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INTRODUCTION

This thesis is devoted to the investigation of some aspects of the connection between the theory of Siegel modular forms and the study of the geometry of Siegel modular varieties. We will show how one can use this connection to study polarized abelian varieties and their moduli spaces.

In order to understand abelian varieties one has to understand first complex tori since abelian varieties turns out to be complex tori that admit an immersion in some projective space. The simpler example of an abelian variety is an elliptic curve. We will present an introduction to the basic theory of complex tori and complex abelian varieties (see Chapter 1) in order to highlight the deep relationship between this subject and the theory of Siegel modular varieties. Indeed these varieties arise naturally as compactifications of moduli spaces of complex abelian varieties.

We will mostly talk about Siegel modular forms as tools for the study of complex abelian varieties and their moduli spaces, but they also represent an interesting and rich subject in the theory of automorphic forms. We will develop the theory of Siegel modular forms in Chapter 2 where we will also give many examples of Siegel modular forms. These modular forms will have a prominent role in the exposition of the original results of the thesis which are mostly based on my papers [39], [8], [40].

Let $\mathbb{H}_g$ denote the Siegel space of degree g. This is the space of $g \times g$ symmetric complex matrices with positive definite imaginary part. The group of integral symplectic matrices $\Gamma_g := \mathrm{Sp}(2g, \mathbb{Z})$ acts properly discontinuously on $\mathbb{H}_g$ as follows:

$$\gamma \cdot \tau = (A\tau + B)(C\tau + D)^{-1},\ \forall \gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_g,$$

where A, B, C, D are $g \times g$ matrices. If $\Gamma \subset \Gamma_g$ is a group acting properly discontinuously on the Siegel space, the quotient $\mathbb{H}_g/\Gamma$ is called a modular variety. It has the structure of a normal analytic space and it is a quasi-projective variety.

Clearly the Siegel space and Γ_g are a natural generalization of the upper-half plane of complex numbers and the special linear group of degree 2 acting on the complex plane by Möbius transformation respectively. Since isomorphism classes of elliptic curves are in bijection with the quotient of the upper-half plane of complex numbers by such an action we get a first hint of the connection between modular varieties and moduli spaces of abelian varieties. In Section 1.4.1 and 1.4.2 we will make this connection precise by giving explicit construction of some moduli spaces of polarized abelian varieties. For example the moduli space of principally polarized abelian varieties is a modular variety since the set isomorphism classes of these varieties is in bijection with the points of the quotient space $\mathbb{H}_g/\Gamma_g$.

Let us briefly give the definition of both vector-valued and scalar-valued Siegel modular forms. If (ρ, V_ρ) is a finite dimensional rational representation of $GL_g(\mathbb{C})$, a vector-valued Siegel modular form with respect to ρ and a subgroup $\Gamma \subset \Gamma_g$ is a holomorphic function $f : \mathbb{H}_g \to V_\rho$ such that

$$f(\gamma \cdot \tau) = \rho(C\tau + D)\, f(\tau),\ \forall \gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma, \forall \tau \in \mathbb{H}_g.$$

If $\rho(C\tau + D) = \det(C\tau + D)^{k/2}$ for some $k \in \mathbb{N}$, f is said to be a scalar-valued Siegel modular form of weight $k/2$ with respect to Γ. For technical reasons, in order to consider half-integer weights we shall introduce the notion of multiplier system.

The most important examples of scalar-valued Siegel modular forms we will work with are theta functions with characteristics. For any $m = \left[\begin{smallmatrix} m' \\ m'' \end{smallmatrix}\right]$, m', $m'' \in \mathbb{Z}^g$, the theta function with characteristic m is defined by the series

$$\vartheta_m(\tau, z) = \sum_{n \in \mathbb{Z}^g} e^{\pi i \left({}^t(n+m'/2)\tau(n+m'/2) + 2\,{}^t(n+m'/2)(z+m''/2)\right)},\ \ \tau \in \mathbb{H}_g,\, z \in \mathbb{C}^g.$$

We will first introduce these functions in Section 1.3.1 as theta functions for suitable line bundles on principally polarized abelian varieties. The theta function with characteristic m is a holomorphic function in the two variables τ and z which is an even or odd function of z if ${}^t m' m''$ is even or odd respectively. Correspondingly, the characteristic m is called even or odd. We will see that one can reduce to the case where $m \in \{0, 1\}^{2g}$.

The function $\vartheta_m(\tau, 0)$ is a holomorphic function on $\mathbb{H}_g$ which is not identically zero if and only if the characteristic m is even. These functions are usually called theta constants. They are scalar-valued Siegel modular forms of weight $1/2$ and a suitable multiplier system with respect to a subgroup of Γ_g.

Regarding theta functions with odd characteristics, they give rise to vector-valued Siegel modular forms by taking gradients with respect to the variable z and then evaluating in $z = 0$. We will present examples of both scalar-valued and vector-valued Siegel modular forms arising from theta functions with characteristics in Section 2.4.

In Chapter 3 we will investigate the role of both scalar-valued and vector-valued Siegel modular forms in the study of the geometry of Siegel modular varieties.

In particular scalar-valued Siegel modular forms can be used to give a compactification of these varieties. If the ring of scalar-valued Siegel modular forms with respect to a subgroup Γ is denoted by $A(\Gamma)$, the Satake compactification of $\mathbb{H}_g/\Gamma$, also called Siegel modular variety associated to Γ, is defined as $\mathrm{Proj}(A(\Gamma))$.

In Section 3.1 we will explain in more details the construction of the Satake compactification of a modular variety and study in details the Satake compactification of the moduli space of principally polarized abelian surfaces with level 2 structure. This is known to be a quartic hypersurface in $\mathbb{P}^4$ usually called the Igusa quartic. We will also present a different modular interpretation of the Igusa quartic involving the Kummer variety of an abelian variety.

In [32] the Igusa quartic has been characterized as a Steiner hyperquartic. As such it has a degree 8 endomorphism. By means of this characterization the Satake compactification of the moduli space of principally polarized abelian surfaces with Göpel triples is isomorphic to the Igusa quartic [32]. Both compactifications can be thought as Siegel modular threefolds (Siegel modular varieties of degree 2) and the latter isomorphism can also be given by means of Siegel modular forms (cf. [6, section 11]).

In Section 3.3 we will prove that many modular threefolds share the property of the Igusa quartic of having a degree 8 endomorphism. With this result we give also an alternative proof of the existence of a degree 8 endomorphism of the Igusa quartic exploiting Siegel modular forms. The construction of a degree 8 endomorphism on suitable Siegel modular threefolds will be done via an isomorphism of graded rings of scalar-valued Siegel modular forms and a degree 8 map between two given Siegel modular threefolds. The results of this section are based on my paper [39].

We will prove that the degree 8 map

$$\begin{array}{c} \mathbb{P}^3 \to \mathbb{P}^3 \\ [x_0, x_1, x_2, x_3] \mapsto [x_0^2, x_1^2, x_2^2, x_3^2] \end{array} , \tag{1}$$

is a map between two Siegel modular varieties. This is the part that will give the right degree of the desired self map between some modular threefolds. For the other part, that is the isomorphism of suitable rings of scalar-valued Siegel modular forms, we shall prove that there is an isomorphism

$$\Gamma_0(2)/\Gamma_2^2(2,4) \cong \Gamma_0^0(2)/\Gamma_2(2,4), \tag{2}$$

equivariant with respect to the action of the groups on the two copies of $\mathbb{P}^3$ in (1). For the definition of the groups see (59), (56), (60), (15). Denoting the group in (2) by G we shall prove the following theorem.

Theorem. *For any subgroup $H \subset G$ there exists an isomorphism of graded rings of modular forms*

$$\Phi_H : A(\Gamma) \to A(\Gamma'),$$

where $\Gamma_2(2,4) \subset \Gamma \subset \Gamma_0^0(2)$, $\Gamma_2^2(2,4) \subset \Gamma' \subset \Gamma_0(2)$ *and the quotients* $\Gamma/\Gamma_2(2,4)$ *and* $\Gamma'/\Gamma_2^2(2,4)$ *are both isomorphic to* H.

With a suitable choice of the subgroup H one can actually recover the isomorphism between the Satake compactification of the moduli space of abelian surfaces with level 2 structure and with Göpel triples respectively which is proven in [32, 6].

By this theorem and the modular interpretation of the map (1) we will prove the following result.

Theorem. *For any subgroup* $\Gamma_2(2,4) \subset \Gamma' \subset \Gamma_0^0(2)$ *the Siegel modular variety* $\mathrm{Proj}(A(\Gamma'))$ *has a map of degree 8 onto itself.*

We can recover the degree 8 endomorphism of the Igusa quartic if we set

$$\Gamma' = \{\, \gamma \in \Gamma_2 \mid \gamma \equiv \mathbb{1}_4 \pmod 2 \,\}.$$

The theorem about the existence of a degree 8 endomorphism can be further extended to other modular threefolds. This will be achieved by studying the action of the Fricke involution

$$J_2 = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & \mathbb{1}_g \\ -2\,\mathbb{1}_g & 0 \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{R})$$

on the groups considered so far and consequently on the rings of modular forms. The last section of Chapter 3 is dedicated to the case of degree three, with a view toward Siegel modular varieties in higher dimensions. We will see that the arguments exploited in degree two do not generalize directly.

Regarding the role of vector-valued Siegel modular forms in the study of the geometry of Siegel modular threefolds, in Section 3.2 we will explain their relationship with the definition of holomorphic differential forms on modular varieties.

If X is a complex manifold, denote by $\Omega^n(X)$ the space of holomorphic differential forms on X of degree n. If $g \geqslant 2$ and $n < g(g+1)/2$, there is a natural isomorphism

$$\Omega^n(X_\Gamma^0) \cong \Omega^n(\mathbb{H}_g)^\Gamma,$$

where X_Γ^0 is the set of regular points of $\mathbb{H}_g/\Gamma$ and $\Omega^n(\mathbb{H}_g)^\Gamma$ is the space of Γ-invariant holomorphic differential forms on $\mathbb{H}_g$ of degree n (cf. [17]). For suitable degrees some of these spaces are known to be trivial. The possible non-trivial spaces are identified with vector spaces of vector-valued modular forms (cf. Theorem 3.2.1). For example, for $N = g(g+1)/2$ the identification of Γ-invariant holomorphic differential forms of degree $N-1$ with some vector-valued Siegel modular forms is given in the following way. Let $\{ d\check{\tau}_{ij} \}_{i,j=1}^{g}$ be the basis of holomorphic differential forms on $\mathbb{H}_g$ of degree $N-1$ given by

$$d\check{\tau}_{ij} = \pm\, e_{ij} \bigwedge_{\substack{1\leqslant k\leqslant l\leqslant g\\ (k,l)\neq(i,j)}} d\tau_{kl}; \quad e_{ij} = \frac{1+\delta_{ij}}{2},$$

where the sign is chosen in such a way that $d\check{\tau}_{ij} \wedge d\tau_{ij} = e_{ij} \bigwedge_{1\leqslant k\leqslant l\leqslant g} d\tau_{kl}$. By [14] a differential form $\omega \in \Omega^{N-1}(\mathbb{H}_g)$ is Γ-invariant if and only if

$$\omega = \mathrm{Tr}(A(\tau) d\check{\tau}),$$

where $A(\tau)$ is a vector-valued modular form satisfying the transformation rule

$$A(\gamma\cdot\tau) = \det(C\tau + D)^{g+1}\, {}^{t}(C\tau + D)^{-1}\, A(\tau)\, (C\tau + D)^{-1}, \tag{3}$$

for any $\gamma \in \Gamma$ and $\tau \in \mathbb{H}_g$. We are interested in some explicit constructions of Γ-invariant holomorphic differential forms of degree $N-1$.

In [12] the author uses some differential operators applied to scalar-valued modular forms of suitable weight to define such holomorphic differential forms. For any $f, h \in [\Gamma, (g-1)/2]$, we will denote by $\omega_{f,h}$ the holomorphic differential form constructed in this way. This method produces Γ_g-invariant holomorphic differential forms for suitable values of g. Indeed we will prove the following proposition.

Proposition. *Let*

$$f = \sum_{m\ \mathrm{even}} \vartheta_m(\tau)^{g-1}.$$

Then $\omega_{f,f} \in \Omega^{N-1}(\mathbb{H}_g)^{\Gamma_g}$ *and does not vanish for* $g = 8k+1$, $k \geqslant 1$.

The result for $g \equiv 1 \pmod 8$, $g > 9$ is well known and it is proven in [12]. Recently in my join work [8] we extended the result to $g = 9$.

A second method of building elements of $\Omega^{N-1}(\mathbb{H}_g)^{\Gamma_g}$ is examined in [44]. There the author starts from gradients of odd theta functions and produces holomorphic differential forms invariant under the action of the full modular group for $g \equiv 0 \pmod 4$, $g \neq 5, 13$.

These two methods seemed to be totally unrelated until [8] provided a link between them. The key point in the proof of this result is that theta functions satisfy the heat equation

$$\frac{\partial^2}{\partial z_j \partial z_k}\vartheta_m(\tau, z) = 2\pi i(1+\delta_{jk})\, \frac{\partial}{\partial \tau_{jk}}\vartheta_m(\tau, z), \quad j, k = 1, \ldots, g.$$

In Chapter 4 we will present some new results on the construction of vector-valued Siegel modular forms starting from scalar-valued ones.

Some constructions of vector-valued Siegel modular forms from scalar-valued Siegel modular forms have been recently investigated in [5] and [4]. In the first paper the authors consider the restriction of a scalar-valued modular form f with respect to Γ_g to the diagonally embedded $\mathbb{H}_j \times \mathbb{H}_{g-j}$ for some $1 \leqslant j \leqslant g-1$. If such a restriction vanishes, one can develop f in the normal bundle of $\mathbb{H}_j \times \mathbb{H}_{g-j}$ inside $\mathbb{H}_g$. As lowest non-zero term one finds a sum of tensor products of vector-valued Siegel modular forms with respect to Γ_j and Γ_{g-j}. In the second paper the authors focus on degree two and extend the correspondence given by Igusa between invariants of binary sextics and scalar-valued Siegel modular forms with respect to Γ_2 to a correspondence between covariants of the action of $\mathrm{SL}(2, \mathbb{C})$ on the space of binary sextics and vector-valued Siegel modular forms with respect to Γ_2.

In this thesis we will construct vector-valued Siegel modular forms with respect to a congruence subgroup $\Gamma \subset \Gamma_g$ from singular scalar-valued Siegel modular forms, where $f \in [\Gamma, r/2]$ is singular if and only if $r < g$. This new construction comes from a development of the ideas in [8, Section 5], where we provided a link between the two above-mentioned method to build holomorphic differential forms of degree $N-1$ on suitable modular varieties.

The details of the construction are presented in Sectionr 4.1 which is based on my work [40]. For f, $h \in [\Gamma, 1/2]$ define

$$A_{f,h} = f(\partial h) - (\partial f)h,$$

where $\partial := (\partial_{ij})$ is the $g \times g$ matrix of differential operators

$$\partial_{ij} = \begin{cases} \frac{\partial}{\partial \tau_{ij}} & i = j \\ \frac{1}{2}\frac{\partial}{\partial \tau_{ij}} & i \neq j \end{cases}.$$

It is easy to prove that $A_{f,h}$ is a vector-valued modular form that satisfies the transformation rule

$$A_{f,h}(\gamma \cdot \tau) = \det(C\tau + D)\,(C\tau + D)\,A_{f,h}(\tau)\,{}^t(C\tau + D),$$

for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$ and $\tau \in \mathbb{H}_g$. If f_i, h_i, with $1 \leqslant i \leqslant k < g$, are in $[\Gamma, 1/2]$ we will construct a vector-valued Siegel modular form in the following way. We will define a product $*$ (see (68)) such that

$$A_{f_1,h_1} * \cdots * A_{f_k,h_k} \in [\Gamma, \rho_k] \tag{4}$$

where ρ_k is a suitable irreducible representation. The representation we consider here is interesting because ρ_{g-1} turns out to be the representation appearing in the transformation formula (3) for vector-valued Siegel modular forms that define Γ-invariant holomorphic differential forms of degree $N-1$.

For f and h scalar-valued Siegel modular forms with respect to a subgroup Γ of weight $k/2$, with $1 \leqslant k < g$, we will define two pairings involving the product $*$ that generalize the pairing defined for $k = g-1$ in [12] for the construction of Γ-invariant holomorphic differential forms of degree $N-1$. We will denote them by $\{f, h\}_k$ and $[f, h]_k$. If f, $h \in [\Gamma, (g-1)/2]$, the Γ-invariant holomorphic differential form $\omega_{f,h}$ can be written as follows:

$$\omega_{f,h} = \{f, h\}_{g-1} \sqcap d\check{\tau} = \mathrm{Tr}([f, h]_{g-1}\, d\check{\tau}),$$

where $\sqcap$ is a suitably defined product (see (54)).

If $f = \prod_{i=1}^{k} f_i$ and $h = \prod_{i=1}^{k} h_i$ with f_i, $h_i \in [\Gamma, 1/2]$ for $1 \leqslant i \leqslant k$, we will prove that

$$[f, h]_k = \sum_{\sigma \in S_k} A_{f_1, h_{\sigma(1)}} * \cdots * A_{f_k, h_{\sigma(k)}},$$

where S_k is the group of permutations of the set $\{1, \dots, k\}$. Hence our new vector-valued Siegel modular forms (4) appear as generalizations of the method in [12]. More precisely they appear in the construction of vector-valued modular forms with our new method applied to singular scalar-valued modular forms of a suitable type, namely the ones that can be expressed as products of weight 1/2 scalar-valued modular forms.

We will prove that the relationship between the two methods in [12] and [44] given in [8] is not only at the level of holomorphic differential forms but also at the level of vector-valued modular forms (cf. Section 4.1.3). Gradients of odd theta functions can be used to construct vector-valued modular forms and not only holomorphic differential forms. Such a construction is presented in [45], generalizing the construction of holomorphic differential forms of degree $N-1$ presented in [44]. In order to find this relationship we will apply our new method to second order theta constants. For $\varepsilon \in \{0, 1\}^g$ the second order theta constant with characteristic ε is defined as

$$\Theta[\varepsilon](\tau) = \vartheta \left[\begin{smallmatrix} \varepsilon \\ 0 \end{smallmatrix}\right](\tau, 0).$$

Theorem. *Denote by V_{grad} the vector space generated by the vector-valued modular forms constructed with gradients of odd theta functions and by V_Θ the vector space generated by the vector-valued modular forms constructed with our new method applied to second order theta constants. Then $V_{\mathrm{grad}} = V_\Theta$.*

Finally in Section 4.2 we will apply these constructions to study principally polarized abelian varieties. In particular we will give a new characterization of the locus of decomposable principally polarized abelian varieties. This is part of my joint work [8].

An abelian variety is decomposable if it is a product of lower dimensional abelian varieties. The simpler example of reducible abelian variety is an abelian surface which is a product of two elliptic curves. The analytic characterization of the locus of decomposable abelian varieties involving second order theta constants is well known (cf. [49] and [48]).

In [8] we give a characterization of a decomposable principally polarized abelian variety by looking at the Gauss map of its the theta divisor. Any $\tau \in \mathbb{H}_g$ defines a principally polarized abelian variety (X_τ, Θ_τ). By taking the gradient with respect to z of the holomorphic function $\theta_0(\tau, z)$, we get the Gauss map

$$G : \Theta_\tau \dashrightarrow \mathbb{P}^{g-1}.$$

The base locus of G_τ is equal to the singular locus of the theta divisor Θ_τ. Via the Gauss map, the gradients at $z = 0$ of odd theta functions can be thought of as the images of the 2-torsion points in X_τ that are smooth points of the theta divisor. With this we will prove the following theorem:

Theorem. *A principally polarized abelian variety is decomposable if and only if the images under the Gauss map of all smooth 2-torsion points in the theta divisor lie on a quadric in $\mathbb{P}^{g-1}$.*

CONTENTS

1

COMPLEX ABELIAN VARIETIES

The history of abelian varieties goes back to the beginning of the 19th century when N. Abel and C. Jacobi started to investigate what we now call the hyperelliptic integrals. Essential progress in the subject was made by Riemann who used heavily theta functions in his investigation of the problem.

Towards the end of the 19th century geometers started to study the theory of abelian and theta functions by geometric methods. Originally an "abelian variety" of dimension g meant a hypersurface in $\mathbb{P}^{g+1}$ given as the image of $\mathbb{C}^g$ under the map defined by $g+2$ suitable theta functions (cf. [11]). Since these variety often have unpleasant singularities and do not admit a group structure, the language of complex tori turned out to be more fruitful for this purpose. For the modern mathematician, who is working with $\mathbb{C}$ as ground field, an abelian variety is a complex torus that is a projective variety. It was only after the work of Lefschetz that this point of view was generally accepted.

Today abelian varieties play an important role in many areas of mathematics. Their importance in algebraic geometry lies in the fact that there are natural ways to associate to any smooth projective algebraic variety Y an abelian variety X and investigate properties of Y by studying X. Examples of this are the Picard variety, the Albanese variety and certain intermediate Jacobians.

Apart from this, geometric properties of abelian varieties are interesting for their own sake and this is the subject of this chapter. After a short review of some properties of elliptic curves we will introduce complex tori and study in details the definition of line bundles on them via factors of automorphy. In Section 1.3 we will start talking about abelian varieties. Here we will introduce Riemann's theta function as a global section of a suitable line bundle on a principally polarized abelian variety. Finally in Section 1.4.1 we will present the construction of some moduli spaces of abelian varieties. These will represent in the following examples of modular varieties.

A comprehensive reference for the theory of complex tori and abelian varieties is [3].

In this section we briefly present the simplest examples of complex abelian varieties: elliptic curves. These are the natural setting for the theory of elliptic functions.

For $f : \mathbb{C} \to \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ an analytic function, the group of the periods of is defined as

$$\Lambda_f = \{\, \omega \in \mathbb{C} \mid f(z+\omega) = f(z) \ \forall z \in \mathbb{C} \,\}.$$

It is easy to see that Λ_f is a discrete subgroup of $\mathbb{C}$. If Λ_f is a lattice, i.e. a maximal rank subgroup of $\mathbb{C}$, then f is said to be elliptic or doubly periodic. This terminology is justified by the fact that a discrete subgroup $\Lambda \subset \mathbb{C}$ is a lattice if and only if there are non-zero λ_1, $\lambda_2 \in \mathbb{C}$ with $\mathrm{Im}(\lambda_2/\lambda_1) \neq 0$ such that $\Lambda = \lambda_1\mathbb{Z} \oplus \lambda_2\mathbb{Z}$. As a consequence

$$f(z+\lambda) = f(\lambda) \ \forall \lambda \in \Lambda \quad \Leftrightarrow \quad f(z+\lambda_1) = f(z+\lambda_2) = f(z).$$

Clearly holomorphic elliptic functions are constant by Liouville's theorem, so in order to consider non-constant elliptic functions one must admit some poles and consider meromorphic elliptic functions.

Let Λ be a lattice and consider $\mathcal{M}_\Lambda = \left\{ f : \mathbb{C} \to \hat{\mathbb{C}} \text{ analytic } \mid \Lambda_f = \Lambda \right\}$. It is easily checked that for z, $w \in \mathbb{C}$

$$z - w \in \Lambda \Rightarrow f(z) = f(w) \ \forall f \in \mathcal{M}_\Lambda.$$

So elliptic functions in $\mathcal{M}_\Lambda$ actually live on the quotient group $\mathbb{C}/\Lambda$. This group is usually called complex torus or elliptic curve.

Any lattice can be normalized to one of the form $\Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$, where $\mathrm{Im}(\tau) > 0$. This can be done in such a way that $\mathbb{C}/\Lambda \simeq \mathbb{C}/\Lambda_\tau$. In the following we will consider only lattices of this type.

One of the most important elliptic function is the Weierstrass $\wp$ function defined for a given lattice Λ_τ by the series

$$\wp(z) = \frac{1}{z^2} + \sum_{n,m \in \mathbb{Z}\setminus\{0\}} \left(\frac{1}{(z-n-m\tau)^2} - \frac{1}{(n+m\tau)^2} \right).$$

Its importance is related to the fact that the field $\mathcal{M}(\mathbb{C}/\Lambda_\tau)$ of meromorphic functions on the elliptic curve $\mathbb{C}/\Lambda_\tau$ can be completely described in terms of $\wp(z)$ and its first derivative. Indeed we have that

$$\mathcal{M}(\mathbb{C}/\Lambda_\tau) = \mathbb{C}(\wp(z)) + \mathbb{C}(\wp(z))\wp'(z),$$

where $\mathbb{C}(T)$ is the field of rational functions in T with coefficients in $\mathbb{C}$.

Via the $\wp$ function it is well defined the embedding of $\mathbb{C}/\Lambda_\tau$ in $\mathbb{P}^2$ as

$$\begin{aligned} \mathbb{C}/\Lambda_\tau &\xrightarrow{\varphi} \mathbb{P}^2 \\ z &\mapsto \left[1, \wp(\tau,z), \wp'(\tau,z)\right]. \end{aligned}$$

The functions $\wp$ and $\wp'$ have a pole of order 2 and 3 respectively in the origin and by periodicity in every point of the lattice Λ_τ. The map extends holomorphically onto the poles if we put $\varphi(z) = [0,0,1]$ for $z \in \Lambda$.

It is well known that the Weierstrass $\wp$ function satisfies the following differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

where

$$g_2 = 60 \sum_{n,\, m \in \mathbb{Z}\setminus\{0\}} \frac{1}{(n+m\tau)^4}, \quad g_3 = 140 \sum_{n,\, m \in \mathbb{Z}\setminus\{0\}} \frac{1}{(n+m\tau)^6}.$$

Because of this the image $\varphi(\mathbb{C}/\Lambda_\tau)$ is the projective curve given by the equation:

$$x_0 x_2^2 = 4x_1^3 - g_2 x_0^2 x_1 - g_3 x_0^3.$$

1.2 COMPLEX TORI

In this section we will introduce complex tori which are the higher dimensional version of elliptic curves. If any elliptic curve admits an embedding in projective space, this is not the case for higher dimensional complex tori. We will be mainly interested in the ones that are projective, namely we will be interested in complex abelian varieties.

For V a complex vector space of dimension g, a lattice Λ in V is a discrete subgroup of maximal rank. Equivalently there are non-zero $\lambda_1, \dots, \lambda_{2g} \in \mathbb{C}$ independent over $\mathbb{R}$ such that $\Lambda = \lambda_1\mathbb{Z} \oplus \cdots \oplus \lambda_{2g}\mathbb{Z}$. Hence a lattice is a free abelian group of rank $2g$. The quotient group $X = V/\Lambda$ is called a complex torus of dimension g. For $v \in V$ we denote by $[v]$ its equivalence class in X.

A complex torus is a connected compact Lie group. The vector space V may be considered as the universal covering space of X via the canonical projection $\pi : V \to X$. The lattice Λ can be identified with the fundamental group $\pi_1(X) := \pi_1(X, 0)$. Moreover, since Λ is abelian, $\pi_1(X)$ is canonically isomorphic to $H_1(X, \mathbb{Z})$. As the torus is locally isomorphic to V we can regard it as the tangent space T_0X of X in 0 so that the universal covering map is nothing but the exponential map $\pi : T_0X \to X$.

The period matrix of a complex torus determines it completely. Choose $e_1, \dots, e_g$ and $\lambda_1, \dots, \lambda_{2g}$ bases of V and Λ respectively. For $i = 1, \dots, 2g$ write $\lambda_i = \sum_{j=1}^{g} \lambda_{ji} e_j$.

The period matrix of X with respect to these basis is the $g \times 2g$ matrix $\Pi = (\lambda_{ji})$. The columns of this matrix generate a lattice that we denote by $\Pi\mathbb{Z}^{2g}$. With these choices $X \cong \mathbb{C}^g/\Pi\mathbb{Z}^{2g}$. Conversely one may ask when a $g \times 2g$ complex matrix is a period matrix of a complex torus.

Proposition 1.2.1. *A matrix $\Pi \in M_{g\times 2g}(\mathbb{C})$ is a period matrix of a complex torus if and only if the matrix $\binom{\Pi}{\overline{\Pi}} \in M_{2g}(\mathbb{C})$ is non singular, where $\overline{\Pi}$ is the complex conjugate matrix of Π.*

Proof. A matrix $\Pi \in M_{g\times 2g}(\mathbb{C})$ is a period matrix if and only if its columns vectors span a lattice in $\mathbb{C}^g$ or equivalently if its columns are linearly independent over $\mathbb{R}$. If the columns of Π are dependent over $\mathbb{R}$ then there is a non zero $x \in \mathbb{R}^{2g}$ with $Px = 0$. This implies $\det P = 0$. Conversely, if P is singular there are non zero vectors $x, y \in \mathbb{R}^{2g}$ such that $P(x+iy) = 0$. Hence the columns of Π are linearly dependent over $\mathbb{R}$ since $\Pi(x+iy) = 0$ and $\Pi(x-iy) = \overline{\overline{\Pi}(x+iy)} = 0$ imply $\Pi x = \Pi y = 0$. $\square$

If $X = V/\Lambda$ and $X' = X'/\Lambda'$ are two complex tori of dimension g and g', a homomorphism of X to X' is a holomorphic map $f : X \to X'$ compatible with the group structures. The connected component $(\ker f)_0$ of $\ker f$ containing 0 is a closed subtorus of X of finite index in $\ker f$. The following Proposition shows that every holomorphic map between complex tori is the composition of an homomorphism and a translation.

Proposition 1.2.2. *If $h : X \to X'$ is a holomorphic map of complex tori then there is a unique homomorphism $f : X \to X'$ such that $h(x) = f(x) + h(0)$ for all $x \in X$ and there is a unique $\mathbb{C}$-linear map $F : V \to V'$ with $F(\Lambda) \subset F(\Lambda')$ inducing the homomorphism f.*

Remark 1.2.3. *The $\mathbb{C}$-linear map $F : V \to V'$ fits in the commutative diagram*

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[d, "f\cdot\pi"'] & V' \arrow[dl, "\pi'"] \\ X' & \end{tikzcd}.$$

It is useful to define also the analytic and rational representation of the group of homomorphisms $\mathrm{Hom}(X, X')$ for X and X' complex tori. The analytic representation is given as

$$\rho_a : \mathrm{Hom}(X, X') \to \mathrm{Hom}_{\mathbb{C}}(V, V')$$
$$f \mapsto F.$$

By Proposition 1.2.2 ρ_a is an injective homomorphism of abelian groups.

The restriction $F_\Lambda : \Lambda \to \Lambda'$ of F to the lattices is $\mathbb{Z}$-linear and determines F and f completely. So the rational representation of $\mathrm{Hom}(X, X')$ is given as

$$\rho_r : \mathrm{Hom}(X, X') \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \Lambda')$$
$$f \mapsto F_\Lambda.$$

The map ρ_r is again an injective homomorphism of groups.

Fixing bases of V, V' and Λ, Λ' then for any $f \in \mathrm{Hom}(X, X')$ the associated homomorphisms $\rho_a(f)$ and $\rho_r(f)$ are given by a matrix $A \in M_{g' \times g}(\mathbb{C})$ and $R \in M_{2g' \times 2g}(\mathbb{Z})$ respectively. If Π and Π' are period matrices for X and X' respectively, with respect to these bases, then $\rho_a(f)(\Lambda) \subset \Lambda'$ if and only if $A\Pi = \Pi' R$.

