{2q}} \||x|^{\frac{\sigma}{2}}\eta u\|_2^{1-\frac{(N+\sigma)(q-2)}{2q}} \\
&\le c_1^{-\frac{1}{q}-\frac{1}{2}} C_q \|\eta\|_{W^{1,\infty}} \Big(\|u\|_{L^2_\nu} + \|\nabla u\|_{L^2_\nu}\Big)^{\frac{(N+\sigma)(q-2)}{2q}} \|u\|_{L^2_\nu}^{1-\frac{(N+\sigma)(q-2)}{2q}}
\end{aligned}$$

and

$$\begin{aligned}
\|(1-\eta)u\|_{L^q_{d\nu}} &\le c_2^{-\frac{1}{q}}\|(1-\eta)u\|_q \\
&\le c_2^{-\frac{1}{q}}\widetilde{C}_q\,\|\nabla((1-\eta)u)\|_2^{\frac{N(q-2)}{2q}}\,\|(1-\eta)u\|_2^{1-\frac{N(q-2)}{2q}} \\
&\le c_2^{-\frac{1}{q}-\frac{1}{2}}\widetilde{C}_q\|\eta\|_{W^{1,\infty}}\Big(\|u\|_{L^2_\nu}+\|\nabla u\|_{L^2_\nu}\Big)^{\frac{N(q-2)}{2q}}\,\|u\|_{L^2_\nu}^{1-\frac{N(q-2)}{2q}} \\
&\le c_2^{-\frac{1}{q}-\frac{1}{2}}\widetilde{C}_q\|\eta\|_{W^{1,\infty}}\Big(\|u\|_{L^2_\nu}+\|\nabla u\|_{L^2_\nu}\Big)^{\frac{(N+\sigma)(q-2)}{2q}}\,\|u\|_{L^2_\nu}^{1-\frac{(N+\sigma)(q-2)}{2q}}
\end{aligned}$$

(observe that $(A+B)^r A^{1-r}$ is increasing in $r\in[0,1]$, for $A,B\ge 0$). Combining the above estimates, we deduce

$$\|u\|_{L^q_{d\nu}} \le C\Big(\|u\|_{L^2_\nu}+\|\nabla u\|_{L^2_\nu}\Big)^{\frac{(N+\sigma)(q-2)}{2q}}\,\|u\|_{L^2_\nu}^{1-\frac{(N+\sigma)(q-2)}{2q}}.$$

Using (3.12) we complete the proof. □

Using the equivalence between ultracontractivity and Gagliardo-Nirenberg type inequalities given by Theorem C.1.1, we obtain some pointwise kernel estimates for $\tilde{L}$.

Proposition 3.3.3 *Let $\Omega=\mathbb{R}^N\setminus\{0\}$ and let $\tilde{L}$ be the operator defined by Proposition 3.1.5. Then there exists $p_\nu(z,\cdot,\cdot)\in L^\infty(\mathbb{R}^N\times\mathbb{R}^N)$ such that for $z\in\mathbb{C}_+$*

$$e^{z\tilde{L}}f(x)=\int_{\mathbb{R}^N}p_\nu(z,x,y)f(y)\,d\nu,\quad f\in L^1_{d\nu}\cap L^\infty(\mathbb{R}^N).$$

p_ν has a representation for which, for every $z\in\mathbb{C}_+$, $x,y\in\Omega$, $p_\nu(\cdot,x,y)$ is analytic on $\mathbb{C}_+$, $p_\nu(z,\cdot,\cdot)$ is symmetric on $\Omega\times\Omega$ and for which p_ν is continuous on $\mathbb{C}_+\times\Omega\times\Omega$.

$p_\nu(t,\cdot,\cdot)\ge 0$, for real positive t, and, furthermore, there exist two positive constants $C_1,\ C_2$ such that

$$|p_\nu(z,x,y)|\le\begin{cases} C_1\,(\mathrm{Re}\,z)^{-\frac{N}{2}}\exp\left(C_2\mathrm{Re}\,z\right), & \text{if } 0\le D\le\left(\frac{N-2}{2}\right)^2,\\[2ex] C_1\,(\mathrm{Re}\,z)^{-\frac{N+\gamma-2s_1}{2}}\exp\left(C_2\mathrm{Re}\,z\right), & \text{if } D>\left(\frac{N-2}{2}\right)^2,\end{cases}$$

for every $z\in\mathbb{C}_+$, $x,y\in\mathbb{R}^N$.

Proof. Let us suppose, preliminarily, $0\le D\le\left(\frac{N-2}{2}\right)^2$.

We apply Theorem C.1.1 to the semigroup $e^{-C_0t}e^{t\tilde{L}}$ whose generator is associated to the symmetric and accretive form

$$\mathfrak{b}+C_0\langle\ ,\ \rangle_{L^2_\nu}$$

and which is, by Proposition 3.1.5, L^∞-contractive. Lemma 3.3.1, Theorem C.1.1 and (C.2) give, for some $C>0$, the ultracontractivity estimate

$$\|e^{-C_0t}e^{t\tilde{L}}\|_{1\to\infty}\le Ct^{-\frac{N}{2}}.$$

Proposition C.1.3 then proves

$$\|e^{-C_0 z}e^{z\tilde{L}}\|_{1\to\infty} \le C\,(Re\, z)^{-\frac{N}{2}}$$

and the existence of a kernel p_ν such that, for $z \in \mathbb{C}_+$,

$$e^{z\tilde{L}}f(x) = \int_{\mathbb{R}^N} p_\nu(z,x,y)f(y)\,d\nu, \quad f \in L^1_\nu \cap L^\infty(\mathbb{R}^N),$$

with

$$\operatorname*{ess\,sup}_{x,y\in\mathbb{R}^N\setminus\{0\}} |p_\nu(z,x,y)| \le C\,(Re\, z)^{-\frac{N}{2}}\, e^{C_0 Re\, z},$$

and such that $p_\nu(\cdot,x,y)$ is analytic on $\mathbb{C}_+$, for every $x,y\in\Omega$. The symmetry, in the space variables, of p_ν and its positivity for $t>0$, are consequences of the positivity and self-adjointness of the semigroup.
To prove the continuity, we observe that, for fixed $y \in \mathbb{R}^N\setminus\{0\}$, $p_\nu(\,\cdot\,,\,\cdot\,,y)$ is a distributional solution of the equation $\left(\partial_t - \tilde{L}\right)p_\nu = 0$, where the coefficients of the operator $\tilde{L}$ are C^∞ far away from the origin . Then, by local parabolic regularity, $p_\nu(t,x,y)$ is a continuous function of $t>0$ and $x\in\mathbb{R}^N\setminus\{0\}$; for the same reason, since the constant of ellipticity of $\tilde{L}$ does not depend on y, $p_\nu(\cdot,\,\cdot\,,y)$ is locally Hölder continuous uniformly in $y\in\Omega$. The last property, combined with the symmetry in the space variables, implies the continuity of p_ν for $t>0$, $x,y\in\Omega$. The uniqueness theorem for holomorphic functions then proves the continuity for $z\in\mathbb{C}_+$ as well.
The case $D > \left(\frac{N-2}{2}\right)^2$ follows similarly using the form

$$\mathfrak{b} + (1+C_0)\,\langle\ ,\ \rangle_{L^2_\nu}$$

and Lemma 3.3.2 instead of Lemma 3.3.1.

$\square$

Remark 3.3.4 The continuity of $p_\nu(z,\cdot,\cdot)$ on $\Omega\times\Omega$, for every $z\in\mathbb{C}_+$, can be proved in a more direct way. Indeed using (3.22) and the expansion in spherical harmonics of the heat kernel of L proved in Proposition 4.2.7 (see also [50, Proposition 6.7]), we obtain for $t>0$, $x=r\omega$, $y=\rho\eta$, $r,\ \rho>0$, $|\omega|=|\eta|=1$,

$$p_\nu(t,x,y) = \frac{1}{\phi(x)\phi(y)}\frac{1}{2at}\,(r\rho)^{-s_1-\sqrt{D}}\exp\left\{-\frac{r^2+\rho^2}{4at}\right\}\sum_{n=0}^\infty I_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right)\mathbb{Z}^{(n)}_\omega(\eta). \quad (3.16)$$

Here I_α is the modified Bessel function of order $\alpha>0$, $\mathbb{Z}^{(n)}$ is the spherical harmonic of order $n\in\mathbb{N}_0$ and $D_n := D+\frac{n(n+N-2)}{a}$ (we refer the reader to Chapter 4 and to [50] and references therein for further details and for the proof of the convergence of the series involved). Fixing

$x, y \in \Omega$, the functions on both sides of the equation (3.16) are holomorphic on $\mathbb{C}_+$: the same equality, therefore, extends to $z \in \mathbb{C}_+$ producing

$$p_\nu(z,x,y) = \frac{1}{\phi(x)\phi(y)} \frac{1}{2az} (r\rho)^{-s_1-\sqrt{D}} \exp\left\{-\frac{r^2+\rho^2}{4az}\right\} \sum_{n=0}^{\infty} I_{\sqrt{D_n}}\left(\frac{r\rho}{2az}\right) \mathbb{Z}_\omega^{(n)}(\eta).$$

From the last equality one can deduce, in particular, the continuity of p_ν.

3.4 Davies-Gaffney estimates

In order to add a Gaussian term in the estimates found in Proposition 3.3.3, we need to prove that the contractive semigroup generated by $\tilde{L}$ satisfies the so-called Davies-Gaffney estimates.

Definition 3.4.1 *Let $(\mathcal{M}, d, \mu)$ be a metric measure space. Suppose that, for every $z \in \mathbb{C}_+$, $T(z)$ is a bounded linear operator acting on $L^2(\mathcal{M}, d\mu)$ and that $T(z)$ is an analytic function of z. Assume in addition that*

$$\|T(z)\|_{2\to 2} \le 1 \qquad \forall\ z \in \mathbb{C}_+.$$

For U_1, U_2 open subsets of $\mathcal{M}$, let $d(U_1, U_2) = \inf_{x\in U_1,\ y\in U_2} d(x,y)$. We say that the family $\{T(z):\ z \in \mathbb{C}_+\}$ satisfies the Davies-Gaffney estimate if

$$|\langle T(t)f_1, f_2\rangle| \le exp\left(-\frac{r^2}{4t}\right)\|f_1\|_2\|f_2\|_2 \tag{3.17}$$

for all $t > 0$, U_1, U_2 open subsets of $\mathcal{M}$, $f_1 \in L^2(U_1, d\mu)$, $f_2 \in L^2(U_2, d\mu)$ and $r := d(U_1, U_2)$.

In our case $\mathcal{M}$ is the Riemannian manifold $(\mathbb{R}^N \setminus \{0\}, g)$ associated to the operator L, where

$$g = \sum_{i,j=1}^{N} \left(\delta_{ij} + \left(\frac{1}{a} - 1\right)\frac{x_i x_j}{|x|^2}\right) dx_i dx_j,$$

and which is studied in details in Chapter 2. We recall that, by Proposition 2.1.4, as reflection of the singularity which appears in the leading term of the operator L, $(\mathbb{R}^N\setminus\{0\}, g)$ is non-complete for $a \neq 1$.

The reference measure is $d\nu = \phi^2\ d\mu$ and the distance is the Riemannian distance d_g induced by the metric g. Theorem 2.2.1 proves that

$$d_g(p,q) = \sqrt{\frac{1}{a}\Big[r_p^2 + r_q^2 - 2r_p r_q \cos\Big(\pi \wedge \sqrt{a}\arccos\left(\langle\omega_p, \omega_q\rangle\right)\Big)\Big]} \tag{3.18}$$

where $p = r_p\omega_p, q = r_q\omega_q \in R^N \setminus \{0\}$ and $r_p = |p|, \omega_p = p/|p|$ (and correspondingly for q). Recalling (2.11), d_g is equivalent to the euclidean distance:

$$\left(1 \wedge \frac{1}{\sqrt{a}}\right) |p-q| \leq d_g(p,q) \leq \left(1 \vee \frac{1}{\sqrt{a}}\right) |p-q|. \tag{3.19}$$

Proposition 2.3.5 shows also that

$$d_g(x,y) := \sup\{\psi(x) - \psi(y) \; ; \; \psi \in \Psi\}, \quad x, y \in \mathbb{R}^N \setminus \{0\},$$

where

$$\Psi := \left\{\psi \in C^\infty(\mathbb{R}^N \setminus \{0\}) \; : \; \langle \overline{a}\nabla\psi, \nabla\psi\rangle \leq 1\right\}.$$

Before stating the next Theorem we recall that Proposition 3.1.5 gives the contractivity in L^2_ν of $e^{z\tilde{L}}$ for $z \in C_+$.

Theorem 3.4.2 *The family $\{e^{z\tilde{L}} \; : \; z \in C_+\}$ satisfies the Davies-Gaffney estimate that is*

$$|\langle e^{t\tilde{L}} f_1, f_2\rangle| \leq \exp\left(-\frac{r^2}{4t}\right) \|f_1\|_{L^2_\nu} \|f_2\|_{L^2_\nu}$$

for all $t > 0$, U_1, U_2 open subsets of $\mathbb{R}^N \setminus \{0\}$, f_i in $L^2(U_i, d\nu)$ and $r := d_g(U_1, U_2)$.

PROOF. For $f \in L^2_\nu$, $t > 0$, $x \in \mathbb{R}^N \setminus \{0\}$, we put $f_t(x) = e^{t\tilde{L}} f(x)$. Let $k > 0$ and $\xi \in Lip(\mathbb{R}^N \setminus \{0\}, d_g)$, both to be chosen later, such that $Lip(\xi, d_g) \leq k$. We recall that, by (2.6), the last requirement is equivalent to

$$\sup_{x \in \mathbb{R}^N \setminus \{0\}} |\nabla_g \xi(x)|_g^2 = \|\langle \overline{a}\nabla\xi, \nabla\xi\rangle\|_\infty \leq k^2,$$

where $\nabla\xi$ is the weak gradient of ξ. For every $t \geq 0$ let us define

$$E(t) := \int_{\mathbb{R}^N} |f_t(x)|^2 e^{\xi(x)} d\nu \leq \infty.$$

We want to prove that

$$E(t) = \int_{\mathbb{R}^N} |f_t(x)|^2 e^{\xi(x)} d\nu \leq \exp\left\{\left(\frac{k^2}{2} + 2C_0\right) t\right\} E(0),$$

where C_0 is the constant which appears in (3.12). To this aim let us set, for $n \in \mathbb{N}$, $\xi_n = \xi \wedge n$, and

$$E_n(t) = \int_{\mathbb{R}^N} |f_t(x)|^2 e^{\xi_n(x)} d\nu.$$

Let us prove, preliminarily, that $E_n(t) \leq \exp\left\{\left(\frac{k^2}{2} + 2C_0\right) t\right\} E_n(0)$.
By a density argument, we can suppose, without loss of generality, $f \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$.

Otherwise we apply the latter inequality to a sequence of functions $(f_m)_{m\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^N\setminus\{0\})$ converging to f in L^2_ν and then take the limit for $m \to \infty$. Supposed $f \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, we have, in particular, $f_t, f_t e^{\xi_n} \in W_c^{1,\infty}(\mathbb{R}^N \setminus \{0\}) \subseteq D(\tilde{\mathfrak{b}})$. Then, differentiating under the integral and recalling (3.13), we get

$$\begin{aligned}
\frac{E_n'(t)}{2} &= \int_{\mathbb{R}^N} \partial_t f_t(x) f_t(x) e^{\xi_n(x)} d\nu = \int_{\mathbb{R}^N} (\tilde{L} f_t(x)) f_t(x) e^{\xi_n(x)} d\nu \\
&= -\tilde{\mathfrak{b}}(f_t, f_t e^{\xi_n}) = -\int_{\mathbb{R}^N} \Big(\langle \overline{a}\nabla f_t, \nabla(f_t e^{\xi_n})\rangle \phi^2 - (L\phi)\phi|f_t|^2 e^{\xi_n} \Big) d\mu \\
&= -\int_{\mathbb{R}^N} \Big(\langle \overline{a}\nabla f_t, \nabla f_t\rangle e^{\xi_n}\phi^2 + \langle \overline{a}\nabla f_t, f_t\nabla\xi_n\rangle e^{\xi_n}\phi^2 - (L\phi)\phi|f_t|^2 e^{\xi_n} \Big) d\mu.
\end{aligned}$$

Using (iii) of Lemma 3.1.3 and the CauchySchwarz inequality, we obtain

$$\begin{aligned}
\frac{E_n'(t)}{2} &\le \int_{\mathbb{R}^N} \left(-\langle \overline{a}\nabla f_t, \nabla f_t)\rangle - \langle \overline{a}\nabla f_t, f_t\nabla\xi_n\rangle + C_0|f_t|^2 \right) e^{\xi_n}\phi^2 d\mu \\
&\le \int_{\mathbb{R}^N} \Big(-\langle \overline{a}\nabla f_t, \nabla f_t\rangle + \langle \overline{a}\nabla f_t, \nabla f_t)\rangle^{\frac{1}{2}} \langle \overline{a} f_t\nabla\xi_n, f_t\nabla\xi_n)\rangle^{\frac{1}{2}} + C_0|f_t|^2 \Big) e^{\xi_n} d\nu \\
&\le \int_{\mathbb{R}^N} \left(\frac{1}{4}\langle \overline{a}\nabla\xi_n, \nabla\xi_n\rangle |f_t|^2 \ + C_0|f_t|^2 \right) e^{\xi_n} d\nu \\
&\le \left(\frac{1}{4}k^2 + C_0 \right) E_n(t).
\end{aligned}$$

We therefore deduce

$$E_n(t) \le \exp\left\{ \left(\frac{k^2}{2} + 2C_0 \right) t \right\} E_n(0).$$

Taking, in the last relation, the limit as n goes to infinity, we get, by monotone convergence, the required inequality

$$E(t) = \int_{\mathbb{R}^N} |f_t(x)|^2 e^{\xi(x)} d\nu \le \exp\left\{ \left(\frac{k^2}{2} + 2C_0 \right) t \right\} E(0).$$

Consider now two disjoint open sets U_1 and U_2 in $\mathbb{R}^N \setminus \{0\}$ and choose $\xi := k d_g(\cdot, U_1)$. Then $\xi \in Lip(\mathbb{R}^N \setminus \{0\}, d_g)$, $Lip(\xi, d_g) \le k$, $\xi = 0$ on U_1 and for every $g \in L^2_{loc}(\mathbb{R}^N, d\nu)$

$$\int_{U_2} |g|^2 e^{\xi} d\nu \ge e^{kr} \int_{U_2} |g|^2 d\nu,$$

where $r = d(U_1, U_2)$. Therefore, for $f \in L^2_\nu$ with $supp\, f \subseteq U_1$ and for $g = f_t$ we get, recalling that $\xi = 0$ on U_1,

$$\begin{aligned}
\int_{U_2} |f_t|^2 d\nu \le e^{-kr} E(t) &\le \exp\left\{ \left(\frac{k^2}{2} + 2C_0 \right) t - kr \right\} E(0) \\
&= \exp\left\{ \left(\frac{k^2}{2} + 2C_0 \right) t - kr \right\} \int_{U_1} |f|^2 d\nu.
\end{aligned}$$

By choosing $k = \frac{r}{t}$, we obtain

$$\int_{U_2} |e^{t\tilde{L}} f(x)|^2 d\nu \le \exp\left\{ \frac{r^2}{2t} - \frac{r^2}{t} + 2C_0 t \right\} \int_{U_1} |f|^2 d\nu = \exp\{2C_0 t\} \exp\left\{ -\frac{r^2}{2t} \right\} \int_{U_1} |f|^2 d\nu.$$

We finally deduce that, for every $f \in L^2(U_1, d\nu)$,

$$\sup_{g\in L^2(U_2,d\nu),\ \|g\|_{L^2_\nu}=1} |\langle e^{t\tilde{L}} f, g\rangle| = \left(\int_{U_2} |e^{t\tilde{L}} f(x)|^2 d\nu\right)^{\frac{1}{2}} \le \exp\{C_0 t\} \exp\left(-\frac{r^2}{4t}\right)\|f\|_{L^2_\nu}.$$

Using Lemma C.3.3, we can eliminate, in the last estimate, the exponential term $\exp\{C_0 t\}$ proving the desired claim.

□

Using the isometry $u \in L^2_\nu \to \phi u \in L^2_\mu$, we deduce, immediately, the same result for the semigroup generated by L.

Corollary 3.4.3 *The family $\{e^{zL} : \ z \in C_+\}$ satisfies the Davies-Gaffney estimate in $(\mathbb{R}^N \setminus \{0\}, d_g, d\mu)$ that is*

$$|\langle e^{tL} f_1, f_2\rangle| \le \exp\left(-\frac{r^2}{4t}\right) \|f_1\|_{L^2_\mu}\|f_2\|_{L^2_\mu}$$

for all $t > 0$, U_1, U_2 open subsets of $\mathbb{R}^N \setminus \{0\}$, f_i in $L^2(U_i, d\mu)$ and $r := d_g(U_1, U_2)$.

3.5 Kernel estimates for e^{zL}

Using Theorem C.4.1 we can, finally, prove the announced kernel estimates for the semigroup generated by L. We start by adding a Gaussian term in the bounds found in Proposition 3.3.3.

Lemma 3.5.1 *There exist two positive constants $C_1,\ C_2 > 0$ such that the heat kernel p_ν of $\tilde{L}$, with respect to the measure $d\nu$, satisfies the following estimates.*

(i) If $0 \le D \le \left(\frac{N-2}{2}\right)^2$ then, for every $z \in \mathbb{C}_+$, $x, y \in \mathbb{R}^N \setminus \{0\}$,

$$|p_\nu(z, x, y)| \le C_1(\mathrm{Re}\, z)^{-\frac{N}{2}} \left(1 + \mathrm{Re}\,\frac{d_g^2(x, y)}{4z}\right)^{\frac{N}{2}} \exp\left(C_2 \mathrm{Re}\, z - \mathrm{Re}\,\frac{d_g(x, y)^2}{4z}\right). \tag{3.20}$$

(ii) If $D > \left(\frac{N-2}{2}\right)^2$ then, for every $z \in \mathbb{C}_+$, $x, y \in \mathbb{R}^N \setminus \{0\}$,

$$|p_\nu(z, x, y)| \le C_1(\mathrm{Re}\, z)^{-\frac{N+\gamma-2s_1}{2}} \left(1 + \mathrm{Re}\,\frac{d_g^2(x, y)}{4z}\right)^{\frac{N+\gamma-2s_1}{2}} \exp\left(C_2 \mathrm{Re}\, z - \mathrm{Re}\,\frac{d_g(x, y)^2}{4z}\right). \tag{3.21}$$

Here d_g is the metric defined in $R^N \setminus \{0\}$ by

$$d_g(p, q) = \sqrt{\frac{1}{a}\left[r_p^2 + r_q^2 - 2r_p r_q \cos\left(\pi \wedge \sqrt{a} \arccos\left(\langle \omega_p, \omega_q\rangle\right)\right)\right]},$$

where $p = r_p\omega_p, q = r_q\omega_q \in R^N \setminus \{0\}$ and $r_p = |p|, \omega_p = p/|p|$ (and correspondingly for q).

Proof. Let us suppose, preliminarily, $0 \le D \le \left(\frac{N-2}{2}\right)^2$.

As before we consider the semigroup $e^{-C_0 t}e^{t\tilde{L}}$ whose generator is associated to the form

$$\mathfrak{b} + C_0\langle\ ,\ \rangle_{L^2_\nu}.$$

Proposition 3.3.3 gives the ultracontractivity estimate

$$\|e^{-C_0 t}e^{t\tilde{L}}\|_{1\to\infty} \le Ct^{-\frac{N}{2}},$$

which in terms of kernels reads as $0 \le e^{-C_0 t}p_\nu(t,x,y) \le Ct^{-\frac{N}{2}}$. Furthermore, for every $z \in \mathbb{C}_+$, $p(z,\cdot,\cdot)$ is continuous on $\mathbb{R}^N \setminus \{0\} \times \mathbb{R}^N \setminus \{0\}$.

Theorem 3.4.2 assures that the family $\{e^{z\tilde{L}}\ :\ z \in C_+\}$ satisfies the Davies-Gaffney estimate in $\left(\mathbb{R}^N \setminus \{0\}, d_g, d\nu\right)$ and so does $\{e^{-C_0 z}e^{z\tilde{L}}\ :\ z \in C_+\}$ since $|e^{-C_0 z}| \le 1$ for $z \in \mathbb{C}_+$.

Applying Theorem C.4.1 to the operator $C_0 - \tilde{L}$, we get, for every $z \in \mathbb{C}_+$, $x, y \in \mathbb{R}^N \setminus \{0\}$,

$$|p_\nu(z,x,y)| \le eC(\mathrm{Re}\, z)^{-\frac{N}{2}}\left(1 + \mathrm{Re}\,\frac{d_g^2(x,y)}{4z}\right)^{\frac{N}{2}} \exp\left(C_0 \mathrm{Re}\, z - Re\,\frac{d_g(x,y)^2}{4z}\right).$$

The case $D > \left(\frac{N-2}{2}\right)^2$ follows similarly using the semigroup $e^{-(1+C_0)t}e^{t\tilde{L}}$ associated to the form

$$\mathfrak{b} + (1 + C_0)\langle\ ,\ \rangle_{L^2_\nu}.$$

□

Remark 3.5.2 *The different exponents in* (3.20) *and* (3.21) *are due to the different parameters θ which occur in Lemmas 3.3.1 and 3.3.2.*

We can finally deduce the estimates for the heat kernel associated to the initial operator L.

For $z \in \mathbb{C}_+$, $x, y \in \mathbb{R}^N \setminus \{0\}$ let us define

$$p(z,x,y) := \phi(x)\phi(y)p_\nu(z,x,y). \tag{3.22}$$

By Lemma 3.1.5 and Proposition 3.3.3 we have, for every $f \in L^2_\mu$,

$$\begin{aligned} e^{zL}f(x) = \phi(x)\left[e^{z\tilde{L}}(\phi^{-1}f)\right](x) &= \int_{\mathbb{R}^N} \phi(x)p_\nu(z,x,y)\phi(y)^{-1}f(y)\, d\nu(y) \\ &= \int_{\mathbb{R}^N} p(z,x,y)f(y)d\mu(y). \end{aligned} \tag{3.23}$$

The scaling property (3.7) of e^{zL}, taking into account that the reference measure is $d\mu = |x|^\gamma dx$, implies that, for $z \in \mathbb{C}_+$, $s > 0$, $x, y \in \Omega$,

$$p(s^2 z, x, y) = s^{-N-\gamma}p\left(z, \frac{x}{s}, \frac{y}{s}\right). \tag{3.24}$$

We can state and prove, now, the main Theorem. Note that, using the scaling (3.24), the distinction between the cases $0 \le D \le \left(\frac{N-2}{2}\right)^2$ and $D > \left(\frac{N-2}{2}\right)^2$ appears only in the polynomial term and that $s_1 = N/2 - 1 - \sqrt{D} + \gamma/2$.

Theorem 3.5.3 *Let $\Omega = \mathbb{R}^N \setminus \{0\}$. There exists $p(z,\cdot,\cdot) \in L^\infty(\Omega \times \Omega)$ such that, for $z \in \mathbb{C}_+$,*

$$e^{zL} f(x) = \int_{\mathbb{R}^N} p(z,x,y) f(y)\, d\mu, \quad f \in L^2_\mu.$$

p is continuous on $\mathbb{C}_+ \times \Omega \times \Omega$ and, for every $z \in \mathbb{C}_+$ and $x, y \in \Omega$, $p(z,\cdot,\cdot)$ is symmetric on $\Omega \times \Omega$ and $p(\cdot,x,y)$ is analytic on $\mathbb{C}_+$.

Furthermore, for real positive t, $p(t,\cdot,\cdot) \ge 0$ and there exists a positive constant $C > 0$ such that the following estimates hold.

(i) If $0 \le D \le \left(\frac{N-2}{2}\right)^2$, then for every $z \in \mathbb{C}_+$ and $(x,y) \in \Omega \times \Omega$,

$$|p(z,x,y)| \le C\,(\mathrm{Re}\, z)^{-\frac{N}{2}} \left(1 + \mathrm{Re}\,\frac{d_g^2(x,y)}{4z}\right)^{\frac{N}{2}} |x|^{-\frac{\gamma}{2}} |y|^{-\frac{\gamma}{2}}$$
$$\times \left[\left(\frac{|x|}{(\mathrm{Re}\, z)^{\frac{1}{2}}} \wedge 1\right)\left(\frac{|y|}{(\mathrm{Re}\, z)^{\frac{1}{2}}} \wedge 1\right)\right]^{-\frac{N}{2}+1+\sqrt{D}} \exp\left(-\mathrm{Re}\,\frac{d_g(x,y)^2}{4z}\right).$$

(ii) If $D > \left(\frac{N-2}{2}\right)^2$, then for every $z \in \mathbb{C}_+$ and $(x,y) \in \Omega \times \Omega$,

$$|p(z,x,y)| \le C\,(\mathrm{Re}\, z)^{-\frac{N}{2}} \left(1 + \mathrm{Re}\,\frac{d_g^2(x,y)}{4z}\right)^{\frac{N+\gamma-2s_1}{2}} |x|^{-\frac{\gamma}{2}} |y|^{-\frac{\gamma}{2}}$$
$$\times \left[\left(\frac{|x|}{(\mathrm{Re}\, z)^{\frac{1}{2}}} \wedge 1\right)\left(\frac{|y|}{(\mathrm{Re}\, z)^{\frac{1}{2}}} \wedge 1\right)\right]^{-\frac{N}{2}+1+\sqrt{D}} \exp\left(-\mathrm{Re}\,\frac{d_g(x,y)^2}{4z}\right).$$

Here d_g is the metric defined in $R^N \setminus \{0\}$ by

$$d_g(p,q) = \sqrt{\frac{1}{a}\left[r_p^2 + r_q^2 - 2r_p r_q \cos\left(\pi \wedge \sqrt{a}\arccos\left(\langle \omega_p, \omega_q\rangle\right)\right)\right]},$$

where $p = r_p\omega_p, q = r_q\omega_q \in R^N \setminus \{0\}$ and $r_p = |p|, \omega_p = p/|p|$ (and correspondingly for q).

Proof. The existence of p as well as its regularity properties follows from (3.22) and (3.23) above. Consider $\zeta \in \mathbb{C}_+$ satisfying $Re\,\zeta = 1$. Then Proposition 3.3.3, Lemma 3.1.3 and the relation between the kernels yield

$$|p(\zeta,x,y)| \le C_1 \varrho_{s_1,\frac{\gamma}{2}}(x)\varrho_{s_1,\frac{\gamma}{2}}(y)|p_\nu(\zeta,x,y)|.$$

On the other hand, for $z \in \mathbb{C}_+$, $x, y \in \Omega$, the scaling (3.24) implies that, setting $\zeta = (Re\, z)^{-1} z$,

$$p(z,x,y) = (Re\, z)^{-\frac{N+\gamma}{2}} p\left(\zeta, \frac{x}{(Re\, z)^{\frac{1}{2}}}, \frac{y}{(Re\, z)^{\frac{1}{2}}}\right).$$

Therefore, using the previous estimates we have

$$|p(z,x,y)| = (Re\, z)^{-\frac{N+\gamma}{2}} \left| p\left(\zeta, \frac{x}{(Re\, z)^{\frac{1}{2}}}, \frac{y}{(Re\, z)^{\frac{1}{2}}}\right)\right|$$
$$\leq C_1 (Re\, z)^{-\frac{N+\gamma}{2}} \varrho_{s_1,\frac{\gamma}{2}}\left(\frac{x}{(Re\, z)^{\frac{1}{2}}}\right) \varrho_{s_1,\frac{\gamma}{2}}\left(\frac{y}{(Re\, z)^{\frac{1}{2}}}\right) \left| p_\nu\left(\zeta, \frac{x}{(Re\, z)^{\frac{1}{2}}}, \frac{y}{(Re\, z)^{\frac{1}{2}}}\right)\right|. \tag{3.25}$$

Recalling (i) of Lemma 3.1.3

$$\varrho_{s_1,\frac{\gamma}{2}}\left(\frac{x}{(Re\, z)^{\frac{1}{2}}}\right) = \left(\frac{|x|}{(Re\, z)^{\frac{1}{2}}}\right)^{-\frac{\gamma}{2}} \left(\frac{|x|}{(Re\, z)^{\frac{1}{2}}} \wedge 1\right)^{-\frac{N}{2}+1+\sqrt{D}}. \tag{3.26}$$

Furthermore, from (3.18), we easily deduce

$$d_g\left(\frac{x}{(Re\, z)^{\frac{1}{2}}}, \frac{y}{(Re\, z)^{\frac{1}{2}}}\right) = (Re\, z)^{-\frac{1}{2}} d_g(x,y).$$

The claim then follows by using the estimates of p_ν proved in Lemma 3.5.1. For example, if $0 \leq D \leq \left(\frac{N-2}{2}\right)^2$, then using (3.20) to estimate (3.25) we get

$$|p(z,x,y)| \leq C_1 (Re\, z)^{-\frac{N+\gamma}{2}} \varrho_{s_1,\frac{\gamma}{2}}\left(\frac{x}{(Re\, z)^{\frac{1}{2}}}\right) \varrho_{s_1,\frac{\gamma}{2}}\left(\frac{y}{(Re\, z)^{\frac{1}{2}}}\right)$$
$$\times \left(1 + \mathrm{Re}\, \frac{d_g^2(x,y)}{4z}\right)^{\frac{N}{2}} \exp\left(C_2 - Re\, \frac{d_g(x,y)^2}{4z}\right).$$

Recalling (3.26), we get the desired bound. The case $D > \left(\frac{N-2}{2}\right)^2$ can be treated in a similar way.

□

By absorbing the term $\left(1 + \mathrm{Re}\, \frac{d_g^2(x,y)}{4z}\right)^k$ into the exponential we get the following Corollary, in which there is no distinction between the cases $0 \leq D \leq \left(\frac{N-2}{2}\right)^2$ and $D > \left(\frac{N-2}{2}\right)^2$.

Corollary 3.5.4 *Let $D \geq 0$. For every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that, for every $z \in \mathbb{C}_+$ and $(x,y) \in \Omega \times \Omega$,*

$$|p(z,x,y)| \leq C_\epsilon\, (\mathrm{Re}\, z)^{-\frac{N}{2}}\, |x|^{-\frac{\gamma}{2}} |y|^{-\frac{\gamma}{2}} \left[\left(\frac{|x|}{(\mathrm{Re}\, z)^{\frac{1}{2}}} \wedge 1\right)\left(\frac{|y|}{(\mathrm{Re}\, z)^{\frac{1}{2}}} \wedge 1\right)\right]^{-\frac{N}{2}+1+\sqrt{D}}$$
$$\times \exp\left(-\mathrm{Re}\, \frac{d_g(x,y)^2}{(4+\epsilon)z}\right).$$

Moreover there exists $C'_\varepsilon > 0$ such that for $z \in \mathbb{C}_+$ satisfying $|\arg z| \le \frac{\pi}{2} - \varepsilon$, and $(x,y) \in \Omega \times \Omega$

$$|p(z,x,y)| \le C'_\epsilon |z|^{-\frac{N}{2}} |x|^{-\frac{\gamma}{2}} |y|^{-\frac{\gamma}{2}} \left[\left(\frac{|x|}{|z|^{\frac{1}{2}}} \wedge 1\right)\left(\frac{|y|}{|z|^{\frac{1}{2}}} \wedge 1\right)\right]^{-\frac{N}{2}+1+\sqrt{D}} \exp\left(-\frac{d_g(x,y)^2}{m_\epsilon |z|}\right),$$

where $m_\epsilon = \frac{4+\epsilon}{\cos(\frac{\pi}{2}-\epsilon)}$.

Proof. The first claim is consequence of Theorem 3.5.3 and of the inequality

$$(1+x)^k \exp\left\{-\frac{x}{4}\right\} \le C(\epsilon,k) \exp\left\{-\frac{x}{4+\epsilon}\right\}, \quad x \ge 0.$$

For the second it is enough to observe that, for $z \in \mathbb{C}_+$ satisfying $|\arg z| \le \frac{\pi}{2} - \varepsilon$, one has $cos(\frac{\pi}{2} - \epsilon) \le \frac{Re\, z}{|z|} \le 1$.

□

Using the equivalence (3.19) between d_g and the euclidean distance, we can improve [57, Theorem 4.14]. We point out that lower estimates of p for $z = t \in \mathbb{R}^+$ are proved in [50].

Corollary 3.5.5 *Let $\Omega = \mathbb{R}^N \setminus \{0\}$. For every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for $z \in \mathbb{C}_+$ satisfying $|\arg z| \le \frac{\pi}{2} - \varepsilon$, and $(x,y) \in \Omega \times \Omega$*

$$|p(z,x,y)| \le C_\varepsilon |z|^{-\frac{N}{2}} |x|^{-\frac{\gamma}{2}} |y|^{-\frac{\gamma}{2}} \left[\left(\frac{|x|}{|z|^{\frac{1}{2}}} \wedge 1\right)\left(\frac{|y|}{|z|^{\frac{1}{2}}} \wedge 1\right)\right]^{-\frac{N}{2}+1+\sqrt{D}} \exp\left(-\frac{|x-y|^2}{m'_\varepsilon |z|}\right),$$

where $m_\epsilon = \frac{4+\epsilon}{\cos(\frac{\pi}{2}-\epsilon)} \left(1 \wedge \frac{1}{a}\right)^{-1} = \frac{4+\epsilon}{\cos(\frac{\pi}{2}-\epsilon)}(1 \vee a)$.

The previous kernel estimate can be rewritten into the following equivalent form. For simplicity we only consider real positive t.

Corollary 3.5.6 *There exist $C > 0$ and $m > 0$ such that for $t > 0$ and $(x,y) \in \Omega \times \Omega$*

$$0 \le p(t,x,y) \le C t^{-\frac{N}{2}} |y|^{-\gamma} \left(\frac{|x|}{t^{\frac{1}{2}}} \wedge 1\right)^{-s_1} \left(\frac{|y|}{t^{\frac{1}{2}}} \wedge 1\right)^{-s_1^*} \exp\left(-\frac{|x-y|^2}{mt}\right) \tag{3.27}$$

where s_1 is defined in (1.7) and s_1^* in (1.29).

Proof. Let us recall that

$$s_1 = \frac{N}{2} - 1 - \sqrt{D} + \frac{\gamma}{2}, \quad s_1^* := \frac{N}{2} - 1 - \sqrt{D} - \frac{\gamma}{2}.$$

We prove that

$$t^{-\frac{N}{2}} \left(\frac{|x|}{|y|}\right)^{-\frac{\gamma}{2}} \left[\left(\frac{|x|}{t^{\frac{1}{2}}} \wedge 1\right)\left(\frac{|y|}{t^{\frac{1}{2}}} \wedge 1\right)\right]^{-\frac{N}{2}+1+\sqrt{D}} \exp\left(-\frac{|x-y|^2}{mt}\right) \le$$
$$\le C_\varepsilon t^{-\frac{N}{2}} \left(\frac{|x|}{t^{\frac{1}{2}}} \wedge 1\right)^{-s_1} \left(\frac{|y|}{t^{\frac{1}{2}}} \wedge 1\right)^{-s_1^*} \exp\left(-\frac{(1-\varepsilon)|x-y|^2}{mt}\right).$$

By scaling it is enough to show that

$$\left(\frac{|x|}{|y|}\right)^{-\frac{\gamma}{2}} \leq C \left(\frac{|x| \wedge 1}{|y| \wedge 1}\right)^{-\frac{\gamma}{2}} \exp\left(\delta |x-y|^2\right). \tag{3.28}$$

If $|x| \leq 1$ and $|y| \leq 1$ this is clearly true. Assume that $|y| \leq 1 \leq |x|$. Then $|x-y|^2 \geq (|x|-1)^2$ and

$$|x|^{-\frac{\gamma}{2}} \leq C \exp\left\{\delta(|x|-1)^2\right\} \leq C \exp\left\{\delta(|x-y|)^2\right\}$$

and (3.28) holds. If $|x| \leq 1 \leq |y|$ we argue in similar way. Finally, when $|x| \geq 1$, $|y| \geq 1$ we write $x = r\omega, y = \rho\eta$ with $|\omega| = |\eta| = 1$ and we may assume that $r \geq \rho \geq 1$. Writing $r = s\rho$ with $s \geq 1$, the inequality $s^{-\frac{\gamma}{2}} \leq Ce^{\delta(s-1)^2} \leq Ce^{\delta(s-1)^2\rho^2}$ $(s, \rho \geq 1)$ implies that

$$\left(\frac{r}{\rho}\right)^{-\frac{\gamma}{2}} \leq Ce^{\delta|r-\rho|^2} \leq Ce^{\delta|x-y|^2}.$$

$\square$

Chapter 4

Gaussian lower bound

In this chapter we find the following two-side estimates for the heat kernel p of L with respect to the measure $|y|^\gamma \, dy$

$$p(t,x,y) \simeq c_1 t^{-\frac{N}{2}} |x|^{-\frac{\gamma}{2}} |y|^{-\frac{\gamma}{2}} \left[\left(\frac{|x|}{t^{\frac{1}{2}}} \wedge 1 \right) \left(\frac{|y|}{t^{\frac{1}{2}}} \wedge 1 \right) \right]^{-\frac{N}{2}+1+\sqrt{D}} \exp\left(-\frac{c_2|x-y|^2}{t} \right).$$

Since upper estimates were already proved in Chapter 3, we focus on lower estimates for real positive t. Section 4.1 is devoted to the decomposition of the heat kernel of L as the (infinite) sum of heat kernels of one-dimensional Bessel operators. In Section 4.2 we get the main result by combining kernel estimates near the origin, obtained thanks to the explicit formula of one-dimensional Bessel operators, with Gaussian estimates faraway from the origin already known for uniformly elliptic operators.

The Chapter is mainly based on [50].

4.1 Decomposition of the N-dimensional operator

Let us consider, for $a > 0$ and $b,\ c$ real coefficients, the elliptic operator

$$L = \Delta + (a-1) \sum_{i,j=1}^{N} \frac{x_i x_j}{|x|^2} D_{ij} + c\frac{x}{|x|^2} \cdot \nabla - \frac{b}{|x|^2}.$$

Let $\gamma = \frac{N-1+c}{a} - N + 1$ and let us consider the weighted space $L^2_\mu = L^2(\mathbb{R}^N, d\mu)$ with $d\mu = |x|^\gamma dx$. Setting $\overline{a}(x) := I + (a-1)\frac{x \otimes x}{|x|^2}$ and recalling (1.10), L can be written as

$$L = |x|^{-\gamma} \mathrm{div}(|x|^\gamma \overline{a}\, \nabla) - \frac{b}{|x|^2}.$$

As usual, let us assume $D := \frac{b}{a} + \left(\frac{N-1+c-a}{2a}\right)^2 \geq 0$.

Following the construction of Section 1.2, $-L$ is the operator associated with the closure $\tilde{\mathfrak{a}}$

of the nonnegative and symmetric form in L^2_μ

$$\mathfrak{a}(u,v) := \int_{\mathbb{R}^N} \left(\langle \overline{a}\nabla u, \nabla v\rangle + \frac{b}{|x|^2} u\overline{v} \right) d\mu, \qquad D(\mathfrak{a}) := C_c^\infty(\mathbb{R}^N \setminus \{0\}).$$

Let us introduce some notation.

