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Evolution of microstructures for a damage model

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Evolution of microstructures for a damage model

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Abstract

This thesis is devoted to the analytical study of rate-independent damage process in physically linearly two-phases elastic materials. The study is done through an energetic approach with particular attention to threshold properties. The evolution of the systems is driven by an external force $f(t)$ or by a time-dependent boundary conditions $g(t)$ that can produce deformations and damage of the material, which physically means a weakening of elastic properties. We associate to the system an energy $\mathcal{E}$, which consists in the elastic internal stored energy of the material and the amount of energy generated by external loadings (with an inertial term in last chapter), and a dissipation $\mathcal{D}$, which represents the amount of energy dissipated when changing from a damage configuration to another. The notion of solution that we consider is the *quasi-static evolution* (q.s.e.) which means a couple deformation-damage that for each time satisfies a minimality condition among all the admissible competitors, an energy balance and an irreversibility property of the damage.
The main interesting point related to the considered model is given to the possibility to combining efficiently quasi-static evolution with non classical problem as homogenization, both in the (quasi)static case and in the dynamic framework. Moreover for this model we can also give two different natural notion of evolutions, based on an energetic criterion and on a threshold one.

The main original results of the thesis are the following:

- *Quasi-static damage evolution and homogenization, existence and convergence results.*
  We consider a family of q.s.e. for oscillating energy E^ε and dissipation D^ε describing a mixture of two one-dimensional elastic two-phase materials. We show that this family converges to a q.s.e. related to the Γ-limit of $E^\varepsilon + D^\varepsilon$ (which is shown to be different from the sum of the Γ-limits, but nevertheless interpretable as sum of an energy F_{hom} and dissipation D_{hom}). Moreover we characterize the limit relaxed evolution as the one corresponding to a double-damage material (i.e. homogeneous material with two possibility of damaged states and related dissipations). This results contribute to the analysis of interaction between Γ-convergence and q.s.e..
  This is treated in Chapter 2.

- *Quasi-static damage evolution for a perimeter-regularized energy, convergence result.*
  We consider the energy and dissipation for a linear two-phase isotropic elastic material with penalization given by the perimeter of the damage. We show that in case that the penalization goes to zero, the q.s.e. of the system converges to a q.s.e. for the relaxed energy according to the definition given in [43], and moreover that some threshold properties are satisfied. This results contribute to the study of the interplay between q.s.e. and homogenization and show the stability of the threshold solution defined in [43] w.r.t. singularity perturbations.
  This is treated in Chapter 3.

- *Dynamic evolution of the damage through energetic and threshold approach.*
  We consider the energy and dissipation for a linear two-phase isotropic elastic material for discrete time with a kinetic (inertial) term given by discretization of the second derivative (in time) of the deformation. We show that the limit (in the time step) of minimizing sequences satisfies the monotonicity of the damage, an energy inequality, a threshold condition and a "relaxed" elasto-dynamic equation. Moreover starting directly from the momentum equation for the system and considering damage obtained, step by step, through a threshold condition, we show that, also with this approach, the limit solution of the equation and the limit of the damage satisfy a "relaxed" elasto-dynamic equation. We stress that there is no viscosity term in the model which usually helps for the limit passages.
  This results contribute to the study of the homogenization process in a dynamic framework.
  This is treated in Chapter 4.

Contents

Introduction

In nature when an external force acts on a material it can produce some modifications to the state of equilibrium of it, producing effects as damage, fracture, phase transitions or plasticity. Our interest is in the damaging of an elastic material, where elastic means that when the external forces disappear the body goes back to its original position without maintain permanent deformation. Usually we can think at damaging as a process that modifies the state of a region of the body weakening its properties but without producing a macroscopic fracture on it. We can imagine that fractures can happen at a microscopic level breaking connections between different particles but at a macroscopic level if these microscopic fractures are not so frequent we will see in that region just a weakening of the material. The interest in these problems is to describe how the damage (or the fracture or other relevant properties of the material) changes during the time.
Historically the necessity of a damage theory came from an engineering environment and started as phenomenological observations (see [48], [54] and [55]) to try to describe (and hopefully) predict the evolution of the damage as tool for mechanical engineering applications. In the framework of the damage of an elastic material it is postulate that there exists a space-time internal scalar (or vectorial) variable of the model $\theta(x,t)$ which usually corresponds to a characteristic function or a density function of the damaged set. It describes the 'amount' of damage at the point $x \in \Omega \subset \mathbb{R}^n$ (with $n = 1,2,3$) of the material (occupying the bounded region Ω) at the time $t \geq 0$ and it is considered intrinsically part of the system through the tensor $A(x,t)$ according to a constitutive relation

$$A(x,t) = \hat{A}(\theta(x,t)).$$

The choice of the function $\hat{A}(\theta)$ is a matter of great debate and can take different forms depending on the assumption made on the system although it must reflect the physical idea that greater is the damage (i.e. increasing θ) then weaker is the material (i.e. decreasing is $\hat{A}$). In the case of *small* strains of the material the response of the material is assumed to follow the linear Hooke's law which leads to the linear elastic internal energy density

$$W(e) = \frac{1}{2} \sum_{i,j,k,l}^{n} A_{ijkl} e_{ij} e_{kj} \tag{0.0.1}$$

where $A \in \mathcal{F}(\alpha, \beta)$ which is the set of the 4-th order bounded symmetric tensors (see (1.2.15) for scalar definition), and $e(u)$ denotes the linearized strain tensor

$$e(u) := \frac{(\nabla u)^T + (\nabla u)}{2}$$

with

$$u : \Omega \longrightarrow \mathbb{R}^m, \quad m = 1,2,3,$$

the displacement of the material.

From thermodynamics it is classical (see [44]) to write a constitutive law that relates the thermodynamic forces

$$\mathcal{F} := -\frac{\partial}{\partial \theta} W(e, \theta)$$

to a dissipation potential $\mathcal{D} = \mathcal{D}(\dot{\theta}(t))$ (which represents the energy density dissipated when changing from a damage configuration to another because of external loading) by the subdifferential relation

$$F(t) \in \partial \mathcal{D}(\dot{\theta}(t))$$

for each point of the domain $\Omega \subset \mathbb{R}^n$.
The processes in which we are interested are the ones called *rate-independent* processes namely the processes such that with a reparametrization of the time we obtain just the time-reparametrized evolution of the damage and deformation, i.e., the evolution doesn't depend on the *velocity* of the external loadings . This physical property is mathematically expressed requiring that the dissipation potential is 1-homogeneous in the **velocity** of the damage variable, i.e.

$$\mathcal{D}(\alpha \dot{\theta}(t)) = \alpha \mathcal{D}(\dot{\theta}(t)) \quad \text{if} \quad \dot{\theta}(t) \geq 0, \tag{0.0.2}$$

with $\alpha > 0$, while, imposing also that

$$\mathcal{D}(\dot{\theta}(t)) = \infty \quad \text{if} \quad \dot{\theta}(t) < 0 \tag{0.0.3}$$

we recover the physical non-decreasing property of the damage. A way to study the evolution of the displacement and the damage of the material (due to external loadings $f(t)$ on the domain and/or to boundary condition $g(t)$) is to assume that at each time the material reaches the elastic equilibrium and so satisfies the Euler-Lagrange equation related to the energy density. In so doing the analysis of the evolution of the displacement and damage becomes to find a pair solution $(u(t), \theta(t))$ of the problem

$$\begin{cases} -div\big(\frac{\partial}{\partial e} W(e(u(t)), \theta(t))\big) = f(t) & \Omega \\ -\frac{\partial}{\partial \theta} W(e(u(t)), \theta(t)) \in \partial \mathcal{D}(\dot{\theta}(t)), & \theta(0) = \theta_0 \end{cases} \tag{0.0.4}$$

where $u(t)$ satisfies some boundary conditions (in case depending on time through a function $g(t)$). From a classical standpoint, this model requires great regularity and to be solved it is necessary to introduced some regularization in the form of the gradient of the damage variable (see [41, 39, 47, 53]). On the other hand in the last decades another powerful approach for rate-independent processes to study damage evolutions (but also other processes as fracture, plasticity and phase transition) was introduced in [66, 67] and it is based on an energetic formulation instead of a partial differential equations approach which leads to the definition of *Quasi-static Evolution*. The idea is to associate an energy $\mathcal{E}(t, u, \theta)$ to the system

$$\mathcal{E}(t, u, \theta) = \int_\Omega W(x, \nabla u, \theta(x)) dx - \langle l(t), u \rangle \tag{0.0.5}$$

where $l(t) = f(t) + g(t)$, and a dissipation potential satisfying (0.0.2)-(0.0.3). Then a dissipation energy is introduced as

$$Diss(\theta, [t_0, t_1]) := \int_{t_0}^{t_1} \int_\Omega \mathcal{D}(x, \theta(x,t), \dot{\theta}(x,t)) dx dt$$

with a dissipation distance between two damage states

$$\tilde{\mathcal{D}}(\theta_0, \theta_1) = \inf\{Diss(\theta, [0,1]) : \theta \in C^1([0,1] \times \Omega), \quad \theta(0) = \theta_0, \ \theta(1) = \theta_1\},$$

and by this looking for a quasi-static evolution defined as follows:

Definition 0.1. A pair $(u(t), \theta(t)) : [0,T] \longrightarrow \mathcal{Q} := \mathcal{F} \times \mathcal{Z}$, with $\mathcal{F}$ and $\mathcal{Z}$ Banach spaces (e.g. $H^1_0(\Omega) \times L^\infty(\Omega)$) is called *Quasi-static Evolution* of the rate-independent problem associate to an energy $\mathcal{E}$ and a dissipation distance $\tilde{D}$ if it holds

- **Minimality property**: For all $t \in [0,T]$ and all admissible competitors $(\tilde{u}, \tilde{\theta}) \in \mathcal{Q}$ we have
$$\mathcal{E}(t, u(t), \theta(t)) \leq \mathcal{E}(t, \tilde{u}, \tilde{\theta}) + \tilde{\mathcal{D}}(\theta(t), \tilde{\theta}) \tag{0.0.6}$$
- **Energy Balance**: For all $t \in [0,T]$ we have
$$\mathcal{E}(t, u(t), \theta(t)) + Diss(\theta, [0,1]) = \mathcal{E}(0, u(0), \theta(0)) - \int_0^t \langle \dot{l}(s), u(s) \rangle ds \tag{0.0.7}$$
- **Monotonicity**: The damage variable $\theta(t)$ is increasing in time, for each $x \in \Omega$.

For a more general abstract framework see [57] and [60, 66, 68] in which it was shown that the quasi-static formulation can be derived from the mechanical one given by (0.0.4) satisfying properties (0.0.2)-(0.0.3), and that, with appropriate convexity hypothesis on the energy density, the two formulation are equivalent. The energy model that we consider in this thesis is the one proposed in [33] (see (0.0.11) below) where the simplest density dissipation potential satisfying (0.0.2)-(0.0.3) is taken, i.e.
$$\mathcal{D}(s) := ks \quad \text{with} \quad s \geq 0, \qquad \text{and} \qquad \mathcal{D}(s) := \infty \quad \text{with} \quad s < 0 \tag{0.0.8}$$
with k the dissipation constant. Although this is a very simple model, it allows to study q.s.e. in which complex processes can occurs as homogenization and (at least in the one dimensional case) it is possible to make explicit computations. Moreover, as will be clear, the (homogenized) q.s.e. from this model neither need to impose an a-priori form for the elastic matrix nor a particular microstructure for the damage must be prescribed at the beginning. The notion of quasi-static evolution does not require the solutions to be smooth in time and space and, moreover, such energetic approach allows for the usage of the powerful tools of the modern theory of the calculus of variation, such as lower semi-continuity, quasi/poly convexity, non-smooth techniques and, as we will see, to study homogenization processes through $G-convergence$ notions. All this tools have been much more adequate to study many mechanical system and by this energetic approach of rate-independent process has been developed for a variety of evolution problems in the material science as for damage ([33, 35, 43, 65]), plasticity ([22, 25, 59, 71]), fracture ([26, 28, 34, 36]), phase transitions ([5, 6, 37, 49, 67]) but also for the interplay between damage and fracture (e.g. [10]) and between fracture and plasticity (e.g. [29]).
The usual strategy to obtain quasi-static evolution for rate-independent materials is through the resolution of time-parametrized minimization problems: the first part is to perform a step by step minimization of the total energy, then let the time-step go to zero and show that the limit of such minimizing sequences satisfies the assumption of a quasi static evolution. More precisely, it is considered a space $\mathcal{Q}$ of all possible configurations of the system, the energy functional $\mathcal{E}(t,q)$ with the associated dissipation distance $\tilde{\mathcal{D}}(q_1, q_2)$ and, given an initial datum $q_0 \in \mathcal{Q}$ and a time step τ, the first part consists in solving for each $j = 1, 2 \dots \lfloor T/\tau \rfloor$
$$\min\Big\{ \mathcal{E}(j\tau, q) + \tilde{\mathcal{D}}(q, q^\tau_{j-1}) : q \in \mathcal{Q} \Big\}. \tag{0.0.9}$$
with $q^\tau_0 := q_0$, which can be done, for example, applying direct methods.
Then a piecewise-constant trajectory $q^\tau(t)$ is usually defined by
$$q^\tau(t) = q^\tau_j \text{ if } t \in [j\tau, (j+1)\tau),$$
and taking its limit in τ (by some compactness property) the idea is to show that it satisfies a minimality property, an energy balance, and some monotonicity property, namely, it is a quasi-static evolution.

In this thesis we follow this fruitful approach to study a model for rate-independent damage process in linear two-phases elastic materials and homogenization effects that can arise.
We focus our analysis considering a bidimensional isotropic material (i.e. such that for each point the elastic properties are the same in each direction) whose deformation is permitted only in the orthogonal way of the domain of such material (antiplane case).
The damaged region D is characterized by the matrix αI, with $\alpha > 0$ and I the identity matrix in $\mathbb{R}^2$, while the undamaged region by the matrix βI, with $\beta > \alpha$ and hence the elastic properties of the material are described by the matrix

$$\sigma_D(x) := \alpha I \chi_D(x) + \beta I(1 - \chi_D(x)). \tag{0.0.10}$$

This model for a brittle damage of a material with two phases (not necessarily isotropic) in the vectorial case was introduced by Francfort and Marigo in [35] (and then numerically implemented in [1] and [2]) in which the dissipation is given by (0.0.8).
In the scalar isotropic case their total energy takes the form

$$E_{tot}(t, u, D) := \frac{1}{2}\int_\Omega \sigma_D |\nabla u|^2 dx - \langle f(t), u\rangle + k|D| \tag{0.0.11}$$

where $f(t) \in H^{-1}(\Omega)$ and $\langle \cdot, \cdot \rangle$ is the duality between H^{-1} and H^1_0.
The evolution according to such energy and dissipation describes a damage process with a non-decreasing damage zone, driven by the external force $f(t)$ and the competition between the internal energy, which is characterized by the elastic matrix in (0.0.10) which is lower in the damaged region D, and the dissipation, which accounts for the amount of damaged material.
We remark that minimize (0.0.11) in (u, D) is equivalent to minimize in u the energy

$$E_{tot}(u) := \frac{1}{2}\int_\Omega W(\nabla u) dx - \langle f(t), u\rangle \tag{0.0.12}$$

with

$$W(\xi) = \min\{\frac{1}{2}\beta\xi^2, \frac{1}{2}\alpha\xi^2 + k\}$$

and since $W(\xi)$ is not a convex function (neither quasiconvex in the vectorial framework of [35]) it implies that during the minimization process we need to relax the problem. This was exactly the most important point noticed in [35] where Francfort and Marigo showed that in this case (without further restriction on the damage set) we can not expect time per time a well localized damage region but, in the process of minimization (at the first time step!), the material could prefer to create a finer and finer mixture of the damaged and undamaged region.
In the multi-dimensional case the right framework for the relaxation is that of the G-convergence for the coefficients σ_D (see [35], [33] and [43]), and this will be treated in Chapter 3 and Chapter 4.

In Chapter 2 we focus on the one-dimensional evolution case (driven by a time-dependent boundary condition $g(t)$) for which *weak evolution* can be easily expressed in terms of the weak limits of characteristic functions χ_D and for which it can be seen it is always possible to construct *strong evolutions* of the form $(u(t), D(t))$ and we explicitly compute this kind of solution. In particular we will treat a heterogeneous case, with a parameter depending energy functional given by

$$\mathcal{E}^\varepsilon(u, D) = \int_D \alpha\Big(\frac{x}{\varepsilon}\Big)|u'|^2\, dx + \int_{\Omega\setminus D} \beta\Big(\frac{x}{\varepsilon}\Big)|u'|^2\, dx \tag{0.0.13}$$

associated to a dissipation term of the form

$$\tilde{\mathcal{D}}^\varepsilon(D) = \int_D \gamma\Big(\frac{x}{\varepsilon}\Big)\,dx\,, \tag{0.0.14}$$

where α, β, and γ are 1-periodic functions taking, for sake of simplicity, only two values so that $\mathcal{E}^\varepsilon$ can be interpreted as describing a mixture of two materials with coefficients β_1 and β_2 when undamaged, and α_1 and α_2 when damaged. In this case the same scheme adopted in (0.0.9) can be followed. In so doing for fixed ε it can be defined a quasi-static evolution q_ε and after that the limit as $\varepsilon \to 0$ can be studied. Conversely, with fixed τ and ε, we may consider discrete trajectories $q_j^{\tau,\varepsilon}$ defined iteratively as solutions of

$$\min\Big\{\mathcal{E}^\varepsilon(j\tau, q) + \tilde{\mathcal{D}}^\varepsilon(q, q_{j-1}^{\tau,\varepsilon}) : q \in \mathcal{Q}\Big\} \tag{0.0.15}$$

and take the limit as $\varepsilon \to 0$ first instead of τ. Under some coerciveness and continuity assumptions (that guarantees the possibility to apply the Fundamental Theorem of Γ-convergence, see Section 1.1) these trajectories converge as $\varepsilon \to 0$ to $q_j^{\tau,0}$, which solves an effective limit problem

$$\min\Big\{F(j\tau, q, q_{j-1}^{\tau,0}) : q \in \mathcal{Q}\Big\}, \tag{0.0.16}$$

where $F(\cdot, \tilde{q}, t)$ is the Γ-limit of

$$q \mapsto \mathcal{E}^\varepsilon(t, q) + \tilde{\mathcal{D}}^\varepsilon(q, \tilde{q}).$$

with respect to the topology in $\mathcal{Q}$. We underline that, while the properties of Γ-convergence easily imply the existence of a limit functional F and the convergence of minimizers, the actual form of $F(t, \cdot, \tilde{q})$ may depend in a non-trivial way on $\tilde{q}$, and is not immediately written as a sum of an energy and a dissipation. After this passage as $\varepsilon \to 0$, we can define the piecewise constant functions

$$q^{\tau,0}(t) = q_j^{\tau,0} \text{ if } t \in [j\tau, (j+1)\tau)$$

and again, using some compactness property, take the limit as $\tau \to 0$ to obtain (up to subsequences) a continuous in time limit $q^0(t)$. In general it is not clear whether this last trajectory $q^0(t)$ agrees with the *effective energetic solution* given by the limit of $q^\varepsilon(t)$. In other words it is not obvious, and in general is false, that a quasi-static evolution related to an energy and dissipation (depending by a parameter) converges to a quasi-static evolution for the Γ-limit of energy and dissipation. Mielke et al. [64] proved that this property of commutability of Γ-convergence and quasi-static evolution happens when $\mathcal{E}^\varepsilon$ and $\tilde{\mathcal{D}}^\varepsilon$ separately Γ-converge to some $\mathcal{E}$ and $\tilde{\mathcal{D}}$ and suitable additional assumptions are satisfied (as the existence of a *mutual recovery sequence* (see Lemma 2.1. in [64]) which in particular implies that

$$F(q, \tilde{q}, t) = \mathcal{E}(q, t) + \tilde{\mathcal{D}}(q, \tilde{q})\,, \tag{0.0.17}$$

and hence the limiting trajectory can be again regarded as an energetic solution. This however is a restrictive hypothesis, and in general neither the form of the limit F may be immediately interpreted as the sum of an internal energy and a dissipation, nor we may deduce from the Γ-convergence of $\mathcal{E}^\varepsilon$ and $\mathcal{D}^\varepsilon$ enough information on the convergence of their sum.
We will illustrate that for the energy and dissipation given in (0.0.13)-(0.0.14) we do have separate Γ-convergence, but the Γ-limit of the sum does not agree with the sum of the Γ-limits. Nevertheless, the limit F can be viewed as the sum of an internal energy and a dissipation and the main result that we will prove is that the corresponding quasi-static evolution is the limit of the quasi-static evolutions for $\mathcal{E}^\varepsilon$ and $\tilde{\mathcal{D}}^\varepsilon$. This result contributes to the analysis of the interaction between Γ-convergence and variational evolution which has recently attracted much interest both in the framework of energetic solutions and in the theory of gradient flows (see [4, 14, 16, 62, 63, 64]).

In Chapter 3 we will consider the energy in (0.0.11) in the multi-dimensional case for which, as noticed, we need G-convergence tools. In [33] Francfort and Garroni showed (in the vectorial framework) the existence of a relaxed form of the energetic quasi-static evolution (see Definition 3.1 with Remark 3.2 in Chapter 3) for the relaxed functional of the energy given in (0.0.12) which is (see [35, 38, 33, 70])

$$E^*_{tot}(u) := \frac{1}{2}\int_\Omega W^*(\nabla u)dx - \langle f(t), u\rangle \tag{0.0.18}$$

with

$$W^*(\xi) = \min_{\theta\in[0,1]} \min_{A\in \bar{G}_\theta(\alpha I,\beta I)} \{\frac{1}{2}A\xi\xi + k\theta\}$$

where $G_\theta(\alpha I, \beta I)$ is the so called G-closure of α and β with volume fractions θ and $1-\theta$ which is the set of matrices which represents every possible fine periodic mixture of healthy material with proportion $(1-\theta)$ and of damaged material with proportion θ (see Chapter 1). As noticed, the necessity to relaxing the problem arises in the minimizing process at the first time step and leads to consider density damage functions instead of characteristic functions for damage sets. By this it is needed to decide how to impose the irreversibility of the damage for the next time steps and how to impose the minimality condition. In this relaxed framework the minimality condition at time t considered in [33] was given in terms of adding further damage to the material obtained at time t by previous mixture process, i.e. it was expressed by a couple $(A(t), \theta(t))$ minimizing the energy (0.0.18) with respect to competitors (A', θ') with $A' \in \bar{G}_{\theta'}(\alpha I, A(t))$. Moreover they proved that the evolution satisfying such minimality condition can be approximated by a sequence of (increasing) sets $D_n(t)$, such that

$$\alpha I \chi_{D_n(t)} + \beta I(1-\chi_{D_n(t)}) \xrightarrow{G} A(t) \quad \text{ and } \quad \chi_{D_n(t)} \overset{*}{\rightharpoonup} \theta(t).$$

However in [43] was proved that $D_n(t)$ (or any such approximating sequence) does not have good optimality properties, since if $D'_n \supset D_n(t)$ it is not generally true that the G-limit of $\sigma_{D'_n}$ is in the set of the competitors $\bar{G}_{\theta'}(\alpha I, A(t))$ (see Remark 4 in [43]). By this a new minimality condition (with larger class of competitors) was defined to take in account good properties also for approximating set (see Definition 3.1).
The existence of a (relaxed) quasi-static evolution with this larger class of competitors for the minimality condition is proved in [43] and moreover it is proved that this quasi-static evolution is also a (relaxed) threshold solution, where by this they mean that the damage satisfies a relaxed version of the following definition of Threshold solution (see Definition 3.4):

Definition 0.2. The set function $t \longrightarrow D(t)$ is a threshold solution if there exists a scalar $\lambda > 0$ such that

- **Monotonicity**: $t \longrightarrow D(t)$ is increasing (in the sense that $D(s) \supseteq D(t)$ if $s > t$);
- **Threshold condition**: given $u(t) = u(t,x)$ the solution of

  $$-div(\sigma_{D(t)}\nabla v) = f(t),$$

  we have $|\nabla u(t)| \le \lambda$ in $\Omega\backslash D(t)$;
- **Necessity of Damage:**
  -$\forall E \subset D(T)$ with $|E| > 0$, and all Δt sufficiently small, $\exists \tau < t - \Delta t$ such that considering the solution v of

  $$-div(\sigma_{D(\tau+\Delta t)\backslash \Delta E}\nabla v) = f(\tau + \Delta t),$$

  where $\Delta E := E \cap [D(\tau+\Delta t)\backslash D(\tau)]$ we have $|\nabla v| > \lambda$ in on a set of positive measure of ΔE.

-If $D(t)$ is not continuous at T, then we also require that $\forall E \subset D(T) \backslash D(T^-)$ with $|E| > 0$ and $D(T^-) := \cup_{t<T} D(t)$, the solution v of

$$-div(\sigma_{D(T)\backslash E}\nabla v) = f(T)$$

satisfies

$$|\nabla v(x)| > \lambda$$

in a subset of E with positive measure.