If $f : X \to X'$ is a surjective homomorphism, then the Stein factorization of f is defined by the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & X' , \\ X/(\ker f)_0 \arrow[ur, "h"'] & \end{tikzcd}$$

where g is a surjective homomorphism with a complex torus as kernel and h is a surjective homomorphism with finite kernel. A homomorphism with the same properties as h is usually called an isogeny. It is easily seen that a homomorphism $X \to X'$ is an isogeny if and only if it is surjective and $\dim X = \dim X'$. The degree of a homomorphism f is defined as the order of the group $\ker f$, if it is finite, and 0 otherwise. Hence if $f : V/\Lambda \to V'/\Lambda'$ is an isogeny we have that

$$\deg f = [\Lambda' : \rho_r(f)(\Lambda)],$$

where $[\Lambda' : \rho_r(f)(\Lambda)]$ is the order of the subgroup $\rho_r(f)(\Lambda)$ in Λ'. If f is an endomorphism then $\deg f = \det \rho_r(f)$.

For any non zero integer n we have a remarkable example of endomorphism which is an isogeny:

$$\begin{aligned} n_X : X &\to X \\ x &\mapsto nx. \end{aligned}$$

If $\dim X = g$, then $X_n := \ker(n_X) \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$ is finite and it is called the group of n-torsion points of X. Then n_X is an isogeny of degree n^{2g}. Isogenies define an equivalence relation on the set of complex tori. Two complex tori are isogenous if they are in the same coset for this equivalence relation.

We are interested in complex tori $X = V/\Lambda$ that can be embedded in some projective space. There is more than one way to define a map from a complex torus to a projective space. Suppose for example that $f_i : V \to \mathbb{C}$, $i = 0, \dots, n$, are holomorphic functions with no common zeros with the property that for every $\lambda \in \Lambda$ there exists $e_\lambda : V \to \mathbb{C} \setminus \{0\}$ such that

$$f_i(v + \lambda) = e_\lambda(v) f_i(v), \ \forall v \in V, \forall \lambda \in \Lambda.$$

Then the map $\bar{f} : X \to \mathbb{P}^n$ with $\bar{f}([v]) = [f_0(v), \dots, f_n(v)]$ is holomorphic. We will see that these maps arise naturally as the maps defined by sections of line bundles on complex tori.

1.2.1 *Line bundles and factors of automorphy*

For a complex manifold M let $\pi : \widetilde{M} \to M$ be the universal covering. Denote by $\pi_1(M)$ the fundamental group of M, where we omit the base point in the notation. Holomorphic line bundles on M whose pullback on $\widetilde{M}$ is trivial can be described in therms of the cohomology of the fundamental group $\pi_1(M)$ acting on $\widetilde{M}$.

The action of $\pi_1(M)$ on $\widetilde{M}$ induces a $\pi_1(M)$-module structure on $H^0(\mathcal{O}^*_{\widetilde{M}})$, where $\mathcal{O}^*_{\widetilde{M}}$ is the sheaf of invertible holomorphic functions on $\widetilde{M}$. We are interested in the first cohomology group of $\pi_1(M)$ with values in $H^0(\mathcal{O}^*_{\widetilde{M}})$

$$H^1(\pi_1(M), H^0(\mathcal{O}^*_{\widetilde{M}})) = Z^1(\pi_1(M), H^0(\mathcal{O}^*_{\widetilde{M}}))/B^1(\pi_1(M), H^0(\mathcal{O}^*_{\widetilde{M}})),$$

where as usual Z^1 is the abelian group of cocycles and B^1 is the subgroup of boundaries.

More explicitly a cocycle is a holomorphic map $f : \pi_1(M) \times \widetilde{M} \to \mathbb{C}^*$ satisfying the cocycle relation

$$f(\lambda\mu, \tilde{x}) = f(\lambda, \mu\tilde{x}) f(\mu, \tilde{x})$$

for all λ, $\mu \in \pi_1(M)$ and $\tilde{x} \in \widetilde{M}$. Functions of this type are also called factors of automorphy. A boundary is a factor of the form $f(\lambda, \tilde{x}) = h(\lambda\tilde{x})h(\tilde{x})^{-1}$ for some $h \in H^0(\mathcal{O}^*_{\widetilde{M}})$.

For any factor of automorphy we can define a line bundle on M starting from the trivial line bundle on $\widetilde{M}$. Let $\pi_1(M)$ act on the trivial line bundle $\widetilde{M} \times \mathbb{C} \to \widetilde{M}$ by

$$\lambda \cdot (\tilde{x}, z) = (\lambda\tilde{x}, f(\lambda, \tilde{x})z), \ \forall \lambda \in \pi_1(M).$$

Since the action is free and properly discontinuous, the quotient $L = \widetilde{M} \times \mathbb{C}/\pi_1(M)$ is a complex manifold. It is easily checked that L is a holomorphic line bundle on M by considering the projection $p : L \to M$ induced by the canonical projection $\widetilde{M} \times \mathbb{C} \to \widetilde{M}$. The following Proposition shows that there is an isomorphism between $H^1(\pi_1(M), H^0(\mathcal{O}^*_{\widetilde{M}}))$ and the group of line bundles on M whose pullback on $\widetilde{M}$ is trivial.

Proposition 1.2.4. *There is a canonical isomorphism*

$$\psi : \ker\left(H^1(M, \mathcal{O}^*_M) \xrightarrow{\pi^*} H^1(\widetilde{M}, \mathcal{O}^*_{\widetilde{M}})\right) \xrightarrow{\simeq} H^1(\pi_1(M), H^0(\mathcal{O}^*_{\widetilde{M}})).$$

Proof. We will only give details on the definition of the map ψ. That is we will explain how one can associate a factor of automorphy to a line bundle L with trivial pullback on $\widetilde{M}$. Let $\alpha : \pi^*L \to \widetilde{M} \times \mathbb{C}$ be a trivialization for π^*L. The action of $\pi_1(M)$ on $\widetilde{M}$ induces holomorphic automorphisms of π^*L over this action. Via α we get for every $\lambda \in \pi_1(M)$ an automorphism φ_λ of the trivial line bundle $\widetilde{M} \times \mathbb{C}$. Necessarily φ_λ is of the form $\varphi_\lambda(\tilde{x}, z) = (\lambda\tilde{x}, f(\lambda, \tilde{x})z)$ with a map $f : \pi_1(M) \times \widetilde{M} \to \mathbb{C}$ holomorphic in $\tilde{x}$. The equation $\varphi_{\lambda\mu} = \varphi_\lambda \varphi_\mu$ implies that $f \in Z^1(\pi_1(M), H^0(\mathcal{O}^*_{\widetilde{M}}))$. Suppose $\alpha' : \pi^*L \to \widetilde{M} \times \mathbb{C}$ is a different trivialization. Then there is an $h \in H^0(\mathcal{O}^*_{\widetilde{M}})$ such that $\alpha'\alpha^{-1}(\tilde{x}, z) = (\tilde{x}, h(\tilde{x})z)$ for all $(\tilde{x}, z) \in \widetilde{M} \times \mathbb{C}$. If φ'_λ denotes the automorphism of $\widetilde{M} \times \mathbb{C}$ associated to $\lambda \in \pi_1(M)$ with respect to the trivialization α', then

$$\varphi'_\lambda(\tilde{x}, z) = (\alpha'\alpha^{-1})\varphi_\lambda(\alpha'\alpha^{-1})^{-1}(\tilde{x}, z) = (\lambda\tilde{x}, h(\lambda\tilde{x})f(\lambda, \tilde{x})h^{-1}(\tilde{x})z).$$

Hence the class of the cocycle f does not depend on the trivialization $\pi^*L \to \widetilde{M} \times \mathbb{C}$ and we get a canonical map $\ker\left(H^1(M, \mathcal{O}^*_M) \xrightarrow{\pi^*} H^1(\widetilde{M}, \mathcal{O}^*_{\widetilde{M}})\right) \to H^1(\pi_1(M), H^0(\mathcal{O}^*_{\widetilde{M}}))$. For the rest of the proof the reader may refer to [3, Proposition B.1]. $\square$

Also the global sections of line bundles on M with trivial pullback on $\widetilde{M}$ can be described in terms of a factor of automorphy for L. For any line bundle L on M there is a canonical isomorphism between the space of global sections of L and the space $H^0(\pi^*L)^{\pi_1(M)}$ of global sections of π^*L that are invariant under the action of $\pi_1(M)$ on $\widetilde{M}$. A trivialization $\alpha : \pi^*L \to \widetilde{M} \times \mathbb{C}$ induces an isomorphism between $H^0(\pi^*L)^{\pi_1(M)}$ and $H^0(\widetilde{M} \times \mathbb{C})^{\pi_1(M)}$. If f is the factor of automorphy associated to L with respect to the same trivialization, then the elements of $H^0(\widetilde{M} \times \mathbb{C})^{\pi_1(M)}$ are holomorphic functions $\vartheta : \widetilde{M} \to \mathbb{C}$ satisfying

$$\vartheta(\lambda\tilde{x}) = f(\lambda, \tilde{x})\vartheta(\tilde{x}),$$

for $\tilde{x} \in \widetilde{M}$ and $\lambda \in \pi_1(M)$. We will call them theta functions for the factor of automorphy f. In this way the sections of L are identified with the space of theta functions for a factor of automorphy associated to L. This identification depends on the trivialization of π^*L. If f and f' are two factors of automorphy for L associated to two different trivialization, then

$$f'(\lambda, \tilde{x}) = h(\lambda\tilde{x})f(\lambda, \tilde{x})h^{-1}(\tilde{x}),$$

for some $h \in H^0(\mathcal{O}^*_{\widetilde{M}})$. So if one changes the trivialization the factor of automorphy for a line bundle L is multiplied by a boundary.

1.2.2 *Canonical factor of automorphy on complex tori*

In this section we turn back to line bundles on complex tori. The projection $\pi : V \to V/\Lambda$ is the universal covering map of the complex torus V/Λ and $\pi_1(V/\Lambda) = \Lambda$. Since any line bundle on a complex vector space is trivial, from Section 1.2.1 we get a description of all line bundles on a complex torus in terms of an action of the lattice on the trivial line bundle on V. We will see that any line bundle is defined by a factor of automorphy, which will be called canonical, that is related to its first Chern class.

We briefly recall the definition of the first Chern class of a line bundle on a complex manifold. Consider the exponential exact sequence for a complex manifold M

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_M \xrightarrow{e^{2\pi i \cdot}} \mathcal{O}_M^* \longrightarrow 0,$$

where $\mathbb{Z}$ is the constant sheaf with values in $\mathbb{Z}$, $\mathcal{O}_M$ is the sheaf of holomorphic functions on M and $\mathcal{O}_M^*$ is the sheaf of invertible holomorphic functions on M. The induced long exact sequence in cohomology gives a map

$$\mathrm{Pic}(M) \xrightarrow{c_1} H^2(M, \mathbb{Z}),$$

where $\mathrm{Pic}(M)$ is the group of line bundles on M. For $L \in \mathrm{Pic}(M)$ the class $c_1(L)$ is called the first Chern class of the line bundle L. The image $c_1(\mathrm{Pic}(M)) = NS(M)$ is the Neron-Severi group of M. If M is a complex torus, then the Neron-Severi group can be described in terms of Hermitian and alternating forms on the universal cover:

$$NS(V/\Lambda) = \left\{ \begin{array}{l} E : V \times V \to \mathbb{R} \text{ alternating form} \\ E(\Lambda, \Lambda) \subset \mathbb{Z},\ E(iv, iw) = E(v, w) \end{array} \right\} = \left\{ \begin{array}{l} H : V \times V \to \mathbb{C} \text{ hermitian form} \\ \text{with } \operatorname{Im} H(\Lambda, \Lambda) \subset \mathbb{Z} \end{array} \right\}.$$

The correspondence is given by

$$\begin{array}{ccc} E & \to & E(iv, w) + iE(v, w) \\ \operatorname{Im} H & \leftarrow & H. \end{array}$$

Let $X = V/\Lambda$. A semi-character for $H \in NS(X)$ is a map $\alpha : \Lambda \to S^1 := \{ z \in \mathbb{C} \mid |z| = 1 \}$ such that

$$\alpha(\lambda + \mu) = \alpha(\lambda)\alpha(\mu) e^{\pi i \operatorname{Im} H(\lambda, \mu)}, \ \forall \lambda, \ \mu \in \Lambda.$$

For any couple (H, α) where $H \in NS(X)$ and α is a semi-character for H one can define the factor of automorphy

$$e^{H,\alpha}(\lambda, v) = \alpha(\lambda) e^{\pi(H(v,\lambda) + H(\lambda,\lambda)/2)}, \ \forall \lambda \in \Lambda, \ \forall v \in V. \tag{5}$$

Denote by $L(H, \alpha)$ the line bundle defined by this factor of automorphy. By Section 1.2.1 it follows that

$$L(H, \alpha) = V \times \mathbb{C}/\Lambda,$$

where the action of the lattice on $V \times \mathbb{C}$ is given by

$$\lambda \cdot (v, z) = (v + \lambda, e^{H,\alpha}(\lambda, v)z), \ \forall\, v \in V, \forall\, z \in \mathbb{C}.$$

The following theorem states that for any line bundle on X we can distinguish a canonical factor of automorphy for it.

Theorem 1.2.5 (Appell-Humbert). *If $L \in \mathrm{Pic}(X)$ with $c_1(L) = H$, one can distinguish a semi-character for H such that*

$$L \simeq L(H, \alpha).$$

The factor of automorphy $e^{H,\alpha}$ as in (5) is called the canonical factor of automorphy for L.

With this description of line bundles in terms of canonical factors of automorphy we can easily prove the following theorem.

Theorem of the square. *For any $x \in X$ denote by t_x the translation by x. For any $v, w \in X$ and $L \in \mathrm{Pic}(X)$*

$$t^*_{v+w}L \simeq t^*_v L \otimes t^*_w L \otimes L^{-1}.$$

Proof. For any $x \in X$

$$t^*_x L(H, \alpha) \simeq L(H, \alpha\, e^{2\pi i \operatorname{Im} H(-,x)}). \tag{6}$$

Comparing hermitian forms and semi-characters, the theorem easily follows. □

Formula (6) in the proof of the theorem also shows that two line bundles that differ by a translation have the same first Chern class. If the first Chern class of L is non degenerate the vice versa is also true.

Lemma 1.2.6. *Let L be a line bundle with $c_1(L)$ non degenerate. If $L' \in \mathrm{Pic}(X)$ has the same Chern class of L then there exists $x \in X$ such that $L' \simeq t^*_x L$.*

1.3 ABELIAN VARIETIES

We now come to the main subject of this chapter. An abelian variety is a polarized complex torus, i.e. a complex torus X with a positive definite Hermitian form $H \in \mathrm{NS}(X)$. Since the elements of the Neron-Severi group are first Chern classes of line bundles on X, we will also call polarization a line bundle with positive definite first Chern class. We will denote by (X, L) or (X, H) the abelian variety X with polarization given by a line bundle L or with polarization given by a positive definite $H \in NS(X)$ respectively. If we think of a polarization as given by a line bundle, by Lemma 1.2.6 the polarization is defined up to translation.

If $X = \mathbb{C}/\Lambda_\tau$ is an elliptic curve then the Hermitian form $H : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined as

$$H(v, w) = \frac{v \cdot \overline{w}}{\operatorname{Im} \tau}$$

is a polarization on X. Then every one dimensional complex torus is an abelian variety. In higher dimension it is not true that any complex torus is an abelian variety. The following theorem gives explicit examples of 2-dimensional complex tori which are not abelian varieties.

Theorem 1.3.1 ([10], Appendix). *Let* $X = \mathbb{C}^2/\Lambda$, *where* Λ *is the lattice generated by the columns of the matrix*

$$\begin{pmatrix} 1 & 0 & ip & ir \\ 0 & 1 & iq & is \end{pmatrix}$$

with $p, q, r, s, \in \mathbb{R}$. *Then*

$$\operatorname{rank}_{\mathbb{Z}}(NS(X)) = 4 - \operatorname{rk}_{\mathbb{Q}}(p, q, r, s) + \begin{cases} 1 & \textit{if} \quad ps - qr \in \mathbb{Q} \\ 0 & \textit{if} \quad ps - qr \notin \mathbb{Q} \end{cases}.$$

So if we take $p, q, r, s \in \mathbb{R}$ independent over $\mathbb{Q}$ such that $ps - qr \notin \mathbb{Q}$, then the $\operatorname{rank}_{\mathbb{Z}}(NS(X)) = 0$. In these cases the complex torus X is not an abelian variety. For a very explicit example we can take $(p, q, r, s) = (1, \sqrt{3}, \sqrt{2}, \sqrt{5})$.

Even if not all complex tori are abelian varieties, for any complex torus X there exist an abelian variety X_{ab}, called the abelianization of X, and a surjective holomorphic map $\rho : X \to X_{ab}$ such that any holomorphic map of X into projective space factors trough ρ (cf. [9]). The morphism ρ induces isomorphisms $\mathcal{M}(X_{ab}) \simeq \mathcal{M}(X)$ and $\operatorname{Div}(X_{ab}) \simeq \operatorname{Div}(X)$, where $\mathcal{M}(X)$ is the field of meromorphic functions on X and $\operatorname{Div}(X)$ is the group of divisors on X. As the field of functions on an abelian variety of dimension g is an extension of finite type of $\mathbb{C}$ with transcendence degree g, the field of functions on X is an extension of finite type of $\mathbb{C}$ with transcendence degree equal to $\dim X_{ab}$. This degree equals the dimension of X if and only if X is an abelian variety.

If (X, H) is an abelian variety, the type of the polarization is defined in the following way. The elementary divisor theorem states that there is a basis of the lattice, called a symplectic basis of Λ for H, with respect to which the alternating form $E := \operatorname{Im} H$ is given by the matrix

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

where $D = \operatorname{diag}(d_1, \ldots, d_g)$ with $d_i \in \mathbb{Z}_{>0}$ and $d_i | d_{i+1}$ for $i = 1, \ldots, g-1$. The elementary divisors $d_1, \ldots, d_g$ are uniquely determined by E and the lattice. The g-tuple $(d_1, \ldots, d_g)$, or the matrix D, is called the type of the polarization. If D is the identity matrix the polarization is called principal and X is called a principally polarized abelian variety.

If $f : Y \to X$ is a homomorphism of complex tori with finite kernel and $L \in \operatorname{Pic}(X)$ is a polarization on X, then the line bundle f^*L gives a polarization on Y which is called the induced polarization. Any polarization on a complex torus is induced by a principal polarization via an isogeny (cf. [9]). More precisely, if (X, L) is an abelian variety, then there exists a principally polarized abelian variety (Y, L') and an isogeny $f : X \to Y$ such that $L \simeq f^*L'$.

If L is a polarization, the dimension of the space of global section depends only on its type. Indeed if L is a polarization of type $(d_1, \ldots, d_g)$ then

$$\dim H^0(L) = \prod_{i=1}^{g} d_i.$$

A polarization is what we need to define an embedding of a complex torus in some projective space. A line bundle L on a compact complex variety M is said to be generated by global sections if for every $x \in M$ there exists a global section of L which does not vanish at x. If $s_0, \ldots, s_m$ is a basis of $H^0(L)$, one can define a map

$$\begin{aligned} \varphi_L : M &\to \mathbb{P}(H^0(L))^\vee \\ p &\mapsto [s_0(p), \ldots, s_m(p)]. \end{aligned}$$

The line bundle L is said to be very ample if φ_L is an embedding. If there exist $n \in \mathbb{N}$ such that L^n is very ample then L is said to be ample.

A line bundle on a complex torus defines a polarization if and only if it is ample (cf [3, Proposition 4.5.2]). The following theorem states that a polarization is not so far from being very ample.

Theorem 1.3.2 (Lefschetz). *If L is a polarization on X, then L^n is very ample for $n \geqslant 3$.*

1.3.1 *Riemann's theta functions*

In this section we will introduce Riemann's theta function as a theta function for a suitable line bundle on a principally polarized abelian variety. We will start by looking at the period matrix of an abelian variety.

Theorem 1.3.3 (Riemann's relations). *A complex torus* X *with period matrix* Π *is an abelian variety if and only if there is a non degenerate alternating matrix* $A \in M_{2g}(\mathbb{Z})$ *such that*

$$\Pi A^{-1}\,{}^t\Pi = 0 \ \textit{and} \ \ i\Pi A^{-1}\,{}^t\overline{\Pi} > 0.$$

If the polarization is of type D and $\Pi = (\Pi_1, \Pi_2)$ is the period matrix with respect to a symplectic basis of the lattice then Riemann's relations give:

$$\begin{aligned} \Pi_2 D^{-1}\,{}^t\Pi_1 - \Pi_1 D^{-1}\,{}^t\Pi_2 &= 0, \\ i\Pi_2 D^{-1}\,{}^t\overline{\Pi}_1 - i\Pi_1 D^{-1}\,{}^t\overline{\Pi}_2 &= 0. \end{aligned} \tag{7}$$

Let $X = V/\Lambda$ and denote by $\lambda_1, \dots, \lambda_{2g}$ a symplectic basis of Λ for the polarization. Then $e_i = \frac{1}{d_i}\lambda_{g+i}$, $i = 1, \dots, g$, is a $\mathbb{C}$-basis for V (cf. [3, Lemma 3.2.1]). With respect to these bases the period matrix of X is of the form $\Pi = (\tau, D)$ for some $\tau \in M_g(\mathbb{C})$. By (7) the matrix τ satisfies the identities ${}^t\tau = \tau$, $\mathrm{Im}(\tau) > 0$. Hence a complex torus is an abelian variety with polarization of type D if and only if there are basis with respect to which the period matrix $\Pi = (\tau, D)$, for some τ in

$$\mathbb{H}_g = \left\{ \tau \in M_g(\mathbb{C}) \mid {}^t\tau = \tau,\ \mathrm{Im}(\tau) > 0 \right\}. \tag{8}$$

This is called the Siegel space of degree g. So if one fixes the type D of the polarization, any abelian variety with a polarization of type D defines an element of $\mathbb{H}_g$ by taking the first g columns of the period matrix $\Pi = (\tau, D)$.

Conversely, any matrix in the Siegel space defines a polarized abelian variety of type D with a symplectic basis in the following way. For $\tau \in \mathbb{H}_g$ let $X_\tau = \mathbb{C}^g/\tau\mathbb{Z}^g \oplus D\mathbb{Z}^g$, where $\tau\mathbb{Z}^g \oplus D\mathbb{Z}^g$ is the lattice generated by the columns of the matrices τ and D. The Hermitian form $H_\tau = (\mathrm{Im}\,\tau)^{-1}$ is a polarization of type D for X_τ and the symplectic basis of the lattice $\tau\mathbb{Z}^g \oplus D\mathbb{Z}^g$ for H_τ is given just by the column of the matrix (τ, D). In this way we can think of the Siegel space of degree g as a moduli space for polarized abelian varieties of type D with symplectic basis.

Riemann's theta functions arise as sections of a principal polarization on a complex torus X_τ. The factor of automorphy

$$e(\tau m + n, z) = e^{-\pi i({}^t m\tau m + 2\,{}^t m z)},\ m,\, n \in \mathbb{Z}^g,\, z \in \mathbb{C},$$

defines a principal polarization L_τ on X_τ. Up to scalar the unique theta function for L_τ, in the sense of Section 1.2.1, is defined by the series

$$\sum_{m \in \mathbb{Z}^g} e^{\pi i({}^t m\tau m + 2\,{}^t m z)}.$$

The Riemann theta function $\vartheta : \mathbb{H}_g \times \mathbb{C}^g \to \mathbb{C}$ is defined as

$$\vartheta(\tau, z) = \sum_{m \in \mathbb{Z}^g} e^{\pi i ({}^t m \tau m + 2 {}^t m z)}.$$

By formula (6), for any $x \in X_\tau$ the line bundle $t_x^* L_\tau$ defines again a principal polarization on X_τ. The sections of the line bundles for x a 2-torsion point are extremely interesting in the theory of modular forms. If $m = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, $a, b \in \{0, 1\}^g$, the point $x_m = a\frac{\tau}{2} + b\frac{1}{2}$ is a 2-torsion point in X_τ. The unique non-zero section up to scalar of $t_{x_m}^* L_\tau$ is the theta function with characteristic

$$\vartheta_m(\tau, z) = \vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](\tau, z) = \sum_{n \in \mathbb{Z}^g} e^{\pi i [{}^t (n + a/2) \tau (n + a/2) + 2 {}^t (n + a/2)(z + b/2)]}.$$

The function ϑ_m is an even function of z if ${}^t a\, b \equiv 0 \pmod 2$ and it is an odd function of z if ${}^t a\, b \equiv 1 \pmod 2$. Correspondingly the characteristic m is called even or odd. This actually means that the line bundles $t_{x_m}^* L_\tau$ are symmetric, that is they are invariant under pullback by the involution

$$\begin{aligned} \iota : X &\to X \\ x &\mapsto -x. \end{aligned}$$

If we write a line bundle L in terms of Appell-Humbert data $L = L(H, \alpha)$, then it is symmetric if and only if $\alpha(\Lambda) \subset \{\pm 1\}$. It is called totally symmetric if $\alpha(\lambda) = 1$ for all $\lambda \in \Lambda$.

The line bundle L_τ^2 is totally symmetric and a basis of sections is given by the second order theta functions

$$\Theta[\sigma](\tau, z) = \vartheta \left[\begin{smallmatrix} \sigma \\ 0 \end{smallmatrix}\right](2\tau, 2z), \ \sigma \in \{0, 1\}^g.$$

By the following Lemma the line bundles $t_{x_m}^* L_\tau$ are all the possible symmetric principal polarizations on X_τ.

Lemma 1.3.4 ([30])**.** *For a complex g-dimensional complex torus X and a given $H \in NS(X)$ there are 2^{2g} semi-characters for H such that $L(H, \alpha)$ is a symmetric line bundle.*

1.4 Moduli spaces of abelian varieties

Moduli spaces arise as solutions to classification problems. Given a collection of interesting geometric objects (e.g. polarized abelian varieties with a given polarization up to isomorphisms), a moduli space is roughly speaking a geometric space (scheme or algebraic stack) whose points are in some natural one to one correspondence with the elements of the set. If the moduli space has such a structure then one can parametrize the objects to classify by introducing coordinates on the moduli space.

In this section we will present the analytic construction of some moduli space of polarized abelian varieties.

1.4.1 *Moduli space of polarized abelian varieties*

We want to classify abelian varieties with polarization of a given type D and a given dimension g. The space to start with is the Siegel space $\mathbb{H}_g$ defined in (8). We can think of $\mathbb{H}_g$ as a moduli space for triples

$$\left(X, H, \{\, \lambda_1, \ldots, \lambda_{2g} \,\}\right),$$

where X is an abelian variety, $H \in NS(X)$ is a polarization of type D on X and $\{\, \lambda_1, \ldots, \lambda_{2g} \,\}$ is a symplectic basis of Λ for H (see Section 1.3.1).

For any $\tau \in \mathbb{H}_g$ denote by (X_τ, H_τ) the corresponding polarized abelian variety of type D. To construct a moduli space for these varieties we need to know when two elements in $\mathbb{H}_g$ define isomorphic abelian varieties.

Define the integral symplectic group of type D as

$$\Gamma_D = \left\{\, M \in M_{2g}(\mathbb{Z}) \mid M \left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right) {}^t M = \left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right) \,\right\}.$$

It acts on $\mathbb{H}_g$ by the formula

$$R \cdot \tau = (a\tau + bD)(D^{-1}c\tau + D^{-1}dD)^{-1}, \quad R = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_D. \tag{9}$$

Theorem 1.4.1. *Two polarized abelian varieties (X_τ, H_τ) and $(X_{\tau'}, H_{\tau'})$ of type D are isomorphic if and only if $\tau' = R \cdot \tau$ for some $R \in \Gamma_D$.*

Proof. Let

$$G_D = \left\{\, M \in Sp(2g, \mathbb{Q}) \mid {}^t M \Lambda_D \subset \Lambda_D \,\right\},$$

where Λ_D is the lattice generated by the columns of the matrix $(\mathbb{1}_g, D)$. We can let G_D act on $\mathbb{H}_g$ by

$$M * \tau = (a\tau + b)(c\tau + d)^{-1},$$

where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, with $g \times g$ blocks. One can prove that (X_τ, H_τ) and $(X_{\tau'}, H_{\tau'})$ are isomorphic if and only if $\tau' = M * \tau$ for some $M \in G_D$. For this part of the proof we refer to [3].

Let $Sp^D(2g,\mathbb{R}) = \left\{ M \in M_{2g}(\mathbb{R}) \mid M \left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right) {}^tM = \left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right) \right\}$ and define

$$\begin{aligned} \sigma_D : Sp^D(2g,\mathbb{R}) &\to Sp(2g,\mathbb{R}) \\ M &\mapsto \begin{pmatrix} \mathbb{1}_g & 0 \\ 0 & D \end{pmatrix}^{-1} M \begin{pmatrix} \mathbb{1}_g & 0 \\ 0 & D \end{pmatrix}^{-1}. \end{aligned}$$

This is an isomorphism such that $\sigma_D(\Gamma_D) = G_D$. If $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Sp^D(2g,\mathbb{R})$ then $\sigma_D \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & bD \\ D^{-1}c & D^{-1}dD \end{smallmatrix}\right) \in Sp(2g,\mathbb{R})$. So the action of G_D on $\mathbb{H}_g$ induce the action of Γ_D given in formula (9). This ends the proof of the theorem. □

So there is a one-to-one correspondence between the quotient $\mathcal{A}_g^D = \mathbb{H}_g/\Gamma_D$ and the isomorphism classes of polarized abelian varieties of type D. The action of Γ_D on $\mathbb{H}_g$ is properly discontinuous, so that $\mathcal{A}_g^D$ is a normal complex variety of dimension $g(g+1)/2$.

The principally polarized case is of great interest for us. If $D = \mathbb{1}_g$, the moduli space of principally polarized abelian varieties is

$$\mathcal{A}_g = \mathbb{H}_g/\Gamma_g,$$

where Γ_g is the integral symplectic group.

1.4.2 *Moduli space of abelian varieties with level structures*

In this section we will consider abelian varieties with some extra structure and give a description of their moduli spaces. If X is a complex torus, denote by $\mathrm{Pic}^0(X)$ the group of line bundles with trivial first Chern class. It is a complex torus of the same dimension of X. Because of the theorem of the square (see page 9), any line bundle $L \in \mathrm{Pic}(X)$ defines a homomorphism of complex tori by

$$\begin{aligned} \phi_L : X &\to \mathrm{Pic}^0(X) \\ x &\mapsto t_x^* L \otimes L^{-1}. \end{aligned} \tag{10}$$

This homomorphism is an isogeny if and only if $c_1(L)$ is non degenerate. The kernel of the homomorphism ϕ_L will be denoted by $H(L)$. If L defines a polarization of type $D = (d_1, \dots, d_g)$ on X then $H(L)$ is a finite group of order equal to $\deg \phi_L = \prod_{i=1}^g d_i^2$. This group satisfies the following properties.