For fixed $n \in \mathbb{N}_0$ let $\{P_i^n, i = 1, ..., a_n\}$ be an orthonormal basis of spherical harmonics of degree n and let $\mathcal{H}_n$ denote the space of spherical harmonics of degree n, with $\dim\mathcal{H}_n = a_n$. Let us define the subspace of L^2_μ

$$\begin{aligned} L^2_n : &= L^2((0,\infty), r^{\frac{N-1+c}{a}}\, dr) \otimes \mathcal{H}_n \\ &= \bigoplus_{i=1}^{a_n} \left(L^2((0,\infty), r^{\frac{N-1+c}{a}}\, dr) \otimes P_j^n \right) = \bigoplus_{i=1}^{a_n} L^2_{P_i^n}, \end{aligned} \tag{4.1}$$

where

$$L^2_P = L^2((0,\infty), r^{\frac{N-1+c}{a}}\, dr) \otimes P. \tag{4.2}$$

Let $\mathbb{Z}^{(n)}_\omega$ be the zonal harmonic of degree n with pole $\omega \in \mathbb{S}^{N-1}$ defined by

$$\mathbb{Z}^{(n)}_\omega(\eta) := \mathbb{Z}^{(n)}(\omega,\eta) = \sum_{i=1}^{a_n} P_i^n(\omega) P_i^n(\eta),$$

where $\omega, \eta \in \mathbb{S}^{N-1}$.

Zonal harmonics provide a simple way to describe the orthogonal projection of L^2_μ onto L^2_n. We refer the reader to Section B.2 for further details.

Proposition 4.1.1 *The following properties hold.*

(i) $L^2\left(\mathbb{R}^N, |x|^\gamma dx\right) = \bigoplus_{n=0}^\infty L^2_n$.

(ii) for every $u \in L^2\left(\mathbb{R}^N, |x|^\gamma dx\right)$ *the orthogonal projection on* L^2_n *is given by*

$$Q_n(u)(r,\eta) := \sum_{i=1}^{a_n} P_i^n(\eta) \int_{\mathbb{S}^{N-1}} u(r,\omega) P_i^n(\omega) d\omega = \int_{\mathbb{S}^{N-1}} u(r,\omega) Z^{(n)}_\omega(\eta) d\omega$$

and moreover $u = \sum_{n=0}^\infty Q_n(u)$ *in* $L^2\left(\mathbb{R}^N, |x|^\gamma dx\right)$.

Proof. The result follows by setting $\alpha = \gamma = \frac{N-1+c}{a} - N + 1$ in Proposition B.2.3 in Appendix B. $\square$

Note that, if P is a normalized spherical harmonic of degree n, the projection on L^2_P is given by

$$Q_P(u)(r,\eta) := P(\eta) \int_{\mathbb{S}^{N-1}} u(r,\omega) P(\omega) d\omega.$$

Moreover $Q_n = \sum_{i=1}^{a_n} Q_{P_i^n}$, Q_P is symmetric and, for every $u \in C_c^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$, Q_P commutes with the radial derivative that is

$$(Q_P u)_r = P(\eta) \int_{\mathbb{S}^{N-1}} u_r(r,\omega) P(\omega)\, d\omega = Q_p u_r.$$

This follows, since $u_r = \nabla u \cdot \frac{x}{|x|}$, by differentiating under the integral sign.

In this section we prove that on each L_n^2, defined in (4.1), L coincides with a one-dimensional Bessel operator. Since $L_\mu^2 = \bigoplus_{n=0}^\infty L_n^2$, this provides us a complete decomposition of the N-dimensional resolvent and kernel in terms of its one-dimensional counterparts. We write

$$L = aD_{rr} + \frac{N-1+c}{r} D_r - \frac{b-\Delta_0}{r^2}$$

where Δ_0 is the Laplace-Beltrami on S^{N-1} and recall that $\Delta_0 P = -\lambda_n P$ if P is a spherical harmonic of degree n.

If $v(x) = u(r)P(\omega) \in C_c^\infty(\mathbb{R}^N \setminus \{0\}) \cap L_n^2$ then, from Proposition 1.1.2,

$$Lv = \left(a u_{rr} + \frac{N-1+c}{r} u_r - \frac{b+\lambda_n}{r^2} u \right) P(\omega) := (L_n u)(r) P(\omega). \tag{4.3}$$

The one dimensional Bessel operator $-L_n$ is associated to the form

$$\mathfrak{a}_n(u,v) = \int_0^\infty \left(a u_r \overline{v}_r + \frac{b+\lambda_n}{r^2} u \overline{v} \right) r^{\frac{N-1+c}{a}} dr$$

considered in Section 1.3. Observe that if $uP, vP \in C_c^\infty(\mathbb{R}^N \setminus \{0\}) \cap L_n^2$, with $u,\ v \in C_c^\infty(0,\infty)$, then (4.3) can be written in the equivalent form

$$\mathfrak{a}(uP, vP) = \int_0^\infty \left(a u_r \overline{v}_r + \frac{b+\lambda_n}{r^2} u \overline{v} \right) r^{\frac{N-1+c}{a}} dr = \mathfrak{a}_n(u,v).$$

However, this is not yet sufficient to conclude that the part of L into L_n^2 is the Bessel operator L_n, since domain questions arise (both at the level of the domains of the operators and of the closures of the forms).

In the following we fix a normalized spherical harmonic P of degree n. First we show that L and the projection Q_P, defined in Section 2, commute.

Lemma 4.1.2 *Let $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$. Then $LQ_P u = Q_P L u$.*

PROOF. Let $w(r\omega) = (Q_P u)(r\omega) = v(r)P(\omega)$, where

$$v(r) = \int_{S^{N-1}} u(r\omega) P(\omega)\, d\omega.$$

Since Δ_0 is self-adjoint in $L^2(S^{N-1})$ and $\Delta_0 P = -\lambda_n P$ we get

$$av_{rr} - \frac{N-1+c}{r} v_r - \frac{b+\lambda_n}{r^2} v = \int_{S^{N-1}} \left(au_{rr} - \frac{N-1+c}{r} u_r - \frac{b+\lambda_n}{r^2} u \right) P(\omega)\, d\omega$$
$$= \int_{S^{N-1}} \left(Lu - \frac{\lambda_n}{r^2} u - \frac{\Delta_0 u}{r^2} \right) P(\omega)\, d\omega = \int_{S^{N-1}} \left(Lu - \frac{\lambda_n}{r^2} u \right) d\omega - \frac{1}{r^2} \int_{S^{N-1}} u \Delta_0 P\, d\omega$$
$$= \int_{S^{N-1}} Lu\, d\omega.$$

Since

$$Lw = P(\omega) \left(av_{rr} - \frac{N-1+c}{r} v_r - \frac{b+\lambda_n}{r^2} v \right),$$

the claim follows. □

We prove now the continuity of Q_P with respect to the norm

$$\|u\|_{\mathfrak{a}}^2 = \|u\|_2^2 + \mathfrak{a}(u,u) = \|u\|_2^2 - (Lu, u)$$

Lemma 4.1.3 *Let $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$. Then $\|Q_P u\|_{\mathfrak{a}}^2 \le \|u\|_{\mathfrak{a}}^2$.*

Proof. We write $u = u_1 + u_2$ where $u_1 = Q_P u$ and $u_2 = (I - Q_P)u$. Then

$$\mathfrak{a}(u,u) = -(Lu,u) = -(Lu_1,u_1) - (Lu_2,u_2) - (Lu_1,u_2) - (Lu_2,u_1).$$

By observing that, by Lemma 4.1.2,

$$\begin{aligned}
(-Lu_1, u_1) &= (-LQ_P u, Q_P u) = \mathfrak{a}(Q_P u, Q_P u);\\
(-Lu_2, u_2) &= (-L(I-Q_P)u, (I-Q_P)u) = \mathfrak{a}((I-Q_P)u, (I-Q_P)u);\\
(-Lu_1, u_2) &= (-LQ_P u, (I-Q_P)u) = -(Q_P Lu, (I-Q_P)u) = 0;\\
(-Lu_2, u_1) &= (-L(I-Q_P)u, Q_P u) = -((I-Q_P)Lu, Q_P u) = 0,
\end{aligned}$$

we get

$$\mathfrak{a}(u,u) = \mathfrak{a}(Q_P u, Q_P u) + \mathfrak{a}((I-Q_P)u, (I-Q_P)u).$$

The thesis follows from the positivity of the form and the boundedness of Q_P in L^2_μ. □

Remark 4.1.4 *Observe that the above proof yields*

$$\mathfrak{a}(u,u) = \sum_{i=1}^n \mathfrak{a}(u_i P_i, u_i P_i)$$

if $u = \sum_{i=1}^n u_i(r) P_i(\omega)$ with P_i spherical harmonics.

Lemma 4.1.5 *Let $u,\ v \in D(\tilde{\mathfrak{a}})$. Then $Q_P u,\ Q_P v \in D(\tilde{\mathfrak{a}})$ and $\tilde{\mathfrak{a}}(Q_P u, v) = \tilde{\mathfrak{a}}(u, Q_P v)$.*

PROOF. If $u,\ v \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, the claim follows by Lemma 4.1.2. Let $u,\ v \in D(\tilde{\mathfrak{a}})$. There exist $(u_n)_{n\in\mathbb{N}},\ (v_n)_{n\in\mathbb{N}} \in C_c^\infty(C_c^\infty(\mathbb{R}^N \setminus \{0\}))$ such that $u_n \to u$ and $v_n \to v$ in L^2_μ and $\mathfrak{a}(u_n,u_n)$, $\mathfrak{a}(v_n,v_n)$ are Cauchy sequences. By Lemma 4.1.3, $(Q_P u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $D(\tilde{\mathfrak{a}})$. Since $Q_P u_n \to Q_P u$ in L^2_μ we get $Q_P u$ in $D(\tilde{\mathfrak{a}})$ and $Q_P u_n \to Q_P u$ in $D(\tilde{\mathfrak{a}})$. The same applies to v. Since, by Lemma 4.1.2,

$$\tilde{\mathfrak{a}}(Q_P u_n, v_n) = \tilde{\mathfrak{a}}(u_n, Q_P v_n),$$

the claim follows by letting n to infinity. □

Lemma 4.1.6 *Let $u \in D(L)$, then $Q_P u \in D(L)$ and $LQ_P u = Q_P L u$. In particular, for $f \in L^2_\mu$, $(\lambda - L)^{-1} Q_P f = Q_P(\lambda - L)^{-1} f$, $\lambda > 0$, and $e^{tL} Q_P f = Q_P e^{tL} f$.*

PROOF. By assumption

$$\tilde{\mathfrak{a}}(u,v) = -(Lu,v), \quad \forall\ v \in D(\tilde{\mathfrak{a}}).$$

The above lemma yields

$$\tilde{\mathfrak{a}}(Q_P u, v) = \tilde{\mathfrak{a}}(u, Q_P v) = -(Lu, Q_P v) = -(Q_P L u, v).$$

Therefore $Q_P u \in D(L)$ and $LQ_P u = Q_P L u$. The last assertion follows immediately. □

Finally let us prove that the part of L in L^2_n is L_n, by showing that the restriction of $\tilde{\mathfrak{a}}$ onto L^2_n coincide with $\tilde{\mathfrak{a}}_n$. Note that this last form is defined on functions of one variable $u = u(r)$ However, for a fixed P, we identify u with uP and use $\tilde{\mathfrak{a}}_n(uP, vP)$ for $\tilde{\mathfrak{a}}_n(u,v)$.

Lemma 4.1.7 *The forms $\tilde{\mathfrak{a}}$ and $\tilde{\mathfrak{a}}_n$ coincide on L^2_P. It follows that $u(r)P(\omega)$ belongs to $D(L)$ if and only if $u \in D(L_n)$ and, in this case, $L(u(r)P(\omega)) = P(\omega)L_n u(r)$. Finally, $(\lambda - L)^{-1}(u(r)P(\omega)) = P(\omega)(\lambda - L_n)^{-1}u(r)$ and $e^{tL}(u(r)P(\omega)) = P(\omega)e^{tL_n}u(r)$.*

PROOF. Let $u,\ v \in C_c^\infty(0,\infty)$. Then, as shown at the beginning of this section,

$$\tilde{\mathfrak{a}}_n(u,v) = \tilde{\mathfrak{a}}(uP, vP).$$

Let now $u,\ v \in D(\tilde{\mathfrak{a}}_n)$. There exist $(u_k)_{k\in\mathbb{N}}$, $(v_k)_{k\in\mathbb{N}}$ such that $u_k \to u$ in $L^2(0,\infty)$ and $\mathfrak{a}_n(u_k,u_k)$ is a Cauchy sequence. Then $u_k P \to uP$ in L^2_μ and $\mathfrak{a}(u_k P, u_k P)$ is a Cauchy sequence. This implies that $uP \in D(\tilde{\mathfrak{a}})$. In similar way we can argue for v. Since $\tilde{\mathfrak{a}}_n(u_k, v_k) = \tilde{\mathfrak{a}}(u_k P, v_k P)$ the equality $\tilde{\mathfrak{a}}_n(u,v) = \tilde{\mathfrak{a}}(uP,vP)$ follows letting k to infinity.

Conversely, let $uP \in D(\tilde{\mathfrak{a}}) \cap L^2_P$ and let $u_k \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ such that u_k goes to uP in L^2_μ and $\mathfrak{a}(u_k,u_k)$ is a Cauchy sequence. Then $Q_P u_k \to uP$ in L^2_μ and, by Lemma 4.1.3, $\mathfrak{a}(Q_P u_k, Q_P u_k)$ is a Cauchy sequence. Since

$$\mathfrak{a}(Q_P u_k, Q_P u_k) = \tilde{\mathfrak{a}}_n(Q_P u_k, Q_P u_k),$$

then $uP = Q_p(uP) \in D(\tilde{\mathfrak{a}}_n)$.

The last statements now follow using also Lemma 4.1.5, since for $v \in D(\tilde{\mathfrak{a}})$, $Q_P v \in D(\tilde{\mathfrak{a}}_n)$ and

$$\tilde{\mathfrak{a}}(uP, v) = \tilde{\mathfrak{a}}(Q_P(uP), v) = \tilde{\mathfrak{a}}(uP, Q_P v) = \tilde{\mathfrak{a}}_n(uP, Q_P v).$$

□

The above two lemmas yield

$$\begin{aligned}(\lambda - L)^{-1}(fP) &= P(\lambda - L_n)^{-1} f, \qquad \text{for } \lambda > 0,\\ e^{tL}(fP) &= P e^{tL_n} f,\end{aligned}$$

when $f = f(r)$ and P is a normalized spherical harmonic of degree n.

This fact, together with the decomposition $L^2_\mu = \bigoplus_{n=0}^\infty L^2_n$, allows to factorize e^{tL} as the direct sum of the one dimensional semigroups e^{tL_n}. We consider, using Proposition 4.1.1, the projection onto L^2_n given by

$$Q_n(f)(r, \omega) = \sum_{i=1}^{a_n} P_i^n(\omega) \int_{S^{N-1}} f(r, \eta) P_i^n(\eta) d\eta.$$

If $f = \sum_{i=1}^{a_n} f_i(r) P_i^n(\omega)$, then Lemma 4.1.7 gives

$$e^{tL} f = Q_n e^{etL} f = \sum_{i=1}^{a_n} P_i^n(\omega) e^{tL_n} f_i(r)$$

which we shorten to $e^{tL_n} f$, with a little abuse of notation. Then we can prove the announced decomposition of the semigroup generated by L.

Proposition 4.1.8 *For every $f \in L^2_\mu$ one has*

$$e^{tL} f = \sum_{n=0}^\infty Q_n e^{tL_n} Q_n f.$$

Proof. Let $f \in L^2_\mu$, using Lemmas 4.1.6, 4.1.7 we obtain

$$\begin{aligned} e^{tL} f &= e^{tL} \sum_{n=0}^\infty Q_n(f) = \sum_{n=0}^\infty e^{tL} Q_n(f) = \sum_{n=0}^\infty e^{tL} \sum_{i=1}^{a_n} \left[P_n^i(\omega) \int_{S^{N-1}} f(r, \eta) P_n^i(\eta) d\eta \right] = \\ &= \sum_{n=0}^\infty \sum_{i=1}^{a_n} P_n^i(\omega) e^{tL_n} \left[\int_{S^{N-1}} f(r, \eta) P_n^i(\eta) d\eta \right] = \sum_{n=0}^\infty Q_n e^{tL_n} Q_n f. \end{aligned}$$

□

4.2 Kernel estimates

In order to state and prove the main result of this Chapter we recall that the formal adjoint of L, with respect to the Lebesgue measure, is given by

$$L^* = \Delta + (a-1)\sum_{i,j=1}^{N} \frac{x_i x_j}{|x|^2} D_{ij} + c^* \frac{x}{|x|^2}\cdot\nabla - b^*|x|^{-2}$$

where $c^* = 2(N-1)(a-1) - c$ and $b^* = b + (N-2)(c-(N-1)(a-1))$. Let us compute the numbers s_1^*, s_2^*, D^* defined as in (1.7), (1.8) and relative to L^*. We have

$$D^* := \frac{b^*}{a} + \left(\frac{N-1+c^*-a}{2a}\right)^2 = D,$$

$$s_{1,2}^* := \frac{N-1+c^*-a}{2a} \mp \sqrt{D^*} = s_{1,2} + \frac{(a-1)(N-1)-c}{a} = N-2-s_{2,1}.$$

Recalling that $\gamma = \frac{N-1+c}{a} - N + 1$ we have also

$$s_1 = \frac{N}{2} - 1 - \sqrt{D} + \frac{\gamma}{2}, \quad s_1^* := \frac{N}{2} - 1 - \sqrt{D} - \frac{\gamma}{2}.$$

Since upper and lower bounds for $N=1$ have been already deduced in Proposition 1.3.15, we assume $N \geq 2$.
We write $f(x) \simeq g(x)$ if for some $C_1, C_2 > 0$, $C_1\, g(x) \leq f(x) \leq C_2\, g(x)$.

Before stating the main result, we recall that in Chapter 3 we proved that the semigroup is analytic in the right half plane and satisfies the following complex upper bounds.

Proposition 4.2.1 *(Corollary 3.5.5) Let $\Omega = \mathbb{R}^N\setminus\{0\}$. For every $\varepsilon > 0$, there exist $C_\varepsilon > 0$ and $m_\varepsilon > 0$ such that the heat kernel p of L, with respect to the measure $d\mu = |y|^\gamma dy$, satisfies for $z \in \mathbb{C}_+$ with $|\arg z| \leq \frac{\pi}{2} - \varepsilon$ and $(x,y) \in \Omega\times\Omega$*

$$|p(z,x,y)| \leq C_\varepsilon |z|^{-\frac{N}{2}} |x|^{-\frac{\gamma}{2}} |y|^{-\frac{\gamma}{2}} \left[\left(\frac{|x|}{|z|^{\frac12}} \wedge 1\right)\left(\frac{|y|}{|z|^{\frac12}} \wedge 1\right)\right]^{-\frac{N}{2}+1+\sqrt{D}} \exp\left(-\frac{|x-y|^2}{m_\varepsilon |z|}\right) \tag{4.4}$$

The main result of this chapter consists in showing that the above upper bound admits a lower bound for positive t.

Theorem 4.2.2 *Let $\Omega = \mathbb{R}^N \setminus \{0\}$. The heat kernel p of L, with respect to the measure $d\mu = |y|^\gamma dy$, satisfies for $(x,y) \in \Omega\times\Omega$*

$$p(t,x,y) \simeq t^{-\frac{N}{2}} |x|^{-\frac{\gamma}{2}} |y|^{-\frac{\gamma}{2}} \left[\left(\frac{|x|}{t^{\frac12}} \wedge 1\right)\left(\frac{|y|}{t^{\frac12}} \wedge 1\right)\right]^{-\frac{N}{2}+1+\sqrt{D}} \exp\left(-\frac{c|x-y|^2}{t}\right). \tag{4.5}$$

The constant $c > 0$ may differ in the upper and lower bounds.

Clearly the upper bound follows from Theorem 4.2.1

Remark 4.2.3 *Using Lemma 1.3.14 one can replace*

$$\left[\left(\frac{|x|}{t^{\frac{1}{2}}}\wedge 1\right)\left(\frac{|y|}{t^{\frac{1}{2}}}\wedge 1\right)\right]^{-\frac{N}{2}+1+\sqrt{D}} \quad \text{with} \quad \left(\frac{|x||y|}{t}\wedge 1\right)^{-\frac{N}{2}+1+\sqrt{D}}$$

in the above Theorem, slightly changing the constant c.

The previous kernel estimate can be rewritten in the following equivalent form, as in Corollary 3.5.6.

Corollary 4.2.4

$$p(t,x,y) \simeq t^{-\frac{N}{2}}|y|^{-\gamma}\left(\frac{|x|}{t^{\frac{1}{2}}}\wedge 1\right)^{-s_1}\left(\frac{|y|}{t^{\frac{1}{2}}}\wedge 1\right)^{-s_1^*}\exp\left(-\frac{c|x-y|^2}{t}\right)$$

Remark 4.2.5 *We remark that the estimate in* (4.5) *becomes*

$$p(t,x,y) \simeq Ct^{-1-\sqrt{D}}|x|^{-s_1}|y|^{-s_1}, \qquad \frac{|x|}{\sqrt{t}} \le 1, \frac{|y|}{\sqrt{t}} \le 1$$

and, using Corollary 4.2.4,

$$p(t,x,y) \simeq Ct^{-\frac{N}{2}}|y|^{-\gamma}\exp\left(-\frac{m|x-y|^2}{t}\right), \qquad \frac{|x|}{\sqrt{t}} \ge 1, \frac{|y|}{\sqrt{t}} \ge 1.$$

We need some further preparation for the proof of the lower bound.

Let $p_n(t,r,\rho)$ be the parabolic kernels of the Bessel operators L_n with respect to the measure $\rho^{\frac{N-1+c}{a}}\,d\rho$. Theorem 1.3.13 yields

$$p_n(t,r,\rho) = \frac{1}{2at}(r\rho)^{-\frac{N-1+c-a}{2a}} I_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right)\exp\left\{-\frac{r^2+\rho^2}{4at}\right\},$$

where $D_n = \frac{b+\lambda_n}{a} + \left(\frac{N-1+c-a}{2a}\right)^2$ and we write D for D_0.

In order to show that the heat kernel of L is the sum of the heat kernels of L_n we need the following lemma which will be also useful to prove lower bounds for small values of $|x|, |y|$. For this reason, we do not make any attempt to improve the bounds below for large r, ρ.

Lemma 4.2.6 *There exists $h \in C([0,\infty[)$, with $h(0) \neq 0$ such that*

$$\left|\sum_{n\ge1} p_n(t,r,\rho)\mathbb{Z}_\omega^{(n)}(\eta)\right| \le \sum_{n\ge1} p_n(t,r,\rho)\mathbb{Z}_\omega^{(n)}(\omega) \le Cp_0(t,r,\rho)\left(\frac{r\rho}{4at}\right)^{\sqrt{D_1}-\sqrt{D}} h\left(\frac{r\rho}{4at}\right).$$

In particular for every $t>0$ the series $\sum_{n\ge1} p_n(t,r,\rho)\mathbb{Z}_\omega^{(n)}(\eta)$ converges uniformly on compact sets of $\Omega\times\Omega$.

Proof. We use Proposition B.2.2 (ii) for the estimate $|Z_\omega^{(n)}(\eta)| \le Z_\omega^{(n)}(\omega) \le Cn^{N-2}$. Then

$$\left|\sum_{n\ge 1} p_n(t,r,\rho)\mathbb{Z}_\omega^{(n)}(\eta)\right| \le \frac{1}{2at}(r\rho)^{-\frac{N-1+c-a}{2a}} \exp\left\{-\frac{r^2+\rho^2}{4at}\right\}\sum_{n\ge 1} I_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right)\mathbb{Z}_\omega^{(n)}(\omega). \quad (4.6)$$

We use $\Gamma(\alpha+\beta) \ge C_\delta\Gamma(\alpha)\Gamma(\beta)$ if $\alpha,\beta\ge\delta$ to obtain $\Gamma(m+\sqrt{D}+1+\sqrt{D_n}-\sqrt{D}) \ge C\Gamma(m+1+\sqrt{D})\Gamma(\sqrt{D_n}-\sqrt{D})$ for every $n\ge 1$. Then

$$\sum_{n\ge 1} I_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right)\mathbb{Z}_\omega^{(n)}(\omega) = \sum_{n\ge 1}\mathbb{Z}_\omega^{(n)}(\omega)\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+\sqrt{D_n}+1)}\left(\frac{r\rho}{4at}\right)^{2m+\sqrt{D_n}-\sqrt{D}+\sqrt{D}}$$

$$= \sum_{n\ge 1}\mathbb{Z}_\omega^{(n)}(\omega)\left(\frac{r\rho}{4at}\right)^{\sqrt{D_n}-\sqrt{D}}\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+\sqrt{D_n}+1)}\left(\frac{r\rho}{4at}\right)^{2m+\sqrt{D}}$$

$$\le C\sum_{n\ge 1} n^{N-2}\left(\frac{r\rho}{4at}\right)^{\sqrt{D_n}-\sqrt{D}}\frac{1}{\Gamma(\sqrt{D_n}-\sqrt{D})}\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+1+\sqrt{D})}\left(\frac{r\rho}{4at}\right)^{2m+\sqrt{D}}$$

$$= CI_{\sqrt{D}}\left(\frac{r\rho}{2at}\right)\left(\frac{r\rho}{4at}\right)^{\sqrt{D_1}-\sqrt{D}}\sum_{n\ge 1} n^{N-2}\left(\frac{r\rho}{4at}\right)^{\sqrt{D_n}-\sqrt{D_1}}\frac{1}{\Gamma(\sqrt{D_n}-\sqrt{D})}.$$

Since $\sqrt{D_n}\approx cn$, $c=1/\sqrt{a}$ as $n\to\infty$, by the asymptotic of the Gamma function the series

$$h(s) = \sum_{n\ge 1} n^{N-2}s^{\sqrt{D_n}-\sqrt{D_1}}\frac{1}{\Gamma(\sqrt{D_n}-\sqrt{D})}$$

converges uniformly on compact sets of $[0,\infty[$ and does not vanish at 0. Then (4.6) yields

$$\left|\sum_{n\ge 1} p_n(t,r,\rho)\mathbb{Z}_\omega^{(n)}(\eta)\right| \le \frac{C}{2at}(r\rho)^{-\frac{N-1+c-a}{2a}}\exp\left\{-\frac{r^2+\rho^2}{4at}\right\} I_{\sqrt{D}}\left(\frac{r\rho}{2at}\right)\left(\frac{r\rho}{4at}\right)^{\sqrt{D_1}-\sqrt{D}} h\left(\frac{r\rho}{4at}\right)$$

$$= Cp_0(t,r,\rho)\left(\frac{r\rho}{4at}\right)^{\sqrt{D_1}-\sqrt{D}} h\left(\frac{r\rho}{4at}\right).$$

□

We can now obtain the announced kernel decomposition.

Proposition 4.2.7 *Let p be the heat kernel of L with respect to the measure $d\mu(y)=|y|^\gamma\,dx$. Then for $x=r\omega$, $y=\rho\eta$, $r,\ \rho>0$, $|\omega|=|\eta|=1$ we have*

$$p(t,x,y) = \frac{1}{2at}(r\rho)^{-\frac{N-1+c-a}{2a}}\exp\left\{-\frac{r^2+\rho^2}{4at}\right\}\sum_{n=0}^{\infty} I_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right)\mathbb{Z}_\omega^{(n)}(\eta)$$
$$= \sum_{n\ge 0} p_n(t,r,\rho)\mathbb{Z}_\omega^{(n)}(\eta).$$

Proof. We use Proposition 4.1.8. For $f \in L^2_\mu$

$$e^{tL}f = \sum_{n=0}^{\infty} Q_n e^{tL_n} Q_n f = \sum_{n=0}^{\infty}\sum_{i=1}^{a_n} P_i^n(\omega)\int_0^\infty \int_{S^{N-1}} p_n(t,r,\rho)P_i^n(\eta)f(\rho,\eta)\rho^{\frac{N-1+c}{a}}\,d\eta d\rho$$
$$= \sum_{n=0}^{\infty}\int_0^\infty \int_{S^{N-1}} p_n(t,r,\rho)\mathbb{Z}_\omega^{(n)}(\eta)f(\rho,\eta)\rho^{\frac{N-1+c}{a}}\,d\eta d\rho.$$

If f is continuous with compact support in $\mathbb{R}^N \setminus \{0\}$, since by Proposition 4.2.6 the series $\phi(t,x,y) = \sum_{n\geq 0} p_n(t,r,\rho)\mathbb{Z}_\omega^{(n)}(\eta)$ converges uniformly on compact sets, we interchange the series with the integrals thus obtaining from above

$$e^{tL}f(x) = \int_0^\infty \int_{S^{N-1}} \phi(t,r\omega,\rho\eta)f(\rho,\eta)\rho^{\frac{N-1+c}{a}}\,d\eta d\rho = \int_{\mathbb{R}^N}\phi(t,x,y)f(y)|y|^\gamma dy.$$

On the other hand

$$e^{tL}f(x) = \int_{\mathbb{R}^N} p(t,x,y)f(y)|y|^\gamma dy = \int_0^\infty \int_{S^{N-1}} p(t,r\eta,\rho\omega)f(\rho,\eta)\rho^{\frac{N-1+c}{a}}\,d\eta d\rho.$$

For any fixed $t>0$, $x \in \mathbb{R}^N \setminus \{0\}$ the L^1_{loc}-function $p(t,x,\cdot) - \phi(t,x,\cdot)$ has integral zero against any continuous and compactly supported function. Therefore it vanishes. □

We now start proving the lower estimate (4.5) near the origin, that is for $|x|/\sqrt{t}, |y|/\sqrt{t}$ small, see also Remark 4.2.5. 1. In this case the behaviour of p is the same as its radial part p_0.

Lemma 4.2.8 *There exists $\delta > 0$ such that if $\frac{|x|}{\sqrt{t}} \leq \delta$ and $\frac{|y|}{\sqrt{t}} \leq \delta$, then*

$$p(t,x,y) \geq Cp_0(t,r,\rho) \geq Ct^{-1-\sqrt{D}}|x|^{-s_1}|y|^{-s_1}.$$

Proof. By Proposition 4.2.7

$$p(t,x,y) = p_0(t,r,\rho)\frac{1}{|S^{N-1}|} + \sum_{n\geq 1} p_n(t,r,\rho)\mathbb{Z}_\omega^{(n)}(\eta).$$

Next we choose $\delta > 0$ such that if $\frac{|x|}{\sqrt{t}} \leq \delta$ and $\frac{|x|}{\sqrt{t}} \leq \delta$ then $C\left(\frac{r\rho}{4at}\right)^{\sqrt{D_1}-\sqrt{D}} h\left(\frac{r\rho}{4at}\right) \leq \frac{1}{2}\frac{1}{|S^{N-1}|}$ and use Proposition 4.2.6 to infer that

$$\left|\sum_{n\geq 1} p_n(t,r,\rho)\mathbb{Z}_\omega^{(n)}(\eta)\right| \leq \frac{1}{2}p_0(t,r,\rho)\frac{1}{|S^{N-1}|}.$$

The proof is now completed by the explicit expression of

$$p_0(t,r,\rho) = \frac{1}{2at}(r\rho)^{-\frac{N-1+c-a}{2a}} I_{\sqrt{D}}\left(\frac{r\rho}{2at}\right)\exp\left\{-\frac{r^2+\rho^2}{4at}\right\},$$

taking into account that the exponential term plays no role near the origin and using the behaviour of $I_{\sqrt{D}}$ near 0. □

Remark 4.2.9 *Observe that the above proof works if the product* $\frac{|xy|}{\sqrt{t}}$ *is less than* δ.

The lower bound in (4.5) for large values of $\frac{|x|}{\sqrt{t}}$ and $\frac{|y|}{\sqrt{t}}$ is in the next Proposition. Note that if p is the heat kernel of L with respect to the measure $d\mu = |y|^\gamma\,dy$, then the heat kernel with respect to the Lebesgue measure is $p(t,x,y)|y|^\gamma$. We prove, first, a preliminary lemma which shows a regularity property of the semigroup when applied to test functions.

Lemma 4.2.10 *Let* $\Omega = \mathbb{R}^N \setminus \{0\}$, $f \in C_c^\infty(\Omega)$ *and set* $u(t,x) := e^{tL}f(x)$. *Then* $u \in C^{1,2}(]0,\infty[\times\Omega) \cap C([0,\infty[\times\Omega)$ *and for every* $\delta > 0$ *there exists* $C_\delta > 0$ *s.t.* $|u(t,x)| \le C_\delta$ *for every* $t \ge 0$ *and* $|x| \ge \delta$.

Proof. By using (4.5) we immediately have the required boundedness of u for $t \ge 0$ and $|x| \ge \delta$. To prove the regularity properties we preliminarily observe that by Proposition 1.2.6 we have $C_c^\infty(\Omega) \hookrightarrow D(L^n) \hookrightarrow H^{2n}_{\mathrm{loc}}(\Omega)$ and so fixing a sufficiently large $n \in \mathbb{N}$ we get $D(L^n) \hookrightarrow C^2(\Omega)$. Let us consider now the semigroup in $D(L^n)$; we have obviously $u(t,\cdot) \in D(L^n) \subseteq C^2(\Omega)$ and since u is a solution of $\frac{d}{dt}u(t,\cdot) = Lu(t,\cdot)$ in $D(L^n)$, the embedding $D(L^n) \hookrightarrow C^2(\Omega)$ yields that the time derivative is a classical derivative and we have also $\frac{d}{dt}u(t,x) = Lu(t,x)$ pointwise. This proves $u \in C^{1,2}(]0,\infty[\times\Omega)$. Analogously $u(t,\cdot) \to f$ in $D(L^n)$ as $t \to 0$ and so pointwise and this implies $u \in C([0,\infty[\times\Omega)$. □

Proposition 4.2.11 *Let* $\delta > 0$ *be fixed. Then there exist positive constants* C, c *such that for* $\frac{|x|}{\sqrt{t}} \ge \delta$ *and* $\frac{|y|}{\sqrt{t}} \ge \delta$

$$p(t,x,y) \ge Ct^{-\frac{N}{2}}|y|^{-\gamma}\exp\left(-c\frac{|x-y|^2}{t}\right).$$

Proof. Given $\delta > 0$ let us set $\omega_0 := 0$ if $b \le 0$ and $\omega_0 := \frac{4b}{\delta^2}$ if $b \ge 0$. We consider the uniformly elliptic operator $L_0 := L + \frac{b}{|x|^2}$ in $C_b\left(\mathbb{R}^N \setminus B_{\frac{\delta}{2}}\right)$ with Dirichlet boundary conditions. The generated semigroup e^{tL_0} is represented by a kernel $q_0(t,x,y)$ satisfying

$$q_0(t,x,y) \ge C\left(1 \wedge \frac{|x|-\frac{\delta}{2}}{\sqrt{t}}\right)\left(1 \wedge \frac{|y|-\frac{\delta}{2}}{\sqrt{t}}\right)t^{-\frac{N}{2}}\exp\left(-c\frac{|x-y|^2}{t}\right) \tag{4.7}$$

for some positive constants C, c and $t > 0$, $|x|, |y| \ge \frac{\delta}{2}$, see [18, Theorem 3.8]. Given $0 \le f \in C_c^\infty\left(\mathbb{R}^n \setminus \bar{B}_{\frac{\delta}{2}}\right)$, let $u(t,\cdot) = e^{\omega_0 t}e^{tL}f(\cdot)$ and $v(t,\cdot) = e^{tL_0}f(\cdot)$. Then both u and v are positive and satisfy

$$\begin{cases} \frac{d}{dt}u(t,x) = (L_0 - \frac{b}{|x|^2} + \omega_0)u(t,x) & t > 0,\ x \in \Omega, \\ u(0,x) = f(x) & x \in \Omega, \end{cases}$$

and

$$\begin{cases} \frac{d}{dt}v(t,x) = L_0 v(t,x) & t>0,\ |x|>\frac{\delta}{2}, \\ v(0,x)=f(x) & |x|\ge \frac{\delta}{2}, \\ v(t,x)=0 & |x|=\frac{\delta}{2}, \end{cases}$$

respectively. Both u, v are bounded classical solution, continuous up to $t=0$, see Lemma 4.2.10 for u.

Now we observe that $(\partial_t - L_0)(u-v) = (\omega_0 - \frac{b}{|x|^2})u \ge 0$ in $]0,1]\times\mathbb{R}^N\setminus\bar{B}_{\frac{\delta}{2}}$, $u(0,x)-v(0,x)=0$ for $|x|\ge\frac{\delta}{2}$ and $u(t,x)-v(t,x)=u(t,x)\ge 0$ for $0\le t\le 1$, $|x|=\frac{\delta}{2}$. By the maximum principle $u(t,x)\ge v(t,x)$ for $0\le t\le 1$, $|x|\ge\frac{\delta}{2}$, that is

$$\int_{\mathbb{R}^N\setminus B_\delta} e^{\omega_0 t} p(t,x,y)|y|^\gamma f(y)dy \ge \int_{\mathbb{R}^N\setminus B_\delta} q_0(t,x,y)f(y)dy.$$

By the arbitrariness of $f\in C_c^\infty\left(\mathbb{R}^n\setminus\bar{B}_{\frac{\delta}{2}}\right)$ and using (4.7), we get for $t=1$, $|x|$, $|y|\ge\delta$

$$p(1,x,y)|y|^\gamma \ge q_0(1,x,y)e^{-\omega_0} \ge C\exp\left(-c|x-y|^2\right).$$

The scaling equality (1.16) now gives immediately the statement. $\square$

Finally, we can prove the lower bound in Theorem 4.2.2

(Proof of Theorem 4.2.2) By the scaling property (1.16) we may assume that $t=1$ and prove that

$$p(1,x,y) \ge C(|x||y|)^{-\frac{\gamma}{2}}\left((|x|\wedge 1)(|y|\wedge 1)\right)^{-\frac{N}{2}+1+\sqrt{D}}\exp\left\{-c|x-y|^2\right\}. \tag{4.8}$$

We use Proposition 4.2.7 to write

$$p(1,x,y) = \frac{1}{2a}(r\rho)^{-\frac{N-1+c-a}{2a}}\exp\left\{-\frac{r^2+\rho^2}{4a}\right\}\sum_{n=0}^{\infty} I_{\sqrt{D_n}}\left(\frac{r\rho}{2a}\right)\mathbb{Z}_\omega^{(n)}(\eta). \tag{4.9}$$

Since by Lemma 4.2.8 and Proposition 4.2.11 inequality (4.8) holds if $|x|\le\delta$, $|y|\le\delta$ or $|x|\ge\delta$, $|y|\ge\delta$, then it holds whenever $|x|=|y|$. Let therefore $x=r\omega$, $y=r\eta$. Then we obtain

$$\frac{1}{2a}r^{-\frac{N-1+c-a}{a}}\exp\left\{-\frac{r^2}{2a}\right\}\sum_{n=0}^{\infty} I_{\sqrt{D_n}}\left(\frac{r^2}{2a}\right)\mathbb{Z}_\omega^{(n)}(\eta) \ge Cr^{-\gamma}\left(r\wedge 1\right)^{-N+2+2\sqrt{D}}e^{-cr^2|\omega-\eta|^2}.$$

Since $(N-1+c-a)/a-\gamma = N-2$ we obtain

$$\sum_{n=0}^{\infty} I_{\sqrt{D_n}}\left(\frac{r^2}{2a}\right)\mathbb{Z}_\omega^{(n)}(\eta) \ge Cr^{N-2}\left(r\wedge 1\right)^{-N+2+2\sqrt{D}}\exp\left\{\frac{r^2}{2a}\right\}\exp\left\{-cr^2|\omega-\eta|^2\right\}$$

and, changing r^2 to $r\rho$,

$$\sum_{n\ge 0} I_{\sqrt{D_n}}\left(\frac{r\rho}{2a}\right)\mathbb{Z}_\omega^{(n)}(\eta) \ge C(r\rho)^{\frac{N-2}{2}}\left((r\rho)^{\frac{1}{2}}\wedge 1\right)^{-N+2+2\sqrt{D}}\exp\left\{\frac{r\rho}{2a}\right\}\exp\left\{-cr\rho|\omega-\eta|^2\right\}.$$

By putting the last estimate in (4.9) we deduce

$$
\begin{aligned}
p(1,x,y) \geq\ & C\,(r\rho)^{\frac{N-2}{2}-\frac{N-1+c-a}{2a}}\left((r\rho)^{\frac{1}{2}}\wedge 1\right)^{-N+2+2\sqrt{D}} \\
& \times \exp\left\{-\frac{r^2+\rho^2}{4a}\right\}\exp\left\{\frac{r\rho}{2a}\right\}\exp\left\{-cr\rho|\omega-\eta|^2\right\} \\
=\ & C\,(r\rho)^{\frac{N-2}{2}-\frac{N-1+c-a}{2a}}\left((r\rho)^{\frac{1}{2}}\wedge 1\right)^{-N+2+2\sqrt{D}}\exp\left\{-\frac{(r-\rho)^2}{4a}\right\}\exp\left\{-cr\rho|\omega-\eta|^2\right\} \\
\geq\ & C\,(r\rho)^{-\frac{\gamma}{2}}\left((r\rho)^{\frac{1}{2}}\wedge 1\right)^{-N+2+2\sqrt{D}}\exp\left\{-m|x-y|^2\right\},
\end{aligned}
$$

with $m \geq c, \frac{1}{4a}$. To complete the proof of (4.8) it suffices now to apply Lemma 1.3.14. $\square$

Chapter 5

Green function estimates

In this chapter we prove sharp upper and lower bounds for the Green function G_λ defined by

$$G_\lambda(x,y) := \int_0^\infty e^{-\lambda t} p(t,x,y)\, dt, \quad x,y \in \mathbb{R}^N \setminus \{0\}\,.$$

For $\lambda > 0$ the integral converges pointwise, due to Theorem 4.2.2, and defines the resolvent of $\lambda - L$ (the kernel being written with respect to the measure $|y|^\gamma\, dy$). However we consider also the case $\lambda = 0$ when the integral converges, that is when $D > 0$.

All the results presented in this Chapter are collected in [50].

For $l \geq 1$, $m \geq 1$, $\alpha \geq 0$, $\beta \in \mathbb{R}$ let

$$F(t) := t^{-l} \left(\frac{\alpha}{t} \wedge 1\right)^{-l+m} \exp\left(-\frac{\beta^2}{t}\right).$$

We write also $F(l,m,\alpha,\beta,t) := F(t)$ in order to emphasize the explicit dependence on the parameters. With this notation the estimates in Theorem 4.2.2 (see also Remark 4.2.3) take the form

$$p(t,x,y) \simeq (|x||y|)^{-\frac{\gamma}{2}} F\left(\frac{N}{2}, 1+\sqrt{D}, |x||y|, c|x-y|, t\right),$$

with the understanding that the constant c may differ in the upper and lower bounds. Defining

$$I(\alpha,\beta) := \int_0^\infty e^{-\lambda t} F(t) dt = \int_0^\alpha e^{-\lambda t} t^{-l} e^{-\frac{\beta^2}{t}} dt + \alpha^{-l+m} \int_\alpha^\infty e^{-\lambda t} t^{-m} e^{-\frac{\beta^2}{t}} dt \qquad (5.1)$$

we have for $l = \frac{N}{2}$ and $m = 1+\sqrt{D}$

$$G_\lambda(x,y) \simeq (|x||y|)^{-\frac{\gamma}{2}} I\left(|x||y|, c|x-y|\right). \qquad (5.2)$$

We treat separately the cases $\lambda = 0$ and $\lambda > 0$ for clarity and also because of technical details.