The first two properties are very clear, the last one needs an explanation: it is a condition that codifies the fact that if a region has been damaged then there exists a previous time such that if that region had not been damaged then it would be exceeded the threshold λ in a set of positive measure of the undamaged region.
The importance of a threshold approach is that it is, intuitively, more physical than the energetic one because the evolution of the damage is not related to the energy of the system (which can be something abstract) but it follows a computational criterion: in the region in which the (gradient of the) deformation of the material exceeds a certain threshold then a damage occurs. Moreover it is a local criterion so we don't need to know the state of the whole material to know if in some point the damage will happen. This approach seems to have a natural extension in dynamic processes, and in Chapter 4 a definition of threshold solution is investigated. Our starting point in Chapter 3 is to avoid, in the process of damaging, the homogenization of the material (mixture of healthy and damage phase) and to this aim we add in the energy a penalization for the damaged set to have a compactness property, and, considering the relative (not relaxed) quasi-static evolution we study what happens in the limit when such penalization term goes to zero. Precisely, we will start with the energy (0.0.11) penalized by the perimeter of the damaged region

$$E^{\varepsilon}_{tot}(u, D) := \frac{1}{2}\int_{\Omega} \sigma_D |\nabla u|^2 dx - \langle f(t), u\rangle + k|D| + \varepsilon Per(D) \tag{0.0.19}$$

where $Per(D)$ is the perimeter given by the total variation of the characteristic function χ_D

$$Per(D) := \sup\left\{ \int_{\mathrm{D}} div\varphi dx \ : \ \varphi \in C^1_0(\Omega), \quad \|\varphi\|_\infty \leq 1 \right\}.$$

The minimization problem for such energy (with $\varepsilon = 1$) was studied by Ambrosio and Buttazzo in [3] where they proved the existence of a minimizer couple (u, D) where u turns to be Holder continuous in space and D is (equivalent to) an open set. The space regularity of u was improved in [56] and in [50] it was proved that, in the two dimensional case, components of D have C^1 boundary.
The existence of a quasi-static evolution (without homogenization effects) for (0.0.19) was proved in [80] in which a more general framework has been analyzed.
Starting from this point our main result is in proving that, under the assumption that the perimeter term goes to zero in ε, the quasi-static evolution for the energy (0.0.19) converges to the quasi-static evolutions in the relaxed version of [43] whose class of competitors is bigger than the one in [33] (and so to a threshold solution in the sense of Definition 3.4). In this sense, this result contributes to the study of homogenization process for quasi-static evolution, moreover it confirms the validity of the new physical idea of damaging processes introduced in [43], and finally that the threshold definition is, in some sense, stable under perturbation at finite scale.

In Chapter 4 we will study a damaging process in an elastodynamic framework. This means that, considering the elasticity matrix $A(t,x)$, the general equations that describe the deformation u of the material (with, e.g. zero boundary conditions), with initial data $(p(x), q(x))$ are given by

$$\begin{cases} \partial_t(\rho(t,x)\partial_t u) - div(A(t,x)\nabla u) = f \text{ in } \Omega \\ u = 0 \qquad \text{in } \partial\Omega \\ u(0,x) = p(x), \\ \rho(x,0)\partial_t u(x,0) = q(x) \end{cases} \tag{0.0.20}$$

where $\rho(t,x)$ represents the density of the material and f the external loading. The problem (0.0.20) has been widely studied both in case of given $A(t,x)$ and in case it is not given a-priori and satisfying some (growth) conditions. Homogenization and corrector results for a parameter depending matrix $A_n(t,x)$ can be found respectively in [20, 11] and in [18] while conditions for existence and non-existence of solutions are studied in [19, 21, 46].
The case that we will analyze is given by $A(t,x) := \sigma_{D(t)}(x)$ and $\rho \equiv 1$ that is covered by the result of Casado-Diaz et al. ([17]) in which it is proved an existence and uniqueness result for a parameter depending $\rho_n \in BV(0,T;L^\infty)$ and $A_n \in BV(0,T;M^{2\times 2}_{sym})$, and an homogenization and corrector result.
The study of the evolution of the damage variable $\theta(x) \in [0,1]$ coupled to the displacement given by the solution of an hyperbolic equation like in (0.0.20) started from [40] in the framework of complete damage. In the one-dimensional case they proved a local in time weak solution for the displacement equation with a viscosity term and for the associated damage variable equation (see also [8]) while the problem without viscosity term was solved (locally in time) in [9] in the elliptic and parabolic case. In the case of incomplete damage global existence results, considering viscosity and temperature, can be found in [45] and in [76] in which is used the notion of entropic solution and energy inequality results was proved.
The energetic approach for rate-independent processes coupled with inertia (but also viscosity and temperature) has first been analyzed in two pioneering papers [77, 78] and by these has been used in different materials problems like fracture (e.g. [51]) and plasticity (e.g. [30]). Recently using this approach was studied in [52] a rate-independent damage model in thermo-visco-elastic materials with inertia where, differently from [77, 78], the damage was considered unidirectional.
The dynamic model from which we start is not considering viscosity effects or temperature, as done in [9], and it is in the framework of incomplete damage of a two-phase material. Differently from the results cited before, we will obtain (as in the previous Chapter) evolutions for the damage in which neither the microstructure of the damage set nor the form of the elastic matrix is chosen a priori but they will be defined as consequence of minimizing problems. The model that we have in mind is then

$$\begin{cases} \ddot{u} - div(\sigma_D \nabla u) = f(t) \quad \text{ in } \Omega \\ u(0) = p(x), \\ \dot{u}(0) = q(x), \end{cases} \tag{0.0.21}$$

where u has zero boundary condition and σ_D is given by (0.0.10), where D has to satisfies an irreversibility property of damage. The interesting point of our results is related to the possibility to give different natural notions of dynamic evolution to study the homogenization that arises in such problems. The first one is driven by the classical energetic approach while the second follows a threshold criterion. We stress that by the lack of uniqueness in such kind of problems the two limit solutions can be different and by this we will suggest a definition of *threshold solution* for the dynamic problem (extended from the (quasi)static framework) that could help in some way to select physically "good" solutions.

In the first part of Chapter 4 following the energetic approach we use a discretization of time to find a sequence of solution (u_n, D_n) for a discrete version of the minimum problem associated to the equation (0.0.21). The main result is given in Theorem 4.1 in which we prove that the limit (as the time step tends to zero) of such sequence of solution satisfies (in weak sense) the elasto-dynamic equation for a homogenized material, a monotonicity condition (regarding to the density of the damage), a one-side energy inequality and a threshold property.
In the second part we will present a different way to study the evolution based in considering the solution of the discrete in time version of problem (0.0.21). This approach consists in solving the equation till the time in which the gradient of the solution exceeds a certain threshold λ in a set of measure $1/n$, then to damage this region and to modify the equation with the new damage and iterate the argument. We will show, using results in [**17**], that also with this approach we obtain a limit of such sequence that, also in this case, solves (in the weak sense) the elasto-dynamic equation for a homogenized material.

CHAPTER 1

Preliminaries

1.1. Γ- convergence

In many physical situations the energy that describes the model can depend by a parameter j, that could represent, for example, a microscopic space scale in which some properties of a material appears. In this case the minimum problem in which we are interested is of the form

$$\min\{F_j(u) : u \in X\} \tag{1.1.1}$$

The condition to have solution for (1.1.1) are given by direct methods. It could happen that the problem in (1.1.1) has no solution (but only minimizing sequence) or that even in case of existence of solution it is hard to solve the problem. The Γ-convergence, introduced by De Giorgi and Franzoni in [**31**], is a kind of convergence to obtain a *some kind of limit* $\bar{F}$ of F_j in such a way that considering the *limit* problem

$$\min\{\bar{F}(u) : u \in X\} \tag{1.1.2}$$

we have that minimizing sequences of problem in (1.1.1) converge to minimum points of the problem in (1.1.2) and viceversa (see Theorem 1.4). We recall here definition and some well known property of Γ-convergence, for further details see [**24**] or [**13**].

Definition 1.1.1. Let X a metric space. A functional $\bar{F} : X \longrightarrow \bar{\mathbb{R}}$ is the Γ-limit of F_j if

1) ($\Gamma-$**liminf inequality**) for all $u \in X$ and for all $u_j \longrightarrow u$ it holds
$$\bar{F}(u) \leq \liminf_{j \longrightarrow \infty} F_j(u_j)$$
2) ($\Gamma-$**limsup inequality**) for all $u \in X$ there exists a **recovery sequence** $\bar{u}_j \longrightarrow u$ it holds
$$\bar{F}(u) \geq \liminf_{j \longrightarrow \infty} F_j(u_j)$$

Remark 1.1. *In case of a family F_ε of functionals indexed by continuous parameter we say that F_0 is the Γ-limit of F_ε and we write*

$$F_0 = \Gamma - \lim_{\varepsilon} F_\varepsilon$$

if F_0 is the Γ-limit of F_{ε_j} for every sequence $\{\varepsilon_j\}_j$ converging to zero.

Remark 1.2. *Let note that the Γ-limit depends on the convergence used on X but it easy to prove that once this is fixed then the Γ-limit, if exists, is unique. Moreover the Γ-limit satisfies the following properties:*

1) *The Γ-limit is lower semicontinuos.*
2) *A sequence of functional Γ-converges if and only if every subsequence Γ-converges to the same limit.*
3) *If $F_\varepsilon \xrightarrow{\Gamma} \bar{F}$ and G is a continuous functional on X then $F_\varepsilon + G \xrightarrow{\Gamma} \bar{F} + G$*
4) *If $F_\varepsilon \equiv F$, then $F_\varepsilon \xrightarrow{\Gamma} F^*$ where F^* is the lower semicontinuous envelope of F.*

An important result about Γ-convergence is the fact that, up to subsequences, there always exists the Γ-limit of a sequence of functional.

THEOREM 1.3. *Let X a metric space and $F_\varepsilon : X \longrightarrow \bar{X}$, $\varepsilon > 0$. There exists a subsequence F_{ε_j} and a functional $\bar{F} : X \longrightarrow \bar{R}$ such that $F_{\varepsilon_j} \xrightarrow{\Gamma} \bar{F}$.*

We now remind the well known fundamental theorem of Γ-convergence which precise the idea introduced to study problems (1.1.1) and (1.1.2).

THEOREM 1.4. ***(Fundamental Theorem of Γ-convergence)***
Let X a metric space and $F_\varepsilon : X \longrightarrow \bar{\mathbb{R}}$ equi-coercive[1] *and such that*

$$\bar{F} = \Gamma - \lim_{\varepsilon} F_\varepsilon$$

then

1) *F admits minimum, and $\min F = \lim_{\varepsilon \longrightarrow 0} \inf F_\varepsilon$;*
2) *if u_{ε_j} is a minimizing sequence for some subsequence F_{ε_j} which converges to some u then its limit point is a minimizer for F;*
3) *every minimizers of F is the limit of a converging minimizing sequence of F_ε*

PROOF. We focus just on the proof of point 1), which is the most important in our purposes.
Let u_ε be a minimizing sequence for F_ε, that is

$$\liminf_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon) = \liminf_{\varepsilon \to 0} \inf_X F_\varepsilon.$$

Since F_ε is equi-coercive, there exists $u \in X$ such that, up to a subsequence, $u_\varepsilon \longrightarrow u$. By the Γ-liminf inequality we have immediately

$$\inf_X F \leq F(u) \leq \liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon \to 0} \inf_X F_\varepsilon. \tag{1.1.3}$$

Moreover for any $v \in X$ let $\{\bar{u}_{\varepsilon,v}\}$ be a recovery sequence for u according with Γ-limsup inequality. Then, by 1.1.3, we get for any v

$$F(u) \leq \liminf_{\varepsilon \to 0} \inf_X F_\varepsilon \leq \limsup_{\varepsilon \to 0} F_\varepsilon(\bar{u}_{\varepsilon,v}) \leq F(v).$$

It follows that $F(u) = \min_X F$ and that

$$F(u) = \liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon).$$

□

1.1.1. Homogenization Formula. For special kind of family of functional it is possible to characterized the Γ-limit. Let consider the family of functional $F_\varepsilon : W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ with $p \in (1, \infty)$ defined by

$$F_\varepsilon(u) \int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u\right) dx \tag{1.1.4}$$

where $u \in W^{1,p}(\Omega; \mathbb{R})$ and Ω open set in $\mathbb{R}^n$. Let assume also that

$$f \text{ is a Caratheodory function} \tag{1.1.5}$$

$$f(\cdot, \xi) \text{ is 1-periodic for each } \xi \in \mathbb{R}^n \tag{1.1.6}$$

$$c_1(|\xi|^p - 1) \leq f(s, \xi) \leq c_2(1 + |\xi|^p) \qquad \forall \xi \in \mathbb{R}^n \text{ and } \forall s \tag{1.1.7}$$

with $c_2 > c_1 > 0$. Then the following Representation Theorem holds

[1] i.e. such that for any sequence $u_\varepsilon \in X$ with $\sup_\varepsilon F_\varepsilon(x_\varepsilon) \leq C$ for some $C > 0$, there exists $\bar{u} \in X$ such that, up to a subsequences, $u_\varepsilon \longrightarrow \bar{u}$.

THEOREM 1.5. *(**Homogenization Theorem**)*
Let F_ε a family as in (1.1.4) with $f : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying (1.1.5), (1.1.6) and (1.1.7). Then there exists a convex function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$, satisfying (1.1.7) such that, defined $F_{hom} : W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ as

$$F_{hom}(u) = \int_\Omega \varphi(\nabla u)dx \tag{1.1.8}$$

it holds

$$F_\varepsilon \xrightarrow{\Gamma} F_{hom}.$$

Moreover φ is characterized by the following representation formula

$$\varphi(\xi) = \inf\left\{ \int_{(0,1)^n} f(x, \nabla u + \xi)dx : u \in W^{1,p}_{\#}((0,1)^n)\right\} \tag{1.1.9}$$

$$= \lim_{T\longrightarrow\infty} \left[inf\left\{ \frac{1}{T^n}\int_{(0,1)^n} f(x, \nabla u + \xi)dx : u \in W^{1,p}_0((0,T)^n)\right\}\right] \tag{1.1.10}$$

$$= \lim_{\varepsilon\longrightarrow 0} \left[inf\left\{ \int_{(0,1)^n} f\left(\frac{x}{\varepsilon}, \nabla u + \xi\right)dx : u \in W^{1,p}_0((0,1)^n)\right\}\right]. \tag{1.1.11}$$

where $W^{1,p}_{\#}((0,1)^n) := \{\varphi \in W^{1,p}_{loc}((0,1)^n)$ periodic in the unit cell $(0,1)^n\}$.

We conclude this section reminding a characterization Theorem in scalar case that will be useful in Chapter 2 (see [13], Theorem 2.35).

DEFINITION 1.6. Given a function $f : (a,b) \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function, we define the *conjugate function* of f as

$$f^*(x, z^*) = \sup\{z^*z - f(t,z) : z \in \mathbb{R}\}$$

for all $x \in (a,b)$ and $z^* \in \mathbb{R}$.

THEOREM 1.7. *Let F_j a family of function of the form in (1.1.12) with integrand f_j satisfying (1.1.5), (1.1.6) and such that $f_j(x,\cdot)$ is convex for all $x \in (a,b)$. Then the following statements are equivalent:*

1) *for all I open subintervals of (a,b), $F(\cdot, I)$ given by*

$$F(u, I) = \int_I f(x, u'(x))dx \tag{1.1.12}$$

is the Γ-limit on $W^{1,p}(I)$ w.r.t $L^p(I)$ topology

2) *for all $z^*\mathbb{R}$, $f^*(\cdot, z^*)$ is the weak*-limit of the sequence $f_j^*(\cdot, z^*)$.*
Moreover, both conditions are compact.

EXAMPLE 1.8. As a particular case, take $f_j(t,z) = \alpha_j(t)|z|^2$ with $0 < c_1 \leq \alpha_j \leq c_2 < \infty$. Then

$$f_j^*(x, z^*) = \frac{(z^*)^2}{4\alpha_j(t)}.$$

So, $f_j^*(\cdot, z^*)$ converges *weakly** if and only if

$$\frac{1}{\alpha_j(x)} \overset{*}{\rightharpoonup} \frac{1}{\beta(x)} \text{ for some } \beta \in L^\infty(a,b),$$

and in this case we get

$$\Gamma - lim_j \int_a^b \alpha_j(x)|u'(x)|dx = \int_a^b \beta(x)|u'(x)|dx.$$

As a particular case we can take $\alpha_j(x) = \alpha(jx)$ with α 1-periodic. in this case β is constant and

$$\beta = \left(\int_0^1 \frac{1}{a(x)} dx \right)^{-1},$$

the *harmonic mean* of $\alpha(x)$.

1.2. G-convergence

The notion of G-convergence, introduced by De Giorgi and Spagnolo in [32] and [79], is one of the most powerful tool to study homogenization in material science. Let consider a sequence of problem in a bounded domain $\Omega \subset \mathbb{R}^n$

$$\begin{cases} -div(A^j \nabla u) = f \text{ in } \Omega \\ u = 0 \text{ in } \partial\Omega \end{cases} \tag{1.2.13}$$

where, in the framework of elastic material, $A^n(x) \in \mathbb{R}^{n\times n}$ describes the elastic properties of an heterogeneous material, $u : \Omega \longrightarrow \mathbb{R}$ the displacement of the point x of the material and $f \in H^{-1}(\Omega)$ an external force. As for Γ- convergence sometimes it could be more easier and useful not to solve the problem (1.2.13) but to solve a some kind of *limit problem*

$$\begin{cases} -div(A^* \nabla u) = f \text{ in } \Omega \\ u = 0 \text{ in } \partial\Omega \end{cases} \tag{1.2.14}$$

for which the solution of (1.2.13) converges to the solution of (1.2.14). Roughly speaking G-convergence is, by definition, the convergence of the elastic matrices A^n (tensors in vectorial case) that assures this kind of approximation.
The notion of G-convergence can be given in the general vectorial case (see, e.g., [32], or [69] for the more general case of nonsymmetric linear operators and H-convergence), but we focus just in the case of symmetric scalar operator. In this case we recall, the definition and some well known properties.

Considering a sequence $A^j \in L^\infty(\Omega; \mathcal{F}(\alpha,\beta))$, where $\beta > \alpha > 0$ and

$$\mathcal{F}(\alpha,\beta) \; := \{ \; B \in Sym^{2\times 2}(\mathbb{R}) \text{ such that } \alpha|\xi|^2 \le B\xi\cdot\xi \le \beta|\xi|^2, \;\; \xi \in \mathbb{R}^n\}. \tag{1.2.15}$$

with $Sym^{2\times 2}(\mathbb{R})$ the set of 2×2, real valued and symmetric matrices, we give the following definition

Definition 1.9. Given $A^j \in L^\infty(\Omega; \mathcal{F}(\alpha,\beta))$, we say that A^j G-converges to $A \in L^\infty(\Omega; \mathcal{F}(\alpha,\beta))$, and we write $A^j \xrightarrow{G} A$, if for every body force $f \in H^{-1}(\Omega)$, the solutions u^j of the equilibrium equation

$$-\mathrm{div}(A^j \nabla u^j) = f, \quad u^j \in H^1_0(\Omega;\mathbb{R}),$$

is such that

$$u^j \rightharpoonup u, \text{ weakly in } H^1 \tag{1.2.16}$$

where u is the solution of

$$-\mathrm{div}(A\nabla u) = f.$$

Now, considering B and C matrices in $\mathcal{F}(\alpha,\beta)$, and so representing the elastic properties of two phases, we can look for any mixture of those two phases, that is, for any characteristic function χ_D of a set $D \subset \Omega$ of, say, phase B, we look at a new elastic material with elastic matrix

$$\sigma_D := \chi_D B + (1 - \chi_D)C.$$

Considering a sequence of characteristic functions $\chi_{D^j} \overset{*}{\rightharpoonup} \theta$, we investigate the possible G-limits of $\sigma_{\chi_{D^j}}$. The main important properties of G-convergence are the followings

- *Compactness*: for any sequence $A^j \in L^\infty(\Omega; \mathcal{F}(\alpha, \beta))$, there exists a subsequence, $A^{k(j)}$, and $A \in L^\infty(\Omega; \mathcal{F}(\alpha, \beta))$ such that $A^{k(j)} \xrightarrow{G} A$;
- *Convergence of the energy*: if $A^j \xrightarrow{G} A$, then
$$\int_\Omega A^j \nabla u^j \nabla u^j \, dx \to \int_\Omega A \nabla u \nabla u \, dx;$$
with u^j and u defined as above.
- *Metrizability*: G-convergence is associated to a metrizable topology on $L^\infty(\Omega; \mathcal{F}(\alpha, \beta))$;
- *Ordering*: if $B^j \leq A^j$ and $B^j \xrightarrow{G} B$, $A^j \xrightarrow{G} A$, then $B \leq A$ (the inequalities are in the sense of quadratic forms);
- *Locality*: if $B^j \xrightarrow{G} B$, $A^j \xrightarrow{G} A$, and χ_D is a characteristic function of a set $D \subset \Omega$, then $\chi_D B^j + (1 - \chi_D)A^j \xrightarrow{G} \chi_D B + (1 - \chi_D)A$;
- *Periodicity*: if $A^j(x) := A(jx)$, with $A \in L^\infty([0,1]^2; \mathcal{F}(\alpha, \beta))$ periodic, then the whole sequence A^j G-converges to A^0, which is the constant matrix given by
$$A^0 \xi \cdot \xi = \inf_{\varphi \text{ periodic}} \int_{[0,1]^2} A(y)(\xi + \nabla\varphi) \cdot (\xi + \nabla\varphi) \, dy. \tag{1.2.17}$$
Note that it coincides with the homogenization formula (1.1.9), obtain with Γ-convergence in case that $f(x, \xi + \nabla u) = A(x)(\xi + \nabla u) \cdot (\xi + \nabla u)$.

In the case of a two-phase periodic material we consider $\sigma_{\chi_{D^j}}(x) = \chi_D(jx)B + (1 - \chi_D(jx))C$, with χ_D a characteristic function of a set $D \subset [0,1]^2$, and we speak of periodic mixtures with volume fraction

$$\theta := \frac{1}{|D|} \int_{[0,1]^2} \chi_D(y) \, dy \tag{1.2.18}$$

of material B. The set of all G-limits resulting from the periodic mixture of B and C with volume fractions θ and $1 - \theta$ is denoted by $G_\theta(B, C)$ i.e.

$$G_\theta(B, C) := \{A \in \mathcal{F}(\alpha, \beta) \text{ such that } \exists \chi_D(x) : \sigma_D(jx) \xrightarrow{G} A \text{ and } \chi_D(jx) \overset{*}{\rightharpoonup} \theta\}$$

The relevance of this set is clarified by a famous unpublished result of localization due to Dal Maso and Kohn (see [73] for the nonlinear case). It claims that the range of all possible mixtures of B and C is given by periodic homogenization. More precisely if $\theta \in L^\infty(\Omega)$ with values in $[0,1]$ and we denote by $\mathcal{G}_\theta(B, C)$ the set of all possible G-limits of $\sigma_{D^j}(x)$, with $\chi_{D^j} \overset{*}{\rightharpoonup} \theta$ as

$$\mathcal{G}_\theta(B, C) := \{A \in L^\infty(\Omega; \mathcal{F}(\alpha, \beta)) \text{ such that } \exists \chi_{D^j}(x) : \sigma_{D^j} \xrightarrow{G} A \text{ and } \chi_{D^j} \overset{*}{\rightharpoonup} \theta\}$$

then is proved that

$$\mathcal{G}_\theta(B, C) = \{A \in L^\infty(\Omega; \mathcal{F}(\alpha, \beta)) : A(x) \in \overline{G}_{\theta(x)}(B, C), \text{ a.e. in } \Omega\}. \tag{1.2.19}$$

The set of all possible mixtures of B and C, as the volume fraction varies from point to point, is the G-closure of B and C and will be denoted by $\mathcal{G}(B, C)$ and as consequence of the localization result mentioned above is given by

$$\mathcal{G}(B,C)=\{A\in L^\infty(\Omega;\mathcal{F}(\alpha,\beta))\ :\ \exists\theta\in L^\infty(\Omega),\ \text{such that}\\ A(x)\in\overline{G}_{\theta(x)}(B,C),\ \text{a.e. in }\Omega\}\,.$$

We finally recall an useful result which links G-convergence and Γ-convergence (see [**24**] pag.233, for details):

THEOREM 1.10. *Let A and A^j in $L^\infty(\Omega;\mathcal{F}(\alpha,\beta))$, G_j and G the functionals:*

$$G_j(u)=\int_\Omega A^j(x)\nabla u\nabla u dx,\qquad G(u)=\int_\Omega A(x)\nabla u\nabla u dx.$$

Then the following conditions are equivalent:
1) $A^j\xrightarrow{G}A$
2) $G_j\xrightarrow{\Gamma}G$ (with respect to the H^1 weak convergence).