Lemma 1.4.2 ([3]). *For any* $L \in \mathrm{Pic}(X)$

1. $H(L \otimes P) = H(L)$ *for any* $P \in \mathrm{Pic}^0(X)$,
2. $H(L) = X$ *if and only if* $L \in \mathrm{Pic}^0(X)$,

3. $H(L^n) = n_X^{-1} H(L)$ *for any* $n \in \mathbb{Z}$,

4. $H(L) = n_X H(L^n)$ *for any* $n \in \mathbb{Z}$, $n \neq 0$

5. $X_n \subset H(L)$ *if and only if* $L = M^n$ *for some* $M \in \mathrm{Pic}(X)$.

The Weyl pairing $e^L : H(L) \times H(L) \to \mathbb{C}^*$ on $H(L)$ is defined as

$$e^L(w_1, w_2) = e^{-2\pi i \operatorname{Im} H(w_1, w_2)}.$$

This is a multiplicative alternating form on $H(L)$ with values in $\mathbb{C}^*$, i.e.

$$\begin{aligned} &e^L(w_1 + w_2, w) = e^L(w_1, w) e^L(w_2, w), \\ &e^L(w_1, w_2) = e^L(w_2, w_1)^{-1}, \\ &e^L(w, w) = 1, \end{aligned}$$

for all $w_1, w_2, w \in H(L)$.

Define $H(D) = (\oplus_{i=1}^g \mathbb{Z}/d_i\mathbb{Z})^2$. If $f_1, \ldots, f_{2g}$ is the standard basis of $H(D)$ define an alternating form $e^D : H(D) \times H(D) \to \mathbb{C}^*$ as

$$e^D(f_i, f_j) = \begin{cases} e^{-\frac{2\pi i}{d_i}} & \text{if } j = g + i \\ e^{\frac{2\pi i}{d_i}} & \text{if } i = g + j \\ 1 & \text{otherwise.} \end{cases}$$

A level D structure on X is a symplectic isomorphism between $H(L)$ and $H(D)$ with respect to the alternating forms e^L and e^D. If (X, H) is a principally polarized abelian variety, a level $n\mathbb{1}_g$ structure on (X, nH) is called a level n structure for (X, H). By Lemma 1.4.2 a level n structure on a principally polarized abelian variety X is a symplectic isomorphism $X_n \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$.

Because of the modular interpretation of $\mathbb{H}_g$, polarized abelian varieties with level D structure are essentially triples

$$\left(X_\tau, H_\tau, \{ \lambda_1/d_1, \ldots, \lambda_g/d_g, \lambda_{g+1}/d_1, \ldots, \lambda_{2g}/d_g \}\right),$$

where $\{ \lambda_1, \ldots, \lambda_{2g} \}$ is the symplectic basis given by the columns of the matrix (τ, D).

Theorem 1.4.3 ([3]). *Two matrices* τ *and* τ' *determine isomorphic polarized abelian varieties of type* D *with level* D *structure if and only if* $\tau' = R \cdot \tau$ *for some*

$$R \in \Gamma_D(D) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_D \mid a - \mathbb{1}_g \equiv b \equiv c \equiv d - \mathbb{1}_g \equiv 0 \pmod{D} \right\},$$

where $a \equiv 0 \pmod{D}$ *means* $a \in DM_g(\mathbb{Z})$.

A generalized level n structure on an abelian variety (X, H) with a polarization of arbitrary type D is a basis of X_n coming from a symplectic basis of Λ. If $\lambda_1, \ldots, \lambda_{2g}$ is a symplectic basis of Λ for H then the corresponding basis of X_n is given by $\frac{1}{n}\lambda_1, \ldots, \frac{1}{n}\lambda_{2g}$.

Theorem 1.4.4 ([3]). *Denote by* $\Gamma_D(n) = \left\{ R \in \Gamma_D \mid M \equiv \mathbb{1}_{2g} \pmod{n} \right\}$*. The normal complex analytic space* $\mathbb{H}_g/\Gamma_D(n)$ *is a moduli space of polarized abelian varieties of type* D *with generalized level* n *structure.*

2

SIEGEL MODULAR FORMS

In this chapter we will develop in some details the general theory of Siegel modular forms. In connection with his famous investigation of the analytic theory of quadratic form, C. Siegel pioneered the generalization of the theory of elliptic modular forms to modular forms in more variables now named after him.

Siegel modular forms represent a rich subject in the theory of automorphic forms and are of great importance in number theory and algebraic geometry. Concerning the theory of Siegel modular varieties we will see that these are the right coordinates to view these as projective varieties.

In this Chapter we will review the classical definitions and properties of both vector-valued and scalar-valued modular forms. We will mostly refer to [14].

In Section 2.4 we will present and discuss many examples of Siegel modular forms. Most of them are strictly related to the classical Riemann's theta function.

2.1 THE SIEGEL MODULAR GROUP ACTING ON THE SIEGEL SPACE

For any $g \in \mathbb{N}$ let

$$\mathrm{Sp}(2g, \mathbb{R}) = \left\{ \gamma \in M_{2g\times 2g}(\mathbb{R}) \mid {}^t\gamma\, J\, \gamma = J \right\},$$

where $J = \left(\begin{smallmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{smallmatrix}\right)$. This is called the real symplectic group of degree g. We will use a standard block notation for the elements of $\mathrm{Sp}(2g, \mathbb{R})$. Any $M \in \mathrm{Sp}(2g, \mathbb{R})$ can be written in block notation as $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where A, B, C, D are $g \times g$ matrices. We will keep this block notation throughout the thesis. It is easy to see that $M \in \mathrm{Sp}(2g, \mathbb{R})$ if and only if the following relations are satisfied:

$$\begin{cases} A\,{}^tB = B\,{}^tA, \\ C\,{}^tD = D\,{}^tC, \\ A\,{}^tD - B\,{}^tC = 1_g \end{cases} ; \quad \begin{cases} {}^tA\,C = {}^tC\,A, \\ {}^tB\,D = {}^tD\,B, \\ {}^tA\,D - {}^tC\,B = 1_g \end{cases} .$$

The group $\mathrm{Sp}(2g,\mathbb{R})$ acts continuously and transitively on the Siegel space $\mathbb{H}_g$ defined in (8). This action is given by the formula

$$\gamma \cdot \tau = (A\tau + B)(C\tau + D)^{-1}, \text{ for } \gamma \in \mathrm{Sp}(2g,\mathbb{R}),\ \tau \in \mathbb{H}_g. \tag{11}$$

Via this action any element $\gamma \in \mathrm{Sp}(2g,\mathbb{R})$ defines a biholomorphic automorphism

$$\begin{aligned} T_\gamma : \mathbb{H}_g &\to \mathbb{H}_g \\ \tau &\mapsto \gamma \cdot \tau \quad . \end{aligned} \tag{12}$$

If we denote by $\mathrm{Aut}(\mathbb{H}_g)$ the group of biholomorphic automorphisms of $\mathbb{H}_g$, the map

$$\begin{aligned} \mathrm{Sp}(2g,\mathbb{R}) &\to \mathrm{Aut}(\mathbb{H}_g) \\ \gamma &\mapsto T_\gamma \quad , \end{aligned}$$

is a group homomorphism with kernel $\{\pm \mathbb{1}_g\}$. Since it is a surjective homomorphism by [51], we get that $\mathrm{Aut}(\mathbb{H}_g)$ can be completely described in terms of the action of the symplectic group on the Siegel space. Namely $\mathrm{Aut}(\mathbb{H}_g) \simeq \mathrm{Sp}(2g,\mathbb{R})/\{\pm \mathbb{1}_g\}$.

It is easy to describe $\mathbb{H}_g$ as homogeneous space. The stabilizer of $i\mathbb{1}_g$ is the subgroup

$$U(g) := \left\{ \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right) \in \mathrm{Sp}(2g,\mathbb{R}) \mid A\,{}^tA + B\,{}^tB = \mathbb{1}_g \right\}.$$

Hence by a classical result for group actions on a topological space we get

$$\mathbb{H}_g \simeq \mathrm{Sp}(2g,\mathbb{R})/U(g). \tag{13}$$

As a consequence, each discrete subgroup of $\mathrm{Sp}(2g,\mathbb{R})$ acts properly discontinuously on $\mathbb{H}_g$. The most important example of discrete subgroup of $\mathrm{Sp}(2g,\mathbb{R})$ we will consider is

$$\Gamma_g := \mathrm{Sp}(2g,\mathbb{Z}) = \mathrm{Sp}(2g,\mathbb{R}) \cap M_{2g\times 2g}(\mathbb{Z}). \tag{14}$$

This is usually called the Siegel modular group. It is generated by the matrix J and by the matrices $\gamma_S = \left(\begin{smallmatrix} \mathbb{1}_g & S \\ 0 & \mathbb{1}_g \end{smallmatrix}\right)$, where S is an integral symmetric matrix.

A fundamental domain $\mathcal{F}_g$ for the action of Γ_g on $\mathbb{H}_g$ is determined in [51]. It is the set of $\tau = x + iy \in \mathbb{H}_g$ such that

i) $|\det(C\tau + D)| \geqslant 1,\ \forall \gamma \in \Gamma_g$;

ii) for all primitive vectors $n \in \mathbb{Z}^g$, $n^t y n \geqslant y_{kk}$ for $1 \leqslant k \leqslant g$ and $y_{k,k+1} \geqslant 0$ for $1 \leqslant k \leqslant g-1$;

iii) $|x_{ij}| \leqslant 1/2$ for $1 \leqslant i,j \leqslant g$.

For $g = 1$ we have an easier description of the fundamental domain. Indeed

$$\mathcal{F}_1 = \{ \tau = x + iy \in \mathbb{H}_1 \mid |x| \leqslant 1/2, |\tau| \geqslant 1 \}.$$

For any natural number n let $\Gamma_g(n)$ denote the principal congruence subgroup of level n of Γ_g. It is defines as the kernel of the natural homomorphism $\mathrm{Sp}(2g, \mathbb{Z}) \to \mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ induced by the canonical projection $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$:

$$\Gamma_g(n) = \left\{ \gamma \in \Gamma_g \;\middle|\; \gamma \equiv \mathbb{1}_{2g} \pmod{n} \right\}.$$

These are normal finite index subgroups of Γ_g. If $n \geqslant 3$ the action of $\Gamma_g(n)$ on $\mathbb{H}_g$ is free (cf. [50]). A subgroup $\Gamma \subset \Gamma_g$ such that $\Gamma_g(n) \subset \Gamma$ as a finite index subgroup for some $n \in \mathbb{N}$ is called a congruence subgroup. Such a subgroup is said to be of level n if n is the least integer such that $\Gamma_g(n) \subset \Gamma$. For $g = 1$, the symplectic group is nothing but the special linear group $\mathrm{SL}(2, \mathbb{Z})$ and there are examples of finite index subgroups that are not congruence subgroups (see [41, 37]). If $g > 1$ any subgroup of finite index in Γ_g is a congruence subgroup of some level (cf. [38]).

Examples of level $2n$ subgroups are given by the groups

$$\Gamma_g(n, 2n) = \left\{ \gamma \in \Gamma_g(n) \;\middle|\; \mathrm{diag}({}^t\!A\, C) \equiv \mathrm{diag}({}^t\!B\, D) \equiv 0 \pmod{2n} \right\}. \tag{15}$$

For even values of n these are normal subgroups of Γ_g. It is easily seen that for $\gamma \in \Gamma_g(2m)$ the following congruences hold:

$$\begin{cases} \mathrm{diag}({}^t\!A\, C) \equiv \mathrm{diag}(C) \pmod{4m} \\ \mathrm{diag}({}^t\!B\, D) \equiv \mathrm{diag}(B) \pmod{4m} \end{cases}.$$

Then we get the simpler description

$$\Gamma_g(2m, 4m) = \{\gamma \in \Gamma_g(2m) \mid \mathrm{diag}(B) \equiv \mathrm{diag}(C) \equiv 0 \pmod{4m}\}.$$

Furthermore, $[\Gamma_g(2m) : \Gamma_g(2m, 4m)] = 2^{2g}$ for any $m \in \mathbb{N}$.

2.2 VECTOR-VALUED SIEGEL MODULAR FORMS

First of all we need to introduce the notion of multiplier system, since we will consider not only modular forms of integral weight but also half-integral weight ones. A multiplier system of weight $r \in \mathbb{R}$ for a congruence subgroup $\Gamma \subset \Gamma_g$ is a function $\nu : \Gamma \to \mathbb{C} \setminus \{0\}$ such that $j_r(\gamma, \tau) := \nu(\gamma) \det(C\tau + D)^r$ is holomorphic in τ and satisfies the following conditions:

(i) $j_r(\gamma_1\gamma_2, \tau) = j_r(\gamma_1, \gamma_2 \cdot \tau)\, j_r(\gamma_2, \tau)$ for all $\gamma_1, \gamma_2 \in \Gamma$ and $\tau \in \mathbb{H}_g$;

(ii) If $-\mathbb{1}_{2g} \in \Gamma$ then $j_r(-\mathbb{1}_{2g}, \tau) = 1$ for all $\tau \in \mathbb{H}_g$.

We will first introduce vector-valued Siegel modular forms. For this, we need to recall some basic facts about rational representations of the group $\mathrm{GL}_g(\mathbb{C})$. Proofs can be found in [35].

Let V be a finite dimensional complex vector space. A representation of $\mathrm{GL}_g(\mathbb{C})$ is a homomorphism $\rho : \mathrm{GL}_g(\mathbb{C}) \to \mathrm{GL}(V)$. It is called rational if the entries of $\rho(A)$ are polynomials in the entries of A and $\det(A)^{-1}$. In the following we will always work with rational representations if not otherwise stated. A representation is called irreducible if $V \neq \{0\}$ and the only invariant subspaces of V are 0 and V itself. It is well known that each representation of $\mathrm{GL}_g(\mathbb{C})$ is isomorphic to a finite direct sum of irreducible representations. The isomorphisms classes of the irreducible addends are uniquely determined up to the order.

A vector of V is called a highest weight vector of ρ if it is invariant under the group of strictly upper triangular matrices. It can be shown that highest weight vectors always exists if $V \neq \{0\}$. Moreover, a representation of $\mathrm{GL}_g(\mathbb{C})$ is irreducible if and only if the space of highest weight vectors is one dimensional. If v is a highest weight vector for the irreducible representation ρ, then there exist integers $\lambda_i \in \mathbb{Z}$ with $\lambda_1 \geqslant \cdots \geqslant \lambda_g$ such that

$$\rho\begin{pmatrix} a_1 & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & a_g \end{pmatrix} \cdot v = a_1^{\lambda_1} \cdots a_g^{\lambda_g}\, v.$$

The vector $(\lambda_1, \dots, \lambda_g)$ is called the highest weight of ρ. Two irreducible representations are isomorphic if and only if their highest weights agree, so an irreducible representation is uniquely identified up to isomorphism by its highest weight. It is interesting to note that each vector $(\lambda_1, \dots, \lambda_g)$ with $\lambda_1 \geqslant \cdots \geqslant \lambda_g$ occurs as a highest weight. We will write $\rho = (\lambda_1, \dots, \lambda_g)$ if $(\lambda_1, \dots, \lambda_g)$ is the highest weight of ρ. The dual representation of ρ is $\rho^\vee : \mathrm{GL}_g(\mathbb{C}) \to \mathrm{GL}(V^\vee)$ with $\rho^\vee(A) = {}^t\rho(A^{-1})$. If $\rho = (\lambda_1, \dots, \lambda_g)$ then $\rho^\vee = (-\lambda_g, \dots, -\lambda_1)$.

In the following we will work with representations that involve the "determinant representation". Denote by $\det : \mathrm{GL}_g(\mathbb{C}) \to \mathbb{C} \setminus \{0\}$ the representation such that $\rho(A) \cdot v = \det(A)\, v$. Clearly $\det = (1, \dots, 1)$ and the dual is $\det^{-1} = (-1, \dots, -1)$. For any $k \in \mathbb{Z}$ the representation $\rho \otimes \det^k = (\lambda_1 + k, \dots, \lambda_g + k)$.

The weight $w(\rho)$ of a representation is defined as the biggest integer k such that $\det^{-k} \otimes \rho$ is a polynomial representation. If $\rho = (\lambda_1, \dots, \lambda_g)$ then $w(\rho) = \lambda_g$. An irreducible representation is called reduced if its weight is 0. The co-rank of an irreducible representation $\rho = (\lambda_1, \ldots, \lambda_g)$ is defined as

$$\text{co-rank}(\rho) = \#\{\, i, 1 \leqslant i \leqslant g \mid \lambda_i = \lambda_g \,\}.$$

For $r \in \mathbb{Z}$ we will consider irreducible representations of the form $\rho = \det^{r/2} \otimes \rho_0$, where ρ_0 is a reduced irreducible representation of $\mathrm{GL}_g(\mathbb{C})$. We will call these representations half-integral weight representations. A half-integral weight representation is called singular if $2w(\rho) < g$.

If ν is a multiplier system of weight $r/2$ for a subgroup Γ and ρ is a half-integral weight representation, a holomorphic function $f : \mathbb{H}_g \to V$ is a vector-valued Siegel modular form with respect to Γ, ρ and ν if

$$f(\gamma \cdot \tau) = \nu(\gamma)\rho(C\tau + D)\, f(\tau),\ \forall\, \gamma \in \Gamma,\ \forall\, \tau \in \mathbb{H}_g,$$

where the action of Γ_g on $\mathbb{H}_g$ is defined in (11). If $g = 1$ we need to require also that f is holomorphic at ∞. This condition is always satisfied for $g > 2$ by the Köcher principle [14, Hilfssatz 4.11]. Denote by $[\Gamma, \rho, \nu]$ the complex vector space of such modular forms. If ν is trivial it will be omitted in the notation. Each $[\Gamma, \rho, \nu]$ is a finite dimensional vector space. Clearly if $\rho = \rho_1 \oplus \rho_2$, then $[\Gamma, \rho, \nu] = [\Gamma, \rho_1, \nu] \oplus [\Gamma, \rho_2, \nu]$.

Lemma 2.2.1 ([13])**.** *Let $\rho = (\lambda_1, \ldots, \lambda_g)$ be a non-trivial irreducible representation of $\mathrm{GL}_g(\mathbb{C})$. If $[\Gamma, \rho] \neq \{\, 0 \,\}$ for a congruence subgroup $\Gamma \subset \Gamma_g$ then $\lambda_g \geqslant 1$.*

Under certain conditions, the space $[\Gamma, \rho]$ is known to be trivial.

Vanishing theorem ([52])**.** *Let $\rho = (\lambda_1, \ldots, \lambda_g)$ be an irreducible representation with $r :=$ co-rank$(\rho) < g - \lambda_g$. Then*

$$\#\{\, i \mid 1 \leqslant i \leqslant g,\ \lambda_i = \lambda_g + 1 \,\} < 2(g - \lambda_g - r) \implies [\Gamma, \rho] = (0).$$

2.3 Scalar-valued Siegel modular forms

In this section we will focus on scalar-valued Siegel modular forms. We will denote by $[\Gamma, r/2, \nu]$ the vector space of modular forms with respect to the group Γ, the representation $\rho = \det^{r/2}$ and the multiplier system ν. We will refer to its elements as scalar-valued Siegel modular forms of weight $r/2$ with multiplier. Lemma 2.2.1 implies that scalar-valued modular forms of negative weight vanish, hence the ring of scalar-valued modular forms with respect to a given subgroup Γ is positively graded. It is well known that this is a normal integral domain of finite type over $\mathbb{C}$.

A symmetric $g \times g$ matrix $S \in GL(g, \mathbb{Q})$ is called half-integral if $2S$ is integral and $\mathrm{diag}(2S)$ is even. Every half-integral matrix gives a linear form with integral coefficients in the coordinates of $\tau = (\tau_{ij}) \in \mathbb{H}_g$ by

$$\mathrm{Tr}(S\tau) = \sum_{i=1}^{g} S_{ii}\tau_{ii} + 2 \sum_{1 \leqslant i < j \leqslant g} S_{ij}\tau_{ij}.$$

Let Γ be a level n congruence subgroup of Γ_g. Hence $\gamma_{nT} = \begin{pmatrix} \mathbb{1}_g & nT \\ 0 & \mathbb{1}_g \end{pmatrix} \in \Gamma$ for any integral symmetric $g \times g$ matrix T. By definition, any scalar-valued modular form f with respect to Γ satisfies $f(\gamma_{nT} \cdot \tau) = f(\tau + nT) = f(\tau)$. Then f is periodic with period n in each variable τ_{ij} and therefore admits a Fourier expansion

$$f(\tau) = \sum_{S} a(S) e^{\frac{2\pi i}{n} \mathrm{Tr}(S\tau)}, \tag{16}$$

where S runs over the set of all symmetric half-integral matrices and $a(S) \in \mathbb{C}$.

In particular the series (16) converges absolutely on $\mathbb{H}_g$ and uniformly on each compact set in $\mathbb{H}_g$. Moreover it can be shown that $a(S) = 0$ for each half-integral symmetric matrix S which is not positive semi-definite (cf. [14]).

A scalar-valued modular form is called singular if the matrices that appear in its Fourier expansion are singular matrices, that is $a(S) \neq 0$ implies that $\det S = 0$. The rank of a scalar-valued Siegel modular form f is defined as follows:

$$\mathrm{rank}(f) = \max\{\, \mathrm{rank}(T) \mid a(T) \neq 0 \,\}.$$

Clearly $0 \leqslant \mathrm{rank}(f) \leqslant g$

Proposition 2.3.1 ([15, 16]). *A scalar-valued Siegel modular form $f \in [\Gamma, r/2]$ is singular if and only if $r < g$. If f is a non-vanishing singular modular form, then $r \in \mathbb{N}$. Moreover,* $\mathrm{rank}(f) = r$.

We can characterize these properties by means of suitable differential operators. Let $\partial_{\tau_{ij}} := \frac{\partial}{\partial \tau_{ij}}$ and define the $g \times g$ matrix

$$\partial := (\partial_{ij}), \qquad \partial_{ij} = \frac{1 + \delta_{ij}}{2} \, \partial_{\tau_{ij}}.$$

For $1 \leqslant k \leqslant g$ define the differential operator acting on a singular scalar-valued modular form f as

$$\partial^{[k]} f = (\det(\partial^I_J)\, f)_{I,J \in P^*_k(X_g)},$$

where $P^*_k(X_g)$ is the collection of increasingly ordered subset of $\{1, \ldots, g\}$ of cardinality k and ∂^I_J is the submatrix of ∂ obtained by taking rows with indexes in I and columns with indexes in J. If

$$f(\tau) = \sum_{S} a(S) e^{\pi i \,\mathrm{Tr}(S\tau)}$$

with $a(S) \in \mathbb{C}$, then

$$\partial^{[k]} f = \sum a(S)\, S^{[k]} e^{\pi i \operatorname{Tr}(S\tau)},$$

where $S^{[k]}$ is the matrix of $k \times k$ minors of S (cf. [14]). Then it follows by definition that f is singular if and only if $\partial^{[g]} f = 0$. Moreover, since $\operatorname{rank}(S) < k$ if and only if $S^{[k]} = 0$, for $1 \leqslant n \leqslant g$ one has

$$\operatorname{rank}(f) = n \Leftrightarrow \partial^{[g]} f = \partial^{[g-1]} f = \cdots = \partial^{[n+1]} f = 0 \text{ and } \partial^{[n]} f \neq 0. \tag{17}$$

2.4 THETA FUNCTIONS AND THETA SERIES

In Section 1.3.1 we have introduced Riemann's theta functions with characteristic as theta functions for suitable line bundles on principally polarized abelian varieties. In this section we present scalar-valued and vector-valued modular forms arising from these functions.

Recall that for a vector $m = \left[\begin{smallmatrix} m' \\ m'' \end{smallmatrix}\right]$, m', $m'' \in \mathbb{Z}^g$, the theta function with characteristic (or theta-characteristic) m is defined by the series

$$\vartheta_m(\tau, z) = \sum_{n \in \mathbb{Z}^g} e^{\pi i \left({}^t(n+m'/2)\tau(n+m'/2) + 2\,{}^t(n+m'/2)(z+m''/2)\right)}.$$

This series converges absolutely and uniformly in every compact subset of $\mathbb{H}_g \times \mathbb{C}^g$. Then it defines a holomorphic function of the two variables τ and z.

Since

$$\vartheta_m(\tau, -z) = (-1)^{{}^t m' m''} \vartheta_m(\tau, z),$$

the theta function with characteristic m is an even or odd function of z if ${}^t m' m''$ is even or odd respectively. Correspondingly, the characteristic m is called even or odd. It follows by definition that

$$\vartheta_{m+2n}(\tau, z) = (-1)^{{}^t m' n''} \vartheta_m(\tau, z),$$

for any $n \in \mathbb{Z}^g$. Therefore up to sign there are 2^{2g} theta functions and we can normalize a characteristic by the condition that its coefficients are either zero or one. We have defined theta functions with characteristic in this way for completeness, but in the following we will consider only normalized characteristics and characteristics. The number of even characteristics is $2^{g-1}(2^g + 1)$ and the number of odd ones is $2^{g-1}(2^g - 1)$.

We will now define an action of the Siegel modular group (14) on theta functions. We have already seen that this group acts on the the Siegel upper-half space by (11).

We are going to see that there is an action of Γ_g also on the set of theta characteristics. For any $\gamma \in \Gamma_g$ and $m \in \{0,1\}^{2g}$ set

$$\gamma \cdot \left[\begin{smallmatrix} m' \\ m'' \end{smallmatrix}\right] = \left[\left(\begin{smallmatrix} D & -C \\ -B & A \end{smallmatrix}\right)\left(\begin{smallmatrix} m' \\ m'' \end{smallmatrix}\right) + \left(\begin{smallmatrix} \operatorname{diag}(C\,{}^tD) \\ \operatorname{diag}(A\,{}^tB) \end{smallmatrix}\right) \right] \pmod 2, \tag{18}$$

where we think of the elements of $\mathbb{Z}/2\mathbb{Z}$ as zeroes and ones. The action defined in this way is neither linear nor transitive. Indeed, the action preserves the parity of the characteristics. Clearly the action of the principal congruence subgroup $\Gamma_g(2)$ on the set of theta characteristics is trivial.

Theta functions with characteristics satisfy the following transformation law for any $\gamma \in \Gamma_g$ (see [27]):

$$\vartheta_{\gamma\cdot m}(\gamma\cdot\tau, {}^t(C\tau+D)^{-1}z) = \kappa(\gamma)e^{\pi i[2\phi_m(\gamma)+{}^tz(C\tau+D)^{-1}Cz]}\det(C\tau+D)^{1/2}\vartheta_m(\tau,z), \tag{19}$$

where $\kappa(\gamma)$ is an 8^{th} root of the unity, with the same sign ambiguity as in $\det(C\tau+D)^{1/2}$, and

$$\begin{aligned}\phi_m(\gamma) = -\frac{1}{8}({}^tm'\,{}^tBDm' + {}^tm''\,{}^tACm'' - 2\,{}^tm'\,{}^tBCm'')+ \\ + \frac{1}{4}{}^t\operatorname{diag}(A\,{}^tB)(Dm' - Cm'').\end{aligned}$$

Evaluating a theta function with characteristic m at $z = 0$ we get a holomorphic function on $\mathbb{H}_g$ which is not identically zero if and only if the characteristic m is even (cf. [24]). These functions are usually called theta constants and are denoted by

$$\vartheta_m(\tau) = \vartheta\left[\begin{smallmatrix} m' \\ m'' \end{smallmatrix}\right](\tau, 0).$$

Acting with elements of $\Gamma_g(2)$ and evaluating the formula (19) in $z = 0$, we get a simple transformation formula for theta constants:

$$\vartheta_m(\gamma\cdot\tau) = \kappa(\gamma)e^{2\pi i\phi_m(\gamma)}\det(C\tau+D)^{1/2}\vartheta_m(\tau), \quad \forall\,\gamma\in\Gamma_g(2). \tag{20}$$

It is easy to see that whenever $\gamma \in \Gamma_g(4,8)$ we have $e^{2\pi i\phi_m(\gamma)} = 1$, thus the theta constants with even characteristics are scalar-valued modular forms of weight $1/2$ with a multiplier with respect to the group $\Gamma_g(4,8)$.

By Section 2.3 we already know that $\operatorname{rank}(\vartheta_m) = 1$ for even m, but this is also a straightforward consequence of the following system of equations, usually called "heat equation":

$$\frac{\partial^2}{\partial z_j \partial z_k}\vartheta_m(\tau,z) = 2\pi i(1+\delta_{jk})\frac{\partial}{\partial\tau_{jk}}\vartheta_m(\tau,z), \tag{21}$$

for $j,k = 1,\dots,g$.

We will need a different formulation of the transformation rule for a product of theta constants $\vartheta_m \vartheta_n$ in order to examine some modularity properties of suitable products of theta constants.

We will first recall the construction of a set of generators for the subgroup $\Gamma_g(2)$. If $1 \leqslant i \neq j \leqslant g$ are positive integers, let a_{ij} be the matrix obtained by replacing the (i,j)-coefficient of $\mathbb{1}_g$ by 2. If $1 \leqslant i \leqslant g$ let a_{ii} be the matrix obtained by replacing the (i,i)-coefficient of $\mathbb{1}_g$ by -1. Then

$$A_{ij} = \begin{pmatrix} a_{ij} & 0 \\ 0 & {}^t a_{ij}^{-1} \end{pmatrix} \in \Gamma_g(2).$$

If $1 \leqslant i < j \leqslant g$ are positive integers, let b_{ij} be the matrix obtained by replacing the (i,j)-coefficient and the (j,i)-coefficient of the zero matrix by 2. If $1 \leqslant i \leqslant g$ let b_{ii} be the matrix obtained by replacing the (i,i)-coefficient of the zero matrix by 2. Then

$$B_{ij} = \begin{pmatrix} \mathbb{1}_g & b_{ij} \\ 0 & \mathbb{1}_g \end{pmatrix} \in \Gamma_g(2).$$

For $1 \leqslant i \leqslant j \leqslant g$ let $C_{ij} = {}^t B_{ij}$. Then $C_{ij} \in \Gamma_g(2)$.

By [24, Theorem 1] the $g(2g+1)$ matrices A_{ij} for $1 \leqslant i, j \leqslant g$, B_{ij} and C_{ij} for $1 \leqslant i \leqslant j \leqslant g$ are a set of generators for $\Gamma_g(2)$. This implies that any $\gamma \in \Gamma_g(2)$ can be written as

$$\gamma = \left(\prod_{1 \leqslant i, j \leqslant g} A_{ij}^{p_{ij}} \right) \cdot \left(\prod_{1 \leqslant i \leqslant j \leqslant g} B_{ij}^{q_{ij}} \right) \cdot \left(\prod_{1 \leqslant i \leqslant j \leqslant g} C_{ij}^{r_{ij}} \right) \cdot \gamma', \tag{22}$$

where p_{ij}, q_{ij}, $r_{ij} \in \mathbb{Z}$ and γ' is in the commutator subgroup of $\Gamma_g(2)$. Note that by [24, Lemma 1], $\gamma' \in \Gamma_g(4,8)$. Denote by p the $g \times g$ matrix with entries p_{ij} and by q and r the symmetric matrices with entries q_{ij} and r_{ij} respectively.