5.1 The Green function G_0

Our main result is the following.

Theorem 5.1.1 *Let $\Omega = \mathbb{R}^N \setminus \{0\}$ and let us suppose that $D > 0$. For $(x, y) \in \Omega \times \Omega$ with $x \neq y$, the Green function G_0 of L, with respect to the measure $d\mu = |y|^\gamma dy$, satisfies the estimates*
if $N > 2$

$$(|x||y|)^{\frac{\gamma}{2}} G_0(x, y) \simeq |x-y|^{2-N} \left(1 \wedge \frac{|x||y|}{|x-y|^2}\right)^{\sqrt{D}-\frac{N-2}{2}} \tag{5.3}$$

and if $N = 2$

$$(|x||y|)^{\frac{\gamma}{2}} G_0(x, y) \simeq \begin{cases} \dfrac{(|x||y|)^{\sqrt{D}}}{|x-y|^{2\sqrt{D}}}, & \text{if } \frac{|x-y|^2}{|x||y|} \geq 1; \\[2ex] 1 - \log\left(\dfrac{|x-y|^2}{|x||y|}\right), & \text{if } \frac{|x-y|^2}{|x||y|} \leq 1. \end{cases}$$

Remark 5.1.2 *We remark that* (5.3) *becomes*

$$(|x||y|)^{\frac{\gamma}{2}} G_0(x, y) \simeq |x-y|^{2-N}, \qquad \frac{|x-y|^2}{|x||y|} \leq 1$$

and

$$(|x||y|)^{\frac{\gamma}{2}} G_0(x, y) \simeq \frac{(|x||y|)^{\sqrt{D}-\frac{N-2}{2}}}{|x-y|^{2\sqrt{D}}}, \qquad \frac{|x-y|^2}{|x||y|} \geq 1.$$

The asymptotic behaviour of G_0 depends on the sign of $\sqrt{D} - (N-2)/2$, see Remark 5.1.4 below.

The proof is an immediate consequence of (5.2) and the lemma below, recalling that $\alpha = |x||y|$ and $\beta = |x - y|$. We use the incomplete Gamma functions defined by

$$\Gamma(a, r) := \int_r^\infty e^{-t} t^{a-1}\, dt, \qquad \gamma(a, r) := \int_0^r e^{-t} t^{a-1}\, dt.$$

Clearly $\Gamma(a, r) + \gamma(a, r) = \Gamma(a)$, moreover $\Gamma(a, r) \simeq r^{a-1} e^{-r}$ as $r \to \infty$ and $\Gamma(0, r) \simeq -\log r$, $\gamma(a, r) \approx \frac{r^a}{a}$ as $r \to 0$. In particular,

$$\frac{\Gamma(a, r)}{\Gamma(b, r)} \simeq r^{a-b}, \quad \text{for } r \geq 1, a, b \geq 0 \quad \frac{\gamma(a, r)}{\gamma(b, r)} \simeq r^{a-b}, \quad \text{for } r \leq 1, a, b > 0. \tag{5.4}$$

Lemma 5.1.3 *Let $l \geq 1$, $m > 1$, $\alpha, \beta > 0$ and*

$$F(t) = t^{-l} \left(\frac{\alpha}{t} \wedge 1\right)^{-l+m} \exp\left(-\frac{\beta^2}{t}\right).$$

Then if $l > 1$

$$\int_0^\infty F(t)\,dt \simeq \beta^{2-2l}\left(1 \wedge \frac{\alpha}{\beta^2}\right)^{m-l}$$

and if $l = 1$

$$\int_0^\infty F(t)\,dt \simeq \begin{cases} \left(\frac{\beta^2}{\alpha}\right)^{1-m}, & \textit{if } \frac{\beta^2}{\alpha} \geq 1; \\ 1 - \log\left(\frac{\beta^2}{\alpha}\right), & \textit{if } \frac{\beta^2}{\alpha} \leq 1. \end{cases} \tag{5.5}$$

Proof. By the change of variables $s = \frac{\beta^2}{t}$ we have

$$\begin{aligned}\int_0^\infty F(t)\,dt &= \beta^{-2l+2}\int_{\frac{\beta^2}{\alpha}}^\infty s^{l-2}\exp(-s)\;ds + \alpha^{-(l-m)}\beta^{-2m+2}\int_0^{\frac{\beta^2}{\alpha}} s^{m-2}\exp(-s)\;ds \\ &= \beta^{-2l+2}\,\Gamma\left(l-1,\frac{\beta^2}{\alpha}\right) + \alpha^{-l+m}\beta^{-2m+2}\,\gamma\left(m-1,\frac{\beta^2}{\alpha}\right).\end{aligned}$$

If $\frac{\beta^2}{\alpha} \geq 1$, (5.4) yields

$$\Gamma\left(l-1,\frac{\beta^2}{\alpha}\right) \backsimeq \left(\frac{\beta^2}{\alpha}\right)^{l-m}\Gamma\left(m-1,\frac{\beta^2}{\alpha}\right).$$

Then

$$\begin{aligned}\int_0^\infty F(t)\,dt &\simeq \alpha^{-l+m}\beta^{-2m+2}\,\gamma\left(m-1,\frac{\beta^2}{\alpha}\right) + \beta^{-2l+2}\left(\frac{\beta^2}{\alpha}\right)^{l-m}\Gamma\left(m-1,\frac{\beta^2}{\alpha}\right) \\ &= \alpha^{-l+m}\beta^{-2m+2}\left[\gamma\left(m-1,\frac{\beta^2}{\alpha}\right) + \Gamma\left(m-1,\frac{\beta^2}{\alpha}\right)\right] \\ &= \alpha^{-l+m}\beta^{-2m+2}\,\Gamma\,(m-1)\,.\end{aligned}$$

The case $\frac{\beta^2}{\alpha} \leq 1$ and $l > 1$ is similar. Using

$$\gamma\left(m-1,\frac{\beta^2}{\alpha}\right) \simeq \left(\frac{\beta^2}{\alpha}\right)^{m-l}\gamma\left(l-1,\frac{\beta^2}{\alpha}\right)$$

we get

$$\begin{aligned}\int_0^\infty F(t)\,dt &\backsimeq \alpha^{-l+m}\beta^{-2m+2}\left(\frac{\beta^2}{\alpha}\right)^{m-l}\gamma\left(l-1,\frac{\beta^2}{\alpha}\right) + \beta^{-2l+2}\Gamma\left(l-1,\frac{\beta^2}{\alpha}\right) \\ &= \beta^{-2l+2}\left[\gamma\left(l-1,\frac{\beta^2}{\alpha}\right) + \Gamma\left(l-1,\frac{\beta^2}{\alpha}\right)\right] = \beta^{-2l+2}\,\Gamma\,(l-1)\,.\end{aligned}$$

Finally, if $\frac{\beta^2}{\alpha} \leq 1$ and $l = 1$, then

$$\gamma(m-1,\frac{\beta^2}{\alpha}) \simeq \left(\frac{\beta^2}{\alpha}\right)^{m-1}, \quad \Gamma(0,\frac{\beta^2}{\alpha}) \simeq -\log\left(\frac{\beta^2}{\alpha}\right)$$

and

$$\int_0^\infty F(t)\,dt \simeq c_1\alpha^{-1+m}\beta^{-2m+2}\left(\frac{\beta^2}{\alpha}\right)^{m-1} - c_2\log\left(\frac{\beta^2}{\alpha}\right) \simeq 1 - \log\left(\frac{\beta^2}{\alpha}\right).$$

□

Remark 5.1.4 *Observe that when $m \geq l > 1$, then $\int_0^\infty F(t)\,dt$ is bounded from above by β^{2-2l} and tends to 0 as α^{m-l} for $\alpha \to 0$. When $m < l$, instead, it blows up as $\alpha \to 0$ like α^{m-l} and is bounded if $\alpha, \beta \geq \delta > 0$. If $l = 1$, the integral behaves as $\log \alpha$ for $\alpha \to \infty$.*

5.2 The Green function G_λ, $\lambda > 0$

Let us consider now $\lambda > 0$. Using again (5.2) we look for estimates of $I(\alpha, \beta)$.

We observe that if $M_{\sqrt{\lambda}}$ is the dilation defined by $M_{\sqrt{\lambda}}u(x) = u(\sqrt{\lambda}\,x)$, the scaling property $M_{\sqrt{\lambda}}LM_{\sqrt{\lambda}^{-1}} = \frac{L}{\lambda}$ implies that

$$\begin{aligned}
\left[(\lambda - L)^{-1}f\right](x) &= \frac{1}{\lambda}\left[M_{\sqrt{\lambda}}(I-L)^{-1}M_{\sqrt{\lambda}^{-1}}f\right](x) \\
&= \frac{1}{\lambda}\int_{\mathbb{R}^N} G_1(\sqrt{\lambda}\,x, w) f\left(\frac{w}{\sqrt{\lambda}}\right)|w|^\gamma\,dw \\
&= \lambda^{\frac{\gamma+N-2}{2}}\int_{\mathbb{R}^N} G_1(\sqrt{\lambda}\,x, \sqrt{\lambda}\,y) f(y)\,|y|^\gamma\,dy.
\end{aligned}$$

This proves that

$$G_\lambda(x,y) = \lambda^{\frac{\gamma+N-2}{2}} G_1(\sqrt{\lambda}\,x, \sqrt{\lambda}\,y) \tag{5.6}$$

and allows us to treat only the case $\lambda = 1$ by estimating the integral

$$I(\alpha,\beta) = \int_0^\alpha e^{-t}t^{-l}e^{-\frac{\beta^2}{t}}\,dt + \alpha^{-l+m}\int_\alpha^\infty e^{-t}t^{-m}e^{-\frac{\beta^2}{t}}\,dt.$$

For $l, m \geq 1$ let

$$h(t) := e^{-t}t^{-l}e^{-\frac{\beta^2}{t}}, \quad g(t) := t^{l-m}h(t)$$

and

$$H(\alpha,\beta) := \int_0^\alpha h(t)\,dt, \quad G(\alpha,\beta) := \int_0^\alpha g(t)\,dt.$$

We have the following identities, see [28], formula (29), pag. 146,

$$\begin{aligned}
H(\beta) &:= \int_0^\infty h(t)\,dt = 2\beta^{1-l}K_{l-1}(2\,\beta), \\
G(\beta) &:= \int_0^\infty g(t)\,dt = 2\beta^{1-m}K_{m-1}(2\,\beta)
\end{aligned} \tag{5.7}$$

where the K_ν are the modified Bessel functions. With this notation the integral in (5.1) takes the form

$$I(\alpha,\beta) := H(\alpha,\beta) + \alpha^{m-l}\left(G(\beta) - G(\alpha,\beta)\right).$$

Let us observe that $I(\alpha,\beta)$ is decreasing with respect to β and, considered as a function of α, is increasing when $m - l \geq 0$ and decreasing otherwise, since

$$\frac{\partial}{\partial\alpha}I(\alpha,\beta) = (m-l)\,\alpha^{m-l-1}\left(G(\beta) - G(\alpha,\beta)\right).$$

We split the proof between the cases $0 \le D \le (N-2)^2/4$ and $D > (N-2)^2/4$ and note that for Schrödinger operators the above conditions correspond to $b \le 0$, $b > 0$, respectively.

The case $0 \le D \le \left(\frac{N-2}{2}\right)^2$

Since $l = N/2$ and $m = 1 + \sqrt{D}$, this corresponds to $m \le l$.

Theorem 5.2.1 *Let $\lambda > 0$, $\Omega = \mathbb{R}^N \setminus \{0\}$ and let us suppose $D \ge 0$. For $(x,y) \in \Omega \times \Omega$ with $x \neq y$, the Green function G_λ, with respect to the measure $d\mu = |y|^\gamma dy$, satisfies the estimates*

(i) if $N > 2$ and $D > 0$

$$(|x||y|)^{\frac{\gamma}{2}}\, G_\lambda(x,y) \simeq e^{-c\sqrt{\lambda}|x-y|}\, |x-y|^{2-N} \left(1 \wedge \frac{|x||y|}{|x-y|^2}\right)^{\sqrt{D}-\frac{N-2}{2}}.$$

(ii) If $N = 2$ and $D = 0$

$$(|x||y|)^{\frac{\gamma}{2}}\, G_\lambda(x,y) \simeq \begin{cases} e^{-c\sqrt{\lambda}\,|x-y|} & , \quad \text{if} \quad \sqrt{\lambda}\,|x-y| \ge 1; \\ 1 - \log\left(\sqrt{\lambda}\,|x-y|\right) & , \quad \text{if} \quad \sqrt{\lambda}\,|x-y| < 1. \end{cases}$$

(iii) If $N > 2$ and $D = 0$:

For $\sqrt{\lambda}|x-y| \ge 1$

$$(|x||y|)^{\frac{\gamma}{2}}\, G_\lambda(x,y) \simeq e^{-c\sqrt{\lambda}|x-y|}\left((|x||y|) \wedge \frac{|x-y|}{\sqrt{\lambda}}\right)^{\frac{2-N}{2}}.$$

For $\sqrt{\lambda}|x-y| < 1$

$$(|x||y|)^{\frac{\gamma}{2}}\, G_\lambda(x,y) \simeq |x-y|^{2-N} \vee \left((|x||y|)^{\frac{2-N}{2}}(1 - \log(\sqrt{\lambda}\,|x-y|)\right).$$

All the constants appearing in the above estimates, including those hidden in the symbol $\simeq$, do not depend on λ; the generic constants c in the exponentials may differ in the upper and lower bounds.

The logarithmic term is due either to the dimension $N = 2$ or to the degeneracy of the discriminant, $D = 0$. Note that when $N > 2$ but $D = 0$ the estimates are influenced both by the terms $|x-y|^{2-N}$ and $(|x||y|)^{\frac{2-N}{2}}(1 - \log(\sqrt{\lambda}\,|x-y|)$.

The proof of the theorem follows immediately from the scaling property (5.6) and from the following lemma, by noticing that powers of $|x-y|$ can be absorbed into the exponential, when $|x-y|$ is large.

Lemma 5.2.2 *If $m \le l$ we have*

$$\max\left\{H(\beta), \alpha^{m-l}G(\beta)\right\} \le I(\alpha,\beta) \le 2\max\left\{H(\beta), \alpha^{m-l}G(\beta)\right\} \tag{5.8}$$

and therefore

(i) if $1 < m \le l$ we have

$$I(\alpha,\beta) \simeq \begin{cases} e^{-2\beta}\,\alpha^{m-l}\beta^{\frac{1}{2}-m}\left(1\wedge\frac{\beta}{\alpha}\right)^{m-l} & , \quad \textit{if} \quad \beta \ge 1; \\ \beta^{2-2l}\left(1\wedge\frac{\alpha}{\beta^2}\right)^{m-l} & , \quad \textit{if} \quad \beta < 1. \end{cases}$$

(ii) If $m = l = 1$

$$I(\alpha,\beta) \simeq 2K_0\,(2\beta) \simeq \begin{cases} \beta^{-\frac{1}{2}}e^{-2\beta} & , \quad \textit{if} \quad \beta \ge 1; \\ 1-\log\beta & , \quad \textit{if} \quad \beta < 1. \end{cases}$$

(iii) If $1 = m < l$ we have

$$I\,(\alpha,\beta) \simeq \begin{cases} e^{-2\beta}\,\beta^{-\frac{1}{2}}\,(\alpha\wedge\beta)^{1-l} & , \quad \textit{if} \quad \beta \ge 1; \\ \beta^{2-2l}\vee\alpha^{1-l}(1-\log\beta) & , \quad \textit{if} \quad \beta < 1. \end{cases}$$

Proof. We have

$$\begin{aligned} I(\alpha,\beta) &= \int_0^\alpha h(t)dt + \alpha^{m-l}\int_\alpha^\infty g(t)dt \le \int_0^\infty h(t)dt + \alpha^{m-l}\int_0^\infty g(t)dt \\ &= H(\beta) + \alpha^{m-l}G(\beta) \le 2\max\left\{H(\beta), \alpha^{m-l}G(\beta)\right\}. \end{aligned}$$

On the other hand

$$I(\alpha,\beta) = \int_0^\alpha h(t)dt + \alpha^{m-l}\int_\alpha^\infty t^{l-m}h(t)dt \ge \int_0^\alpha h(t)dt + \int_\alpha^\infty h(t)dt = H(\beta)$$

and

$$\begin{aligned} I(\alpha,\beta) &= \int_0^\alpha t^{m-l}g(t)dt + \alpha^{m-l}\int_\alpha^\infty g(t)dt \\ &\ge \alpha^{m-l}\left[\int_0^\alpha g(t)dt + \int_\alpha^\infty g(t)dt\right] = \alpha^{m-l}G(\beta). \end{aligned}$$

It follows that $I(\alpha,\beta) \ge \max\left\{H(\beta), \alpha^{m-l}G(\beta)\right\}$ and this proves (5.8). It follows from (5.7) that

$$\max\left\{H(\beta), \alpha^{m-l}G(\beta)\right\} = \max\left\{2\beta^{1-l}K_{l-1}\,(2\beta)\,,\ \alpha^{m-l}\,2\beta^{1-m}K_{m-1}\,(2\beta)\right\}.$$

If $m = l$ then $I(\alpha, \beta) = 2\beta^{1-l} K_{l-1}(2\beta)$ and Lemma 1.3.5 gives the result. The same Lemma applies in the other cases and we get for $\beta \geq 1$

$$I(\alpha, \beta) \simeq e^{-2\beta} \beta^{-\frac{1}{2}} \max\left\{\beta^{1-l}, \alpha^{m-l}\beta^{1-m}\right\} = e^{-2\beta}\, \beta^{\frac{1}{2}-m} (\alpha \wedge \beta)^{m-l}.$$

For $\beta \leq 1$ we have, if $m > 1$,

$$I(\alpha, \beta) \simeq \max\left\{\beta^{2-2l}, \alpha^{m-l}\beta^{2-2m}\right\} = \beta^{2-2l}\left(1 \wedge \frac{\alpha}{\beta^2}\right)^{m-l}$$

and if $1 = m$

$$I(\alpha, \beta) \simeq \max\left\{\beta^{2-2l}, \alpha^{1-l}(1 - \log \beta)\right\}.$$

$\square$

The case $D > \left(\frac{N-2}{2}\right)^2$

Since $l = N/2$ and $m = 1 + \sqrt{D}$, this corresponds to $m > l$.

Theorem 5.2.3 *Let $\lambda > 0$, $\Omega = \mathbb{R}^N \setminus \{0\}$ and let us suppose $D > \left(\frac{N-2}{2}\right)^2$. For $(x, y) \in \Omega \times \Omega$ with $x \neq y$, the Green function G_λ, with respect to the measure $d\mu = |y|^\gamma dy$, satisfies the estimates*

(i) if $N > 2$

$$(|x||y|)^{\frac{\gamma}{2}} G_\lambda(x, y) \simeq e^{-c\sqrt{\lambda}|x-y|} |x-y|^{2-N} \left(1 \wedge \frac{|x||y|}{|x-y|^2}\right)^{\sqrt{D}-\frac{N-2}{2}}.$$

(ii) If $N = 2$:

For $\sqrt{\lambda}\,|x-y| \geq 1$

$$(|x||y|)^{\frac{\gamma}{2}} G_\lambda(x, y) \simeq e^{-c\sqrt{\lambda}|x-y|} \left(1 \wedge \lambda\,|x||y|\right)^{\sqrt{D}}.$$

For $\sqrt{\lambda}\,|x-y| < 1 \leq \lambda\,|x||y|$

$$(|x||y|)^{\frac{\gamma}{2}} G_\lambda(x, y) \simeq 1 - \log\left(\sqrt{\lambda}\,|x-y|\right).$$

For $\lambda\,|x||y|,\ \sqrt{\lambda}\,|x-y| < 1,\ \frac{|x-y|^2}{|x||y|} \geq 1$

$$(|x||y|)^{\frac{\gamma}{2}} G_\lambda(x, y) \simeq \left(\frac{|x-y|^2}{|x||y|}\right)^{-\sqrt{D}}.$$

For $\lambda\,|x||y|,\ \sqrt{\lambda}\,|x-y| < 1,\ \frac{|x-y|^2}{|x||y|} \leq 1$

$$(|x||y|)^{\frac{\gamma}{2}} G_\lambda(x, y) \simeq 1 - \log\left(\frac{|x-y|^2}{|x||y|}\right).$$

All the constants appearing in the above estimates, including those hidden in the symbol $\simeq$, do not depend on λ; the generic constants c in the exponentials may differ in the upper and lower bounds.

As before, the proof follows from some elementary but tedious lemmas on the integrals $I(\alpha,\beta)$.

Lemma 5.2.4 *If $m > l$ we have*

$$I(\alpha,\beta) \le \min\left\{H(\beta), \alpha^{m-l}G(\beta)\right\}.$$

In particular

(i) if $l > 1$

$$I(\alpha,\beta) \le C\, e^{-\beta}\beta^{2-2l}\left(1 \wedge \frac{\alpha}{\beta^2}\right)^{m-l}.$$

(ii) If $l = 1$

$$I(\alpha,\beta) \le C \begin{cases} e^{-2\beta}\,\alpha^{m-1}\beta^{\frac{1}{2}-m}\left(1\wedge\frac{\beta}{\alpha}\right)^{m-1} & , \quad \textit{if} \quad \beta \ge 1; \\ \min\left\{\left(\frac{\beta^2}{\alpha}\right)^{1-m}, 1-\log\beta\right\} & , \quad \textit{if} \quad \beta < 1. \end{cases}$$

Proof. We have

$$I(\alpha,\beta) = \int_0^\alpha h(t)dt + \alpha^{m-l}\int_\alpha^\infty g(t)dt = \int_0^\alpha t^{m-l}g(t)dt + \alpha^{m-l}\int_\alpha^\infty g(t)dt \le \alpha^{m-l}G(\beta)$$

and

$$I(\alpha,\beta) = \int_0^\alpha h(t)dt + \alpha^{m-l}\int_\alpha^\infty t^{l-m}h(t)dt \le H(\beta).$$

It follows that

$$\begin{aligned} I(\alpha,\beta) &\le \min\left\{H(\beta), \alpha^{m-l}G(\beta)\right\} \\ &= \min\left\{2\beta^{1-l}K_{l-1}(2\beta)\,,\ \alpha^{m-l}\,2\beta^{1-m}K_{m-1}(2\beta)\right\}. \end{aligned}$$

Using Lemma 1.3.5 we have for $\beta \ge 1$

$$\begin{aligned} I(\alpha,\beta) &\le Ce^{-2\beta}\beta^{-\frac{1}{2}}\min\left\{\beta^{1-l}, \alpha^{m-l}\beta^{1-m}\right\} \\ &= Ce^{-2\beta}\,\alpha^{m-l}\beta^{\frac{1}{2}-m}\left(1\wedge\frac{\beta}{\alpha}\right)^{m-l} \le C_1\, e^{-\beta}\alpha^{m-l}\beta^{2-2m}\left(1\wedge\frac{\beta^2}{\alpha}\right)^{m-l}. \end{aligned}$$

For $\beta \le 1$, by Lemma 1.3.5 again we have, if $l > 1$,

$$I(\alpha,\beta) \le C\min\left\{\beta^{2-2l}, \alpha^{m-l}\beta^{2-2m}\right\} = C\alpha^{m-l}\beta^{2-2m}\left(1\wedge\frac{\beta^2}{\alpha}\right)^{m-l}$$

and if $l = 1$

$$I(\alpha,\beta) \le C\min\left\{1-\log\beta,\ \beta^{2-2m}\alpha^{m-1}\right\} = C\min\left\{1-\log\beta,\ \left(\frac{\beta^2}{\alpha}\right)^{1-m}\right\}.$$

$\square$

The lower bound for $I(\alpha,\beta)$, in the case $l>1$, is proved in the following Lemma.

Lemma 5.2.5 *If $m>l>1$ we have for some constant $C=C(\lambda,m,l),\ c>0$*

$$I(\alpha,\beta) \geq C\, e^{-c\beta}\beta^{2-2l}\left(1\wedge\frac{\alpha}{\beta^2}\right)^{m-l}.$$

PROOF.

1. Case $\alpha,\ \beta\leq 1$.
   Assume first $\beta^2\leq 2\alpha$. Then

$$\begin{aligned} I(\alpha,\beta) &\geq \int_0^\alpha e^{-t}t^{-l}e^{-\frac{\beta^2}{t}}dt \geq e^{-\alpha}\beta^{2-2l}\int_0^{\frac{\alpha}{\beta^2}} s^{-l}e^{-\frac{1}{s}}ds \\ &\geq e^{-\alpha}\beta^{2-2l}\int_0^{\frac{1}{2}} s^{-l}e^{-\frac{1}{s}}ds \geq C\beta^{2-2l}. \end{aligned}$$

   Assume now $\beta^2\geq 2\alpha$. Then

$$\begin{aligned} I(\alpha,\beta) &\geq \int_{\frac{\beta^2}{2}}^{\beta^2} e^{-t}\alpha^{m-l}t^{-m}e^{-\frac{\beta^2}{t}}dt \geq e^{-\beta^2}\alpha^{m-l}\int_{\frac{\beta^2}{2}}^{\beta^2} t^{-m}e^{-\frac{\beta^2}{t}}dt \\ &\geq e^{-\beta^2}\alpha^{m-l}e^{-2}\int_{\frac{\beta^2}{2}}^{\beta^2} t^{-m}dt \geq C\alpha^{m-l}\beta^{2-2m}. \end{aligned}$$

2. Case $\alpha\leq 1,\ \beta\geq 1$.
   Since $I(\alpha,\beta)$ is decreasing in β we may assume that $\frac{\beta}{2}\geq 1\geq\alpha$. Then

$$\begin{aligned} I(\alpha,\beta) &\geq \int_\alpha^\beta e^{-t}\alpha^{m-l}t^{-m}e^{-\frac{\beta^2}{t}}dt \geq e^{-\beta}\alpha^{m-l}\int_{\frac{\beta}{2}}^{\beta} t^{-m}e^{-\frac{\beta^2}{t}}dt \geq \alpha^{m-l}e^{-\beta}\int_{\frac{\beta}{2}}^{\beta} t^{-m}dt \\ &\geq Ce^{-2\beta}\alpha^{m-l}. \end{aligned}$$

3. Case $\beta\leq 1,\ \alpha\geq 1$.

$$I(\alpha,\beta) \geq \int_{\frac{\beta^2}{2}}^{\beta^2} e^{-t}t^{-l}e^{-\frac{\beta^2}{t}}dt \geq e^{-\beta^2}e^{-2}\int_{\frac{\beta^2}{2}}^{\beta^2} t^{-l}dt \geq C\beta^{2-2l}.$$

4. Case $\alpha\geq 1,\ \beta\geq 1$.
   Since $I(\alpha,\beta)$ is increasing with respect to α, we have $I(\alpha,\beta)\geq I(1,\beta)\geq Ce^{-2\beta}$.

□

Finally we treat the case $l=1$.

Lemma 5.2.6 *If $m > l = 1$ we have for some constant $c > 0$*

$$I(\alpha,\beta) \simeq \begin{cases} e^{-c\beta}\,(1\wedge\alpha)^{m-1} & , \quad \textit{if} \quad \beta \geq 1; \\ 1 - \log\beta & , \quad \textit{if} \quad \beta < 1 \leq \alpha; \\ \left(\frac{\beta^2}{\alpha}\right)^{1-m} & , \quad \textit{if} \quad \alpha,\beta < 1,\ \beta^2 \geq \alpha; \\ 1 - \log\left(\frac{\beta^2}{\alpha}\right) & , \quad \textit{if} \quad \alpha,\beta < 1,\ \beta^2 \leq \alpha. \end{cases}$$

PROOF. The upper estimates follow immediately by applying Lemma 5.2.4 (ii) in the cases $\beta \geq 1$ and $\beta < 1 \leq \alpha$ and, for $\alpha, \beta < 1$, by applying (5.5) after observing that $I(\alpha,\beta) \leq \int_0^\infty F(t)\,dt$.

Let us prove, now, the lower estimates. We write $I(l,m,\alpha,\beta)$ to make explicit the dependence on the parameters. With this notation we have from Lemma 5.2.5

$$-\frac{\partial}{\partial\beta} I(1,m,\alpha,\beta) = 2\beta\, I(2,m+1,\alpha,\beta) \geq C\, e^{-c\beta}\beta^{-1}\left(1\wedge\frac{\alpha}{\beta^2}\right)^{m-1}. \tag{5.9}$$

1. Case $\beta \geq 1$.
   This follows as in cases 2 and 4 of Lemma 5.2.5.

2. Case $\beta < 1 \leq \alpha$.
   This follows as in case 3 of Lemma 5.2.5, integrating between β and 1, instead of $\beta^2/2$ and β^2.

3. Case $\alpha, \beta < 1$, $\beta^2 \geq \alpha$.
   Observing that $\lim_{\beta\to\infty} I(\alpha,\beta) = 0$, we integrate (5.9) between β and ∞. Then
   $$I(1,m,\alpha,\beta) \geq C\alpha^{m-1}\int_\beta^\infty e^{-cs}s^{-2m+1}\,ds \geq C\alpha^{m-1}\beta^{-2m+2}.$$

4. Case $\alpha, \beta < 1$, $\beta^2 \leq \alpha$.
   Integrating (5.9) between β and $\sqrt{\alpha}$ we have
   $$I(1,m,\alpha,\beta) \geq I(1,m,\alpha\sqrt{\alpha}) + C\int_\beta^{\sqrt{\alpha}} \frac{1}{s}\,ds \geq C\log\frac{\sqrt{\alpha}}{\beta} = -\frac{C}{2}\log\left(\frac{\beta^2}{\alpha}\right).$$

□

By comparing the estimates for G_λ and G_0 we obtain the following Corollary.

Corollary 5.2.7 *If $N > 2$ and $D > 0$, then $G_\lambda(x,y) \simeq e^{-c\sqrt{\lambda}|x-y|}G_0(x,y)$.*

Note that the above corollary does not hold for $N = 2$, $D > 0$, since G_λ is bounded when $|x||y| \to \infty$, whereas G_0 is not, see Remark 5.1.4.

Using the estimates proved in the previous theorems we obtain the following result (note that $(|x||y|)^{\frac{\gamma}{2}}\, G_\lambda(x,y)$ is symmetric).

Corollary 5.2.8 *Assume that $\lambda, D \geq 0$ and that $D > 0$ when $\lambda = 0$. For any fixed $y \neq 0$ the following asymptotic relations hold.*

(i) As $x \to 0$

$$(|x||y|)^{\frac{\gamma}{2}}\, G_\lambda(x,y) \simeq |x|^{\sqrt{D}-\frac{N-2}{2}}.$$

(ii) As $|x| \to \infty$

$$(|x||y|)^{\frac{\gamma}{2}}\, G_\lambda(x,y) \simeq \begin{cases} e^{-c\sqrt{\lambda}|x|}, & \text{if} \quad \lambda > 0; \\ |x|^{-\sqrt{D}-\frac{N-2}{2}}, & \text{if} \quad \lambda = 0. \end{cases}$$

(iii) As $x \to y$

$$(|x||y|)^{\frac{\gamma}{2}}\, G_\lambda(x,y) \simeq \begin{cases} |x-y|^{2-N}, & \text{if} \quad N > 2; \\ \log|x-y|, & \text{if} \quad N = 2. \end{cases}$$

5.3 Resolvent and spectrum of L

We start by proving that the spectrum of L coincides with $(-\infty, 0]$.

Proposition 5.3.1 *The operator L generates a bounded positive analytic semigroup of angle $\pi/2$ in L^2_μ and its spectrum is the half-line $(-\infty, 0]$.*

Proof. The analyticity of angle $\pi/2$ and the inclusion $\sigma(L) \subset (-\infty, 0]$ follow from the selfadjointness in L^2_μ proved in Section 1.2. To prove that the equality holds, let us assume that the resolvent set $\rho(L)$ contains a point in the negative real axis. If $M_s u(x) = u(s\,x)$, the scaling property $M_s L M_{s^{-1}} = s^{-1} L$ implies that

$$(s^2\lambda - L)^{-1} = s^{-2} M_s (\lambda - L)^{-1} M_{s^{-1}} \tag{5.10}$$

It follows that the resolvent set contains the point -1, hence the unit circle S^1 and the resolvent estimate $\|(\lambda - L)^{-1}\| \leq C|\lambda|^{-1}$ holds for every $\lambda \neq 0$. This implies that $\lambda(\lambda - L)^{-1}$ is a bounded entire function and then it coincides with a constant operator A. Letting $\lambda \to \infty$, we get $A = I$ and hence $L = 0$, which is a contradiction. $\square$

Next we prove that, for $\lambda \neq 0$, the resolvent $(\lambda - L)^{-1}$ is given by an integral kernel $K_\lambda(x,y)$, which we still call the Green function. Clearly, $K_\lambda(x,y) = G_\lambda(x,y)$ whenever the integral defining G_λ converges. This happens if $\operatorname{Re}\lambda > 0$, since

$$|G_\lambda(x,y)| = \left|\int_0^\infty e^{-\lambda t} p(t,x,y)\,dt\right| \leq \int_0^\infty e^{-(\operatorname{Re}\lambda)t} p(t,x,y)\,dt = G_{\operatorname{Re}\lambda}(x,y).$$

By using the upper estimates for the Green function G_λ for positive λ proved in the previous section and the kernel estimates of the semigroup for complex z provided by Theorem 4.2.1, we show bounds for the Green function K_λ for every $\lambda \neq 0$.

Theorem 5.3.2 *Let $\Omega = \mathbb{R}^N \setminus \{0\}$. For every $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, the resolvent $(\lambda - L)^{-1}$ can be represented trough an integral kernel $K_\lambda(x,y)$ with respect to the measure $d\mu = |y|^\gamma dy$. Moreover for every $\varepsilon > 0$, there exist C_ε, $m_\varepsilon > 0$ such that, for $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ with $|\arg\lambda| \leq \pi - \varepsilon$ and $(x,y) \in \Omega \times \Omega$,*

$$|K_\lambda(x,y)| \leq C_\varepsilon G_{m_\varepsilon|\lambda|}(x,y).$$

Proof. Let $\varepsilon > 0$, and let us consider the sector $\Sigma_{\pi-\epsilon} := \{\lambda \in \mathbb{C} \mid |\arg\lambda| \leq \pi - \epsilon\}$. Let us fix $\lambda \in \Sigma_{\pi-\epsilon} \setminus \{0\}$ and let θ be the angle defined by $\theta = \frac{\pi}{2} - \frac{\epsilon}{2}$ if $\arg\lambda \geq 0$ and by $\theta = -\frac{\pi}{2} + \frac{\epsilon}{2}$ if $\arg\lambda < 0$. Let us denote by $T(e^{i\theta}t)$ the semigroup generated by $e^{i\theta}L$ in L^2_μ. Setting $\mu := e^{-i\theta}\lambda = |\lambda|e^{i(\arg\lambda - \theta)}$, since $|\arg\mu| \leq \frac{\pi}{2} - \frac{\epsilon}{2}$, we have obviously $\operatorname{Re}\mu > 0$. Let us define

$$G_{\theta,\mu}(x,y) = \int_0^\infty e^{-\mu t} p(e^{i\theta}t,x,y)\,dt.$$

We observe, preliminarily, that by Theorems 4.2.1 and 4.2.2 there exist $C_\varepsilon, \tilde{C}_\varepsilon, m_\varepsilon > 0$ such that, after a suitable choice of a constant $\tilde{m}_\varepsilon > 0$ in the argument of the real kernel, we have

$$\begin{aligned}|p(e^{i\theta}t,x,y)| &\leq C_\varepsilon |x|^{-\frac{\gamma}{2}}|y|^{-\frac{\gamma}{2}} t^{-\frac{N}{2}} \left[\left(\frac{|x|}{t^{\frac{1}{2}}} \wedge 1\right)\left(\frac{|y|}{t^{\frac{1}{2}}} \wedge 1\right)\right]^{-\frac{N}{2}+1+\sqrt{D}} \exp\left(-\frac{|x-y|^2}{m_\varepsilon t}\right) dt \\ &\leq \tilde{C}_\varepsilon\, p(\tilde{m}_\varepsilon t,x,y).\end{aligned}$$

From now on let C_ε and m_ε be different at each occurrence. After a change of variable in the integral and observing that $\cos(\arg\mu) \geq \cos(\frac{\pi}{2} - \frac{\epsilon}{2}) = \sin(\frac{\epsilon}{2})$, it follows from the previous relation that $G_{\theta,\mu}(x,y)$ satisfies

$$\begin{aligned}|G_{\theta,\mu}(x,y)| &\leq C_\varepsilon \int_0^\infty e^{-\operatorname{Re}\mu\, t} p(\tilde{m}_\varepsilon t,x,y)\,dt \leq C_\varepsilon \int_0^\infty e^{-m_\varepsilon|\lambda|\sin(\frac{\epsilon}{2})\,t} p(t,x,y)\,dt \\ &= C_\varepsilon G_{m_\varepsilon \sin(\frac{\epsilon}{2})|\lambda|}(x,y). \qquad (5.11)\end{aligned}$$

Since

$$(\mu - e^{i\theta}L)^{-1} = \int_0^\infty e^{-\mu t} T(e^{i\theta}t)\,dt$$

it follows that $G_{\theta,\mu}(x, y)$ is the integral kernel of of $(\mu - e^{i\theta}L)^{-1}$ (the kernel being written with respect to the measure $|y|^\gamma\, dy$). By multiplying by $e^{i\theta}$ we deduce that the Green function $K_\lambda(x, y)$ of $(\lambda - L)^{-1}$ satisfies

$$K_\lambda(x, y) = e^{i\theta} G_{\theta,\mu}(x, y)$$

and the proof is concluded using (5.11). $\square$

Chapter 6

Gradient estimates

In this chapter we prove some Gaussian bounds for the time derivative and for the space gradient of the heat kernel of the operator

$$L = \Delta + (a-1)\sum_{i,j=1}^{N} \frac{x_i x_j}{|x|^2} D_{ij} + c\frac{x}{|x|^2} \cdot \nabla - \frac{b}{|x|^2},$$

where $a > 0$, $b,\ c \in \mathbb{R}$. The holomorphy of the heat kernel as function of t and the Cauchy formula for the derivatives of holomorphic functions, combined with the kernel estimates previously proved, easily yield estimates for the time derivative. For the spatial gradient more effort is needed and an essential role is played by the series decomposition of the kernel proved in Chapter 4 and in [50]. After differentiating term by term and by using the well known properties of the derivatives of Bessel functions and of zonal harmonics, we will get the gradient estimates in some space-time regions. Interior a-priori estimates for L with precise coefficients will allow to cover all the cases.

In the following we write $f(x) \simeq g(x)$ if for some $C_1, C_2 > 0$, $C_1\, g(x) \le f(x) \le C_2\, g(x)$. The results of this chapter are collected in [51].

6.1 Estimates for the time derivatives of p

In order to state and prove the main result we recall, for the reader's convenience, some properties concerning the kernel decomposition and its estimates proved in Chapter 4, see also [50].

Let $p_n(t, r, \rho)$ be the parabolic kernel, with respect to the measure $\rho^{\frac{N-1+c}{a}}\, d\rho$, of the one-dimensional Bessel operators

$$L_n := aD_{rr} + \frac{N-1+c}{r} D_r - \frac{b+\lambda_n}{r^2}.$$

It has been shown in Section 1.3 that

$$p_n(t,r,\rho) = \frac{1}{2at}\,(r\rho)^{-\frac{N-1+c-a}{2a}}\, I_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right)\exp\left\{-\frac{r^2+\rho^2}{4at}\right\}, \tag{6.1}$$

where $D_n = \frac{b+\lambda_n}{a} + \left(\frac{N-1+c-a}{2a}\right)^2$ and we write D for D_0.

Let p be the heat kernel of L with respect to the measure $d\mu(y) = |y|^\gamma\, dx$. Then for $x = r\omega$, $y = \rho\eta$, $r,\ \rho > 0$, $|\omega| = |\eta| = 1$ we have

$$\begin{aligned} p(t,x,y) &= \frac{1}{2at}\,(r\rho)^{-\frac{N-1+c-a}{2a}}\exp\left\{-\frac{r^2+\rho^2}{4at}\right\}\sum_{n=0}^{\infty} I_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right)\mathbb{Z}_\omega^{(n)}(\eta) \qquad (6.2)\\ &= \sum_{n\geq 0} p_n(t,r,\rho)\mathbb{Z}_\omega^{(n)}(\eta). \end{aligned}$$

Moreover, setting $\Omega = \mathbb{R}^N \setminus \{0\}$, for $(x,y) \in \Omega\times\Omega$, the heat kernel p satisfies

$$p(t,x,y) \simeq t^{-\frac{N}{2}}|x|^{-\frac{\gamma}{2}}|y|^{-\frac{\gamma}{2}}\left[\left(\frac{|x|}{t^{\frac12}}\wedge 1\right)\left(\frac{|y|}{t^{\frac12}}\wedge 1\right)\right]^{-\frac{N}{2}+1+\sqrt{D}}\exp\left(-\frac{c|x-y|^2}{t}\right).$$

The constant $c > 0$ may differ in the upper and lower bounds.

The Cauchy formula for the derivatives of holomorphic functions allows to estimates the time derivative of p.

Proposition 6.1.1 *Let $\Omega = \mathbb{R}^N \setminus \{0\}$. The heat kernel p of L, with respect to the measure $d\mu = |y|^\gamma dy$, satisfies*

$$\left|\frac{\partial p}{\partial t}(t,x,y)\right| \leq Ct^{-\frac{N}{2}-1}|x|^{-\frac{\gamma}{2}}|y|^{-\frac{\gamma}{2}}\left[\left(\frac{|x|}{t^{\frac12}}\wedge 1\right)\left(\frac{|y|}{t^{\frac12}}\wedge 1\right)\right]^{-\frac{N}{2}+1+\sqrt{D}}\exp\left(-\frac{c|x-y|^2}{t}\right)$$

for $t > 0$, $(x,y) \in \Omega\times\Omega$ and for some constants $C, c > 0$. In particular, up to a modification of the constants involved,

$$\left|\frac{\partial p}{\partial t}(t,x,y)\right| \leq C\frac{1}{t}p(ct,x,y).$$

Proof. By Theorem 4.2.1, the kernel $p(z,x,y)$ is holomorphic as a function of z. Let us fix $t > 0$. Using the Cauchy formula for the derivatives of holomorphic functions in $B\left(t,\frac{t}{2}\right)$, we get for some constants $C, c > 0$ and for $x, y \in \Omega$

$$\left|\frac{\partial p}{\partial t}(t,x,y)\right| \leq C\frac{1}{t}\max_{|z|=\frac{t}{2}}|p(z,x,y)|.$$

Applying the estimate (4.4) with $\epsilon = \frac{\pi}{3}$, we obtain

$$\left|\frac{\partial p}{\partial t}(t,x,y)\right| \leq C_\epsilon t^{-\frac{N}{2}-1}|x|^{-\frac{\gamma}{2}}|y|^{-\frac{\gamma}{2}}\left[\left(\frac{|x|}{t^{\frac12}}\wedge 1\right)\left(\frac{|y|}{t^{\frac12}}\wedge 1\right)\right]^{-\frac{N}{2}+1+\sqrt{D}}\exp\left(-\frac{c|x-y|^2}{t}\right).$$

The last claim is a consequence of (4.5).