1.2.1. G-closure of two dimensional isotropic materials. In the case of a two dimensional composites made by the mixture of two isotropic materials is it possible to give a clear characterization of the set of the G-closure, even geometrically. Considering a mixture of two phases described by the matrix αI and βI (where I is the identity matrix and $\beta>\alpha>0$) with volume fraction $\theta\in[0,1]$, all the possible mixtures of such components (i.e. the G-closure set) are characterized by the eigenvalues (λ_1,λ_2) of the matrices A obtained as limit

$$\alpha I\chi_{D^j}+\beta I(1-\chi_{D^j})\xrightarrow{G}A$$

where D^j describes every possible way to mixture the components at a very small scale and satisfying (1.2.18). It can be proved that the greater eigenvalue is always less than the arithmetic mean of the coefficients α and β and the smaller eigenvalue is bigger than the harmonic mean, i.e. (supposing $\lambda_1\le\lambda_2$)

$$\lambda_1\le\alpha\theta+(1-\theta)\beta\quad\text{and}\quad\lambda_2\ge\frac{1}{\frac{\theta}{\alpha}+\frac{1-\theta}{\beta}}\tag{1.2.20}$$

Combining this two inequalities we in

$$\alpha\le\lambda_1\le\frac{\alpha\beta}{\alpha+\beta-\lambda_2}\le\lambda_2\le\beta\tag{1.2.21}$$

The last relation characterizes completely the G-closure, and can be easily represented graphically in Figure 1, in which the boundary, describes the so called laminates and is obtained taking equalities in (1.2.21).

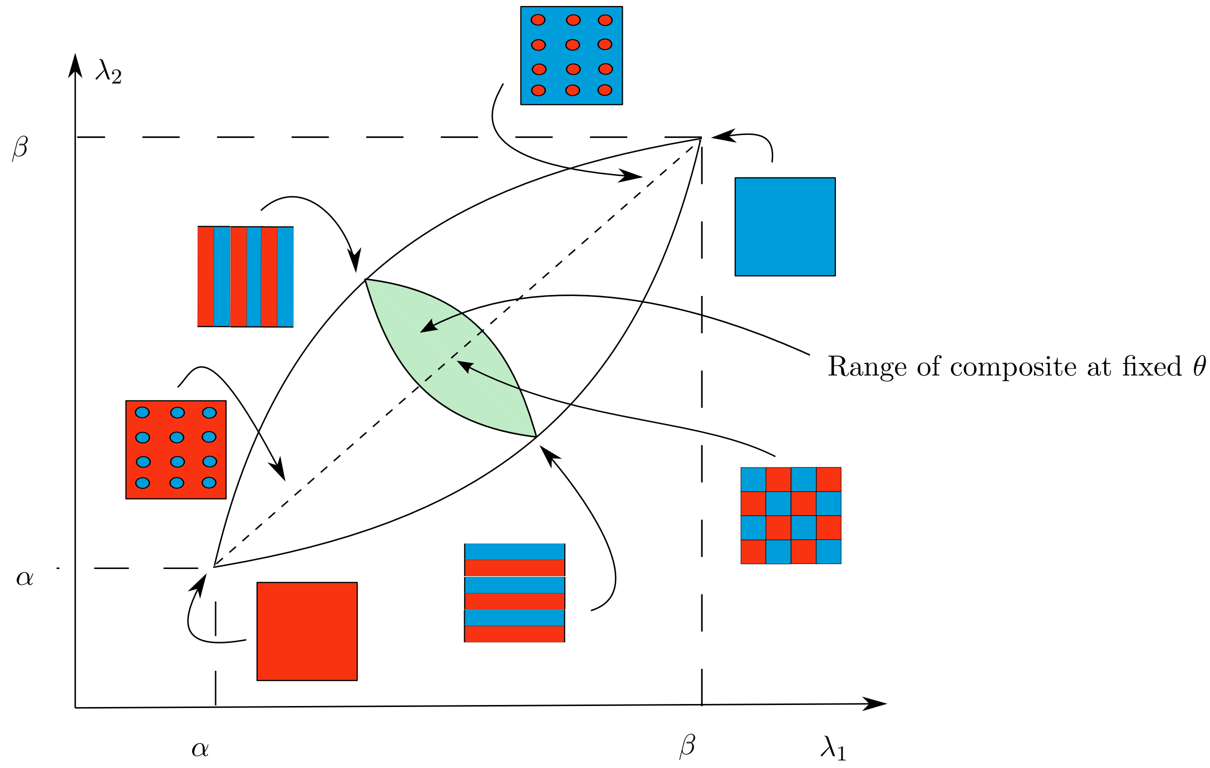

FIGURE 1. G-closure of two-dimensional isotropic materials

CHAPTER 2

Quasi-static damage evolution and Homogenization: a 1-dimensional (non)commutative result

2.1. Main results

In this chapter our original goal is to show that every *approximable* quasi-static evolution for the energy and dissipation given in (0.0.13) and (0.0.14) converges to an *approximable* quasi-static evolution for the homogenized energy limit and viceversa (Theorem 2.18). Moreover we will give a characterization for such quasi-static evolutions (Theorem 2.9 and Theorem 2.17) and by these results we will show the interesting fact that (at least in one dimensional case) the effective evolution of a mixture of two homogeneous two-phase materials can be interpreted as the relaxed evolution of a homogeneous three-phase material (Theorem 2.19). This chapter is from [**15**].

2.2. Introduction

It is well known that for fixed D the energy given in (0.0.13) Γ-converge to

$$\underline{E}(t,u,D) = \underline{\alpha}\int_D |u'|^2\,dx + \underline{\beta}\int_{\Omega\setminus D}|u'|^2\,dx, \tag{2.2.1}$$

where $\underline{\alpha}$ and $\underline{\beta}$ are the harmonic means of α and β, respectively. This is immediate by the example 1.8 using as recovery sequence $\bar{u}^\varepsilon = \bar{u}^\varepsilon_\alpha\chi_D + \bar{u}^\varepsilon_\beta(1-\chi_D)$ where $\bar{u}^\varepsilon_\alpha$ and $\bar{u}^\varepsilon_\beta$ are the recovery sequences respectively for the first part and for the second part of right side of (2.2.1).

First of all, we note that the Γ-limit of $\mathcal{E}^\varepsilon(u,D)+\tilde{\mathcal{D}}^\varepsilon(D)$ always requires a relaxation process. In fact, minimizing sequences of D will never be compact as sets, and their limit (precisely, the weak limit of their characteristic functions) must be described by a density function $\theta\in[0,1]$. Hence, the limit evolution must be expressed in terms of the relaxed variable (u,θ). In these variables the Γ-limit of $\mathcal{E}^\varepsilon(t,u,D)+\tilde{\mathcal{D}}^\varepsilon(D)$ takes the form (see Theorem 2.11)

$$\int_{(0,1)} f^{\hom}(\theta)|u'|^2 dx + \int_{(0,1)} \gamma^{\hom}(\theta)\,dx\,,$$

so that a weak quasi-static evolution can be constructed for this energy. We show that this agrees with the limit of the corresponding strong ε-quasi-static evolutions (see Theorem 2.18). We show that an equivalent formulation can be given in terms of a *three-phase material* model: the effective evolution can itself be seen as a relaxed evolution of a homogenized energy of the form

$$E(t,u,D_1,D_2) = \underline{\alpha}\int_{D_2}|u'|^2\,dx + C(\alpha,\beta)\int_{D_1}|u'|^2\,dx + \underline{\beta}\int_{\Omega\setminus(D_1\cup D_2)}|u'|^2\,dx$$

with $D_1\cap D_2=\emptyset$. The sets D_2 and D_1 can be interpreted, respectively, as the zone where either both materials are damaged, or one of the two (which is uniquely determined by the values of α_i and β_i) is damaged. $C(\alpha,\beta)$ is the corresponding harmonic mean in the latter case.

2.3. Quasi-static evolution for composite materials

In this section we give the definition of quasi-static evolution related to the elastic energy and dissipation in (0.0.13) and (0.0.14) for fixed ε, and show explicitly the existence of such evolution.
For fixed $\varepsilon > 0$ we consider the functional

$$E^{\varepsilon}_{\mathrm{Tot}}(u, D) = E^{\varepsilon}(u, D) + \tilde{\mathcal{D}}^{\varepsilon}(D)\,, \tag{2.3.2}$$

where

$$E^{\varepsilon}(u, D) = \int_{(0,1)} \sigma^{\varepsilon}_D(x)|u'(x)|^2 dx \qquad \text{and} \qquad \tilde{\mathcal{D}}^{\varepsilon}(D) = \int_D \gamma\Big(\frac{x}{\varepsilon}\Big)dx, \tag{2.3.3}$$

with $u \in H^1(0,1)$, $D \subset (0,1)$,

$$\sigma^{\varepsilon}_D(x) = \alpha\Big(\frac{x}{\varepsilon}\Big)\chi_D(x) + \beta\Big(\frac{x}{\varepsilon}\Big)(1 - \chi_D(x)), \tag{2.3.4}$$

and

$$\alpha(y) = \begin{cases} \alpha_1 & \text{if } y \in [0, \frac{1}{2}) \\ \alpha_2 & \text{if } y \in [\frac{1}{2}, 1) \end{cases} \qquad \beta(y) = \begin{cases} \beta_1 & \text{if } y \in [0, \frac{1}{2}) \\ \beta_2 & \text{if } y \in [\frac{1}{2}, 1) \end{cases} \tag{2.3.5}$$

$$\gamma(y) = \begin{cases} \gamma_1 & \text{if } y \in [0, \frac{1}{2}) \\ \gamma_2 & \text{if } y \in [\frac{1}{2}, 1) \end{cases} \tag{2.3.6}$$

with

$$\beta_i > \alpha_i > 0, \qquad \gamma_i > 0, \qquad \text{for } i = 1, 2. \tag{2.3.7}$$

Moreover, we will denote in the following

$$\underline{\alpha} := \Big(\frac{1}{2\alpha_1} + \frac{1}{2\alpha_2}\Big)^{-1} \qquad \text{and} \qquad \underline{\beta} := \Big(\frac{1}{2\beta_1} + \frac{1}{2\beta_2}\Big)^{-1}, \tag{2.3.8}$$

the *harmonic means* of α_i and β_i, respectively.

As noticed this energy describes a 1-dimensional elastic heterogeneous material, for example a bar, made periodically by two homogeneous materials characterized, in the undamaged regions, by the elastic constant β_1 for the first material and β_2 for the second one, and, in the damaged region by α_1 for the first material and by β_2 for the second one .

We suppose that

$$\varepsilon^{-1} \in \mathbb{N}; \tag{2.3.9}$$

the general case can be always reduced to this assumption up to a negligible error in the energy (2.3.2) (as $\varepsilon \to 0$).

As remarked before the first usual step in the quasi-static evolution approach to study an evolution of a system described by an energy varying in time is to look for existence of minimizers for the energy.

2.3.1. Minimum problems for the ε-energy. In the following proposition and in the next corollary we show explicitly, by a characterization, the existence of a solution for the minimum problem for the energy functional in (2.3.2) with prescribed boundary data.

Proposition 2.1. *Let $t \in \mathbb{R}$; then there exists a minimizer $(u^{\varepsilon}, D^{\varepsilon})$ of*

$$m(t) := \min\{E^{\varepsilon}_{\mathrm{Tot}}(u, D) : u(0) = 0, u(1) = t, D \subset (0,1)\}. \tag{2.3.10}$$

Moreover, $m(t)$ can be computed explicitly and it is independent of ε. If

$$p_1 := \sqrt{\frac{\alpha_1\beta_1\gamma_1}{\beta_1 - \alpha_1}} < \sqrt{\frac{\alpha_2\beta_2\gamma_2}{\beta_2 - \alpha_2}} =: p_2, \tag{2.3.11}$$

(which we may suppose without loss of generality) then

$$m(t)=\begin{cases}\underline{\beta}t^2 & \text{if } |t|\le \dfrac{p_1}{\underline{\beta}}\\ 2p_1t-\dfrac{p_1^2}{\underline{\beta}} & \text{if } \dfrac{p_1}{\underline{\beta}}<|t|\le \dfrac{p_1(\beta_2+\alpha_1)}{2\beta_2\alpha_1}\\ \dfrac{2\beta_2\alpha_1}{\beta_2+\alpha_1}t^2+\dfrac{\gamma_1}{2} & \text{if } \dfrac{p_1(\beta_2+\alpha_1)}{2\beta_2\alpha_1}<|t|\le \dfrac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1}\\ 2p_2t+\dfrac{\gamma_1+\gamma_2}{2}-\dfrac{p_2^2}{\underline{\alpha}} & \text{if } \dfrac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1}<|t|\le \dfrac{p_2}{\underline{\alpha}}\\ \underline{\alpha}t^2+\dfrac{\gamma_1+\gamma_2}{2} & \text{if } t\ge \dfrac{p_2}{\underline{\alpha}}.\end{cases} \tag{2.3.12}$$

The function $m(t)$ is plotted in Fig. 1.

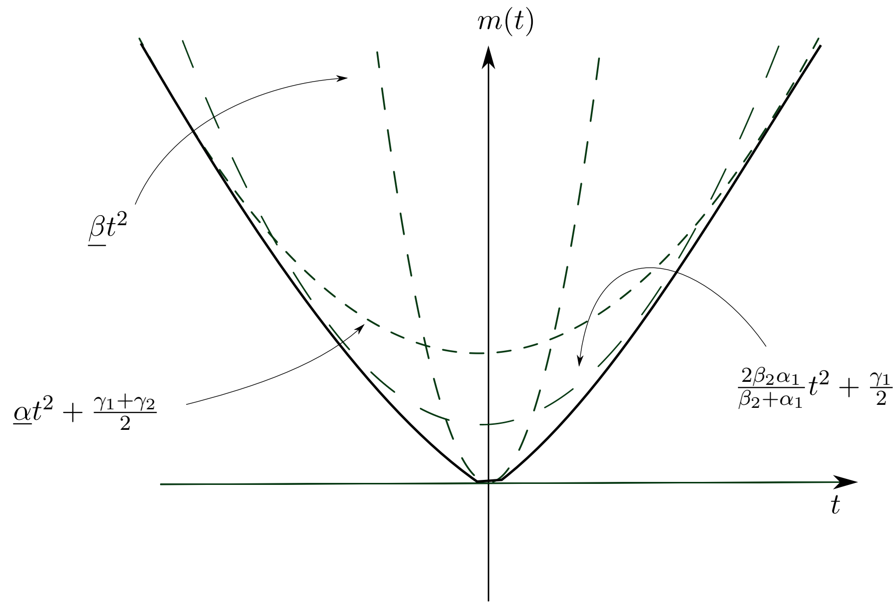

FIGURE 1. The minimal value m

REMARK 2.2. *As will be clear (see Corollary 2.3) the assumption (2.3.11) only implies, once we have a quasi-static evolution, that the first material that goes damaged is the one defined by elastic constants α_1 and β_1. When all this material will be completely damaged then the second one will start to become damaged.*

Proof. For $D \subset (0,1)$, we set

$$\begin{aligned} &D_1^\varepsilon := D \cap \Big(\Big[0, \frac{\varepsilon}{2}\Big) + \varepsilon\mathbb{N}\Big), \quad D_2^\varepsilon := D \cap \Big(\Big[\frac{\varepsilon}{2}, \varepsilon\Big) + \varepsilon\mathbb{N}\Big), \\ &B_1^\varepsilon := ((0,1)\setminus D) \cap \Big(\Big[0, \frac{\varepsilon}{2}\Big) + \varepsilon\mathbb{N}\Big), \quad B_2^\varepsilon := ((0,1)\setminus D) \cap \Big(\Big[\frac{\varepsilon}{2}, \varepsilon\Big) + \varepsilon\mathbb{N}\Big). \end{aligned} \tag{2.3.13}$$

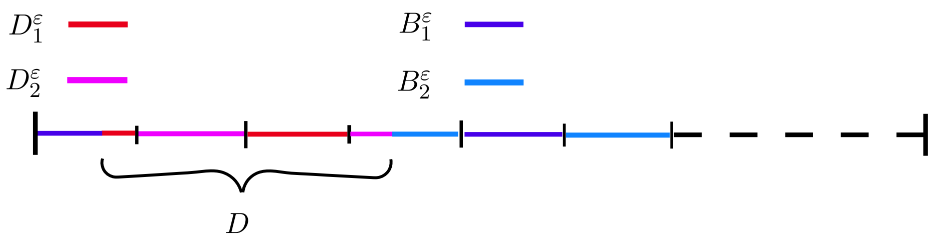

Figure 2.

Note that $(0,1) = D_1^\varepsilon \cup D_2^\varepsilon \cup B_1^\varepsilon \cup B_2^\varepsilon$, $\alpha\big(\frac{x}{\varepsilon}\big) = \alpha_i$ for $x \in D_i^\varepsilon$, and $\beta\big(\frac{x}{\varepsilon}\big) = \beta_i$ for $x \in B_i^\varepsilon$ $(i = 1,2)$. It means that considering D as a damaged set of a heterogeneous bar made periodically by two homogeneous materials, we have that D_1^ε represents the damaged part of the first homogeneous material and D_2^ε the damaged part of the second one.

We observe that the value

$$m_D(t) := \min\Big\{E^\varepsilon_{\mathrm{Tot}}(u,D) : u(0) = 0, u(1) = t\Big\} \tag{2.3.14}$$

depends on D only through the measures $|D_1^\varepsilon|$ and $|D_2^\varepsilon|$. Indeed, by Jensen's inequality and (2.3.13), for all test functions u we have

$$\begin{aligned} &\int_D \alpha\Big(\frac{x}{\varepsilon}\Big)|u'|^2\,dx + \int_{(0,1)\setminus D} \beta\Big(\frac{x}{\varepsilon}\Big)|u'|^2 dx \\ &\qquad \geq \alpha_1|D_1^\varepsilon||z_{11}|^2 + \beta_1|B_1^\varepsilon||z_{12}|^2 + \alpha_2|D_2^\varepsilon||z_{21}|^2 + \beta_2|B_2^\varepsilon||z_{22}|^2, \end{aligned}$$

where

$$z_{i1} := \frac{1}{|D_i^\varepsilon|}\int_{D_i^\varepsilon} u'\,dx, \quad z_{i2} := \frac{1}{|B_i^\varepsilon|}\int_{B_i^\varepsilon} u'\,dx, \quad i = 1,2, \tag{2.3.15}$$

with a strict inequality unless u' is constant on D_i^ε and B_i^ε. Hence, each minimizer must have a constant value of the derivative on each of the four sets D_i^ε and B_i^ε.
This observation allows to reduce the computation of $m(t)$ to a finite-dimensional minimization. To that end, denote

$$\lambda_i := 2\big|D_i^\varepsilon\big|, \quad i = 1,2. \tag{2.3.16}$$

Observing that $|B_i^\varepsilon| = \frac{1}{2} - |D_i^\varepsilon| = \frac{1}{2}(1-\lambda_i)$, we have that

$$\begin{aligned} m(t) = \min_{z_{ij},\lambda_k} \Big\{&\frac{1}{2}(\lambda_1\alpha_1 z_{11}^2 + (1-\lambda_1)\beta_1 z_{12}^2) + \frac{1}{2}(\lambda_2\alpha_2 z_{21}^2 + (1-\lambda_2)\beta_2 z_{22}^2) \\ &+ \frac{1}{2}\gamma_1\lambda_1 + \frac{1}{2}\gamma_2\lambda_2 : \frac{1}{2}(\lambda_1 z_{11} + (1-\lambda_1)z_{12}) + \frac{1}{2}(\lambda_2 z_{21} + (1-\lambda_2)z_{22}) = t\Big\}. \end{aligned} \tag{2.3.17}$$

A solution $\lambda_{i,min}$, $z_{ij,min}$ $(i,j = 1,2)$ provides a description of all minimizers of problem (2.3.10) as follows: the set D^ε is any set D such that $2|D_i^\varepsilon| = \lambda_{i,min}$, and u^ε is the unique solution of (2.3.15), which gives (see Figure 3)

$$u' = z_{i1,min} \text{ on } D_i^\varepsilon \qquad \text{and} \qquad u' = z_{i2,min} \text{ on } B_i^\varepsilon, \quad i = 1,2. \tag{2.3.18}$$

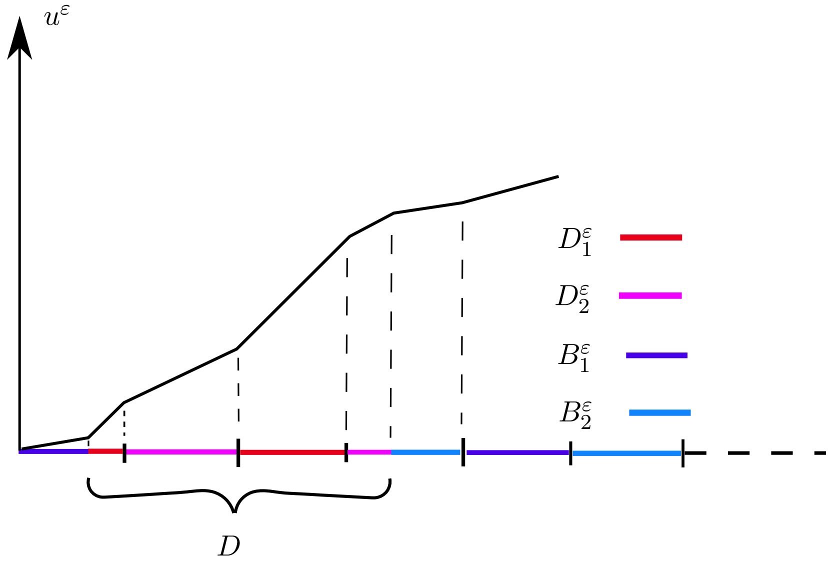

FIGURE 3. Example of minimizer u^ε

We can explicitly compute the minimum in (2.3.17). We conclude that $m(t)$ is independent on ε and satisfies

$$m(t) = \frac{1}{2}\min\Big\{m_1(t_1) + m_2(t_2) : \frac{t_1+t_2}{2} = t\Big\}, \tag{2.3.19}$$

where

$$m_i(t) := \min_{z_{i1}, z_{i2}, \lambda_i} \Big\{\lambda_i \alpha_i z_{i1}^2 + (1-\lambda_i)\beta_i z_{i2}^2 + \gamma_i \lambda_i : \lambda_i z_{i1} + (1-\lambda_i) z_{i2} = t\Big\},$$

whose explicit form is given by

$$m_i(t) = \begin{cases} \beta_i t^2 & \text{if } |t| \le \sqrt{\frac{\alpha_i\gamma_i}{\beta_i(\beta_i-\alpha_i)}} = \frac{p_i}{\beta_i} \\ \alpha_i t^2 + \gamma_i & \text{if } |t| \ge \sqrt{\frac{\beta_i\gamma_i}{\alpha_i(\beta_i-\alpha_i)}} = \frac{p_i}{\alpha_i} \\ 2t\sqrt{\dfrac{\alpha_i\beta_i\gamma_i}{\beta_i-\alpha_i}} - \dfrac{\gamma_i\alpha_i}{\beta_i-\alpha_i} = 2tp_i - \dfrac{p_i^2}{\beta_i} & \text{otherwise.} \end{cases} \tag{2.3.20}$$

Using (2.3.20) and solving (2.3.19) we obtain the expression of $m(t)$ as in (2.3.12). $\square$

We can also explicitly compute the minimum values $\lambda_{i,\min}$ in (2.3.17) and by the characterization of the minimizers $(u^\varepsilon, D^\varepsilon)$ given by (2.3.16) and (2.3.18)we obtain immediately the following corollary.

COROLLARY 2.3. *Each minimizer* $(u^\varepsilon, D^\varepsilon)$ *for the problem (2.3.10) is characterized as follows (assuming* (2.3.11)*, i.e.,* $p_1 < p_2$*) :*
consider

$$D_1^\varepsilon := D^\varepsilon \cap \Big(\Big[0, \frac{\varepsilon}{2}\Big) + \varepsilon\mathbb{N}\Big), \quad D_2^\varepsilon := D^\varepsilon \cap \Big(\Big[\frac{\varepsilon}{2}, \varepsilon\Big) + \varepsilon\mathbb{N}\Big)$$

then

$$\lambda_{\min}(t) := |D^\varepsilon| = \begin{cases} \dfrac{\lambda_{1,\min}(t)}{2} & \textit{if } |t| < \dfrac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1} \\ \dfrac{1}{2} + \dfrac{\lambda_{2,\min}(t)}{2} & \textit{if } |t| \ge \dfrac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1} \end{cases} \tag{2.3.21}$$

where

$$\lambda_{1,\min}(t) := 2|D_1^\varepsilon| = \begin{cases} 0 & \textit{if } 0 \le |t| \le \dfrac{p_1}{\underline{\beta}} \\ \dfrac{2p_1}{\gamma_1}\left(|t| - \dfrac{p_1}{\underline{\beta}}\right) & \textit{if } \dfrac{p_1}{\underline{\beta}} \le |t| \le \dfrac{p_1(\beta_2+\alpha_1)}{2\beta_2\alpha_1} \\ 1 & \textit{if } |t| \ge \dfrac{p_1(\beta_2+\alpha_1)}{2\beta_2\alpha_1}, \end{cases} \tag{2.3.22}$$

and

$$\lambda_{2,\min}(t) := 2|D_2^\varepsilon| = \begin{cases} 0 & \textit{if } 0 \le |t| \le \dfrac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1} \\ \dfrac{2p_2}{\gamma_2}\left(|t| - \dfrac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1}\right) & \textit{if } \dfrac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1} \le |t| \le \dfrac{p_2}{\underline{\alpha}} \\ 1 & \textit{if } |t| \ge \dfrac{p_2}{\underline{\alpha}}. \end{cases} \tag{2.3.23}$$

For such D^ε, u^ε is the unique minimizer of

$$\min_u \Big\{ E^\varepsilon_{\mathrm{Tot}}(u, D^\varepsilon) : u(0) = 0, u(1) = t \Big\}$$

The value of $\lambda_{\min}$ is plotted in Fig. 4.