With this notations, by [24] one has that

$$\gamma \in \Gamma_g(2,4) \Leftrightarrow \operatorname{diag}(q) \equiv \operatorname{diag}(r) \equiv 0 \pmod 2, \tag{23}$$

$$\gamma \in \Gamma_g(4,8) \Leftrightarrow p,\ q,\ r \equiv 0 \pmod 2; \ \ \operatorname{diag}(q) \equiv \operatorname{diag}(r) \equiv 0 \pmod 4. \tag{24}$$

We are ready to write down the transformation formula we need. For any $\gamma \in \Gamma_g(2)$ written in the form (22), we have the following transformation formula (cf. [24, Theorem 3]):

$$(\vartheta_m \vartheta_n)(\gamma \cdot \tau) = \kappa(\gamma)^2 \, (-1)^A \, \mathsf{exp}(-1/4)^B \det(C\tau + D)(\vartheta_m \vartheta_n)(\tau), \tag{25}$$

where $\mathsf{exp}(t) = e^{2\pi i t}$,

$$\kappa(\gamma)^2 = (-1)^{\sum_i p_{ii}}, \tag{26}$$

and

$$A = \sum_{1 \leqslant i,j \leqslant g} p_{ij}(m_i' m_j'' + n_i' n_j'') + \sum_{1 \leqslant i<j \leqslant g} q_{ij}(m_i' m_j' + n_i' n_j') + \\ + \sum_{1 \leqslant i<j \leqslant g} r_{ij}(m_i'' m_j'' + n_i'' n_j'') \; ,$$

$$B = \sum_{1 \leqslant i \leqslant g} q_{ii}((m_i')^2 + (n_i')^2) + \sum_{1 \leqslant i \leqslant g} r_{ii}((m_i'')^2 + (n_i'')^2) \; .$$

Then by (24) and (25) it is easy to see that $\vartheta_m \vartheta_n$ is a scalar-valued modular form of weight 1 and trivial multiplier system with respect to $\Gamma_g(4,8)$.

Regarding the congruence subgroup $\Gamma_g(2,4)$, the modularity condition can be expressed in terms of some equations satisfied by the entries of the characteristics m and n. By (23) and (25) it is easy to see that $\vartheta_m \vartheta_n$ is a scalar-valued modular form of weight 1 and trivial multiplier system with respect to $\Gamma_g(2,4)$ if for $1 \leqslant i, j \leqslant g$

$$m_i' m_j'' + n_i' n_j'' \equiv \begin{cases} 1 \pmod 2 & \text{if } i = j \\ 0 \pmod 2 & \text{if } i \neq j \end{cases}, \tag{27}$$

$$m_i' m_j' + n_i' n_j' \equiv m_i'' m_j'' + n_i'' n_j'' \equiv 0 \pmod 2. \tag{28}$$

Denote by $\Gamma_g(2,4)^*$ the index two subgroup of $\Gamma_g(2,4)$ where $\kappa(\gamma)^2 = 1$. Clearly if $\vartheta_m \vartheta_n \in [\Gamma_g(2,4), 1]$ then it also a scalar-valued modular form of weight 1 with respect to $\Gamma_g(2,4)^*$. Moreover by equation (26), $\vartheta_m \vartheta_n$ is a scalar-valued modular form of weight 1 and trivial multiplier system with respect to $\Gamma_g(2,4)^*$ if for $1 \leqslant i, j \leqslant g$

$$m_i' m_j'' + n_i' n_j'' \equiv 0 \pmod 2, \tag{29}$$

$$m_i' m_j' + n_i' n_j' \equiv m_i'' m_j'' + n_i'' n_j'' \equiv 0 \pmod 2. \tag{30}$$

If we define the matrix $M = (m, n)$, then we can reformulate the modularity conditions given by equations (27), (28), (29) and (30) in the following way

$$\vartheta_m \vartheta_n \in [\Gamma_g(2,4), 1] \;\text{ if }\; M\,{}^tM \equiv \left(\begin{smallmatrix} 0 & \mathbb{1}_g \\ \mathbb{1}_g & 0 \end{smallmatrix}\right) \pmod 2,$$

$$\vartheta_m \vartheta_n \in [\Gamma_g(2,4)^*, 1] \;\text{ if }\; M\,{}^tM \equiv 0 \pmod 2 \;\text{ or }\; M\,{}^tM \equiv \left(\begin{smallmatrix} 0 & \mathbb{1}_g \\ \mathbb{1}_g & 0 \end{smallmatrix}\right) \pmod 2.$$

Now we can easily understand what happens if we take a product of an even number of theta constants. Let $M = (m_1, \ldots, m_{2k})$ be a matrix of even characteristics and let

$$\vartheta_M = \vartheta_{m_1} \cdots \vartheta_{m_{2k}}. \tag{31}$$

It is easily seen that

$$\vartheta_M \in [\Gamma_g(2,4), k] \;\text{ if }\; M^tM \equiv k \left(\begin{smallmatrix} 0 & \mathbb{1}_g \\ \mathbb{1}_g & 0 \end{smallmatrix}\right) \pmod 2, \tag{32}$$

$$\vartheta_M \in [\Gamma_g(2,4)^*, k] \;\text{ if }\; M^tM \equiv 0 \pmod 2 \;\text{ or }\; M\,{}^tM \equiv \left(\begin{smallmatrix} 0 & \mathbb{1}_g \\ \mathbb{1}_g & 0 \end{smallmatrix}\right) \pmod 2. \tag{33}$$

We will be also interested in modular forms constructed with second order theta functions. We have already introduced them as theta functions for a line bundle which is twice a principal polarization on abelian varieties parametrized by points in $\mathbb{H}_g$ (see Section 1.3.1). Recall that for $\varepsilon \in \{0,1\}^g$ the second order theta functions are defined as

$$\Theta[\varepsilon](\tau, z) = \vartheta\left[\begin{smallmatrix}\varepsilon\\ 0\end{smallmatrix}\right](2\tau, 2z); \quad \tau \in \mathbb{H}_g,\ z \in \mathbb{C}^g. \tag{34}$$

These are all even functions of z. As for theta constants with characteristic, denote by $\Theta[\varepsilon] = \Theta[\varepsilon](\tau, 0)$ the second order theta constant with characteristic ε. These are related to theta constants with characteristic by Riemann's addition formula (cf. [27]):

$$\Theta[\sigma](\tau)\,\Theta[\sigma+\varepsilon](\tau) = \frac{1}{2^g}\sum_{\delta\in\{0,1\}^g}(-1)^{\sigma\cdot\delta}\vartheta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau)^2, \tag{35}$$

$$\vartheta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau)^2 = \sum_{\sigma\in\{0,1\}^g}(-1)^{\sigma\cdot\delta}\Theta[\sigma](\tau)\,\Theta[\sigma+\varepsilon](\tau). \tag{36}$$

For every $\gamma \in \Gamma_g$ let $\tilde{\gamma} \in \Gamma_g$ be such that $2(\gamma\cdot\tau) = \tilde{\gamma}\cdot(2\tau)$, that is $\tilde{\gamma} = \left(\begin{smallmatrix}A & 2B\\ C/2 & D\end{smallmatrix}\right)$. By the above transformation formula for theta constants we get

$$\Theta[\varepsilon](\gamma\cdot\tau) = \kappa(\tilde{\gamma})\det(C\tau+D)^{1/2}\Theta[\varepsilon](\tau),\ \forall\,\gamma \in \Gamma_g(2,4). \tag{37}$$

Second order theta constants are then modular forms of weight $1/2$ with respect to the congruence subgroup $\Gamma_g(2,4)$ and the multiplier system $v_\Theta(\gamma) := k(\tilde{\gamma})$. By equations (35) and (36) it is easy to see that $\kappa(\tilde{\gamma})^2 = \kappa(\gamma)^2$.

We will now give some examples of scalar-valued and vector-valued Siegel modular forms constructed by taking derivatives of theta functions with odd characteristics. If n is an odd characteristic, denote by

$$v_n(\tau) := \operatorname{grad}_z \theta_n(\tau, z)\big|_{z=0},$$

the gradient of the odd theta function with characteristic n. Differentiating the transformation formula for theta function (19) with respect to the variable z_i and evaluating it at $z = 0$, we get the following transformation rule for the gradient of an odd theta function:

$$v_{\gamma\cdot n}(\gamma\cdot\tau) = \kappa(\gamma)e^{2\pi i\phi_n(\gamma)}\det(C\tau+D)^{1/2}\,(C\tau+D)\,v_n(\tau),\ \forall\,\gamma\in\Gamma_g. \tag{38}$$

Hence for an odd characteristic n we have that $v_n(\tau)$ is a vector-valued Siegel modular form with a multiplier with respect to the congruence subgroup $\Gamma_g(4,8)$ and the half-integral weight representation ρ such that $\rho(A) = \det(A)^{1/2}A$.

We will now present vector-valued modular forms associated to a set of $1 \leqslant k < g$ odd characteristics. For $N = (n_1, \dots, n_k) \in M_{2g\times k}$ where $\{n_i\}_{i=1,\dots,k}$ is a set of distinct odd characteristics, define

$$W(N)(\tau) = \pi^{-2k}\,(\nu_{n_1}(\tau) \wedge \dots \wedge \nu_{n_k}(\tau))\,{}^t(\nu_{n_1}(\tau) \wedge \dots \wedge \nu_{n_k}(\tau)). \tag{39}$$

By [45] for every $\gamma \in \Gamma_g$ one has the following transformation formula:

$$W(\gamma \cdot N)(\gamma \cdot \tau) = \kappa(\gamma)^{2k}\, e^{4\pi i \sum_i \phi_{n_i}(\gamma)}\, \rho_k(C\tau + D)\, W(N)(\tau),$$

where $\gamma \cdot N = (\gamma \cdot n_1, \dots, \gamma \cdot n_k)$ and $\rho_k = (k+2, \dots, k+2, k, \dots, k)$ with co-rank$(\rho_k) = g - k$. Clearly $W(N)$ is a vector-valued modular form with respect to the subgroup $\Gamma_g(4, 8)$ and the representation ρ_k.

We are also interested in the modularity of $W(N)$ with respect to the subgroups $\Gamma_g(2, 4)$ and $\Gamma_g(2, 4)^*$. In order to study this modularity we can make the same reasoning we made before in the case of products of theta constants (see (32) and (33)). Indeed if we take a matrix $\widetilde{M} = (M, M)$ where M is a matrix of k even characteristics, then with the notations as in (31)

$$\vartheta_{\widetilde{M}}(\gamma \cdot \tau) = \kappa(\gamma)^{2k}\, e^{4\pi i \sum_{i=1}^{k} \phi_{m_i}(\gamma)} \det(C\tau + D)^k \vartheta_{\widetilde{M}}(\tau),$$

for any $\gamma \in \Gamma_g(2)$.

If $\widetilde{N} = (N, N)$ with N a matrix of k odd characteristics, then

$$\widetilde{N}\,{}^t\widetilde{N} = 2\sum_{i=1}^{k} n_i\,{}^t n_i \equiv 0 \pmod 2.$$

So we can conclude that $W(N) \in [\Gamma_g(2, 4)^*, \rho_k]$ for any k. Moreover, if k is even then $W(N) \in [\Gamma_g(2, 4), \rho_k]$.

If $k = g$ then (39) defines a scalar-valued modular form which can be expressed in term of the long studied "Jacobian determinant" of odd theta functions. For $N = (n_1, \dots, n_k) \in M_{2g\times g}$ a matrix of g odd characteristics define

$$D(N)(\tau) = \pi^{-g} \det(\partial(\vartheta_{n_1} \dots \vartheta_{n_g})/\partial(z_1 \dots z_g))(\tau, 0) = \pi^{-g}\, \nu_{n_1}(\tau) \wedge \dots \wedge \nu_{n_g}(\tau). \tag{40}$$

By [43] this is a scalar-valued Siegel modular form of weight $g/2 + 1$ with respect to the subgroup $\Gamma_g(4, 8)$ that never vanishes identically provided that $n_i \neq n_j$ for $i \neq j$. Clearly if N is a matrix of g distinct odd characteristics then

$$W(N) = D(N)^2.$$

We end this section with a brief introduction to the theory of theta series. We will present two kinds of theta series: theta series with respect to positive definite quadratic forms and theta series with harmonic polynomial coefficients.

Let S denote a positive definite integral matrix of degree $k \equiv 0 \pmod 8$ which is unimodular (i.e. $\det(S) = 1$) and even (i.e. $\mathrm{diag}(S)$ is even). The theta series with respect to S is defined as

$$\vartheta_S(\tau) = \sum_{G \in M_{g\times k}(\mathbb{Z})} e^{\pi i \operatorname{Tr}(GS^tG\tau)}.$$

By [47] $\vartheta_S(\tau) \in [\Gamma_g, k/2]$.

In some cases, this theta series can be written in terms of theta constants. The theta series that have such an expression are studied in [46, Section 3]. One example is the theta series $\Theta^{(g)}_{E_8}$ with respect to the quadratic form of the lattice E_8. For a suitable choice of the basis the matrix ζ_{E_8} of this quadratic form is given as follows:

$$\zeta_{E_8} = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \tag{41}$$

By [28] we have that

$$\Theta^{(g)}_{E_8} = \frac{1}{2^g} \sum_{m\, even} \vartheta^8_m(\tau). \tag{42}$$

Another kind of theta series we would like to present are theta series with harmonic polynomial coefficients. We will see in Section 4.1.3 that they are related to some generalization of the remarkable Jacobi's derivative formula.

Let A denote an even positive definite integral matrix of degree p. Assume that p is even. Denote by l the smallest positive integer such that $B = lA^{-1}$ is even. Let $B(x) = \frac{1}{2}\operatorname{Tr}({}^t xBx)$ for $x \in M_{p\times g}(\mathbb{R})$. Choose a polynomial $h(x)$ satisfying

$$B(\partial/\partial x)h(x) = 0, \quad h(\nu x) = \det(\nu)^q h(x),$$

for every invertible $g \times g$ matrix ν and for some non-negative integer q. Given $\xi_0 \in BM_{p\times g}(\mathbb{Z})$, the theta series with harmonic polynomial coefficient h and basic quadratic form A is defined as

$$\vartheta_{\xi_0}(\tau; A, h) = \sum_{\xi \equiv \xi_0 \pmod l} h(\xi) e^{\pi i l^2 \operatorname{Tr}(\tau^t \xi A \xi)}, \ \tau \in \mathbb{H}_g. \tag{43}$$

These theta series span a finite dimensional vector space $\Theta(A, h)$ which is stable under the action of Γ_g on holomorphic functions on $\mathbb{H}_g$ defined as

$$(\gamma^{-1} \cdot f)(\tau) = \det(C\tau + D)^{-p/2-q} f(\gamma \cdot \tau).$$

It is interesting to note that both Jacobian determinants (40) and products of $g+2$ even theta constants can be expressed in terms of theta series with suitable harmonic coefficients. Indeed by [29, Section 11] the $\mathbb{C}$-span of Jacobian determinants is the space of theta series with "det" as harmonic polynomial coefficient and with $4(x_1^2+\cdots+x_g^2)$ as basic even quadratic form while the $\mathbb{C}$-span of products of $g+2$ even theta constants is the space of theta series with "1" as harmonic coefficient and with $4(y_1^2+\cdots+y_{g+2}^2)$ as basic even quadratic form.

3

GEOMETRY OF SIEGEL MODULAR VARIETIES

For any discrete subgroup Γ of $\mathrm{Sp}(2g,\mathbb{R})$, the modular variety $X_\Gamma = \mathbb{H}_g/\Gamma$ with its natural quotient structure is a normal complex analytic space of dimension $N = g(g+1)/2$. It is smooth if Γ acts freely and it has at most finite quotient singularities in any case. If all the isotropy groups of the action of Γ are generated by reflections, then the quotient is still non-singular even though there are fixed points.

Our interest in the theory of Siegel modular varieties is mostly due to its connection with the theory of moduli spaces of complex abelian varieties (see Section 1.4.1 and Section 1.4.2). They also represent an interesting setting where Siegel modular forms can be used to investigate geometric problems. In Section 3.1 and Section 3.2 we present some geometric properties of modular varieties that can be investigated by means of scalar-valued and vector-valued Siegel modular forms respectively.

Section 3.1 is about the construction of the Satake compactification $\overline{X}_\Gamma$ of a modular variety X_Γ. We will call such a compactification the Siegel modular variety associated to Γ. This a projective and normal variety that contains X_Γ as a Zariski open subset. The space of scalar-valued Siegel modular forms with respect to Γ gives the projective embedding of $\overline{X}_\Gamma$. We will explicitly present the Satake compactification of the moduli space of abelian surfaces with a level 2 structure as a quartic hypersurface in $\mathbb{P}^4$. In Section 3.1.2 we will analyze the relationship between a point x of this quartic and the Kummer variety of the abelian surface whose moduli point is x.

The topic of Section 3.2 is the construction of holomorphic differential forms on modular varieties by means of vector-valued Siegel modular forms. In Section 3.2.1 we will give methods to define holomorphic differential forms of degree $N-1$ starting from scalar-valued Siegel modular forms and from gradients of odd theta constants that produce holomorphic differential forms invariant under the action of the full modular group. In Proposition 3.2.2 we will give an explicit construction of non-zero Γ_g-invariant holomorphic differential forms exploiting theta series for suitable values of g. This result for $g = 9$ is part of my joint work [8].

In the following chapter (see Section 4.1) we will generalize these methods to the construction of vector-valued Siegel modular forms and prove that in some remarkable cases they give rise to elements of the same vector space (see Theorem 4.1.10).

In the last two sections of this chapter we will focus on modular varieties of low degree. In Section 3.3 we will focus on modular varieties of degree 2. We will present the results of my paper [39] about the construction of Siegel modular threefolds with a degree 8 endomorphism, generalizing the result proven in [32] for the Satake compactification of the moduli space of abelian surfaces with a level 2 structure. In Section 3.4 we will briefly study the situation for degree 3.

3.1 THE SATAKE COMPACTIFICATION OF A SIEGEL MODULAR VARIETY

In this section we begin to investigate the role of modular forms in the study of the geometry of Siegel modular varieties. We will focus on the construction of the Satake or Baily-Borel compactification of these varieties. This kind of compactification arise more generally in the theory of locally symmetric spaces.

By (13) $\mathbb{H}_g$ is a homogeneous space. Moreover it is a symmetric space, that is each point of $\mathbb{H}_g$ admits a symmetry. For this consider the automorphism T_J defined in (12) for $J = \left(\begin{smallmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{smallmatrix}\right)$. In particular T_J is an involution of $\mathbb{H}_g$ and $T_J(i\mathbb{1}_g) = i\mathbb{1}_g$. Hence T_J is a symmetry for the point $i\mathbb{1}_g$. Since the action of $\mathrm{Sp}(2g, \mathbb{R})$ on $\mathbb{H}_g$ is transitive, for each $\tau \in \mathbb{H}_g$ there exists $\eta \in \mathrm{Sp}(2g, \mathbb{R})$ such that $\tau = \eta \cdot (i\mathbb{1}_g)$. Then the automorphism $T_{\eta J \eta^{-1}}$ is a symmetry for τ.

If Γ is a finite index subgroup of Γ_g then X_Γ is a locally symmetric space. An embedding theorem proved by Borel and Harish-Chandra (cf. [1]) states that every symmetric domain can be realized as a bounded domain in a complex affine space of the same dimension if and only if it does not admit a direct factor, which is isomorphic to $\mathbb{C}^n$ modulo a discrete group of translations. Since $\mathrm{Sp}(2g, \mathbb{R})$ is a simple Lie group, the picture of $\mathbb{H}_g$ as a homogeneous space implies that $\mathbb{H}_g$ does not admit such a factor. The Borel and Harish-Chandra theorem applies and the embedding is given as follows. Let

$$D_g := \left\{ M \in M_{g\times g}(\mathbb{C}) \;\middle|\; M = {}^tM,\, M\overline{M} - 1 < 0 \right\}.$$

Then

$$\begin{aligned} \varphi : \mathbb{H}_g &\to D_g \\ \tau &\mapsto (\tau - i\mathbb{1}_g)(\tau + i\mathbb{1}_g)^{-1}. \end{aligned}$$

is an embedding and displays the Siegel space as a bounded domain.

Let

$$\overline{D}_g = \{ M \in M_{g\times g}(\mathbb{C}) \mid M = {}^tM,\, M\overline{M} - 1 \leqslant 0 \},$$

be the closure of D_g. The action of $Sp(2g, \mathbb{R})$ on $\mathbb{H}_g$ then defines, via φ, an action on D_g which extends to $\overline{D}_g$. We say that two points in $\overline{D}_g$ are equivalent if and only if they are connected by finitely many holomorphic arcs. More precisely, for $z, w \in D_g$ we write $z \sim w$ if and only if there exists finitely many holomorphic maps $f_1, \ldots, f_k : D_1 \to \overline{D}_g$ such that $f_1(0) = z$, $f_k(0) = w$ and $f_i(D_1) \cap f_{i+1}(D_1) \neq \emptyset$ for $i = 1, \ldots, k$. Under this equivalence relation all points in D_g are equivalent. The equivalence classes with respect to the relation $\sim$ are usually called the boundary components of $\overline{D}_g$. The equivalence classes of $\overline{D}_g \setminus D_g$ are called the proper boundary components of $\overline{D}_g$.

There is a bijection between the proper boundary components of $\overline{D}_g$ and the non-trivial isotropic subspaces of $\mathbb{R}^{2g}$ with respect to the standard symplectic form J. For this, for any $z \in \overline{D}_g$ define

$$\begin{aligned} \psi(z) : \mathbb{R}^{2g} &\to \mathbb{C}^g \\ x &\mapsto x \begin{pmatrix} i(\mathbb{1}_g + z) \\ \mathbb{1} - z \end{pmatrix}. \end{aligned}$$

The real subspace of $\mathbb{R}^{2g}$

$$U(z) := \ker \psi(z)$$

is an isotropic subspace of $\mathbb{R}^{2g}$ with respect to the standard symplectic form J. It has the property that $U(z) \neq \{0\}$ if and only if $z \notin D_g$ and $U(z_1) = U(z_2)$ if and only if $z_1 \sim z_2$.

If we consider the action of $Sp(2g, \mathbb{R})$ on these isotropic subspaces we get

$$U(\gamma \cdot z) = U(z)\gamma^{-1},\ \forall z \in \overline{D}_g,\ \forall \gamma \in Sp(2g, \mathbb{R}).$$

So we have also an action of $Sp(2g, \mathbb{R})$ on the set of boundary components. A boundary component is called rational if its stabilizer subgroup in $Sp(2g, \mathbb{R})$ is defined over $\mathbb{Q}$. Let $\overline{D}_g^{rc}$ be the set of rational boundary components of $\overline{D}_g$ and consider the rational closure $D_g^{rat} = D_g \cup \overline{D}_g^{rc}$. The space D_g^{rat} can be equipped with the cylindrical topology (cf. [36]). With this topology D_g^{rat} is a Hausdorff space and D_g is an open dense subset in D_g^{rat}. The Satake compactification of X_Γ is then defined as

$$\overline{X}_\Gamma = D_g^{rat}/\Gamma.$$

For example, if $\Gamma = \Gamma_g$ we have that set theoretically

$$\overline{X}_{\Gamma_g} = \mathbb{H}_g/\Gamma_g \cup \mathbb{H}_{g-1}/\Gamma_{g-1} \cup \cdots \cup \mathbb{H}_1/\Gamma_1 \cup \{pt\},$$

and the boundary components are given by

$$D_{g,r} = \left\{ \begin{pmatrix} N & 0 \\ 0 & \mathbb{1}_{g-r} \end{pmatrix} \mid N \in D_r \right\} \subset \overline{D}_g,$$

for $0 \leqslant r \leqslant g-1$.

For any finite index subgroup Γ, $\overline{X}_\Gamma$ is a projective and normal variety that contains X_Γ as a Zariski open subset. If $\Gamma' \subset \Gamma$ are two finite index subgroup of Γ_g with $[\Gamma : \Gamma'] < \infty$, there is a canonical finite holomorphic map $\overline{X}_{\Gamma'} \to \overline{X}_\Gamma$ extending the natural finite map $X_{\Gamma'} \to X_\Gamma$. The compactification $\overline{X}_\Gamma$ is highly singular along the boundary which is also of codimension g.

A partial desingularization of $\overline{X}_\Gamma$ can be found by blowing up along the boundary, this is usually called Igusa's compactification. This procedure gives a resolution of $\overline{X}_{\Gamma_g(n)}$ for $n \geqslant 3$ and $g \leqslant 3$ but not for $g \geqslant 4$ (cf. [26]).

The ideas of Igusa and the work of Hirzebruch on Hilbert modular surfaces lead to the general theory of toroidal compactifications developed by Mumford in [34]. One of the good properties of toroidal compactifications that complement the bad properties of Satake compactifications is that the boundary is a divisor. Nevertheless to construct a toroidal compactification one has to choose some additional data, so this kind of compactification is not unique. There are however criterion for smoothness or projectivity of the compactification in terms of some properties of the chosen data. In particular one can construct in this way a smooth compactification of a modular variety. Igusa's compactification is a toroidal compactification in Mumford sense (cf. [36]).

Despite the high codimension of the boundary and its singularities, one of the main features of the Satake compactification is that one can describe it by means of Siegel modular forms. We will work precisely with this description in order to study geometric properties of Siegel modular varieties.

The idea is to define an explicit embedding of X_Γ into projective space by means of modular forms and then compactify the image. If the subgroup Γ acts freely on $\mathbb{H}_g$, then the factor of automorphy $\det(C\tau + D)^k$ defines a line bundle L^k on $\mathbb{H}_g/\Gamma$ whose global sections are weight k modular forms (for a detailed discussion on line bundles and factors of automorphy see Section 1.2.1.). Since any element with fixed points is torsion and the order of all torsion elements is bounded, even if Γ does not act freely, the modular forms of weight nk_0 for some integer k_0 and $n \geqslant 1$ are sections of a line bundle L^{nk_0} on $\mathbb{H}_g/\Gamma$. As global sections of a line bundle, the elements of $[\Gamma, nk_0]$ define a rational map $\varphi : \mathbb{H}_g/\Gamma \to \mathbb{P}^N$ for some N (see Section 1.3). If n is sufficiently large, then φ is an immersion and the Satake compactification of $\mathbb{H}_g/\Gamma$ is the projective closure of the image of φ.

In other words, the Satake compactification of X_Γ is

$$\overline{X}_\Gamma = \mathrm{Proj}(A(\Gamma)),$$

where $A(\Gamma)$ is the graded ring of scalar-valued Siegel modular forms with respect to Γ. If one is working with scalar-valued Siegel modular forms with respect to a multiplier system ν of weight $1/2$, one can consider the ring of Siegel modular forms with respect to the group Γ and the multiplier system ν which is the graded ring

$$A(\Gamma, \nu) = \bigoplus_{k \in \mathbb{N}} [\Gamma, k/2, \nu^k],$$

and define in the same way

$$\overline{X}_\Gamma = \mathrm{Proj}(A(\Gamma, \nu)).$$

The compactification $\overline{X}_\Gamma$ indeed does not depend on the multiplier system chosen. Moreover, if we let

$$A(\Gamma, \nu)^{(d)} = \bigoplus_{k \equiv 0 \pmod d} [\Gamma, k/2, \nu^k],$$

then $\mathrm{Proj}(A(\Gamma, \nu)) \cong \mathrm{Proj}(A(\Gamma, \nu)^{(d)})$.

3.1.1 *The Igusa quartic*

We will give an explicit description of the Satake compactification of the modular variety $X_{\Gamma_2(2)}$ as a quartic hypersurface in $\mathbb{P}^4$. This is usually called the Igusa quartic and gives a compactification of the moduli space of abelian surfaces with a level 2 structure (see Section 1.4.2).

We will present this compactification in two ways. For the first one, consider the ring of modular forms of even weight with respect to $\Gamma_2(2)$

$$A(\Gamma_2(2))^{ev} = \sum_{k \in \mathbb{N}} [\Gamma_2(2), 2k].$$

The structure of this ring is given in [25] where the author proves that $A(\Gamma_2(2))^{ev}$ is generated over $\mathbb{C}$ by the five scalar-valued modular forms

$$y_0 = \vartheta\left[\begin{smallmatrix} 01 \\ 10 \end{smallmatrix}\right](\tau)^4, \quad y_1 = \vartheta\left[\begin{smallmatrix} 01 \\ 00 \end{smallmatrix}\right](\tau)^4, \quad y_2 = \vartheta\left[\begin{smallmatrix} 00 \\ 00 \end{smallmatrix}\right](\tau)^4,$$

$$y_3 = \vartheta\left[\begin{smallmatrix} 10 \\ 00 \end{smallmatrix}\right](\tau)^4 - \vartheta\left[\begin{smallmatrix} 01 \\ 10 \end{smallmatrix}\right](\tau)^4, \quad y_4 = -\left(\vartheta\left[\begin{smallmatrix} 11 \\ 00 \end{smallmatrix}\right](\tau)^4 + \vartheta\left[\begin{smallmatrix} 01 \\ 10 \end{smallmatrix}\right](\tau)^4\right).$$

These generators satisfy the relation

$$(y_0y_1 + y_0y_2 + y_1y_2 - y_3y_4)^2 - 4\, y_0y_1y_2(y_0 + y_1 + y_2 + y_3 + y_4) = 0. \tag{44}$$

Since

$$\overline{X}_{\Gamma_2(2)} = \mathrm{Proj}(A(\Gamma_2(2))) = \mathrm{Proj}(A(\Gamma_2(2))^{ev}),$$

the Satake compactification of $X_{\Gamma_2(2)}$ is the quartic hypersurface in $\mathbb{P}^4$ defined by the equation (44).

For the second one, we want to give explicitly the embedding of the Satake compactification by a map defined by scalar-valued Siegel modular forms. By (20) and (26) it is easy to see that $\vartheta_m(\tau)^4$ is a modular form of weight 2 with respect to the group $\Gamma_2(2)$ for any even characteristic m. Hence we can define a map

$$\begin{aligned} \overline{X}_{\Gamma_2(2)} \to \quad & \mathbb{P}^9 \\ \tau \mapsto & [\ldots, \vartheta_m(\tau)^4, \ldots]. \end{aligned}$$

There are linear relations between these modular forms and the vector space of weight 2 modular forms is 5 dimensional. A set of independent relations are the following ones (cf. [20]):

$$\begin{aligned}
&\vartheta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right](\tau)^4 - \vartheta\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right](\tau)^4 - \vartheta\left[\begin{smallmatrix}11\\11\end{smallmatrix}\right](\tau)^4 - \vartheta\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right](\tau)^4 = 0,\\
&\vartheta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](\tau)^4 - \vartheta\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right](\tau)^4 - \vartheta\left[\begin{smallmatrix}01\\10\end{smallmatrix}\right](\tau)^4 - \vartheta\left[\begin{smallmatrix}11\\00\end{smallmatrix}\right](\tau)^4 = 0,\\
&\vartheta\left[\begin{smallmatrix}01\\10\end{smallmatrix}\right](\tau)^4 - \vartheta\left[\begin{smallmatrix}00\\10\end{smallmatrix}\right](\tau)^4 - \vartheta\left[\begin{smallmatrix}11\\11\end{smallmatrix}\right](\tau)^4 + \vartheta\left[\begin{smallmatrix}00\\11\end{smallmatrix}\right](\tau)^4 = 0,\\
&\vartheta\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right](\tau)^4 - \vartheta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](\tau)^4 + \vartheta\left[\begin{smallmatrix}10\\01\end{smallmatrix}\right](\tau)^4 + \vartheta\left[\begin{smallmatrix}00\\11\end{smallmatrix}\right](\tau)^4 = 0,\\
&\vartheta\left[\begin{smallmatrix}01\\00\end{smallmatrix}\right](\tau)^4 - \vartheta\left[\begin{smallmatrix}10\\00\end{smallmatrix}\right](\tau)^4 + \vartheta\left[\begin{smallmatrix}00\\01\end{smallmatrix}\right](\tau)^4 - \vartheta\left[\begin{smallmatrix}00\\10\end{smallmatrix}\right](\tau)^4 = 0.
\end{aligned}$$

Hence the modular forms of weight 2 gives a map

$$\begin{aligned} \overline{X}_{\Gamma_2(2)} \to & \mathbb{P}^4 \subset \mathbb{P}^9 \\ \tau \mapsto & [\ldots, \vartheta_m(\tau)^4, \ldots]. \end{aligned}$$

In [20] the author shows that this map is an embedding and the image is the quartic hypersurface given by

$$\mathcal{I} = \left\{ \left(\sum_{m\ even} \vartheta_m(\tau)^8 \right)^2 - 4 \sum_{m\ even} \vartheta_m(\tau)^{16} = 0 \right\}. \tag{45}$$

The singular locus of (45) is the boundary of the Baily-Borel embedding of $X_{\Gamma_2(2)}$. There are 15 1-dimensional boundary components and 15 0-dimensional boundary components.