□

Since the kernel satisfies the equation $\dfrac{\partial p}{\partial t} = Lp$, Proposition 6.1.1 implies

$$|Lp(t,x,y)| \le Ct^{-\frac{N}{2}-1}|x|^{-\frac{\gamma}{2}}|y|^{-\frac{\gamma}{2}} \left[\left(\frac{|x|}{t^{\frac12}} \wedge 1\right)\left(\frac{|y|}{t^{\frac12}} \wedge 1\right)\right]^{-\frac{N}{2}+1+\sqrt{D}} \exp\left(-\frac{c|x-y|^2}{t}\right), \tag{6.3}$$

for $t>0$, $(x,y) \in \Omega \times \Omega$

6.2 Estimates for the space derivatives of p

The main result of this chapter is the following estimate for the gradient of p.

Theorem 6.2.1 *Let $\Omega = \mathbb{R}^N \setminus \{0\}$. The heat kernel p of L, with respect to the measure $d\mu = |y|^\gamma dy$, satisfies*

$$|\nabla p(t,x,y)| \le C\left(1 \wedge \frac{|x|}{\sqrt{t}}\right)^{-1} t^{-\frac{N+1}{2}}|x|^{-\frac{\gamma}{2}}|y|^{-\frac{\gamma}{2}} \left[\left(\frac{|x|}{t^{\frac12}} \wedge 1\right)\left(\frac{|y|}{t^{\frac12}} \wedge 1\right)\right]^{-\frac{N}{2}+1+\sqrt{D}} \times \exp\left(-\frac{c|x-y|^2}{t}\right) \tag{6.4}$$

for $t>0$, $(x,y)\in\Omega\times\Omega$ and for some constants $C,c>0$. In particular, up to a modification of the constants involved,

$$|\nabla p(t,x,y)| \le C\frac{1}{\sqrt{t}}\left(1 \wedge \frac{|x|}{\sqrt{t}}\right)^{-1} p(ct,x,y). \tag{6.5}$$

Here the gradient is taken with respect the x variable. Similar results hold, by symmetry, for the gradient respect the y variable.

Remark 6.2.2 *The scaling property of the kernel*

$$p(t,x,y) = t^{-\frac{N+\gamma}{2}} p\left(1, \frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right)$$

implies the analogous scaling for its gradient:

$$\nabla p(t,x,y) = t^{-\frac{N+\gamma+1}{2}} \nabla p\left(1, \frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right). \tag{6.6}$$

From the last relation, in the proof of Theorem 6.2.1, one may assume $t=1$ without loss of generality.

Remark 6.2.3 *Recalling Corollary 4.2.4, the estimate* (6.4) *becomes, for some possibly different constants $C,c>0$,*

$$|\nabla p(t,x,y)| \le Ct^{-1-\sqrt{D}}|x|^{-s_1-1}|y|^{-s_1}, \qquad \frac{|x|}{\sqrt{t}} \le \delta, \frac{|y|}{\sqrt{t}} \le \delta,$$

$$|\nabla p(t,x,y)| \le Ct^{-\frac12-\frac{s_1}{2}-\sqrt{D}}|x|^{-s_1-1}e^{-c\frac{|y|^2}{t}}, \qquad \frac{|x|}{\sqrt{t}} \le \delta, \frac{|y|}{\sqrt{t}} \ge \delta,$$

$$|\nabla p(t,x,y)| \le Ct^{-\frac{N}{2}-\frac12}|y|^{-\gamma}\left(\frac{|y|}{t^{\frac12}} \wedge 1\right)^{-s_1^*} \exp\left(-\frac{c|x-y|^2}{t}\right), \qquad \frac{|x|}{\sqrt{t}} \ge \delta,$$

where $s_1^* = \frac{N}{2} - 1 - \sqrt{D} - \frac{\gamma}{2}$ *and* $\delta > 0$ *is fixed.*

Let us employ spherical coordinates to write, for x, $y \in \mathbb{R}^n$, $x = r\omega$, $y = \rho\eta$ and let p_r, $\nabla_\tau p$ be respectively the radial and the tangential component of the gradient. We, formally, differentiate the series in (6.2) obtaining the following expression for the derivatives of the heat kernel

$$\begin{aligned}\frac{\partial}{\partial t}p(t,x,y) =& \frac{1}{2at}(r\rho)^{-\frac{N-1+c-a}{2a}} \exp\left\{-\frac{r^2+\rho^2}{4at}\right\}\left(-\frac{r\rho}{2at^2}\right)\sum_{n=0}^{\infty} I'_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right)\mathbb{Z}_\omega^{(n)}(\eta) \qquad (6.7)\\ &+\left(-\frac{1}{t}+\frac{r^2+\rho^2}{4at^2}\right)p(t,x,y),\end{aligned}$$

$$\begin{aligned}p_r(t,x,y) =& \frac{1}{2at}(r\rho)^{-\frac{N-1+c-a}{2a}} \exp\left\{-\frac{r^2+\rho^2}{4at}\right\}\frac{\rho}{2at}\sum_{n=0}^{\infty} I'_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right)\mathbb{Z}_\omega^{(n)}(\eta) \qquad (6.8)\\ &+\left(-\frac{N-1+c-a}{2ar}-\frac{2r}{4at}\right)p(t,x,y),\end{aligned}$$

$$\nabla_\tau p(t,x,y) = \frac{1}{2at}(r\rho)^{-\frac{N-1+c-a}{2a}} \exp\left\{-\frac{r^2+\rho^2}{4at}\right\}\sum_{n=0}^{\infty} I_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right)\nabla_\tau\mathbb{Z}_\omega^{(n)}(\eta), \qquad (6.9)$$

$$\nabla p = p_r\frac{x}{|x|}+\frac{\nabla_\tau p}{r}. \qquad (6.10)$$

In order to rigorously prove (6.7), (6.8), (6.9) we need to assure the convergence of the series involved. Some basic properties about the derivative of I_ν and $\mathbb{Z}^{(n)}$ are needed.

Lemma 6.2.4 *For every $\nu \geq 0$, the modified Bessel function I_ν is a regular function and its derivative I'_ν satisfies the following relations:*

(i) $I'_\nu(r) = \frac{\nu}{r}I_\nu(r) + I_{\nu+1}(r)$;

(ii) $0 \leq I'_\nu(r) \leq \left(1+\frac{\nu^2}{r^2}\right)^{\frac{1}{2}} I_\nu(r) \leq \left(1+\frac{\nu}{r}\right) I_\nu(r)$.

Proof. See e.g., [2, 9.6 and 9.7] for (i) and [12] for (ii). □

Lemma 6.2.5 *The tangential derivative of the zonal harmonics $\mathbb{Z}^{(n)}$ satisfies, for some constant $C = C(N) > 0$,*

$$\|\nabla_\tau \mathbb{Z}^{(n)}\|_\infty \leq C n^{\frac{3n-4}{2}}.$$

Proof. See corollary B.4.7 in the appendix. □

Lemma 6.2.6 *Let $(c_n)_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers such that $c_n \approx cn^\alpha$ for some $\alpha > 0$. Then there exists $h \in C([0,\infty[)$, with $h(0) \neq 0$, such that for every $s > 0$*

$$\sum_{n=1}^{\infty} c_n I_{\sqrt{D_n}}(s) \leq C I_{\sqrt{D}}(s)\left(\frac{s}{2}\right)^{\sqrt{D_1}-\sqrt{D}} h\left(\frac{s}{2}\right).$$

In particular, the series $\sum_{n=1}^{\infty} c_n I_{\sqrt{D_n}}(s)$ converges uniformly on compact sets of $]0,\infty[$.

PROOF. We use $\Gamma(\alpha+\beta) \geq C_\delta \Gamma(\alpha)\Gamma(\beta)$ if $\alpha, \beta \geq \delta$ to obtain $\Gamma(m+\sqrt{D}+1+\sqrt{D_n}-\sqrt{D}) \geq C\Gamma(m+1+\sqrt{D})\Gamma(\sqrt{D_n}-\sqrt{D})$ for every $n \geq 1$. Then

$$\sum_{n\geq 1} c_n I_{\sqrt{D_n}}(s) = \sum_{n\geq 1} c_n \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\sqrt{D_n}+1)} \left(\frac{s}{2}\right)^{2m+\sqrt{D_n}-\sqrt{D}+\sqrt{D}}$$

$$= \sum_{n\geq 1} c_n \left(\frac{s}{2}\right)^{\sqrt{D_n}-\sqrt{D}} \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\sqrt{D_n}+1)} \left(\frac{s}{2}\right)^{2m+\sqrt{D}}$$

$$\leq C \sum_{n\geq 1} n^\alpha \left(\frac{s}{2}\right)^{\sqrt{D_n}-\sqrt{D}} \frac{1}{\Gamma(\sqrt{D_n}-\sqrt{D})} \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+1+\sqrt{D})} \left(\frac{s}{2}\right)^{2m+\sqrt{D}}$$

$$= C I_{\sqrt{D}}(s) \left(\frac{s}{2}\right)^{\sqrt{D_1}-\sqrt{D}} \sum_{n\geq 1} n^\alpha \left(\frac{s}{2}\right)^{\sqrt{D_n}-\sqrt{D_1}} \frac{1}{\Gamma(\sqrt{D_n}-\sqrt{D})}.$$

Since $\sqrt{D_n} \approx cn$, $c = 1/\sqrt{a}$ as $n \to \infty$, by the asymptotic of the Gamma function the series

$$h(s) = \sum_{n\geq 1} n^\alpha s^{\sqrt{D_n}-\sqrt{D_1}} \frac{1}{\Gamma(\sqrt{D_n}-\sqrt{D})}$$

converges uniformly on compact sets of $[0,\infty[$ and it does not vanish at 0. $\square$

Corollary 6.2.7 *There exists $h \in C([0,\infty[)$, with $h(0) \neq 0$ such that*

$$\sum_{n=1}^{\infty} I'_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right) |\mathbb{Z}_\omega^{(n)}(\eta)| \leq C I_{\sqrt{D}}\left(\frac{r\rho}{2at}\right) \left(\frac{r\rho}{4at}\right)^{\sqrt{D_1}-\sqrt{D}} \left(1+\frac{2at}{r\rho}\right) h\left(\frac{r\rho}{4at}\right),$$

$$\sum_{n=1}^{\infty} I_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right) |\nabla_\tau \mathbb{Z}_\omega^{(n)}(\eta)| \leq C I_{\sqrt{D}}\left(\frac{r\rho}{2at}\right) \left(\frac{r\rho}{4at}\right)^{\sqrt{D_1}-\sqrt{D}} h\left(\frac{r\rho}{4at}\right).$$

In particular the series $\sum_{n=1}^\infty I'_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right) \mathbb{Z}_\omega^{(n)}(\eta)$, $\sum_{n=1}^\infty I_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right) \nabla_\tau \mathbb{Z}_\omega^{(n)}(\eta)$ converges uniformly on compact sets of $]0,\infty[\times\Omega\times\Omega$.

PROOF. We use (ii) of Proposition B.2.2, for the estimate $|Z_\omega^{(n)}(\eta)| \leq Z_\omega^{(n)}(\omega) \leq Cn^{N-2}$, and Lemmas 6.2.4 and 6.2.5 to obtain

$$\sum_{n=1}^{\infty} I'_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right) |\mathbb{Z}_\omega^{(n)}(\eta)| \leq C \sum_{n\geq 1} \left(1+\sqrt{D_n}\frac{2at}{r\rho}\right) I_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right) n^{N-2},$$

$$\sum_{n=1}^{\infty} I_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right) |\nabla_\tau \mathbb{Z}_\omega^{(n)}(\eta)| \leq C \sum_{n\geq 1} I_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right) n^{\frac{3N-4}{2}}.$$

Recalling $\sqrt{D_n} \approx cn$, the claim follows by applying Lemma 6.2.6. $\square$

We now start proving the estimate (6.4) near the origin, that is for $|x|/\sqrt{t}, |y|/\sqrt{t}$ small, see Remark 6.2.2. We recall that, in this case, the behaviour of p is the same as the radial part p_0 (see Remark 4.2.5 and Lemma 4.2.8):

$$p(t,x,y) \simeq C p_0(t,r,\rho) \simeq C t^{-1-\sqrt{D}} |x|^{-s_1} |y|^{-s_1}. \tag{6.11}$$

Proposition 6.2.8 *The heat kernel p is a C^1 function of its arguments and its derivatives are given by formulas (6.7), (6.8), (6.9), (6.10). Moreover for every fixed $\delta > 0$, there exists $C > 0$ such that for $\frac{|x|}{\sqrt{t}}, \frac{|y|}{\sqrt{t}} < \delta$*

$$|\nabla p(t,x,y)| \le Ct^{-1-\sqrt{D}}|x|^{-s_1-1}|y|^{-s_1}.$$

Proof. The first sentence easily follows once observed that Lemma 6.2.6 allows us to differentiate the series in (6.2). Let $\delta > 0$; applying Corollary 6.2.7 we can choose a constant C_δ such that for $\frac{|x|}{\sqrt{t}}, \frac{|y|}{\sqrt{t}} < \delta$

$$\sum_{n=1}^{\infty} I'_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right)|\mathbb{Z}_\omega^{(n)}(\eta)| \le C_\delta I_{\sqrt{D}}\left(\frac{r\rho}{2at}\right)\left(1+\frac{2at}{r\rho}\right),$$

$$\sum_{n=1}^{\infty} I_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right)|\nabla_\tau\mathbb{Z}_\omega^{(n)}(\eta)| \le C_\delta I_{\sqrt{D}}\left(\frac{r\rho}{2at}\right).$$

Recalling (6.8) we have

$$\begin{aligned}p_r(t,x,y) =&\frac{1}{2at}(r\rho)^{-\frac{N-1+c-a}{2a}}\exp\left\{-\frac{r^2+\rho^2}{4at}\right\}\frac{\rho}{2at}\sum_{n=0}^{\infty} I'_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right)\mathbb{Z}_\omega^{(n)}(\eta)\\ &+\left(-\frac{N-1+c-a}{2ar}-\frac{2r}{4at}\right)p(t,x,y).\end{aligned}$$

We use, now, (ii) of Lemma 6.2.4 to estimate the first term of the series and we apply the previous relations for the remaining terms; recalling (6.1) we obtain for some constant C (that may vary from line to line)

$$\begin{aligned}|p_r(t,x,y)| \le&\frac{1}{2at}(r\rho)^{-\frac{N-1+c-a}{2a}}\exp\left\{-\frac{r^2+\rho^2}{4at}\right\}\frac{\rho}{2at}I_{\sqrt{D}}\left(\frac{r\rho}{2at}\right)\\ &\times\left[\left(1+\sqrt{D}\frac{2at}{r\rho}\right)\frac{1}{|S^{N-1}|}+C_\delta\left(1+\frac{2at}{r\rho}\right)\right]+C\left(\frac{1}{r}+\frac{r}{t}\right)p(t,x,y)\\ \le& C\left(1+\frac{t}{r\rho}\right)\frac{\rho}{t}p_0(t,r,\rho)+C\left(\frac{1}{r}+\frac{r}{t}\right)p(t,x,y).\end{aligned}$$

Since $\frac{r}{\sqrt{t}}, \frac{\rho}{\sqrt{t}} < \delta$ it follows from (6.11) that

$$|p_r(t,x,y)| \le C\left(\frac{\rho}{t}+\frac{1}{r}+\frac{r}{t}\right)p_0(t,r,\rho) \le C\frac{1}{r}t^{-1-\sqrt{D}}r^{-s_1}\rho^{-s_1}.$$

The same argument applied to (6.9) proves

$$|\nabla_\tau p(t,x,y)| \le C\,t^{-1-\sqrt{D}}r^{-s_1}\rho^{-s_1}.$$

Since $\nabla p = p_r\frac{x}{|x|}+\frac{\nabla_\tau p}{r}$, the last two estimates prove the thesis.

□

We now focus on the estimate (6.4) in the range $|x|/\sqrt{t} \leq \delta, |y|/\sqrt{t} \geq \delta$. In this case, up to a small perturbation of the constant in the exponential factor, the behaviour of p takes the form

$$p(t,x,y) \simeq t^{-\frac{N}{2}}|x|^{-s_1}|y|^{-\frac{\gamma}{2}} \exp\left(-\frac{c|x-y|^2}{t}\right) \simeq t^{-\frac{N}{2}}|x|^{-s_1} \exp\left(-\frac{c'|y|^2}{t}\right). \tag{6.12}$$

As before we need some preparation.
We observe, preliminarily, that $D_n = D + \frac{\lambda_n}{a} = D + \frac{n(n+N-2)}{a} \leq D + \frac{(n+\frac{N-2}{2})^2}{a}$; taking square roots we have

$$\sqrt{D_n} \leq \sqrt{D} + \frac{N-2}{2\sqrt{a}} + \frac{n}{\sqrt{a}} \tag{6.13}$$

and moreover, from the asymptotic expansion of the Gamma function, the following asymptotic behaviour holds:

$$\sqrt{D_n} \approx \frac{n}{\sqrt{a}}, \qquad \Gamma\left(\sqrt{D_n}\right)^{\frac{1}{n}} \approx n^{\frac{1}{\sqrt{a}}}, \quad \text{as } n \to \infty. \tag{6.14}$$

The Lemma below is an application of a classic result related to the growth of entire functions.

Lemma 6.2.9 *Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be an entire function. If for some constant c_0 the coefficients satisfy the condition $|c_n|^{\frac{1}{n}} n^k \leq c_0$ then there exist $C, c > 0$ such that for every $z \in \mathbb{C}$*

$$|f(z)| \leq C e^{c|z|^{\frac{1}{k}}}$$

Proof. See for example [42, Lemma 2, pag 5].

□

Lemma 6.2.10 *Let $(c_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $c_n \approx cn^\alpha$ for some $\alpha > 0$. Then there exist $C, c > 0$ such that for every $r < \delta$, $\rho > \delta$*

$$\sum_{n=0}^{\infty} c_n I_{\sqrt{D_n}}\left(\frac{r\rho}{2a}\right) \leq C\rho^{\sqrt{D}+\frac{N-2}{2\sqrt{a}}} r^{\sqrt{D}} e^{c\rho}.$$

In particular we have, for every $|x| < \delta, |y| > \delta$, $x = r\omega, y = \rho\eta$

$$\sum_{n=0}^{\infty} \sqrt{D}_n I_{\sqrt{D_n}}\left(\frac{r\rho}{2a}\right) |\mathbb{Z}_\omega^{(n)}(\eta)| \leq C\rho^{\sqrt{D}+\frac{N-2}{2\sqrt{a}}} r^{\sqrt{D}} e^{c\rho}$$

$$\sum_{n=1}^{\infty} I_{\sqrt{D_n}}\left(\frac{r\rho}{2a}\right) |\nabla_\tau \mathbb{Z}_\omega^{(n)}(\eta)| \leq C\rho^{\sqrt{D}+\frac{N-2}{2\sqrt{a}}} r^{\sqrt{D}} e^{c\rho}.$$

Proof. In analogy with the proof of Lemma 6.2.6 we use $\Gamma(\alpha+\beta) \geq C\Gamma(\alpha)\Gamma(\beta)$ to obtain

$$\begin{aligned} \sum_{n\geq 1} c_n I_{\sqrt{D_n}}\left(\frac{r\rho}{2a}\right) &= \sum_{n\geq 1} c_n \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\sqrt{D_n}+1)} \left(\frac{r\rho}{4a}\right)^{2m+\sqrt{D_n}} \\ &\leq C \sum_{m=0}^{\infty} \frac{1}{m!m!} \left(\frac{r\rho}{4a}\right)^{2m} \left(\frac{r\rho}{4a}\right)^{\sqrt{D}} \sum_{n\geq 0} c_n \frac{1}{\Gamma(\sqrt{D_n})} \left(\frac{r\rho}{4a}\right)^{\sqrt{D_n}-\sqrt{D}}. \end{aligned} \tag{6.15}$$

Since $\left(\frac{1}{m!}\right)^{\frac{1}{m}} \approx \frac{1}{m}$, Lemma 6.2.9 yields for some constants $C, C_\delta, c > 0$

$$\sum_{m=0}^{\infty} \frac{1}{m!m!} \left(\frac{r\rho}{4a}\right)^{2m} \leq Ce^{cr\rho} \leq C_\delta e^{c\rho}. \tag{6.16}$$

Recalling (6.13), since $r < \delta$, $\rho > \delta$ we get

$$\begin{aligned}\sum_{n\geq 0} c_n \frac{1}{\Gamma(\sqrt{D_n})} \left(\frac{r\rho}{4a}\right)^{\sqrt{D_n}-\sqrt{D}} &\leq \sum_{n\geq 0} c_n \frac{1}{\Gamma(\sqrt{D_n})} \left(\frac{\delta^2}{4a}\right)^{\sqrt{D_n}-\sqrt{D}} \left(\frac{\rho}{\delta}\right)^{\sqrt{D_n}-\sqrt{D}} \\ &\leq \sum_{n\geq 0} c_n \frac{1}{\Gamma(\sqrt{D_n})} \left(\frac{\delta^2}{4a}\right)^{\sqrt{D_n}-\sqrt{D}} \left(\frac{\rho}{\delta}\right)^{\frac{n}{\sqrt{a}}+\frac{N-2}{2\sqrt{a}}}.\end{aligned}$$

It follows from (6.14) and the hypothesis on c_n that

$$\left[c_n \frac{1}{\Gamma(\sqrt{D_n})} \left(\frac{\delta^2}{4a}\right)^{\sqrt{D_n}-\sqrt{D}}\right]^{\frac{1}{n}} \approx \left(\frac{1}{n}\right)^{\frac{1}{\sqrt{a}}}.$$

Lemma 6.2.9 again yields, for some other constants $C, c > 0$,

$$\sum_{n\geq 0} c_n \frac{1}{\Gamma(\sqrt{D_n})} \left(\frac{r\rho}{4a}\right)^{\sqrt{D_n}-\sqrt{D}} \leq C \left(\frac{\rho}{\delta}\right)^{\frac{N-2}{2\sqrt{a}}} e^{c\rho}. \tag{6.17}$$

Inserting (6.16) and (6.17) in (6.15) we prove the first required estimate. The remaining part of the proof is a consequence of the first part and of the behaviour of $\sqrt{D_n}$, $\mathbb{Z}_\omega^{(n)}$ and $\nabla_\tau \mathbb{Z}_\omega^{(n)}(\eta)$.

□

We can, now, prove the required estimates for $\frac{|x|}{\sqrt{t}} < \delta$, $\frac{|y|}{\sqrt{t}} > \delta$.

Proposition 6.2.11 *For every fixed $\delta > 0$ there exist $C, m > 0$ such that, if $\frac{|x|}{\sqrt{t}} < \delta$, $\frac{|y|}{\sqrt{t}} > \delta$, then*

$$|\nabla p(t,x,y)| \leq Ct^{-\frac{1}{2}-\frac{s_1}{2}-\sqrt{D}} |x|^{-s_1-1} e^{-m\frac{|y|^2}{t}}.$$

Proof. By the scaling property (6.6), we may assume that $t = 1$ and prove that, for $x = r\omega, y = \rho\eta$ such that $|x| < \delta, |y| > \delta$,

$$|\nabla p(1,x,y)| \leq Cr^{-s_1-1} e^{-m\frac{\rho^2}{t}}.$$

From (6.8) we have

$$\begin{aligned}p_r(1,x,y) =& \frac{1}{2a} (r\rho)^{-\frac{N-1+c-a}{2a}} \exp\left\{-\frac{r^2+\rho^2}{4a}\right\} \frac{\rho}{2a} \sum_{n=0}^{\infty} I'_{\sqrt{D_n}} \left(\frac{r\rho}{2a}\right) \mathbb{Z}_\omega^{(n)}(\eta) \\ &+ \left(-\frac{N-1+c-a}{2ar} - \frac{2r}{4a}\right) p(1,x,y).\end{aligned}$$

Using Lemma 6.2.4 and recalling (6.2) and (6.1), we obtain for some constant C that may vary from line to line

$$\begin{aligned}
|p_r(1,x,y)| \leq & \frac{1}{2a}(r\rho)^{-\frac{N-1+c-a}{2a}} \exp\left\{-\frac{r^2+\rho^2}{4a}\right\} \frac{\rho}{2a}\sum_{n=0}^{\infty}\left(1+\sqrt{D}_n\frac{2a}{r\rho}\right) I_{\sqrt{D_n}}\left(\frac{r\rho}{2a}\right)\mathbb{Z}_{\omega}^{(n)}(\omega) \\
& + C\left(\frac{1}{r}+r\right)p(1,r\omega,\rho\eta) \\
= & \frac{\rho}{2a}p(1,r\omega,\rho\omega) + \frac{1}{2a}(r\rho)^{-\frac{N-1+c-a}{2a}} \exp\left\{-\frac{r^2+\rho^2}{4a}\right\}\frac{1}{r}\sum_{n=0}^{\infty}\sqrt{D}_n I_{\sqrt{D_n}}\left(\frac{r\rho}{2a}\right)\mathbb{Z}_{\omega}^{(n)}(\omega) \\
& + C\left(\frac{1}{r}+r\right)p(1,r\omega,\rho\eta).
\end{aligned}$$

Since $r<\delta, \rho>\delta$, using (6.12) we can estimate the first and the third addendum as

$$\frac{\rho}{2a}p(1,r\omega,\rho\omega) + C\left(\frac{1}{r}+r\right)p(1,r\omega,\rho\eta) \leq C\frac{1}{r}r^{-s_1}\rho e^{-c\rho^2} \leq Cr^{-s_1-1}e^{-c_1\rho^2}$$

with $c_1 < c$. Analogously, using Lemma 6.2.10 and $s_1 = \frac{N-1+c-a}{2a} - \sqrt{D}$,

$$\begin{aligned}
& \frac{1}{2a}(r\rho)^{-\frac{N-1+c-a}{2a}} \exp\left\{-\frac{r^2+\rho^2}{4a}\right\}\frac{1}{r}\sum_{n=0}^{\infty}\sqrt{D}_n I_{\sqrt{D_n}}\left(\frac{r\rho}{2a}\right)\mathbb{Z}_{\omega}^{(n)}(\omega) \\
& \leq C(r\rho)^{-\frac{N-1+c-a}{2a}} \exp\left\{-\frac{\rho^2}{4a}\right\}\frac{1}{r}\rho^{\sqrt{D}+\frac{N-2}{2\sqrt{a}}} r^{\sqrt{D}}e^{c\rho} \\
& \leq Cr^{-s_1-1}e^{-c_2\rho^2}
\end{aligned}$$

with $c_2 < \frac{1}{4a}$. Setting $m = \min\{c_1, c_2\}$ we have

$$|p_r(1,x,y)| \leq Cr^{-s_1-1}e^{-m\frac{\rho^2}{t}}.$$

The same reasoning applied to (6.9) proves

$$|\nabla_\tau p(t,x,y)| \leq Cr^{-s_1}e^{-m\frac{\rho^2}{t}}.$$

Since $\nabla p = p_r\frac{x}{|x|} + \frac{\nabla_\tau p}{r}$, the last two estimates prove the thesis.

□

In the following proposition, we prove some interpolative interior estimates for the gradient of C^2 functions.

Proposition 6.2.12 *Let $\delta > 0$, $x \in \mathbb{R}^N \setminus \{0\}$, $r > 0$ such that $\overline{B(x,2r)} \subseteq \mathbb{R}^N \setminus \overline{B_\delta}$. Then there exists a constant $C = C(\delta) > 0$ such that, for every $u \in C^2(\mathbb{R}^N \setminus \{0\})$, one has*

$$\|\nabla u\|_{L^\infty(B(x,r))} \leq C\left(r\|Lu\|_{L^\infty(B(x,2r))} + \left(\frac{1}{r}+1\right)\|u\|_{L^\infty(B(x,2r))}\right).$$

Proof. Let $x \in \mathbb{R}^N \setminus \{0\}$, $r > 0$ such that $\overline{B(x,2r)} \subseteq \mathbb{R}^N \setminus \overline{B_\delta}$ and let $u \in C^2(\mathbb{R}^N \setminus \{0\})$. Let us define, for $n \in \mathbb{N}_0$

$$r_n = r\sum_{j=0}^{n} 2^{-j}, \quad r_0 = r, \quad r_\infty = 2r.$$

Let $\eta_n \in C_c^\infty(B(x, r_{n+1}))$ such that

$$0 \le \eta_n \le 1, \quad \eta_n = 1 \text{ in } B(x,r_n), \quad |\nabla \eta_n|_\infty \le M\frac{2^n}{r}, \quad |D^2\eta_n|_\infty \le M\frac{4^n}{r^2}, \tag{6.18}$$

for some constant $M > 0$.

Let U be an open set such that $\overline{B(x,2r)} \subseteq U \subseteq \mathbb{R}^N \setminus \overline{B_\delta}$ and let $\psi \in C_c^\infty(U)$ be such that $0 \le \psi \le 1$, $\psi = 1$ in $B(x,2r)$. We observe that, when restricted to $\mathrm{supp}(\psi) \subset \mathbb{R}^N \setminus \overline{B_\delta}$, L is a uniformly elliptic operator with bounded coefficients. Therefore, if we consider the operator

$$\tilde{L} := \psi L + (1-\psi)\Delta,$$

then $\tilde{L}$ is a uniformly elliptic operator with bounded coefficients which is globally defined and coincides with L over $B(x,2r)$.

From now on we write $C = C(\delta) > 0$ to indicate a positive constant that depends on the bound of the coefficients of $\tilde{L}$ (and so on the fixed radius δ) and which may vary from line to line.

We apply [46, Theorem 3.16, page 77] to $\eta_n u$ and $\tilde{L}$ and we deduce the existence of $C = C(\delta) > 0$, $\epsilon_0 > 0$ such that, for every $n \in \mathbb{N}_0$ and $\epsilon \le \epsilon_0$

$$\|\nabla(\eta_n u)\|_{L^\infty(\mathbb{R}^N)} \le \epsilon \|\tilde{L}(\eta_n u)\|_{L^\infty(\mathbb{R}^N)} + \frac{C}{\epsilon}\|\eta_n u\|_{L^\infty(\mathbb{R}^N)}. \tag{6.19}$$

If $\tilde{L} = \mathrm{tr}(\tilde{A}D^2) + \langle \tilde{c}, \nabla\rangle + \tilde{b}$, then $\tilde{L}(\eta_n u) = \eta_n \tilde{L}(u) + 2\langle \tilde{A}\nabla\eta_n, \nabla u\rangle + u\,\mathrm{tr}(\tilde{A}D^2\eta_n) + u\tilde{c}\cdot\nabla\eta_n$. Recalling (6.18) we get

$$\begin{aligned}
\|\tilde{L}(\eta_n u)\|_{L^\infty(\mathbb{R}^N)} \le& \|\tilde{\eta}_n L(u)\|_{L^\infty(\mathbb{R}^N)} + \|2\langle \tilde{A}\nabla\eta_n, \nabla u\rangle\|_{L^\infty(\mathbb{R}^N)} \\
&+ \|u\,\mathrm{tr}(\tilde{A}D^2\eta_n)\|_{L^\infty(\mathbb{R}^N)} + \|u\langle \tilde{c}, \nabla\eta_n\rangle\|_{L^\infty(\mathbb{R}^N)} \\
\le& \|Lu\|_{L^\infty(B(x,2r))} + C\frac{M2^n}{r}\|\nabla u\|_{L^\infty(B(x,r_{n+1}))} + C\frac{M4^n}{r^2}\|u\|_{L^\infty(B(x,2r))} \\
&+ C\frac{M2^n}{r}\|u\|_{L^\infty(B(x,2r))} \\
\le& \|Lu\|_{L^\infty(B(x,2r))} + C\frac{2^n}{r}\|\nabla(\eta_{n+1}u)\|_{L^\infty(\mathbb{R}^N)} \\
&+ C\frac{2^n}{r}\left(1 + \frac{2^n}{r}\right)\|u\|_{L^\infty(B(x,2r))}.
\end{aligned}$$

Inserting the last relation in (6.19) we obtain

$$\begin{aligned}\|\nabla(\eta_n u)\|_{L^\infty(\mathbb{R}^N)} \leq & \epsilon\|Lu\|_{L^\infty(B(x,2r))} + \epsilon C\frac{2^n}{r}\|\nabla(\eta_{n+1}u)\|_{L^\infty(\mathbb{R}^N)} \\ & + \left[\epsilon C\frac{2^n}{r}\left(1+\frac{2^n}{r}\right)+\frac{C}{\epsilon}\right]\|u\|_{L^\infty(B(x,2r))}.\end{aligned}$$

The last estimate holds for all $n \in \mathbb{N}_0$ and all $\epsilon \leq \epsilon_0$. Let us set $\gamma = \epsilon C\frac{2^n}{r}$ and let us choose ϵ sufficiently small such that γ is independent of n and $\gamma < \frac{1}{4}$. Then

$$\begin{aligned}\|\nabla(\eta_n u)\|_{L^\infty(\mathbb{R}^N)} \leq & \frac{\gamma r}{C2^n}\|Lu\|_{L^\infty(B(x,2r))} + \gamma\|\nabla(\eta_{n+1}u)\|_{L^\infty(\mathbb{R}^N)} \\ & + \left[\gamma\left(1+\frac{2^n}{r}\right)+\frac{C2^n}{\gamma r}\right]\|u\|_{L^\infty(B(x,2r))} \\ \leq & \frac{Cr}{2^n}\|Lu\|_{L^\infty(B(x,2r))} + \gamma\|\nabla(\eta_{n+1}u)\|_{L^\infty(\mathbb{R}^N)} \\ & + C\left(1+\frac{2^n}{r}\right)\|u\|_{L^\infty(B(x,2r))}.\end{aligned}$$

Multiplying both terms of the inequality by γ^n and summing up for $n \in \mathbb{N}_0$, we get

$$\begin{aligned}\sum_{n=0}^\infty \gamma^n\|\nabla(\eta_n u)\|_{L^\infty(\mathbb{R}^N)} \leq & Cr\sum_{n=0}^\infty\left(\frac{\gamma}{2}\right)^n\|Lu\|_{L^\infty(B(x,2r))} + \sum_{n=0}^\infty \gamma^{n+1}\|\nabla(\eta_{n+1}u)\|_{L^\infty(\mathbb{R}^N)} \\ & + C\sum_{n=0}^\infty\left(\gamma^n+\frac{(2\gamma)^n}{r}\right)\|u\|_{L^\infty(B(x,2r))} \\ = & Cr\|Lu\|_{L^\infty(B(x,2r))} + \sum_{n=0}^\infty \gamma^{n+1}\|\nabla(\eta_{n+1}u)\|_{L^\infty(\mathbb{R}^N)} \\ & + C\left(1+\frac{1}{r}\right)\|u\|_{L^\infty(B(x,2r))}. \qquad (6.20)\end{aligned}$$

The series in (6.20) converge since $\gamma < \frac{1}{4}$ and, by the hypothesis on η_n,

$$\|\nabla(\eta_n u)\|_{L^\infty(\mathbb{R}^N)} \leq C4^n\left(\|u\|_{L^\infty(B(x,2r))} + \|\nabla u\|_{L^\infty(B(x,2r))}\right).$$

Deleting the equal terms in both side of (6.19) we get

$$\|\nabla(\eta_0 u)\|_{L^\infty(\mathbb{R}^N)} \leq C\left[r\|Lu\|_{L^\infty(B(x,2r))} + \left(1+\frac{1}{r}\right)\|u\|_{L^\infty(B(x,2r))}\right].$$

The required claim, then follows once observed that $\|\nabla u\|_{L^\infty(B(x,r))} \leq \|\nabla(\eta_0 u)\|_{L^\infty(\mathbb{R}^N)}$.

□

Let us prove, finally, Theorem 6.2.1 for $\frac{|x|}{\sqrt{t}} > 2$.

Proposition 6.2.13 *There exist $C, c > 0$ such that, if $\frac{|x|}{\sqrt{t}} > 2$,*

$$|\nabla p(t,x,y)| \leq Ct^{-\frac{N}{2}-\frac{1}{2}}|y|^{-\gamma}\left(\frac{|y|}{t^{\frac{1}{2}}}\wedge 1\right)^{-s_1^*}\exp\left(-\frac{c|x-y|^2}{t}\right).$$

Proof. Recalling the scaling property (6.6), we may assume, without loss of generality, $t=1$. Let $x\in\Omega$ with $|x|>2$ and let us fix $y\in\Omega$. Let us observe, preliminarily, that as in Remark 6.2.2, the estimates (6.3) becomes for $|\omega|>1$,

$$|Lp(1,\omega,y)|\leq C|y|^{-\gamma}\left(|y|\wedge 1\right)^{-s_1^*}\exp\left(-c|\omega-y|^2\right). \tag{6.21}$$

Let $r>0$, to be later specified, such that $\overline{B(x,2r)}\subseteq\mathbb{R}^N\setminus\overline{B_1}$. Using Proposition 6.2.12, we have, for some $C>0$

$$\begin{aligned}&\|\nabla p(1,\cdot,y)\|_{L^\infty(B(x,r))}\\&\qquad\leq C\left(r\|Lp(1,\cdot,y)\|_{L^\infty(B(x,2r))}+\left(\frac{1}{r}+1\right)\|p(1,\cdot,y)\|_{L^\infty(B(x,2r))}\right).\end{aligned}$$

We treat separately the cases $|x-y|>1$ and $|x-y|\leq 1$.

In the first one we choose $r=\inf\left\{\frac{|x|-1}{2},\ \frac{|x-y|}{4}\right\}$ and we observe that, if $w\in\overline{B(x,2r)}$ then $|w|>1$ and

$$|w-y|\geq -|w-x|+|x-y|\geq -\frac{|x-y|}{2}+|x-y|=\frac{|x-y|}{2}.$$

Recalling (6.21), this imply, for some (possibly different) constants $C,c>0$,

$$\begin{aligned}\|Lp(1,\cdot,y)\|_{L^\infty(B(x,2r))}&\leq C|y|^{-\gamma}\left(|y|\wedge 1\right)^{-s_1^*}\exp\left(-c|x-y|^2\right),\\ \|p(1,\cdot,y)\|_{L^\infty(B(x,2r))}&\leq C|y|^{-\gamma}\left(|y|\wedge 1\right)^{-s_1^*}\exp\left(-c|x-y|^2\right).\end{aligned}$$

Therefore, since $\frac{1}{4}\leq r\leq\frac{|x-y|}{4}$, up to a modification of the constants involved, we get

$$\begin{aligned}|\nabla p(1,x,y)|&\leq\|\nabla p(1,\cdot,y)\|_{L^\infty(B(x,r))}\\&\leq C\left(r\|Lp(1,\cdot,y)\|_{L^\infty(B(x,2r))}+\left(\frac{1}{r}+1\right)\|p(1,\cdot,y)\|_{L^\infty(B(x,2r))}\right)\\&\leq C\left(|y|\wedge 1\right)^{-s_1^*}\exp\left(-c|x-y|^2\right).\end{aligned}$$

If $|x-y|\leq 1$, we choose $r=\frac{1}{2}$ and we argue as before obtaining a similar estimate. That proves the claim for all the cases.

□

Finally, we can prove the bound in Theorem 6.2.1

(Proof of Theorem 6.2.1) Using Remark 6.2.2, the proof follows by combining together the estimates proved in Propositions 6.2.8, 6.2.11 and 6.2.13.

□

In the following corollary we deduce boundedness properties of e^{tL} and ∇e^{tL}.

Corollary 6.2.14 *Let $f \in L^2(\mathbb{R}^N, d\mu)$. Then $e^{tL}f$ is differentiable in $\mathbb{R}^N \setminus \{0\}$ and satisfies*

$$\nabla e^{tL} f(x) = \int_{\mathbb{R}^N} \nabla p(t,x,y) f(y)\, d\mu,$$
$$|\nabla e^{tL} f(x)| \le \frac{C}{\sqrt{t}} \left(1 \wedge \frac{|x|}{\sqrt{t}}\right)^{-1} e^{ctL}|f|(x),$$

for every $x \in \mathbb{R}^N \setminus \{0\}$. Moreover, e^{tL} and ∇e^{tL} are bounded from the spaces indicated below

$$e^{tL} : L^1(\mathbb{R}^N, \phi d\mu) \to \phi L^\infty(\mathbb{R}^N),$$
$$\nabla e^{tL} : L^1(\mathbb{R}^N, \phi d\mu) \to \frac{\phi}{1 \wedge |x|} L^\infty(\mathbb{R}^N, \mathbb{R}^N),$$

where $\phi(x) = |x|^{-\frac{\gamma}{2}}(1 \wedge |x|)^{-\frac{N}{2}+1+\sqrt{D}}$ and $gL^\infty(\mathbb{R}^N)$ (resp. $gL^\infty(\mathbb{R}^N, \mathbb{R}^N)$) is the set of (vectorial) functions f such that $\|fg^{-1}\|_\infty < \infty$.

Proof. Let $f \in L^2(\mathbb{R}^N d\mu)$. Then

$$e^{tL} f(x) = \int_{\mathbb{R}^N} p(t,x,y) f(y)\, d\mu \tag{6.22}$$

for every $x \in \mathbb{R}^N \setminus \{0\}$. By scaling properties we can assume $t = 1$. By the gradient kernel estimates and by the lower kernel estimates we have

$$\begin{aligned}\nabla p(1,x,y) &\le C(1 \wedge |x|)^{-1}|x|^{-\frac{\gamma}{2}}|y|^{-\frac{\gamma}{2}}[(1 \wedge |x|)(1 \wedge |y|)]^{-\frac{N}{2}+1+\sqrt{D}} \exp\{-m|x-y|^2\}\\ &\le C(1 \wedge |x|)^{-1} p(c,x,y).\end{aligned}$$

Let now $x_0 \in \mathbb{R}^N \setminus \{0\}$ and let $r > 0$ such that $0 \notin \overline{B_r(x_0)}$. For every $x \in B_r(x_0)$, $y \in \mathbb{R}^N \setminus \{0\}$,

$$p(c,x,y) \le C|y|^{-\frac{\gamma}{2}}(1 \wedge |y|)^{-\frac{N}{2}+1+\sqrt{D}} \exp\{-m(|y|^2 - |y|)\}$$

with C and m depending on r and x_0. By observing that $|y|^{2(-\frac{N}{2}+1+\sqrt{D})}$ is integrable near the origin and that the exponential term insures the integrability at infinity, we get that $p(c,x,\cdot)$ is uniformly dominated by a function in $L^2(\mathbb{R}^N, d\mu)$ and therefore it is possible to differentiate under the integral sign in (6.22). It follows that

$$\nabla e^{tL} f(x) = \int_{\mathbb{R}^N} \nabla p(t,x,y) f(y)\, d\mu$$

and, by (6.5),

$$|\nabla e^{tL} f(x)| \le \frac{C}{\sqrt{t}} \left(1 \wedge \frac{|x|}{\sqrt{t}}\right)^{-1} e^{ctL}|f|(x). \tag{6.23}$$

By arguing in a similar way one shows that the above gradient formula holds also for $f \in L^1(\mathbb{R}^N, \phi\, d\mu)$.

Observe now that, by the kernel estimates,

$$\begin{aligned}\left\| \phi\left(\frac{|x|}{\sqrt{t}}\right)^{-1} e^{tL} f(x) \right\|_\infty &\leq C t^{-\frac{\gamma}{2}} \left\| G(mt) * \left(f\phi\left(\frac{\cdot}{\sqrt{t}}\right) |\cdot|^\gamma \right) \right\|_\infty \\ &\leq C t^{-\frac{\gamma}{2}} \|G(mt)\|_\infty \left\| f\phi\left(\frac{\cdot}{\sqrt{t}}\right) |\cdot|^\gamma \right\|_1,\end{aligned}$$

where $G(t)$ is the Gauss-Weierstrass kernel defined by $G(t)(x) := G(t,x) = t^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right)$. Therefore

$$e^{tL} : L^1(\mathbb{R}^N, \phi d\mu) \to \phi L^\infty(\mathbb{R}^N).$$

The last part of the claim follows then from (6.23).