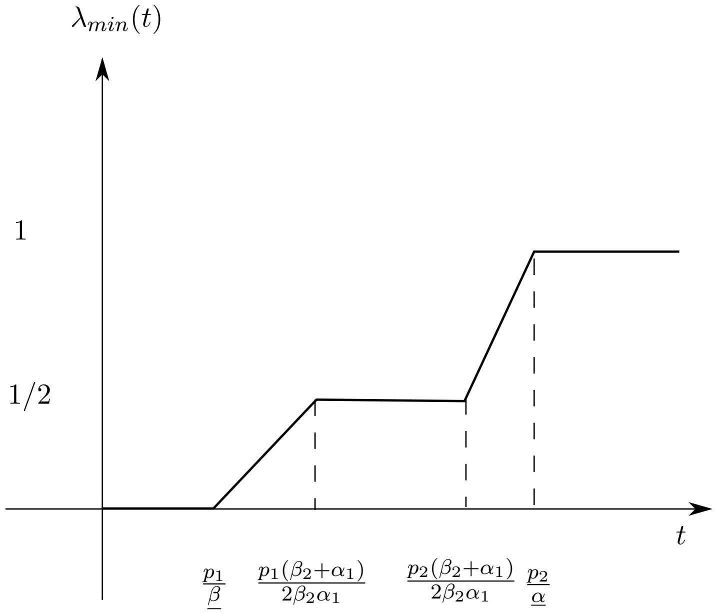

FIGURE 4. The value of $\lambda_{\min}$.

REMARK 2.4.
1)Let note that by Corollary 2.3 we don't have a characterization of the *shape* of D^ε but only about the measure and by this we have the existence of infinitely-many minimizers, except in the cases when both $\lambda_{i,\min} \in \{0,1\}$, for which the minimizing pair is unique. Under condition (2.3.11), i.e., $p_1 < p_2$, this corresponds to $D^\varepsilon = \emptyset$, $D^\varepsilon = (0,1) \cap ([0, \frac{\varepsilon}{2}) + \varepsilon\mathbb{N})$ or $D^\varepsilon = (0,1)$. Note that the minimality conditions for (2.3.17) give the relations

$$\alpha_1 z_{11} = \beta_1 z_{21} = \alpha_2 z_{12} = \beta_2 z_{22}. \tag{2.3.24}$$

2) Among all the minimizers $(u^\varepsilon, D^\varepsilon)$ we have those with

$$D_1^\varepsilon := (0,1) \cap \Big(\Big[0, \lambda_{1,\min}\frac{\varepsilon}{2}\Big) + \varepsilon\mathbb{N} \Big), \quad D_2^\varepsilon = (0,1) \cap \Big(\Big[\frac{\varepsilon}{2}, (1+\lambda_{2,\min})\frac{\varepsilon}{2}\Big) + \varepsilon\mathbb{N} \Big),$$

for which the damage is "uniformly distributed" in $(0,1)$ (see Figure 5). In this case the weak

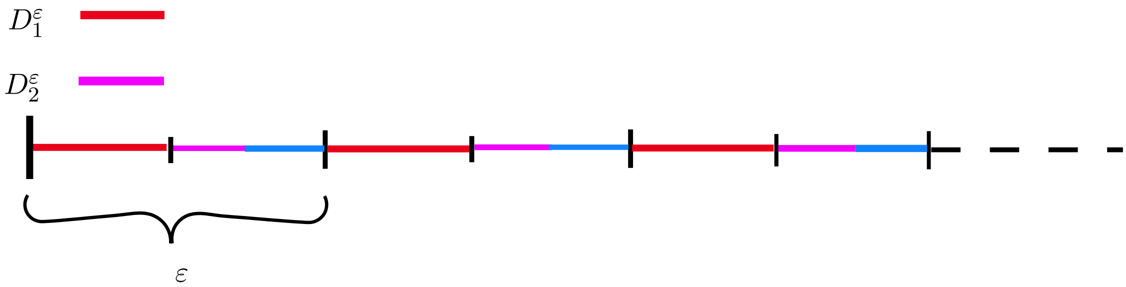

FIGURE 5. Example of minimizer D^ε with $\lambda_{1,min} = 1$ and $\lambda_{2,min} = 1/2$.

limit of the characteristic functions of the sets D_i^ε is the constant $\frac{1}{2}\lambda_{i,\min}$. So from a damaging point of view, this possible microscopical structure represents on a macroscopical framework a uniform damage along the bar with constant density (at fixed t) equal to $\frac{1}{2}(\lambda_{1,min}+\lambda_{2,min})$.

Another family of minimizers are those with

$$D_1^\varepsilon := (0, \lambda_{1,\min}^\varepsilon) \cap \Big(\Big[0, \frac{\varepsilon}{2}\Big) + \varepsilon\mathbb{N} \Big), \quad D_2^\varepsilon = (0, \lambda_{2,\min}^\varepsilon) \cap \Big(\Big[\frac{\varepsilon}{2}, \varepsilon\Big) + \varepsilon\mathbb{N} \Big),$$

where $\lambda_{i,\min}^\varepsilon$ is such that $2|D_i^\varepsilon| = \lambda_{i,\min}$, for which the damage is "concentrated towards 0" (see Figure 6).

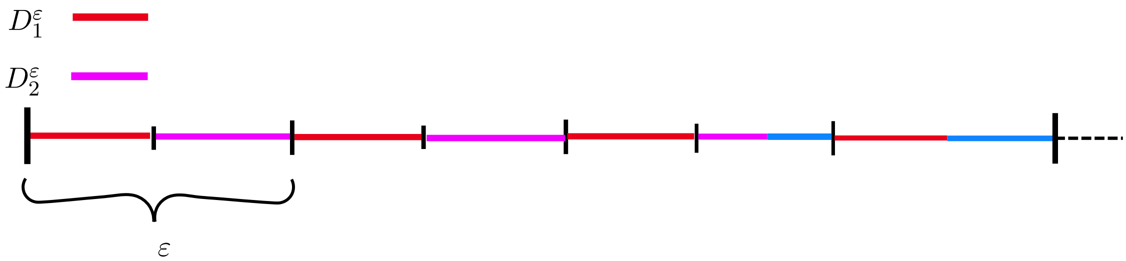

FIGURE 6. Example of minimizer D^ε with $\lambda_{1,min} = 1$ and $\lambda_{2,min} > 0$.

Note that in this case we have $|\lambda_{i,\min}^\varepsilon - \lambda_{i,\min}| \le \varepsilon$ and hence the weak limit of the characteristic functions of the sets D_i^ε is the function $\frac{1}{2}\chi_{[0,\lambda_{i,\min}]}$, which from a macroscopical point of view means that the damage is localized in $[0, \lambda_{min}]$.

REMARK 2.5. Let note that we can rewrite the minimization problem (2.3.17) in a more compact way that will be useful for what follows, i.e.

$$m(t) = \min_{\lambda_1,\lambda_2} \Big\{ f^{\hom}\Big(\frac{\lambda_1}{2}, \frac{\lambda_2}{2}\Big)t^2 + \frac{1}{2}\gamma_1\lambda_1 + \frac{1}{2}\gamma_2\lambda_2 \Big\}, \tag{2.3.25}$$

where

$$f^{\hom}\Big(\frac{\lambda_1}{2},\frac{\lambda_2}{2}\Big) := \min_{z_{ij}}\Big\{\frac{1}{2}(\lambda_1\alpha_1 z_{11}^2+(1-\lambda_1)\beta_1 z_{12}^2)+\frac{1}{2}(\lambda_2\alpha_2 z_{21}^2+(1-\lambda_2)\beta_2 z_{22}^2) : \\ \frac{1}{2}(\lambda_1 z_{11}+(1-\lambda_1)z_{12})+\frac{1}{2}(\lambda_2 z_{21}+(1-\lambda_2)z_{22})=1\Big\}, \tag{2.3.26}$$

which is, solving the minimum problem (2.3.26), defined by

$$f^{\hom}(\eta_1,\eta_2) := \Big[\frac{1}{\alpha_1}\eta_1+\frac{1}{\beta_1}\Big(\frac{1}{2}-\eta_1\Big)+\frac{1}{\alpha_2}\eta_2+\frac{1}{\beta_2}\Big(\frac{1}{2}-\eta_2\Big)\Big]^{-1}, \tag{2.3.27}$$

that we call *homogenized coefficient* related to η_1 and η_2.

REMARK 2.6. Let

$$G(\lambda_1,\lambda_2,t) = f^{\hom}\Big(\frac{\lambda_1}{2},\frac{\lambda_2}{2}\Big)t^2+\frac{\gamma_1}{2}\lambda_1+\frac{\gamma_2}{2}\lambda_2\,, \tag{2.3.28}$$

where $f^{\hom}$ is defined by (2.3.27). Then for fixed s and t with $0\le s\le t$, the unique minimizer of the function $G(\cdot,\cdot,s)$ on $[\lambda_{1,\min}(t),1]\times[\lambda_{2,\min}(t),1]$ is $(\lambda_{1,\min}(t),\lambda_{2,\min}(t))$. This follows from a straightforward calculation.
As will be clear in the proof of Theorem 2.9 the importance of Remark 2.6 is the following. Let consider the minimum problem in (2.3.10) in which instead of the increasing boundary condition t we have a non-increasing function $g(t)$. Since t has the role of a parameter all the computations done change only in the substitution of t with $g(t)$. The evolution that we are interested in must satisfy the property of increasing of damage in the case that the external force $g(t)$ increases. This remark, on the other hand, assure that in the case that $g(t)$ decreases in time, then the measure of the damage remains unchanged during the decreasing of the body force.
This observation will be fundamental in Theorem 2.9 to prove the characterization result for quasi-static evolution related to the energy in (2.3.2), and in particular to show the *approximability* of such quasi-static evolution in case of a non-increasing body force $g(t)$.

2.3.2. Quasi-static evolution for the ε-energy. We state now the definition of quasi-static evolution for the energy functional (2.3.2) and we describe explicitly the behaviour of such motions in Theorem 2.9 through a characterization result.

From now on we will consider $u=u(t,x)$, with $u(t,\cdot)\in H^1(0,1)$ parametrized by $t\in[0,T]$. As a shorthand we will write $u(t)=u(t,\cdot)$.

DEFINITION 2.3.1. Given $g\in AC([0,T])$, with $g(0)=0$, and $\varepsilon>0$, we say that $(u(t),D(t))$ is a *(strong) quasi-static evolution* (for the energy (2.3.2) subjected to the boundary condition g) if for all $t\in[0,T]$ we have $u(t)\in H^1(0,1)$, $u(t,0)=0$, $u(t,1)=g(t)$, $D(t)\subset(0,1)$, and the following properties hold:

- **Damage Irreversibility**: $D(t_1)\subset D(t_2)$ if $t_1<t_2$;
- **Energy Balance**: for all $t\in[0,T]$ we have

$$E^{\varepsilon}_{\mathrm{Tot}}(u(t),D(t)) = E^{\varepsilon}_{\mathrm{Tot}}(u(0),D(0))+2\int_0^t \dot g(s)\int_{(0,1)}\sigma^{\varepsilon}_{D(s)}(x)u'(s,x)\,dxds; \tag{2.3.29}$$

- **Minimality Condition**: for all $t\in[0,T]$

$$E^{\varepsilon}_{\mathrm{Tot}}(u(t),D(t))\le E^{\varepsilon}_{\mathrm{Tot}}(v,B) \tag{2.3.30}$$

for all $v\in H^1(0,1)$ with $v(0)=0$, $v(1)=g(t)$, and $D(t)\subset B\subset(0,1)$.

Moreover, we say that $(u(t), D(t))$ is an *approximable (strong) quasi-static evolution* (for the energy (2.3.2) subjected to the boundary condition $g(t)$) if it satisfies the conditions above and if considering the solution (u_k^τ, D_k^τ) of the discrete problem (for $k \geq 1$)

$$\min\Big\{E_{\rm Tot}^\varepsilon(v, D) : v(0) = 0,\ v(1) = g(k\tau),\ D_{k-1}^\tau \subset D\Big\} \tag{2.3.31}$$

and defined

$$u_\tau(t) := u_k^\tau \text{ if } t \in [k\tau, (k+1)\tau)$$

and

$$D_\tau(t) := D_k^\tau \text{ if } t \in [k\tau, (k+1)\tau)$$

with $u_0^\tau = 0$ and $D_0^\tau = \emptyset$, it holds (up to subsequences)

$$u_\tau(t) \overset{H^1}{\rightharpoonup} u(t)$$

and

$$\chi_{D_\tau(t)} \overset{*}{\rightharpoonup} \chi_{D(t)}$$

when $\tau \longrightarrow 0$.

Remark 2.7. Let note that in general, given an energy E and a dissipation D, is not true that a quasi-static evolution related to $E + D$ is approximable. In [62] is given a counterexample. In Theorem 2.9 we will prove that if $(u^\varepsilon(t), D^\varepsilon(t))$ satisfies some properties (i.e. (i)-(ii)-(iii) in the Theorem) then $(u^\varepsilon(t), D^\varepsilon(t))$ is a quasi-static evolution, and moreover that is approximable. To prove the converse, and in particular the property (ii), we will use the approximability condition. Considering our particular 1-dimensional case one could wonder if in this case every quasi-static evolution for the energy in (0.0.13) and dissipation in (0.0.14) is approximable. Actually this is not completely clear, and we will not address this issue in the following.

Remark 2.8. By the minimality condition (2.3.30), with $B = D(t)$, we deduce that $u(t)$ is the unique minimizer of the quadratic energy $E^\varepsilon(v, D(t))$ satisfying the boundary condition $u(t, 0) = 0$ and $u(t, 1) = g(t)$. Testing the Euler-Lagrange equation with $u(t, x) - g(t)x$ we deduce the identity

$$\int_{(0,1)} \sigma_{D(t)}^\varepsilon(x) u'(t, x)\, dx = f^\varepsilon(D(t)) g(t)\,, \tag{2.3.32}$$

where

$$f^\varepsilon(D) := \min\Big\{E^\varepsilon(v, D)\ :\ v \in H^1(0,1)\,,\ v(0) = 0\,,\ v(1) = 1\Big\}\,. \tag{2.3.33}$$

This remark will be useful in the following Theorem.

Theorem 2.9. *Let $g \in AC([0, T])$, with $g(0) = 0$. Assume (without loss of generality) that (2.3.11) holds. Then each approximable strong quasi-static evolution $(u^\varepsilon(t), D^\varepsilon(t))$ (in the sense of Definition 2.3.1) for the energy in (2.3.2), subjected to the boundary condition g, is characterized by*

(i) *$D^\varepsilon(t)$ is increasing in t;*

(ii) *if $D_1^\varepsilon(t) := D^\varepsilon(t) \cap \big([0, \frac{\varepsilon}{2}) + \varepsilon\mathbb{N}\big)$ and $D_2^\varepsilon(t) := D^\varepsilon(t) \setminus D_1^\varepsilon(t)$, then*

$$2|D_1^\varepsilon(t)| = \lambda_{1,\min}(\overline{g}(t)) \qquad \text{and} \qquad 2|D_2^\varepsilon(t)| = \lambda_{2,\min}(\overline{g}(t)),$$

where $\overline{g}$ is the non-decreasing envelope *of the function g (see Figure 7), defined by*

$$\overline{g}(t) := \inf_h \{h(t) : h \geq g \text{ on } [0, T], h \text{ non decreasing}\}; \tag{2.3.34}$$

(iii) *the function $u^\varepsilon(t)$ is the unique minimizer of $E^\varepsilon(\cdot, D^\varepsilon(t))$ under the boundary condition $u^\varepsilon(t, 0) = 0$ and $u^\varepsilon(t, 1) = g(t)$.*

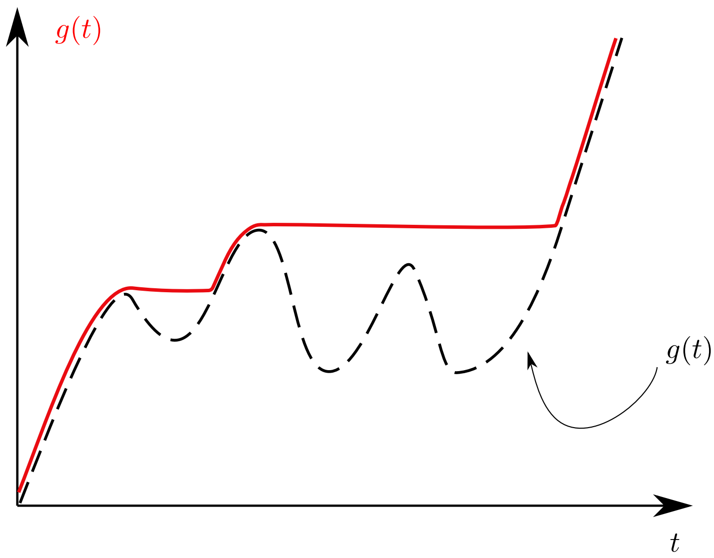

FIGURE 7. Non-decreasing envelop of $g(t)$.

PROOF. Note that the approximability condition in general implies the minimality and the energy balance. This can be derived from [57], upon a relaxation argument in order to fulfil the abstract framework therein. Here we give a direct proof that highlights the homogenization process through the explicit description of the solutions, using $\lambda_{1,\min}$ and $\lambda_{2,\min}$. We consider the case of g non-decreasing first, and then the general case. If g is non-decreasing, we can assume without loss of generality that $g(t) = t$ for all $t \in \mathbb{R}$, since t has just the role of a parameter. Let $(u^\varepsilon(t), D^\varepsilon(t))$ satisfy (i)–(iii). By the characterization of minimizers in Proposition 2.1 and Corollary 2.3 such a pair is a solution to

$$\min\{E^\varepsilon_{\mathrm{Tot}}(v, B) : v(0) = 0, v(1) = t, B \subset (0, 1)\}. \tag{2.3.35}$$

and hence satisfies the minimality condition in Definition 2.3.1. Damage irreversibility is property (i). It remains to prove the energy balance. To that end, we first note that by (2.3.12) the function $s \mapsto E^\varepsilon_{\mathrm{Tot}}(u^\varepsilon(s), D^\varepsilon(s))$ is absolutely continuous and its a.e. derivative is given by

$$\partial_s E^\varepsilon_{\mathrm{Tot}}(u^\varepsilon(s), D^\varepsilon(s)) = m'(s) = \begin{cases} 2\underline{\beta}t & \text{if } 0 < t < \dfrac{p_1}{\underline{\beta}} \\ 2p_1 & \text{if } \dfrac{p_1}{\underline{\beta}} < t < \dfrac{p_1(\beta_2+\alpha_1)}{2\beta_2\alpha_1} \\ \dfrac{4\beta_2\alpha_1}{\beta_2+\alpha_1}t & \text{if } \dfrac{p_1(\beta_2+\alpha_1)}{2\beta_2\alpha_1} < t < \dfrac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1} \\ 2p_2 & \text{if } \dfrac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1} < t < \dfrac{p_2}{\underline{\alpha}} \\ 2\underline{\alpha}t & \text{if } t > \dfrac{p_2}{\underline{\alpha}}. \end{cases} \tag{2.3.36}$$

Using this equality we now prove (2.3.29), rewritten as

$$\int_0^t \partial_s E^\varepsilon_{\mathrm{Tot}}(u^\varepsilon(s), D^\varepsilon(s))\, ds = 2\int_0^t \int_{(0,1)} \sigma^\varepsilon_{D^\varepsilon(s)}(x)(u^\varepsilon(x,s))'\, dxds.$$

In order to conclude we show that for all $s \in \mathbb{R}$

$$m'(s) = 2\int_{(0,1)} \sigma^\varepsilon_{D^\varepsilon(s)}(x)(u^\varepsilon(x,s))'\, dx. \tag{2.3.37}$$

Note that, by Remark 2.4(1) we have (in the notation of Proposition 2.1)

$$(u^\varepsilon)' = z_{i1} \text{ on } D_i^\varepsilon, \qquad (u^\varepsilon)' = z_{i2} \text{ on } B_i^\varepsilon\,.$$

Taking into account conditions (2.3.24) and the boundary condition

$$\frac{1}{2}(\lambda_{1,\min}(s)z_{11} + (1-\lambda_{1,\min}(s))z_{12}) + \frac{1}{2}(\lambda_{2,\min}(s)z_{21} + (1-\lambda_{2,\min}(s))z_{22}) = s\,,$$

this allows us to conclude that the right-hand side of (2.3.37) equals

$$\frac{4\alpha_1\alpha_2\beta_1\beta_2}{\alpha_2\beta_2(\beta_1-\alpha_1)\lambda_{1,\min}(s) + \alpha_1\beta_1(\beta_2-\alpha_2)\lambda_{2,\min}(s) + \alpha_1\alpha_2(\beta_2+\beta_1)}s.$$

Using (2.3.22), (2.3.23), (2.3.36) it is immediate to check that this expression is equal to the one for $m'(s)$ above which concludes the proof of the energy balance property and so the fact that $(u^\varepsilon(t), D^\varepsilon(t))$ is a quasi-static evolution.

By (i)–(iii) and recalling the minimality properties of $\lambda_{1,\min}$ and $\lambda_{2,\min}$, we have that, for every τ and k, $(u_k^\tau, D_k^\tau) = (u^\varepsilon(k\tau), D^\varepsilon(k\tau))$ is a minimizer for (2.3.31). Moreover since

$$u_\tau^\varepsilon(t) \overset{H^1}{\rightharpoonup} u^\varepsilon(t) \qquad \text{and} \qquad \chi_{D_\tau^\varepsilon(t)} \overset{*}{\rightharpoonup} \chi_{D^\varepsilon(t)}$$

where

$$u_\tau^\varepsilon(t) := u^\varepsilon(k\tau) \text{ if } t \in [k\tau, (k+1)\tau)$$

and

$$D_\tau^\varepsilon(t) := D^\varepsilon(k\tau) \text{ if } t \in [k\tau, (k+1)\tau)$$

we conclude the approximability of $(u^\varepsilon(t), D^\varepsilon(t))$ and the proof of the first part of theorem in the case of g increasing. In the general case, we define $\overline{g}$ by (2.3.34) (see Figure 7) and consider $(u^\varepsilon(t), D^\varepsilon(t))$ satisfying (i)–(iii). If we denote by $\overline{u}^\varepsilon$ the minimizer of

$$\min\{E^\varepsilon_{\text{Tot}}(v, D^\varepsilon(t)) : v \in H^1(0,1), v(0) = 0, v(1) = \overline{g}(t)\}, \tag{2.3.38}$$

then, by the previous step, the pair $(\overline{u}^\varepsilon(t), D^\varepsilon(t))$ is an approximable quasi-static evolution for the boundary condition $\overline{g}$. In order to show that $(u^\varepsilon(t), D^\varepsilon(t))$ is an approximable quasi-static evolution for the boundary condition g we first examine the minimality condition. It is enough to consider t such that $g(t) < \overline{g}(t)$ (otherwise $g(t)$ is increasing in t and so the result is true by the previous step). Suppose by contradiction that there exists $B \supset D^\varepsilon(t)$ such that

$$E^\varepsilon_{\text{Tot}}(u^\varepsilon(t), D^\varepsilon(t)) > \min\big\{E^\varepsilon_{\text{Tot}}(v, B)\ :\ v(0) = 0\,,\ v(1) = g(t)\big\}\,.$$

Then, noting that $f^\varepsilon(D)$, as defined in (2.3.33), is decreasing by inclusion, we have

$$\begin{aligned} E^\varepsilon_{\text{Tot}}(\overline{u}^\varepsilon(t), D^\varepsilon(t)) &= E^\varepsilon_{\text{Tot}}(u^\varepsilon(t), D^\varepsilon(t)) + f^\varepsilon(D^\varepsilon(t))\big(\overline{g}^2(t) - g^2(t)\big) \\ &> \min\big\{E^\varepsilon_{\text{Tot}}(v, B)\ :\ v(0) = 0\,,\ v(1) = g(t)\big\} \\ &\qquad + f^\varepsilon(B)\big(\overline{g}^2(t) - g^2(t)\big) \\ &= \min\big\{E^\varepsilon_{\text{Tot}}(v, B)\ :\ v(0) = 0\,,\ v(1) = \overline{g}(t)\big\}\,, \end{aligned}$$

contradicting the minimality condition for $(\overline{u}^\varepsilon(t), D^\varepsilon(t))$. As for the energy balance, it is enough to check it between two points s and t such that $\overline{g}(\tau) = \overline{g}(s) = \overline{g}(t)$ for all $\tau \in (s,t)$; i.e.,

$$E^\varepsilon_{\text{Tot}}(u(t), D) - E^\varepsilon_{\text{Tot}}(u(s), D) = 2\int_s^t \dot{g}(\tau)\int_{(0,1)} \sigma_D^\varepsilon(x) u'(\tau, x)\, dx d\tau, \tag{2.3.39}$$

where $D = D(t) = D(s)$. This is easily verified by noting that, in view of Remark 2.8, we can rewrite (2.3.39) as

$$f^\varepsilon(D)\big(g^2(t) - g^2(s)\big) = 2\int_s^t \dot g(\tau) g(\tau) f^\varepsilon(D)\, d\tau\,.$$

The approximability is obtained as in the non-decreasing case above, after recalling the constrained minimality properties of $\lambda_{i,\min}(t)$ in Remark 2.6 which allow to characterize the minimum values of the energy as in Remark 2.5. This conclude the first part of Theorem in general case.
It now remains to prove that every approximable quasi-static evolution $(u^\varepsilon(t), D^\varepsilon(t))$ satisfies properties (i)–(iii). Properties (i) and (iii) are immediately implied by the definition. Let (u_k^τ, D_k^τ) be as in Definition 2.3.1. We define the piecewise-constant function g_τ by

$$g_\tau(t) = g(k\tau) \text{ if } t \in [k\tau, (k+1)\tau),$$

and $\overline{g}_\tau$ as its non-decreasing envelope in the notation (2.3.34). The sets D_k^τ satisfy

$$\begin{aligned} 2\left|D_k^\tau \cap \left(\left[0, \frac{\varepsilon}{2}\right) + \varepsilon\mathbb{N}\right)\right| &= \lambda_{1,\min}(\overline{g}_\tau(k\tau)) \;\text{ and} \\ 2\left|D_k^\tau \setminus \left(\left[0, \frac{\varepsilon}{2}\right) + \varepsilon\mathbb{N}\right)\right| &= \lambda_{2,\min}(\overline{g}_\tau(k\tau)). \end{aligned} \tag{2.3.40}$$

This can be proved by induction. Indeed, (2.3.40) is satisfied for $k = 0$, since $g(0) = 0$ and $D_0^\tau = \emptyset$. Assume it holds true with $k-1$ in the place of k. We have two cases: if $\overline{g}_\tau(k\tau) > \overline{g}_\tau((k-1)\tau)$ then $\overline{g}(k\tau) = g_\tau(k\tau)$ and the validity of (2.3.40) follows by the minimality properties of $\lambda_{1,\min}$ and $\lambda_{2,\min}$; if otherwise $\overline{g}_\tau(k\tau) = \overline{g}_\tau((k-1)\tau)$ then the conclusion follows by noting that $D_k^\tau = D_{k-1}^\tau$ as a consequence of Remark 2.6. Passing to the limit as $\tau \to 0$ we then obtain property (ii), after noting the uniform convergence of $\overline{g}_\tau$ to $\overline{g}$. ☐

Remark 2.10. For any quasi-static evolution $(u^\varepsilon(t), D^\varepsilon(t))$ in the sense of Definition 2.3.1, for fixed t the sets $D^\varepsilon(t)$ do not converge to sets as $\varepsilon \to 0$, except for the trivial cases $\emptyset$ and $(0,1)$. Indeed, for example for $p_1 < p_2$ and $t \in \left[\frac{p_1(\beta_2+\alpha_1)}{2\beta_2\alpha_1}, \frac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1}\right]$, we have that $D^\varepsilon(t) = \varepsilon\big(\mathbb{N} + \big[0, \frac{1}{2}\big)\big)$, whose characteristic functions weakly converge to the constant $\frac{1}{2}$. This suggests that to study the limit behaviour of the quasi-static evolution characterized in Theorem 2.9 we must consider in the limit density function $\theta(t)$ instead of characteristic function $\chi_{A(t)}$ of the damage set.