3.1.2 *Igusa quartic and universal Kummer variety*

In [20] the author also explains a different modular interpretation of the Igusa quartic.

We need to recall some basic facts about the Kummer variety of an abelian variety. Let (X, Θ) be a principally polarized abelian variety of dimension g. Let $\iota : X \to X$ be the involution defined by $\iota(x) = -x$. The fixed point locus of ι is clearly the group of 2-torsion points on X denoted by X_2. We will call the quotient $K_X = X/\langle\iota\rangle$ the "abstract" Kummer variety of X. If $g \geqslant 2$ then K_X is a singular variety of dimension g with 2^{2g} singular points of multiplicity 2^{g-1} corresponding to the image of X_2 via the projection map $\pi : X \to K_X$. If X is an elliptic curve then $K_X \cong \mathbb{P}^1$ and the projection $\pi : X \to \mathbb{P}^1$ is a $2:1$ cover branching over four points, which are the image under π of the 2-torsion points of the elliptic curve. Then we can regard the abstract Kummer variety of an elliptic curve as a $\mathbb{P}^1$ with four marked points.

If (X_1, L_1) and (X_2, L_2) are two abelian varieties, denote by $p_i : X_1 \times X_2 \to X_i$ the projection on the i-th factor for $i = 1, 2$. If $L_1 \boxtimes L_2 := p_1^* L_1 \otimes p_2^* L_2$ then $(X_1 \times X_2, L_1 \boxtimes L_2)$ is an abelian variety, called the product abelian variety with the product polarization. A principally polarized abelian variety is called decomposable if it is a product abelian variety. If (X, Θ) is a decomposable abelian variety with $X = X_1 \times \cdots \times X_s$ and Θ the product polarization, then the map defined by the linear system $|2\Theta|$ gives an embedding of $K_{A_1} \times \cdots \times K_{A_s}$ in projective space:

$$\begin{tikzcd} A \arrow[r, "|2\Theta|"] \arrow[d] & \mathbb{P}^{2^g-1} \\ K_{A_1} \times \cdots \times K_{A_s} \arrow[ur, hook] & \end{tikzcd}$$

We will refer to $K_{X_1} \times \cdots \times K_{X_s} \hookrightarrow \mathbb{P}^{2^g-1}$ as the "embedded" Kummer variety of X. Note that if X is indecomposable the abstract and embedded Kummer variety of X coincide, we will call it just the Kummer variety of X.

The map $|2\Theta|$ can be easily given in terms of theta functions. For $\tau \in \mathbb{H}_g$ denote as usual by (X_τ, L_τ) the principally polarized abelian variety where $X_\tau = \mathbb{C}^g / \tau\mathbb{Z}^g \oplus \mathbb{Z}^g$ and L_τ is the principal polarization whose only section, up to scalar, is given by the theta function with characteristic $m = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ (see Section 1.3.1). Then a basis of sections of the line bundle L_τ^2 is given by the second order theta functions (34). So the map $|2\Theta|$ is given by

$$\begin{aligned} |2\Theta| : X_\tau &\to \mathbb{P}^{2^g-1} \\ z &\mapsto [\ldots, \Theta[\sigma](\tau, z), \ldots], \end{aligned} \tag{46}$$

where the coordinates on $\mathbb{P}^{2^g-1}$ are indexed by $\sigma \in \{0, 1\}^g$.

If X is an indecomposable abelian surface then the Kummer surface of X is a quartic hypersurface in $\mathbb{P}^3$ with 16 nodes. If $X = E_1 \times E_2$ is the product of two elliptic curves, then the image of $K_{E_1} \times K_{E_2}$ in $\mathbb{P}^3$ is a non-singular quadric isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The following theorem gives a relation between suitable points of the Igusa quartic and their Kummer varieties.

Theorem 3.1.1 ([20]). *Let x be a non-singular point of $\mathcal{I}$ not lying on the intersection of $\mathcal{I}$ with the 10 coordinate hyperplanes $\{\vartheta_m(\tau)^4 = 0\}_{m\,\mathrm{even}}$. The intersection of $\mathcal{I}$ with the tangent space of $\mathcal{I}$ at x is a quartic surface with 16 nodes. If x is a non-singular point lying on $\mathcal{I} \cap \{\vartheta_m(\tau)^4 = 0\}$ then the intersection of $\mathcal{I}$ with the tangent space of $\mathcal{I}$ at x is a quadric surface with multiplicity 2.*

The quadrics in the theorem corresponds to the embedded Kummer varieties of abelian surfaces which are products of two elliptic curves. Indeed the locus of reducible abelian surfaces in the moduli space of principally polarized abelian surfaces is given by the θ_{null} divisor. It has 10 irreducible components each corresponding to the vanishing of a single theta constant with even characteristic. In $X_{\Gamma_2(2)}$ these components are given by the 10 coordinates hyperplanes $\{\vartheta_m(\tau)^4 = 0\}$ with m even. These quadrics must be counted twice to preserve the degree.

We can see this situation in a different light by introducing the universal Kummer variety. Define the map

$$\begin{aligned} \phi : \mathbb{H}^g \times \mathbb{C}^g &\to \mathbb{P}^{2^g-1} \times \mathbb{P}^{2^g-1} \\ (\tau, z) &\mapsto [\Theta_\tau(0), \Theta_\tau(z)]. \end{aligned} \tag{47}$$

The image of this map is a quasi-projective variety of dimension $g + g(g+1)/2$. The closure of the image is called the universal Kummer variety. This variety is indeed a modular family of embedded Kummer varieties. If we fix a point $\tau \in \mathbb{H}_g$, the (reduced) image of ϕ is the embedded Kummer variety of the abelian variety A_τ defined by the point $\tau \in \mathbb{H}_g$.

There is a lot of beautiful geometry in the study of the universal Kummer variety. One of the main features is that it can be studied by means of computer algebra systems like *Macaulay2* [21] since there are many known equations for the universal Kummer variety and some remarkable sub-loci.

For example, for $g = 2$ the closure of the image of ϕ is given by a single equation $F(u, x)$ (cf. [19]). Let $x_\sigma = \Theta[\sigma](\tau, z)$ and $u_\sigma = \Theta[\sigma](\tau, 0)$. Then set $\mathbf{u} = (u_{00}, u_{01}, u_{10}, u_{11})$ and $\mathbf{x} = (x_{00}, x_{01}, x_{10}, x_{11})$ and take $(\mathbf{u}, \mathbf{x})$ as the coordinates in $\mathbb{P}^3 \times \mathbb{P}^3$. Set

$$\begin{aligned}
P_0 &= x_{00}^4 + x_{01}^4 + x_{10}^4 + x_{11}^4;\\
P_1 &= 2(x_{00}^2 x_{01}^2 + x_{10}^2 x_{11}^2);\\
P_2 &= 2(x_{00}^2 x_{10}^2 + x_{01}^2 x_{11}^2);\\
P_3 &= 2(x_{00}^2 x_{11}^2 + x_{01}^2 x_{10}^2);\\
P_4 &= 4 x_{00} x_{01} x_{10} x_{11}.
\end{aligned} \tag{48}$$

The equation of the universal Kummer variety in degree 2 is then given by:

$$F(\mathbf{u}, \mathbf{x}) = \det \begin{pmatrix}
P_0 & P_1 & P_2 & P_3 & P_4 \\
u_{00}^3 & u_{00} u_{01}^2 & u_{00} u_{10}^2 & u_{00} u_{11}^2 & u_{01} u_{10} u_{11} \\
u_{01}^3 & u_{00}^2 u_{01} & u_{01} u_{11}^2 & u_{01} u_{10}^2 & u_{00} u_{10} u_{11} \\
u_{10}^3 & u_{10} u_{11}^2 & u_{00}^2 u_{10} & u_{01}^2 u_{10}^2 & u_{00} u_{01} u_{11} \\
u_{11}^3 & u_{10}^2 u_{11} & u_{01}^2 u_{11} & u_{00}^2 u_{11} & u_{00} u_{01} u_{10}
\end{pmatrix} \tag{49}$$

By Riemann's addition formula (36), the irreducible components of θ_{null} are given by the zero locus of the 10 quadrics:

$$\begin{aligned}
Q_1 &= u_{00}^2 + u_{01}^2 + u_{10}^2 + u_{11}^2,\\
Q_2 &= u_{00}^2 - u_{01}^2 + u_{10}^2 - u_{11}^2,\\
Q_3 &= u_{00}^2 + u_{01}^2 - u_{10}^2 - u_{11}^2,\\
Q_4 &= u_{00}^2 - u_{01}^2 - u_{10}^2 + u_{11}^2,\\
Q_5 &= u_{00} u_{01} + u_{10} u_{11},\\
Q_6 &= u_{00} u_{01} - u_{10} u_{11},\\
Q_7 &= u_{00} u_{10} + u_{01} u_{11},\\
Q_8 &= u_{00} u_{10} - u_{01} u_{11},\\
Q_9 &= u_{01} u_{10} + u_{00} u_{11},\\
Q_{10} &= u_{00} u_{11} - u_{01} u_{10}.
\end{aligned}$$

Let $f_i = F(u,x)_{|\{Q_i=0\}}$, $i = 1,\dots,10$. By direct computations in *Macaulay2* [21] we have:

$$\begin{aligned}
f_1 &= p_1(\mathbf{u})(x_{00}^2 + x_{01}^2 + x_{10}^2 + x_{11}^2)^2,\\
f_2 &= p_2(\mathbf{u})(x_{00}^2 - x_{01}^2 + x_{10}^2 - x_{11}^2)^2,\\
f_3 &= p_3(\mathbf{u})(x_{00}^2 + x_{01}^2 - x_{10}^2 - x_{11}^2)^2,\\
f_4 &= p_4(\mathbf{u})(x_{00}^2 - x_{01}^2 - x_{10}^2 + x_{11}^2)^2,\\
f_5 &= p_5(\mathbf{u})(x_{00}x_{01} + x_{10}x_{11})^2,\\
f_6 &= p_6(\mathbf{u})(x_{00}x_{01} - x_{10}x_{11})^2,\\
f_7 &= p_7(\mathbf{u})(x_{00}x_{10} + x_{01}x_{11})^2,\\
f_8 &= p_8(\mathbf{u})(x_{00}x_{10} - x_{01}x_{11})^2,\\
f_9 &= p_9(\mathbf{u})(x_{01}x_{10} + x_{00}x_{11})^2,\\
f_{10} &= p_{10}(\mathbf{u})(x_{00}x_{11} - x_{01}x_{10})^2,
\end{aligned}$$

where p_i are suitable polynomials in the second order theta constants. So on any irreducible component of the reducible locus of principally polarized abelian surfaces, which has a quadric equation in the moduli space, the universal Kummer variety is essentially the same quadric counted twice.

We want to end this section with a short discussion on theta structures on abelian varieties. We will explain how the datum of a theta-structure allows to identify a canonical map from an abelian variety to a projective space. We will show that indeed the map (46) is one of these. Moreover, this will give a deeper understanding of the map (47) for the definition of the universal Kummer variety.

Recall that if X is a complex torus, any line bundle $L \in \mathrm{Pic}(X)$ defines a homomorphism of complex tori

$$\begin{aligned}
\phi_L : X &\to \mathrm{Pic}^0(X)\\
x &\mapsto t_x^*L \otimes L^{-1}.
\end{aligned}$$

This homomorphism is an isogeny if and only if $c_1(L)$ is non degenerate. The kernel of the homomorphism ϕ_L will be denoted by $H(L)$. If L defines a polarization of type $D = (d_1,\dots,d_g)$ on X then $H(L)$ is a finite group of order equal to $\deg\phi_L = \prod_{i=1}^g d_i^2$.

Another remarkable group attached to a polarization L on X is the theta-group $\mathcal{G}(L)$. It is defined as follows

$$\mathcal{G}(L) = \left\{ (x,\varphi) \mid x \in X,\ \varphi : L \xrightarrow{\simeq} t_x^*L \right\}.$$

In other words we are considering points in $H(L)$ and we also remember the datum of the isomorphism between the line bundles. The group law on $\mathcal{G}(L)$ is given as follows:

$$(y,\psi)\cdot(x,\varphi) = (x+y, t_x^*\psi \circ \varphi)$$

for any $(x, \varphi), (y, \psi) \in \mathcal{G}(L)$. Define $\alpha : \mathcal{G}(L) \to H(L)$ as $\alpha(x, \varphi) = x$. Then α is surjective by definition of $H(L)$ and $\ker(\alpha)$ is the group of isomorphisms of L with itself. Hence $\mathcal{G}(L)$ fits in the exact sequence

$$0 \to \mathbb{C}^* \to \mathcal{G}(L) \xrightarrow{\alpha} H(L) \to 0, \tag{50}$$

where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

These groups can be completely described in terms of the type of the polarization. If the polarization L is of type $D = (d_1, \dots, d_g)$, define $K(D) = \oplus_{i=1}^{g} \mathbb{Z}/d_i\mathbb{Z}$, the dual $\widehat{K(D)} = \mathrm{Hom}(K(D), \mathbb{C}^*)$ and $\mathcal{H}(D) = K(D) \oplus \widehat{K(D)}$. Let $\mathcal{G}(D) = \mathbb{C}^* \times K(D) \times \widehat{K(D)}$ as a set. Define a group law on $\mathcal{G}(D)$ by

$$(t, x, l) \cdot (t', x', l') = (t\, t'\, l'(x), x + x', l + l').$$

Then it is easy to prove that the sequence (50) is isomorphic to the sequence

$$0 \to \mathbb{C}^* \to \mathcal{G}(D) \to \mathcal{H}(D) \to 0.$$

Proposition 3.1.2 ([33])**.** *The group $\mathcal{G}(D)$ has a unique irreducible representation in which $\mathbb{C}^*$ acts by multiplication by scalars. Denote this representation by $V(D)$. If V is any representation of $\mathcal{G}(D)$ in which $\mathbb{C}^*$ acts by multiplication by scalars, then V is isomorphic to the direct sum of $V(D)$ with itself r-times for some r.*

There is a simple way to describe this unique irreducible representation. Let $V(D)$ be the vector space of functions on $K(D)$ with values in $\mathbb{C}$. The representation of $\mathcal{G}(D)$ on $V(D)$ is usually called the Schrödinger representation and is given as follows. Any $w = (t, x, l) \in \mathcal{G}(D)$ acts on $V(D)$ by $U_w : V(D) \to V(D)$ where

$$(U_w(f))(y) = t\, l(y)\, f(x + y).$$

The most important feature of the group $\mathcal{G}(L)$ is that it acts on $H^0(L)$, the space of sections of the line bundle L. For any $z = (x, \varphi) \in \mathcal{G}(L)$ define

$$\begin{aligned} U_z : H^0(L) &\to H^0(L) \\ s &\mapsto t_{-x}^*(\varphi(s)) \end{aligned}$$

Also, the center $\mathbb{C}^*$ of $\mathcal{G}(L)$ acts on $H^0(L)$ by multiplication by scalars.

Theorem 3.1.3 ([33])**.** *The space $H^0(L)$ is an irreducible $\mathcal{G}(L)$-module for any polarization L.*

A theta-structure is an isomorphism $\alpha : \mathcal{G}(L) \to \mathcal{G}(D)$ which is the identity on $\mathbb{C}^*$. If L is a very ample line bundle, a theta-structure determines in a canonical way one projective embedding of X (not just an equivalence class of projectively equivalent embeddings) in the following way (cf. [33]). Since $H^0(L)$ is the unique irreducible representation of $\mathcal{G}(L)$ and $V(D)$ is the unique irreducible representation of $\mathcal{G}(D)$ in which $\mathbb{C}^*$ acts by multiplication by scalars, there is an isomorphism

$$\overline{\alpha} : H^0(L) \to V(D),$$

which is unique up to scalar multiples and such that $\overline{\alpha}(U_z(s)) = U_{\alpha(z)}(\overline{\alpha}(s))$ for all $z \in \mathcal{G}(L)$ and $s \in H^0(L)$. Then $\overline{\alpha}$ induces a unique isomorphism

$$\mathbb{P}(\overline{\alpha}) : \mathbb{P}\, H^0(L) \to \mathbb{P}\, V(D).$$

Fixing a basis of $V(D)$ we define an isomorphism

$$\gamma : \mathbb{P}\, V(D) \to \mathbb{P}^{m-1},$$

where m is the order of the group $K(D)$. Finally, since L is very ample there is a canonical embedding

$$F : X \hookrightarrow \mathbb{P}\, H^0(L).$$

Then the composition

$$\mathcal{F}_\alpha = \gamma \circ \mathbb{P}(\overline{\alpha}) \circ F$$

is the canonical embedding of X in $\mathbb{P}^{m-1}$ we where looking for. In this way we can also define a canonical point of $\mathbb{P}^{m-1}$, namely $\mathcal{F}_\alpha(0)$ where 0 is the identity point on X. If the line bundle L is just globally generated, by similar arguments we get a unique canonical morphism $X \to \mathbb{P}^{m-1}$.

To construct the map that gives the embedded Kummer variety of X we need to consider a polarization of type $2\mathbb{1}_g = (2, \dots, 2)$. It is given by a line bundle which is twice a principal polarization. If (X, L^2) is a polarized abelian variety with a polarization of type $2\mathbb{1}_g$ it is easy to see that $H(L^2) = X_2$. Moreover $\dim H^0(2\Theta) = 2^g$. Regarding the theta group of L^2 this is isomorphic to the group

$$\mathcal{G} := \mathcal{G}(2\mathbb{1}_g) = \mathbb{C}^* \times (\mathbb{Z}/2\mathbb{Z})^g \times (\mathbb{Z}/2\mathbb{Z})^g.$$

Indeed the image of $1 \in \mathbb{Z}/2\mathbb{Z}$ by a homomorphism $f : \mathbb{Z}/2\mathbb{Z} \to \mathbb{C}^*$ must be 2-torsion, so $f(1) = (-1)^n$ for a unique $n \in \mathbb{Z}/2\mathbb{Z}$. So we have an isomorphism

$$\begin{aligned} \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*) &\to \mathbb{Z}/2\mathbb{Z} \\ f &\mapsto n. \end{aligned}$$

The group $\mathcal{G}$ is generated by $\mathbb{C}^*$ and the two subgroups

$$\begin{aligned} K &= \{\, (1, 0, y),\ y \in (\mathbb{Z}/2\mathbb{Z})^g \,\}, \\ \tilde{K} &= \{\, (1, x, 0),\ x \in (\mathbb{Z}/2\mathbb{Z})^g \,\}. \end{aligned}$$

The vector space $V := V(2\mathbb{1}_g) = \mathrm{Functions}((\mathbb{Z}/2\mathbb{Z})^g, \mathbb{C})$ has a natural basis of "delta functions". For any $\sigma \in (\mathbb{Z}/2\mathbb{Z})^g$ let

$$\delta_\sigma(u) = \begin{cases} 0 & u \neq \sigma \\ 1 & u = \sigma \end{cases}.$$

Then given a theta structure α on $(X, 2\Theta)$ we get the morphism

$$\begin{aligned} \mathcal{F}_\alpha : X &\to \mathbb{P}^{2^g - 1} \\ x &\mapsto [\dots, \overline{\alpha}^{-1}(\delta_\sigma)(x), \dots]. \end{aligned}$$

Since $(t, 0, 0) \in \mathcal{G}$ acts by scalar multiplication on V it acts trivially on $\mathbb{P}V$ and thus the Shrödinger representation induces a representation of $\mathcal{G}/\mathbb{C}^* \simeq (\mathbb{Z}/2\mathbb{Z})^{2g}$ on $\mathbb{P}V$. Let $w = (t, x, l) \in \mathcal{G}$ map to $\tilde{w} \in \mathcal{G}/\mathbb{C}^*$ and denote by $P(\tilde{w}) : \mathbb{P}V \to \mathbb{P}V$ the projective linear map induced by the action of w on V.

For any $x \in X_2$, two elements (x, φ) and (x, φ') in $\mathcal{G}(L^2)$ are related by $\varphi' = t\varphi$ for some $t \in \mathbb{C}^*$. Thus a theta-structure induces an isomorphism

$$\tilde{\alpha} : X_2 \simeq \mathcal{G}(L^2)/\mathbb{C}^* \to \mathcal{G}/\mathbb{C}^* \simeq (\mathbb{Z}/2\mathbb{Z})^{2g}.$$

We then get that the translation by two torsion points on X is given by projective transformations of $\mathbb{P}V$. Indeed one has the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\mathcal{F}_\alpha"] \arrow[d, "t_{\tilde{\alpha}^{-1}(\tilde{w})}"'] & \mathbb{P}V \arrow[d, "P(\tilde{w})"] \\ X \arrow[r, "\mathcal{F}_\alpha"] & \mathbb{P}V \end{tikzcd}$$

IF (X_τ, L_τ) is the principally polarized abelian variety corresponding to a point of $\mathbb{H}_g$, the polarization L_τ^2 on X_τ is a polarization of type $2\mathbb{1}_g$. There is a natural theta-structure $\alpha_\tau : \mathcal{G}(L_\tau^2) \to \mathcal{G}$ is defined as $\alpha_\tau(((a\tau + b)/2, \varphi)) = (t_\varphi, [a], [b])$, where $a, b \in \mathbb{Z}^g$ and $[a]$ $[b]$ are the corresponding classes in $(\mathbb{Z}/2\mathbb{Z})^g$. Then the isomorphism $\overline{\alpha}_\tau$ is given as

$$\begin{aligned} \overline{\alpha}_\tau : H^0(L_\tau^2) &\to V \\ \Theta[\sigma] &\mapsto \delta_\sigma. \end{aligned}$$

It follows that the induced morphism in projective space is given as

$$\begin{aligned} \mathcal{F}_{\alpha_\tau} : X_\tau &\to \mathbb{P}V \\ z &\mapsto [\dots, \Theta[\sigma](\tau, z), \dots]. \end{aligned}$$

So in the definition of the map (47) we see that we need the theta constants coordinates to determine the class of the abelian variety in the moduli space $\mathcal{A}_g(2,4) = X_{\Gamma_g(2,4)}$.

Once we have this, we also have the natural theta structure that let us define the map $\mathcal{F}_{\alpha_\tau}$ via second order theta functions.

The map $\mathcal{F}_{\alpha_\tau}$ is also equivariant for the action of $\mathcal{G}$ on theta functions. This action is explicitly given by

$$(t, x, y) \cdot \Theta[\sigma](\tau, z) = t(-1)^{(\sigma+x)\cdot y}\Theta[\sigma + x](\tau, z).$$

It then gives an action of the Heisenberg group $\mathcal{H} = \mathcal{G}/\mathbb{C}^*$ on $\mathbb{P}V$.

The polynomials appearing in (48) are invariant for this action. Indeed the equation (49) of the universal Kummer variety for $g = 2$ is also invariant.

3.2 HOLOMORHIC DIFFERENTIAL FORMS ON SIEGEL MODULAR VARIETIES

We will look at the role of vector-valued Siegel modular forms in the definition of holomorphic differential forms on modular varieties.

For any complex manifold X denote by $\Omega^n(X)$ the sections of the sheaf of holomorphic differential forms on X of degree n. For a congruence subgroup Γ denote by X^0_Γ the set of regular points of the Satake compactification $\overline{X}_\Gamma$ and by $\widetilde{X_\Gamma}$ a desingularization of $\overline{X}_\Gamma$. If N is the dimension of $\mathbb{H}_g$, by [17] every holomorphic differential form $\omega \in \Omega^n(X^0_\Gamma)$ of degree $n < N$ extends holomorphically to $\widetilde{X_\Gamma}$. Moreover, if $g \geqslant 2$ and $n < N$ there is a natural isomorphism

$$\Omega^n(X^0_\Gamma) \cong \Omega^n(\mathbb{H}_g)^\Gamma,$$

where $\Omega^n(\mathbb{H}_g)^\Gamma$ is the space of Γ-invariant holomorphic differential forms on $\mathbb{H}_g$ of degree n.

We will see that any non-zero Γ-invariant holomorphic differential form on $\mathbb{H}_g$ can be described in terms of a vector-valued Siegel modular form. Let us start with a description the Γ-invariant holomorphic 1-forms. Any $\omega \in \Omega^1(\mathbb{H}_g)$ can be written as

$$\omega = \mathrm{Tr}(f(\tau)\, d\tau),$$

where $f(\tau)$ is a symmetric matrix of holomorphic functions on $\mathbb{H}_g$. By [14],

$$\gamma^*(d\tau) = {}^t(C\tau + D)^{-1}\, d\tau\, (C\tau + D)^{-1},\ \forall \gamma \in \Gamma_g.$$

Since the trace is invariant under cyclic permutations we get

$$\begin{aligned}\gamma^*\omega &= \mathrm{Tr}(f(\gamma \cdot \tau)\gamma^*(d\tau)) = \\ &= \mathrm{Tr}((C\tau + D)^{-1} f(\gamma \cdot \tau)^t (C\tau + D)^{-1}\, d\tau).\end{aligned}$$

Hence if follows that

$$\gamma^* \omega = \omega \Leftrightarrow f(\gamma \cdot \tau) = (C\tau + D) f(\tau)^t (C\tau + D) \Leftrightarrow f \in [\Gamma, \mathrm{Sym}^2(\mathbb{C}^g)] ,$$

where $\mathrm{Sym}^2(\mathbb{C}^g)$ is the symmetric power of the standard representation of $GL_g(\mathbb{C})$. Then $\Omega^1(\mathbb{H}_g)^\Gamma \simeq [\Gamma, \mathrm{Sym}^2(\mathbb{C}^g)]$ and consequently

$$\Omega^n(\mathbb{H}_g)^\Gamma \simeq [\Gamma, \wedge^n \mathrm{Sym}^2(\mathbb{C}^g)] . \tag{51}$$

For suitable degrees, depending only on g, some of these spaces are known to be trivial a priori. The following theorem gives the list of the non-trivial representations appearing in (51) and the list of degrees n for which $\Omega^n(\mathbb{H}_g)^\Gamma$ is trivial.

Theorem 3.2.1 ([52]). *For* $1 \leqslant \alpha \leqslant g$ *let* $\rho_\alpha = (g+1, \dots, g+1, g-\alpha, \dots, g-\alpha)$ *with* co-rank$(\rho_\alpha) = \alpha$. *For* $\alpha = -1$ *let* $\rho_{-1} = (g+1, \dots, g+1)$. *Then*

$$\Omega^n(\mathbb{H}_g)^\Gamma = \begin{cases} [\Gamma, \rho_\alpha] & \text{if } n = g(g+1)/2 - \alpha(\alpha+1)/2 \\ 0 & \text{otherwise} \end{cases} \tag{52}$$

Proof. According to [31] the representation

$$\wedge^* \mathrm{Sym}^2 \mathbb{C}^g = \bigoplus_{\rho \in R} \hat{\rho},$$

where R is the set of representations of the form $w\delta - \delta$ with $\delta = (g, g-1, \dots, 2, 1)$ and w in the set W_0 of Kostant representatives. Hence

$$[\Gamma, \wedge^* \mathrm{Sym}^2 \mathbb{C}^g] = [\Gamma, \oplus_{\rho \in R} \hat{\rho}].$$

If $\hat{\rho} = (\lambda_1, \dots, \lambda_g)$ with $\lambda_g = g - \alpha$, then $w\delta = (\alpha, *, \dots, *)$. If α is the largest integer that occurs among the entries of the highest weight of $w\delta$, then either $\alpha = -1$ or $1 \leqslant \alpha \leqslant g$. If $\alpha \neq -1$ then

$$w\delta = (\alpha, *, \dots, *, -(\alpha+1), -(\alpha+2), \dots, -(g-1), -g),$$

and $\lambda_{g-\alpha} = g+1$. Let $r = $ co-rank$(\hat{\rho})$ and $c = \#\{i, 1 \leqslant i \leqslant g \mid \lambda_i = \lambda_g + 1\}$. Because $\lambda_{g-\alpha} = g+1$ it follows that $r + c \leqslant \alpha$ and $r \leqslant \alpha$. The Vanishing Theorem at page 23 now implies that

$$c \geqslant 2(a - r) \text{ or } r \geqslant \alpha.$$

Hence it follows that $r = \alpha$. □

By this theorem a vector-valued Siegel modular form with respect to Γ and the irreducible representation with highest weight $(g+1,\dots,g+1,g-1)$ determines a Γ-invariant holomorphic differential form of degree $N-1$. Define

$$d\check{\tau}_{ij} = \pm\, e_{ij} \bigwedge_{\substack{1\leqslant k\leqslant l\leqslant g\\(k,l)\neq(i,j)}} d\tau_{kl};\quad e_{ij} = \frac{1+\delta_{ij}}{2}, \tag{53}$$

where the sign is chosen such that $d\check{\tau}_{ij}\wedge d\tau_{ij} = e_{ij}\bigwedge_{1\leqslant k\leqslant l\leqslant g} d\tau_{kl}$. Then each holomorphic differential form $\omega\in\Omega^{N-1}(\mathbb{H}_g)$ can be written in the form

$$\omega = \mathrm{Tr}(f(\tau)d\check{\tau}) = \sum_{1\leqslant i,j\leqslant g} f_{ij}(\tau)d\check{\tau}_{ij}.$$

It is easy to see by similar arguments exploited for 1-forms that $\omega\in\Omega^{N-1}(\mathbb{H}_g)^{\Gamma}$ if and only if

$$f(\gamma\cdot\tau) = \det(C\tau+D)^{g+1}\,{}^t(C\tau+D)^{-1}f(\tau)(C\tau+D)^{-1}.$$

3.2.1 *Differential forms of degree $g(g+1)/2-1$ invariant for the full modular group*

We are going to present two methods for constructing Γ_g-invariant holomorphic differential forms of degree $N-1$, where $N=g(g+1)/2$. The first one exploit scalar-valued Siegel modular forms and in particular theta constants, while the second one exploits gradients of odd theta functions.

Let us start from the construction of a holomorphic differential form invariant under the action of a congruence subgroup $\Gamma\subset\Gamma_g$ starting from two suitable scalar-valued modular forms. By [12] for any f and h in $[\Gamma,(g-1)/2]$, possibly with a multiplier system, it is possible to define a holomorphic differential form $\omega_{f,h}\in\Omega^{N-1}(\mathbb{H}_g)^{\Gamma}$. The definition of $\omega_{f,h}$ exploits suitable differential operators applied to the two scalar-valued modular forms.