$\square$

Chapter 7

Applications and examples

In this chapter we present some applications to some special cases, including Schödinger operators and homogeneous operators with unbounded coefficients studied in [52]. In Section 7.4 we derive a new proof of the Gegenbauers generalization of the Poisson integral representation of Bessel functions $I_{\nu+n}$, where $\nu = \left(\frac{N-2}{2}\right)$ and $n \in N_0$.

The Chapter is mainly based on [50].

7.1 Schrödinger operators with inverse square potential

If $a = 1$, $c = 0$ then $\gamma = 0$, $D = b + \left(\frac{N-2}{2}\right)^2 \geq 0$ and $L = \Delta - \frac{b}{|x|^2}$ is the Schrödinger operator with inverse square potential.

Kernel estimates for the Schrödinger operator have already been widely investigated in the literature. Using Theorem 3.5.3 and Theorem 4.2.2 we can obtain the sharp bounds of [58] including the critical case $D = b + \left(\frac{N-2}{2}\right)^2 = 0$.

Theorem 7.1.1 *The heat kernel p of L, with respect to the Lebesgue measure, satisfies*

$$p(t,x,y) \simeq t^{-\frac{N}{2}} \left[\left(\frac{|x|}{t^{\frac{1}{2}}} \wedge 1\right)\left(\frac{|y|}{t^{\frac{1}{2}}} \wedge 1\right)\right]^{-\frac{N}{2}+1+\sqrt{D}} \exp\left(-\frac{c|x-y|^2}{t}\right).$$

The upper bound holds with any $c < \frac{1}{4}$ and can be improved to

$$p(t,x,y) \leq Ct^{-\frac{N}{2}} \left(1 + \frac{|x-y|^2}{4t}\right)^{\alpha} \left[\left(\frac{|x|}{|z|^{\frac{1}{2}}} \wedge 1\right)\left(\frac{|y|}{|z|^{\frac{1}{2}}} \wedge 1\right)\right]^{-\frac{N}{2}+1+\sqrt{D}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

where $\alpha = \frac{N}{2}$ if $b \leq 0$ and $\alpha = 1 + \sqrt{D}$ if $b > 0$.

Using the results of Chapter 5 we obtain sharp bounds for the Green function (also in the critical case $D = b + \left(\frac{N-2}{2}\right)^2 = 0$) which we state for $N > 2$.

Theorem 7.1.2 *For $N > 2$, the Green function G_λ, with respect to the Lebesgue measure, satisfies the estimates*

(i) if $D > 0$, $\lambda \geq 0$,

$$G_\lambda(x,y) \simeq e^{-c\sqrt{\lambda}|x-y|}\,|x-y|^{2-N}\left(1 \wedge \frac{|x||y|}{|x-y|^2}\right)^{\sqrt{D}-\frac{N-2}{2}}. \tag{7.1}$$

(ii) If $D = 0$, and $\lambda > 0$,

$$G_\lambda(x,y) \simeq \begin{cases} e^{-c\sqrt{\lambda}|x-y|}\left((|x||y|) \wedge \frac{|x-y|}{\lambda}\right)^{\frac{2-N}{2}} & , \quad \text{if} \quad \sqrt{\lambda}\,|x-y| \geq 1; \\ |x-y|^{2-N} \vee \left((|x||y|)^{\frac{2-N}{2}}(1-\log(\sqrt{\lambda}\,|x-y|)\right) & , \quad \text{if} \quad \sqrt{\lambda}\,|x-y| < 1. \end{cases}$$

Remark 7.1.3 *Note that $G_\lambda \to 0, \infty$ in (7.1) as $|x| \to 0$ and $y \neq 0$ fixed (or conversely), according to $D > (N-2)^2/4$ or $D < (N-2)^2/4$, that is when $b > 0$ or $b < 0$. We refer also to [61, Theorem 3.11] for the local behavior of the Green function when $D > 0$. The above estimates in the critical case $D = 0$ seem to be new.*

7.2 Purely second order operators

If $b = c = 0$, then $D = \left(\frac{N-1-a}{2a}\right)^2$, $\gamma = \frac{(N-1)(1-a)}{a}$ and

$$L = \Delta + (a-1)\sum_{i,j=1}^{N} \frac{x_i x_j}{|x|^2} D_{ij}.$$

Using Theorem 4.2.2 we deduce the following kernel estimates.

Theorem 7.2.1 *Let $\Omega = \mathbb{R}^N \setminus \{0\}$. The heat kernel p of L, with respect to the measure $d\mu = |y|^{\frac{(N-1)(1-a)}{a}}dy$, satisfies*

$$p(t,x,y) \simeq t^{-\frac{N}{2}}|x|^{-\frac{(N-1)(1-a)}{2a}}|y|^{-\frac{(N-1)(1-a)}{2a}}\left[\left(\frac{|x|}{t^{\frac{1}{2}}} \wedge 1\right)\left(\frac{|y|}{t^{\frac{1}{2}}} \wedge 1\right)\right]^{-\frac{N}{2}+1+\left|\frac{N-1-a}{2a}\right|} \times \exp\left(-\frac{c|x-y|^2}{t}\right).$$

Using Theorem 5.1.1, 5.2.1 and Theorem 5.2.3 we obtain sharp bounds for the Green function and the resolvent operator.

Observe that the condition $D \leq \left(\frac{N-2}{2}\right)^2$ becomes $a \geq 1$ if $N \geq 3$ and $a = 1$ if $N = 2$. We omit the case $N = 2$ and $\lambda > 0$ to focus the estimates near the origin but let $\lambda \geq 0$ if $N \geq 3$ in order to treat the critical case $a = N-1$ (when $N = 2$ the critical case $a = 1$ corresponds to the Laplacian).

Theorem 7.2.2 *Let $\Omega = \mathbb{R}^N \setminus \{0\}$. For $(x,y) \in \Omega \times \Omega$ with $x \neq y$, the Green function G_λ, with respect to the measure $d\mu = |y|^{\frac{(N-1)(1-a)}{a}} dy$, satisfies the estimates*

(i) if $N > 2$, $\lambda \geq 0$ and $a \neq N-1$

$$(|x||y|)^{\frac{(N-1)(1-a)}{2a}} G_\lambda(x,y) \simeq e^{-c\sqrt{\lambda}|x-y|}\, |x-y|^{2-N} \left(1 \wedge \frac{|x||y|}{|x-y|^2}\right)^{\left|\frac{N-1-a}{2a}\right| - \frac{N-2}{2}}.$$

(ii) If $N > 2$, $a = N-1$ and $\lambda > 0$:

For $\sqrt{\lambda}\,|x-y| \geq 1$

$$(|x||y|)^{\frac{2-N}{2}} G_\lambda(x,y) \simeq e^{-c\sqrt{\lambda}|x-y|} \left((|x||y|) \wedge \frac{|x-y|}{\sqrt{\lambda}}\right)^{\frac{2-N}{2}}.$$

For $\sqrt{\lambda}\,|x-y| < 1$

$$(|x||y|)^{\frac{2-N}{2}} G_\lambda(x,y) \simeq |x-y|^{2-N} \vee \left((|x||y|)^{\frac{2-N}{2}} (1 - \log(\sqrt{\lambda}\,|x-y|))\right).$$

(iii) If $N = 2$, $a \neq 1$, $\lambda = 0$

$$(|x||y|)^{\frac{1-a}{2a}} G_0(x,y) \simeq \begin{cases} \left(\frac{|x||y|}{|x-y|^2}\right)^{\left|\frac{1-a}{2a}\right|}, & \text{if } \frac{|x-y|^2}{|x||y|} \geq 1; \\ 1 - \log\left(\frac{|x-y|^2}{|x||y|}\right), & \text{if } \frac{|x-y|^2}{|x||y|} \leq 1. \end{cases}$$

The Green function with respect to the Lebesgue measure is $G'_\lambda(x,y) = |y|^\gamma G_\lambda(x,y)$. It follows that $G'_\lambda(x,y) \simeq |x|^{\left(1-\frac{N-1}{a}\right)_+}$ as $|x| \to 0$ for $y \neq 0$ fixed and $G'_\lambda(x,y) \simeq |y|^{2-N+\left(\frac{N-1}{a}-1\right)_+}$ as $|y| \to 0$ for $x \neq 0$ fixed. The joint behavior in $|x||y|$ is more complicated and when $N = 2$ or $a = N-1$ logarithmic terms also appear.

7.3 Operators with unbounded coefficients

Consider operators of the form

$$\mathcal{S} = |x|^\alpha \Delta + \tilde{c}|x|^{\alpha-1} \frac{x}{|x|} \cdot \nabla - \tilde{b}|x|^{\alpha-2}, \quad x \in \mathbb{R}^N \setminus \{0\}$$

with $\alpha \neq 2$, $\tilde{c}, \tilde{b} \in \mathbb{R}$. Note that α can be positive or negative but the case $\alpha = 2$ is special and easier, see [52], and will be not treated here. Generation properties and domain characterization have already been studied in [52]. Here we follow the same method as in [57], where upper bounds are proved, to deduce lower bounds. Since the proofs are similar we do not repeat them and refer the reader to the above paper. The crucial point consists in observing that the operators previously studied and the last ones are related by an isometry in $L^p(\mathbb{R}^N)$.

Lemma 7.3.1 *Let $1 \le p \le \infty$, $J_p : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ given by*

$$J_p u(x) := \left|\frac{\alpha}{2} - 1\right|^{\frac{1}{p}} |x|^{-\frac{N\alpha}{2p}} u(|x|^{-\frac{\alpha}{2}} x).$$

Then J_p is an isometry in $L^p(\mathbb{R}^N)$ and

$$\begin{aligned} J_p^{-1}\big(|x|^\alpha \Delta + \tilde{c}|x|^{\alpha-1}\frac{x}{|x|} \cdot \nabla - \tilde{b}|x|^{\alpha-2}\big) J_p \\ = \Delta + (a-1)\sum_{i,j=1}^N \frac{x_i x_j}{|x|^2} D_{ij} + c\frac{x}{|x|^2} \cdot \nabla - b|x|^{-2}, \end{aligned}$$

where

$$\begin{aligned} a &= \left(\frac{\alpha}{2} - 1\right)^2; \quad b = \tilde{b} + \frac{N\alpha}{2p}\left(N - 2 + \tilde{c} - \frac{N\alpha}{2p}\right); \\ c &= (N-1)\left(\left(\frac{\alpha}{2} - 1\right)^2 - 1\right) + \left(1 - \frac{\alpha}{2}\right)\left[\tilde{c} - \alpha + \frac{N\alpha}{2}\left(1 - \frac{2}{p}\right)\right]. \end{aligned}$$

From Theorem 4.2.2 we deduce sharp kernel estimates for the heat kernel $p_{\mathcal{S}}$ associated to $\mathcal{S}$ which we write this time with respect to the Lebesgue measure.

Theorem 7.3.2

$$\begin{aligned} p_{\mathcal{S}}(t,x,y) \simeq t^{-\frac{N}{2}} |x|^{-\frac{\tilde{c}-\alpha}{2} - \frac{N\alpha}{4}} |y|^{\frac{\tilde{c}-\alpha}{2} - \frac{N\alpha}{4}} \left[\left(\frac{|x|^{\frac{2-\alpha}{2}}}{t^{\frac{1}{2}}} \wedge 1\right)\left(\frac{|y|^{\frac{2-\alpha}{2}}}{t^{\frac{1}{2}}} \wedge 1\right)\right]^{-\frac{N}{2}+1+\left|1-\frac{\alpha}{2}\right|^{-1}\sqrt{\tilde{D}}} \\ \times \exp\left(-\frac{c\big||x|^{-\frac{\alpha}{2}}x - |y|^{-\frac{\alpha}{2}}y\big|^2}{t}\right). \end{aligned}$$

Similarly, by Theorems 5.1.1, 5.1.1, 5.2.1 and 5.2.3, we can deduce the estimates for the Green function $G_\lambda^{\mathcal{S}}(x,y) = \int_0^\infty e^{-\lambda t} p_{\mathcal{S}}(t,x,y)\,dt$. We state them when $\tilde{D} = \tilde{b} + \big(\frac{N-2+\tilde{c}}{2}\big)^2 > 0$, $N > 2$.

Theorem 7.3.3 *Let $\lambda > 0$, $\Omega = \mathbb{R}^N \setminus \{0\}$ and let us suppose $\tilde{D} > 0$, $N > 2$. For $(x,y) \in \Omega \times \Omega$ with $x \ne y$, the Green function G_λ, with respect to the Lebesgue measure, satisfies the estimates*

$$\begin{aligned} G_\lambda(x,y)^{\mathcal{S}} \simeq & e^{-c\sqrt{\lambda}\,|x|x|^{-\frac{\alpha}{2}} - y|y|^{-\frac{\alpha}{2}}|} \frac{|x|^{-\frac{\alpha N}{2p}} |y|^{-\frac{\alpha N}{2p'} + \gamma\left(1-\frac{\alpha}{2}\right)}}{\left(|x|^{-\frac{\alpha}{2}+1}|y|^{-\frac{\alpha}{2}+1}\right)^s \, |x|x|^{-\frac{\alpha}{2}} - y|y|^{-\frac{\alpha}{2}}|^{2\sqrt{\left(1-\frac{\alpha}{2}\right)^{-2}\tilde{D}}}} \\ & \times \left(1 \wedge \frac{|x|x|^{-\frac{\alpha}{2}} - y|y|^{-\frac{\alpha}{2}}|^2}{|x|^{-\frac{\alpha}{2}+1}|y|^{-\frac{\alpha}{2}+1}}\right)^{\sqrt{\left(1-\frac{\alpha}{2}\right)^{-2}\tilde{D}} - \frac{N-2}{2}} \end{aligned}$$

where

$$s = \left(1 - \frac{\alpha}{2}\right)^{-1}\left(\frac{N-2+\tilde{c} - \frac{N\alpha}{p}}{2}\right) - \left|1 - \frac{\alpha}{2}\right|^{-1}\sqrt{\tilde{b} + \left(\frac{N-2+\tilde{c}}{2}\right)^2}$$

and

$$\gamma = \frac{N-1+(N-1)\left(\left(\frac{\alpha}{2}-1\right)^2-1\right)+\left(1-\frac{\alpha}{2}\right)\left[\tilde{c}-\alpha+\frac{N\alpha}{2}\left(1-\frac{2}{p}\right)\right]}{\left(\frac{\alpha}{2}-1\right)^2}-N+1.$$

PROOF. The proof follows by the equality

$$G^{\mathcal{S}}_\lambda(x,y)=\left(1-\frac{\alpha}{2}\right)|x|^{-\frac{\alpha N}{2p}}|y|^{-\frac{\alpha N}{2p'}+\gamma\left(1-\frac{\alpha}{2}\right)}G_\lambda(x|x|^{-\frac{\alpha}{2}},y|y|^{-\frac{\alpha}{2}})$$

and by observing that, since

$$\frac{N-1+c-a}{2a}=\frac{N-2+\tilde{c}-\frac{N\alpha}{p}}{2\left(1-\frac{\alpha}{2}\right)},$$

we get

$$D=\left(1-\frac{\alpha}{2}\right)^{-2}\tilde{D}$$

and the number s_1 for the operator L becomes

$$s=\left(1-\frac{\alpha}{2}\right)^{-1}\left(\frac{N-2+\tilde{c}-\frac{N\alpha}{p}}{2}\right)-\left|1-\frac{\alpha}{2}\right|^{-1}\sqrt{\tilde{b}+\left(\frac{N-2+\tilde{c}}{2}\right)^2}.$$

□

7.4 A special case

Theorem 4.2.7 provided us a complete decomposition of the N-dimensional kernel in terms of its one-dimensional counterparts

$$p(t,x,y)=\frac{1}{2at}(r\rho)^{-\frac{N-1+c-a}{2a}}\exp\left\{-\frac{r^2+\rho^2}{4at}\right\}\sum_{n=0}^{\infty}I_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right)\mathbb{Z}^{(n)}_{\omega}(\eta) \tag{7.2}$$

where p is the heat kernel of L with respect to the measure $d\mu=|y|^\gamma\,dy$, $x=r\omega$, $y=\rho\eta$, $r,\ \rho>0$, $|\omega|=|\eta|=1$, $(-\lambda_n)_{n\in N_0}$ are the eigenvalues of Δ_0 and $D_n=\frac{b+\lambda_n}{a}+\left(\frac{N-1+c-a}{2a}\right)^2$.

Setting $a=1$, and $b=c=0$, the operator L becomes the Laplace operator and $D_n=\left(\frac{N-2}{2}\right)^2+\lambda_n=\left(\frac{N-2}{2}+n\right)^2$. Inserting in (7.2) the expression of the Gauss-Weierstass kernel of Δ we find

$$\sum_{n=0}^{\infty}I_{\left(\frac{N-2}{2}+n\right)}\left(\frac{r\rho}{2t}\right)\mathbb{Z}^{(n)}_{\eta}(\omega)=\frac{1}{(4\pi)^{\frac{N}{2}}}\left(\frac{r\rho}{4t}\right)^{\frac{N-2}{2}}\exp\left\{\frac{r\rho}{2t}\,\omega\cdot\eta\right\} \tag{7.3}$$

Formula (7.3) allows us to write explicitly the heat kernel of L in some other special cases.

Proposition 7.4.1 *Assume that $a = 1$ and $b + \frac{c^2}{4} + \frac{c}{2}(N-2) = 0$. Then $\gamma = c$ and*

$$p(t,x,y) = \frac{1}{(4\pi t)^{\frac{N}{2}}} |x|^{-\frac{c}{2}} |y|^{-\frac{c}{2}} \exp\left\{-\frac{|x-y|^2}{4t}\right\}$$

where p is the heat kernel of L with respect to the measure $d\mu = |y|^c\, dy$.

Proof. It is enough to observe that, under the assumption on the parameters a, b, c,

$$D_n = \left(\frac{N-2}{2}\right)^2 + \lambda_n = \left(\frac{N-2}{2} + n\right)^2$$

is the same as for the Laplacian and therefore (7.3) holds. Inserting this in (7.2), the proof follows. □

Note that the parameter c is unrestricted but b ranges from $-\infty$ to $(N-2)^2/4$, attained when $c = 2 - N$.

We point out that the same result can be proved in a more direct way. With this choice of parameters, given $u, v \in D(\tilde{\mathfrak{a}})$ and setting $u = u_1|x|^{-\frac{c}{2}}$ and $v = v_1|x|^{-\frac{c}{2}}$ the form becomes

$$\begin{aligned}
\tilde{\mathfrak{a}}(u,v) &= \int_{\mathbb{R}^N} \left(u_r\overline{v}_r + \frac{\nabla_\tau u\, \nabla_\tau \overline{v}}{r^2} + \frac{b}{r^2}u\overline{v}\right)|x|^c\, dx \\
&= \int_{\mathbb{R}^N} \left[(u_1)_r\overline{(v_1)}_r + \frac{\nabla_\tau u_1\, \nabla_\tau \overline{v_1}}{|x|^2} + \frac{b + \frac{c^2}{4} + \frac{c}{2}(N-2)}{|x|^2}u_1\overline{v_1}\right] dx \\
&= \int_{\mathbb{R}^N} \left[(u_1)_r\overline{(v_1)}_r + \frac{\nabla_\tau u_1\, \nabla_\tau \overline{v_1}}{|x|^2}\right] dx = \int_{\mathbb{R}^N} \nabla u_1\, \nabla \overline{v_1}\, dx.
\end{aligned}$$

This shows that, with the isometry

$$J : L^2(\mathbb{R}^N, dx) \to L^2_\mu, \quad Jv = v|x|^{-\frac{c}{2}},$$

the equality $J^{-1}LJ = \Delta$ holds, hence $J^{-1}e^{tL}J = e^{t\Delta}$ and the heat kernel of L is readily obtained by that of the Laplacian.

□

We come back, now, to formula (7.2) and define the function

$$q\left(\frac{r\rho}{2at}, \omega, \eta\right) := \sum_{n=0}^{\infty} I_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right) \mathbb{Z}_\eta^{(n)}(\omega). \tag{7.4}$$

The last sum, recalling Proposition 4.2.6, converges uniformly on compact sets of $\mathbb{R}^N$. We observe that, fixing the values $\frac{r\rho}{2at}$ and recalling property (i) in Proposition B.2.2, the function q is invariant under the action of orthogonal transformations i.e.

$$q\left(\frac{r\rho}{2at}, T\omega, T\eta\right) = q\left(\frac{r\rho}{2at}, \omega, \eta\right)$$

for every ω, $\eta \in \mathbb{S}^{N-1}$ and $T \in O(N)$. Therefore, recalling Definition B.3.1, q is, as a function of ω, a zonal function of pole η (we refer the reader to section B.3 in Appendix B for the basic properties about zonal functions). If we denote by $\phi\left(\frac{r\rho}{2at}, \cdot\right)$ the profile function of $q(\frac{r\rho}{2at}, \cdot, \eta)$ (see formula (B.8)), then

$$q\left(\frac{r\rho}{2at}, \omega, \eta\right) = \phi\left(\frac{r\rho}{2at}, \eta \cdot \omega\right).$$

Using Proposition B.3.3 we can expand $q\left(\frac{r\rho}{2at}, \omega, \eta\right)$ in zonal harmonics obtaining

$$q\left(\frac{r\rho}{2at}, \omega, \eta\right) = \sum_{n=0}^{\infty} d_n\left(\frac{r\rho}{2at}\right) \mathbb{Z}_{\eta}^{(n)}(\omega), \tag{7.5}$$

where the coefficients $d_n\left(\frac{r\rho}{2at}\right)$ are given by

$$d_n\left(\frac{r\rho}{2at}\right) = \frac{|S^{N-2}|}{P_n^{\frac{N-2}{2}}(1)} \int_{-1}^{1} \phi\left(\frac{r\rho}{2at}, s\right) P_n^{\frac{N-2}{2}}(s)\left(1-s^2\right)^{\frac{N-3}{2}} ds.$$

Thus, comparing (7.5) with (7.4), we derive

$$I_{\sqrt{D_n}}\left(\frac{r\rho}{2at}\right) = \frac{|S^{N-2}|}{P_n^{\frac{N-2}{2}}(1)} \int_{-1}^{1} \phi\left(\frac{r\rho}{2at}, s\right) P_n^{\frac{N-2}{2}}(s)\left(1-s^2\right)^{\frac{N-3}{2}} ds. \tag{7.6}$$

The last formula gives an integral representation of the Modified Bessel function $I_{\sqrt{D_n}}$ in terms of the Gegenbauer polynomials and the profile function ϕ. Using (7.3) we can provide a new proof of the Gegenbauers generalization of the Poisson integral representation of Bessel functions $I_{\nu+n}$ where $\nu = \left(\frac{N-2}{2}\right)$ and $n \in N_0$, (see also [80, Section 3.32, page 50]).

Proposition 7.4.2 *For every $n \in N_0$, setting $C_n := \frac{|S^{N-2}|}{4\pi^{\frac{N}{2}} P_n^{\frac{N-2}{2}}(1)}$, one has for every $x > 0$*

$$I_{\left(\frac{N-2}{2}+n\right)}(x) = C_n\left(\frac{x}{2}\right)^{\frac{N-2}{2}} \int_{-1}^{1} e^{xs} P_n^{\frac{N-2}{2}}(s)\left(1-s^2\right)^{\frac{N-3}{2}} ds.$$

Proof. Let us fix $\eta \in \mathbb{S}^{N-1}$ and let us set, in equation (7.3), $x := \frac{r\rho}{2t} > 0$. It follows that the zonal function

$$q(x, \omega, \eta) = \sum_{n=0}^{\infty} I_{\left(\frac{N-2}{2}+n\right)}(x)\, \mathbb{Z}_{\eta}^{(n)}(\omega)$$

has profile function

$$\phi(x, s) = \frac{1}{4\pi^{\frac{N}{2}}}\left(\frac{x}{2}\right)^{\frac{N-2}{2}} e^{xs}.$$

Using (7.6) we obtain

$$I_{\left(\frac{N-2}{2}+n\right)}(x) = \frac{|S^{N-2}|}{P_n^{\frac{N-2}{2}}(1)} \frac{1}{4\pi^{\frac{N}{2}}}\left(\frac{x}{2}\right)^{\frac{N-2}{2}} \int_{-1}^{1} e^{xs} P_n^{\frac{N-2}{2}}(s)\left(1-s^2\right)^{\frac{N-3}{2}} ds.$$

$\square$

Appendix A

A brief introduction to Riemannian Geometry

This Appendix is devoted to the presentation of the main notions and results of Riemannian geometry used throughout the dissertation. Many fundamental theorems will be quoted without proofs since they are available in classical textbooks on Riemannian geometry. A good survey on the subject can be found in [37, Chapter 1] and [7, Chapter 1] whereas for a deeper discussion on the topic and for the proofs of the results presented here, we refer to [13], [16], [17], [66], [39] and [68].

A.1 Manifold Theory

A.1.1 Differentiable Manifolds

Definition A.1.1 *A* ***manifold*** $\mathcal{M}$ *of dimension* $N = dim\mathcal{M}$ *is a connected Hausdorff topological space such that each point of* $\mathcal{M}$ *has a neighbourhood homeomorphic to* $\mathbb{R}^N$*.*

A *local chart* on $\mathcal{M}$ is a pair (U, ϕ) where U is an open set of $\mathcal{M}$ and $\phi : U \to D,\ p \mapsto \phi(p) = (x_1(p), \ldots, x_N(p))$ is a homeomorphism of U onto an open set D of $\mathbb{R}^N$. $(x_1(p), \ldots, x_N(p))$ are the *local coordinates* of $p \in U$ related to the given chart (U, ϕ) and when needed, in order to emphasize the role of the local coordinates, we will often write $(U, \phi, (x_i))$ to refer to (U, ϕ).

An *atlas* is a family $(U_i, \phi_i)_{i \in I}$ of charts for which $(U_i)_{i \in I}$ constitutes an open covering of $\mathcal{M}$.
$(U_i)_{i \in I}$ is called *differentiable* if all changes of coordinates are C^∞ i.e. for any choose of local charts (U, ϕ) and (V, ψ) with $U \cap V \neq \emptyset$, the map $\phi \circ \psi^{-1} : \psi(U \cap V) :\to \phi(U \cap V)$ is a diffeomorphism of class C^∞. Two differentiable atlases are said to be *equivalent* if their union is again a differentiable atlas.

By definition, a *differentiable manifold* is a manifold together with an equivalence class of differentiable atlases.

Remark A.1.2

(i) Since any differentiable atlas is contained in a maximal one, in order to construct a differentiable manifolds, it is sufficient to assign a differentiable atlas.

(ii) In all the above definitions one can require weaker differentiability property than C^∞ or even different type of regularity on the chart transitions; for example one can require them to be continuous, affine, algebraic or real analytic and thereby define a class of manifolds with that particular structure. Moreover, changing $\mathbb{R}^N$ with $\mathbb{C}^N$ and considering holomorphic chart transitions, one has the notion of Complex Manifold.

Example A.1.3

(i) $\mathbb{R}^N$ is a N-dimensional differentiable manifold and a differentiable atlas is given by the global chart $\big(\mathbb{R}^N, id\big)$.

(ii) The sphere $\mathbb{S}^{N-1} = \big\{x \in \mathbb{R}^N \, : \, |x| = 1\big\}$ is a differentiable manifold of dimension $N-1$. Charts can be given by means of the stereographic projections as follows: on $U_1 = \mathbb{S}^{N-1} \setminus \{0, \dots, 0, 1\}$ we define

$$f_1(x_1, \dots, x_N) = \left(\frac{x_1}{1 - x_N}, \dots, \frac{x_{N-1}}{1 - x_N}\right)$$

and on $U_2 = \mathbb{S}^{N-1} \setminus \{0, \dots, 0, -1\}$

$$f_2(x_1, \dots, x_N) = \left(\frac{x_1}{1 + x_N}, \dots, \frac{x_{N-1}}{1 + x_N}\right).$$

(iii) Let $\mathcal{M}$ be a differentiable manifold. Any open subset U of $\mathcal{M}$ is again a differentiable manifold with $dim\, U = dim\, \mathcal{M}$.

(iv) If $\mathcal{M}$ and $\mathcal{N}$ are differentiable manifolds, the Cartesian product $\mathcal{M} \times \mathcal{N}$ naturally carries the structure of a differentiable manifold. Namely, if $(U_i, \phi_i)_{i \in I}$ and $(V_j, \psi_j)_{j \in J}$ are differentiable atlases for $\mathcal{M}$ and $\mathcal{N}$ respectively, then $\Big(U_i \times V_j, (\phi_i, \psi_j)\Big)_{(i,j) \in I \times J}$ is a differentiable atlas for $\mathcal{M} \times \mathcal{N}$.

A map $f : \mathcal{M} \to \mathcal{M}'$ between two differentiable manifolds $\mathcal{M}$ and $\mathcal{M}'$ is called *differentiable* (respectively differentiable at $p \in \mathcal{M}$) if for every choice of charts (U, ϕ) of $\mathcal{M}$ and (V, ψ) of $\mathcal{M}'$, all the maps $\psi \circ f \circ \phi^{-1}$ are C^∞ (respectively differentiable at $\phi(p)$). Analogous definitions hold for f to be of class C^k.

f is called a *diffeomorphism* if it is bijective with differentiable inverse function.

If $\mathcal{M}' = \mathbb{R}^N$ and U is an open subset of $\mathcal{M}$, we denote with $\mathcal{F}(U) := C^\infty(\mathcal{M})$ the set of differentiable functions $f : U \to \mathbb{R}^N$.

A.1.2 Tangent space

Let $p \in \mathcal{M}$ and let $\mathcal{F}(p)$ be the set of functions which are defined and C^∞ in a neighbourhood of p .

Definition A.1.4 *A* ***tangent vector*** X_p *at* p *is a map* $X_p : \mathcal{F}(p) \to \mathbb{R},\ f \mapsto X_p(f)$ *which satisfies for every* $\lambda, \mu \in \mathbb{R}$ *and* $f, g \in \mathcal{F}(p)$*:*

(i) $X_p(\lambda f + \mu g) = \lambda X_p(f) + \mu X_p(g)$*;*

(ii) $X_p(fg) = f(P)X_p(g) + g(P)X_p(f)$*.*

One can easily shows, starting from the definition, that $X_p(f) = 0$ for every constant function f and that, in particular, if $f, g \in \mathcal{F}(p)$ coincide on a neighbourhood of p, then $X_p(f) = X_p(g)$.

Definition A.1.5 *The* ***tangent space*** $T_p(M)$ *at* p *is the set of all the tangent vectors at* p *and it has a natural vector space structure.*

Given a local chart $(U, \phi, (x_i))$, with $p \in U$, we define the *coordinate tangent vectors*

$$\left(\frac{\partial}{\partial x_i}\right)_p : \mathcal{F}(p) \to \mathbb{R},\ f \mapsto \frac{\partial f}{\partial x_i}(p) := \frac{\partial (f \circ \phi^{-1})}{\partial x_i}(\phi^{-1}(p)).$$

Proposition A.1.6 *Let* $\mathcal{M}$ *be a differentiable manifolds with* $dim\,\mathcal{M} = N$*,* $p_0 \in \mathcal{M}$ *and let* $(U, \phi, (x_i))$ *be a local chart with* $p_0 \in U$*. The following properties hold.*

(i) For every $X_{p_0} \in T_{p_0}(\mathcal{M})$*,* $X_{p_0} = \displaystyle\sum_{i=1}^{N} X_{p_0}(x_i) \left(\frac{\partial}{\partial x_i}\right)_{p_0}$.

(ii) The coordinate tangent vectors $\left(\frac{\partial}{\partial x_1}\right)_{p_0}, \ldots, \left(\frac{\partial}{\partial x_N}\right)_{p_0}$ *form a basis of* $T_{p_0}(\mathcal{M})$*.*

(iii) $dim\,T_{p_0}(\mathcal{M}) = dim\,\mathcal{M}$

Proof. Let $X_{p_0} \in T_{p_0}(\mathcal{M})$ and $f \in \mathcal{F}(p_0)$. Due to the localness of the tangent vector X_{p_0}, we may assume, for simplicity, $\phi(U) = B_r(x_0)$, where $r > 0$ and $x_0 = (x_1^0, \ldots, x_N^0) = \phi(p_0)$. Setting $\tilde{f} = f \circ \phi^{-1}$, we have for every $x \in B_r(x_0)$

$$\begin{aligned}\tilde{f}(x) - \tilde{f}(x_0) &= \int_0^1 \frac{d}{dt}\tilde{f}\,(x_0 + t(x - x_0))\;dt \\ &= \sum_{i=1}^{N}(x_i - x_i^0)\int_0^1 \frac{\partial \tilde{f}}{\partial x_i}\big(x_0 + t(x - x_0)\big)\,dt = \sum_{i=1}^{N}(x_i - x_i^0)\tilde{f}_i(x),\end{aligned}$$

where $\tilde{f}_i = \int_0^1 \frac{\partial \tilde{f}}{\partial x_i}(x_0 + t(\,\cdot\, - x_0))\, dt \in C^\infty(B_r(x_0))$ and $\tilde{f}_i(x_0) = \frac{\partial \tilde{f}}{\partial x_i}(x_0)$. Then, defining $f_i = \tilde{f}_i \circ \phi$, one has $f_i \in \mathcal{F}(p_0)$, $f_i(p_0) = \frac{\partial f}{\partial x_i}(p_0)$ and

$$f(p) - f(p_0) = \sum_{i=1}^{N} \Big(x_i(p) - x_i(p_0)\Big) f_i(p), \qquad p \in \mathcal{M}.$$

Applying, now, X_{p_0} to the above equation we get $X_{p_0}(f) = \displaystyle\sum_{i=1}^{N} X_{p_0}(x_i) \frac{\partial f}{\partial x_i}(p_0)$ which proves (i). (ii) and (iii) are now immediate consequences of (i) just proved and of $\frac{\partial x_i}{\partial x_j} = \delta_i^j$.

□

Remark A.1.7 *In the euclidean case, for each $p \in \mathbb{R}^N$, there is a canonical linear isomorphism from $\mathbb{R}^N$ to $T_p(\mathbb{R}^N)$ that, in terms of Cartesian coordinates, sends $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ to $X_p = \sum_i x_i \left(\frac{\partial}{\partial x_i}\right)_p \in T_p(\mathbb{R}^N)$. So, in what follows, we will always make the identification $T_p(\mathbb{R}^N) \equiv \{p\} \times \mathbb{R}^N \equiv \mathbb{R}^N$.*

Let U be an open subset of $\mathcal{M}$ and let us define $T(U) = \dot{\bigcup}_{p \in U} T_p(\mathcal{M})$. $T(U)$ has a natural vector fiber bundle structure and $T(\mathcal{M})$ is the *tangent bundle space* of $\mathcal{M}$.

Definition A.1.8 *A **vector field** X over U is an application*

$$X : U \to T(U),\ p \mapsto X(p) = X_p \in T_p(\mathcal{M}).$$

X is called differentiable if, for every $f \in \mathcal{F}(U)$, the map

$$X(f) : U \to \mathbb{R},\ p \mapsto X(f)(p) := X_p(f)$$

is differentiable. The set $\mathfrak{X}(U)$ of all the differentiable vector fields on U is a module over the ring $\mathcal{F}(U)$ and every $X \in \mathfrak{X}(U)$ can be equivalently identified with the *derivation*

$$X : \mathcal{F}(U) \to \mathcal{F}(U),\ f \mapsto X(f)$$

which satisfies, for every $f, g \in \mathcal{F}$,

(i) $X(\lambda f + \mu g) = \lambda X(f) + \mu X(g)$;

(ii) $X(fg) = fX(g) + gX(f)$.

This interpretation allows to consider a Lie Bracket operation on $\mathfrak{X}(U)$ defined, for every $X, Y \in \mathfrak{X}(U)$, by $[X, Y] = X \circ Y - Y \circ X$.

Let $(U, \phi, (x_i))$ be a local chart of $\mathcal{M}$ and let us define the coordinate vector fields

$$\frac{\partial}{\partial x_i} : U \to T(U),\ p \mapsto \frac{\partial}{\partial x_i}(p) = \left(\frac{\partial}{\partial x_i}\right)_p \in T_p(\mathcal{M}).$$

Proposition A.1.6 shows that X is given in local coordinates by $X = \sum_{i=1}^N X(x_i) \frac{\partial}{\partial x_i}$.

A.1.3 Differentiable maps

Let $\mathcal{M}$ be a N-dimensional differentiable manifold, $p \in \mathcal{M}$ and $f \in \mathcal{F}(\mathcal{M})$.

Definition A.1.9 *The* ***differential of f in p*** *is the form $(df)_p \in T_p(\mathcal{M})^*$ defined by*

$$(df)_p : T_p(\mathcal{M}) \to \mathbb{R},\ X_p \mapsto (df)_p(X_p) := X_p(f).$$

If $(U, \phi, (x_i))$ is a local chart, with $p \in U$, the differentials $(dx_1)_p, \ldots, (dx_N)_p$ form a basis of $T_p(\mathcal{M})^*$ which is dual to the coordinate basis $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_N}\right)_p$ of $T_p(\mathcal{M})$. Therefore

$$(df)_p = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(p)(dx_i)_p.$$

Definition A.1.10 *If $f \in \mathcal{F}(U)$ the* ***differential of f*** *is the $\mathcal{F}(U)$-linear map*

$$df : \mathfrak{X}(U) \to \mathcal{F}(U),\ X \mapsto df(X) := X(f).$$

The set $\mathfrak{X}(U)^*$ of all the differential forms is the dual $\mathcal{F}(U)$-module of $\mathfrak{X}(U)$. The differentials $dx_1, \ldots, dx_N$ constitute a basis of $\mathfrak{X}(U)^*$ which is dual to the coordinate basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N}$ of $\mathfrak{X}(U)$ and

$$df = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i} dx_i.$$

Let, now, $\mathcal{N}$ be a differentiable manifolds with $\dim \mathcal{N} = m$ and let $F : \mathcal{M} \to \mathcal{N}$ be a differentiable function.

Definition A.1.11 *The* ***differential of F in p*** $\in \mathcal{M}$ *is the linear map*

$$F_{*p} : T_p(\mathcal{M}) \to T_{F(p)}(\mathcal{N}),\ X_p \mapsto F_{*p}(X_p)\ :\quad F_{*p}(X_p)(g) = X_p(g \circ F) \quad \forall g \in \mathcal{F}(F(p)).$$

Let $(U, \phi, (x_i))$ and $(V, \psi, (y_j))$ be two coordinated neighbourhood of p and $F(p)$ respectively. Considering the basis $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_N}\right)_p$ of $T_p(\mathcal{M})$ and $\left(\frac{\partial}{\partial y_1}\right)_p, \ldots, \left(\frac{\partial}{\partial y_m}\right)_p$ of $T_{F(p)}(\mathcal{N})$, F_{*p} is described by the *Jacobian matrix* $\left(\frac{\partial(y_j \circ F)}{\partial x_i}(p)\right)_{\substack{i=1,\ldots,N \\ j=1,\ldots,m}}$ i.e.

$$F_{*p}(X_p) = \sum_{\substack{i=1,\ldots,N \\ j=1,\ldots,m}} \frac{\partial(y_j \circ F)}{\partial x_i} X_p(x_i), \quad \forall X_p \in T_p(\mathcal{M}). \tag{A.1}$$

A.1.4 Curves

Let $\mathcal{M}$ be a N-dimensional differentiable manifold.

Definition A.1.12 *A* ***differentiable curve*** *in $\mathcal{M}$ is a differentiable map $\alpha : I \to \mathcal{M}$, where I is an open interval of $\mathbb{R}$.*

If $t_o \in I$, the **tangent vector** at $\alpha(t_0)$ is

$$\dot{\alpha}(t_0) = (\alpha_*)_{t_0}\left(\frac{d}{dt}\right)_{t_0}.$$

If $(U, \phi, (x_i))$ is a local chart with $\alpha(t_0) \in U$, $\tilde{\alpha} = \phi \circ \alpha$ and $\tilde{x}_i = x_i \circ \alpha$ are the coordinates of α in the local chart, then

$$\dot{\alpha}(t_0) = \sum_{i=1}^{N} \frac{d\tilde{x}_i}{dt}(t_0)\left(\frac{\partial}{\partial x_i}\right)_{\alpha(t_o)}.$$

It follows immediately from the definition that for any $f \in \mathcal{F}(\alpha(t_0))$, setting $\tilde{f} = f \circ \alpha$, we have $\dot{\alpha}(t_0)(f) = \frac{d\tilde{f}}{dt}(t_0)$.

Remark A.1.13 *Let $p \in \mathcal{M}$. $T_p(\mathcal{M})$ is the set of all the tangent vector to curves in $\mathcal{M}$ at p. Indeed, for every $V_p \in \mathcal{M}$, there exists, for a sufficiently small $\epsilon > 0$, a differentiable curve $\alpha : (-\epsilon, \epsilon) \to \mathcal{M}$ such that $\alpha(0) = p$, $\dot{\alpha}(0) = V_p$. In a coordinated neighbourhood $(U, \phi, (x_i))$ of p, setting $V_p = \sum_{i=1}^{N} v_i \left(\frac{\partial}{\partial x_i}\right)_p$, α can be chosen as $t \mapsto \alpha(t) = \phi^{-1}(\phi(p) + t(v_1, \ldots, v_N))$, where .*

A.2 Riemannian Manifolds

A.2.1 Riemannian Manifolds

Let $\mathcal{M}$ be a N-dimensional differentiable manifold.

Definition A.2.1 *A **Riemannian metric** on $\mathcal{M}$ is a symmetric and positive definite second order covariant tensor on $\mathfrak{X}(\mathcal{M})$, that is a $\mathcal{F}(\mathcal{M})$-linear application*

$$g : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathcal{F}(\mathcal{M})$$

which satisfies, for every X, $Y \in \mathfrak{X}(\mathcal{M})$, $p \in \mathcal{M}$,

(i) $g(X, Y) = g(Y, X)$;

(ii) $g(X, X) \geq 0, \quad g(X, X)(p) = 0 \Rightarrow X_p = 0.$

*The couple $(\mathcal{M}, g)$ is called a **Riemannian manifold**.*

Let $(U, \phi, (x_i))$ be a local chart of $\mathcal{M}$. In this coordinates the metric tensor g is represented by the symmetric and positive definite matrix $(g_{ij}(p))$, where

$$g_{ij} := g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \in \mathcal{F}(U).$$

With this notation g is expressed by

$$g = \sum_{i,j=1}^{N} g_{ij}\, dx_i \otimes dx_j$$

and if $X = \sum_{i=1}^{N} X^i \frac{\partial}{\partial x_i}$, $Y = \sum_{i=1}^{N} Y^i \frac{\partial}{\partial x_i} \in \mathfrak{X}(U)$, then

$$g\,(X,Y) = \sum_{ij=1}^{N} g_{ij}\, X^i Y^j.$$

At each point $p \in \mathcal{M}$, the Riemannian metric g induces an inner product g_p on the tangent space $T_p(\mathcal{M})$. g_p is defined, for every $X_p, Y_p \in T_p(\mathcal{M})$, with $X_p = \sum_{i=1}^{N} X^i \left(\frac{\partial}{\partial x_i}\right)_p$, $Y_p = \sum_{i=1}^{N} Y^i \left(\frac{\partial}{\partial x_i}\right)_p$, by

$$g_p\,(X_p, Y_p) := \sum_{ij=1}^{N} g_{ij}(p)\, X^i Y^j.$$

It follows immediately by the definition that

$$g_p(X_p, Y_p) = g(X,Y)(p), \quad \forall\, X, Y \in \mathfrak{X}(\mathcal{M}).$$

Remark A.2.2 *Equivalently a Riemannian metric g can be defined as a family of inner products $\{g_p : T_p(\mathcal{M}) \times T_p(\mathcal{M}) \to \mathbb{R} \,:\, p \in \mathcal{M}\}$ such that for every $X, Y \in \mathfrak{X}(\mathcal{M})$ the map $p \mapsto g(X,Y)(p) = g_p(X_p, Y_p)$ is differentiable.*

Theorem A.2.3 *Each paracompact differentiable manifold can be equipped with a Riemannian metric.*

PROOF. The metric is simply obtained by piecing together the Euclidean metrics defined on the coordinate images of the local charts together with the help of a partition of unity. For a detailed proof see [13, Theorem 4.5].