2.4. Quasi-static evolution for non-homogeneous materials

In this section we compute the Γ-limit of the energy functionals $E^\varepsilon_{\mathrm{Tot}}$, we give the definition of (approximable) quasi-static evolution for such limit and we show that the approximable quasi-static evolutions related to the energy functionals $E^\varepsilon_{\mathrm{Tot}}$ converge, up to subsequences, to the approximable quasi-static evolutions related to the Γ-limit of such energy functionals. Vice versa, any approximable quasi-static evolution for the Γ-limit of the functionals (2.3.2) is the limit of the corresponding approximable quasi-static evolutions. From this point of view it is a commutative result about Γ-convergence a and quasi-static evolutions.

2.4.1. Relaxed homogenization. We start computing the Γ-limit of the family of functionals $E^\varepsilon_{\mathrm{Tot}}$. We tacitly identify sets with the characteristic functions as elements of $L^1(0,1)$.

Theorem 2.11 (relaxed homogenization). *Let (2.3.11) hold. Then the family $E^\varepsilon_{\mathrm{Tot}}$ in (2.3.2) Γ-converges, in the $L^2 \times L^1$-weak topology, to the functional*

$$E^{\mathrm{hom}}_{\mathrm{Tot}}(u,\theta) = E^{\mathrm{hom}}(u,\theta) + \mathcal{D}^{\mathrm{hom}}(\theta)\,, \tag{2.4.41}$$

where
$$E^{\hom}(u,\theta) = \int_{(0,1)} f_{\hom}(\theta)|u'|^2 dx, \tag{2.4.42}$$
with
$$f_{\hom}(\theta) = \begin{cases} \left[\frac{\beta_1+\beta_2}{2\beta_1\beta_2} + \frac{(\beta_1-\alpha_1)}{\beta_1\alpha_1}\theta\right]^{-1} & \textit{if} \quad \theta \in [0,\frac{1}{2}) \\ \left[\frac{\beta_2+\alpha_1}{2\beta_2\alpha_1} + \frac{(\beta_2-\alpha_2)}{2\beta_2\alpha_2}(2\theta-1)\right]^{-1} & \textit{if} \quad \theta \in [\frac{1}{2},1) \end{cases} \tag{2.4.43}$$
and
$$\mathcal{D}^{\hom}(\theta) = \int_{(0,1)} \gamma_{\hom}(\theta)\,dx, \quad \gamma_{\hom}(\theta) = \begin{cases} \gamma_1\theta & \textit{if } \theta \in [0,\frac{1}{2}) \\ \frac{\gamma_1}{2} + \gamma_2(\theta-\frac{1}{2}) & \textit{if } \theta \in [\frac{1}{2},1). \end{cases} \tag{2.4.44}$$

Proof. This is a particular case of homogenization in L^p spaces, where the cell-problem formula rewrites as
$$\begin{aligned} \phi(\theta,z) \quad := \quad \min\Big\{ & \int_D \alpha(y)|v|^2\,dy + \int_{(0,1)\setminus D} \beta(y)|v|^2\,dy + \int_D \gamma(y)\,dy \ : \\ & D \subset (0,1),\ |D| = \theta, \int_0^1 v\,dx = z\Big\}. \end{aligned}$$
Note that, minimizing first in v, and denoting by $\eta_1 = |D \cap [0,\frac{1}{2}]|$ and $\eta_2 = |D \setminus [0,\frac{1}{2}]|$, we obtain
$$\phi(\theta,z) = \min\Big\{ f^{\hom}(\eta_1,\eta_2)z^2 + \gamma_1\eta_1 + \gamma_2\eta_2 \ : \eta_1+\eta_2 = \theta\Big\}, \tag{2.4.45}$$
with $f^{\hom}(\eta_1,\eta_2)$ defined in (2.3.27). By a direct computation we get the unique minimal
$$\eta_1 = \min\Big\{\theta, \frac{1}{2}\Big\}, \qquad \eta_2 = \max\Big\{\Big(\theta - \frac{1}{2}\Big), 0\Big\} \tag{2.4.46}$$
and
$$\phi(\theta,z) = f_{\hom}(\theta)z^2 + \gamma_{\hom}(\theta),$$
and the desired characterization. □

As done in Proposition 2.1 and in Corollary 2.3 we are interested at the minimum value and at the minimum points of such limit energy. Let remark that, by Corollary 2.3, considering $(u^\varepsilon(t), \chi_{D^\varepsilon(t)})$ minimum point for $E^\varepsilon_{\mathrm{Tot}}$ it holds the followings

1) $D^\varepsilon(t)$ is such that
$$|D^\varepsilon(t)| \le \frac{1}{2} \qquad \text{if } t < \frac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1}$$
and
$$|D^\varepsilon(t)| \ge \frac{1}{2} \qquad \text{if } t \ge \frac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1}$$
2) $\lambda_{min}(t)$ satisfies
$$\int_{(0,1)} \chi_{D^\varepsilon} dx = \lambda_{min}(t)$$
3) u^ε is the unique minimum point of
$$\min\{E^\varepsilon_{\mathrm{Tot}}(u, D^\varepsilon) : u(0) = 0, u(1) = t\}.$$

In the Corollary 2.12 we show that analogous properties are true for (u,θ) minimum points for $E^{\hom}_{\mathrm{Tot}}$.

Corollary 2.12. *For all $t \geq 0$ we have*

$$\min\Big\{E^{\mathrm{hom}}_{\mathrm{Tot}}(v,\theta) : v \in H^1(0,1),\ v(0)=0,\ v(1)=t,\ 0\leq\theta\leq 1\Big\} = m(t), \tag{2.4.47}$$

where m is given by (2.3.12). Furthermore, the minimizers (u,θ) for this problem are characterized by the following properties:

(i) *either $\theta \geq \frac{1}{2}$ a.e. or $\theta \leq \frac{1}{2}$ a.e.;*

(ii) *we have*

$$\int_{(0,1)} \theta\, dx = \lambda_{\min}(t), \tag{2.4.48}$$

where $\lambda_{\min}(t)$ is given by (2.3.21);

(iii) *u is the unique minimizer of*

$$\min\Big\{E^{\mathrm{hom}}(v,\theta) : v \in H^1(0,1),\ v(0)=0,\ v(1)=t\Big\}. \tag{2.4.49}$$

Proof. The corollary follows from a direct computation, or from the previous theorem, Proposition 2.1 and the Fundamental Theorem of Γ-convergence. To that end, note that the characterization of m in the proof of Proposition 2.1 guarantees that sequences $(u_\varepsilon, D_\varepsilon)$ such that $u_\varepsilon(0)=0$, $u_\varepsilon(1)=t$ and $E^\varepsilon_{\mathrm{Tot}}(u_\varepsilon, D_\varepsilon) = m(t)+o(1)$ as $\varepsilon\to 0$ have the same cluster points as the sequences of minimizers of (2.4.47).

If $(u(t),\theta(t))$ satisfy (i)–(iii) then we can define $D_\varepsilon(t)$ such that $\chi_{D_\varepsilon(t)}$ weakly converges to $\theta(t)$, $|D_\varepsilon(t)| = \lambda_{\min}(t)$, $D_\varepsilon(t) \supset [0,\frac{\varepsilon}{2})+\varepsilon\mathbb{N}$ or $D_\varepsilon(t)\subset[0,\frac{\varepsilon}{2})+\varepsilon\mathbb{N}$, and u_ε is the corresponding solution of $\min E^\varepsilon_{\mathrm{Tot}}(u, D_\varepsilon(t))$ with $u(0)=0$ and $u(1)=t$. By Proposition 2.1 and Corollary 2.3 $(u_\varepsilon(t), D_\varepsilon(t))$ is a minimizer of $E^\varepsilon_{\mathrm{Tot}}(u,D)$ with $u(0)=0$ and $u(1)=t$ and then converges to a minimizer of $E^{\mathrm{hom}}_{\mathrm{Tot}}(u,\theta)$ subject to the same boundary conditions. □

Remark 2.13. Note that we do not have the separate Γ-convergence of E^ε and $\mathcal{D}^\varepsilon$ to E^{hom} and $\mathcal{D}^{\mathrm{hom}}$. This is evident from the dependence of the form of the limit functionals on inequality (2.3.11). This fact has an important consequence. Considering the separate Γ-limit $E^{\mathrm{hom},2}$ of E^ε and $\mathcal{D}^{\mathrm{hom},2}$ of $\tilde{\mathcal{D}}^\varepsilon$ (which exist up to subsequence) and suppose to have a quasi-static evolution for this limit energy $E^{\mathrm{hom},2}+\mathcal{D}^{\mathrm{hom},2}$. One could wonder if the (approximable) quasi-static evolution $(u^\varepsilon, D^\varepsilon)$ for the energy $E^\varepsilon+\tilde{\mathcal{D}}^\varepsilon$ converge to (u,θ) (approximable) quasi-static evolution for the energy $E^{\mathrm{hom},2}+\mathcal{D}^{\mathrm{hom},2}$ and viceversa. This kind of commutability result is the one analyzed in an abstract framework by Mielke, Roubichek and Stefanelli in [64] using the so called *mutual recovery sequence* . But in our case since we will prove in Theorem 2.18 that every (approximable) quasi-static evolution at step $\epsilon>0$ converges to a (approximable) quasi-static evolution for the energy $E^{\mathrm{hom}}+\mathcal{D}^{\mathrm{hom}} \neq E^{\mathrm{hom},2}+\mathcal{D}^{\mathrm{hom},2}$ and viceversa, we conclude that from this point of view there is not commutability between (separated) Γ-convergence and quasi-static evolutions.

Proposition 2.14 (compatibility of constraints). *Let B_ε be a family of subsets of $(0,1)$ and $\varphi\in L^1(0,1)$, such that $\chi_{B_\varepsilon} \overset{*}{\rightharpoonup} \varphi$ and*

$$\Gamma\text{-}\lim_{\varepsilon\to 0} E^\varepsilon_{\mathrm{Tot}}(\cdot, B_\varepsilon) = E^{\mathrm{hom}}_{\mathrm{Tot}}(\cdot,\varphi) \tag{2.4.50}$$

with respect to the L^2-convergence, then the Γ-limit of

$$E^\varepsilon_{\mathrm{Tot}}(u,D;B_\varepsilon) := \begin{cases} E^\varepsilon(u,D)+\tilde{\mathcal{D}}^\varepsilon(D) & \text{if} \quad D\supset B_\varepsilon \\ +\infty & \text{otherwise} \end{cases} \tag{2.4.51}$$

with respect to the $L^2\times L^1$-weak convergence is

$$E^{\mathrm{hom}}_{\mathrm{Tot}}(u,\theta;\varphi) := \begin{cases} E^{\mathrm{hom}}(u,\theta)+\mathcal{D}^{\mathrm{hom}}(\theta) & \text{if} \quad \theta\geq\varphi \\ +\infty & \text{otherwise.} \end{cases} \tag{2.4.52}$$

REMARK 2.15. Let show that condition (2.4.50) is equivalent to requiring that

$$\chi_{B_\varepsilon\cap([0,\frac{\varepsilon}{2})+\varepsilon\mathbb{N})} \rightharpoonup \varphi_1 := \max\left\{\varphi, \frac{1}{2}\right\} \tag{2.4.53}$$

or, equivalently, that

$$\chi_{B_\varepsilon\cap([\frac{\varepsilon}{2},\varepsilon)+\varepsilon\mathbb{N})} \rightharpoonup \varphi_1 := \min\left\{\varphi - \frac{1}{2}, 0\right\}. \tag{2.4.54}$$

In order to check (2.4.53), denote with φ_1 the weak limit of the sequence on the left-hand side of (2.4.53), which exists up to subsequences, and $\varphi_2 = \varphi - \varphi_1$, which is the weak limit of the sequence on the left-hand side of (2.4.54). Note that (we do not relabel the subsequence)

$$\Gamma\text{-}\lim_{\varepsilon\to 0} E^{\varepsilon}_{\mathrm{Tot}}(u, B_\varepsilon) = F(u, \varphi_1, \varphi_2), \tag{2.4.55}$$

where

$$F(u, \varphi_1, \varphi_2) := \int_{(0,1)} f^{\mathrm{hom}}(\varphi_1, \varphi_2)|u'|^2 dx + \gamma_1 \int_{(0,1)} \varphi_1\, dx + \gamma_2 \int_{(0,1)} \varphi_2\, dx \tag{2.4.56}$$

and f^{hom} is defined in (2.3.27). This immediately follows from the convergence of the dissipation term, and the characterization of one-dimensional Γ-convergence (see Theorem 1.7 and Example 1.8).
So by (2.4.45) and (2.4.46) we have that $F(u, \varphi_1, \varphi_2) = E^{\mathrm{hom}}_{\mathrm{Tot}}(u, \varphi)$,i.e., we have the 2.4.50) if and only if φ_1 and φ_2 are as in (2.4.53) and (2.4.54).

PROOF OF PROPOSITION 2.14. The lower bound inequality is trivial since the constraint is closed. As for the upper bound, with fixed $\theta \geq \varphi$, we use a diagonal argument, first constructing a recovery sequence of sets for a sequence of θ^σ converging to θ.

With fixed $\sigma > 0$, for all x Lebesgue point of φ, φ_1 (as defined in Remark 2.15) and θ, we consider the family

$$\begin{aligned} \mathcal{I}^\sigma_x \;=\; & \Big\{ I = (x-\delta, x+\delta) \subset (0,1) : \delta < \sigma, \\ & \int_I |\varphi(x) - \varphi|\, dy + \int_I |\varphi_1(x) - \varphi_1|\, dy + \int_I |\theta(x) - \theta|\, dy < \sigma |I| \Big\}. \end{aligned}$$

Since $\mathcal{I}^\sigma = \bigcup_x \mathcal{I}^\sigma_x$ is a fine cover of the set of Lebesgue points of $(0,1)$ we can find a finite family of disjoint intervals $\{I^\sigma_k\}$ of $\mathcal{I}^\sigma$ such that

$$\Big|(0,1) \setminus \bigcup_k I^\sigma_k\Big| < \sigma.$$

We construct subsets D^σ_ε of $(0,1)$ defining them on each such interval

$$I^\sigma_k = (x^\sigma_k - \delta^\sigma_k, x^\sigma_k + \delta^\sigma_k)$$

as follows:

(i) $D^\sigma_\varepsilon \cap I^\sigma_k \supset B_\varepsilon \cap I^\sigma_k$;

(ii) $|D^\sigma_\varepsilon \cap I^\sigma_k| = \int_{I^\sigma_k} \theta\, dy$.

If $\varphi(x^\sigma_k) > \frac{1}{2}$ conditions (i) and (ii) are the only ones required in our construction; otherwise, if $\varphi(x^\sigma_k) \leq \frac{1}{2}$, we have to require some additional conditions. In order to specify such conditions we introduce the notation

$$D^\sigma_{\varepsilon,1} = D^\sigma_\varepsilon \cap \left(\left[0, \frac{\varepsilon}{2}\right) + \varepsilon\mathbb{N}\right), \qquad D^\sigma_{\varepsilon,2} = D^\sigma_\varepsilon \setminus D^\sigma_{\varepsilon,1}$$

and

$$B_{\varepsilon,1} = B_\varepsilon \cap \left(\left[0, \frac{\varepsilon}{2}\right) + \varepsilon\mathbb{N}\right), \qquad B_{\varepsilon,2} = B_\varepsilon \setminus B_{\varepsilon,1}.$$

(iiia) if $\theta(x_k^\sigma) \le \frac{1}{2}$ then

$$D^\sigma_{\varepsilon,2} \cap I^\sigma_k = B_{\varepsilon,2} \cap I^\sigma_k; \tag{2.4.57}$$

(iiib) if $\theta(x_k^\sigma) > \frac{1}{2}$ then

$$|D^\sigma_{\varepsilon,2} \cap I^\sigma_k| = \max\left\{|B_{\varepsilon,2} \cap I^\sigma_k|, \int_{I^\sigma_k} \Big(\theta - \frac{1}{2}\Big)\,dy\right\}. \tag{2.4.58}$$

We finally include in the sets D^σ_ε the complement of $\bigcup_k I^\sigma_k$.
Up to a subsequence we have that

$$\chi_{D^\sigma_\varepsilon} \rightharpoonup \theta^\sigma, \qquad \text{and} \qquad \chi_{D^\sigma_{\varepsilon,2}} \rightharpoonup \theta^\sigma_2$$

as $\varepsilon \to 0$, for some θ^σ_2 and θ^σ.
By the fact that I^σ_k belong to $\mathcal{I}^\sigma_{x^\sigma_k}$, that B_ε satisfy the optimality condition (2.4.54), and by the properties of D^σ_ε and $D^\sigma_{\varepsilon,2}$, we have: for all intervals $I \subset (0,1)$

$$\Big|\int_I \theta^\sigma_2\,dy - \int_I \widetilde{\theta}^\sigma\,dy\Big| \le 4\sigma,$$

where

$$\widetilde{\theta}^\sigma(x) = \begin{cases} \Big(\theta(x^\sigma_k) - \frac{1}{2}\Big)^+ & \text{if } x \in I^\sigma_k \\ 0 & \text{otherwise.} \end{cases}$$

Since $\widetilde{\theta}^\sigma$ converges in $L^1(0,1)$ to $\Big(\theta - \frac{1}{2}\Big)^+$, we deduce that

$$\theta^\sigma \rightharpoonup \theta \qquad \text{and} \qquad \theta^\sigma_2 \rightharpoonup \Big(\theta - \frac{1}{2}\Big)^+$$

as $\sigma \to 0$.
By a diagonal argument, we can construct $D_\varepsilon = D^{\sigma(\varepsilon)}_\varepsilon \supseteq B_\varepsilon$ which thanks to (2.4.54) satisfies

$$\Gamma\text{-}\lim_{\varepsilon\to 0} E^\varepsilon_{\mathrm{Tot}}(\cdot, D_\varepsilon) = E^{\mathrm{hom}}_{\mathrm{Tot}}(\cdot, \theta), \tag{2.4.59}$$

which implies the desired upper bound. □

We now prove a result which is the analogous of the one stated in Remark 2.6 but in a homogenized situation.

Corollary 2.16. *Given $s \in [0,T]$. Assume that $\varphi : [0,1] \to [0,1]$ satisfies $\varphi \le \frac{1}{2}$ a.e. or $\varphi \ge \frac{1}{2}$ a.e. and*

$$\int_{(0,1)} \varphi\,dx > \lambda_{\min}(s). \tag{2.4.60}$$

Then

$$\min\{E^{\mathrm{hom}}_{\mathrm{Tot}}(u,\varphi) : u(0) = 0,\ u(1) = s\} \le \min\{E^{\mathrm{hom}}_{\mathrm{Tot}}(u,\theta) : u(0) = 0,\ u(1) = s\}$$

for all $\theta \ge \varphi$.

Proof. This is a direct consequence of the Γ-convergence result above, combined with Remark 2.6. Indeed, denoting by $t > s$ the time value such that

$$\int_{(0,1)} \varphi\,dx = \lambda_{\min}(t), \tag{2.4.61}$$

we have, using Corollary 2.12, that φ can be approximated by a sequence χ_{B_ε}, with B_ε satisfying the assumption of Proposition 2.14 and

$$2|B_\varepsilon \cap ([0,\varepsilon/2) + \varepsilon\mathbb{N})| = \lambda_{1,\min}(t)$$

and

$$2|B_\varepsilon \setminus ([0,\varepsilon/2)+\varepsilon\mathbb{N}))| = \lambda_{2,\min}(t).$$

Then by Remark 2.6 we get that

$$\min\{E^\varepsilon_{\rm Tot}(u,B_\varepsilon) : u(0)=0,\ u(1)=s\} \le \min\{E^\varepsilon_{\rm Tot}(u,D):\ u(0)=0,\ u(1)=s\}$$

for all $D \supset B_\varepsilon$. We conclude applying Proposition 2.14. $\square$

2.4.2. Quasi-static evolution. As done for the (approximable) quasi-static evolution at step $\varepsilon>0$ we state now the definition of quasi-static evolution for the homogenized energy functional in (2.4.52) and describe explicitly the behaviour of such motions in Theorem 2.17 through a characterization result.

DEFINITION 2.4.1. Given $g \in AC([0,T])$, with $g(0)=0$, we say that $(u(t),\theta(t))$ is a *(weak) quasi-static evolution* (for the energy (2.4.41)) if for all $t\in[0,T]$ we have $u(t)\in H^1(0,1)$, $u(0)=0$, $u(1)=g(t)$, $\theta(t)\in L^\infty(0,1)$, $0\le\theta\le 1$, and the following properties hold:

- **Damage Irreversibility**: $\theta(t)$ is non-decreasing in time;
- **Energy Balance**:

$$E^{\rm hom}_{\rm Tot}(u(t),\theta(t)) = E^{\rm hom}_{\rm Tot}(u(0),\theta(0)) + 2\int_0^t \dot g(s)\int_{(0,1)} f_{\rm hom}(\theta)\,u'(s,x)\,dxds; \tag{2.4.62}$$

- **Minimality Condition**:

$$E^{\rm hom}_{\rm Tot}(u(t),\theta(t)) \le E^{\rm hom}_{\rm Tot}(v,\psi), \tag{2.4.63}$$

  for all $v\in H^1(0,1)$ $v(0)=0$, $v(1)=g(t)$ and $\psi\in L^\infty(0,1)$, $\psi\ge\theta(t)$.

Moreover, we say that $(u(t),D(t))$ is an *approximable (strong) quasi-static evolution* $(u(t),\theta(t))$ is an *approximable (weak) quasi-static evolution* (for the energy (2.4.41) subjected to the boundary condition g) if it satisfies the conditions above and if considering the solution (u^τ_k,θ^τ_k) of the discrete problem (for $k\ge1$)

$$\min\Big\{E^{\rm hom}_{\rm Tot}(v,\theta) : v(0)=0,\ v(1)=g(k\tau),\ \theta^\tau_{k-1}\subset\theta\Big\} \tag{2.4.64}$$

and defined

$$u_\tau(t) := u^\tau_k \text{ if } t\in[k\tau,(k+1)\tau)$$

and

$$\theta_\tau(t) := \theta^\tau_k \text{ if } t\in[k\tau,(k+1)\tau)$$

with $u^\tau_0=0$ and $\theta^\tau_0=\emptyset$, it holds (up to subsequence)

$$u_\tau(t) \overset{H^1}{\rightharpoonup} u(t)$$

and

$$\theta_\tau(t) \overset{*}{\rightharpoonup} \theta(t)$$

when $\tau\longrightarrow 0$.