Let us first fix some notation. For $X\subset\mathbb{N}$ of finite cardinality, denote by $P^*_k(X)$ the collection of the increasingly ordered subsets of X with fixed cardinality k. If $I\in P^*_k(X)$ set $I^c := X\setminus I\in P^*_{n-k}(X)$, where n is the cardinality of X. Denote by X_g the ordered set $\{1,\dots,g\}$.

If V is a g-dimensional complex vector space with a given basis, one can choose a basis of the exterior product $\bigwedge^p V$ which is indexed by suitable sets of indexes of length p. A linear map $L:\bigwedge^p V\to\bigwedge^p V$ is then given by a matrix $\left(L^I_J\right)$ for $I,J\in P^*_p(X_g)$. If $A:\wedge^p V\to\wedge^p V$ and $B:\wedge^q V\to\wedge^q V$ define the linear map $A\sqcap B$

$$A\sqcap B:\wedge^{p+q}V\to\wedge^{p+q}V$$

given by the following matrix (cf. [12])

$$(A \sqcap B)^H_K = \frac{1}{\binom{p+q}{p}} \sum_{\substack{I \in P^*_p(H) \\ J \in P^*_p(K)}} (-1)^{I+J} A^I_J \, B^{I^c}_{J^c}, \quad H, K \in P^*_{p+q}(X_g). \tag{54}$$

Let $\partial_{\tau_{ij}} := \frac{\partial}{\partial \tau_{ij}}$ and define the $g \times g$ matrix of differential operators

$$\partial := (\partial_{ij}), \qquad \partial_{ij} = \frac{1}{2}(1+\delta_{ij})\partial_{\tau_{ij}}, \; i, j = 1, \dots, g.$$

For any $1 \leqslant k \leqslant g$ let $\partial^{[k]} = \partial \sqcap \cdots \sqcap \partial$, where we take the $\sqcap$ product k times. Hence for $I, J \in P^*_k(X_g)$

$$\left(\partial^{[k]}\right)^I_J = \det\left(\partial(I,J)\right),$$

where by $\partial(I,J)$ we denote the $k \times k$ submatrix of ∂ obtained by taking rows with indexes in I and columns with indexes in J. If f is a modular form we denote by $\partial^{[k]} f$ the matrix such that

$$\left(\partial^{[k]} f\right)^I_J = \det(\partial(I,J)) \cdot f, \;\; I, J \in P^*_k(X_g).$$

If ν a given multiplier system, there exists a linear pairing

$$[\Gamma, (g-1)/2, \nu] \times \left[\Gamma, (g-1)/2, \nu^{-1}\right] \to \Omega^{N-1}(\mathbb{H}_g)^\Gamma$$
$$(f, h) \mapsto \omega_{f,h}\,,$$

with

$$\omega_{f,h} = \sum_{p+q=g-1} (-1)^p \partial^{[p]} f \sqcap \partial^{[q]} h \sqcap d\check{\tau}.$$

By definition one can easily see that

$$\omega_{f,h} = \mathrm{Tr}(B(\tau) d\check{\tau}),$$

where

$$B(\tau)_{ij} := (-1)^{i+j} \sum_{k=0}^{g-1} \frac{(-1)^k}{\binom{g-1}{k}} \sum_{\substack{I \in P^*_k(X_g \setminus \{i\}) \\ J \in P^*_k(X_g \setminus \{j\})}} (-1)^{I+J} \det\left(\partial(I,J)\right) \cdot f \det\left(\partial(I^c, J^c)\right) \cdot h\,,$$

where $I + J$ means the sum of all the indexes in I and J.

By [12, eq. 61], the Fourier coefficient with respect to a matrix T of the entry $B(\tau)_{gg}$ is given by

$$b(T) = \sum_{k=1}^{g} \frac{(-1)^k}{\binom{g-1}{k-1}} \sum_{I,J \in P^*_{k-1}(X_{g-1})} \sum_{T_1+T_2=T} \epsilon \det\left(T_1(I,J)\right) \det\left(T_2(I^c, J^c)\right) a_f(T_1) a_h(T_2), \tag{55}$$

where $\epsilon = (-1)^{I+J}$, $T_1(I,J)$ (resp. $T_2(I^c,J^c)$) is the submatrix of T_1 (resp. T_2) obtained by taking rows in I (resp. I^c) and columns in J (resp. J^c), $a_f(T_1)$ and $a_h(T_2)$ are the Fourier coefficients of f and h corresponding to the matrices T_1 and T_2 respectively.

We are interested in the explicit construction of the Γ_g-invariant holomorphic differential forms obtained with this method.

Proposition 3.2.2 ([12, 8]). *Let*

$$f = \sum_{m\ even} \vartheta_m(\tau)^{g-1}.$$

Then $\omega_{f,f} \in \Omega^{N-1}(\mathbb{H}_g)^{\Gamma_g}$ *and does not vanish for* $g = 8k+1$, $k \geqslant 1$.

Proof. The result for $k \geqslant 2$ is classical and it is proven in [12]. Here we will prove that if $f = \sum_{m\ even} \vartheta_m(\tau)^8$, then $\omega_{f,f} \in \Omega^{35}(\mathbb{H}_9)^{\Gamma_9}$ does not vanish identically (cf. [8]).

Recall by (42) that

$$\sum_{m\ even} \vartheta_m(\tau)^8 = 2^g \Theta^{(g)}_{E_8},$$

where $\Theta^{(g)}_{E_8}$ is the degree g theta series with respect to the quadratic form on E_8. Then we only need to show the non vanishing of the form $\omega_{F,F}$ for $F = \Theta^{(9)}_{E_8}$. Let $A(\tau)$ be such that $\omega_{F,F} = \sum_{1\leqslant i,j\leqslant g} A(\tau)_{ij}\, d\check{\tau}_{ij}$. Let $a(T)$ be the Fourier coefficient with respect to a matrix T of the entry $A(\tau)_{99}$. The expression of $a(T)$ is given in general by (55), but it can be greatly simplified by suitably choosing the matrix T. To prove the non vanishing of the form $\omega_{F,F}$ we will prove that the Fourier coefficient $a(T)$ does not vanish for

$$T := \begin{pmatrix} \zeta_{E_8} & 0 \\ 0 & 0 \end{pmatrix},$$

where ζ_{E_8} is the matrix (41). By Köcher principle, only the terms with even positive semi-definite T_1 and T_2 produce non-zero summands in the expression of $a(T)$. The unique decompositions of this type for the chosen T are $T_1 = T$, $T_2 = 0$ and $T_1 = 0$, $T_2 = T$.

So we have to study the Fourier coefficients of $\Theta^{(9)}_{E_8}$ for the chosen matrix T. Recall that

$$\Theta^{(9)}_{E_8}(\tau) = \sum_{p\in M_{9\times 8}(\mathbb{Z})} e^{\pi i \operatorname{Tr}(p\,\zeta_{E_8}\,{}^t p\,\tau)} = \sum_M N_M \prod_{j\leqslant k} e^{\pi i m_{jk}\tau_{jk}},$$

where, for $M = (m_{jk})$ a symmetric $g\times g$ integral matrix, $N_M \in \mathbb{N}$ is the number of integral matrix solutions of the Diophantine system $p\,\zeta_{E_8}\,{}^t p = M$. Setting $M = T$ and writing $p = \left(\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right)$, where p_1 and p_2 are 8×8 and 1×8 integer matrices respectively, it follows that for all solutions $p_2 = 0$, while p_1 satisfies $p_1\,\zeta_{E_8}\,{}^t p_1 = \zeta_{E_8}$. The number of solutions of this latter equation equals the order of the group $U(\zeta_{E_8})$ of automorphisms of the E_8 lattice. By [7, page 121] $\#(U(\zeta_{E_8})) = 4!6!8!$. Thus we finally have that there is a non-empty set of summands in the expression of $a(T)$. Since all of the summands are positive it follows that $A(T)_{99}$ is non-zero and the Proposition is proven.

Using the modular form $\Theta_{E_8}(\tau)^k$, the argument above easily generalizes to give an alternative proof of the classical result in [12] for $k \geqslant 2$. $\square$

A second method of building elements of $\Omega^{N-1}(\mathbb{H}_g)^{\Gamma_g}$ is examined in [44]. Here the author starts from the vector-valued modular forms constructed from a set of $g-1$ odd characteristics defined as in (39) and produces holomorphic differential forms invariant under the action of the full modular group for $g \equiv 0 \pmod 4$, $g \neq 5, 13$. We will see in Section 4.1.3 that in some remarkable cases the two constructions agree (cf. [8, 40]).

3.3 Siegel modular threefolds with a degree 8 endomorphism

In Section 3.1.1 we have seen that the Satake compactification of the moduli space of principally polarized abelian surfaces with level 2 structure is a quartic hypersurface in $\mathbb{P}^4$. In [32] the Igusa quartic has been characterized as a Steiner hyperquartic and as such it has a degree 8 endomorphism.

In this section we will show that the existence of a degree 8 endomorphism of the Igusa quartic is indeed a part of a more general result. We will construct a degree 8 endomorphism on suitable Siegel modular threefolds via an isomorphism of graded rings of scalar-valued Siegel modular forms and a degree 8 map between two given Siegel modular threefolds.

First we will consider the rings of scalar-valued Siegel modular forms (with multiplier) with respect to the subgroup $\Gamma_2(2,4)$, defined as in (15), and the subgroup

$$\Gamma_2^2(2,4) = \left\{ \gamma \in \Gamma_2 \mid \gamma \equiv \begin{pmatrix} \mathbb{1}_2 & * \\ 0 & \mathbb{1}_2 \end{pmatrix} \pmod 2,\ \operatorname{diag}(2B) \equiv \operatorname{diag}(C) \equiv 0 \pmod 4 \right\}. \quad (56)$$

The computation of such rings give us that the degree 8 map

$$\begin{gathered} \mathbb{P}^3 \to \mathbb{P}^3 \\ [x_0, x_1, x_2, x_3] \mapsto [x_0^2, x_1^2, x_2^2, x_3^2] \end{gathered} \quad (57)$$

is indeed a map between the modular varieties related to these two groups:

$$\psi : \operatorname{Proj}(A(\Gamma_2(2,4))) \to \operatorname{Proj}(A(\Gamma_2^2(2,4))). \quad (58)$$

The above-mentioned isomorphism of suitable rings of modular forms involves the following two subgroups:

$$\Gamma_0(2) = \{\gamma \in \Gamma_2 \mid C \equiv 0 \pmod 2\}, \tag{59}$$

$$\Gamma_0^0(2) = \{\gamma \in \Gamma_2 \mid C \equiv B \equiv 0 \pmod 2\}. \tag{60}$$

We shall prove that there is indeed an isomorphism

$$\Gamma_0(2)/\Gamma_2^2(2,4) \cong \Gamma_0^0(2)/\Gamma_2(2,4),$$

equivariant with respect to the action of the groups on the two copies of $\mathbb{P}^3$ in (58). If $G = \Gamma_0(2)/\Gamma_2^2(2,4) \cong \Gamma_0^0(2)/\Gamma_2(2,4)$, we will establish the following theorem.

Theorem. *For any subgroup* $H \subset G$ *there exists an isomorphism of graded rings of modular forms*

$$\Phi_H : A(\Gamma) \to A(\Gamma'),$$

where $\Gamma_2(2,4) \subset \Gamma \subset \Gamma_0^0(2)$, $\Gamma_2^2(2,4) \subset \Gamma' \subset \Gamma_0(2)$ *and the quotients* $\Gamma/\Gamma_2(2,4)$ *and* $\Gamma'/\Gamma_2^2(2,4)$ *are both isomorphic to* H.

By this theorem we will give a degree 8 endomorphism of the Satake compactification $\overline{X}_{\Gamma'}$ for any subgroup $\Gamma_2(2,4) \subset \Gamma' \subset \Gamma_0^0(2)$. Studying the action of the Fricke involution we will find other modular threefolds with a degree 8 endomorphism.

In the next section we will see to what extent it is possible to generalize these results to Siegel modular varieties of degree 3. Section 3.3 and 3.4 are based on my paper [39].

3.3.1 *Degree 8 map between two modular threefolds*

We are going to prove that the map (57) is indeed a morphism of modular varieties. For the sake of simplicity we will denote the four second order theta constants in degree 2 as follows:

$$f_{00} := \Theta[0\,0],\ f_{01} := \Theta[0\,1],\ f_{10} := \Theta[1\,0],\ f_{11} := \Theta[1\,1].$$

By [42] we have that

$$A(\Gamma_2(2,4), v_\Theta) = \mathbb{C}[f_{00}, f_{01}, f_{10}, f_{11}], \tag{61}$$

where v_Θ is the multiplier system appearing in the transformation formula (37) for second order theta constants. Hence $\overline{X}_{\Gamma_2(2,4)}$ isomorphic to $\mathbb{P}^3$.

It is easily seen that the quotient $\Gamma_2^2(2,4)/\Gamma(2,4)$ is isomorphic to the vector space $\mathbb{F}_2^3$, where $\mathbb{F}_2$ is the finite field with two elements. Indeed, consider the map

$$\begin{array}{ccc} \Gamma_2^2(2,4) & \xrightarrow{\varphi} & \mathbb{F}_2^3 \\ \gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) & \longmapsto & \left(\dfrac{b_{11}}{2}, b_{12}, \dfrac{b_{22}}{2}\right) \quad \pmod 2, \end{array}$$

where $B = \left(\begin{smallmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{smallmatrix}\right)$. Since γ is a symplectic matrix, the condition $A \equiv 0 \pmod 2$ implies that B is a symmetric matrix once we reduce it modulo 2, so the classes of three entries b_{11}, b_{12}, b_{22} in $\mathbb{F}_2$ determine the matrix B. Moreover the condition $\mathrm{diag}(B) \equiv 0 \pmod 2$ implies that b_{11}, b_{22} are even integers so the map φ is well defined. By the conditions $A \equiv D \equiv 1_2 \pmod 2$ it is easily checked that φ is a group homomorphism. We want to show that the map φ is surjective and its kernel is the group $\Gamma(2,4)$. We have that

$$\mathrm{diag}(B) \equiv 0 \pmod 2 \implies B = \left(\begin{smallmatrix} 2a & b \\ c & 2d \end{smallmatrix}\right), \text{ with } a, b\, c, d \in \mathbb{Z},$$

so the surjectivity of the map follows. Furthermore we have that

$$\begin{aligned} \varphi(\gamma) = (0,0,0) &\Leftrightarrow \mathrm{diag}(B) \equiv 0 \pmod 4 \quad \text{and} \quad b_{12} \equiv 0 \pmod 2 \Leftrightarrow \\ &\Leftrightarrow \mathrm{diag}(B) \equiv 0 \pmod 4 \quad \text{and} \quad B \equiv 0 \pmod 2 \Leftrightarrow \\ &\Leftrightarrow \gamma \in \Gamma(2,4). \end{aligned}$$

So we have that $\Gamma^1(2,4)/\Gamma(2,4)$ is abelian and the index $[\Gamma^1(2,4) : \Gamma(2,4)] = 8$.

Any symmetric 2×2 integer matrix S determines an element $\gamma_S \in \Gamma_2$, namely

$$\gamma_S = \begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix}.$$

In particular, if we put

$$B_1 = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix}\right), \; B_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right), \; B_3 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right),$$

then the matrices $M_i := \gamma_{B_i}$ belong to $\Gamma_2^2(2,4)$, and the M_i^2 belong to its index 8 normal subgroup $\Gamma_2(2,4)$. By taking $\{M_1, M_2, M_3\}$ as a basis we thus identify $\Gamma_2^2(2,4)/\Gamma_2(2,4)$ with $\mathbb{F}_2^3$.

We will discuss the action of the quotient $\Gamma_2^2(2,4)/\Gamma(2,4)$ on the second order theta constants f_a. Focus on the action of the matrices M_i on theta constants. From [42, p. 59] we have

$$\vartheta\left[\begin{smallmatrix} m' \\ m'' \end{smallmatrix}\right](\gamma_S \cdot \tau) = \vartheta\left[\begin{smallmatrix} m' \\ m'' \end{smallmatrix}\right](\tau + S) = \varepsilon^{-{}^t m'(Sm' + 2\,\mathrm{diag}(S))}\vartheta\left[\begin{smallmatrix} m' \\ m'' + Sm' + \mathrm{diag}(S) \end{smallmatrix}\right](\tau),$$

with $\varepsilon = \frac{1+i}{\sqrt{2}}$ a primitive 8^{th} root of unity. For the second order theta constants this gives

$$\Theta[a](\gamma_S \cdot \tau) = i^{\,{}^t a S a}\, \Theta[a](\tau).$$

Thus, for $a = (a_1, a_2) \in \{0, 1\}^2$ it follows that

$$\begin{aligned} f_a(M_1 \cdot \tau) &= (-1)^{a_1} f_a(\tau), \\ f_a(M_2 \cdot \tau) &= (-1)^{a_2} f_a(\tau), \\ f_a(M_3 \cdot \tau) &= (-1)^{a_1 a_2} f_a(\tau). \end{aligned}$$

So the group $\Gamma_2^2(2,4)/\Gamma_2(2,4)$ acts by changes of sign on the f_a. Therefore it acts trivially on the f_a^2.

Proposition 3.3.1. *The ring* $\mathbb{C}[f_{00}^2, \dots, f_{11}^2]$ *is equal to the subring* $A_{\mathbb{N}}(\Gamma_2^2(2,4), \nu_\Theta^2) \subset A(\Gamma_2^2(2,4), \nu_\Theta^2)$ *of scalar-valued Siegel modular forms with integral weight.*

Proof. We have just seen that $\mathbb{C}[f_a^2] \subset A_{\mathbb{N}}(\Gamma_2^2(2,4), \nu_\Theta^2)$. For the opposite inclusion, since both rings are integrally closed it is enough to show that they have the same field of fractions. This is also immediate, because we have already seen that they both have $\mathbb{C}(f_a)$ as an extension of degree 8. $\square$

Thus the degree 8 endomorphism of $\mathbb{P}^3$ given by $[x_0, \dots, x_3] \mapsto [x_0^2, \dots, x_3^2]$ can be seen as a map between the two modular varieties

$$\psi : \mathrm{Proj}(A(\Gamma_2(2,4))) \to \mathrm{Proj}(A(\Gamma_2^2(2,4))).$$

Here we omit the multipliers since the modular variety is independent of the choice of the multiplier system.

3.3.2 *Isomorphic modular threefolds and degree 8 endomorphisms*

In this section we will prove an isomorphism of graded ring of scalar-valued modular forms. It is easily checked that the groups $\Gamma_2^2(2,4)$ and $\Gamma_2(2,4)$ are normal subgroups of $\Gamma_0(2)$ and $\Gamma_0^0(2)$ respectively. Moreover $[\Gamma_0(2) \colon \Gamma_2^2(2,4)] = 96$ and the quotient group is isomorphic to the semidirect product $\mathbb{F}_2^4 \ltimes S_3$ where S_3 is the symmetric group of degree 3.

We can construct an isomorphism

$$\varphi \colon \Gamma_0(2)/\Gamma_2^2(2,4) \to \Gamma_0^0(2)/\Gamma_2(2,4),$$

as follows. For a class $\gamma \in \Gamma_0(2)/\Gamma_2^2(2,4)$ we can choose a representative (which we also call γ) of the form

$$\gamma \equiv \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ CA^{-1} & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & A^{-1}B \\ 0 & 1 \end{smallmatrix}\right).$$

Define

$$\varphi(\gamma) = \left(\begin{smallmatrix} 1 & 0 \\ CA^{-1} & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & A^{-1}2B \\ 0 & 1 \end{smallmatrix}\right).$$

Roughly speaking, the map φ sends "B" to "2B". Set

$$G := \Gamma_0(2)/\Gamma_2^2(2,4) \cong \Gamma_0^0(2)/\Gamma_2(2,4).$$

From [18, section 2] we know that $\Gamma_0(2)$ is generated by matrices of the form ${}^t\gamma_{2S}$, $\gamma' = \left(\begin{smallmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{smallmatrix}\right)$ and γ_S, where S is a symmetric 2×2 matrix with integer coefficients. The classes of these matrices are then generators for the group $\Gamma_0(2)/\Gamma_2^2(2,4)$ and their images under φ are generators for the group $\Gamma_0^0(2)/\Gamma_2(2,4)$.

An easy computation gives

$$f_a({}^t\gamma_{2S} \cdot \tau)^2 = f_{a-\mathrm{diag}(S)}(\tau)^2, \quad f_a({}^t\gamma_{2S} \cdot \tau) = f_{a-\mathrm{diag}(S)}(\tau).$$

$$f_a(\gamma' \cdot \tau)^2 = f_{Aa}(\tau)^2, \qquad f_a(\gamma' \cdot \tau) = f_{Aa}(\tau),$$

$$f_a(\gamma_S \cdot \tau)^2 = i^{{}^ta2Sa} f_a(\tau)^2, \qquad f_a(\gamma_{2S} \cdot \tau) = i^{{}^ta2Sa} f_a(\tau).$$

This shows that via the isomorphism φ the action of the group G on the two polynomial rings is the same and the map $f_a \mapsto f_a^2$ is an isomorphism of G-modules.

Theorem 3.3.2. *For any subgroup $H \subset G$ there exist two groups Γ, Γ' such that*

$$\Gamma_2(2,4) \subset \Gamma \subset \Gamma_0^0(2), \quad \Gamma_2^2(2,4) \subset \Gamma' \subset \Gamma_0(2)$$

and the quotients $\Gamma/\Gamma_2(2,4)$ and $\Gamma'/\Gamma_2^2(2,4)$ are both isomorphic to H via the map induced by φ. This also induces an isomorphism

$$\Phi_H : A(\Gamma, v_\Theta) \to A_{\mathbb{N}}(\Gamma', v_\Theta^2),$$

such that if $f \in [\Gamma, k/2, v_\Theta]$ then $\Phi_H(f) \in [\Gamma', k, v_\Theta^2]$.

As an immediate consequence we have the following Theorem.

Theorem 3.3.3. *For every subgroup Γ such that $\Gamma_2(2,4) \subset \Gamma \subset \Gamma_0^0(2)$ the projective variety $\overline{X}_\Gamma$ has a degree 8 endomorphism.*

Proof. We will follow the notation of Theorem 3.3.2. Directly from the inclusion of groups we have that

$$A(\Gamma, v_\Theta) \subset A(\Gamma_2(2,4), v_\Theta) = \mathbb{C}[f_a]$$
$$A_{\mathbb{N}}(\Gamma', v_\Theta^2) \subset A(\Gamma_2^2(2,4), v_\Theta^2) = \mathbb{C}[f_a^2].$$

Let

$$\begin{aligned} \psi : \mathbb{C}[f_a] &\to \mathbb{C}[f_a^2] \\ f_a &\mapsto f_a^2 \end{aligned},$$

then

$$A(\Gamma, v_\Theta) \xrightarrow{\psi} A_{\mathbb{N}}(\Gamma', v_\Theta^2) \xrightarrow[\cong]{\Phi_H} A(\Gamma, v_\Theta),$$

has degree 8 and as a consequence the modular variety associated to Γ has a degree 8 endomorphism. $\square$

Since $\Gamma_2(2,4) \subset \Gamma_2(2) \subset \Gamma_0^0(2)$, Theorem 3.3.3 gives a degree 8 endomorphism of the Igusa quartic.

By means of the characterization of the Igusa quartic as a Steiner hypersurface, the Satake compactification of the moduli space of principally polarized abelian surfaces with Göpel triples is shown to be isomorphic to the Igusa quartic (cf. [32]). This isomorphism is given by means of scalar-valued Siegel modular forms in [6, section 11]. With the result of Theorem 3.3.2 we will give a different proof of this isomorphism between this two Siegel modular varieties. The moduli space of principally polarized abelian surfaces with Göpel triples is the modular variety with respect to the subgroup

$$\Gamma^1(2) = \{\, \gamma \in \Gamma_2 \mid A \equiv D \equiv 1_2 \pmod 2,\ C \equiv 0 \pmod 2 \,\}.$$

It is readily seen that both $\Gamma_2(2)/\Gamma_2(2,4)$ and $\Gamma^1(2)/\Gamma_2^2(2,4)$ are isomorphic to the group H generated by M_1, M_2, ${}^tM_1$ and ${}^tM_2$. Therefore the isomorphism φ_H of Theorem 3.3.2 induces an isomorphism between $\overline{X}_{\Gamma_2(2)}$ and $\overline{X}_{\Gamma^1(2)}$.

3.3.3 *Action of the Fricke involution*

In this section we will see that Theorem 3.3.3 can be extended to other modular threefolds by studying the action of the Fricke involution.

First note that by Riemann's addition formula (see (35) and (36)) the vector space of modular forms spanned by $\{\, \Theta[a]^2 \,\}_{a \in \{0,1\}^g}$ coincides with the one spanned by $\left\{ \vartheta \left[\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right]^2 \right\}_{b \in \{0,1\}^g}$ where $\vartheta_b := \vartheta \left[\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right]$ for $b \in \{0,1\}^g$. By the action of the Fricke involution we will see that in the arguments of the previous section we can actually replace the rings $\mathbb{C}[f_a]$ and $\mathbb{C}[f_a^2]$ by the rings $\mathbb{C}[\vartheta_b]$ and $\mathbb{C}[\vartheta_b^2]$ thus finding other modular threefolds with a degree 8 endomorphism.

The Fricke involution on $\mathbb{H}_g$ is the involution given by the matrix

$$J_g = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1_g \\ -2\,1_g & 0 \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{R}),$$

acting on $\mathbb{H}_g$ as in (11), so that $J_g \cdot \tau = (2\tau)^{-1}$. We are interested in the case $g = 2$ and the action of J_2 on the functions f_a with $a \in \{0, 1\}^2$.

Although formula (18) does not define an action of $\mathrm{Sp}(2g, \mathbb{R})$ on theta characteristics, it is still possible to use the classical transformation formula for theta functions to compute the action of the matrix J_2 on theta constants. An easy computation shows that

$$f_a(J_2 \cdot \tau) = v_\Theta(J_2) \det(\tau)^{1/2} \vartheta_a(\tau), \tag{62}$$

where we define $v_\Theta(J_2)$ to be equal to $v_\vartheta(J)$ with $J = \left(\begin{smallmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{smallmatrix}\right)$. By v_ϑ we denote the multiplier system appearing in the transformation formula for theta constants. For any $\gamma \in \mathrm{Sp}(4, \mathbb{R})$ we write γ^{J_2} for the conjugate $J_2 \gamma J_2^{-1}$. Then

$$\gamma^{J_2} = \begin{pmatrix} D & -C/2 \\ -2B & A \end{pmatrix}, \quad \forall \gamma \in \Gamma_2. \tag{63}$$

In particular, if $\gamma \in \Gamma_2$, then $\gamma^{J_2} \in \Gamma_2$ if and only if $C \equiv 0 \pmod 2$.

From (63) we can compute that

$$\Gamma_2^2(2,4)^{J_2} = \Gamma_2^2(2,4) \ \text{ and } \ \Gamma_0(2)^{J_2} = \Gamma_0(2),$$

whereas

$$\Gamma_0^0(2)^{J_2} = \Gamma_0(4) := \{ \gamma \in \Gamma_2 \mid C \equiv 0 \pmod 4 \},$$

and

$$\Gamma_2(2,4)^{J_2} = \left\{ \gamma \in \Gamma_2 \ \text{ s.t. } \begin{array}{c} A \equiv D \equiv 1_2 \pmod 2, \\ C \equiv 0 \pmod 4, \ \mathrm{diag}(C) \equiv 0 \pmod 8, \\ \mathrm{diag}(B) \equiv 0 \pmod 2 \end{array} \right\}.$$

We can exploit this action to compute the ring of scalar-valued Siegel modular forms with respect to the group $\Gamma_2(2,4)^{J_2}$. From (61) and (62) it follows that

$$A(\Gamma_2(2,4)^{J_2}, v_\vartheta) = \mathbb{C}[\vartheta_b].$$

Moreover, since the f_a^2 are linear combination of the ϑ_b^2 and vice versa, by (35) and (36), the polynomial ring $\mathbb{C}[f_a^2] = \mathbb{C}[\vartheta_b^2]$ is invariant under the action of the Fricke involution.

Thus, we have another modular interpretation of the endomorphism (57) of $\mathbb{P}^3$. Set $G' := \Gamma_0(4)/\Gamma_2(2,4)^{J_2}$. With the same arguments that led us to Theorem 3.3.2, we have an isomorphism $\varphi' : \Gamma_0(2)/\Gamma_2^2(2,4) \to \Gamma_0(4)/\Gamma_2(2,4)^{J_2}$ such that via this isomorphism the action of the group G' on the rings $\mathbb{C}[\vartheta_b]$ and $\mathbb{C}[\vartheta_b^2]$ is the same and the map $\vartheta_b \mapsto \vartheta_b^2$ is an isomorphism of G'-modules.

Theorem 3.3.4. *For any subgroup $H' \subset G'$ there exist two groups Δ, Δ' such that*

$$\Gamma_2(2,4)^{J_2} \subset \Delta \subset \Gamma_0(4), \;\; \Gamma_2^2(2,4) \subset \Delta' \subset \Gamma_0(2),$$

and the quotients $\Delta/\Gamma_2(2,4)^{J_2}$ and $\Delta'/\Gamma_2^2(2,4)$ are both isomorphic to H'. This isomorphism is induced by φ'. Therefore it is also induced an isomorphism of graded ring of modular forms

$$\Psi_{H'} : A(\Delta, \nu_\vartheta) \to A_{\mathbb{N}}(\Delta', \nu_\vartheta^2)$$

such that if $f \in [\Delta, k/2, \nu_\vartheta]$ then $\Psi_{H'}(f) \in [\Delta', k, \nu_\vartheta^2]$.

Note that since the groups $\Gamma_2^2(2,4)$ and $\Gamma_0(2)$ are fixed by the Fricke involution the set of groups between them is also fixed, but the individual groups need not be. As before we get the following statement about the existence of a degree 8 endomorphism on suitable Siegel modular threefolds.

Theorem 3.3.5. *For every subgroup Δ such that $\Gamma_2(2,4)^{J_2} \subset \Delta \subset \Gamma_0(4)$ the projective variety $\overline{X}_\Delta$ has a degree 8 endomorphism.*

3.4 Siegel modular varieties in degree 3

In this section we will examine the modular varieties associated to the degree 3 version of the subgroups appearing in (58). We will investigate some properties of these modular varieties in order to show that the arguments of the previous section do not generalize directly in the higher dimensional case.

Define the group

$$\Gamma_3^2(2,4) = \left\{ \gamma \in \Gamma_3 \;\middle|\; \gamma \equiv \begin{pmatrix} 1_2 & * \\ 0 & 1_2 \end{pmatrix} \text{ (mod 2), } \operatorname{diag}(2B) \equiv \operatorname{diag}(C) \equiv 0 \text{ (mod 4)} \right\}.$$

We will show that both $\overline{\mathbb{H}_3/\Gamma_3(2,4)}$ and $\overline{\mathbb{H}_3/\Gamma_3^2(2,4)}$ are not unirational. A necessary condition for unirationality is that there are no non-trivial holomorphic differential forms in any degree. Exploiting the construction of holomorphic differential forms by means of gradients of odd theta functions, we will show that $\Omega^5(\mathbb{H}_3)^{\Gamma_3(2,4)}$ and $\Omega^5(\mathbb{H}_3)^{\Gamma_3^2(2,4)}$ are not trivial.