□

Remark A.2.4 *Also the converse is true i.e. every Riemannian manifold is paracompact and, moreover, separable (see for example [76, page 459]).*

If p is a point in $(\mathcal{M}, g)$, we define the norm of any tangent vector $X_p \in T_p(\mathcal{M})$ to be $\|X_p\|_g = g_p(X_p, X_p)^{\frac{1}{2}}$ and, analogously, for a given vector field $X \in \mathfrak{X}(U)$ we write $\|X\|_g = g(X,X)^{\frac{1}{2}}$.

A.2.2 Metric space and the Riemannian distance

Let $(\mathcal{M}, g)$ be a N-dimensional Riemannian manifold and let $\alpha : I = [a, b] \to \mathcal{M}$ be a differentiable curve in $\mathcal{M}$.

Definition A.2.5 *The* ***length*** $\mathcal{L}(\alpha)$ *of* α *is defined as*

$$\mathcal{L}(\alpha) := \int_a^b \|\dot{\alpha}(t)\|_g \, dt.$$

Let $(U, \phi, (x_i))$ be a local chart of $\mathcal{M}$ and let $g = \sum_{i,j=1}^N g_{ij} \, dx_i \otimes dx_j$ on U. If $\alpha(I) \subseteq U$ and $\phi(\alpha(t)) = (\alpha_1(t), \ldots, \alpha_N(t))$ then

$$\mathcal{L}(\alpha) = \int_a^b \left(\sum_{i,j=1}^N g_{ij}(\alpha(t)) \frac{d\alpha_i}{dt} \frac{d\alpha_j}{dt} \right)^{\frac{1}{2}} dt.$$

One verifies, easily, that the definition of $\mathcal{L}(\alpha)$, as well as the last integral, depends neither on the choice of the local chart, nor on a change of parametrization $s = s(t)$ with $\frac{ds}{dt} \neq 0$.

We also remark that the length of a piecewise differentiable curve may be defined as the sum of the lengths of the smooth pieces and that the above definition makes sense also for curves which are only (piecewise) C^1.

Since $\mathcal{M}$ is connected, any two points $p, q \in \mathcal{M}$ can be joined by a piecewise differentiable curve. This leads to the following definition.

Definition A.2.6 *On a Riemannian manifold* $(\mathcal{M}, g)$*, the* ***Riemannian distance*** *between two points* $p, q \in \mathcal{M}$ *is defined by*

$$d_g(p, q) := inf \left\{ \mathcal{L}(\alpha) \ : \ \alpha : I = [a, b] \to \mathcal{M} \ \textit{piecewise differentiable curve from}\, p \,\textit{to}\, q \right\}.$$

Theorem A.2.7 *The function* $d_g : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ *defines a distance on* $\mathcal{M}$ *and the topology induces by* d_g *is the same topology of* $\mathcal{M}$ *as a manifold.*

Proof. See [13, Chapter V, Theorem 3.1]

□

A.2.3 Connection and Curvature

Let $\mathcal{M}$ be a N-dimensional differentiable manifold. Whereas the differentiation of functions on $\mathcal{M}$ is naturally determined by the differential structure, the differentiation of vector fields, on the other hand, is not naturally determined but involves the choice of a rule which associates, to every vector fields X, Y, a sort of directional derivative $D_{X_p}Y$ which gives the rate of change of Y at $p \in \mathcal{M}$ in the direction of X_p.

Definition A.2.8 *A* ***Connection*** *on $\mathcal{M}$ (also called* covariant derivative*) is a map $D : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$, $(X, Y) \mapsto D(X, Y) := D_X Y$ which satisfies for every $f, g \in \mathcal{F}(\mathcal{M})$, $X, Y, Z \in \mathfrak{X}(\mathcal{M})$:*

(i) $D_{fX+gY}Z = fD_X Z + gD_Y Z$.

(ii) $D_X(Y + Z) = D_X Y + D_X Z$.

(iii) $D_X(fY) = fD_X Y + X(f)Y$.

Let $(U, \phi, (x_i))$ be a local chart and let us denote $\partial_i = \frac{\partial}{\partial x_i}$.
D is completely determined in U by the *Christoffel symbols* Γ^k_{ij} which are defined through the relation

$$D_{\partial_i}\partial_j = \sum_{k=1}^{N} \Gamma^k_{ij}\,\partial_k.$$

If $X = \sum_{i=1}^N X^i\partial_i$ and $Y = \sum_{i=1}^N Y^i\partial_i$, then it follows from the definition that

$$D_X Y = \sum_{k=1}^{N}\left(X(Y^k) + \sum_{ij=1}^{N} X^i Y^j \Gamma^k_{ij}\right)\partial_k.$$

The *Torsion* T of D is the $\mathcal{F}(\mathcal{M})$-linear map $T : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ defined by

$$(X, Y) \mapsto T(X, Y) = D_X Y - D_Y X - [X, Y].$$

T estimates the asymmetry of the connection and D is said *symmetric* if $T = 0$.

Definition A.2.9 *The* ***Curvature*** *of D is the $\mathcal{F}(\mathcal{M})$-linear map $R : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ defined by*

$$(X, Y, Z) \mapsto R(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z.$$

R estimates the asymmetry of the second covariant derivative and one verifies that $R(X, Y)Z$ at $p \in \mathcal{M}$ depends only upon the values of X, Y, Z at p.
Locally R is completely determined in U by the functions R^l_{ijk} which are defined through the relation

$$R(\partial_i, \partial_j)\partial_k = \sum_{l=1}^{N} R^l_{ijk}\partial_l$$

and are given by

$$R^s_{ijk} = \partial_j\Gamma^s_{ik} - \partial_i\Gamma^s_{jk} + \sum_{h=1}^{N}\left(\Gamma^h_{ik}\Gamma^s_{jh} - \Gamma^h_{jk}\Gamma^s_{ih}\right).$$

Let now $(\mathcal{M}, g)$ be a N-dimensional Riemannian manifold. The Riemannian metric on $\mathcal{M}$ does determine a canonical choice of a connection.

Theorem A.2.10 [66, Theorem 11, Chapter 3] *On a Riemannian manifold $(\mathcal{M}, g)$ there is a unique connection D which is symmetric and compatible with the metric tensor g, i.e. for every $X, Y, Z \in \mathfrak{X}(\mathcal{M})$,*

(i) $D_X Y - D_Y X = [X, Y]$,

(ii) $X g(Y, Z) = g(D_X Y, Z) + g(Y, D_X Z)$.

D is characterized by the ***Koszul formula***

$$\begin{aligned} 2g(D_X Y, Z) =& X g(Y, Z) + Y g(Z, X) - Z g(X, Y) \\ & - g(X, [Y, Z]) + g(Y, [Z, X]) - g(Z, [X, Y]). \end{aligned}$$

Definition A.2.11 *The* ***Levi-Civita connection*** *(or the Riemannian connection) of $(\mathcal{M}, g)$ is the unique connection D which is determined by Theorem A.2.10. Its curvature tensor is called the* ***Riemannian curvature*** *of $(\mathcal{M}, g)$.*

Let $g = \sum_{i,j=1}^{N} g_{ij}\, dx_i \otimes dx_j$ in the local chart $(U, \phi, (x_i))$ and let $(g^{ij})_{i,j=1,\dots,N}$ be the inverse matrix of $(g_{ij})_{i,j=1,\dots,N}$. Using the Koszul formula, the Christoffel symbols of D take the form

$$\Gamma_{ij}^{k} = \Gamma_{ji}^{k} = \frac{1}{2} \sum_{l=1}^{N} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right).$$

Definition A.2.12 *A Riemannian manifold $(\mathcal{M}, g)$is said to be* ***flat*** *if its Riemannian curvature is zero at every point.*

It is immediate to check that the Euclidean manifold $\mathbb{R}^N$ and every one-dimensional Riemannian manifold are flat.

Definition A.2.13 *Let $(\mathcal{M}, g)$ be a Riemannian manifold with Riemannian curvature R and let $E_1, \dots, E_n$ be an orthonormal basis of $\mathfrak{X}(\mathcal{M})$. The* ***Ricci curvature*** Ric *of $(\mathcal{M}, g)$ is the symmetric bilinear form defined as the trace of R:*

$$\mathrm{Ric}(X, Y) = \sum_{i=1}^{n} g\Big(R(E_i, X)Y, E_i\Big) = \sum_{i=1}^{n} g\Big(R(X, E_i)E_i, Y\Big), \quad X, Y \in \mathfrak{X}(\mathcal{M}).$$

Equivalently Ric *can be identified with the symmetric $(1,1)$-tensor*

$$\mathrm{Ric}(X) = \sum_{i=1}^{n} R(X, E_i)E_i, \quad X \in \mathfrak{X}(\mathcal{M}),$$

which satisfies

$$\mathrm{Ric}(X, Y) = g\left(\mathrm{Ric}(X), Y\right), \quad X, Y \in \mathfrak{X}(\mathcal{M}).$$

Definition A.2.14 *We write* Ric $\geq k$ *(respectively* Ric $\leq k$*) and we say that* Ric *is bounded from below if all eigenvalues of* Ric(X) *are* $\geq k$ *(respectively* $\leq k$*) for every* $X \in \mathfrak{X}(\mathcal{M})$*. In particular we say that* $(\mathcal{M}, g)$ *has non-negative Ricci curvature if and only if* Ric$(X, X) \geq 0$ *for every* $X \in \mathfrak{X}(\mathcal{M})$*.*

A.2.4 Geodesics

Let $\mathcal{M}$ be a N-dimensional differentiable manifold with connection D and let $\gamma : I = [a, b] \to \mathcal{M}$ be a differentiable curve in $\mathcal{M}$.

Definition A.2.15 *A vector field* $X \in \mathfrak{X}(\mathcal{M})$ *is said to be **parallel** along* γ *if its covariant derivative in the direction of* $\dot{\gamma}$ *is zero:*

$$D_{\dot{\gamma}} X = 0$$

Let $(U, \phi, (x_i))$ be a local chart of $\mathcal{M}$ and let Γ_{ij}^k be the *Christoffel symbols* of D. If $\gamma(I) \subseteq U$ and $X = \sum_i X^i \partial_i$, we write $\phi(\gamma(t)) = (\gamma^1(t), \ldots, \gamma^N(t))$ and $X^i(t) := X^i(\gamma(t))$. With this notation we have

$$D_{\dot{\gamma}(t)} X = \sum_k \left(\frac{dX^k}{dt} + \sum_{i,j} \Gamma_{ij}^k \frac{d\gamma^j}{dt} X^i \right) \partial_k.$$

Thus X is parallel along γ if and only if

$$\frac{dX^k}{dt} + \sum_{i,j} \Gamma_{ij}^k \frac{d\gamma^j}{dt} X^i = 0, \quad k = 1, \ldots, N.$$

Definition A.2.16 γ *is called a **geodesic** if its field of tangent vectors is parallel along* γ*. Thus* γ *is a geodesic if and only if*

$$\frac{d^2\gamma^k}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} = 0, \quad k = 1, \ldots, N.$$

The Cauchy theorem applied to the above equations, implies that, for given $p \in \mathcal{M}$ and $0 \neq X_p \in T_p(\mathcal{M})$, there exists a unique geodesic $\gamma : (-\epsilon, \epsilon) \to \mathcal{M}$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = X_p$. This geodesic depends smoothly on the initial conditions p and X_p.

If $(\mathcal{M}, g)$ is a Riemannian manifold and D is its Levi-Civita Connection, $\gamma : (a, b) \to \mathcal{M}$ is a geodesic if and only if it is locally distance minimizing i.e., for any $t \in (a, b)$ and for all s close enough to t, the curve $\gamma_{|[t,s]}$ is a shortest path between the points $\alpha(t)$ and $\alpha(s)$ (see [68, Chapter 5, Theorem 5.1 and Corollary 5.5]).

We state the following theorem which proves the equivalence between the notion of geodesical and metrical completeness.

Theorem A.2.17 (H. Hopf-Rinow)[68, Chapter 5, Theorem 7.1] The following statement are equivalent:

(i) $\mathcal{M}$ is geodesically complete, i.e. all geodesics are defined for all time.

(ii) $\mathcal{M}$ is geodesically complete at $p \in \mathcal{M}$, i.e. all geodesics through p are defined for all time.

(iii) Every closed bounded set is compact.

(iv) $\mathcal{M}$ is metrically complete.

Furthermore, any one of the above implies that given any two points $p, q \in \mathcal{M}$, there exists a length minimizing geodesic connecting p, q.

A.2.5 Riemannian measure

Let $(\mathcal{M}, g)$ be a Riemannian manifold of dimension N. We say that a set $E \subseteq \mathcal{M}$ is measurable if, for any chart (U, ϕ), $\phi(E \cap U)$ is Lebesgue measurable in $\phi(U) \subseteq \mathbb{R}^N$. The family of all measurable sets in $\mathcal{M}$ forms a σ-algebra which we denote by $\Lambda(\mathcal{M})$ and we define the Borel σ-algebra $\mathcal{B}(\mathcal{M})$ as the σ-algebra containing all open subset of $\mathcal{M}$. Since any open subset of $\mathcal{M}$ is measurable, $\mathcal{B}(\mathcal{M})$ is contained in $\Lambda(\mathcal{M})$.

$(\mathcal{M}, g)$ is naturally provided with a canonical measure μ_g on $\Lambda(\mathcal{M})$, which is called the *Riemannian measure* (or volume) of $\mathcal{M}$ and it is defined by means of the following theorem.

Theorem A.2.18 *Let $(\mathcal{M}, g)$ a Riemannian manifold of dimension N. Then there exists a unique measure μ_g on $\Lambda(\mathcal{M})$ such that, in any chart $(U, \phi, (x_i))$,*

$$d\mu_g = \sqrt{\det g}\, d\mathcal{L}.$$

Here $g = \sum_{i,j=1}^N g_{ij}\, dx_i \otimes dx_j$ on U, $det g := \det(g_{ij})$ and $\mathcal{L}$ is the Lebesgue measure in U.

Furthermore the measure μ_g is complete and regular.

Proof. On any local chart $(U, \phi, (x_i))$ the formula

$$\mu_U(E) = \int_{\phi(E)} \sqrt{\det g}\, d\mathcal{L}, \quad \forall E \in \Lambda(\mathcal{M}),\ E \subseteq U,$$

defines a measure on $\Lambda(\mathcal{M}) \cap U$. The measure μ_g is then simply obtained by piecing together the measures μ_U defined on the local charts together with the help of a partition of unity. For a detailed proof see, for example, [35, Theorem 3.11].

□

A.3 Some differential operators

Let $(\mathcal{M}, g)$ be a N-dimensional Riemannian manifold and let D be its Levi-Civita connection. On $(\mathcal{M}, g)$ there are natural generalizations of the well-known differential operators of vector calculus on $\mathbb{R}^N$: gradient, divergence, and Laplacian.

Let us fix, preliminarily, some notation. Let $(U, \phi, (x_i))$ be a local chart of $\mathcal{M}$ and let us denote $\partial_i = \frac{\partial}{\partial x_i}$. Let $g = \sum_{i,j=1}^N g_{ij}\, dx_i \otimes dx_j$ on U and let $(g^{ij})_{i,j=1,\dots,N}$ be the inverse matrix of $(g_{ij})_{i,j=1,\dots,N}$ and $\det g := \det(g_{ij})$.

Definition A.3.1 *The **Gradient** $\nabla_g f$ of a function $f \in \mathcal{F}(\mathcal{M})$ is the vector field dual of the differential $df \in \mathfrak{X}(\mathcal{M})^*$. Thus*

$$g(\nabla_g f, X) = df(X) = X(f), \quad \textit{for all } X \in \mathfrak{X}(\mathcal{M}).$$

In local coordinates on U, if $df = \sum_{i=1}^N \frac{\partial f}{\partial x_i} dx_i$, then

$$\nabla_g f = \sum_{i,j=1}^N g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}. \tag{A.2}$$

In particular, for Cartesian coordinates on the Euclidean manifold $\mathbb{R}^N$, the above formula, recalling the canonical identification $T_p(\mathbb{R}^N) \equiv \mathbb{R}^N$, reduces to $\nabla_g f = \sum_{i,j=1}^N \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i} \equiv \nabla f$ which is the usual formula for the gradient in $\mathbb{R}^N$.
Recalling Remark A.1.13, $\nabla_g f$ is characterized by

$$\frac{d}{dt}(f \circ \alpha) = g(\nabla_g f, \dot{\alpha}), \quad \text{for every differentiable curve } \alpha : I \to \mathcal{M}. \tag{A.3}$$

Remark A.3.2 *The above definition obviously makes sense only requiring $f \in C^1(\mathcal{M})$.*

Definition A.3.3 *The **Divergence** of a vector field $X \in \mathfrak{X}(\mathcal{M})$ is defined as*

$$\operatorname{div} X := \operatorname{trace}\,(Y \mapsto D_Y X)\,.$$

In local coordinates on U, if $X = \sum_{i=1}^N X^i \frac{\partial}{\partial x_i} \in \mathfrak{X}(\mathcal{M})$, then it can be shown that

$$\operatorname{div} X = \sum_{i=1}^N \left(\partial_i X^i + \sum_{j=1}^N \Gamma^i_{ij} X^j \right) = \frac{1}{\sqrt{det\, g}} \sum_{i=1}^N \partial_i \left(X^i \sqrt{det\, g} \right).$$

Employing Theorem A.2.18, it can be seen that $\operatorname{div} X$ is characterized by satisfying the Divergence Theorem

$$\int_{\mathcal{M}} (\operatorname{div} X)\, f\, d\mu_g = - \int_{\mathcal{M}} g(X, \nabla_g f)\, d\mu_g, \quad \forall f \in C_c^\infty(\mathcal{M}).$$

Definition A.3.4 *The **Laplacian** $\Delta_g f$ (frequently referred to as the Laplace-Beltrami operator) of a function $f \in \mathcal{F}(\mathcal{M})$ is the divergence of its gradient:*

$$\Delta_g f := \operatorname{div}(\nabla_g f) \in \mathcal{F}(\mathcal{M}).$$

In local coordinates on U

$$\Delta_g f = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{N} \partial_i \left(g^{ij} \sqrt{\det g}\, \partial_j f \right). \tag{A.4}$$

Since the matrix (g_{ij}) is symmetric and positive definite, the operator Δ_g is an elliptic second order operator in the divergence form.
In particular, employing Cartesian coordinates on the Euclidean manifold $\mathbb{R}^N$, one recognize that the classical Laplace operator on $\mathbb{R}^N$ is a particular case of the Laplace-Beltrami operator.

$\Delta_g f$ is characterized by satisfying the Green formula

$$\int_{\mathcal{M}} \Delta_g f u \, d\mu_g = -\int_{\mathcal{M}} g(\nabla_g f, \nabla_g u)\, d\mu_g = \int_{\mathcal{M}} f \Delta_g u \, d\mu_g, \quad \forall u \in C_c^\infty(\mathcal{M}).$$

Remark A.3.5 *The above definition makes sense only requiring $f \in C^2(\mathcal{M})$.*

A.4 Some constructions

A.4.1 Immersions and Submanifolds

Let $\overline{\mathcal{M}}$ be a differentiable manifold. A *submanifold* (or *immersed submanifold*) of $\overline{\mathcal{M}}$ is a differentiable manifold $\mathcal{M}$ together with an injective differentiable *immersion* $f : \mathcal{M} \to \overline{\mathcal{M}}$ i.e. an injective differentiable map such that, for every $p \in \mathcal{M}$, all differentials f_{*p} are injective.

The most important type of submanifold is that in which the immersion map f is an *embedding*, which means that it is a homeomorphism onto its image $f(\overline{\mathcal{M}})$ with the subspace topology. In that case, $\mathcal{M}$ is called an *embedded submanifold* or a regular submanifold. An application of the implicit function theorem proves that locally any differentiable immersion is a differentiable embedding.

Proposition A.4.1 [39, Lemma 1.3.1] *Let $f : \mathcal{M} \to \overline{\mathcal{M}}$ be a differentiable immersion, dim $\mathcal{M} = N$, dim $\overline{\mathcal{M}} = \overline{N}$, $p \in \mathcal{M}$. Then there exist a neighbourhood $(U, x, (x_i))$ of p and a chart $(V, y, (y_i))$ on $\overline{\mathcal{M}}$ with $f(p) \in V$, such that*

(i) $f_{|U}$ is a differentiable embedding;

(ii) $f(U) = \left\{ q \in V \ : \ y_{N+1}(q) = \cdots = y_{\overline{N}}(q) = 0 \right\}$ and $x_i(q) = y_i(f(q))$ for all $q \in U \cap V$ and $i = 1, \ldots N$.

Since all the properties presented in this section are local, the previous proposition allows us to assume, in what follows, f to be an embedding from $\mathcal{M}$ to $\overline{\mathcal{M}}$.

Let, now, $(\bar{\mathcal{M}}, \bar{g})$ be a $\bar{N}$-dimensional Riemannian manifold and $\mathcal{M}$ a N-dimensional submanifold of $\overline{\mathcal{M}}$ with $f : \mathcal{M} \to \bar{\mathcal{M}}$ being the immersion map.

Definition A.4.2 *The metric $\bar{g}$ induces a Riemannian metric on $\mathcal{M}$ given by the* ***pull-back*** *$g := f^*\bar{g}$ defined by*

$$g_p(X_p, Y_p) := \bar{g}_{f(p)}\left(f_{*p}(X_p), f_{*p}(Y_p)\right), \quad \forall p \in \mathcal{M},\ \forall X, Y \in \mathfrak{X}(\mathcal{M}).$$

$(\mathcal{M}, g)$ is called a ***Riemannian submanifold*** *of $(\bar{\mathcal{M}}, g)$ and f is an isometric immersion.*

Let $(U, \phi, (y_i))$, $(V, \psi, (x_\alpha))$ be two local charts of $\mathcal{M}$ and $\bar{\mathcal{M}}$ respectively, with $f(U) \subseteq V$. Let

$$\bar{g} = \sum_{\alpha\beta=1}^{\bar{N}} \bar{g}_{\alpha\beta}\, dx_\alpha \otimes dx_\beta \quad \text{on } V.$$

g in U is determined by its components $g_{ij} = g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right)$: using (A.1) it follows by the definition of the pull-back that

$$g = \sum_{i=1}^{N} g_{ij}\, dy_i \otimes dy_j, \qquad g_{ij} = \sum_{\alpha,\beta} (\bar{g}_{\alpha\beta} \circ f) \frac{\partial(x^\alpha \circ f)}{\partial y_i} \frac{\partial(x^\beta \circ f)}{\partial y_j}. \tag{A.5}$$

Denoting by $Jf = \left(\frac{\partial(x^\alpha \circ f)}{\partial y_i}\right)$ the Jacobian matrix of f in the local coordinates, (A.5) takes the compact form

$$(g_{ij}) = Jf^t(\bar{g}_{\alpha\beta} \circ f)Jf. \tag{A.6}$$

Since every differentiable curve α of $\mathcal{M}$ induces a differentiable curve $f \circ \alpha$ of $\overline{\mathcal{M}}$, it follows immediately by the definition of $f^*\bar{g}$ that $\mathcal{L}(\alpha) = \mathcal{L}(f \circ \alpha)$.

If $N = \overline{N}$ and f is a diffeomorphism, f is called an *isometry* and $(\mathcal{M}, g)$ and $(\overline{\mathcal{M}}, g)$ are said to be *isometric*.
The following proposition is an immediate consequence of the definition of the distance function.

Proposition A.4.3 *Let $f : \mathcal{M} \to \overline{\mathcal{M}}$ be an isometric immersion between two Riemannian manifolds $(\mathcal{M}, g)$ and $(\overline{\mathcal{M}}, g)$ and let us denote by d and $\bar{d}$ the respectively distance functions. Then one has*

$$\bar{d}(f(p), f(q)) \le d(p, q), \quad \forall\, p, q \in \mathcal{M}.$$

If, in addition, f is an isometry then

$$d(p, q) = \bar{d}(f(p), f(q)), \quad \forall\, p, q \in \mathcal{M}.$$

$f(\mathcal{M}) \subseteq \bar{\mathcal{M}}$ naturally inherits a structure of a Riemannian submanifold of $(\bar{\mathcal{M}}, g)$: identifying, therefore, $\mathcal{M}$ with its image $f(\mathcal{M}) \subseteq \overline{\mathcal{M}}$, we can consider $\mathcal{M}$ as a subset of $(\bar{\mathcal{M}}, g)$ and $f = i : \mathcal{M} \hookrightarrow \bar{\mathcal{M}}$ to be the inclusion map.

Definition A.4.4 *Let $\mathcal{M} \subseteq \overline{\mathcal{M}}$ be a submanifold of $(\overline{\mathcal{M}}, \overline{g})$ and let $i : \mathcal{M} \hookrightarrow \bar{\mathcal{M}}$ be the inclusion map. The pull-back metric $g = i^*\overline{g}$ defined by Definition A.4.2 is called the **First fundamental form** of $\mathcal{M}$.*

Any $f \in \mathcal{F}(\mathcal{M})$ and any vector field $X \in \mathfrak{X}(\mathcal{M})$ can be extended on a neighbourhood of $\mathcal{M}$ in $\overline{\mathcal{M}}$; with the help of Proposition A.4.1, this is most easily done, in local coordinates around $p \in \mathcal{M}$ that locally map $\mathcal{M}$ to $\mathbb{R}^N \hookrightarrow \mathbb{R}^{\overline{N}}$. We use, moreover, the same letter to denote both functions and vector fields on $\mathcal{M}$ and their extensions (also called lifts) to $\overline{\mathcal{M}}$.

Since each $i_{*p} : T_p(\mathcal{M}) \to T_p\left(\bar{\mathcal{M}}\right)$ is injective, we can ignore i_{*p} and consider $T_p(\mathcal{M})$ as a vector subspace of $T_p(\overline{\mathcal{M}})$. At each $p \in \mathcal{M}$ the tangent space $T_p\overline{\mathcal{M}}$ splits as an orthogonal direct sum

$$T_p\overline{\mathcal{M}} = T_p\mathcal{M} \oplus (T_p\mathcal{M})^\perp$$

where $(T_p\mathcal{M})^\perp$ is the *normal space* at p with respect the inner product $\overline{g}_p$. For any $p \in \mathcal{M}$, $X_p \in T_p\overline{\mathcal{M}}$, we denote the projection of X_p onto $T_p\mathcal{M}$ and $(T_p\mathcal{M})^\perp$ by, respectively, $X_p^\top$ and $X_p^\perp$.

Last formula implies immediately that, for any $f \in \mathcal{F}(\mathcal{M})$, the g-gradient of f is the projection of its $\overline{g}$-gradient i.e.

$$\nabla_g f = (\nabla_{\overline{g}} f)^\top = \nabla_{\overline{g}} f - (\nabla_{\overline{g}} f)^\perp . \tag{A.7}$$

An analogous decomposition can be proved for the Levi-Civita connections D and $\overline{D}$ of $\mathcal{M}$ and of $\overline{\mathcal{M}}$ respectively. It is not hard to prove, indeed, that for every X, $Y \in \mathfrak{X}(\mathcal{M})$ $\left(\overline{D}_X Y\right)^\top$ satisfies all the requirement of Theorem A.2.10. This proves

$$D_X Y = \left(\overline{D}_X Y\right)^\top = \overline{D}_X Y - \left(\overline{D}_X Y\right)^\perp .$$

Let $\nu(\mathcal{M}) = \dot{\bigcup}_{p \in \mathcal{M}} (T_p\mathcal{M})^\perp$ be the *normal bundle* of $\mathcal{M}$ in $T(\overline{\mathcal{M}})$ and let us denote with $\mathfrak{X}(\mathcal{M})^\perp$ the set of vector fields $X : \mathcal{M} \to \nu(\mathcal{M})$ such that for any point p, $X_p \in (T_p\mathcal{M})^\perp$. We remark that not every Riemannian submanifold $\mathcal{M} \subseteq \overline{\mathcal{M}}$ admits the existence of a smooth unit normal vector fields defined on all of $\mathcal{M}$: the Möbius band in $\mathbb{R}^N$ is one example.

Definition A.4.5 *Let D and $\overline{D}$ be the Levi-Civita connections of $(\mathcal{M}, g)$ and of $(\overline{\mathcal{M}}, \overline{g})$ respectively. The **Second fundamental form** of $\mathcal{M}$ is the symmetric and $\mathcal{F}(\mathcal{M})$-bilinear application defined by*

$$II : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \nu(\mathcal{M}), \quad (X, Y) \mapsto II(X, Y) := \left(\overline{D}_X Y\right)^\perp .$$

One easily proves that II is well defined and satisfies the *Gauss Formula*

$$\overline{D}_X Y = D_X Y + II(X,Y), \quad \forall\, X,Y \in \mathfrak{X}(\mathcal{M}).$$

Although the second fundamental form is defined in terms of covariant derivatives of vector fields tangent to $\mathcal{M}$, it can also be used to evaluate covariant derivatives of normal vector fields. Indeed, letting $\xi \in \mathfrak{X}(\mathcal{M})^\perp$, one has, for any $X, Y \in \mathfrak{X}(\mathcal{M})$,

$$\overline{g}(\overline{D}_X \xi, Y) = X\overline{g}(\xi, Y) - \overline{g}(\xi, \overline{D}_X Y) = -\overline{g}(\xi, \overline{D}_X Y).$$

Therefore II satisfies the *Weingarten equation*

$$\overline{g}(\overline{D}_X \xi, Y) = -\overline{g}(\xi, II(X,Y)), \quad X, Y \in \mathfrak{X}(\mathcal{M}),\ \xi \in \mathfrak{X}(\mathcal{M})^\perp.$$

Definition A.4.6 *Let $\xi \in \mathfrak{X}(\mathcal{M})^\perp$ and $p \in \mathcal{M}$. The* ***Weingarten map*** *(or* ***Shape Operator****) at p is the self-adjoint linear transformation S_{ξ_p} implicitly determined by the relation*

$$\overline{g}_p(S_{\xi_p} X_p, Y_p) = \overline{g}_p(II_p(X_p, Y_p), \xi_p), \quad X_p, Y_p \in T_p\mathcal{M}.$$

Using the Weingarten equation, S_{ξ_p} is defined as the map

$$S_{\xi_p} : T_p\mathcal{M} \to T_p\mathcal{M}, \quad X_p \mapsto S_{\xi_p} X_p := -\left(\overline{D}_{X_p}\xi\right)^\top.$$

The *principal curvatures* of $\mathcal{M}$ in $\overline{\mathcal{M}}$ at $p \in \mathcal{M}$, relative to the normal direction ξ, are the eigenvalues of S_{ξ_p}. The associated eigenvectors are referred to as the *principal directions*.

A straightforward computation shows, moreover, the following relation between the Riemannian curvatures R and $\overline{R}$ of of $(\mathcal{M}, g)$ and of $(\overline{\mathcal{M}}, \overline{g})$ respectively. In what follows, for $\xi \in \mathfrak{X}(\mathcal{M})^\perp$ and $X \in \mathfrak{X}(\mathcal{M})$, we write $S_\xi X(p) = S_{\xi_p} X_p$.

Theorem A.4.7 (The Gauss Equation) *Let R and $\overline{R}$ be the Riemannian curvatures of $(\mathcal{M}, g)$ and $(\overline{\mathcal{M}}, \overline{g})$ respectively. Then for every $X, Y, Z, W \in \mathfrak{X}(\mathcal{M})$*

$$R(X,Y)Z = \left(\overline{R}(X,Y)Z\right)^\top + S_{II(Y,Z)}X - S_{II(X,Z)}Y,$$

$$\begin{aligned} g(R(X,Y)Z, W) =& \overline{g}\left(\overline{R}(X,Y)Z, W\right) + \overline{g}\left(II(X,W), II(Y,Z)\right) \\ &- \overline{g}\left(II(X,Z), II(Y,W)\right). \end{aligned}$$

Some Riemannian submanifolds have a particular simple shape structure.

Definition A.4.8 *A Riemannian submanifold $(\mathcal{M}, g)$ of $(\overline{\mathcal{M}}, \overline{g})$ is called* ***totally geodesic*** *if its second fundamental form vanishes: $II = 0$.*

We have the following characterization for totally geodesic submanifolds.

Proposition A.4.9 [66, Chapter 4, Proposition 13] *For a Riemannian submanifold* $(\mathcal{M}, g)$ *of* $(\overline{\mathcal{M}}, \bar{g})$ *the following are equivalent.*

(i) $(\mathcal{M}, g)$ *is totally geodesic in* $(\overline{\mathcal{M}}, \bar{g})$.

(ii) Every geodesic of $(\mathcal{M}, g)$ *is also a geodesic of* $(\overline{\mathcal{M}}, \bar{g})$.

The following Proposition gives us a tool to identify totally geodesic submanifolds.

Proposition A.4.10 [68, Chapter 5, Proposition 24] *Let* $\Theta \subseteq Iso(\mathcal{M}, g)$ *be a set of isometries of a Riemannian manifold* $(\mathcal{M}, g)$. *Then each connected component of the fixed point set* $Fix(\Theta) = \{p \in \mathcal{M} \,:\, T(p) = p \quad \forall T \in \Theta\}$ *is a totally geodesic submanifold of* $(\mathcal{M}, g)$.

A.4.2 Submanifolds of Euclidean Space

In this section we specialize some of the results introduced so far to the special case of Riemannian Submanifolds of the Euclidean Space. Actually the study of this class of manifolds is not restrictive since, according to the famous **Nash's Embedding Theorem** [65], every Riemannian manifold can be isometrically embedded in a Euclidean space of sufficiently large dimension.

Let us consider $\mathbb{R}^N$ with its usual inner product $(x, y) \mapsto \sum_i x_i y_i =: \langle x, y \rangle$.
For each $p \in \mathbb{R}^N$, there is a canonical linear isomorphism from $\mathbb{R}^N$ to $T_p(\mathbb{R}^N)$ that, in terms of Cartesian coordinates, sends

$$x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mapsto X_p = \sum_i x_i \left(\frac{\partial}{\partial x_i} \right)_p \in T_p(\mathbb{R}^N). \tag{A.8}$$

The coordinate basis $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_N}\right)_p$ is, so, identified with the standard orthonormal basis $e_1, \ldots, e_N$ of $\mathbb{R}^N$ and, in what follows, when dealing with Submanifolds of Euclidean Space, we will always make the identification

$$T_p(\mathbb{R}^N) \equiv \{p\} \times \mathbb{R}^N \equiv \mathbb{R}^N.$$

The standard Riemannian metric $g_{\mathbb{R}^N}$ on $\mathbb{R}^N$ is so defined by

$$(g_{\mathbb{R}^N})_p(x, y) = \langle x, y \rangle, \quad \text{for every } p \in \mathbb{R}^N, \ x, y \in T_p(\mathbb{R}^N) \equiv \mathbb{R}^N.$$

Let $X, Y \in \mathfrak{X}(\mathbb{R}^N)$ be two vector fields of $\mathbb{R}^N$ with $X = \sum_{i=1}^N X^i \frac{\partial}{\partial x_i} \equiv \left(X^i, \ldots, X^N\right)$ and $Y = \sum_{i=1}^N Y^i \frac{\partial}{\partial x_i} \equiv \left(Y^i, \ldots, Y^N\right)$. A straightforward calculation shows that the Levi-Civita connection on $\mathbb{R}^N$ is the standard derivation on $\mathbb{R}^N$ given by

$$D^0_X Y = \sum_{i=1}^N X(Y^i) \frac{\partial}{\partial x_i} \equiv \left(X(Y^i), \ldots, X(Y^N)\right) = X^t \cdot JY^t,$$

where $X(Y^i) = \sum_{j=1}^{N} X^j \frac{\partial Y^i}{\partial x_j}$ and JY^t is the transpose of the Jacobian matrix of Y.

One easily sees that the Riemann curvature tensor vanishes identically. Thus, $\mathbb{R}^N$ with its standard Riemannian metric is flat. The straight lines of $\mathbb{R}^N$ are shown to be the geodesics and, since they are infinitely extendible in both directions, $\mathbb{R}^N$ is geodesically complete.

Let now $(\mathcal{M}, g)$ be an n-dimensional submanifold of $\mathbb{R}^N$ where $g = i^* g_{\mathbb{R}^N}$ is the metric induced by the immersion i on $\mathbb{R}^N$. Let us assume that we have a local coordinate system given by a local chart $(U, \psi, (u_i))$. The map $\phi := \psi^{-1} : V \to U$ is a parametrization of $\mathcal{M}$, with $V := \psi(U)$, and we have

$$x = \phi(u_1, ..., u_n), \quad \text{for every } x \in V.$$

Under the identification (A.8), the vectors $\phi_{u_1}, \ldots, \phi_{u_n}$ defined by

$$\phi_{u_i} = \left(\frac{\partial \phi_1}{\partial u_i}, \ldots, \frac{\partial \phi_N}{\partial u_i} \right),$$

are linearly independent and generate the tangent space $T_p(\mathcal{M})$.

In these coordinates the first fundamental form $g = i^* g_{\mathbb{R}^N}$ is determined by the matrix of its coefficients $G = (g_{ij})_{i,j=1,\ldots,n}$, where $g_{ij} = \langle \phi_{u_i}, \phi_{u_j} \rangle$.

For every $f \in C^1(\mathbb{R}^N)$, recalling (A.7), the tangential gradient $\nabla_g f$ is, at any point $p \in \mathcal{M}$, the projection of the euclidean gradient ∇f onto the tangent space $T_p(\mathcal{M})$. By (A.3), $\nabla_g f$ is the vector field of $\mathbb{R}^N$ which satisfies

$$\frac{d}{dt}(f \circ \alpha) = \langle \nabla_g f, \alpha' \rangle, \quad \text{for every differentiable curve } \alpha : I \to \mathcal{M} \subset \mathbb{R}^N. \tag{A.9}$$

The Levi-Civita Connection D and the second fundamental form II of $(\mathcal{M}, g)$ are given, for every vector fields X, Y of $\mathcal{M}$, by

$$D_X Y = \left(D^0_X Y\right)^\top, \quad II(X, Y) = \left(D^0_X Y\right)^\perp. \tag{A.10}$$

If ξ is a vector field normal to $\mathcal{M}$ and $X \in \mathfrak{X}(\mathcal{M})$, then the Shape operator at $p \in \mathcal{M}$ takes the form

$$S_{\xi_p} X_p = -\left(D^0_{X_p} \xi\right)^\top = -\left(X^t(p) \cdot J\xi^t(p)\right)^\top. \tag{A.11}$$

Being $\mathbb{R}^N$ flat, Theorem A.4.7 gives us the following expression for the Riemannian curvature R of $(\mathcal{M}, g)$:

$$R(X, Y)Z = S_{II(Y,Z)} X - S_{II(X,Z)} Y, \quad \forall\, X, Y, Z \in \mathfrak{X}(\mathcal{M}). \tag{A.12}$$

Example A.4.11 *Let $(\mathbb{S}^{N-1}(r), g)$ be the $(N-1)$-dimensional sphere of radius r equipped with the metric induced by its immersion on $\mathbb{R}^N$. The vector field $U(x) = \frac{x}{|x|}$ of $\mathbb{R}^N$ is, at each point of $\mathbb{S}^{N-1}(r)$, the outward unit normal on $(\mathbb{S}^{N-1}(r), g)$ and verifies $JU(x) = \frac{I}{|x|} - \frac{x\otimes x}{|x|^3}$, where I is the identity matrix and $x \otimes x = (x_i x_j)$.*
Equation (A.10) and (A.11) imply that , for every $X, Y \in \mathfrak{X}\left(\mathbb{S}^{N-1}(r)\right)$,

$$S_U X = -\frac{1}{r} X, \quad II(X,Y) = -\frac{1}{r}(X,Y)U.$$

Therefore, substituting the last equalities in (A.12), the curvature R of $(\mathbb{S}^{N-1}(r), g)$ is given, for every $X, Y, Z \in \mathfrak{X}(\mathbb{S}^{N-1}(r))$, by

$$R(X,Y)Z = \frac{1}{r^2}\Big[g(Y,Z)X - g(X,Z)Y\Big] = \frac{1}{r^2}\Big[(Y,Z)X - (X,Z)Y\Big].$$

A simple computation implies also

$$\mathrm{Ric}(X) = \frac{N-2}{r^2} X.$$

A.4.3 Warped product

Let (B, g_B), (F, g_F) be two Riemannian manifolds and consider the differentiable manifold $\mathcal{M} := B \times F$ with the canonical projections denoted by $\pi : \mathcal{M} \to B$ and $\sigma : \mathcal{M} \to F$. A rich class of metrics on $B \times F$ can be obtained by homothetically warping the product metric on each fiber $\{p\} \times F$. We refer the reader to [66, p. 204] for further details.

Let $f > 0$ be a C^∞-function on B.

Definition A.4.12 *The* ***warped product*** *$B \times_f F$ is the product manifold $\mathcal{M} := B \times F$ provided with the metric tensor*

$$g = \pi^*(g_B) + (f\circ\pi)^2 \sigma^*(g_F).$$

B is called the ***Base*** *of $B \times_f F$ and F the* ***Fiber****.*

The positivity of f immediately implies that g is a metric tensor. If $f = 1$, then $B \times_f F$ reduces to the *Riemannian product manifold* $\big(B \times F, g = \pi^*(g_B) + \sigma^*(g_F)\big)$.
Explicitly, if $X, Y \in T_{(p,q)}B \times F$ then

$$g_{(p,q)}(X,Y) = g_{B_p}\left(\pi_{*_{(p,q)}}X, \pi_{*_{(p,q)}}Y\right) + f^2(p) g_{F_q}\left(\sigma_{*_{(p,q)}}X, \sigma_{*_{(p,q)}}Y\right).$$

$\pi^{-1}(p) = \{p\} \times F$ and $\sigma^{-1}(q) = B \times \{q\}$ are called, respectively, the *Fibers* and the *Leaves* of the warped product and are both Riemannian submanifolds of $B \times_f F$.
The warped product is characterized by the following properties.

(i) For every $q \in F$, $\pi_{|B\times\{q\}}$ is an isometry onto B.

(ii) For every $p \in B$ $\sigma_{|\{p\}\times F}$ is a homothety onto B with scale factor $\frac{1}{f(p)}$.

(iii) For every $(p,q) \in B \times F$, $B \times \{q\}$ and $\{p\} \times F$ are orthogonal at (p,q).