THEOREM 2.17. *Every approximable (weak) quasi-static evolution $(u(t),\theta(t))$ for $E^{\rm hom}_{\rm Tot}$ is characterized by the following properties:*
(i) *$\theta(t)$ is non-decreasing in t;*
(ii) *$\theta(t)\le\frac12$ a.e. or $\theta(t)\ge\frac12$ a.e., and $\int_{(0,1)}\theta(t)\,dx = \lambda_{\min}(\overline g(t))$;*
(iii) *u is the unique minimizer of*

$$\min\Big\{E^{\rm hom}_{\rm Tot}(v,\theta(t)) : v(0)=0,\ v(1)=g(t)\Big\}. \tag{2.4.65}$$

Proof. By [57] Theorem 4.5, all limits of incremental problems (2.4.64), which exist up to subsequences, are (weak) quasi-static evolutions for the energy $E^{\text{hom}}_{\text{Tot}}$. Then it is enough to show that, for any pair $(u(t), \theta(t))$ satisfying (i)–(iii), we can construct an incremental problem whose solutions converge to $(u(t), \theta(t))$, and that any limit of solutions of incremental problems satisfy (i)–(iii).
Let $(u(t), \theta(t))$ satisfy (i)–(iii) and for every $\tau > 0$, as in the proof of Theorem 2.9, denote by $g_\tau(t)$ the the piecewise-constant function

$$g_\tau(t) = g(k\tau) \text{ if } t \in [k\tau, (k+1)\tau)$$

and let $\overline{g}_\tau(t)$ be its non-decreasing envelope as in (2.3.34). Then we consider a family θ^τ_k, with either $\theta^\tau_k \geq \frac{1}{2}$ a.e. or $\theta^\tau_k \leq \frac{1}{2}$ a.e.,

$$\int_{(0,1)} \theta^\tau_k \, dx = \max\{\lambda_{\min}(g(j\tau)) : j \leq k\} = \lambda_{\min}\big(\overline{g}_\tau(k\tau)\big),$$

and

$$\theta^\tau_{k-1} \leq \theta^\tau_k \leq \theta(k\tau). \tag{2.4.66}$$

This can be done by induction. We also consider the corresponding u^τ_k minimum for the problem

$$\min\Big\{E^{\text{hom}}_{\text{Tot}}(v, \theta^\tau_k) : v(0) = 0,\ v(1) = g(k\tau)\Big\}. \tag{2.4.67}$$

We can show that, by construction, the family $(u^\tau_k, \theta^\tau_k)$ is a solution of the incremental problem

$$E^{\text{hom}}_{\text{Tot}}(u^\tau_k, \theta^\tau_k) \leq E^{\text{hom}}_{\text{Tot}}(v, \varphi)$$

for every $v \in H^1(0,1)$, with $v(0) = 0$ and $v(1) = g(k\tau)$ and for every $\varphi \geq \theta^\tau_{k-1}$. Indeed if

$$\int_{(0,1)} \theta^\tau_{k-1} \, dx \leq \lambda_{\min}(g(k\tau))$$

then by Corollary 2.12 such θ^τ_k minimizes

$$\begin{aligned} \min\Big\{E^{\text{hom}}_{\text{Tot}}(v, \varphi) \ :\ v \in H^1(0,1),\ v(0) = 0,\ v(1) = g(k\tau) \ \text{ and } \varphi \geq \theta^\tau_{k-1}\Big\} \\ = \min\Big\{E^{\text{hom}}_{\text{Tot}}(v, \theta^\tau_k) \ :\ v \in H^1(0,1),\ v(0) = 0,\ v(1) = g(k\tau)\Big\}, \end{aligned} \tag{2.4.68}$$

while if

$$\int_{(0,1)} \theta^\tau_{k-1} \, dx > \lambda_{\min}(g(k\tau)),$$

then, by Corollary 2.16, we deduce that $\theta^\tau_k = \theta^\tau_{k-1}$. By (2.4.66) we deduce that the piecewise-constant functions $(u^\tau(t), \theta^\tau(t)) = (u^\tau_k, \theta^\tau_k)$ if $t \in [k\tau, (k+1)\tau)$ converge to $(u(t), \theta(t))$ for all $t \in [0, T]$, which proves the approximability of $(u(t), \theta(t))$.

On the other hand if $(u(t), \theta(t))$ is an approximable quasi-static evolution, let $(u^\tau_k, \theta^\tau_k)$ be a solution of the incremental problem (2.4.64) which converges to $(u(t), \theta(t))$. We can prove by induction that

$$\theta^\tau_k \leq \frac{1}{2} \text{ a.e., or } \theta^\tau_k \geq \frac{1}{2} \text{ a.e., and } \int_{(0,1)} \theta^\tau_k \, dx = \lambda_{\min}(\overline{g}_\tau(k\tau)). \tag{2.4.69}$$

Indeed, if $k = 0$ this is trivially true. Assume that (2.4.69) holds with k replaced by $k-1$. If $\lambda_{\min}(\overline{g}_\tau(k\tau)) = \lambda_{\min}(\overline{g}_\tau((k-1)\tau))$ then $\lambda_{\min}(\overline{g}_\tau(k\tau)) \geq \lambda_{\min}(g(k\tau))$, and hence, by Corollary 2.16 we have $\theta^\tau_k = \theta^\tau_{k-1}$. Otherwise, if $\lambda_{\min}(\overline{g}_\tau(k\tau)) > \lambda_{\min}(\overline{g}_\tau((k-1)\tau))$ then $\lambda_{\min}(\overline{g}_\tau(k\tau)) = \lambda_{\min}(g(k\tau))$, and the conclusion follows by Corollary 2.12. Properties (i)–(iii) then follow by (2.4.69) taking the limit as $\tau \to 0$. $\square$

We show now that an approximable quasi-static evolution $(u^\varepsilon(t), D^\varepsilon(t))$ for $E^\varepsilon_{\rm Tot}(u, D)$ converges (up to subsequences) to a pair $(u(t), \theta(t))$, approximable quasi-static evolution for $E^{\rm hom}_{\rm Tot}(u, \theta)$ and viceversa.

THEOREM 2.18. *Any approximable quasi-static evolution* $(u^\varepsilon(t), D^\varepsilon(t))$ *for* $E^\varepsilon_{\rm Tot}(u, D)$ *converges (up to subsequences) to a pair* $(u(t), \theta(t))$ *in the* $L^2 \times L^1$*-weak convergence. Moreover,* $(u(t), \theta(t))$ *is an approximable quasi-static evolution for* $E^{\rm hom}_{\rm Tot}(u, \theta)$*.*

Conversely, any approximable quasi-static evolution $(u(t), \theta(t))$ *for* $E^{\rm hom}_{\rm Tot}(u, \theta)$ *is the limit as* $\varepsilon \to 0$ *of an approximable quasi-static evolution* $(u^\varepsilon(t), D^\varepsilon(t))$ *for* $E^\varepsilon_{\rm Tot}(u, D)$*.*

PROOF. By the monotonicity condition of $D^\varepsilon(t)$, using Helly's theorem, we can find a subsequence such that (up to relabelling the apices)

$$\chi_{D^\varepsilon(t)} \overset{*}{\rightharpoonup} \theta(t) \qquad \text{and} \qquad \chi_{D^\varepsilon(t)\cap([0,\frac{\varepsilon}{2})+\varepsilon\mathbb{N})} \rightharpoonup \theta_1(t)$$

in $L^1(0,1)$ for all t.

Since (i)–(iii) of Theorem 2.9 are satisfied for D^ε, then, taking the limit as $\varepsilon \to 0$ we deduce (i)–(iii) of Theorem 2.17 for (u, θ).

On the other hand, let $(u(t), \theta(t))$ be an approximable quasi-static evolution for $E^{\rm hom}_{\rm Tot}(u, \theta)$. By Theorem 2.17 it satisfies (i)-(iii) therein. We then construct for all $t \in [0, T]$ the set $D_\varepsilon(t)$ as follows

$$D_\varepsilon(t) = \bigcup_k \Big(k\varepsilon, k\varepsilon + \int_{(k\varepsilon,(k+1)\varepsilon)} \theta(t)\,dx\Big),$$

and let $u_\varepsilon(t)$ be the corresponding minimizer of $v \mapsto E^\varepsilon_{\rm Tot}(v, D_\varepsilon(t))$ with boundary conditions $v(0) = 0$ and $v(1) = g(t)$. With this definition $(u_\varepsilon(t), D_\varepsilon(t))$ satisfy (i)–(iii) of Theorem 2.9 and hence, it is an approximable quasi-static evolution for $E^\varepsilon_{\rm Tot}(u, D)$, and converge to $(u(t), \theta(t))$. □

2.5. Quasi-static evolution for a three-phase material

In this final section, we use the characterization in Theorem 2.17 to show that the limit evolution can be interpreted as a weak evolution of a three-phase material. To that end, we introduce a double damage set model that generalizes the one introduced by Francfort and Marigo as follows. We consider positive constants $a < b < c$, k_1 and k_2, the energy

$$E^{\rm 3P}(u, D_1, D_2) = a\int_{D_2} |u'|^2\,dx + b\int_{D_1} |u'|^2\,dx + c\int_{(0,1)\setminus(D_1\cup D_2)} |u'|^2\,dx \tag{2.5.70}$$

and the dissipation

$$\mathcal{D}^{\rm 3P}(D_1, D_2) = k_1|D_1| + k_2|D_2|\,,$$

with domain pairs of disjoint subsets D_1, D_2 of $(0,1)$. This can be interpreted as the damage model for a three-phase material, where c is the elastic constant of the undamaged state, b the one of the 'partly damaged' state, and a the one of the 'totally damaged' state. The constant k_1 represents the cost of the partly damaged state and k_2 the one of the totally damaged state. In general, we could consider also an 'intermediate' dissipation $k_{1,2}$ which accounts for the transition from the partly damaged state to the totally damaged state. Our model corresponds to the case

$$k_{1,2} = k_2 - k_1\,.$$

This assumption reflects the fact that the material in order to reach the totally damaged state should pass through the intermediate partly damaged state.

The incremental problem for this model consists in solving iteratively

$$\min_{u,D_1,D_2}\Big\{E^{3P}(u,D_1,D_2)+\mathcal{D}^{3P}(D_1,D_2):\ D_1\cap D_2=\emptyset,\ D_1\cup D_2\supset D_1^{k-1}\cup D_2^{k-1},\\ D_2\supset D_2^{k-1},\ u(0)=0,\ u(1)=g(k\tau)\Big\}. \tag{2.5.71}$$

The monotonicity conditions on the sets correspond to the assumption that the totally damaged state can only increase, while the partially damaged set can become totally damaged.

We first note that problems (2.5.71) may undergo relaxation with respect to the weak convergence in H^1 for u and weak convergence in L^1 for the sets, understood as the weak convergence of their characteristic functions. We are then lead to considering the following relaxed functional

$$E^{3P}_{\mathrm{Tot}}(u,\varphi,\psi):=\int_{(0,1)}H(\varphi,\psi)|u'|^2dx+k_1\int_{(0,1)}\varphi\,dx+k_2\int_{(0,1)}\psi\,dx. \tag{2.5.72}$$

where

$$H(\eta_1,\eta_2)=\left[\frac{1-(\eta_1+\eta_2)}{c}+\frac{\eta_1}{b}+\frac{\eta_2}{a}\right]^{-1}. \tag{2.5.73}$$

The fact that it is the relaxed version of 2.5.70 is an immediate consequence of the characterization of one-dimensional Γ-convergence, once we observe that

$$E^{3P}(u,D_1,D_2)=\int_{(0,1)}\Big(c\chi_{(0,1)\setminus(D_1\cup D_2)}+b\chi_{D_1}+a\chi_{D_2}\Big)|u'|^2\,dx$$

and we can write

$$\frac{1}{c\chi_{(0,1)\setminus(D_1\cup D_2)}+b\chi_{D_1}+a\chi_{D_2}}=\frac{1}{c}\chi_{(0,1)\setminus(D_1\cup D_2)}+\frac{1}{b}\chi_{D_1}+\frac{1}{a}\chi_{D_2}.$$

We give a definition of (weak) quasi-static evolution for these energies as follows. Note that in this definition the monotonicity conditions on D_1 and D_2 given in problems (2.5.71) correspond to conditions on the functions φ and $\varphi+\psi$.

Definition 2.5.1. Given $g\in AC([0,T])$, with $g(0)=0$, we say that $(u(t),\varphi(t),\psi(t))$ is a (three-phase) quasi-static evolution for the energy (2.5.72) if for all $t\in[0,T]$ we have $u(t)\in H^1(0,1)$, $u(0)=0$, $u(1)=g(t)$, $\psi(t)\in L^\infty(0,1)$, $0\le\psi(t)\le 1$, $\varphi(t)\in L^\infty(0,1)$, $0\le\varphi(t)\le 1$, $\varphi(t)+\psi(t)\le 1$, and the following properties hold

- **Damage irreversibility** $\psi(t)$ and $\varphi(t)+\psi(t)$ are increasing in time for each $x\in(0,1)$,
- **Energy Balance**

$$E^{3P}_{\mathrm{Tot}}(u(t),\psi(t),\varphi(t))=E^{3P}_{\mathrm{Tot}}(u(0),\psi(0),\varphi(0))+2\int_0^t\dot g(s)\int_{(0,1)}H(\varphi,\psi)u'dxds$$

  for all $t\in[0,T]$,
- **Minimality Condition**

$$E^{3P}_{\mathrm{Tot}}(u(t),\varphi(t),\psi(t))\le E^{3P}_{\mathrm{Tot}}(v,\tilde\varphi,\tilde\psi)$$

  for all $v:v-u(t)\in H^1_0$, and $(\tilde\varphi,\tilde\psi)$ such that $\tilde\psi\ge\psi(t)$ and $\psi(t)+\varphi(t)\le\tilde\psi+\tilde\varphi\le 1$.

Now we prove that the limit of the quasi-static evolutions considered in Section 2.3 can be seen as a quasi-static evolution of a three-phase homogenized material as in Definition 2.5.1. This will be an immediate consequence of the following proposition.

Proposition 2.19. *If $(u(t), \theta(t))$ is a quasi-static evolution according to the Definition 2.4.1 and we set*

$$(\varphi(t), \psi(t)) = \begin{cases} (2\theta(t), 0) & \text{if } \theta(t) \in [0, \frac{1}{2}) \\ (2(1-\theta(t)), 2\theta(t)-1) & \text{if } \theta(t) \in [\frac{1}{2}, 1), \end{cases} \tag{2.5.74}$$

then $(u(t), \psi(t), \varphi(t))$ is a quasi-static evolution according to Definition 2.5.1, with

$$k_1 = \frac{\gamma_1}{2} \qquad \text{and} \qquad k_2 = \frac{\gamma_1 + \gamma_2}{2},$$

and

$$a = \frac{2\alpha_1\alpha_2}{\alpha_1 + \alpha_2}, \qquad b = \frac{2\alpha_1\beta_2}{\alpha_1 + \beta_2}, \qquad c = \frac{2\beta_1\beta_2}{\beta_1 + \beta_2}.$$

Proof. By the definition of $(\psi(t), \varphi(t))$ the irreversibility of damage is preserved. Moreover, from a direct computation, using the definition of (ψ, φ), k_1 and k_2, a, b, c and the following expression for $f_{\text{hom}}(\theta)$

$$f_{\text{hom}}(\theta) = \begin{cases} \left[\dfrac{\beta_1+\beta_2}{2\beta_1\beta_2}(1-2\theta) + \dfrac{(\beta_2+\alpha_1)}{2\beta_2\alpha_1}2\theta\right]^{-1} & \text{if} \quad \theta \in [0, \frac{1}{2}) \\ \left[\dfrac{\beta_2+\alpha_1}{2\beta_2\alpha_1}2(1-\theta) + \dfrac{(\alpha_1+\alpha_2)}{2\alpha_1\alpha_2}(2\theta-1)\right]^{-1} & \text{if} \quad \theta \in [\frac{1}{2}, 1) \end{cases} \tag{2.5.75}$$

we obtain that

$$f_{\text{hom}}(\theta) = H(\varphi, \psi) \qquad \text{and} \qquad \mathcal{D}^{\text{hom}}(\theta) = k_1 \int_0^1 \varphi \, dx + k_2 \int_0^1 \psi \, dx,$$

which implies immediately the energy balance. It remains to prove the minimality property. To this end we just need to show that for any admissible test pairs $(\tilde{\varphi}, \tilde{\psi})$ for $E^{3\text{P}}_{\text{Tot}}(v, \psi, \varphi)$ (i.e. such that $\tilde{\psi} \geq \psi(t)$ and $\psi(t) + \varphi(t) \leq \tilde{\psi} + \tilde{\varphi} \leq 1$) we can construct an admissible test functions $\tilde{\theta} \geq \theta(t)$ for $E^{\text{hom}}_{\text{Tot}}(v, \theta)$ such that

$$E^{\text{hom}}_{\text{Tot}}(v, \tilde{\theta}) = E^{3\text{P}}_{\text{Tot}}(v, \tilde{\psi}, \tilde{\varphi}).$$

It is enough, given $(\tilde{\varphi}, \tilde{\psi})$ such that $\psi(t) + \varphi(t) \leq \tilde{\psi} + \tilde{\varphi} \leq 1$, to define $\tilde{\theta} = \tilde{\varphi}/2$ if $\tilde{\psi} = 0$ and $\tilde{\theta} = (\tilde{\psi} + 1)/2$ otherwise. This choice allows to conclude. □

Corollary 2.20. *Let $(u^\varepsilon(t), D^\varepsilon(t))$ be a family of approximable quasi-static evolutions for the inhomogeneous two-phase damage energy $E^\varepsilon_{\text{Tot}}(u, D)$. Denoting by $D^\varepsilon_1(t) = D^\varepsilon(t) \cap ([0, \frac{\varepsilon}{2}) + \varepsilon\mathbb{N})$ and $D^\varepsilon_2(t) = D^\varepsilon(t) \setminus ([0, \frac{\varepsilon}{2}) + \varepsilon\mathbb{N})$ the triple $(u^\varepsilon(t), D^\varepsilon_1(t), D^\varepsilon_2(t))$ converges (up to subsequences) to a triple $(u(t), \theta_1(t), \theta_2(t))$ in the $L^2 \times L^1 \times L^1$-weak convergence such that, defining $\varphi(t) = 2(\theta_1(t) - \theta_2(t))$ and $\psi(t) = 2\theta_2(t)$, $(u(t), \psi(t), \varphi(t))$ is a (three-phase) quasi-static evolution in the sense of Definition 2.5.1.*

Proof. The proof is an immediate consequence of Theorem 2.18 and the characterization of $\theta(t)$ in terms of $\theta_1(t)$ and $\theta_2(t)$ (see Remark 2.15). □

CHAPTER 3

Convergence of Regularized Damage Evolutions to Homogenized Threshold Solution

3.1. Main result

The main original result in this chapter is given in Theorem 3.6. We prove that the *strong* quasi-static evolution (i.e. without homogenized effects) associated to the perimeter penalized energy (0.0.19), converges to a *weak* quasi-static evolution (i.e. with homogenization effects) related to the relaxed energy in (3.2.1) (see below). The minimality condition which is satisfied by the limit is proved to be the one introduced in [43] and by this the evolution limit satisfies threshold properties. We restrict our analysis to a bidimensional body occupying the region $\Omega \subset \mathbb{R}^2$ with zero boundary conditions, which can move just in the orthogonal direction to the plane in which the body is embedded (antiplane case) and we consider an external force $f(t) = f(t,x)$ which acts on the body. For simplicity sometimes we identify sets D with associated characteristic function χ_D.

3.2. Introduction

As we explained, when we deal with the energy given by (0.0.11) we can not expect a quasi-static evolution that maintains a well define region of the damage. Moreover we remarked that the process of minimizing of (0.0.11) led to consider the relaxed version

$$E_{tot}(u, A, \theta) = E_{el}(u, A) + k \int_\Omega \theta dx \tag{3.2.1}$$

with

$$E_{el}(u, A) = \frac{1}{2} \int_\Omega A \nabla u \nabla u dx - \langle f(t), u \rangle$$

where $A \in \bar{G}_{\theta(t)}(\alpha I, \beta I)$.
So, it is reasonable to consider for each time t a "density function" $\theta = \theta(x)$ (instead of $\chi_D(x)$) to describe the density damage in x. If we want to maintain the idea that the minimality condition for an optimal set D at time t is given with respect to the sets $\tilde{D}$ that contains D, a possible way to translate this property in the relaxed framework is requiring that this condition is verified by sets D^n that approximate the optimal density function θ (in the sense of characteristic functions).
Precisely, given $A \in L^\infty(\Omega; \mathcal{F}(\alpha, \beta))$ and $\theta \in L^\infty(\Omega; [0,1])$ and a sequence D^n such that

$$\begin{cases} \chi_{D^n} \overset{*}{\rightharpoonup} \theta \\ \sigma_{D^n} \overset{G}{\rightharpoonup} A \end{cases}$$

we define $\hat{G}_{\tilde{\theta}}(\{D^n\})$ as the subset of $\mathcal{G}(\alpha I, \beta I)$ given by all symmetric matrices $\tilde{A}$ that are the G-limit of a subsequence of $\sigma_{\tilde{D}^n}$ with $\tilde{D}^n \supseteq D^n$ and such that $\chi_{\tilde{D}^n} \overset{*}{\rightharpoonup} \tilde{\theta}$. The idea, introduced in [43], is to give a definition of energetic quasi-static evolution for the energy (3.2.1) using the minimality condition with respect to this set as follows

Definition 3.1. Given $f \in W^{1,1}([0,T]; H^{-1}(\Omega))$ we say that $(A(t), \theta(t))$ is an energetic quasi-static evolution for the energy (3.2.1), if for all $t \in [0,T]$ we have $\theta(t) \in L^\infty(\Omega; [0,1])$, $A(t) \in G_{\theta(t)}(\alpha I, \beta I)$ and the following properties hold:

1. **Damage Irreversibility**: $\theta(t)$ is increasing in time and $A(t)$ is decreasing in time;
2. **Energy Balance**:
$$E_{tot}(u(t), A(t), \theta(t)) = E_{tot}(u(0), A(0), \theta(0)) - \int_0^t \langle \dot{f}(s), u(s) \rangle ds;$$
where $u(t)$ is the solution in $H_0^1(\Omega)$ of
$$-div(A(t)\nabla v) = f(t) \text{ in } \ \Omega$$
3. **Minimality Condition**: there exists a family of sequences of sets $D^n(t)$ increasing in time, such that for every $t \in [0,T]$,
$$\begin{cases} \chi_{D^n}(t) \overset{*}{\rightharpoonup} \theta \\ \sigma_{D^n}(t) \xrightarrow{G} A \end{cases}$$
and for every $(\tilde{A}, \tilde{\theta})$ such that $\tilde{A} \in \hat{G}_{\tilde{\theta}}(\{D^n\})$ we have
$$E_{tot}(u(t), A(t), \theta(t)) \le E_{tot}(v, \tilde{A}, \tilde{\theta})$$
for all $v \in H_0^1(\Omega)$.

Remark 3.2. *In a previous paper (see [33]) a different (weaker) definition of energetic quasi-static solution was given. The difference was only in the minimality condition, where the competitors were chosen in a smaller space. Precisely for all $t \in [0,T]$ one required that $\theta(t) \in L^\infty(\Omega; [0,1])$, $A(t) \in G_{\theta(t)}(\alpha I, \beta I)$ satisfied the following property instead of* 3.*:*

3b. Minimality condition*:*
$$E_{el}(u(t), A(t)) \le E_{el}(v, \tilde{A}) + k \int_\Omega (1 - \theta(t))\tilde{\theta} dx$$
for all $v \in H_0^1(\Omega)$, $\tilde{\theta} \in L^\infty(\Omega, [0,1])$ and for all $\tilde{A} \in \bar{G}_{\tilde{\theta}}(\alpha I, A(t))$.

In [33] was proved the existence of an energetic quasi-static evolution with the minimality condition as in Remark 3.2 and that the evolution $(A(t), \theta(t))$ can be approximated by a sequence of (damage) sets $D^n(t)$, increasing in time, such that
$$\sigma_{D^n(t)} \xrightarrow{G} A(t) \ \text{ and } \ \chi_{D^n(t)} \overset{*}{\rightharpoonup} \theta(t)$$
which guarantees the well-posedness of the minimality condition in the Definition 3.1. It is important to remark that the set $\hat{G}_{\tilde{\theta}}(\{D^n\})$ is strictly bigger than the set $G_{\tilde{\theta}}(\alpha I, A(t))$ (see Remark 4 in [43]) which means that Condition 3. implies Condition 3*b*..