We have seen in Section 3.2 that a vector-valued Siegel modular form with respect to a subgroup $\Gamma \subset \Gamma_g$ and the irreducible representation with highest weight

$$(g+1, \dots, g+1, g-1)$$

determines a Γ-invariant holomorphic differential form of degree $N-1$, where $N = g(g+1)/2$. For $\Gamma = \Gamma_3(2,4)$ we can easily find such a modular form among the ones constructed with gradients of odd theta functions. Recall from Section 2.4 that if N is a matrix of two distinct odd characteristics, then $W(N) \in [\Gamma_3(2,4), (4,4,2)]$. Hence the space $\Omega^5(\mathbb{H}_3)^{\Gamma_3(2,4)}$ is non-trivial and so $\overline{\mathbb{H}_3/\Gamma_3(2,4)}$ is not unirational. Actually in this way one can construct at least $\binom{28}{2} = 378$ non-trivial holomorphic differential forms on $\mathbb{H}_3$ invariant under the action of $\Gamma_3(2,4)$, each coming from a vector-valued modular form $W(M)$ where M is a matrix of two distinct odd characteristics.

These vector-valued Siegel modular forms can be also used to construct some non-trivial holomorphic differential forms of degree 5 invariant under the action of the group $\Gamma_3^2(2,4)$ as it is shown in the proof of the following Theorem.

Theorem 3.4.1. *The space $\Omega^5(\mathbb{H}_3)^{\Gamma_3^2(2,4)}$ is non-trivial and so $\overline{\mathbb{H}_3/\Gamma_3^2(2,4)}$ is not unirational.*

Proof. We will prove that this space of holomorphic differential forms is not trivial by exhibiting some elements of $[\Gamma_3^2(2,4), (4,4,2)]$. One way to construct vector-valued Siegel modular forms in this space is to symmetrize suitable vector-valued Siegel modular forms with respect to $\Gamma_3(2,4)$ and the representation $\rho = (4,4,2)$ and then check that the resulting vector-valued Siegel modular form with respect to $\Gamma_3^2(2,4)$ and ρ does not vanish identically.

Given a matrix $M = (m_1, m_2)$ of distinct odd characteristics, consider

$$\begin{aligned}\Phi(M)(\tau) &= \sum_{\gamma \in \Gamma_3^2(2,4)/\Gamma_3(2,4)} \rho(C\tau + D)^{-1} W(M)(\gamma \cdot \tau) \\ &= \sum_{\gamma \in \Gamma_3^2(2,4)/\Gamma_3(2,4)} \kappa(\gamma)^4 \, \mathbf{e}\left(2\varphi_{n_1}(\gamma) + 2\varphi_{n_2}(\gamma)\right) W(N)(\tau),\end{aligned} \tag{64}$$

where $N = (n_1, n_2)$ with $n_i = \gamma^{-1} \cdot m_i$, $i = 1, 2$. If well defined and not identically zero, $\Phi(M)$ is a vector-valued Siegel modular form with respect to $\Gamma_3^2(2,4)$ and the representation ρ by construction.

By [24] we know that $\kappa(\gamma)^4 = (-1)^{\mathrm{Tr}({}^tBC)}$ for $\gamma \in \Gamma_g$. It is easily seen that a set of generators for the quotient group $\Gamma_3^2(2,4)/\Gamma_3(2,4)$ is given by the classes of the matrices $M_1, \dots, M_6$, where $M_i = \begin{pmatrix} 1_3 & B_i \\ 0 & 1_3 \end{pmatrix}$ and

$$B_1 = \begin{pmatrix} 2&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}, B_2 = \begin{pmatrix} 0&0&0 \\ 0&2&0 \\ 0&0&0 \end{pmatrix}, B_3 = \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&2 \end{pmatrix},$$
$$B_4 = \begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&0 \end{pmatrix}, B_5 = \begin{pmatrix} 0&0&1 \\ 0&0&0 \\ 1&0&0 \end{pmatrix}, B_6 = \begin{pmatrix} 0&0&0 \\ 0&0&1 \\ 0&1&0 \end{pmatrix}.$$

Thus, the sum in (64) is finite and $\Phi(M)$ is well defined. Moreover, from the set of generators we can explicitly construct the group $\Gamma_3^2(2,4)/\Gamma_3(2,4)$ and compute (64) in order to see if there are choices of the matrix M such that $\Phi(M)$ does not vanish identically.

A direct computation in *Mathematica* [53] shows that there are only 42 (from the 378 we started with) choices of the matrix M such that $\Phi(M)(\tau)$ does not vanish identically, exactly the ones such that if $M = \left(\begin{smallmatrix} m_1' & m_2' \\ m_1'' & m_2'' \end{smallmatrix}\right)$ then $m_1' = m_2'$. For instance, take

$$M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix},$$

then

$$\Phi(M)(\tau) = 16 \sum_{i=1}^{4} W(N_i)(\tau),$$

where $N_1 = M$ and

$$N_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}, \; N_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \; N_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

□

The results obtained so far can be used to show that the arguments of Section 3.3 do not generalize directly to the degree three case. The first key point in the degree 2 case was that there is a map

$$\psi : \overline{\mathbb{H}_2/\Gamma_2(2,4)} \to \overline{\mathbb{H}_2/\Gamma_2^2(2,4)}$$

which is actually the endomorphism of $\mathbb{P}^3$ given by $[x_0, x_1, x_2, x_3] \mapsto [x_0^2, x_1^2, x_2^2, x_3^2]$. We have shown that both $\overline{\mathbb{H}_3/\Gamma_3(2,4)}$ and $\overline{\mathbb{H}_3/\Gamma_3^2(2,4)}$ are not unirational, and therefore a map between these two modular varieties is not a map between two projective spaces. In fact, it is not even possible to construct a map by "squaring coordinates" as in the degree two case. We will show that the coordinate ring of $\overline{\mathbb{H}_3/\Gamma_3^2(2,4)}$ is not generated by squares of elements of the coordinate ring of $\overline{\mathbb{H}_3/\Gamma_3(2,4)}$.

In degree three there is a non-trivial algebraic relation between second order theta constants. By [42] we know that

$$A(\Gamma_3(2,4), \nu_\Theta) = \mathbb{C}[f_a]/(R_{16}),$$

where

$$R_{16} = P_8(f_{000}^2, \ldots, f_{111}^2) + q \cdot Q_4(f_{000}^2, \ldots, f_{111}^2),$$

with P_8 and Q_4 polynomials in the f_a^2 of degree 8 and 4 respectively and $q = \prod_{a\in\{0,1\}^3} f_a$. Its expression is simpler in terms of theta constants, namely

$$R_{16} = 2^3 \sum_{m \text{ even}} \vartheta_m^{16}(\tau) - \Big(\sum_{m \text{ even}} \vartheta_m^8(\tau) \Big)^2.$$

One can move from one expression to the other by means of the identities (35) and (36), recovering in this way the explicit expression of the polynomials P_8 and Q_4.

It is easily checked that $q \in A(\Gamma_3^2(2,4), \nu_\Theta^2)$ so that this ring contains

$$R := \mathbb{C}[f_a^2, q]/(P_8 + q \cdot Q_4, q^2 - \textstyle\prod_a f_a^2).$$

Hence$A(\Gamma_3^2(2,4), \nu_\Theta^2)$ is not generated by squares of elements of the ring $A(\Gamma_3(2,4), \nu_\Theta^2)$.

4

VECTOR-VALUED MODULAR FORMS AND THE HEAT EQUATION

In this chapter we will present some new results concerning the construction of vector-valued Siegel modular forms and a consequent application to the theory of principally polarized abelian varieties. The chapter is based on my works [8] and [40].

In Section 4.1 we present a new construction of vector-valued Siegel modular forms starting from singular scalar-valued modular forms. Applying this construction to second order theta constants we will prove that the relationship between the two methods in [12] and [44] given in [8] is not only at the level of holomorphic differential forms but also at the level of vector-valued modular forms (cf. Section 4.1.3).

In Section 4.2 we will give an application of this new construction to the characterization of decomposable principally polarized abelian varieties.

4.1 A NEW CONSTRUCTION OF VECTOR-VALUED MODULAR FORMS

The material exposed in this section is a development of the ideas in [8, Section 5]. Here we focus on vector-valued modular forms and not only on invariant holomorphic differential forms and give a new method for constructing vector-valued modular forms from singular scalar-valued ones. We will prove that the relationship between the two methods in [12] and [44] given in [8] is not only at the level of holomorphic differential forms but also at the level of vector-valued Siegel modular forms. Denote by V_{grad} the vector space generated by the vector-valued Siegel modular forms constructed with gradients of odd theta functions as in (39) and by V_{Θ} the vector space generated by the vector-valued modular forms constructed with our new method applied to second order theta constants. We will prove that $V_{\mathrm{grad}} = V_{\Theta}$.

This section is mostly based on my recent work [40].

4.1.1 *Multilinear algebra*

In this section we present some results in multilinear algebra. First we shall fix notations.

If M is a $g \times g$ matrix its elements will be denoted by M^i_j where i is the row index and j is the column index. If $I \in P^*_k(X_g)$ and $J \in P^*_l(X_g)$ denote by $M(I,J)$ the $k \times l$ submatrix of M obtained by taking rows with indexes in I and columns with indexes in J. If $J = \{j_i, \dots, j_l\}$ we will write

$$M(I,J) = (M(I,j_1) \mid \cdots \mid M(I,j_l))$$

to emphasize the columns of the submatrix. If $I = X_g$ we will write M_J for $M(I,J)$.

The following formula is a well known generalization of the Laplace expansion theorem for the determinant of a square matrix. Choose $1 \leqslant k < g$ and fix $J \in P^*_k(X_g)$ then

$$\det(M) = \sum_{I \in P^*_k(X_g)} (-1)^{I+J} \det(\, M(I,J)\,) \det(\, M(I^c, J^c)\,), \tag{65}$$

where $I+J$ means the sum of all the elements of the sets I and J. Here we are fixing a set of columns of M and extracting minors of order k from such columns with the related cofactors, the same formula holds if we fix a set of rows and extract from them minors of order k.

Denote by $M^{(k)}$ the matrix of cofactors of submatrices of order $k < g$ of M. We will index the entries of $M^{(k)}$ by some sets of indexes, that is

$$(\, M^{(k)}\,)^I_J = (-1)^{I+J} \det(\, M(I^c, J^c)\,),$$

for $I, J \in P^*_k(X_g)$. This notation is justified by the relation with exterior powers of linear mapping, relation that we will explain in the following.

Let us make some examples. If

$$M = \begin{pmatrix} M^1_1 & M^1_2 & M^1_3 \\ M^2_1 & M^2_2 & M^2_3 \\ M^3_1 & M^3_2 & M^3_3 \end{pmatrix}$$

is a 3×3 matrix we get

$$M^{(1)} = \begin{pmatrix} \begin{vmatrix} M_2^2 & M_3^2 \\ M_2^3 & M_3^3 \end{vmatrix} & -\begin{vmatrix} M_1^2 & M_3^2 \\ M_1^3 & M_3^3 \end{vmatrix} & \begin{vmatrix} M_1^2 & M_2^2 \\ M_1^3 & M_2^3 \end{vmatrix} \\ -\begin{vmatrix} M_2^1 & M_3^1 \\ M_2^3 & M_3^3 \end{vmatrix} & \begin{vmatrix} M_1^1 & M_3^1 \\ M_1^3 & M_3^3 \end{vmatrix} & -\begin{vmatrix} M_1^1 & M_2^1 \\ M_1^3 & M_2^3 \end{vmatrix} \\ \begin{vmatrix} M_2^1 & M_3^1 \\ M_2^2 & M_3^2 \end{vmatrix} & -\begin{vmatrix} M_1^1 & M_3^1 \\ M_1^2 & M_3^2 \end{vmatrix} & \begin{vmatrix} M_1^1 & M_2^1 \\ M_1^2 & M_2^2 \end{vmatrix} \end{pmatrix}$$

and

$$M^{(2)} = \begin{pmatrix} (M^{(2)})^{\{1,2\}}_{\{1,2\}} & (M^{(2)})^{\{1,2\}}_{\{1,3\}} & (M^{(2)})^{\{1,2\}}_{\{2,3\}} \\ (M^{(2)})^{\{1,3\}}_{\{1,2\}} & (M^{(2)})^{\{1,3\}}_{\{1,3\}} & (M^{(2)})^{\{1,3\}}_{\{2,3\}} \\ (M^{(2)})^{\{2,3\}}_{\{1,2\}} & (M^{(2)})^{\{2,3\}}_{\{1,3\}} & (M^{(2)})^{\{2,3\}}_{\{2,3\}} \end{pmatrix} =$$

$$= \begin{pmatrix} M_3^3 & -M_2^3 & M_1^3 \\ -M_3^2 & M_2^2 & -M_1^2 \\ M_3^1 & -M_2^1 & M_1^1 \end{pmatrix}$$

For $k = 0$ we set $M^{(0)} = \det M$. Moreover, ${}^t(M^{(1)})$ is the adjoint matrix of M, that is the matrix such that

$$M\, {}^t(M^{(1)}) = (\det M)\, \mathbb{1}_g.$$

Let V be a g-dimensional complex vector space and fix a basis $\{ e_i \}_{i=1}^g$. If $L : V \to V$ is a linear map, then for any $1 \leqslant p \leqslant g$ there is an associated linear map $\bigwedge^p L : \bigwedge^p V \to \bigwedge^p V$. If the map L is given by a matrix M with respect to the fixed basis of V, the matrix of the associated map $\bigwedge^p L$ with respect to the basis

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}, \ I = \{i_1, \ldots, i_p\} \in P_p^*(X_g), \tag{66}$$

will be denoted by $\bigwedge^p M$. It can be easily obtained by the matrix M. Indeed

$$(\textstyle\bigwedge^p M)^I_J = \det(M(I, J)), \ I, J \in P_p^*(X_g). \tag{67}$$

Note that when we work with exterior powers of vector spaces, the elements of a matrix representing a linear map are indexed by some set of indexes corresponding to the indexing set of the chosen basis (66). Note that

$$(M^{(k)})^I_J = (-1)^{I+J} (\bigwedge^p M)^{I^c}_{J^c}.$$

Recall by (54) that for $A : \wedge^p V \to \wedge^p V$ and $B : \wedge^q V \to \wedge^q V$ the linear map $A \sqcap B$

$$A \sqcap B : \wedge^{p+q} V \to \wedge^{p+q} V$$

is given by the following matrix

$$(A \sqcap B)^H_K = \frac{1}{\binom{p+q}{p}} \sum_{\substack{I \in P^*_p(H) \\ J \in P^*_p(K)}} (-1)^{I+J} A^I_J \, B^{I^c}_{J^c}, \quad H, K \in P^*_{p+q}(X_g).$$

The following Lemma gives the explicit expression of this product in suitable cases.

Lemma 4.1.1. *Fix* $1 \leqslant k \leqslant g$ *and* $I \in P^*_k(X_g)$ *and* $J = \{j_1, \ldots, j_k\} \in P^*_k(X_g)$*. If* $A_1, \ldots, A_k : V \to V$ *then*

$$(A_1 \sqcap \cdots \sqcap A_k)^I_J = \frac{1}{k!} \sum_{\sigma \in S_k} \epsilon(\sigma) \det(\mathbf{A}_\sigma),$$

where $\epsilon(\sigma)$ *is the sign of the permutation* σ *and*

$$\mathbf{A}_\sigma = \left(A_1(I, j_{\sigma(1)}) \mid \cdots \mid A_k(I, j_{\sigma(k)}) \right).$$

Proof. We proceed by induction on k. The case $k = 2$ follows directly from the definition (54). For $k \geqslant 3$, a direct computation from (54) and the inductive argument gives

$$(A_1 \sqcap \cdots \sqcap A_k)^I_J = \frac{1}{k!} \sum_{\rho \in S_{k-1}} \epsilon(\rho) \sum_{\substack{J' \in P^*_{k-1}(J) \\ I' \in P^*_{k-1}(I)}} \left((-1)^{I'+J'} \det(\mathbf{A}_\rho)(A_k)^{I \setminus I'}_{J \setminus J'} \right).$$

Note that the subsets $I \setminus I'$ and $J \setminus J'$ have only one element, so $(A_k)^{I \setminus I'}_{J \setminus J'}$ is an entry of the matrix A_k. By formula (65) and the properties of the determinant of a matrix it follows that the right-hand side is equal to

$$\frac{1}{k!} \sum_{\rho \in S_{k-1}} \epsilon(\rho) \sum_{J' \in P^*_{k-1}(J)} \epsilon(\rho_{J'}) \det(A_1(I, j_{\rho(1)}) \mid \cdots \mid A_{k-1}(I, j_{\rho(k-1)}) \mid A_k(I, J \setminus J')),$$

where $\rho_{J'} \in S_k$ is the permutation such that $j_{\rho_{J'}(k)} = J \setminus J'$ and fixes all other elements. Since every permutation on k elements is the product of a transposition taking the last element in a given position and a permutation on the others $k-1$ elements, the lemma is proved. $\square$

Corollary 4.1.2. *For* $A : V \to V$ *and* $1 \leqslant k \leqslant g$ *let*

$$A^{[k]} := \underbrace{A \sqcap \cdots \sqcap A}_{k\,\text{times}} .$$

Then we have $A^{[k]} = \bigwedge^k A$*, where* $\bigwedge^k A$ *is defined as in* (67)*.*

For any $A : \wedge^p V \to \wedge^p V$ and $B : \wedge^q V \to \wedge^q V$ we define the linear map

$$A * B : \wedge^{g-(p+q)} V \to \wedge^{g-(p+q)} V$$

given by the matrix

$$(A * B)^I_J = (-1)^{I+J} (A \sqcap B)^{I^c}_{J^c}, \tag{68}$$

for $I, J \in P^*_{g-(p+q)}(X_g)$. If $A_1, \ldots, A_k : V \to V$ are linear maps then the matrix of the map $A_1 * \cdots * A_k$, which we denote with the same symbol, has entries

$$(A_1 * \cdots * A_k)^I_J = (-1)^{I+J} (A_1 \sqcap \cdots \sqcap A_k)^{I^c}_{J^c}, \tag{69}$$

for $I, J \in P^*_{g-k}(X_g)$.

For $A : V \to V$ and $1 < k \leqslant g$,

$$(\underbrace{A * \cdots * A}_{k\,\text{times}})^I_J = (-1)^{I+J} (\underbrace{A \sqcap \cdots \sqcap A}_{k\,\text{times}})^{I^c}_{J^c} = (-1)^{I+J} \left(A^{[k]}\right)^{I^c}_{J^c} = \left(A^{(g-k)}\right)^I_J,$$

for $I, J \in P^*_{g-k}(X_g)$. For example

$$A^{(1)} = \underbrace{A * \cdots * A}_{g-1\,\text{times}}.$$

Lemma 4.1.3. *If* $v_1, \ldots, v_k \in V$*, then*

$$v_1 {}^t v_1 * \cdots * v_k {}^t v_k = \frac{1}{k!} (v_1 \wedge \cdots \wedge v_k)\, {}^t(v_1 \wedge \cdots \wedge v_k).$$

Proof. For any $1 \leqslant k < g$, the Hodge $*$-operator gives an isomorphism

$$*_H : \bigwedge^k V \to \bigwedge^{g-k} V.$$

If e_I is the basis in (66) then the Hodge $*$-operator is defined by

$$*_H(e_I) = \epsilon(I, I^c)\, e_{I^c}, \; I \in P^*_k(X_g),$$

where $\epsilon(I, I^c)$ is the sign of the permutation that turns the set $I \cup I^c$ into the set X_g.

Define A as the matrix whose i-th row is the vector v_i:

$$A = \begin{pmatrix} (v_1)_1 & \ldots & (v_1)_g \\ \vdots & & \vdots \\ (v_k)_1 & \ldots & (v_k)_g \end{pmatrix}.$$

With respect to the basis $\{*_H(e_I)\}_{I\in P^*_k(X_g)}$ the coordinates of the vector $v_1\wedge\cdots\wedge v_k$ are the following

$$(v_1\wedge\cdots\wedge v_k)_J = \epsilon(J,J^c)\det(A_{J^c}),\quad J\in P^*_{g-k}(X_g),$$

where A_{J^c} is the matrix obtained by A by taking columns in J^c.

Let $V_i = v_i{}^t v_i$. A simple computation shows that $\epsilon(I,I^c)\epsilon(J,J^c) = (-1)^{I+J}$, hence by Lemma 4.1.1 it is enough to prove that for I, $J\in P^*_k(X_g)$

$$\sum_{\sigma\in S_k}\epsilon(\sigma)\det(\mathbf{V}_\sigma) = \det(A_I)\,\det(A_J), \tag{70}$$

where

$$\mathbf{V}_\sigma = (V_1(I,j_{\sigma(1)})\mid\cdots\mid V_k(I,j_{\sigma(k)})).$$

Identity (70) easily follows by the fact that

$$V_h(I,j_{\sigma(h)}) = (v_h)_{j_{\sigma(h)}}{}^t(A(h,I)).$$

$\square$

4.1.2 *Vector-valued modular forms from singular scalar-valued modular forms*

In this section we will work with scalar-valued modular forms with trivial multiplier system in order to ease notations. Nevertheless the same arguments work for scalar-valued modular forms with some non-trivial multiplier system with few changes. We will see an example of this in Section 4.1.3.

For f, $h\in[\Gamma,k/2]$ let

$$A_{f,h} = f^2\,\partial\left(\frac{h}{f}\right) = f(\partial h) - (\partial f)h,$$

where $\partial := (\partial_{ij})$ with $\partial_{ij} = \frac{1+\delta_{ij}}{2}\partial_{\tau_{ij}}$. Then $A_{f,h}$ is a vector-valued modular form with respect to the group Γ and the representation $\det^k\otimes\mathrm{Sym}^2(\mathbb{C}^g)$ of highest weight $(k+2,k,\dots,k)$. More explicitly for any $\tau\in\mathbb{H}_g$ and any $\gamma\in\Gamma$ it holds that

$$A_{f,h}(\gamma\cdot\tau) = \det(C\tau+D)^k\,(C\tau+D)\,A_{f,h}(\tau)\,{}^t(C\tau+D).$$

We will be interested in suitable products of this kind of vector-valued modular forms when f and h are weight $1/2$ scalar-valued modular forms. If we let

$$\rho_k = (k+2,\dots,k+2,k,\dots,k)$$

with $\text{co-rank}(\rho_k) = g-k$, then $A_{f,h}\in[\Gamma,\rho_1]$ if f, $h\in[\Gamma,1/2]$.

Proposition 4.1.4. *If* $A_1, \dots, A_k \in [\Gamma, \rho_1]$ *then*

$$A_1 * \cdots * A_k \in [\Gamma, \rho_k],$$

where $*$ *is defined as in* (69).

Proof. By definition

$$(A_1 * \cdots * A_k)(\gamma \cdot \tau) = \big(\rho_1(C\tau + D)A_1(\tau)\big) * \cdots * \big(\rho_1(C\tau + D)A_k(\tau)\big).$$

So we need to prove that

$$\big(\rho_1(C\tau + D)A_1(\tau)\big) * \cdots * \big(\rho_1(C\tau + D)A_k(\tau)\big) = \rho_k(C\tau + D)(A_1 * \cdots * A_k)(\tau).$$

It is enough to check the transformation rule for vector-valued modular forms of a given type. Let $v_i : \mathbb{H}_g \to V$ be such that

$$v_i(\gamma \cdot \tau) = \det(C\tau + D)^{1/2}\,(C\tau + D)\,v_i(\tau),\ \forall \gamma \in \Gamma,$$

then $v_i{}^t v_i \in [\Gamma, \rho_1]$. By [45] we have that

$$(v_1 \wedge \cdots \wedge v_k)\,{}^t(v_1 \wedge \cdots \wedge v_k) \in [\Gamma, \rho_k].$$

The thesis then follows by Proposition 4.1.3. □

By Proposition 4.1.4 it easily follows that if f_i, $h_i \in [\Gamma, 1/2]$, $i = 1, \dots, k$, then

$$A_{f_1, h_1} * \cdots * A_{f_k, h_k}$$

is a vector-valued Siegel modular form with respect to the irreducible representation ρ_k.

We will show that these vector-valued modular forms are related to a generalization of the pairing defined in [12].

For any $1 \leqslant k \leqslant g$ and f, $h \in [\Gamma, k/2]$ we define the pairings

$$\{f, h\}_k = \sum_{p=0}^{k} (-1)^p\, \partial^{[p]} f \sqcap \partial^{[k-p]} h,$$

$$[f, h]_k = \sum_{p=0}^{k} (-1)^p\, \partial^{[p]} f * \partial^{[k-p]} h,$$

Where $\partial := (\partial_{ij})$ with $\partial_{ij} = \frac{1+\delta_{ij}}{2}\,\partial_{\tau_{ij}}$. Note that $\{f, h\}_1 = A_{f,h}$ and

$$[f, h]_k = (\{f, h\}_k)^{(g-k)}.$$

If $f,\, h \in [\Gamma, (g-1)/2]$, the Γ-invariant holomorphic differential form $\omega_{f,h}$ described in [12] is defined as

$$\omega_{f,h} = \{f, h\}_{g-1} \sqcap d\check{\tau} = \mathrm{Tr}([f, h]_{g-1}\, d\check{\tau}), \tag{71}$$

where $d\check{\tau}$ is the basis of $\Omega^{N-1}(\mathbb{H}_g)$ given in (53).

In what follows we will focus on modular forms of half integral weight which are products of weight 1/2 ones.

Lemma 4.1.5. *If* $f, f_1, \dots, f_l \in [\Gamma, 1/2]$, *then for* $k \in \mathbb{N}$

$$\partial^{[k]}(f_1 \cdots f_l) = \begin{cases} 0 & \textit{if} \quad k > l \\ k! \displaystyle\sum_{I=\{i_1<\dots<i_k\}} \frac{f_1 \cdots f_l}{f_{i_1} \cdots f_{i_k}} \partial f_{i_1} \sqcap \dots \sqcap \partial f_{i_k} & \textit{if} \quad 1 \leqslant k \leqslant l \end{cases},$$

and

$$\partial^{[k]} f^l = \begin{cases} 0 & \textit{if} \quad k > l \\ l(l-1) \cdots (l-k+1) f^{l-k} (\partial f)^{[k]} & \textit{if} \quad 1 \leqslant k \leqslant l \end{cases}.$$

Proof. The product $\sqcap$ is bilinear, commutative, associative and distributive with respect to the sum of matrices (cf. [12]). Hence for $A : \wedge^p V \to \wedge^p V$, $B : \wedge^q V \to \wedge^q V$, the following formula holds:

$$(A+B)^{[k]} = \sum_{j=0}^{k} \binom{k}{j} A^{[j]} \sqcap B^{[k-j]}.$$

From this, it easily follows that for $f \in [\Gamma, k]$ and $h \in [\Gamma, l]$ and for every $1 \leqslant p \leqslant g$

$$\partial^{[p]}(fh) = \sum_{j=0}^{p} \binom{p}{j} \partial^{[j]} f \sqcap \partial^{[p-j]} h. \tag{72}$$

Note that here the terms for which $j >$ rank(f) or $p - j >$ rank(h) vanish by (17). So the thesis follows by Lemma 2.3.1 and formula (72). ☐

Proposition 4.1.6. *Let* $1 \leqslant k < g$. *If* $f_i,\, h_i \in [\Gamma, 1/2]$, $i = 1, \dots, k$ *then*

$$[f_1 \cdots f_k, h_1 \cdots h_k]_k = \sum_{\sigma \in S_k} A_{f_1, h_{\sigma(1)}} * \dots * A_{f_k, h_{\sigma(k)}},$$

where S_k *is the group of permutations on* k *elements.*

Proof. It is enough to prove that

$$\{f_1 \cdots f_k, h_1 \cdots h_k\}_k = \sum_{\sigma \in S_k} A_{f_1, h_{\sigma(1)}} \sqcap \dots \sqcap A_{f_k, h_{\sigma(k)}}.$$

If $I = \{i_1, \dots, i_p\} \in P^*_p(X_k)$, with $p \leqslant k$, and $\sigma \in S_p$ denote by $\sigma(I) = \{i_{\sigma(1)}, \dots, i_{\sigma(p)}\}$. Moreover, denote by $f_I = f_{i_1} \cdots f_{i_p}$ and by $\partial f_I = \partial f_{i_1} \sqcap \cdots \sqcap \partial f_{i_p}$. Then by Lemma 4.1.5 we have

$$\partial^{[p]}(f_1 \cdots f_k) = p! \sum_{I \in P^*_p(X_k)} f_{I^c}\, \partial f_I.$$

Since the product $\sqcap$ is bilinear it holds that

$$A_{f_1, h_1} \sqcap \cdots \sqcap A_{f_k, h_k} = \sum_{p=0}^{k} (-1)^p \sum_{I \in P^*_p(X_k)} h_I\, f_{I^c}\, (\partial f_I \sqcap \partial h_{I^c}).$$

Then

$$\begin{aligned} \sum_{\sigma \in S_k} A_{f_1, h_{\sigma(1)}} \sqcap \cdots \sqcap A_{f_k, h_{\sigma(k)}} &= \sum_{p=0}^{k} (-1)^p \sum_{I \in P^*_p(X_k)} f_{I^c} \partial f_I \sqcap \left(\sum_{\sigma \in S_k} h_{\sigma(I)}\, \partial h_{\sigma(I^c)} \right) = \\ &= \sum_{p=0}^{k} (-1)^p \sum_{I \in P^*_p(X_k)} f_{I^c} \partial f_I \sqcap \left(p!(k-p)! \sum_{J \in P^*_p(X_k)} h_J\, \partial h_{J^c} \right) = \\ &= \sum_{p=0}^{k} (-1)^p\, \partial^{[p]}(f_1 \cdots f_k) \sqcap \partial^{[k-p]}(h_1 \cdots h_k). \end{aligned}$$

□

Corollary 4.1.7. *Let* $1 \leqslant k < g$. *If* f, $h \in [\Gamma, 1/2]$ *then*

$$[f^k, h^k]_k = k!(A_{f,h})^{(g-k)}.$$

As a consequence, if $f = F^{g-1}$ and $h = H^{g-1}$ for $F, H \in [\Gamma, 1/2]$ it easily follows that

$$\omega_{f,h} = (g-1)!\, \mathrm{Tr}((A_{F,H})^{(1)} d\check{\tau}),$$

where $\omega_{f,h}$ is defined in (71). So we recover the result in [8, Theorem 14] and actually generalize it to every Γ-invariant holomorphic differential form (71) constructed from two singular scalar-valued Siegel modular forms of weight $(g-1)/2$ which are products of weight $1/2$ ones.