With a little abuse of notation, we confuse all vectors $X \in \mathfrak{X}(B)$, $U \in \mathfrak{X}(F)$ with their respectively lifts on $\mathfrak{X}(B \times F)$ and the same convention will be adopted for the connections D^B, D^F and the Riemannian curvatures R^B, R^F of B and F respectively.

The connection and the Riemannian curvature of $B \times_f F$ can be deduced from those of B and F.

Proposition A.4.13 [66, Chapter 7, Proposition 35] *Let D be the Levi-Civita connection of $B \times_f F$. Then for every X, $Y \in \mathfrak{X}(B)$ and V, $W \in \mathfrak{X}(F)$ one has*

(i) $D_X Y = D^B_X Y$;

(ii) $D_X V = D_V X = \dfrac{X(f)}{f} V$;

(iii) $D_V W = D^F_V W - \dfrac{g(V,W)}{f} \nabla_{g_B} f = D^F_V W - f g_F(V,W) \nabla_{g_B} f$.

Proposition A.4.14 [66, Chapter 7, Proposition 42] *Let R be the Riemannian curvature tensor of $B \times_f F$. Then for every X, Y, $Z \in \mathfrak{X}(B)$ and U, V, $W \in \mathfrak{X}(F)$ one has*

(i) $R(X,Y)Z = R^B(X,Y)Z$;

(ii) $R(X,V)Y = \dfrac{H^f(X,Y)}{f} V$, *where H^f is the Hessian* $H^f(X,Y) := X(Y(f)) - D_X Y(f)$;

(iii) $R(X,Y)V = R(V,W)X = 0$;

(iv) $R(V,X)W = \dfrac{g(V,W)}{f} D^B_X \nabla_{g_B} f$;

(v) $R(V,W)U = R^F(V,W)U - \dfrac{g_B\left(\nabla_{g_B} f, \nabla_{g_B} f\right)}{f^2} \left[g(W,U)V - g(V,U)W\right]$.

Note that in [66], the Riemannian curvature is defined with a minus sign.

Appendix B

Analysis on the sphere

In this appendix we recall the classical theory of spherical harmonics. In Section B.1 we introduce spherical coordinates on $\mathbb{R}^N$ and we calculate a few identities on the Laplace-Beltrami operator Δ_0 on the sphere including is relation with the classical Laplacian on $\mathbb{R}^N$. In Sections B.2 we define the spherical harmonics and prove several properties about them. In B.3 we give a proof of the Henke-Funk theorem and introduce the spherical harmonics expansion for zonal function on $\mathbb{S}^{N-1}$. Finally, in B.4, we introduce some differential operators on $\mathbb{S}^{N-1}$ which allow to derive a bound for $\nabla_\tau \mathbb{Z}^{(n)}$.

B.1 Spherical coordinates in $\mathbb{R}^N$

In this section we describe the spherical coordinates on $\mathbb{R}^N$ and find a decomposition for the Euclidean metric $g_{\mathbb{R}^N}$ and for the Laplace operator Δ in terms, respectively, of the metric $g_{\mathbb{S}^{N-1}}$ and of the Laplace-Beltrami operator Δ_0 on the sphere $\mathbb{S}^{N-1}$. We refer to Appendix A for the geometric notions needed and to [16, Chapter 1, Section 4] and [35, Section 3.9] for further details.

Let us fix, preliminarily, some notations. We write

$$\mathbb{S}^{N-1}(r) := \left\{ x \in \mathbb{R}^N \ : \ |x| = r \right\}, \quad \mathbb{S}^{N-1} = \mathbb{S}^{N-1}(1)$$

and, for every $x \in \mathbb{R}^N$, let $r = |x|$, $\omega = \frac{x}{|x|}$.
We introduce in $\mathbb{R}^N$ the *spherical coordinates*

$$P : [0, \infty[\times \mathbb{S}^{N-1} \to \mathbb{R}^N, \ \ (r, \omega) \mapsto P(r, \omega) = r\omega = x.$$

On $\mathbb{R}^N \setminus \{0\}$ we have the inverse map

$$Q : \mathbb{R}^N \setminus \{0\} \to]0, \infty[\times \mathbb{S}^{N-1}, \ \ x \mapsto Q(x) = \left(|x|, \frac{x}{|x|} \right).$$

Let $(U, u, (u_i))$ be a local chart of $\mathbb{S}^{N-1}$, where $u : U \to u(U) \subseteq \mathbb{R}^{N-1}$ and let us set $f = u^{-1} = (f_1, \ldots, f_N)$. The metric $g_{\mathbb{S}^{N-1}}$ is the metric induced on $\mathbb{S}^{N-1}$ as a submanifold of $\mathbb{R}^N$. Then it follows by (A.5) that $g_{\mathbb{S}^{N-1}}$ takes in U the form

$$g_{\mathbb{S}^{N-1}} = \sum_{h,k=1}^{N-1} \gamma_{hk}\, du_h \otimes du_k, \tag{B.1}$$

where the functions γ_{hk} are determined by

$$\gamma_{hk} = g_{\mathbb{S}^{N-1}} \left(\frac{\partial}{\partial u_h}, \frac{\partial}{\partial u_k} \right) = \sum_{i=1}^{N} \frac{\partial f_i}{\partial u_h} \frac{\partial f_i}{\partial u_k}. \tag{B.2}$$

By means of the spherical coordinates P, any local chart of $\mathbb{S}^{N-1}$ determines, actually, a local chart of $\mathbb{R}^N$. Indeed a chart (V, ϕ) of $\mathbb{R}^N$ is defined, on the open cone $V = P\left((0,\infty) \times U\right)$, by the formula

$$\phi(x) = \left(|x|, u\left(\frac{x}{|x|}\right) \right) = (r, u(\omega)) \,.$$

Therefore $(r, u_1, \ldots, u_{N-1})$ are local coordinates on V and we have for $x \in V$

$$x = r f(u_1, \ldots, u_{N-1}).$$

The last relation implies the announced decomposition for g_{R^N}.

Proposition B.1.1 *The canonical euclidean metric g_{R^N} has the following representation in spherical coordinates $x = r\omega$:*

$$g_{R^N} = dr^2 + r^2 g_{\mathbb{S}^{N-1}}.$$

Here $g_{\mathbb{S}^{N-1}}$ is the canonical spherical metric on $\mathbb{S}^{N-1}$ and $dr^2 := dr \otimes dr$ is the canonical euclidean metric on R^+. In Cartesian coordinates on $\mathbb{R}^N$, dr^2 is the tensor

$$dr^2 = \sum_{i,j=1}^{N} \frac{x_i x_j}{|x|^2} dx_i \otimes dx_j.$$

Proof. The relation $x = r f(u_1, \ldots, u_{N-1})$ implies

(i) $dx_i = f_i dr + r df_i = f_i dr + r \displaystyle\sum_{h=1}^{N-1} \frac{\partial f_i}{\partial u_h} du_j,$

(ii) $dx_i \otimes dx_i = f_i^2 dr \otimes dr + r dr \otimes \left(\displaystyle\sum_{h=1}^{N-1} \frac{\partial f_i^2}{\partial u_h} \right) du_h + r^2 \displaystyle\sum_{h,k=1}^{N-1} \frac{\partial f_i}{\partial u_h} \frac{\partial f_i}{\partial u_k} du_h \otimes du_k.$

Since $\sum_{i=1}^N f_i^2 = 1$ and $\frac{\partial}{\partial u_h}\left(\sum_{i=1}^N f_i^2\right) = 0$, (ii) yields

$$g_{\mathbb{R}^N} = \sum_{i=1}^N dx_i \otimes dx_i = dr \otimes dr + r^2 \sum_{h,k=1}^{N-1} \left(\sum_{i=1}^N \frac{\partial f_i}{\partial u_h} \frac{\partial f_i}{\partial u_k} \right) du_h \otimes du_k,$$

which, recalling (B.1) and (B.2), proves the first claim. To prove the second requirement we observe analogously that

$$dx_i \otimes dx_j = f_i f_j dr \otimes dr + r dr \otimes \left(\sum_{h=1}^{N-1} \frac{\partial (f_i f_j)}{\partial u_h} \right) du_h + r^2 \sum_{h,k=1}^{N-1} \frac{\partial f_i}{\partial u_h} \frac{\partial f_j}{\partial u_k} du_h \otimes du_k$$

which yields

$$\begin{aligned}\frac{x_i x_j}{|x|^2} dx_i \otimes dx_j =& \frac{f_i^2 f_j^2}{r^2} dr \otimes dr + \frac{1}{2} r dr \otimes \left(\sum_{h=1}^{N-1} \frac{\partial (f_i f_j)^2}{\partial u_h} \right) du_h \\ &+ \frac{1}{2} r^2 \sum_{h,k=1}^{N-1} \frac{\partial f_i^2}{\partial u_h} \frac{\partial f_j^2}{\partial u_k} du_h \otimes du_k.\end{aligned}$$

Summing up and using again $\sum_{i=1}^N f_i^2 = 1$, we get

$$\sum_{i,j=1}^N \frac{x_i x_j}{|x|^2} dx_i \otimes dx_j = dr \otimes dr.$$

□

Proposition B.1.1 shows that if G is the matrix of the Riemannian metric g_{R^N} in the chart (V, ϕ), and H is the matrix of $g_{\mathbb{S}^{N-1}}$ in the chart (U, u) of $\mathbb{S}^{N-1}$, then

$$G(r\omega) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & r^2 H(\omega) \end{array} \right), \qquad \sqrt{\det G(r\omega)} = r^{N-1} \sqrt{\det H(\omega)}. \tag{B.3}$$

Remark B.1.2 *Note that the above calculation gives also the expression of the Riemannian metric $g_{\mathbb{S}^{N-1}(r)}$ of $\mathbb{S}^{N-1}(r)$ in terms of the one of $\mathbb{S}^{N-1}$. Indeed it is easily seen that $r^2 H(\omega)$ is the matrix of $g_{\mathbb{S}^{N-1}(r)}$ associated to the chart $\left(rU, u\left(\frac{1}{r}\cdot\right)\right)$ of $\mathbb{S}^{N-1}(r)$.*

We can now prove the decomposition for the gradient and for the Laplace operator in $\mathbb{R}^N$ in terms of the corresponding operator on the sphere $\mathbb{S}^{N-1}$. We denote with ∇_τ and Δ_0 respectively the tangential gradient and the Laplace-Beltrami operator on the sphere $\mathbb{S}^{N-1}$ and $\nabla_{\mathbb{S}^{N-1}(r)}$, $\Delta_{\mathbb{S}^{N-1}(r)}$ will be the corresponding operator on $\mathbb{S}^{N-1}(r)$. We refer to Sections A.3 and A.4.2 and to [79, Chapter IX] for their definitions and further details.

Lemma B.1.3 *Let $f \in C^1\left(\mathbb{R}^N\right)$, $r > 0$, and let us define $g(\omega) = f(r\omega)$ for every $\omega \in \mathbb{S}^{N-1}$. Then one has*

$$\nabla_\tau g(\omega) = r\, \nabla_{\mathbb{S}^{N-1}(r)} f(r\omega).$$

PROOF. *Let* $\omega \in \mathbb{S}^{N-1}$, $v \in T_\omega\left(\mathbb{S}^{N-1}\right) = T_{r\omega}\left(\mathbb{S}^{N-1}(r)\right)$ *and let* $\alpha :]-\epsilon, \epsilon[\to \mathbb{S}^{N-1}$ *be a differentiable curve of* $\mathbb{S}^{N-1}$ *such that* $\alpha(0) = \omega$, $\alpha'(0) = v$. *Using (A.9) we have*

$$(\nabla_\tau g(\omega), v) = \frac{d}{dt}(g \circ \alpha)\bigg|_{t=0} = \frac{d}{dt}\Big(f\big(r\alpha(t)\big)\Big)\bigg|_{t=0} = (\nabla_{\mathbb{S}^{N-1}(r)} f(r\omega), rv).$$

The arbitrariness of v *proves, then, the claim.*

□

Proposition B.1.4 *Let* ∇_τ, Δ_0 *be the gradient and the Laplace-Beltrami operator of* $\mathbb{S}^{N-1}$. *Then, employing the spherical coordinates* $x = r\omega$, *we have the following relations. For every* $f \in C^1\left(\mathbb{R}^N\right)$

$$\nabla f = \partial_r f\,\frac{x}{r} + \frac{1}{r}\nabla_\tau f.$$

For every $f \in C^2\left(\mathbb{R}^N\right)$

$$\Delta f = \partial_{rr} f + \frac{N-1}{r}\partial_r f + \frac{1}{r^2}\Delta_0 f.$$

Here $\nabla_\tau f(r\omega)$ *and* $\Delta_0 f(r\omega)$ *are taken with respect the variable* ω *and*

$$\partial_r f = \sum_{i=1}^N \partial_i f\,\frac{x_i}{r}, \qquad \partial_{rr} f = \sum_{i,j=1}^N \partial_{ij} f\,\frac{x_i x_j}{r^2}.$$

PROOF. The expressions for ∂_r, ∂_{rr} are an immediate consequence of (i) of Proposition A.1.6. Let $f \in C^1\left(\mathbb{R}^N\right)$. Using (A.7) and the previous lemma, we get, for every $x \in \mathbb{R}^N$,

$$\nabla f(x) = \left(\nabla f(x), \frac{x}{r}\right)\frac{x}{r} + \nabla_{\mathbb{S}^{N-1}(r)} f(x) = \partial_r f(x)\,\frac{x}{r} + \frac{1}{r}\nabla_\tau f(r\omega).$$

To prove the second claim, let $f \in C^2\left(\mathbb{R}^N\right)$. It follows, then, from (A.4) and (B.3) that

$$\begin{aligned}
\Delta f =& \frac{1}{r^{N-1}\sqrt{\det H(\omega)}}\partial_r\left(r^{N-1}\sqrt{\det H(\omega)}\partial_r f\right)\\
&+ \frac{1}{r^{N-1}\sqrt{\det H(\omega)}}\sum_{i,j=1}^{N-1}\partial_{u_i}\left(\frac{1}{r^2}h^{ij}r^{N-1}\sqrt{\det H(\omega)}\,\partial_{u_j} f\right)\\
=&\partial_{rr} f + \frac{N-1}{r}\partial_r f + \frac{1}{r^2}\frac{1}{\sqrt{\det H(\omega)}}\sum_{i,j=1}^{N-1}\partial_{u_i}\left(h^{ij}\sqrt{\det H(\omega)}\,\partial_{u_j} f\right)\\
=&\partial_{rr} f + \frac{N-1}{r}\partial_r f + \frac{1}{r^2}\Delta_0 f
\end{aligned}$$

that is the required property.

□

Remark B.1.5 *With a similar proof one can also prove that*

$$\Delta f = \partial_{rr} f + \frac{N-1}{r}\partial_r f + \Delta_{\mathbb{S}^{N-1}(r)} f.$$

B.2 Spherical harmonics

Introducing spherical coordinates $x = r\omega$, where $r = |x|$ and $\omega = x/|x| \in \mathbb{S}^{N-1}$, we write the Laplace operator as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_0,$$

where Δ_0 is the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^{N-1}$, (see Proposition B.1.4).

A spherical harmonic P^n of order n is the restriction to $\mathbb{S}^{N-1}$ of a homogeneous harmonic polynomial of degree n. We recall some results about spherical harmonics, referring to to [60, Chapter II] and [77, Chapter IV.2] for the proofs and other details. We start with the following well-known lemma.

Lemma B.2.1 *The linear span of spherical harmonics coincides with the set of all polynomials and it is dense in $C(\mathbb{S}^{N-1})$, hence in $L^2(\mathbb{S}^{N-1})$. If P is a spherical harmonic of degree n, then*

$$\Delta_0 P = -(n^2 + (N-2)n)P.$$

The values $-\lambda_n := -\big(n^2 + (N-2)n\big) = -n(n+N-2)$ are the eigenvalues of the Laplace-Beltrami operator Δ_0 on $\mathbb{S}^{N-1}$ and the corresponding eigenspaces consist of all spherical harmonics of degree n and have dimension a_n where $a_0 = 1$, $a_1 = N$ and for $n \geq 2$

$$a_n = \binom{N+n-1}{n} - \binom{N+n-3}{n-2}.$$

For fixed $n \in \mathbb{N}$ let $\mathcal{H}_n$ denote the space of spherical harmonics of degree n, which, by the previous Lemma, coincides with the eigenspace corresponding to the eigenvalue $-\lambda_n$.

Let $\mathbb{Z}_\omega^{(n)}$ be the zonal harmonic of degree n with pole $\omega \in \mathbb{S}^{N-1}$ defined by

$$\mathbb{Z}_\omega^{(n)}(\eta) := \mathbb{Z}^{(n)}(\omega, \eta) = \sum_{i=1}^{a_n} P_i^n(\omega) P_i^n(\eta) \tag{B.4}$$

where $\omega, \eta \in \mathbb{S}^{N-1}$ and $\{P_i^n, i = 1, ..., a_n\}$ is an orthonormal basis of spherical harmonics of degree n whose cardinality is given by $a_n = dim\,(\mathcal{H}_n) = \binom{N+n-1}{n} - \binom{N+n-3}{n-2}$. We collect in the following proposition some basic properties of zonal harmonics.

Proposition B.2.2 *For fixed $n \in \mathbb{N}$ the sum in (B.4) is independent of the choice of the orthonormal basis of $\mathcal{H}_n$. The zonal harmonic $\mathbb{Z}_\omega^{(n)}$ is characterized by the relation*

$$P(\omega) = \int_{\mathbb{S}^{N-1}} P(\eta)\mathbb{Z}_\omega^{(n)}(\eta) d\eta$$

valid for all $P \in \mathcal{H}_n$ and $\omega \in \mathbb{S}^{N-1}$. Moreover *(i) for all* $\omega, \eta \in \mathbb{S}^{N-1}$ $\mathbb{Z}_\omega^{(n)}(\eta) = \mathbb{Z}_\eta^{(n)}(\omega)$ *and if* T *is an orthogonal transformation then* $Z_{T\omega}^{(n)}(T\eta) = Z_\omega^{(n)}(\eta)$;

(ii) the following uniform estimates hold

$$\sup_{\eta\in\mathbb{S}^{N-1}} |Z_\omega^{(n)}(\eta)| = Z_\omega^{(n)}(\omega) = \frac{a_n}{|\mathbb{S}^{N-1}|}$$

and, from the asymptotic behaviour $a_n \sim n^{N-2}$ *for* $n \to \infty$, *we have*

$$\sup_{\omega,\eta\in\mathbb{S}^{N-1}} |Z_\omega^{(n)}(\eta)| \approx \frac{n^{N-2}}{|\mathbb{S}^{N-1}|};$$

(iii) for all $P \in \mathcal{H}_n$

$$\sup_{\omega\in\mathbb{S}^{N-1}} |P(\omega)| \leq \sqrt{\frac{a_n}{|\mathbb{S}^{N-1}|}}\|P\|_{L^2(\mathbb{S}^{N-1})}.$$

Let us consider now, for a fixed $\alpha \in \mathbb{R}$, the weighted space $L^2\left(\mathbb{R}^N, |x|^\alpha dx\right)$ and let

$$L_n^2 = L^2((0,\infty), r^{\alpha+N-1}\, dr) \otimes \mathcal{H}_n = \bigoplus_{i=1}^{a_n}\left(L^2((0,\infty), r^{\alpha+N-1}\, dr) \otimes P_j^n\right) = \bigoplus_{i=1}^{a_n} L^2_{P_i^n} \quad \text{(B.5)}$$

where $\{P_i^n, i \in J\}$ is an orthonormal basis of $\mathcal{H}_n$ and

$$L_P^2 = L^2((0,\infty), r^{\alpha+N-1}\, dr) \otimes P. \quad \text{(B.6)}$$

Zonal harmonics provide a simple way to describe the orthogonal projection of $L^2\left(\mathbb{R}^N, |x|^\alpha dx\right)$ onto L_n^2.

Proposition B.2.3 *Let* $\alpha \in \mathbb{R}$. *The following properties hold.*

(i) $L^2\left(\mathbb{R}^N, |x|^\alpha dx\right) = \bigoplus_{n=0}^\infty L_n^2$.

(ii) for every $u \in L^2\left(\mathbb{R}^N, |x|^\alpha dx\right)$ *the orthogonal projection on* L_n^2 *is given by*

$$Q_n(u)(r,\eta) := \sum_{i=1}^{a_n} P_i^n(\eta)\int_{\mathbb{S}^{N-1}} u(r,\omega)P_i^n(\omega)d\omega = \int_{\mathbb{S}^{N-1}} u(r,\omega)Z_\omega^{(n)}(\eta)d\omega$$

and moreover $u = \sum_{n=0}^\infty Q_n(u)$ *in* $L^2\left(\mathbb{R}^N, |x|^\alpha dx\right)$.

Proof. Since the weight $|x|^\alpha$ is radial, the assertion is a simple reformulation of [77, Lemma 4.2.18]. □

Note that, if P is a normalized spherical harmonic of degree n, the projection on L_P^2 is given by

$$Q_P(u)(r,\eta) := P(\eta)\int_{\mathbb{S}^{N-1}} u(r,\omega)P(\omega)d\omega.$$

Moreover $Q_n = \sum_{i=1}^{a_n} Q_{P_i^n}$, Q_P is symmetric and, for every $u \in C_c^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$, Q_P commutes with the radial derivative that is

$$(Q_P u)_r = P(\eta)\int_{\mathbb{S}^{N-1}} u_r(r,\omega)P(\omega)\, d\omega = Q_p u_r.$$

This follows since $u_r = \nabla u \cdot \frac{x}{|x|}$, by differentiating under the integral sign.

B.3 Hecke-Funk formula

Proposition B.2.2 assures that the zonal harmonics $\mathbb{Z}_\omega^{(n)}$ are invariant under orthogonal transformation. We present some basic properties about functions that presents this type of spherical simmetry and their expansion in terms of zonal harmonics referring to [77] and [60] for the proofs and further details.

Let $f \in L^2(\mathbb{S}^{N-1})$. In analogy with the results presented in section B.2, the orthogonal projection of f onto $\mathcal{H}_n$ is given by

$$f_n(\eta) := \int_{\mathbb{S}^{N-1}} f(\omega) Z_\omega^{(n)}(\eta) d\omega$$

and, moreover, one has the spherical harmonics expansion

$$f = \sum_{n=0}^{\infty} f_n \quad \text{in } L^2(\mathbb{S}^{N-1}). \tag{B.7}$$

If $f \in C^\infty(\mathbb{S}^{N-1})$ the last sum converges in the topology of $C^\infty(\mathbb{S}^{N-1})$ (see [60, corollary 2.49]).

Definition B.3.1 *Given a point $\epsilon_0 \in \mathbb{S}^{N-1}$, f is called a* ***zonal function of pole*** ϵ_0 *if and only if it is invariant under the action of orthogonal transformations fixing ϵ_0 i.e. $f(T\omega) = f(\omega)$ for every $\omega \in \mathbb{S}^{N-1}$ and $T \in O(N)$ such that $T\epsilon_0 = \epsilon_0$*

Using Lemma 2.21 in [60], it can be proved that a zonal function f of pole ϵ_0 depends only on the value $\omega \cdot \epsilon_0$ and so it is characterized by a profile function $\phi : [-1, 1] \to \mathbb{C}$ which satisfies

$$f(\omega) = \phi(\omega \cdot \epsilon_0), \quad \omega \in \mathbb{S}^{N-1}. \tag{B.8}$$

Proposition B.2.2 implies that, for a fixed $\omega \in \mathbb{S}^{N-1}$, the zonal harmonics $\mathbb{Z}_\omega^{(n)}$ are zonal functions of pole ω. In order to characterize their profile functions we introduce the system of Gegenbauer polynomials $P_n^{\frac{N-2}{2}}(t)$ defined through the generating function

$$\left(1 - 2rt + r^2\right)^{-\frac{N-2}{2}} = \sum_{n=0}^{\infty} P_n^{\frac{N-2}{2}}(t)\, r^n, \quad 0 \le |r| < 1,\ |t| \le 1.$$

For every $n \in \mathbb{N}_0$, $P_n^{\frac{N-2}{2}}$ is a polynomial of degree n and the system of Gegenbauer polynomials forms an orthogonal basis for the space $L^2\left([-1, 1], (1 - t^2)^{\frac{N-3}{2}} dt\right)$ (see [2, Chapter 22].

With the notation introduced, the profile function of the zonal harmonic $\mathbb{Z}_\omega^{(n)}$ turns out to coincide, up to a normalizing constant b_n, with the Gegenbauer polynomial $P_n^{\frac{N-2}{2}}$ (see [77, Theorem 2.14] and [60, Theorem 2.24]). Therefore we have:

$$\mathbb{Z}_\omega^{(n)}(\eta) = b_n\, P_n^{\frac{N-2}{2}}(\omega \cdot \eta),$$

Recalling property (ii) in Proposition B.2.2, the relation

$$\frac{\dim(\mathcal{H}_n)}{|\mathbb{S}^{N-1}|} = \mathbb{Z}_\omega^{(n)}(\omega) = b_n P_n^{\frac{N-2}{2}}(1)$$

allows to compute

$$b_n = \frac{|\mathbb{S}^{N-1}|}{\dim(\mathcal{H}_n)} \binom{N-3+n}{n} = |\mathbb{S}^{N-1}| \frac{N-2}{N+2n-2}.$$

For a zonal function f of pole $\epsilon_0 \in \mathbb{S}^{N-1}$, the spherical harmonics expansion (B.7) takes the form

$$f(\omega) = \sum_{n=0}^{\infty} d_n \mathbb{Z}_{\epsilon_0}^{(n)}(\omega),$$

where the coefficients d_n depend on the profile function ϕ of f. In order to prove that, we give a preliminary lemma whose proof can be found in [60] (Theorem 2.39). We point out that in [60] the author makes use of the normalized measure of $\mathbb{S}^{N-1}$ and of the Legendre polynomials $P_{n,N} := \frac{1}{P_n^{\frac{N-2}{2}}(1)} P_n^{\frac{N-2}{2}}$.

Lemma B.3.2 (Hecke-Funk formula) *Let $f \in L^2(\mathbb{S}^{N-1})$ be a zonal function of pole ϵ_0 with profile function ϕ. Then, for every $n \in N_0$ and $h \in \mathcal{H}_n$, one has*

$$\int_{\mathbb{S}^{N-1}} f(\omega)h(\omega)\,d\omega = \left[\frac{|S^{N-2}|}{P_n^{\frac{N-2}{2}}(1)} \int_{-1}^{1} \phi(t) P_n^{\frac{N-2}{2}}(t) \left(1-t^2\right)^{\frac{N-3}{2}} dt \right] h(\epsilon_0) \tag{B.9}$$
$$= \left[\frac{|\mathbb{S}^{N-1}|}{\dim(\mathcal{H}_n)} \int_{\mathbb{S}^{N-1}} f(\omega) \mathbb{Z}_{\epsilon_0}^{(n)}(\omega)\,d\omega \right] h(\epsilon_0).$$

We are ready now to prove the following proposition.

Proposition B.3.3 *Let $f \in L^2(\mathbb{S}^{N-1})$ be a zonal function of pole $\epsilon_0 \in \mathbb{S}^{N-1}$ with profile function ϕ. Then f admits the following expansion in terms of zonal harmonics $\mathbb{Z}_{\epsilon_0}^{(n)}$:*

$$f(\eta) = \sum_{n=0}^{\infty} d_n \mathbb{Z}_{\epsilon_0}^{(n)}(\eta). \tag{B.10}$$

The coefficients d_n are given by

$$d_n = \frac{|S^{N-2}|}{P_n^{\frac{N-2}{2}}(1)} \int_{-1}^{1} \phi(t) P_n^{\frac{N-2}{2}}(t) \left(1-t^2\right)^{\frac{N-3}{2}} dt \tag{B.11}$$
$$= \frac{|\mathbb{S}^{N-1}|}{\dim(\mathcal{H}_n)} \int_{\mathbb{S}^{N-1}} f(\omega) \mathbb{Z}_{\epsilon_0}^{(n)}(\omega)\,d\omega.$$

Proof. Let f_n be the orthogonal projection of f on $\mathcal{H}_n$. Using the previous lemma and the symmetry of the zonal harmonics we obtain

$$\begin{aligned} f_n(\eta) &= \int_{\mathbb{S}^{N-1}} f(\omega) Z_\omega^{(n)}(\eta) d\omega \\ &= \frac{|S^{N-2}|}{P_n^{\frac{N-2}{2}}(1)} \int_{-1}^{1} \phi(t) P_n^{\frac{N-2}{2}}(t) \left(1-t^2\right)^{\frac{N-3}{2}} dt\, Z_{\epsilon_0}^{(n)}(\eta) \\ &= \frac{1}{b_n P_n^{\frac{N-2}{2}}(1)} \int_{\mathbb{S}^{N-1}} f(\omega) \mathbb{Z}_{\epsilon_0}^{(n)}(\omega)\, d\omega\, Z_{\epsilon_0}^{(n)}(\eta). \end{aligned}$$

The equality $b_n P_n^{\frac{N-2}{2}}(1) = \frac{\dim(\mathcal{H}_n)}{|\mathbb{S}^{N-1}|}$ proves, then, the claim. □

B.4 Radial and angular derivatives

In this section we study further properties about the differentiability of the spherical harmonics and the Laplace-Beltrami operator Δ_0 on the sphere. For $x \in \mathbb{R}^N$ we use spherical coordinates to write $x = r\omega$, where $r := |x|$, $\omega := \frac{x}{|x|} \in \mathbb{S}^{N-1}$. For every $u \in C^2(\mathbb{R}^N)$ we denote by $D_r u$, $D_{rr} u$ the radial derivatives of u and by $\nabla_\tau u$ the tangential component of its gradient. They are defined, recalling Proposition B.1.4, through the formulas

$$D_r u = \sum_{i=1}^{N} D_i u \frac{x_i}{r}, \qquad D_{rr} u = \sum_{i,j=1}^{N} D_{ij} u \frac{x_i x_j}{r^2}, \qquad \nabla u = D_r u \frac{x}{|x|} + \frac{\nabla_\tau u}{r} \tag{B.12}$$

and, moreover, we have the following relation between the Laplace operator and Δ_0:

$$\Delta = D_{rr} + \frac{N-1}{r} D_r + \frac{\Delta_0}{r^2}.$$

The operator $A := r D_r$ is the Euler operator $\sum_{i=1}^n x_i D_i$ and, for a given function u, Euler's theorem implies that u is α-homogeneous if and only if $Au = \alpha u$.
For $i, j = 1, \ldots, N$ we introduce, moreover, the angular operators S_{ij} defined as

$$S_{ij} := x_i D_j - x_j D_i.$$

They are first-order differential operators and have a central role in the analysis on the sphere, since they allow to decompose the Laplace-Beltrami operator into a sum of second-order angular derivatives. Obviously $S_{ii} = 0$ and $S_{ij} = -S_{ji}$ so it is sufficient to consider S_{ij} for $i < j$.
We collect in the next proposition some basic properties about the angular derivatives S_{ij}.

Proposition B.4.1 *Denote by $[A, B] = AB - BA$ the commutator of A, B and by δ_i^j the Kronecker delta. The following properties hold.*

(i) *For every $i < j$ and $h < k$ we have*

$$[S_{ij}, S_{hk}] = -\delta_i^h S_{jk} + \delta_i^k S_{jh} + \delta_j^h S_{ik} - \delta_j^k S_{ih}.$$

In particular S_{ij} and S_{hk} commute if and only if $(i,j) = (h,k)$ or $\{i,j\} \cap \{h,k\} = \emptyset$

(ii) *For every radial function $u = u(r)$, with $u \in C^2(\mathbb{R}^N)$ and for every $v \in C^2(\mathbb{R}^N)$ we have $S_{ij}u = 0$ and $S_{ij}(uv) = uS_{ij}v$.*

(iii) *For every radial differential operator $E = \sum_{k=0}^{n} a_k(r) \dfrac{d^k}{dr^k}$ we have $S_{ij}E = ES_{ij}$.*

(iv) $\Delta\, S_{ij} = S_{ij}\, \Delta$ *and* $\Delta_0\, S_{ij} = S_{ij}\, \Delta_0$.

Proof. (i) easily follows by a straightforward computation.
Let $u, v \in C^2(\mathbb{R}^N)$ with $u = u(r)$; we have $S_{ij}u = x_i u'(r)\frac{x_j}{r} - x_j u'(r)\frac{x_i}{r} = 0$. Moreover $S_{ij}(uv) = uS_{ij}v + (S_{ij}u)v = uS_{ij}v$ that proves (ii).
To prove (iii) we observe that for $E = a(r)D_r$ we have, using (ii),

$$\begin{aligned}
S_{ij}(a(r)D_r) &= S_{ij}\left(\frac{a(r)}{r}\sum_{k=1}^{N} x_k D_k\right) = \frac{a(r)}{r}\,(x_i D_j - x_j D_i)\sum_{k=1}^{N} x_k D_k \\
&= \frac{a(r)}{r}\left[x_i D_j + x_i \sum_{k=1}^{N} x_k D_{jk} - x_j D_i - x_j \sum_{k=1}^{N} x_k D_{ik}\right] \\
&= \frac{a(r)}{r}\left[S_{ij} + \sum_{k=1}^{N} x_k (x_i D_{jk} - x_j D_{ik})\right].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(a(r)D_r)S_{ij} &= \left(\frac{a(r)}{r}\sum_{k=1}^{N} x_k D_k\right) S_{ij} = \frac{a(r)}{r}\sum_{k=1}^{N} x_k D_k\,(x_i D_j - x_j D_i) \\
&= \frac{a(r)}{r}\left[x_i D_j + \sum_{k=1}^{N} x_k x_i D_{jk} - x_j D_i - \sum_{k=1}^{N} x_k x_j D_{ik}\right] \\
&= \frac{a(r)}{r}\left[S_{ij} + \sum_{k=1}^{N} x_k (x_i D_{jk} - x_j D_{ik})\right].
\end{aligned}$$

Comparing the last expressions we have $S_{ij}E = ES_{ij}$; the general case follows easily by induction.
Finally, to prove (iv), let us suppose, without losing generality, $i = 1$, $j = 2$. We have

$$\begin{aligned}
\Delta S_{12} &= \sum_{i=1}^{N} D_{ii}\,(x_1 D_2 - x_2 D_1) = \sum_{i=1}^{N} (x_1 D_2 D_{ii} + 2D_i x_1 D_i D_2 - x_2 D_1 D_{ii} - 2D_i x_2 D_i D_1) \\
&= \sum_{i=1}^{N} (x_1 D_2 - x_2 D_1)\, D_{ii} + 2D_1 D_2 - 2D_2 D_1 = S_{12}\Delta
\end{aligned}$$

and so, S_{12} commutes with Δ.
Using property (ii) we have also

$$S_{12}(r^2\Delta) = r^2 S_{12}\Delta = (r^2\Delta)S_{12}.$$

Since $r^2\Delta = r^2 D_{rr} + (N-1)rD_r + \Delta_0$ it follows immediately from properties (iii) and the fact that S_{12} commutes with $r^2\Delta$, that $\Delta_0\, S_{ij} = S_{ij}\,\Delta_0$.

$\square$

Before to state the main results it is worth noting that every function $f \in C^2(\mathbb{S}^{N-1})$ is extendible on a neighbourhood of $\mathbb{S}^{N-1}$ preserving the same degree of regularity; therefore, in what follows, we can always suppose f defined on an open set of $\mathbb{R}^N$ containing $\mathbb{S}^{N-1}$. In particular, ∇f, Δf and $S_{ij}f$ are well defined. For instance, if we define $\tilde{f}$ on $\mathbb{R}^N \setminus \{0\}$ by $\tilde{f}(y) := f(\frac{y}{|y|})$, being $\tilde{f}$ constant along every radial direction, it follows immediately by the decomposition of ∇ and Δ that

$$\nabla_\tau f(x) = \nabla\tilde{f}(x), \qquad \Delta_0 f(x) = \Delta\tilde{f}(x), \qquad \text{for every } x \in \mathbb{S}^{N-1}.$$

Let now $\omega \in \mathbb{S}^{N-1}$ and $v \in T_\omega(\mathbb{S}^{N-1})$; the tangential derivative of f in ω in the direction v is given by

$$(\nabla f(\omega), v) = (\nabla_\tau f(\omega), v).$$

Setting $\omega = (\omega_1, ..., \omega_N) \in \mathbb{S}^{N-1}$, let us define for $i < j$

$$\omega_{ij} := (0, .., -\omega_j, 0, .., \omega_i, 0, ..) \in T_\omega(\mathbb{S}^{N-1})$$

i.e the vector who has $-\omega_j$ and ω_i as respectively the i-th and the j-th component and zeros in the others entries. $(\omega_{ij})_{1\le i<j\le N}$ is, by construction, a system of generators of $T_\omega(\mathbb{S}^{N-1})$. The next propositions clarify the role and the interaction among the operators introduced so far. We begin by a geometric Lemma.

Lemma B.4.2 *Let $\omega \in \mathbb{S}^{N-1}$ and $x, y \in \omega^\perp$. Then*

$$(x, y) = \sum_{i<j} (x, \omega_{ij})(y, \omega_{ij}), \qquad |x|^2 = \sum_{i<j} |(x, \omega_{ij})|^2. \tag{B.13}$$

Proof. Since $(\omega_{ij}, x) = \omega_i x_j - \omega_j x_i$, $(\omega_{ij}, y) = \omega_i y_j - \omega_j y_i$, we have

$$\begin{aligned}
\sum_{i<j} (x, \omega_{ij})(y, \omega_{ij}) &= \sum_{i<j} (\omega_i x_j - \omega_j x_i)(\omega_i y_j - \omega_j y_i) \\
&= \sum_{i<j} \left(\omega_i^2 x_j y_j + \omega_j^2 x_i y_i - \omega_j\omega_i x_i y_j - \omega_i\omega_j x_j y_i\right) \\
&= \sum_{i\neq j} \omega_i^2 x_j y_j - \sum_{i\neq j} \omega_j\omega_i x_i y_j.
\end{aligned}$$

Adding and subtracting, in the last formula, the term $\sum_{i=1}^N \omega_i^2 x_i y_i$ we obtain

$$\sum_{i<j}(x,\omega_{ij})(y,\omega_{ij}) = \sum_{i,j=1}^N \omega_i^2 x_j y_j - \sum_{i,j=1}^N \omega_j\omega_i x_i y_j = (x,y)|\omega|^2 - (\omega,x)(\omega,y) = (x,y).$$

Choosing $x=y$ we get $\sum_{i<j}|(x,\omega_{ij})|^2 = |x|^2$.

□

Proposition B.4.3 *Let f, g be C^1 functions defined on a neighbourhood of $\mathbb{S}^{N-1}$. The following properties hold.*

(i) $S_{ij}f(\omega)$ is the tangential derivative of f in ω in the direction ω_{ij} i.e.

$$S_{ij}f(\omega) = (\nabla f(\omega), \omega_{ij}) = (\nabla_\tau f(\omega), \omega_{ij}).$$

(ii) The j-th component of $\nabla_\tau f$ satisfies

$$(\nabla_\tau f(\omega))_j = \sum_{i\neq j}\omega_i\, S_{ij}f(\omega), \qquad \omega\in\mathbb{S}^{N-1}.$$

(iii) For every $\omega\in\mathbb{S}^{N-1}$

$$(\nabla_\tau f(\omega), \nabla_\tau g(\omega)) = \sum_{i<j} S_{ij}f(\omega)S_{ij}g(\omega), \qquad |\nabla_\tau f(\omega)|^2 = \sum_{i<j}|S_{ij}f(\omega)|^2.$$

Proof. The first assertion is an immediate consequence of the definition of S_{ij}. For the second sentence we observe that for $\omega\in\mathbb{S}^{N-1}$ it follows from (B.12) that

$$\begin{aligned}(\nabla_\tau f(\omega))_j &= D_j f(\omega) - \omega_j D_r f(\omega) = D_j f(\omega)\sum_{i=1}^N \omega_i^2 - \omega_j\sum_i \omega_i D_i f(\omega)\\ &= \sum_{i=1}^N \omega_i\left[\omega_i D_j f(\omega) - \omega_j D_i f(\omega)\right] = \sum_{i\neq j}\omega_i\, S_{ij}f(\omega).\end{aligned}$$

(iii) is an immediate consequence of (i) and (B.13). □

We can now show the announced decomposition of the Laplace-Beltrami operator as a sum of second order angular derivatives.

Proposition B.4.4

$$\sum_{i<j} S_{ij}^2 f = \Delta_0 f, \qquad \textit{for every} \quad f\in C^2(\mathbb{R}^N),$$

where, using the spherical coordinates $x=r\omega$, $\Delta_0 f(r\omega)$ acts on the ω-variable.

In particular on $\mathbb{S}^{N-1}$ we have the decomposition $\sum_{i<j} S_{ij}^2 = \Delta_0$.

PROOF. We observe preliminary that

$$r^2 D_{rr} = \sum_{i,j=1}^{N} x_i x_j D_{ij} = 2\sum_{i<j} x_i x_j D_{ij} + \sum_{i=1}^{N} x_i^2 D_{ii} \tag{B.14}$$

and

$$S_{ij}^2 = (x_i D_j - x_j D_i)(x_i D_j - x_j D_i) = x_i^2 D_{jj} + x_j^2 D_{ii} - x_i D_i - x_j D_j - 2x_i x_j D_{ij}.$$

Summing over $i < j$ the last expression and using (B.14) we obtain

$$\begin{aligned}\sum_{i<j} S_{ij}^2 &= \sum_{i<j} \left(x_i^2 D_{jj} + x_j^2 D_{ii}\right) - \sum_{i<j} (x_i D_i + x_j D_j) - 2\sum_{i<j} x_i x_j D_{ij} \\ &= \sum_{i\neq j} x_j^2 D_{ii} - \sum_{i\neq j} x_i D_i - r^2 D_{rr} + \sum_{i=1}^{N} x_i^2 D_{ii} \\ &= \sum_{i=1}^{N} \left(r^2 - x_i^2\right) D_{ii} - (N-1) r D_r - r^2 D_{rr} + \sum_{i=1}^{N} x_i^2 D_{ii} \\ &= r^2 \Delta - (N-1) r D_r - r^2 D_{rr}.\end{aligned}$$

Recalling that $r^2\Delta = r^2 D_{rr} + (N-1) r D_r + \Delta_0$ we get the conclusion.

□

Next we have the following integration by parts formula.

Proposition B.4.5 *For every $i < j$ and for every $u, v \in C^1(\mathbb{S}^{N-1})$*

$$\int_{\mathbb{S}^{N-1}} (S_{ij} u) v \, d\sigma = -\int_{\mathbb{S}^{N-1}} u (S_{ij} v) \, d\sigma.$$

PROOF. Let $i < j$ and $u, v \in C^1(\mathbb{S}^{N-1})$. We have obviously $S_{ij}(uv) = S_{ij} u\, v + u\, S_{ij} v$ and so

$$\int_{\mathbb{S}^{N-1}} (S_{ij} u) v \, d\sigma = -\int_{\mathbb{S}^{N-1}} u\, S_{ij} v \, d\sigma + \int_{\mathbb{S}^{N-1}} S_{ij}(uv) \, d\sigma.$$

Using the Gauss-Green theorem the claim immediately follows by observing that

$$\int_{\mathbb{S}^{N-1}} S_{ij}(uv) \, d\sigma = \int_{\mathbb{S}^{N-1}} \omega_i D_j(uv) - \omega_j D_i(uv) \, d\sigma = \int_{B(0,1)} D_{ij}(uv) - D_{ji}(uv) \, dx = 0.$$

□

Let $\mathcal{H}_n$ be the set of the spherical harmonics of order n. We recall that from Proposition B.2.2, we have

$$\|\phi\|_\infty \le \sqrt{dim \mathcal{H}_n}\, \|\phi\|_{L^2(S^{N-1})}, \qquad \text{for every } \phi \in \mathcal{H}_n.$$

The following proposition shows that $\mathcal{H}_n$ is preserved by S_{ij}.