The first result, due to Garroni and Larsen in [43] is the following

Theorem 3.3. *There exists $(A(t), \theta(t))$ energetic quasi-static evolution for the energy (3.2.1) in the sense of Definition 3.1.*

At this point in [43] was investigated whether this solution was also a threshold solution, and to this aim they introduced a relaxed version of the threshold solution of the Definition 0.2:

DEFINITION 3.4. $(A(t),\theta(t))$ is a threshold solution with threshold $\lambda > 0$ if for every $t \in [0,T]$ there exists a sequence $D^n(t)$ such that $\sigma_{D^n(t)} \xrightarrow{G} A(t)$ and $\chi_{D^n(t)} \overset{*}{\rightharpoonup} \theta(t)$ in L^∞ and the following hold

- **Monotonicity**: $D^n(t)$ is increasing in time;
- **Threshold**: considering the solution u^n of
$$-div(\sigma_{D^n(t)}\nabla v) = f(t)$$
we have that for every $\delta > 0$, the set in which there is no damage but threshold is exceeded by at least δ,
$$U_n := \{x \notin D^n(t) : |\nabla u^n(t)| > \lambda + \delta\}$$
satisfies
$$|U_n| \longrightarrow 0$$
- **Necessity of Damage**: For all $E_n \subset D^n(T)$ with $\liminf |E_n| > 0$, we have that $\forall\delta > 0$ and $\forall\Delta t > 0$ small enough, there exists $\tau < T - \Delta t$ such that, setting v^n to be the solution of
$$-div(\sigma_{D^n(\tau+\Delta t)\backslash\Delta E_n}\nabla v^n) = f(\tau + \Delta t)$$
where $\Delta E_n := E_n \cap [D^n(\tau + \Delta t)\backslash D^n(\tau)]$, we have that the subset of ΔE_n in which the threshold is almost exceeded,
$$\Delta E_n^\delta := \{x \in \Delta E_n : |\nabla v^n(x)| > \lambda - \delta\}$$
satisfies
$$\liminf_{n\longrightarrow\infty} |\Delta E_n^\delta| > 0$$
-If $\int_\Omega \theta(t)dx$ is not continuous at T, then we also require that $\forall t_n \nearrow T$ and $\forall E_n \subset D^n(T)\backslash D(t_n)$ with $\liminf |E_n| > 0$ and for every $\delta > 0$, the solution v^n of
$$-div(\sigma_{D^n(T)\backslash E_n}\nabla v^n) = f(T)$$
satisfies
$$\liminf_{n\longrightarrow\infty} |\{x \in E_n : |\nabla v^n(x)| > \lambda - \delta\}| > 0.$$

An interesting result proved in [43] is the following

THEOREM 3.5. *If $(A(t),\theta(t))$ is a quasi-static evolution (Definition 3.1) for the energy (3.2.1) with dissipation k then it is a threshold solution (Definition 3.4) with threshold λ such that*
$$k = \frac{\lambda^2\beta(\beta-\alpha)}{2\alpha}.$$

In this chapter we start from the results proved in [80] that assure the existence (for each $\varepsilon > 0$) of a quasi-static evolution $(u^\varepsilon(t), D^\varepsilon(t))$ for the total energy
$$E^\varepsilon_{tot}(t,u,D) := E^\varepsilon_{el}(t,u,D) + k|D| + \varepsilon Per(D) \tag{3.2.2}$$
with
$$E^\varepsilon_{el}(t,u,D) := \frac{1}{2}\int_\Omega \sigma_D|\nabla u|^2dx - \langle f(t), u\rangle$$
according to Definition 0.1 in which we consider
$$\theta = \chi_D,$$
$$\mathcal{E}(t,u,\theta) = E^\varepsilon_{el}(t,u,D) + \varepsilon Per(D)$$
and
$$\tilde{\mathcal{D}}(\theta) = k|D|.$$
We study the limit of such solution proving the following theorem:

THEOREM 3.6. *Given a quasi-static evolution* $(u^\varepsilon(t), D^\varepsilon(t))$ *for* (3.2.2), *for all* $t \in [0,T]$ *there exists* $\theta(t) \in L^\infty(\Omega;[0,1])$ *and* $A(t) \in \bar{G}_{\theta(t)}(\alpha I, \beta I)$ *such that (up to subsequences)*

$$\chi_{D^\varepsilon(t)} \overset{*}{\rightharpoonup} \theta(t), \text{ and } \quad \sigma_{D^\varepsilon(t)} \xrightarrow{G} A(t) \quad \text{and} \quad , u^\varepsilon(t) \overset{H^1}{\rightharpoonup} u(t),$$

and it holds

$$\lim_{\varepsilon\to 0} \varepsilon Per(D^\varepsilon(0)) = 0\,. \tag{3.2.3}$$

Moreover assuming that for $t > 0$

$$\lim_{\varepsilon\longrightarrow 0} \varepsilon Per(D^\varepsilon(t)) = 0, \tag{3.2.4}$$

we have that $(A(t), \theta(t))$ *is a quasi-static evolution in the sense of Definition 3.1.*

The rest of the chapter is devoted to the proof of Theorem 3.6. The proof that the quasi-static evolution with perimeter penalization converges (up to a subsequence) and that the limit satisfies the monotonicity condition and the energy balance uses standard arguments. The main difficulty is rather to prove the validity of the minimality condition as in Definition 3.1. We will show it by using a blow-up argument for which it is fundamental the assumption (3.2.4).
Even though we assumed (3.2.4) we believe that it must be true in general; our feeling is that starting from time $t = 0$ the scale of the damage set becomes fixed and by (3.2.3) it should be possible to prove (3.2.4) for each $t > 0$.
We remind a result proved in [43] that it will be useful in what will follow:

LEMMA 3.2.1. Let $E \subset \Omega$ and $S \subset \Omega \backslash E$ be measurable. Consider the solution $u_E \in H^1_0(\Omega)$ of the equation

$$-div(\sigma_E \nabla u_E) = f \quad \text{in} \quad \Omega$$

where $f \in H^{-1}(\Omega)$. Then if we set

$$E_{el}(E,f) := \frac{1}{2}\int_\Omega \sigma_E |\nabla u_E|^2 dx - \langle f, u_E\rangle$$

we have

$$\Delta E_{el} := E_{el}(E,f) - E_{el}(E\cup S, f) \le \frac{(\beta-\alpha)\beta}{2\alpha}\|\nabla u_E\|^2_{L^2(S)}.$$

3.3. Compactness argument and convergence of the Perimeter at $t = 0$

In this section we prove a compactness result for the quasi-static evolution related to the energy (3.2.2), we prove a Γ-limit result and the convergence (3.2.3), which is fundamental with assumption (3.2.4), to obtain an energy balance property for the limit of the quasi-static evolution and to apply a blow up argument to show the minimality condition of the limit.

LEMMA 3.3.1. Given $(u^\varepsilon(t), \chi_{D^\varepsilon(t)})$ a quasi-static evolution for the energy (3.2.2) there exists functions $u(t) \in H^1_0(\Omega)$, $\theta(t) \in L^\infty(\Omega;[0,1])$ and $A(t) \in L^\infty(\Omega; \mathcal{F}(\alpha,\beta))$ such that, up to subsequences,

$$u^\varepsilon(t) \overset{H^1}{\rightharpoonup} u(t), \qquad \chi_{D^\varepsilon(t)} \overset{*}{\rightharpoonup} \theta(t), \qquad \sigma_{D^\varepsilon(t)} \xrightarrow{G} A(t) \tag{3.3.5}$$

with $\theta(t)$ increasing and $A(t)$ decreasing in time.

PROOF. The compactness result is proved by standard techniques. It is enough to note that by the monotonicity in t of $D^\varepsilon(t)$ we deduce that $\sigma_{D^\varepsilon(t)}$ has equibounded total variation with respect to the $L^\infty(\Omega, \mathcal{F}(\alpha,\beta))$ metric. Then using Helly's Theorem (see [57] or [33]), there exists a subsequence of $\sigma_{D^\varepsilon(t)}$ and a matrix $A(t) \in L^\infty(\Omega, \mathcal{F}(\alpha,\beta))$ such that, for each $t \in [0,T]$

$$\sigma_{D^\varepsilon(t)} \xrightarrow{G} A(t)\,.$$

By the properties of G-convergence, $A(t)$ inherit the monotonicity by $\sigma_{D^\varepsilon(t)}$. The same argument can be made for $\chi_{D^\varepsilon(t)}$ in the weak* topology (as noticed in [33]), thus there exists a decreasing function $\theta(t)$, such that, up to subsequence,

$$\chi_{D^\varepsilon(t)} \overset{*}{\rightharpoonup} \theta(t)$$

for each $t \in [0,T]$. Finally, by the minimality of $u^\varepsilon(t)$ and the definition of $G-convergence$, we immediately obtain that there exists a function $u(t) \in H^1(\Omega)$ and a subsequence of $u^\varepsilon(t)$ such that for each $t \in [0,T]$

$$u^\varepsilon(t) \overset{H^1}{\rightharpoonup} u(t).$$

□

We now prove a Γ-convergence result, for the energy in (3.2.2), without constraint, which will imply the perimeter property (3.2.3).

Lemma 3.3.2. Given $u \in H_0^1(\Omega)$ and $D \subset \Omega$ the functional

$$F^\varepsilon(u,\chi_D) := \frac{1}{2}\int_\Omega \sigma_D|\nabla u|^2 dx - \langle f,u\rangle + k|D| + \varepsilon Per(D) \tag{3.3.6}$$

Γ- converges to the functional

$$F(u,A,\theta) := \int_\Omega A\nabla u\nabla u dx - \langle f,u\rangle + k\int_\Omega \theta dx \tag{3.3.7}$$

with respect to the following convergences

$$u^\varepsilon \overset{H^1}{\rightharpoonup} u \qquad \chi_{D^\varepsilon} \overset{*}{\rightharpoonup} \theta, \qquad \sigma_{D^\varepsilon} \xrightarrow{G} A. \tag{3.3.8}$$

with $u \in H_0^1(\Omega)$, $\theta(x) \in [0,1]$ and $A \in L^\infty(\Omega;\mathcal{F}(\alpha,\beta))$.

Proof. Obviously we have that

$$\liminf_\varepsilon \left\{\frac{1}{2}\int_\Omega \sigma_{D^\varepsilon}|\nabla u^\varepsilon|^2 dx - \langle f,u^\varepsilon\rangle + k|D^\varepsilon| + \varepsilon Per(D^\varepsilon)\right\} \geq$$

$$\liminf_\varepsilon \left\{\frac{1}{2}\int_\Omega \sigma_{D^\varepsilon}|\nabla u^\varepsilon|^2 dx - \langle f,u^\varepsilon\rangle + k|D^\varepsilon|\right\}$$

and since by Theorem 1.10

$$\sigma_{D^\varepsilon} \xrightarrow{G} A \quad \text{if and only if} \quad \int_\Omega \sigma_{D^\varepsilon}|\nabla u|^2 dx \xrightarrow{\Gamma} \int_\Omega A\nabla u\nabla u dx \tag{3.3.9}$$

we have by (3.3.8) and the lower semicontinuity of the Γ-limit

$$\liminf_\varepsilon \left\{\frac{1}{2}\int_\Omega \sigma_{D^\varepsilon}|\nabla u^\varepsilon|^2 dx - \langle f,u^\varepsilon\rangle + k|D^\varepsilon|\right\} \geq \tag{3.3.10}$$

$$\frac{1}{2}\int_\Omega A\nabla u\nabla u dx - \langle f,u\rangle + k\int_\Omega \theta dx \tag{3.3.11}$$

So the (3.3.11) is the candidate to be the Γ-limit. Now we want to prove lim sup inequality. Given $\theta \in [0,1]$ and $A \in \bar{G}_\theta(\alpha I,\beta I)$ there exists (by definition of $G_\theta(\alpha I,\beta I)$) a sequence $(\chi_{D^h},\sigma_{D^h})$ such that

$$\chi_{D^h} \overset{*}{\rightharpoonup} \theta \qquad \text{and} \qquad \sigma_{D^h} \xrightarrow{G} A,$$

moreover we can consider the recovery sequence u^h of the Γ convergence

$$\int_\Omega \sigma_{D^h}|\nabla u|^2 dx \xrightarrow{\Gamma} \int_\Omega A\nabla u\nabla u dx \tag{3.3.12}$$

and in so doing we obtain

$$\lim_{\varepsilon}\left\{\frac{1}{2}\int_{\Omega}\sigma_{D^h}|\nabla u^h|^2dx-\langle f,u^h\rangle+k|D^h|+\varepsilon Per(D^h)\right\}= \tag{3.3.13}$$

$$\frac{1}{2}\int_{\Omega}\sigma_{D^h}|\nabla u^h|^2dx-\langle f,u^h\rangle+k|D^h| \tag{3.3.14}$$

and since it is a pointwise convergence it implies that, calling $F''(u,D)$ the Γ-limsup, we have

$$F''(u^h,D^h)\leq\frac{1}{2}\int_{\Omega}\sigma_{D^h}|\nabla u^h|^2dx-\langle f,u^h\rangle+k|D^h|$$

so, passing to the limit in h and using the lower semicontinuity of the Γ-limsup we conclude that

$$\lim_h F''(u^h,D^h)\leq\frac{1}{2}\int_{\Omega}A\nabla u\nabla u dx-\langle f,u^h\rangle+k\int_{\Omega}\theta dx,$$

which conclude the computation of the Γ-limit of (3.3.6). □

By previous Lemma we prove the convergence result (3.2.3).

LEMMA 3.3.3. Given $(u^\varepsilon(t),\chi_{D^\varepsilon(t)})$ quasi-static evolution for the energy (3.2.2) it holds

$$\varepsilon Per(D^\varepsilon(0))\longrightarrow 0 \tag{3.3.15}$$

when ε goes to zero.

PROOF. Since $(u^\varepsilon(t),D^\varepsilon(t))$ is a quasi-static evolution we have in particular that, at time $t=0$, $(u^\varepsilon(0),D^\varepsilon(0))$ minimizes the functional

$$F(u,D):=\frac{1}{2}\int_{\Omega}\sigma_D|\nabla u|^2dx-\langle f,u\rangle+k|D|+\varepsilon Per(D) \tag{3.3.16}$$

without any restriction on the set. Moreover $u^\varepsilon(0)$ is a mininum point for

$$\int_{\Omega}\sigma_{D^\varepsilon(0)}|\nabla v|^2dx-\langle f(0),v\rangle$$

so by the property of G-convergence we have

$$\int_{\Omega}\sigma_{D^\varepsilon(0)}|\nabla u^\varepsilon(0)|^2dx\longrightarrow\int_{\Omega}A(0)\nabla u(0)\nabla u(0)dx$$

So passing to the limit in ε in the functional (3.3.16) with $u^\varepsilon(0)$ and $D^\varepsilon(0)$ we necessarily obtain that

$$\varepsilon Per(D^\varepsilon(0))\longrightarrow 0.$$

□

3.4. Energy Balance and a first minimality property of the limit

In this section we prove that the limit $(u(t),\theta(t))$ of the quasi-static evolution $(u^\varepsilon(t),D^\varepsilon(t))$ satisfies properties (1), (2), and (3b) as in Definition 3.1 and Remark 3.2.

PROPOSITION 3.7. *Given the limit $(u(t),\theta(t))$ and $A(t)$ of the quasi-static evolution $(u^\varepsilon(t),D^\varepsilon(t))$ and $\sigma_{D^\varepsilon(t)}$ for the energy (3.2.2) according to the convergences in Lemma 3.3.1, then, assuming (3.2.4), it satisfies the properties (1), (2), and (3b) of Remark 3.2.*

PROOF. The proof is standard and it follows closely the strategy of Theorem 3.1 in [33]. As proved in Lemma 3.3.1 there exists up to subsequences the limit of $u^\varepsilon(t)$, $D^\varepsilon(t)$ and $\sigma_{D^\varepsilon(t)}$ which we call $u(t)$, $\theta(t)$ and $A(t)$, such that $A(t)$ is decreasing and $\theta(t)$ is increasing and so the property (1) is satisfied.
The energy balance is immediate: it is consequence of the energy balance at ε fixed passing

to the limit and using the perimeter convergence assumption (3.2.4).
Now we focus on the minimality property. The minimality of $(u^\varepsilon(t), D^\varepsilon(t))$ for the energy (3.2.2) gives

$$\frac{1}{2}\int_\Omega \sigma_{D^\varepsilon(t)}|\nabla u^\varepsilon(t)|^2dx - \langle f(t), u^\varepsilon(t)\rangle + k|D^\varepsilon(t)| + \varepsilon Per(D^\varepsilon(t)) \tag{3.4.17}$$

$$\leq \frac{1}{2}\int_\Omega \sigma_{D'}|\nabla v|^2dx - \langle f(t), v\rangle + k|D'| + \varepsilon Per(D') \tag{3.4.18}$$

for all $v \in H^1_0(\Omega)$, for all $D' \supseteq D^\varepsilon(t)$. Testing the above inequality with $D'_\varepsilon = D^\varepsilon(t) \cup E$, for $E \subset \Omega$, and using that

$$Per(D^\varepsilon(t) \cup E) \leq Per(D^\varepsilon(t)) + Per(E)\,,$$

(3.4.17) becomes

$$\frac{1}{2}\int_\Omega \sigma_{D^\varepsilon(t)}|\nabla u^\varepsilon(t)|^2dx - \langle f(t), u^\varepsilon(t)\rangle \tag{3.4.19}$$

$$\leq \frac{1}{2}\int_\Omega \sigma_{D'}|\nabla v|^2dx - \langle f(t), v\rangle + \varepsilon Per(E) + k\int_\Omega (\chi_{D^\varepsilon(t)\cup E} - \chi_{D^\varepsilon})dx. \tag{3.4.20}$$

Since $\sigma_{D^\varepsilon(t)} \xrightarrow{G} A(t)$ and $u^\varepsilon(t)$ satisfies $-\mathrm{div}\,(\sigma_{D^\varepsilon(t)}\nabla u^\varepsilon(t)) = f(t)$, we have

$$\frac{1}{2}\int_\Omega A(t)\nabla u(t)\nabla u(t)dx - \langle f(t), u(t)\rangle \tag{3.4.21}$$

$$= \lim\left\{\frac{1}{2}\int_\Omega \sigma_{D^\varepsilon(t)}|\nabla u^\varepsilon(t)|^2dx - \int_\Omega f(t)u^\varepsilon(t)dx\right\} \tag{3.4.22}$$

Moreover by the locality of the G-convergence we have

$$\sigma_{D^\varepsilon(t)\cup E} \xrightarrow{G} A(t)\chi_{\Omega\setminus E} + \alpha\chi_E \tag{3.4.23}$$

and, by the equivalence with the Γ-convergence (see Theorem 1.10), given $v \in H^1_0(\Omega)$ there exists a (recovery) sequence $v^\varepsilon \in H^1_0(\Omega)$ with $v^\varepsilon \overset{H^1}{\rightharpoonup} v$ such that

$$\limsup\left\{\frac{1}{2}\int_\Omega \sigma_{D^\varepsilon(t)\cup E}|\nabla v^\varepsilon|dx - \langle f(t), v^\varepsilon\rangle\right\} \tag{3.4.24}$$

$$= \frac{1}{2}\int_\Omega (A(t)\chi_{\Omega\setminus E} + \alpha\chi_E)\nabla v\nabla v dx - \langle f(t), v\rangle. \tag{3.4.25}$$

Finally noting that $\varepsilon Per(E) \longrightarrow 0$ and that

$$\lim_{\varepsilon\longrightarrow 0}\left\{\int_\Omega (\chi_{D^\varepsilon(t)\cup E} - \chi_{D^\varepsilon(t)})dx\right\} = \int_\Omega (1-\theta)\chi_E dx$$

we obtain (using also (3.4.21) and (3.4.24)) the following inequality

$$\int_\Omega A(t)\nabla u(t)\nabla u(t)dx - \langle f(t), u(t)\rangle \leq \frac{1}{2}\int_\Omega A'\nabla v\nabla v dx + k\int_\Omega (1-\theta(t))\chi_E dx$$

for all $v \in H^1_0(\Omega)$ and for all $A' = A(t)\chi_{\Omega\setminus E} + \alpha\chi_E$ for all $E \subset \Omega$, and so by density we obtain the minimality condition (3b) of the Remark 3.2.

□

Remark 3.8. *Let note that the possibility to have the energy balance for the limit is equivalent to the convergence of the perimeter (3.2.4). Indeed, let assume that the energy balance holds for the limit. Using the energy balance for the quasi-static evolution $(u^\varepsilon(t), D^\varepsilon(t))$ and* *the fact that $\varepsilon Per(D^\varepsilon(0))$ goes to zero (up to subsequence) we have by the energy balance at ε fixed that*

$$E^\varepsilon_{el}(t) + k|D^\varepsilon(t)| + \varepsilon Per(D^\varepsilon(t)) = E^\varepsilon_{tot}(0) - \int_0^t \langle \dot{f}(t), u^\varepsilon(t)\rangle ds$$

and passing to the $\limsup$ *we have*

$$E_{el}(t) + k\int_\Omega \theta(t)dx + \limsup_\varepsilon(\varepsilon Per(D^\varepsilon(t))) = E_{tot}(0) - \int_0^t \langle \dot{f}(t), u(t)\rangle ds$$

and using the energy balance for the limit of $(u^\varepsilon(t), \chi_{D^\varepsilon(t)})$ we have that the right hand side term is equal to

$$E_{el}(t) + k\int_\Omega \theta(t)dx$$

which implies that up to subsequence

$$\lim_{\varepsilon\longrightarrow 0}(\varepsilon Per(D^\varepsilon(t))) = 0$$

for all $t \in [0,T]$.
Since the convergence (3.2.4), as we will see, is the key point to prove the minimality condition as in the Definition 3.1 it means that if we have an increasing function $\theta(t)$ (and $u(t)$) which is approximated by a quasi-static evolution $D^\varepsilon(t)$ (and $u^\varepsilon(t)$) of the energy in (3.2.2) and such that it satisfies (a priori) the energy balance related to the energy (3.2.1) then it is a quasi-static evolution for this energy.

3.5. Minimality Condition

In this section we will prove the main Theorem of the Chapter, which reduces now to prove the minimality conditions in Definition 3.1. It will be consequence of the fact that $u^\varepsilon(t)$ satisfies a threshold property in the limit (see Step 1 in the proof). This property is implied by fact that, thank to the (3.2.4), we can have a convergent blow-up of $u^\varepsilon(t)$ on a large number of squares $Q^\varepsilon_i \subset \Omega$ whose limit satisfies a sharp threshold property (see Proposition 3.10) which is consequence of a minimality property of that limit (see Lemma 3.5.2).
The first part of this section is devoted to the possibility of make the "blow up argument" which allows us to prove, in the second part of the section, the minimality condition of the Definition 3.1.

3.5.1. The blow-up argument. In this subsection our purpose is to show that we can find a region (square) of Ω in which $(u^\varepsilon(t), D^\varepsilon(t))$, after a change of scale, has a subsequence that converges to a pair $(\tilde{v}_0(t), \tilde{D}_0(t))$ that satisfies a first threshold condition in that region. To do it we need to construct a partition of Ω with good properties. To avoid heavy notation we sometimes neglect the dependence of t underlining the dependence of x.
We need also to introduce some notations: given a square $Q^\varepsilon_i := Q^\varepsilon(x^\varepsilon_i) \subset \Omega$ with center in x^ε_i and length size ρ_ε we define $u^\varepsilon_i(x)$ as the restriction of $u^\varepsilon(x)$ on this square and we define the following blow up on it (see Figure 1)

$$\tilde{v}^\varepsilon_i(x) := \frac{1}{\rho_\varepsilon}(u^\varepsilon_i(\rho_\varepsilon x + x^\varepsilon_i) - \bar{u}^\varepsilon_i), \qquad \tilde{D}^\varepsilon_i := \frac{1}{\rho_\varepsilon}(D^\varepsilon \cap Q^\varepsilon_i - x^\varepsilon_i), \qquad Q^1(0) := \frac{1}{\rho_\varepsilon}(Q^\varepsilon_i - x^\varepsilon_i)$$

where $\bar{u}^\varepsilon_i$ is the mean value of $u^\varepsilon(x)$ on Q^ε_i.

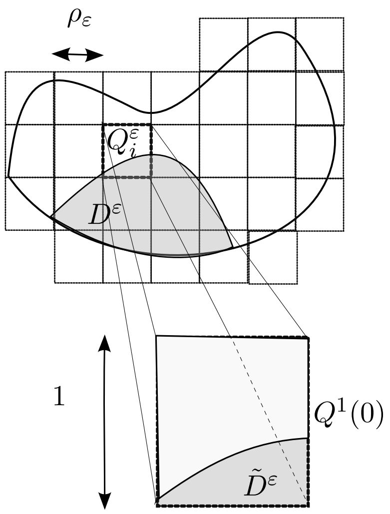

FIGURE 1.