Remark 4.1.8. *For* $k = g$ *the identities in Proposition 4.1.6 and Corollary 4.1.7 still hold. The products* $f_1 \cdots f_g$ *and* $h_1 \cdots h_g$ *are no more singular modular forms and we are constructing scalar-valued Siegel modular forms of weight* $g+2$ *instead of vector-valued modular forms. In particular one of the scalar-valued Siegel modular forms we obtain is*

$$\det(A_{f,h}) = g! \sum_{p=0}^{g} (-1)^p\, \partial^{[p]}(f^g) \sqcap \partial^{[g-p]}(h^g).$$

4.1.3 *An identity of vector spaces of vector-valued modular forms*

In this section we prove that if V_Θ is the vector space generated by the vector-valued Siegel modular forms constructed with our new method applied to second order theta constants and V_{grad} is the vector space generated by the vector-valued Siegel modular forms constructed with gradients of odd theta functions as in (39), then $V_\Theta = V_{grad}$. Hence the two methods, although so different at a first look, give rise to elements of the same vector space of vector-valued Siegel modular forms.

We will first illustrate the construction of the vector-valued Siegel modular forms with the new method applied to second order theta constants. In this way we will also give examples of vector-valued Siegel modular forms constructed as in Section 4.1.2 with scalar-valued Siegel modular forms with some non-trivial multiplier system.

Recall from Section 2.4 that for any $\varepsilon \in \{0,1\}^g$ the second order theta constant with characteristic ε is defined as $\Theta[\varepsilon](\tau) = \vartheta\left[\begin{smallmatrix}\varepsilon\\0\end{smallmatrix}\right](2\tau, 0)$. For every ε, $\delta \in \{0,1\}^g$ denote by $A_{\varepsilon\,\delta} := \{\Theta[\varepsilon], \Theta[\delta]\}_1$. Then it is easy to see that

$$A_{\varepsilon\,\delta}(\gamma\cdot\tau) = \kappa(\gamma)^2 \det(C\tau + D)\,(C\tau + D)\,A_{\varepsilon\,\delta}(\tau)\,{}^t(C\tau + D),\ \forall\gamma \in \Gamma_g(2,4).$$

By this equation and Proposition 4.1.4, for $\varepsilon_1, \dots, \varepsilon_k, \delta_1, \dots, \delta_k \in \{0,1\}^g$, the vector-valued Siegel modular form $A_{\varepsilon_1\,\delta_1} * \cdots * A_{\varepsilon_k\,\delta_k}$ satisfies the following transformation formula for any $\gamma \in \Gamma_g(2,4)$

$$(A_{\varepsilon_1\,\delta_1} * \cdots * A_{\varepsilon_k\,\delta_k})(\gamma\cdot\tau) = \kappa(\gamma)^{2k}\,\rho_k(C\tau + D)(A_{\varepsilon_1\,\delta_1} * \cdots * A_{\varepsilon_k\,\delta_k})(\tau). \tag{73}$$

Note that as we said before, here we are dealing with scalar-valued Siegel modular forms with some non-trivial multiplier system that shows up in the transformation formula (73) for the vector-valued Siegel modular form we are constructing.

Since $\kappa(\gamma)^2 = 1$ for every $\gamma \in \Gamma_g(2,4)^*$, then $A_{\varepsilon_1\,\delta_1} * \cdots * A_{\varepsilon_k\,\delta_k} \in [\Gamma_g(2,4)^*, \rho_k]$ for any k. If k is even, then $A_{\varepsilon_1\,\delta_1} * \cdots * A_{\varepsilon_k\,\delta_k} \in [\Gamma_g(2,4), \rho_k]$ because by formula (26) we have that $\kappa(\gamma)^4 = 1$ for any $\gamma \in \Gamma_g(2,4)$.

Concerning gradients of odd theta functions, recall from Section 2.4 that for $N = (n_1, \dots, n_k) \in M_{2g\times k}$, where $\{n_i\}_{i=1,\dots,k}$ is a set of distinct odd characteristics, we have defined the vector-valued Siegel modular form

$$W(N)(\tau) = \pi^{-2k}\,(v_{n_1}(\tau) \wedge \ldots \wedge v_{n_k}(\tau))\,{}^t(v_{n_1}(\tau) \wedge \ldots \wedge v_{n_k}(\tau)),$$

where $v_n(\tau) := \operatorname{grad}_z \theta_n(\tau, z)|_{z=0}$. We have that $W(N) \in [\Gamma_g(2,4)^*, \rho_k]$ for any k. If k is even the modularity group is bigger, indeed in this case $W(N) \in [\Gamma_g(2,4), \rho_k]$.

A fundamental step in the proof of the identity of the vector spaces V_Θ and V_{grad} is the following Lemma that shows a consequence of the classical Riemann's addition theorem for theta functions.

Proposition 4.1.9. *If v_n is the gradient of an odd theta function with characteristic $n = [\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}]$, then*

$$v_n{}^t v_n = \pi i \sum_{\alpha \in \{0,1\}^g} (-1)^{\alpha \cdot \delta} A_{\varepsilon+\alpha\,\alpha}. \tag{74}$$

Moreover for given ε, $\delta \in \{0,1\}^g$, denote by $n_\alpha = \left[\begin{smallmatrix}\varepsilon+\delta\\ \alpha\end{smallmatrix}\right]$ for $\alpha \in \{0,1\}^g$. Then

$$4\pi i\, A_{\varepsilon\,\delta} = \frac{1}{2^{g-2}} \sum_{\substack{\alpha \in \{0,1\}^g \text{ s.t.} \\ n_\alpha \text{ odd}}} (-1)^{\delta \cdot \alpha}\, v_{n_\alpha}{}^t v_{n_\alpha}. \tag{75}$$

This proposition fits in the big subject of generalizations of Jacobi's derivative formula. For $g = 1$ the classical Jacobi identity states that

$$D\left(\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right) = -\vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\vartheta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\vartheta\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right].$$

Essentially, the problem of generalizing this formula consist in expressing the Jacobian determinant of g distinct odd theta functions as a polynomial in theta constants.

In [28] it is proven that the Jacobian determinant is always a rational function of the theta constants. The question about the possible expression as a polynomial in theta constants is more complicated. For $g = 2$ the formula is still classical and gives the following. If $n_1, \ldots, n_6$ are the six odd characteristics and $m_i = n_1 + n_2 + n_{i+2}$ for $i = 1, \ldots, 4$ then

$$D(n_1, n_2) = \pm\,\vartheta_{m_1} \cdots \vartheta_{m_4}.$$

For $g = 3$ it is known that if n_i, $i = 1, \ldots, 3$ are odd characteristics the Jacobian determinant $D(n_1, n_2, n_3)$ is a polynomial in the theta constants if and only if $n_1 + n_2 + n_3$ is an even characteristic. In higher degree there is a conjectural formula which has been proven only for $g \leqslant 5$.

Nevertheless we can say when a Jacobian determinant is not a polynomial in theta constants by looking at a condition on the characteristics involved. A set of characteristics is called essentially independent if the sum of any even number of them is not congruent to 0 mod 2. A triplet of odd characteristics is called azygetic or syzygetic if their sum is even or odd respectively. A set of odd characteristics is azygetic or syzygetic if all triples in the set are azygetic or syzygetic respectively. By [28] we know that if $n_1, \ldots, n_g$ is a set of odd characteristics which is an essentially independent syzygetic set then $D(n_1, \ldots, n_g)$ is not a polynomial in the theta constants.

A different generalization can be done by looking at higher order derivatives of theta functions. This is the direction taken in our Proposition (recall that by the heat equation (21) one has that $4\pi i \partial_{jk} = \partial_{z_j}\partial_{z_k}$).

Regarding theta series with harmonic polynomial coefficients, we have seen in Section 2.4 that the $\mathbb{C}$-span of Jacobian determinants and the $\mathbb{C}$-span of products of

$g+2$ even theta constants are spaces of theta series with suitable harmonic coefficients. To find an element in the intersection of the two spaces is then equal to find an expression of some linear combinations of Jacobian determinants as a polynomial (of the right degree since the weight of a Jacobian determinant is $g/2+1$) in theta constants. The generalization in the Proposition also fits in this setting. For example in degree 1 we are working with theta series with harmonic polynomial coefficient x^2-y^2 (cf. [22]).

Proof of Proposition 4.1.9. We will follow the proof given in [23]. A spacial case of Riemann's addition theorem for theta functions gives the following formula (cf. [27]):

$$\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](2\tau,2z)\,\vartheta\left[\begin{smallmatrix}\alpha+\varepsilon\\ \beta\end{smallmatrix}\right](2\tau,2x)=\frac{1}{2^g}\sum_{\sigma\in\{0,1\}^g}(-1)^{\alpha\cdot\sigma}\vartheta\left[\begin{smallmatrix}\varepsilon\\ \beta+\sigma\end{smallmatrix}\right](\tau,z+x)\,\vartheta\left[\begin{smallmatrix}\varepsilon\\ \sigma\end{smallmatrix}\right](\tau,z-x),\quad (76)$$

for $\tau\in\mathbb{H}_g$, $z,x\in\mathbb{C}^g$ and $\alpha,\beta,\varepsilon\in\{0,1\}^g$.

Denote by $\mathbf{C}^{\beta}_{\varepsilon\,\delta}(\tau)$ the $g\times g$ matrix with entries

$$\left(\mathbf{C}^{\beta}_{\varepsilon\,\delta}(\tau)\right)^i_j=\partial_{z_i}\vartheta\left[\begin{smallmatrix}\varepsilon\\ \beta+\delta\end{smallmatrix}\right](\tau,0)\,\partial_{z_j}\vartheta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau,0)+\partial_{z_j}\vartheta\left[\begin{smallmatrix}\varepsilon\\ \beta+\delta\end{smallmatrix}\right](\tau,0)\partial_{z_i}\vartheta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau,0),$$

and by $\mathbf{A}^{\beta}_{\varepsilon\,\delta}(\tau)$ the $g\times g$ matrix with entries

$$\left(\mathbf{A}^{\beta}_{\varepsilon\,\delta}(\tau)\right)^i_j=\left(\partial_{z_i}\partial_{z_j}\vartheta\left[\begin{smallmatrix}\delta\\ \beta\end{smallmatrix}\right](2\tau,0)\right)\vartheta\left[\begin{smallmatrix}\varepsilon\\ \beta\end{smallmatrix}\right](2\tau,0)-\vartheta\left[\begin{smallmatrix}\delta\\ \beta\end{smallmatrix}\right](2\tau,0)\left(\partial_{z_i}\partial_{z_j}\vartheta\left[\begin{smallmatrix}\varepsilon\\ \beta\end{smallmatrix}\right](2\tau,0)\right).\quad (77)$$

Note that $\mathbf{A}^{0}_{\varepsilon\,\delta}=4\pi i\,A_{\varepsilon\,\delta}$. Clearly $\mathbf{C}^{\beta}_{\varepsilon\,\delta}(\tau)=0$ unless the characteristics $\left[\begin{smallmatrix}\varepsilon\\ \beta+\delta\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right]$ are odd and $\mathbf{A}^{\beta}_{\varepsilon\,\delta}(\tau)=0$ unless $\left[\begin{smallmatrix}\delta\\ \beta\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\varepsilon\\ \beta\end{smallmatrix}\right]$ are even characteristics.

We are going to prove that if $\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\varepsilon\\ \beta+\delta\end{smallmatrix}\right]$ are odd characteristics then

$$\mathbf{C}^{\beta}_{\varepsilon\,\delta}=\frac{1}{2}\sum_{\alpha\in\{0,1\}^g}(-1)^{\alpha\cdot\delta}\,\mathbf{A}^{\beta}_{\varepsilon+\alpha\,\alpha}.\quad (78)$$

Fix $\delta\in\{0,1\}^g$ and take the sum of the equations (76) for $\alpha\in\{0,1\}^g$ each with coefficient $(-1)^{\alpha\cdot\delta}$. Hence we get

$$\begin{aligned}\sum_{\alpha\in\{0,1\}^g}&(-1)^{\alpha\cdot\delta}\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](2\tau,2z)\,\vartheta\left[\begin{smallmatrix}\alpha+\varepsilon\\ \beta\end{smallmatrix}\right](2\tau,0)=\\ &=\frac{1}{2^g}\sum_{\alpha,\sigma\in\{0,1\}^g}(-1)^{\alpha\cdot(\sigma+\delta)}\vartheta\left[\begin{smallmatrix}\varepsilon\\ \beta+\sigma\end{smallmatrix}\right](\tau,z)\,\vartheta\left[\begin{smallmatrix}\varepsilon\\ \sigma\end{smallmatrix}\right](\tau,z)=\\ &=\vartheta\left[\begin{smallmatrix}\varepsilon\\ \beta+\delta\end{smallmatrix}\right](\tau,z)\,\vartheta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau,z).\end{aligned}$$

Differentiating this relation with respect to z_i and z_j and evaluating at $z=0$ we get the identity (78).

There is also the "inverse" formula, indeed we will prove that

$$\mathbf{A}^{\beta}_{\alpha+\varepsilon\,\alpha}=\frac{1}{2^{g-1}}\sum_{\substack{\sigma\in\{0,1\}^g\text{ s.t.}\\ \left[\begin{smallmatrix}\varepsilon\\ \sigma\end{smallmatrix}\right]\text{ odd}}}(-1)^{\alpha\cdot\sigma}\,\mathbf{C}^{\beta}_{\varepsilon\,\sigma}.\quad (79)$$

Assume that in (76) the characteristics $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\alpha+\varepsilon\\ \beta\end{smallmatrix}\right]$ are both even. Differentiating we get

$$\begin{aligned}&\left(\partial_{z_i}\partial_{z_j}\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](2\tau,0)\right)\vartheta\left[\begin{smallmatrix}\alpha+\varepsilon\\ \beta\end{smallmatrix}\right](2\tau,0)=\\ &\qquad=\frac{1}{2^g}\partial_{z_i}\partial_{z_j}\left(\sum_{\sigma\in\{0,1\}^g}(-1)^{\alpha\cdot\sigma}\,\vartheta\left[\begin{smallmatrix}\varepsilon\\ \beta+\sigma\end{smallmatrix}\right](\tau,z)\,\vartheta\left[\begin{smallmatrix}\varepsilon\\ \sigma\end{smallmatrix}\right](\tau,z)\right)\Bigg|_{z=0}.\end{aligned}$$

Switching α and $\alpha+\varepsilon$ we get

$$\begin{aligned}&\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](2\tau,0)\left(\partial_{z_i}\partial_{z_j}\vartheta\left[\begin{smallmatrix}\alpha+\varepsilon\\ \beta\end{smallmatrix}\right](2\tau,0)\right)=\\ &\qquad=\frac{1}{2^g}\partial_{z_i}\partial_{z_j}\left(\sum_{\sigma\in\{0,1\}^g}(-1)^{(\alpha+\varepsilon)\cdot\sigma}\,\vartheta\left[\begin{smallmatrix}\varepsilon\\ \beta+\sigma\end{smallmatrix}\right](\tau,z)\,\vartheta\left[\begin{smallmatrix}\varepsilon\\ \sigma\end{smallmatrix}\right](\tau,z)\right)\Bigg|_{z=0}.\end{aligned}$$

The identity (79) now follows by subtracting and computing separately for the cases $\left[\begin{smallmatrix}\varepsilon\\ \sigma\end{smallmatrix}\right]$ odd and even.

If $n=\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right]$ then $2v_n{}^tv_n=\mathbf{C}^0_{\varepsilon\,\delta}$. Thus by the heat equation (21) and taking $\beta=0$ in (78) and (79) we get the identity (74) and (75) respectively. □

Now we can establish our result about the identity of vector spaces of vector-valued Siegel modular forms.

Theorem 4.1.10. *Denote by V_{grad} the vector space of vector-valued Siegel modular forms generated by the modular forms $W(N)$, where N is a matrix of k distinct odd characteristics. Denote also by V_Θ the vector space of vector-valued Siegel modular forms generated by the modular forms $A_{\varepsilon_1\,\delta_1}*\cdots*A_{\varepsilon_k\,\delta_k}$ where $\varepsilon_1,\dots,\varepsilon_k,\delta_1,\dots,\delta_k\in\{0,1\}^g$. Then for any $1\leqslant k<g$ one has the identity of vector spaces*

$$V_\Theta=V_{grad}.$$

Proof. We will prove that each vector-valued Siegel modular form $A_{\varepsilon_1\,\delta_1}*\cdots*A_{\varepsilon_k\,\delta_k}$ for some $\varepsilon_i,\ \delta_i\in\{0,1\}^g$, $i=1,\dots,k$ is in V_{grad}. By formula (75) we have that

$$4\pi i\,A_{\varepsilon_i\,\delta_i}=\frac{1}{2^{g-2}}\sum_{\substack{\alpha\in\{0,1\}^g\text{ s.t.}\\ n^i_\alpha\text{ odd}}}(-1)^{\delta\cdot\alpha}\,v_{n^i_\alpha}{}^tv_{n^i_\alpha},$$

where $n^i_\alpha=\left[\begin{smallmatrix}\varepsilon_i+\delta_i\\ \alpha\end{smallmatrix}\right]$, for $i=1,\dots,k$. By the linearity of the product $*$ and by applying Lemma 4.1.3 we see that there exists a computable constant c such that

$$A_{\varepsilon_1\,\delta_1}*\cdots*A_{\varepsilon_k\,\delta_k}=c\sum_{\substack{\alpha_i\in\{0,1\}^g\text{ s.t.}\\ [\varepsilon_i+\delta_i,\alpha_i]\text{ odd}}}(-1)^{\sum_i\delta_i\alpha_i}\,W([\varepsilon_1+\delta_1,\ \alpha_1],\dots,[\varepsilon_k+\delta_k,\ \alpha_k]).$$

On the other hand, by Lemma 4.1.3 and equation (74) it is also easy to prove that each vector-valued Siegel modular form $W(N)$ is in V_Θ. This completes the proof. □

By means of the vector-valued modular forms presented in the previous section we give a new characterization of decomposable principally polarized abelian varieties. In particular we will characterize them in terms of the image of the Gauss map for their theta divisor. This is part of my joint work [8].

For (X_1, L_1) and (X_2, L_2) two polarized abelian varieties denote by $p_i : X_1 \times X_2 \to X_i$ the projection on the i-th factor for $i = 1, 2$. The line bundle $L_1 \boxtimes L_2 := p_1^* L_1 \otimes p_2^* L_2$ defines a polarization on $X_1 \times X_2$. The new abelian variety $(X_1 \times X_2, L_1 \boxtimes L_2)$ is called a product abelian variety with a product polarization.

An abelian variety is called indecomposable if it is not a product abelian variety. Unlike the case of complex tori the indecomposable factors of a decomposable polarized abelian variety are well determined. Let (X, L) be a polarized abelian variety containing abelian subvarieties $(X_1, L_1), \dots, (X_r, L_r)$ and $(Y_1, M_1), \dots, (Y_s, M_s)$ such that

$$X \simeq \left(\prod_{i=1}^{r} X_i, L_1 \boxtimes \cdots \boxtimes L_r \right) \simeq \left(\prod_{j=1}^{s} Y_j, M_1 \boxtimes \cdots \boxtimes M_s \right).$$

If (X_i, L_i), $i = 1, \dots, r$, and (Y_j, M_j), $j = 1, \dots, s$ are indecomposable, then $r = s$ and there exists a permutation σ of $\{1, \dots, s\}$ such that $Y_j = X_{\sigma(j)}$ (cf. [9]).

If X_τ is the principally polarized abelian variety associated to a point $\tau \in \mathbb{H}_g$ (see Section 1.3.1), it is decomposable if and only if there exists $\gamma \in \Gamma_g$ such that

$$\gamma \cdot \tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix},$$

with $\tau_i \in \mathbb{H}_{g_i}$, for some g_1 and g_2 such that $g_1 + g_2 = g$.

Also we can describe the indecomposable factors of an abelian variety X_τ in terms of the irreducible components of the theta divisor

$$\Theta_\tau = \{ z \in X_\tau \mid \vartheta_0(\tau, z) = 0 \}. \tag{80}$$

Denote by $L_\tau = \mathcal{O}(\Theta_\tau)$ the principal polarization defined by the divisor Θ_τ. If $\Theta_1, \dots, \Theta_r$ are the irreducible components of the divisor Θ_τ, let $L_i = \mathcal{O}(\Theta_i)$, $i = 1, \dots, r$. Denote by $H(L_i)$ the kernel of the map ϕ_{L_i} defined as in (10) and by $H(L_i)_0$ the connected component of the identity. Let $\overline{X}_i$ be the complex subtorus $X_\tau / H(L_i)_0$ and denote by $p_i : X_\tau \to \overline{X}_i$ the natural surjection. Then by [9, Corollary 9.4] for any $1 \leqslant i \leqslant r$, there exists a principal polarization $\overline{L}_i$ on $\overline{X}_i$ such that $p_i^* \overline{L}_i \simeq L_i$ and

$$(X_\tau, L_\tau) \simeq \left(\prod_{i=1}^{r} \overline{X}_i, \overline{L}_1 \boxtimes \cdots \boxtimes \overline{L}_r \right).$$

The analytic characterization of the indecomposable principally polarized abelian varieties is well known:

Theorem 4.2.1 ([49],[48]). *Set* $N = g(g+1)/2$ *and define the* $(N+1) \times 2^g$ *matrix* $\mathcal{L}(\tau)$ *as*

$$\mathcal{L}(\tau) := \begin{pmatrix} \Theta[\sigma] \\ \partial_{z_1}\partial_{z_1}\Theta[\sigma] \\ \vdots \\ \partial_{z_i}\partial_{z_j}\Theta[\sigma] \\ \vdots \\ \partial_{z_g}\partial_{z_g}\Theta[\sigma] \end{pmatrix},$$

with entries taken for all $\sigma \in \{0,1\}^g$ *and for all* $1 \leqslant i \leqslant j \leqslant g$. *A principally polarized abelian variety is indecomposable if and only if* $\mathcal{L}(\tau)$ *has maximal rank, i.e. rank equal to* $N+1$.

A remarkable property of the matrix $\mathcal{L}(\tau)$ is given in the following theorem.

Theorem 4.2.2 ([49]).

$$\operatorname{rank}\mathcal{L}(\tau) = \operatorname{rank}\mathcal{L}(\gamma \cdot \tau),\ \forall \gamma \in \Gamma_g.$$

4.2.1 *A new characterization of decomposable principally polarized abelian varieties*

Let (X, L) be an abelian variety of dimension g. For a reduced divisor $D \in \mathbb{P}(H^0(L))^\vee$ denote by D_s the smooth part of the support of D. For every $x \in D_s$ the tangent space $T_{D,x}$ is a $(g-1)$-dimensional vector space and its translation to zero is a well defined $(g-1)$-dimensional vector subspace of the tangent space of X at 0. We will denote this tangent space by V. If we identify the space of global sections of L as the space of theta functions for a factor of automorphy, there is a theta function $\theta \in H^0(L)$, uniquely determined up to a constant, such that $\pi^*D = (\theta)$. If $w_1, \ldots, w_g$ are the coordinate functions with respect to some complex basis of V, the equation of the tangent space $T_{D,x}$ at a point $x \in D$ is

$$\sum_{i=1}^{g} \frac{\partial\theta}{\partial w_i}(x)(w_i - x_i) = 0.$$

So the 1-dimensional subspace of the dual vector space $V^\vee$ determined by $T_{D,x}$ is generated by the vector $\left(\frac{\partial\theta}{\partial w_1}(x), \ldots, \frac{\partial\theta}{\partial w_g}(x)\right)$. The Gauss map of D is then defined as

$$\begin{aligned} G : D_s &\to \mathbb{P}^{g-1} = \mathbb{P}(V^\vee) \\ x &\mapsto \left[\frac{\partial\theta}{\partial w_1}(x), \ldots, \frac{\partial\theta}{\partial w_g}(x)\right] \end{aligned}$$

The Gauss map is holomorphic. It neither depends on the choice of θ nor on the choice of a factor of automorphy for L. If the divisor D is irreducible then the map is dominant.

We will study the Gauss map of the theta divisor of a principally polarized abelian variety in order to give a new characterization of the locus of reducible abelian varieties. For any $\tau \in \mathbb{H}_g$ let $X_\tau = \mathbb{C}^g/\tau\mathbb{Z}^g \otimes \mathbb{Z}^g$ be the principally polarized abelian variety associated to τ and consider the Gauss map G_τ for the theta divisor Θ_τ (see (80)).

In particular we will look at the images of the smooth 2-torsion points on the theta divisor. For $m \in \{0,1\}^{2g}$ the point $x_m = \tau \frac{m'}{2} + \frac{m''}{2}$ is a 2-torsion point of X_τ. Since

$$\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau, z + x_m) = \lambda(z) \vartheta \begin{bmatrix} m' \\ m'' \end{bmatrix} (\tau, z),$$

for some nowhere vanishing function λ, the 2-torsion points x_m on the theta divisor are in bijection with the set of the theta constants $\vartheta_m(\tau)$ vanishing at τ. If m is odd, a theta constant vanishes for any τ so at least the 2-torsion points x_m corresponding to odd m are on the theta divisor. There could be more if the matrix τ is a zero of some even theta constant but the theta divisor cannot contain all the 2-torsion points (cf. [27]). However, these "even" torsion points would not be smooth points of the theta divisor.

Theorem 4.2.3. *A principally polarized abelian variety X_τ is decomposable if and only if the images under the Gauss map G_τ of all smooth 2-torsion points in the theta divisor lie on a quadric in $\mathbb{P}^{g-1}$.*

Proof. If $\tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$, with $\tau_i \in \mathbb{H}_{g_i}$ for $g_1 + g_2 = g$, then it defines a reducible abelian variety

$$(X_\tau, L_\tau) \simeq (X_{\tau_1} \times X_{\tau_2}, L_{\tau_1} \boxtimes L_{\tau_2}).$$

It is well known that in this case any theta function with characteristic splits as a product

$$\theta_m(\tau, z) = \theta_{m_1}(\tau_1, z_1) \cdot \theta_{m_2}(\tau_2, z_2),$$

where $z_i \in \mathbb{C}^{g_i}$ and $m' = \begin{bmatrix} m_1' \\ m_2' \end{bmatrix}$ and $m'' = \begin{bmatrix} m_1'' \\ m_2'' \end{bmatrix}$ with $m_i', m_i'' \in \{0,1\}^{2g_i}$, $i = 1,2$. If $v_m(\tau) := \operatorname{grad}_z \theta_m(\tau, z)|_{z=0}$ for an odd characteristic m, then

$$v_m(\tau) = \Big(v_{m_1}(\tau_1) \cdot \theta_{m_2}(\tau_2, 0),\ \theta_{m_1}(\tau_1, 0) \cdot v_{m_2}(\tau_2) \Big)$$

Since m is odd, precisely one of m_1 and m_2 is odd, and thus only the corresponding g_i entries of the gradient vector are non-zero. Thus if we arrange the gradients for all odd m in a matrix, it will have a block form, with the two non-zero blocks of sizes $g_i \times 2^{g_i - 1}(2^{g_i} - 1)$, and two "off-diagonal" zero blocks. This is just equal to say that the set of gradients of all odd theta functions at a point τ as above lies in the product of coordinate linear spaces $\mathbb{C}^{g_1} \cup \mathbb{C}^{g_2} \subset \mathbb{C}^g$. Since $\operatorname{grad}_z \theta_m(\tau, z)|_{z=0}$ and $\operatorname{grad}_z \theta_0(\tau, z)|_{z=x_m}$ differ by a constant factor and thus give the same point in $\mathbb{P}^{g-1}$, this implies that the images of all the smooth 2-torsion points of Θ_τ under G_τ lie on $g_1 g_2$ reducible quadrics in $\mathbb{P}^{g-1}$ written explicitly as

$$X_i X_j = 0, \qquad \forall\, 1 \leqslant i \leqslant g_1 < j \leqslant g.$$

This is equivalent to these Gauss images all lying on a union of two hyperplanes, and a weaker condition is that they all lie on some quadric (not necessarily a reducible one).

In general if a principally polarized abelian variety is decomposable, its period matrix does not need to have this block shape, and would rather be conjugate to it under Γ_g. Since $v_m(\tau)$ are vector-valued modular forms for the representation $\det^{1/2} \otimes \mathrm{std}$ (see (38)), they transform linearly under the group action, and hence the condition that the images of the odd 2-torsion points under the Gauss map lie on a quadric is preserved under the action of Γ_g. Thus for any decomposable principally polarized abelian variety the images of all smooth 2-torsion points lying on Θ_τ are contained in (many) quadrics.

For the other direction of the theorem we manipulate the gradients to reduce to the characterization of the locus of decomposable principally polarized abelian varieties given by theorem 4.2.1. Indeed, suppose all images of the odd 2-torsion points x_m lie on a quadric with homogeneous equation $Q(x_1, \dots, x_g)$. If B is the matrix of coefficients of Q, then

$$Q(v_m) = v_m^t B v_m = 0$$

for all x_m that are smooth points of Θ_τ. We thus have

$$\operatorname{Tr}(v_m^t B v_m) = \operatorname{Tr}(B v_m v_m^t) = 0$$

for all odd m. Since by (75)

$$v_m{}^t v_m = \frac{1}{4} \sum_{\alpha \in \{0,1\}^g} (-1)^{\alpha \cdot \delta}\, \mathbf{A}_{\varepsilon+\alpha\, \alpha},$$

where $\mathbf{A}_{\varepsilon+\alpha\,\alpha} := \mathbf{A}^0_{\varepsilon+\alpha\,\alpha}$ (see (77)), we also have

$$\operatorname{Tr}(B \mathbf{A}_{\alpha\,\beta}) = 0$$

for all $\alpha, \beta \in \{0,1\}^g$. In particular this implies that the matrix

$$\mathbb{A} := (\mathbb{A}_{\alpha\,\beta})_{\alpha \neq \beta \in \{0,1\}^g},$$

where each $\mathbb{A}_{\alpha\,\beta}$ is a column-vector in $\mathbb{C}^{g(g+1)/2}$, is degenerate.

We will show that the matrix $\mathbb{A}(\tau)$ has non-maximal rank if and only if the matrix $\mathcal{L}(\tau)$ in theorem 4.2.1 has non-maximal rank. This will complete the proof of the theorem.

For $1 \leqslant i \leqslant j \leqslant g$, we denote by $\mathcal{L}_{ij}$ and $\mathbb{A}_{ij}$ the (i, j) rows of the matrices $\mathcal{L}(\tau)$ and $\mathbb{A}(\tau)$ respectively. Denote by $\mathcal{L}_0$ the first row of $\mathcal{L}(\tau)$. We then have

$$\mathcal{L}_0 \wedge \mathcal{L}_{ij} = \mathbb{A}_{ij}$$

where by the wedge we mean taking the row vector whose entries are all two by two minors of the matrix formed by the two row vectors $\mathcal{L}_0$ and $\mathcal{L}_{ij}$. If the vectors $\mathbb{A}_{\alpha\beta}$ are linearly dependent, this means we have some linear relation $0 = \sum a_{ij}\mathbb{A}_{ij}$ among the rows of $\mathbb{A}(\tau)$, which is equivalent to

$$0 = \sum_{i,j} a_{ij}(\mathcal{L}_0 \wedge \mathcal{L}_{ij}) = \mathcal{L}_0 \wedge \left(\sum_{i,j} a_{ij}\mathcal{L}_{ij} \right)$$

and thus $\mathcal{L}_0$ must be proportional to $\sum a_{ij}\mathcal{L}_{ij}$, so that the matrix $\mathcal{L}$ does not have maximal rank. $\square$

The proof above shows that in fact a quadric in $\mathbb{P}^{g-1}$ contains the Gauss images of the 2-torsion points on the theta divisor if and only if it contains the entire image of the Gauss map.

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