Proposition B.4.6 *Let $\phi \in \mathcal{H}_n$. Then, for every $i < j$, $S_{ij}\phi \in \mathcal{H}_n$. Moreover, if $-\lambda_n = -n(n+N-2)$ is the eigenvalue associated with ϕ, then*

$$\|S_{ij}\phi\|_\infty \leq \sqrt{\lambda_n \mathrm{dim}\mathcal{H}_n}\, \|\phi\|_{L^2(S^{N-1})}, \tag{B.15}$$

$$\|\nabla_\tau \phi\|_\infty \leq \sqrt{\frac{N(N-1)}{2}\lambda_n \mathrm{dim}\mathcal{H}_n}\, \|\phi\|_{L^2(S^{N-1})}. \tag{B.16}$$

PROOF. Let $p(x) = r^n\phi(\omega)$; p is a homogeneous harmonic polynomial by construction and $S_{ij}p$ is a polynomial of the same degree by the definition of S_{ij}. Moreover, using (iv) of Proposition B.4.1, we have

$$\Delta S_{ij}p = S_{ij}\Delta p = 0$$

and so $S_{ij}p$ is harmonic; since from (ii) of Proposition B.4.1 $S_{ij}P(r\omega) = r^n S_{ij}\phi(\omega)$ we have $S_{ij}\phi \in \mathcal{H}_n$. In particular,

$$\Delta_0 S_{ij}\phi = S_{ij}\Delta_0\phi = -\lambda_n S_{ij}\phi$$

and

$$\|S_{ij}\phi\|_\infty \leq \sqrt{\mathrm{dim}\mathcal{H}_n}\, \|S_{ij}\phi\|_{L^2(S^{N-1})}. \tag{B.17}$$

Now, using propositions B.4.5 and B.4.4,

$$\begin{aligned}
\sum_{i<j} \|S_{ij}\phi\|^2_{L^2(S^{N-1})} &= \sum_{i<j} \int_{\mathbb{S}^{N-1}} S_{ij}\phi\, S_{ij}\phi\, d\sigma = -\sum_{i<j} \int_{\mathbb{S}^{N-1}} (S^2_{ij}\phi)\phi\, d\sigma \\
&= -\int_{\mathbb{S}^{N-1}} \sum_{i<j} (S^2_{ij}\phi)\phi\, d\sigma = -\int_{\mathbb{S}^{N-1}} \Delta_0\phi\, \phi\, d\sigma \\
&= \lambda_n \|\phi\|^2_{L^2(S^{N-1})}.
\end{aligned}$$

The last inequality implies

$$\|S_{ij}\phi\|^2_{L^2(S^{N-1})} \leq \sqrt{\lambda_n} \|\phi\|_{L^2(S^{N-1})}$$

which, combined with (B.17), yields (B.15). Finally, applying (B.13) and (B.15), it follows for every $\omega \in \mathbb{S}^{N-1}$

$$|\nabla_\tau \phi(\omega)|^2 = \sum_{i<j} |S_{ij}\phi(\omega)|^2 \leq \frac{N(N-1)}{2}\lambda_n \mathrm{dim}\mathcal{H}_n\, \|\phi\|_{L^2(S^{N-1})},$$

which proves (B.16).

□

Corollary B.4.7 *The tangential derivative of the zonal harmonics* $\mathbb{Z}^{(n)}$ *satisfies*

$$\|\nabla_\tau \mathbb{Z}^{(n)}\|_\infty \leq \sqrt{\frac{1}{|\mathbb{S}^{N-1}|}\frac{N(N-1)}{2}\lambda_n}\,(\mathrm{dim}\mathcal{H}_n)^{\frac{3}{2}}\,.$$

In particular for a constant $C = C(N)$ *we have*

$$\|\nabla_\tau \mathbb{Z}^{(n)}\|_\infty \leq C n^{\frac{3N-4}{2}}.$$

PROOF. The first property is an immediate consequence of (B.16) and of the estimate

$$\|\mathbb{Z}^{(n)}\|_{L^2(\mathbb{S}^{N-1})} \leq \sqrt{|\mathbb{S}^{N-1}|}\,\|\mathbb{Z}^{(n)}\|_\infty \leq \sqrt{|\mathbb{S}^{N-1}|}\,\frac{\mathrm{dim}\mathcal{H}_n}{|\mathbb{S}^{N-1}|}.$$

To prove the second inequality it is sufficient to recall the asymptotic behaviours $dim\mathcal{H}_n \sim n^{N-2}$, $\lambda_n \sim n^2$ for $n \to \infty$.

□

Appendix C

Gaussian heat kernel bounds via Phragmén-Lindelöf theorem

Let (M, d, μ) be a metric measure space and let A be a non-negative self-adjoint operator acting on $L^2(M, d\mu)$. In this appendix we expose a technique to prove Gaussian estimates for the analytic semigroup $\{e^{-zA},\, z \in \mathbb{C}_+\}$ generated by A. If $p(z, x, y)$ is the corresponding heat kernel, we show that in presence of some L^2 Gaussian estimates, the so-called Davies-Gaffney estimates (C.24), the on-diagonal upper bounds

$$p(t, x, x) \leq Ct^{-\frac{D}{2}}, \quad \forall\, t > 0,\, x \in M,$$

imply precise off-diagonal Gaussian estimates of the form

$$|p(z, x, y)| \leq C(Re\, z)^{-D/2} \left(1 + Re\, \frac{d^2(x, y)}{4z}\right)^{D/2} \exp\left(-Re\, \frac{d^2(x, y)}{4z}\right),$$

where $z \in \mathbb{C}_+$, $x, y \in M$.

The results presented here are mainly based on [21], where Coulhon and Sikora introduce a method for deducing Gaussian bounds which relies on the Phragmén-Lindelöf theorem and which has the advantage to be applicable to any uniformly bounded analytic family of operators $\{\Psi(z) \colon z \in \mathbb{C}_+\}$ satisfying the Davies-Gaffney estimates (C.24).

In the first section we show the equivalence between the ultracontractive estimate $\|e^{-tA}\|_{1\to\infty} \leq ct^{-\frac{D}{2}}$, $t > 0$, and the Gagliardo-Nirenberg type inequalities (C.3). Sections C.2 and C.3 introduce, respectively, some theorems of Phragmén-Lindelöf type and the Davies-Gaffney estimates; Section C.4 combines together the results of the previous sections and prove the upper Gaussian estimates for the heat kernel p.

In the following (M, μ) will be a σ-finite measure space. For $1 \leq p, q \leq +\infty$, we write $\|f\|_p$ to denote the norm of a function f in $L^p(M, d\mu)$, $\langle ., . \rangle$ to denote the scalar product in $L^2(M, d\mu)$, and, for a bounded linear operator T from $L^p(M, d\mu)$ to $L^q(M, d\mu)$, we write $\|T\|_{p\to q}$ for the operator norm of T.

C.1 Ultracontractivity

Let (M, μ) be a σ-finite measure space and let e^{-tA} be a strongly continuous semigroup on $L^2(M, \mu)$ with generator $-A$. In this section we find sufficient condition on A which guarantee, for some constants $c,\ D > 0$, the existence of L^1-L^∞ estimates of the form

$$\|e^{-tA}\|_{1\to\infty} \le ct^{-\frac{D}{2}}, \quad t > 0. \tag{C.1}$$

We refer the reader to [67] for finding the notion needed and the proof of the results presented here.

Let $\mathfrak{a}$ be a densely defined, symmetric, accretive, and closed form on $L^2(M, \mu)$. Let A be the operator associated to $\mathfrak{a}$, that is

$$\begin{aligned} D(A) &:= \left\{u \in D(\mathfrak{a})\ ;\ \exists v \in L^2(M, \mu)\ \text{ s.t. } \mathfrak{a}(u, w) = \langle v, w\rangle \quad \forall w \in D(\mathfrak{a})\right\}, \\ Au &:= v. \end{aligned}$$

A is, by construction, a non-negative self-adjoint operator and, by standard results, $-A$ generates a contractive analytic C_0-semigroup $\left\{e^{-zA},\ z \in \mathbb{C}_+\right\}$ on $L^2(M, \mu)$ (see, for example, [67, Theorem 1.53]).

We start by observing that, in order to derive (C.1), it is sufficient to prove that

$$\|e^{-tA}\|_{2\to\infty} \le ct^{-\frac{D}{4}}, \quad \forall t > 0.$$

Indeed, since L is self-adjoint, we obtain, by duality, the same estimate for the $L^1 - L^2$ norm, that is

$$\|e^{-tA}\|_{1\to 2} \le ct^{-\frac{D}{4}}, \quad \forall t > 0.$$

Combining together the last inequalities we get, for $K = c^2 2^{\frac{D}{2}}$,

$$\|e^{-tA}\|_{1\to\infty} \le \|e^{-\frac{t}{2}A}\|_{1\to 2}\|e^{-\frac{t}{2}A}\|_{2\to\infty} \le Kt^{-\frac{D}{2}}, \quad \forall t > 0. \tag{C.2}$$

The following Theorem shows that the $L^2 - L^\infty$ polynomial decay of $\left(e^{-tA}\right)_{t\ge 0}$ is equivalent to a Gagliardo-Nirenberg type inequality.

Theorem C.1.1 [67, Theorem 6.2] *Assume that the semigroup $\left(e^{-tA}\right)_{t\ge 0}$ is L^∞-contractive and let $D > 0$. The following assertions are equivalent:*

(i) There exists a constant $c > 0$ such that

$$\|e^{-tA}\|_{2\to\infty} \le ct^{-\frac{D}{4}}, \quad \forall t > 0.$$

(ii) For every $q \in (2, \infty]$ such that $D\frac{q-2}{2q} < 1$, one has, for some $c' > 0$,

$$\|u\|_q \le c'\mathfrak{a}(u, u)^{D\frac{q-2}{4q}} \|u\|_2^{1-D\frac{q-2}{2q}}, \quad \forall u \in D(\mathfrak{a}). \tag{C.3}$$

(iii) There exists $q \in (2,\infty]$ such that $D\frac{q-2}{2q} < 1$, one has, for some $c' > 0$,

$$\|u\|_q \leq c'\mathfrak{a}(u,u)^{D\frac{q-2}{4q}} \|u\|_2^{1-D\frac{q-2}{2q}}, \quad \forall u \in D(\mathfrak{a}).$$

We extend, now, the $L^1 - L^\infty$ estimates (C.1) to the analytic semigroup $\left\{e^{-zA},\ z \in \mathbb{C}_+\right\}$. We state, first, a famous result that allows to deduce from the ultracontractive estimates (C.4), the existence of the integral kernel.

Theorem C.1.2 (Dunford-Pettis) *Any bounded operator T from $L^1(M,\mu)$ to $L^\infty(M,\mu)$ is an integral operator, that is there exists $K \in L^\infty(M \times M)$ such that*

$$Tf(x) = \int_M K(x,y)f(y)d\mu(y), \quad \forall f \in L^1(M,\mu).$$

Moreover

$$\operatorname*{ess\,sup}_{x\in M, y\in M} |K(x,y)| = \|T\|_{1\to\infty}.$$

Proof. For the proof we refer to [26, Theorem 6, p.503] or [5, Theorem 1.3].

□

Proposition C.1.3 *Let A be a self-adjoint operator. Let us assume that, for some constants $D, K > 0$,*

$$\|e^{-tA}\|_{1\to\infty} \leq Kt^{-\frac{D}{2}}, \quad \forall t > 0.$$

Then

$$\|e^{-zA}\|_{1\to\infty} \leq K(\operatorname{Re} z)^{-\frac{D}{2}}, \quad \forall z \in \mathbb{C}_+. \tag{C.4}$$

In particular e^{-zA} is an integral operator for all $z \in \mathbb{C}_+$. This means that there exists a measurable kernel $p(z,x,y)$, such that

$$e^{-zA}f(x) = \int_M p(z,x,y)f(y)\,d\mu(y), \ \text{for a.e. } x \in M, \tag{C.5}$$

and

$$\operatorname*{ess\,sup}_{x\in M, y\in M} |p(z,x,y)| \leq K(\operatorname{Re} z)^{-\frac{D}{2}}, \quad \forall z \in \mathbb{C}_+. \tag{C.6}$$

Furthermore p has a representation such that $p(\cdot,x,y)$ is analytic in $\mathbb{C}_+$ for all $x,\ y \in M$.

Proof. Let us start by observing that, since A is self-adjoint, iA generates a contractive C_0-group $\left(e^{itA}\right)_{t\in\mathbb{R}}$ on $L^2(M,\mu)$ (see for example [27, Theorem 3.24]) that is

$$\|e^{itA}\|_{2\to2} \leq 1, \quad \forall t \in \mathbb{R}.$$

Combining together the hypothesis and the last inequality and using $\|T^*T\|_{1\to\infty} = \|T\|^2_{1\to 2}$, we obtain for $t > 0$, $s \in \mathbb{R}$,

$$\begin{aligned}\|e^{-(t+is)A}\|_{1\to\infty} &\le \|e^{-\frac{t}{2}A}\|_{1\to 2}\|e^{-isA}\|_{2\to 2}\|e^{-\frac{t}{2}A}\|_{2\to\infty}\\ &= \|e^{-\frac{t}{2}A}\|^2_{1\to 2}\|e^{-isA}\|_{2\to 2}\\ &= \|e^{-tA}\|_{1\to\infty}\|e^{-isA}\|_{2\to 2} \le K\ t^{-D/2}.\end{aligned}$$

This proves the desired $L^1 - L^\infty$ estimates

$$\|e^{-zA}\|_{1\to\infty} \le K(Re\, z)^{-\frac{D}{2}}, \quad \forall z \in \mathbb{C}_+.$$

The existence of the kernel, as well as its bound, is then a consequence of the Dunford-Pettis Theorem C.1.2. The holomorphy of p follows from [5, Theorem 3.1] (see also [3, Lemma 4.1]).

$\square$

Remark C.1.4 *Let us assume that, for every $z \in \mathbb{C}_+$, the kernel $p(z,\cdot,\cdot)$ is a continuous complex-valued function defined on $M \times M$. Then* (C.6) *becomes*

$$|p(z,x,y)| \le K(\mathrm{Re}\, z)^{-\frac{D}{2}}, \qquad \textit{for all } z \in \mathbb{C}_+,\ x,\ y, \in M. \tag{C.7}$$

Furthermore we remark that (C.7) *is actually equivalent to the on-diagonal estimate*

$$p(t,x,x) \le Kt^{-\frac{D}{2}}, \quad \forall t > 0,\ x \in M. \tag{C.8}$$

Indeed, by the semigroup property, for $z \in \mathbb{C}_+$, x, $y, \in M$,

$$p(z,x,y) = \int_M p\left(\frac{z}{2},x,u\right) p\left(\frac{z}{2},u,y\right)\, d\mu(u),$$

and the symmetry of e^{-zA} yields

$$\overline{p\left(\frac{z}{2},x,y\right)} = p\left(\frac{\overline{z}}{2},y,x\right).$$

Therefore, if $t = \mathrm{Re}\, z$, then

$$\begin{aligned}|p(z,x,y)| &\le \left(\int_M |p\left(\frac{z}{2},x,u\right)|^2\, d\mu(u)\right)^{\frac{1}{2}} \left(\int_M |p\left(\frac{z}{2},u,y\right)|^2\, d\mu(u)\right)^{\frac{1}{2}}\\ &= \left(\int_M p\left(\frac{z}{2},x,u\right) \overline{p\left(\frac{z}{2},x,u\right)}\, d\mu(u)\right)^{\frac{1}{2}} \left(\int_M p\left(\frac{z}{2},x,u\right) \overline{p\left(\frac{z}{2},u,y\right)}\, d\mu(u)\right)^{\frac{1}{2}}\\ &= \Big(p(t,x,x)p(t,y,y)\Big)^{\frac{1}{2}}.\end{aligned}$$

This shows that

$$\|\exp(-zA)\|_{1\to\infty} = \sup_{x,y\in M} |p(z,x,y)| \le \sup_{x\in M} p(t,x,x),$$

and so (C.7) *follows from* (C.8).

C.2 Theorems of Phragmén-Lindelöf type

We begin by stating the classical Phragmén-Lindelöf theorem for sectors.

Theorem C.2.1 *Let S be the open region in $\mathbb{C}$ bounded by two rays meeting at an angle π/α, for some $\alpha > 1/2$. Suppose that F is analytic on S, continuous on $\bar{S}$, and satisfies $|F(z)| \leq C\exp(c|z|^\beta)$ for some $\beta \in [0, \alpha)$ and for all $z \in S$. Then the condition $|F(z)| \leq B$ on the two bounding rays implies $|F(z)| \leq B$ for all $z \in S$.*

Proof. See [77, Lemma 4.2, p.108].

□

The following Propositions are simple consequences of Theorem C.2.1.

Proposition C.2.2 [21] *Suppose that F is an analytic function on $\mathbb{C}_+$. Assume that, for given numbers $A, B, \gamma > 0$, $a \geq 0$,*

$$|F(z)| \leq B, \qquad \forall\, z \in \mathbb{C}_+, \tag{C.9}$$

$$|F(t)| \leq Ae^{at}e^{-\frac{\gamma}{t}}, \qquad \forall\, t \in \mathbb{R}_+. \tag{C.10}$$

Then

$$|F(z)| \leq B\exp\left(-\mathrm{Re}\,\frac{\gamma}{z}\right), \quad \forall\, z \in \mathbb{C}_+. \tag{C.11}$$

Proof. Let us consider the function defined, for $\zeta \in \mathbb{C}_+$, by

$$g(\zeta) = F\left(\frac{\gamma}{\zeta}\right)e^{\zeta}.$$

By (C.9) we have $|g(\zeta)| \leq Be^{Re\,\zeta}$. In particular, for any $\varepsilon > 0$,

$$\sup_{Re\,\zeta=\varepsilon} |g(\zeta)| \leq Be^{\varepsilon}. \tag{C.12}$$

Using (C.10),

$$\sup_{\zeta\in[\varepsilon,\infty)} |g(\zeta)| \leq Ae^{a\gamma/\varepsilon}. \tag{C.13}$$

Hence, by Phragmén-Lindelöf theorem with angle $\pi/2$ and $\beta = 1$, applied to

$$S_\varepsilon^+ = \{z \in \mathbb{C}\colon\ Re\,z > \varepsilon \quad \text{and} \quad \mathrm{Im} z > 0\}$$

and

$$S_\varepsilon^- = \{z \in \mathbb{C}\colon\ Re\,z > \varepsilon \quad \text{and} \quad \mathrm{Im} z < 0\},$$

one obtains

$$\sup_{Re\,\zeta\geq\varepsilon} |g(\zeta)| \leq \max\{Ae^{a\gamma/\varepsilon}, Be^{\varepsilon}\}, \quad \forall\,\varepsilon > 0.$$

Now by the Phragmén-Lindelöf theorem with angle π and $\beta = 0$,

$$\sup_{Re\,\zeta \geq \varepsilon} |g(\zeta)| \leq Be^{\varepsilon}, \quad \forall\, \varepsilon > 0. \tag{C.14}$$

Letting $\varepsilon \to 0$ we obtain

$$\sup_{Re\,\zeta > 0} |g(\zeta)| \leq B.$$

(C.11) then follows by putting $\zeta = \frac{\gamma}{z}$.

□

Remark C.2.3 *The estimate (C.11) does not depend on the constants A, a of (C.10). This property is the crucial key that allows to derive the upper Gaussian estimates of Section C.4 and the self-improving property of the Davies-Gaffney estimates of Lemma C.3.3.*

Let us define now, for a given $\gamma > 0$,

$$\mathcal{C}_\gamma = \{z \in \mathbb{C} \setminus \{0\} : Re\, \frac{\gamma}{z} \geq 1\}.$$

$\mathcal{C}_\gamma$ is the closed disk in $\mathbb{C}_+$ centred on the real axis, tangent to the imaginary axis and with radius $\gamma/2$.

Proposition C.2.4 [21] *Let F be an analytic function on $\mathbb{C}_+$. Assume that, for given numbers $A, B, \gamma, \nu > 0$,*

$$|F(z)| \leq A, \qquad \forall\, z \in \mathbb{C}_+; \tag{C.15}$$

$$|F(t)| \leq Ae^{-\frac{\gamma}{t}}, \qquad \forall\, t \in \mathbb{R}^+,\ 0 < t \leq \gamma; \tag{C.16}$$

$$|F(z)| \leq B\left(\mathrm{Re}\, z\right)^{-\nu/2}, \qquad \forall\, z \in \mathcal{C}_\gamma. \tag{C.17}$$

Then

$$|F(z)| \leq eB\gamma^{\frac{\nu}{2}}|z|^{-\nu} \exp\left(-\mathrm{Re}\, \frac{\gamma}{z}\right), \qquad \forall\, z \in \mathcal{C}_\gamma. \tag{C.18}$$

Furthermore if one replaces (C.17) *with the stronger condition*

$$|F(z)| \leq B\left(\mathrm{Re}\, z\right)^{-\nu/2}, \qquad \forall\, z \in \mathbb{C}_+. \tag{C.19}$$

then one has

$$|F(z)| \leq eB\left(\mathrm{Re}\, z\right)^{-\frac{\nu}{2}} \left(1 + \mathrm{Re}\, \frac{\gamma}{z}\right)^{\frac{\nu}{2}} \exp\left(-\mathrm{Re}\, \frac{\gamma}{z}\right), \qquad \forall\, z \in \mathbb{C}_+. \tag{C.20}$$

Proof. Consider again the function defined, for $\zeta \in \mathbb{C}_+$, by

$$g(\zeta) = F\left(\frac{\gamma}{\zeta}\right) e^{\zeta}.$$

It satisfies condition (C.12) and (C.13) with $B = A$, $a = 0$ and $\varepsilon = 1$. Hence by (C.14)

$$\sup_{Re\,\zeta \geq 1} |g(\zeta)| \leq eA. \tag{C.21}$$

Consider now the function v defined on $\mathbb{C}_+$ by

$$v(\zeta) = (2\zeta)^{-\nu} g(\zeta).$$

Note that $|v(\zeta)| \leq 2^{-\nu}|g(\zeta)|$ for $Re\,\zeta \geq 1$ so, by (C.21), v is bounded on the set $Re\,\zeta \geq 1$. Now the Phragmén-Lindelöf theorem, with angle π and $\beta = 0$, yields

$$\sup_{Re\,\zeta \geq 1} |v(\zeta)| = \sup_{Re\,\zeta = 1} |v(\zeta)|.$$

Put $\zeta = 1 + is$. By (C.17),

$$\begin{aligned}
\sup_{Re\,\zeta=1} |v(\zeta)| &= \sup_{Re\,\zeta=1} |(2\zeta)^{-\nu} F\left(\frac{\gamma}{\zeta}\right) e^{\zeta}| \\
&\leq \sup_{Re\,\zeta=1} eB|2\zeta|^{-\nu} \left(Re\,\frac{\gamma}{\zeta}\right)^{-\nu/2} \\
&\leq eB\,(4\gamma)^{-\frac{\nu}{2}} \sup_{s\in\mathbb{R}} (1+s^2)^{-\nu/2} \left(\frac{1}{1+s^2}\right)^{-\nu/2} = eB\,(4\gamma)^{-\frac{\nu}{2}}\,.
\end{aligned}$$

Hence

$$\sup_{Re\,\zeta \geq 1} |v(\zeta)| \leq eB\,(4\gamma)^{-\frac{\nu}{2}}\,. \tag{C.22}$$

Now let $\zeta = \frac{\gamma}{z}$ and let us observe that $Re\,\zeta \geq 1$ is equivalent to $z \in \mathcal{C}_\gamma$. Then (C.22) becomes

$$|F(z)| \leq eB\gamma^{\frac{\nu}{2}}|z|^{-\nu} \exp\left(-Re\,\frac{\gamma}{z}\right).$$

This proves (C.18) for all $z \in \mathcal{C}_\gamma$.

For the proof of the last claim, we use the elementary equality $Re\,(\frac{1}{z}) = \frac{Re\,z}{|z|^2}$ to write the last relation as

$$|F(z)| \leq eB\,(Re\,z)^{-\frac{\nu}{2}} \left(Re\,\frac{\gamma}{z}\right)^{\frac{\nu}{2}} \exp\left(-Re\,\frac{\gamma}{z}\right),$$

where $z \in \mathbb{C}^+$ and $Re\,\frac{\gamma}{z} \geq 1$. On the other hand, if we assume (C.19), we have , for every $z \in \mathbb{C}^+$ such that $Re\,\frac{\gamma}{z} \leq 1$,

$$|F(z)| \leq B\,(Re\,z)^{-\frac{\nu}{2}} \leq eB\,(Re\,z)^{-\frac{\nu}{2}} \exp\left(-Re\,\frac{\gamma}{z}\right).$$

Summing the last two inequalities we get (C.20).

□

C.3 Davies-Gaffney estimates

Let (M, d, μ) be a metric measure space, that is μ is a Borel measure with respect to the topology defined by the metric d, and let $B(x, r) = \{y \in M,\, d(x, y) < r\}$ denote the open ball with center $x \in M$ and radius $r > 0$.

Suppose that, for every $z \in \mathbb{C}_+$, $\Psi(z)$ is a bounded linear operator acting on $L^2(M, d\mu)$ and that $\Psi(z)$ is an analytic function of z. Assume in addition that $\Psi(z)$ is contractive over $L^2(M, d\mu)$ that is

$$\|\Psi(z)\|_{2\to 2} \leq 1, \quad \forall\, z \in \mathbb{C}_+. \tag{C.23}$$

Definition C.3.1 *For U_1, U_2 open subsets of M, let $d(U_1, U_2) = \inf_{x\in U_1, y\in U_2} d(x, y)$. We say that the family $\{\Psi(z) \colon z \in \mathbb{C}_+\}$ satisfies the Davies-Gaffney estimate if*

$$|\langle \Psi(t) f_1, f_2\rangle| \leq \exp\left(-\frac{r^2}{4t}\right) \|f_1\|_2 \|f_2\|_2 \tag{C.24}$$

for all $t > 0$, $U_i \subset M$, $f_i \in L^2(U_i, d\mu)$, $i = 1, 2$ and $r = d(U_1, U_2)$. Note that we only assume that (C.24) holds for positive real t.

Remark C.3.2 *Given (C.23) it is enough to test (C.24) for balls $U_i = B(x_i, r_i)$, where x_1, $x_2 \in M$ and r_1, $r_2 > 0$, and for characteristic function $f_i = \chi_{U_i}$ (see [21, Lemma 3.1]).*

Furthermore we remark that the constants in (C.23) *and* (C.24) *have been normalized to one for simplicity. By the way any additional multiplicative constant can be absorbed by multiplying accordingly the family $\Psi(z)$. Indeed let us simply suppose the family $\Psi(z)$ to be bounded over $L^2(M, d\mu)$ that is, for some analytic function $c(z)$ such that $|c(z)| > 0$,*

$$\|\Psi(z)\|_{2\to 2} \leq |c(z)|, \quad \forall\, z \in \mathbb{C}_+.$$

Replacing $\Psi(z)$ with $\frac{1}{c(z)}\Psi(z)$, the definition(C.24) *reads as*

$$|\langle \Psi(t) f_1, f_2\rangle| \leq |c(t)| \exp\left(-\frac{r^2}{4t}\right) \|f_1\|_2 \|f_2\|_2.$$

As a consequence of Proposition C.2.2 and Remark C.3.2, any additional multiplicative constant and exponential factor in (C.24) can be replaced by the constant in (C.23).

Lemma C.3.3 [21] *Suppose that the family $\{\Psi(z) \colon z \in \mathbb{C}_+\}$ satisfies condition* (C.23). *Assume in addition that, for some $C \geq 1$ and some $a > 0$,*

$$|\langle \Psi(t) f_1, f_2\rangle| \leq C e^{at} e^{-\frac{r^2}{4t}} \|f_1\|_2 \|f_2\|_2, \quad \forall t > 0, \tag{C.25}$$

whenever $f_i \in L^2(M, d\mu)$, $\mathrm{supp} f_i \subseteq B(x_i, r_i)$, $i = 1, 2$, and $r = d(B(x_1, r_1), B(x_2, r_2))$. Then the family $\{\Psi(z) \colon z \in \mathbb{C}_+\}$ satisfies condition (C.24).

Let, now, A be a non-negative self-adjoint operator on $L^2(M, d\mu)$; we say that A satisfies the Davies-Gaffney condition if (C.24) holds with $\Psi(t) = e^{-tA}$.

One may wonder what is the justification of the constant 4 in (C.24). It is proved in [74, Theorem 2] (see also [21, Theorem 3.4]) that the Davies-Gaffney estimates (C.24) are equivalent to the fact that the corresponding wave equation $\partial_{tt}u + Au = 0$ has propagation speed 1; roughly speaking 4 is the good normalisation between the operator A and the distance d. If $\cos(t\sqrt{A})$ is the cosine function generated by A (see for example [4, page 203] and [31]), then (C.24) is equivalent to

$$\langle \cos(t\sqrt{A}) f_1, f_2 \rangle = 0$$

for all $0 < t < r$, open sets $U_i \subset M$, $f_i \in L^2(U_i, d\mu)$, $i = 1, 2$, where $r = d(U_1, U_2)$.

C.4 Gaussian upper bounds for the heat kernel

Let (M, d, μ) be a metric measure space and let A be a non-negative self-adjoint operator acting on $L^2(M, d\mu)$; we already saw, in section C.1, that $-A$ generates a contractive analytic C_0-semigroup $\left\{e^{-zA},\ z \in \mathbb{C}_+\right\}$ on $L^2(M, \mu)$; in other words $\Psi(z) = e^{-zA}$, $z \in \mathbb{C}_+$ satisfies condition (C.23).

Let us assume that, for some constant $D, K > 0$,

$$\|e^{-tA}\|_{1\to\infty} \le Kt^{-\frac{D}{2}}, \quad \forall t > 0. \tag{C.26}$$

Then it follows, from Proposition C.1.3, that

$$\|e^{-zA}\|_{1\to\infty} \le K(Re\, z)^{-\frac{D}{2}}, \quad \forall z \in \mathbb{C}_+. \tag{C.27}$$

In particular e^{-zA} is an integral operator for all $z \in \mathbb{C}_+$ whose kernel $p(z, x, y)$ satisfies

$$e^{-zA} f(x) = \int_M p(z, x, y) f(y)\, d\mu(y), \text{ for a.e. } x \in M,$$

and

$$\operatorname*{ess\,sup}_{x\in M, y\in M} |p(z, x, y)| \le K(Re\, z)^{-\frac{D}{2}}, \quad \forall z \in \mathbb{C}_+.$$

Under the hypothesis of continuity of $p(z, \cdot, \cdot)$, we can replace the essential supremum with the supremum obtaining

$$|p(z, x, y)| \le K(Re\, z)^{-D/2},\ \forall\, z \in \mathbb{C}_+,\ x,\ y \in M.$$

Moreover, recalling Remark C.1.4, the last inequality is equivalent to the on-diagonal estimate

$$p(t, x, x) \le K\ t^{-D/2},\ \forall\, t > 0,\ x \in M. \tag{C.28}$$

The next theorem shows how to deduce, from (C.26), precise off-diagonal Gaussian estimates for $p(z,x,y)$.

Theorem C.4.1 [21] *Let (M,d,μ) be a metric measure space and let A be a non-negative self-adjoint operator acting on $L^2(M,d\mu)$. Let us suppose that $\{e^{-zL},\ z\in\mathbb{C}_+\}$ satisfies the Davies-Gaffney condition* (C.24) *and that, for some $K,\ D>0$,*

$$\|e^{-tA}\|_{1\to\infty}\le Kt^{-\frac{D}{2}},\quad \forall t>0.$$

Assume, furthermore, that, for every $z\in\mathbb{C}_+$, the heat kernel $p(z,\cdot,\cdot)$ of e^{-zA} is a continuous complex-valued function defined on $M\times M$. Then

$$|p(z,x,y)|\le eK(\operatorname{Re} z)^{-D/2}\left(1+\operatorname{Re}\frac{d^2(x,y)}{4z}\right)^{D/2}\exp\left(-\operatorname{Re}\frac{d^2(x,y)}{4z}\right)\tag{C.29}$$

for all $z\in\mathbb{C}_+$, $x,y\in M$.

Proof. Let us fix $x,y\in M$ and, for $d(x,y)>2s>0$, let us consider the bounded analytic function $F\colon\mathbb{C}_+\to\mathbb{C}$ defined by the formula

$$F(z)=\langle e^{-zA}f_1,f_2\rangle,$$

where $f_1\in L^1(B(x,s),d\mu)\cap L^2(B(x,s),d\mu)$, $f_2\in L^1(B(y,s),d\mu)\cap L^2(B(y,s),d\mu)$ and $\|f_1\|_1=\|f_2\|_1=1$.

Using the Davies-Gaffney estimates (C.24) and (C.23), we get

$$\begin{aligned}|\langle e^{-zA}f_1,f_2\rangle|&\le\|f_1\|_2\|f_2\|_2, &&\forall\, z\in\mathbb{C}_+;\\ |\langle e^{-tA}f_1,f_2\rangle|&\le\|f_1\|_2\|f_2\|_2\ e^{-\frac{r^2}{4t}}, &&\forall\, t\in\mathbb{R}^+;\end{aligned}$$

So $F(z)=\langle e^{-zA}f_1,f_2\rangle$ satisfies (C.15) and (C.16) with

$$\gamma=r^2/4,\ \text{where } r=d(x,y)-2s,\quad\text{and}\quad A=\|f_1\|_2\|f_2\|_2<\infty.$$

Using (C.27) we get

$$|\langle e^{-zA}f_1,f_2\rangle|\le K(Re\, z)^{-D/2},\ \forall\, z\in\mathbb{C}_+,$$

so that F satisfies (C.19) with $\nu=D$ and $B=K$.

Applying Proposition C.2.4 we, therefore, obtain

$$|\langle e^{-zA}f_1,f_2\rangle|\le eK\,(Re\, z)^{-\frac{D}{2}}\left(1+Re\,\frac{r^2}{4z}\right)^{\frac{D}{2}}\exp\left(-Re\,\frac{r^2}{4z}\right),\qquad\forall\, z\in\mathbb{C}_+.$$

This implies

$$\begin{aligned}|p(z,x,y)| &\leq \sup\left\{|p(z,x',y')| : \quad x' \in B(x,s),\ y' \in B(y,s)\right\}\\ &= \sup\left\{\langle e^{-zA} f_1, f_2\rangle| : \ \|f_1\|_{L^1(B(x,s),d\mu)} = \|f_2\|_{L^1(B(y,s),d\mu)} = 1\right\}\\ &\leq eK\,(Re\, z)^{-\frac{D}{2}}\left(1 + Re\,\frac{r^2}{4z}\right)^{\frac{D}{2}} \exp\left(-Re\,\frac{r^2}{4z}\right).\end{aligned}$$

Taking, in the last inequality, the limit for $s \to 0$ we prove the claim.

□

Remark C.4.2 *For $z = t \in \mathbb{R}_+$, the estimates (C.29) can be improved. It is possible to prove that, if M is a complete Riemannian manifold and $-A$ is its Laplace-Beltrami operator, then*

$$p(t,x,y) \leq Ct^{-D/2}\left(1 + \frac{d^2(x,y)}{4t}\right)^{\frac{D-1}{2}} \exp\left(-\frac{d^2(x,y)}{4t}\right),$$

(see, for example, [75]). This result is sharp: indeed it was shown in [59] that, on the D-dimensional sphere $\mathbb{S}^D$, the following asymptotic relation holds

$$p(t,x,y) \simeq ct^{-\frac{D}{2}}\left(\frac{d(x,y)^2}{t}\right)^{\frac{D-1}{2}} e^{-\frac{d(x,y)^2}{4t}}, \qquad as \qquad t \to 0,$$

where x and y are conjugate points.

The fact that, in (C.29)*, the exponent in the polynomial correction factor in front of the exponential cannot be improved to $(D-1)/2$ is related to the fact that the proof of Theorem C.4.1 does not use the semigroup property of the family $\Psi(z) = e^{-zA}$.*

With the same technique, indeed, it is possible to extend Theorem C.4.1 to any analytic family of operators $\{\Psi(z)\colon z \in \mathbb{C}_+\}$ acting on $L^2(M,d\mu)$ satisfying (C.23), (C.24) *and $\|\Psi(z)\|_{1\to\infty} \leq K(\mathrm{Re}\, z)^{-D/2}$, for $z \in \mathbb{C}_+$ (see [21]).*

Notation

Let V be an open subset of $\mathbb{R}^N$, $1 \leq p, q \leq \infty$.

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$	set of natural numbers including 0
$\mathbb{R}^N$	euclidean N-dimensional space
Ω	$\mathbb{R}^N \setminus \{0\}$
$\mathbb{S}^{N-1}$	unit sphere $\{\|x\| = 1\}$ in $\mathbb{R}^N$
$B(x, r)$	open ball in $\mathbb{R}^N$ centred in x with radius $r > 0$
B_r	$\{x \in \mathbb{R}^n : \|x\| < r\}$
$\overline{B}_r$	$\{x \in \mathbb{R}^n : \|x\| \leq r\}$
$\mathbb{C}$	complex plane
$Re\, z$	real part of $z \in \mathbb{C}$
$\mathbb{C}_+$	$\{z \in \mathbb{C} : \; Re\, z > 0\}$
$\|J\|$	Lebesgue measure of a given set J
J^c	complementary set of J
χ_J	characteristic function of a set J
δ_{ij}, δ_i^j	Kronecker symbol
$D(\mathfrak{a})$	domain of the sesquilinear form $\mathfrak{a}$
$D(A)$	domain of the operator A
$\rho(A)$	resolvent set of the operator A
$\sigma(A)$	spectrum of the operator A
$R(\lambda, A)$	Resolvent $(\lambda - A)^{-1}$ of the operator A
$a \wedge b$	minimum between $a, b \in \mathbb{R}$
$a \vee b$	maximum between $a, b \in \mathbb{R}$
I	identity matrix of $\mathbb{R}^{N \times N}$
$x \otimes y$	the matrix $(x_i y_j)_{i,j=1,\dots N}$, for $x, y \in \mathbb{R}^N$
(x, y), $\langle x, y \rangle$, $x \cdot y$	inner product of $x, y \in \mathbb{R}^N$: $\displaystyle\sum_{i=1}^N x_i y_i$
$\|x\|$	euclidean norm of $x \in \mathbb{R}^N$
supp u	support of a given function u

$f(x) \simeq g(x)$	$\Big(C_1\, g(x) \le f(x) \le C_2\, g(x),\ x \in I\Big)$, for f, g positive functions defined in I, and for some $C_1, C_2 > 0$
$D_t,\ \partial_t,\ \frac{\partial}{\partial t}$	partial derivative with respect to the variable t
$D_i,\ \partial_i,\ \frac{\partial}{\partial x_i}$	partial derivative with respect to the variable x_i
$D_{ij},\ \partial_{ij},\ \frac{\partial^2}{\partial x_i \partial x_j}$	second partial derivative with respect to the variables $x_i,\ x_j$
∇u	space gradient of a real valued function u
$D^2 u$	Hessian matrix of a real valued function u
$x = r\omega$	spherical coordinates of $x \in \mathbb{R}^N$: $r = \lvert x\rvert \ge 0$, $\omega = \frac{x}{\lvert x\rvert} \in \mathbb{S}^{N-1}$
$D_r,\ \partial_r$	radial derivative $\sum_{i=1}^N D_i \frac{x_i}{r}$
$D_{rr},\ \partial_{rr}$	second radial derivative $\sum_{i,j=1}^N D_{ij} \frac{x_i x_j}{r^2}$
∇_τ	tangential component of the gradient $\nabla = D_r \frac{x}{\lvert x\rvert} + \frac{\nabla_\tau}{r}$
Δ_0	Laplace-Beltrami op. on $\mathbb{S}^{N-1}$: $\Delta = D_{rr} + \frac{N-1}{r} D_r + \frac{\Delta_0}{r^2}$
$L^p(M, d\mu)$	L^p space over M with respect the measure $d\mu$
$\lVert f\rVert_p$	norm of $f \in L^p(M, d\mu)$
$\lVert f\rVert_\infty$	sup-norm of f
$\lVert T\rVert_{p\to q}$	operator norm of a bounded linear operator T from $L^p(M, d\mu)$ to $L^q(M, d\mu)$
$\langle f, g\rangle$	inner product of $f, g \in L^2(M, d\mu)$
L^2_μ	$L^2\left(\mathbb{R}^N, d\mu\right)$, $d\mu = \lvert x\rvert^\gamma dx$
L^2_ν	$L^2\left(\mathbb{R}^N, d\nu\right)$, $d\nu = \phi^2 d\mu$, with ϕ defined in (3.8)
$C_b(V)$	Banach space of all continuous and bounded functions in V, endowed with the sup-norm
$C_0(V)$	subspace of $C_b(V)$ consisting of functions vanishing at the boundary of V, including ∞ when V is unbounded
$C_c^\infty(V)$	space of infinitely continuously differentiable functions with compact support in V
$W^{k,p}(V)$	usual Sobolev space
$W^{k,p}_{\text{loc}}(V)$	space of functions belonging to $W^{k,p}(V')$ for all bounded open set V' such that $\overline{V'} \subset V$
$W_c^{1,\infty}(\mathbb{R}^N \setminus \{0\})$	set of functions $u \in W^{1,\infty}(\mathbb{R}^N)$ such that $\operatorname{supp} u$ is compact in $\mathbb{R}^N \setminus \{0\}$
$(\mathcal{M}, g)$	Riemannian manifold $\mathcal{M}$ with metric tensor g
d_g	Riemannian distance induced on $\mathcal{M}$ by g
$\mathrm{Lip}(\mathcal{M}, d_g)$	class of real functions defined on $\mathcal{M}$ which are Lipschitz-continuous with respect to the distance d_g

$\mathrm{Lip}(f, d_g)$	for $f \in \mathrm{Lip}(\mathcal{M}, d_g)$, the best constant L such that $\lvert f(p) - f(q)\rvert \leq L d_g(p,q)$ for every p, $q \in \mathcal{M}$

In order to help reading, we list below the main parameters and formulas we use systematically through the dissertation.

- The second-order elliptic operator

$$L = \Delta + (a-1)\sum_{i,j=1}^{N} \frac{x_i x_j}{|x|^2} D_{ij} + c\frac{x}{|x|^2}\cdot\nabla - \frac{b}{|x|^2},$$

  $a > 0$, b, $c \in \mathbb{R}$.

- The (non-negative) discriminant of the the indicial equation $-as^2+(N-1+c-a)s+b = 0$ is denoted by

$$D = \frac{b}{a} + \left(\frac{N-1+c-a}{2a}\right)^2.$$

- The reference measure is $d\mu = |x|^\gamma\, dx$, where $\gamma = \frac{N-1+c}{a} - N + 1$.

- $|x|^{-s_1}$, $|x|^{-s_2}$ are the radial solutions of $Lu = 0$ where s_1, s_2 are the roots of the indicial equation given by

$$s_1 := \frac{N-1+c-a}{2a} - \sqrt{D}, \quad s_2 := \frac{N-1+c-a}{2a} + \sqrt{D}.$$

  Moreover, $s_1 = \frac{N}{2} - 1 - \sqrt{D} + \frac{\gamma}{2}$.

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