REMARK 3.9. *Let note that, for each $\varepsilon > 0$, we can construct a covering of Ω of disjoint squares such that*

$$Per(D^\varepsilon(t), \Omega) = \sum_i Per(D^\varepsilon(t), Q_i^\varepsilon),$$

indeed the possibility to choose x_i^ε such that this property is satisfied is due to the fact that the perimeter of $D^\varepsilon(t)$ is equibounded. Indeed, fixed arbitrary family x_i^ε, we consider a vector $a = (a_1, a_2) \in [0, \rho_\varepsilon]^2$ and the related partition made by squares

$$Q^\varepsilon(x_i^\varepsilon + a)$$

traslation of Q_i^ε on the vertical line of $a_2 > 0$ and on the horizontal line of $a_1 > 0$.
To prove the property we just need to show that

$$\exists a \in [0, \rho_\varepsilon]^2 \text{ such that } \mathcal{H}^1(\partial D^\varepsilon(t) \cap \partial Q^\varepsilon(x_i^\varepsilon + a)) = 0$$

for every $i \in \mathbb{N}$. By contradiction we suppose that

$$\forall a \in [0, \rho_\varepsilon]^2 \quad \exists i_a : \mathcal{H}^1(\partial D^\varepsilon(t) \cap \partial Q^\varepsilon(x_{i_a}^\varepsilon + a)) > 0,$$

but using the Fubini Theorem we have

$$\mathcal{H}^2(\partial D^\varepsilon(t) \cap \Omega) > \int_0^{\rho_\varepsilon} \int_0^{\rho_\varepsilon} \mathcal{H}^1(\partial D^\varepsilon(t) \cap \partial Q^\varepsilon(x_{i_a}^\varepsilon + a)) da_1 da_2,$$

so we have $\mathcal{H}^2(\partial D^\varepsilon(t) \cap \Omega) > 0$ which implies that $Per(D^\varepsilon(t), \Omega) = \mathcal{H}^1(\partial D^\varepsilon(t) \cap \Omega) = \infty$ that is a contradiction. From now we assume that this property is verified by the family of the squares will consider in the future.

Now we want to show for every $\varepsilon > 0$ that the squares Q_j^ε that define the partition of Ω are "almost all" such that the blow-up function $v_j^\varepsilon(t)$ has finite norm on it and the perimeter of $D_j^\varepsilon(t)$ is equibounded. It will be fundamental to have a convergent subsequence.
So we start defining the set of squares in which, given ρ_ε, there is "too much perimeter"

$$J_\varepsilon := \{i : Per(D^\varepsilon(t), Q_i^\varepsilon) > 4\rho_\varepsilon\} \quad \text{and} \quad UJ_\varepsilon := \bigcup_{i \in J_\varepsilon} Q_i^\varepsilon$$

and we note that if $j \notin J_\varepsilon$ we have

$$Per(\tilde{D}_j^\varepsilon(t), Q^1(0)) = \frac{Per(D^\varepsilon(t), Q_j^\varepsilon)}{\rho_\varepsilon} \leq 4\rho_\varepsilon \frac{1}{\rho_\varepsilon} = 4$$

Moreover we define the set of squares in which, for each t, the norm of $u^\varepsilon(t)$ is "too large" i.e. fixing an arbitrary large constant $M > 0$ we define

$$I_\varepsilon^M := \{i : \|u_i^\varepsilon(t)\|_{H^1(Q_i^\varepsilon)} > M\rho_\varepsilon^2\} \quad \text{and} \quad UI_\varepsilon^M := \bigcup_{i \in I_\varepsilon^M} Q_i^\varepsilon.$$

It is immediate to have an estimate on the measure of UI_ε^M:

$$\sum_{i \in I_\varepsilon^M} \|u^\varepsilon(t)\|_{H^1(Q_i^\varepsilon)} > \#I_\varepsilon^M \cdot M\rho_\varepsilon^2 = M|\bigcup_{i \in I_\varepsilon^M} Q_i^\varepsilon|$$

and since

$$\sum_{i \in I_\varepsilon^M} \|u^\varepsilon(t)\|_{H^1(Q_i^\varepsilon)} < \|u^\varepsilon(t)\|_{H^1(\Omega)} \leq C$$

we obtain

$$|UI_\varepsilon^M| < \frac{C}{M} \tag{3.5.26}$$

Lemma 3.5.1. Assume (3.2.4) holds, then there exists a sequence $\rho_\varepsilon \longrightarrow 0$ such that

- it holds
$$|UJ_\varepsilon| \longrightarrow 0 \tag{3.5.27}$$
when ε goes to zero;
- for every $\delta > 0$ there exists a constant $c > 0$ and a set of indexes
$$\Lambda_\varepsilon^\delta := \{i \notin J_\varepsilon : \ \|u_i^\varepsilon(t)\|_{H^1(Q_i^\varepsilon)} \leq c\rho_\varepsilon^2\}$$
such that, considering ε sufficiently small, it holds
$$|\bigcup_{i \in \Lambda_\varepsilon^\delta} Q_i^\varepsilon| = |\Omega| - \delta \tag{3.5.28}$$
- if $j \in \Lambda_\varepsilon^\delta$ then for every $t > 0$ there exist a constant $c_1 > 0$ such that
$$\|\tilde{v}_j^\varepsilon(t)\|_{H^1(Q^1(0))} \leq c_1 \tag{3.5.29}$$
with c_1 not depending on ε.

Proof. To prove the first property we use Remark 3.9. We have

$$Per(D^\varepsilon(t), \Omega) = \sum_i Per(D^\varepsilon(t), Q_i^\varepsilon) > \sum_{i \in J_\varepsilon} Per(D^\varepsilon(t), Q_i^\varepsilon) > \#J_\varepsilon \cdot 4\rho_\varepsilon$$

and defining $\lambda_\varepsilon := \varepsilon Per(D^\varepsilon(t))$ we have

$$\#J_\varepsilon < \frac{\lambda_\varepsilon}{\varepsilon} \frac{1}{4\rho_\varepsilon}.$$

Then choosing $\rho_\varepsilon := \frac{\varepsilon}{\lambda_\varepsilon^{1/2}}$ (which goes to zero when ε goes to zero) we obtain

$$|UJ_\varepsilon| \leq \#J_\varepsilon \cdot \rho_\varepsilon^2 = \frac{\lambda_\varepsilon^{1/2}}{4}$$

that goes to zero.
To prove the second statement we start defining

$$\Lambda_\varepsilon^M := (J_\varepsilon)^c \cap (I_\varepsilon^M)^c,$$

and so we have

$$\bigcup_{i\in\Lambda_\varepsilon^M} Q_i^\varepsilon = \Omega\backslash\Big[\bigcup_{i\in J_\varepsilon} Q_i^\varepsilon \cup \bigcup_{i\in I_\varepsilon^M} Q_i^\varepsilon\Big]$$

which implies using (3.5.26) and (3.5.27)

$$\Big|\bigcup_{i\in\Lambda_\varepsilon^M} Q_i^\varepsilon\Big| > |\Omega| - |UJ_\varepsilon| - |UI_\varepsilon^M| > |\Omega| - o_\varepsilon(1) - \frac{C}{M}$$

so choosing ε sufficiently small and M sufficiently large we conclude.
Now we prove the third property. Let remark that since $j \in \Lambda_\varepsilon^\delta$ it holds

$$\|u_j^\varepsilon(t)\|_{H^1(Q_j^\varepsilon)} \leq c\rho_\varepsilon^2 \tag{3.5.30}$$

Now by the definition of $\tilde{v}_j^\varepsilon$ it holds (we neglect dependence on t underlining the dependence on x)

$$\int_{Q^1(0)} |\nabla\tilde{v}_j^\varepsilon(x)|^2dx = \int_{Q^1(0)} |\nabla(\frac{1}{\rho_\varepsilon}u_j^\varepsilon(\rho_\varepsilon x + x_j^\varepsilon))|^2dx$$

$$= \int_{Q^1(0)} |(\nabla u_j^\varepsilon)(\rho_\varepsilon x + x_j^\varepsilon)|^2dx = \frac{1}{\rho_\varepsilon^2}\int_{Q_j^\varepsilon} |\nabla u_j^\varepsilon(y)|^2dx \leq c,$$

where the last inequality comes from (3.5.30).
Now, since $\tilde{v}_j^\varepsilon$ is not (a-priori) zero at the boundary of $Q^1(0)$ we need to show a similar bound for the L^2-norm of $\tilde{v}_j^\varepsilon$ to have the boundness in H^1. We have

$$\int_{Q^1(0)} |\tilde{v}_j^\varepsilon(x)|^2dx = \int_{Q^1(0)} |\frac{1}{\rho_\varepsilon}(u_j^\varepsilon(\rho_\varepsilon x + x_j^\varepsilon) - \bar{u}_j^\varepsilon)|^2dx = \frac{1}{\rho_\varepsilon^4}\int_{Q_j^\varepsilon} |u_j^\varepsilon(y) - \bar{u}_j^\varepsilon|^2dy.$$

Now using Poincare-Wirtinger inequality

$$\frac{1}{\rho_\varepsilon^4}\int_{Q_j^\varepsilon} |u_j^\varepsilon(y) - \bar{u}_j^\varepsilon|dy \leq C\rho_\varepsilon^2\frac{1}{\rho_\varepsilon^4}\int_{Q_j^\varepsilon} |\nabla(u_j^\varepsilon(y) - \bar{u}_j^\varepsilon)|dy = C\frac{1}{\rho_\varepsilon^2}\int_{Q_j^\varepsilon} |\nabla u_j^\varepsilon(y)|dy$$

and, again, by (3.5.30) we conclude that $\tilde{v}_j^\varepsilon$ is equibounded also in $L^2(Q^1(0))$.

□

Now we prove that, for each $t \in [0,T]$ we can extract in $Q^1(0)$ a subsequence of the pair $(\tilde{v}_j^\varepsilon(t), \chi_{\tilde{D}_j^\varepsilon}(t))$ such that it satisfies a minimality condition in this square respect to add more damage set and respect to functions with the same value of $\tilde{v}_j^\varepsilon$ on the boundary.
From now the partition $\{Q_i^\varepsilon\}_{i\in\mathbb{N}}$ will be fixed and such that it satisfies properties of previous lemma.

Lemma 3.5.2. Considering the family $\{Q_i^\varepsilon\}_\varepsilon$, and an index $j \in \Lambda_\varepsilon^\delta$, there exists, for every $t \in [0,T]$, a subsequence $(\tilde{v}_j^{\varepsilon_k}(t), \chi_{\tilde{D}_j^{\varepsilon_k}(t)})$ and a pair $(\tilde{v}_0(t), \tilde{D}_0(t))$ s.t.

1. $\tilde{v}_j^{\varepsilon_k}(t) \overset{H^1}{\rightharpoonup} \tilde{v}_0(t)$, and $\chi_{\tilde{D}_j^{\varepsilon_k}(t)} \xrightarrow{L^1} \chi_{\tilde{D}_0(t)}$ in $Q^1(0)$,
2. $(\tilde{v}_0(t), \tilde{D}_0(t))$ satisfies

$$\int_{Q^1(0)} \sigma_{\tilde{D}_0(t)}|\nabla\tilde{v}_0(t)|^2dx + k|\tilde{D}_0(t)| \leq \int_{Q^1(0)} \sigma_D|\nabla v|^2dx + k|D| \tag{3.5.31}$$

for all $D \supseteq \tilde{D}_0(t)$, $D \subset Q^1(0)$ and for all v s.t. $(v - \tilde{v}_0(t)) \in H_0^1(Q^1(0))$.

PROOF. As we have seen before there exists c_1 and c_2 positive constants such that

$$\|\tilde{v}_j^\varepsilon(t)\|_{H^1(Q^1(0))} \le c_1 \text{ and } Per(\tilde{D}_j^\varepsilon(t)) \le c_2$$

and by the weak compactness property of H^1 and compactness property of BV function ($Per(\tilde{D}_j^\varepsilon(t))$ is the total variation of the function $\chi_{\tilde{D}_j^\varepsilon(t)}$) we can extract, for every t, a convergent pair of functions, whose limit we call $(\tilde{v}_0(t), \tilde{D}_0(t))$.
Now we show the minimality property of such limit.
It is useful, for what will follow, to define for $S \subset \Omega$ the duality between a given $f \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$ as

$$\langle f, u\rangle_S := \int_S g\nabla u dx$$

where g is (one of) representative function of f, i.e., a function $g \in L^2(\Omega)$ such that $-div(g) = f$ in weak sense.
We choose as competitor for $(u^\varepsilon(t), D^\varepsilon(t))$ in (3.2.2) a pair (v, D) such that $v = u^\varepsilon(t)$ in $(Q_j^\varepsilon)^c$ and D such that $D = D^\varepsilon(t)$ in $(Q_j^\varepsilon)^c$ and $D \cap Q_j^\varepsilon = (D^\varepsilon(t) \cap Q_j^\varepsilon) \cup E^\varepsilon$ with $E^\varepsilon \subseteq Q_j^\varepsilon$ the rescaling of an arbitrary set $E \subseteq Q^1(0)$. In so doing we obtain by the minimality condition of $u^\varepsilon(t)$ and by these choice of test functions

$$\int_{Q_j^\varepsilon} \sigma_{D^\varepsilon(t)}|\nabla u_j^\varepsilon(t)|^2dx - \langle f(t), u_j^\varepsilon(t)\rangle_{Q_j^\varepsilon} + k|D^\varepsilon(t)\cap Q_j^\varepsilon| + \varepsilon Per(D^\varepsilon(t)\cap Q_j^\varepsilon)$$

$$\le \int_{Q_j^\varepsilon} \sigma_{D(t)}|\nabla v|^2dx - \langle f(t), v\rangle_{Q_j^\varepsilon} + k|D\cap Q_j^\varepsilon| + \varepsilon Per(D\cap Q_j^\varepsilon),$$

which becomes, making the scaling $x = \rho_\varepsilon y + x_i^\varepsilon$, and defining $f^\varepsilon(y) := f(\rho_\varepsilon y + x_j^\varepsilon)$

$$\int_{Q^1(0)} \sigma_{\tilde{D}_j^\varepsilon}(t)|\nabla \tilde{v}_j^\varepsilon(t)|^2dy - \rho_\varepsilon\langle f^\varepsilon(t), \tilde{v}_j^\varepsilon(t)\rangle_{Q_0^1} - \langle f^\varepsilon(t), \bar{u}_j^\varepsilon(t)\rangle_{Q_0^1}$$

$$+k|\tilde{D}_j^\varepsilon(t)\cap Q^1(0)| + \frac{\varepsilon}{\rho_\varepsilon}Per(\tilde{D}_j^\varepsilon(t)\cap Q^1(0)) \le$$

$$\int_{Q^1(0)} \sigma_{\tilde{D}_j^\varepsilon(t)\cup E}|\nabla v|^2dy - \rho_\varepsilon\langle f^\varepsilon(t), v\rangle_{Q_0^1} + k|(\tilde{D}_j^\varepsilon(t)\cup E)\cap Q^1(0)|$$

$$+\frac{\varepsilon}{\rho_\varepsilon}Per((\tilde{D}_j^\varepsilon(t)\cup E)\cap Q^1(0))$$

with v such that $v = \tilde{v}_j^\varepsilon(t) + \bar{u}_i^\varepsilon(t)$ on the boundary of $Q^1(0)$. Now we want to pass to the limit as ε goes to zero. Note that for the mean value $\bar{u}_j^\varepsilon(t)$ we have, using Holder inequality (neglecting dependence on t),

$$\bar{u}_j^\varepsilon = \frac{1}{|Q_j^\varepsilon|}\int_{Q_i^\varepsilon} u_j^\varepsilon(y)dy \le \frac{1}{|Q_j^\varepsilon|}\left(|Q_i^\varepsilon|^{\frac{1}{2}}\left(\int_{Q_i^\varepsilon}|u_j^\varepsilon(y)|^2dy\right)^{1/2}\right)$$

which goes to zero when ε goes to zero since $j \in \Lambda_\varepsilon^\delta$. Moreover (up to subsequences) x_j^ε is a convergent sequence and by the definition on ρ_ε we have that $(\varepsilon/\rho_\varepsilon)$ converges to zero. Then we remark that, for each $t \in [0,T]$ $Per(\tilde{D}_j^\varepsilon(t), Q^1(0)) \le c_2$ and that $\chi_{\tilde{D}_j^\varepsilon(t)} \longrightarrow \chi_{\tilde{D}_0(t)}$ in $L^1(Q^1(0))$. Finally, we have that (up to subsequences) $\sigma_{\tilde{D}_j^\varepsilon}(t)$ strongly converges to $\sigma_{\tilde{D}_0}(t)$ in $L^2(Q^1(0))$ and that $\nabla\tilde{v}_j^\varepsilon(t)$ weakly converges to $\nabla\tilde{v}_0(t)$ in $L^2(Q^1(0))$. So using all these properties we can pass to the limit and obtain the energy minimality (3.5.31). □

Now we will show that $(\tilde{v}_0(t), \tilde{D}_0(t))$ satisfies a threshold condition in $Q^1(0)$.

Proposition 3.10. $(\tilde v_0(t), \tilde D_0(t))$ *satisfies the following threshold property:*

$$|\nabla \tilde v_0(t)| \le \lambda$$

a.e. in $\tilde D_0^c(t)$ *with* $\lambda = \sqrt{\frac{2\alpha k}{\beta(\beta-\alpha)}}$ *for each* $t \in [0,T]$.

Proof. The idea is to prove the claim by contradiction using the minimality property (3.5.31). We suppose (by contradiction) that the set in which $|\nabla \tilde v_0|$ exceeds the threshold outside the damage region has positive measure; we consider a small square $\bar Q$ of this region and we damage it with a lamination. Then we will show that there exists an admissible test function for (3.5.31) such that, considered the damage added by lamination, makes the total energy lower than the one given by $(\tilde v_0, \tilde D_0)$. We will follow the steps used in [43].
We suppose that the set

$$U := \{x \in \tilde D_0^c(t) : |\nabla \tilde v_0(t)| > \lambda\}$$

is such that

$$|U| > \gamma > 0.$$

Now we consider a square $\bar Q \subset Q^1(0)$ with center in a point $\bar x \in U$ such that (see Fig.2)

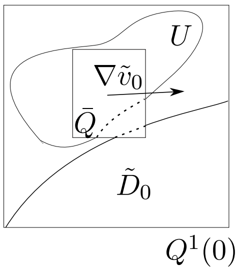

Figure 2.

1) $\bar x$ is a Lebesgue point for $\tilde v_0$ and $\nabla \tilde v_0$, i.e. it holds

$$\lim_{r \longrightarrow 0^+} \frac{1}{|B_r(\bar x)|} \int_{B_r(\bar x)} |\tilde v_0(\bar x) - \tilde v_0(y)|^p dy = 0$$

for all $p \ge 1$ (and the same for $\nabla \tilde v_0$), where $B_r(\bar x)$ is the ball with center $\bar x$ and radius r.
2) two sides of $\bar Q$ are orthogonal to $\nabla \tilde v_0(\bar x)$;
3) defined $\hat v(x) := \tilde v_0(\bar x) + \nabla \tilde v_0(\bar x) \cdot (x - \bar x)$ we have

$$\|\tilde v_0 - \hat v\|^2_{H^1(\bar Q)} \le \varepsilon |\bar Q|$$

$$|\tilde D_0 \cap \bar Q| \le \varepsilon |\bar Q|$$

Note that since we are assuming that in U we have $|\nabla \tilde v_0| > \lambda$ there exists $\delta > 0$ such that $|\nabla \tilde v_0(\bar x)| = \lambda + \delta$. Graphically we have
We can divide the proof in 3 steps.
As first step we will show that considering in $\bar Q$ test functions with the same boundary condition of $\hat v$ (instead of $\tilde v_0$) we can decrease in this square the total energy given by the pair $(\tilde v_0, \emptyset)$ (without the forcing term) using a process of lamination, i.e. we will show that

$$\inf_{w, D'} \left\{ E^{wf}_{tot}(w, D', \bar Q) : (w - \hat v) \in H^1_0(\bar Q), D' \subset \bar Q \right\} \le E^{wf}_{tot}(\hat v, \emptyset, \bar Q) - \frac{1}{2}\beta\delta^2 |\bar Q| \tag{3.5.32}$$

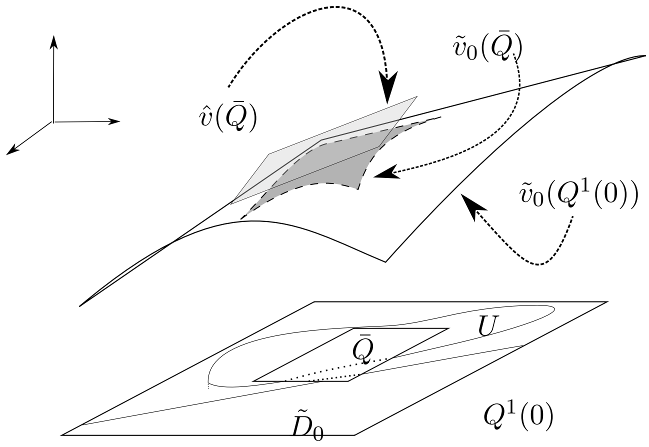

FIGURE 3.

where

$$E^{wf}_{tot}(v, D, \bar{Q}) := \frac{1}{2}\int_{\bar{Q}} \sigma_D |\nabla v|^2 dx + k|D \cap \bar{Q}|$$

To do it we need a technical result to match the boundary conditions of the test function (which will be a piecewise linear function) with the boundary conditions of $\hat{v}$.
In the second step, using the previous one, we will show that in $\bar{Q}$ we can lower the total energy given by $(\tilde{v}_0(t), \tilde{D}_0(t))$ using test functions with the same boundary conditions of $\tilde{v}_0(t)$ and choosing ε sufficiently small, i.e. we will show that

$$\inf_{h,D'} \left\{ E^{wf}_{tot}(h, D', \bar{Q}) : (h - \tilde{v}_0(t)) \in H^1_0(\bar{Q}), D' \subset \bar{Q} \right\} \leq E^{wf}_{tot}(\tilde{v}_0(t), \tilde{D}_0(t), \bar{Q}) - \frac{1}{2}\beta\delta^2|\bar{Q}| \tag{3.5.33}$$

$$+ o_\varepsilon(1)|\bar{Q}| \tag{3.5.34}$$

Finally in the third step we will use the previous steps to construct an admissible pair for the inequality (3.5.31) that has in $Q^1(0)$ a total energy lower than the one given by $(\tilde{v}_0(t), \tilde{D}_0(t))$ and so the contradiction.
Step 1: To avoid heavy notation we can assume that $\bar{x} = 0$ and $\tilde{v}_0(\bar{x}) = 0$ and so we have $\hat{v}(x) = \nabla\tilde{v}_0(\bar{x}) \cdot x$
We consider the continuous periodic function $z(y)$ such that $z(0) = 0$, $z(1) = \lambda + \delta$ and

$$z'(y) = \begin{cases} \frac{\beta}{\alpha}\lambda, & \text{if } y \in (0, d) \\ \lambda, & \text{if } y \in [d, 1) \end{cases} \tag{3.5.35}$$

where d is univocally determined and it turns to be $d = \frac{\delta\alpha}{\lambda(\beta-\alpha)}$ and we define

$$\hat{v}_\varepsilon(x) := z\left(\frac{x}{\varepsilon} \cdot \frac{\nabla\tilde{v}_0(\bar{x})}{|\nabla\tilde{v}_0(\bar{x})|}\right)$$

$$\hat{D}_\varepsilon := \left\{ x \in \bar{Q} : z'(\frac{x}{\varepsilon} \cdot \frac{\nabla\tilde{v}_0(\bar{x})}{|\nabla\tilde{v}_0(\bar{x})|}) = \frac{\beta}{\alpha}\lambda \right\}$$

It is easy to see that $\hat{v}_\varepsilon$ converges strongly to $\hat{v}$ in $L^2(\bar{Q})$ and that it is bounded in $H^1(\bar{Q})$.

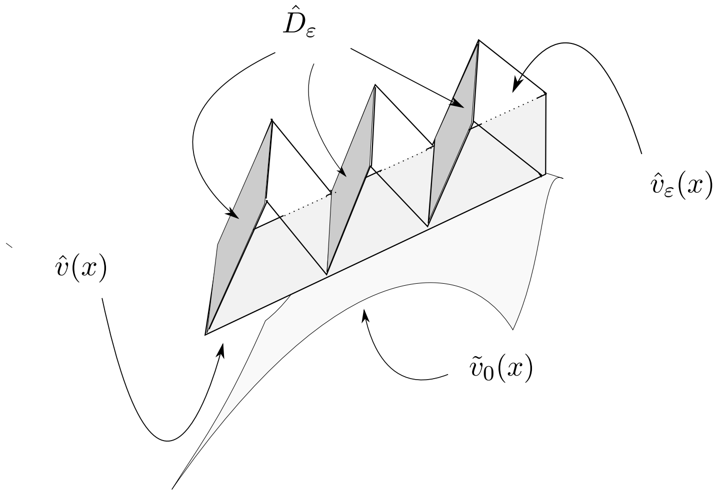

FIGURE 4.

Now we match the boundary conditions of $\hat{v}_\varepsilon$ with the ones of $\hat{v}$ using the cut-off function

$$\phi(y) = \begin{cases} 1, & \text{if } y \in \bar{Q} \backslash Q_R \\ 0, & \text{if } y \in Q_{R-\mu} \end{cases} \tag{3.5.36}$$

and such that $|\nabla \phi| = \frac{1}{\mu}$ in $Q_R \backslash Q_{R-\mu}$ where $R \in (0, |\bar{Q}|^{1/2})$ and $\mu \in (0, R)$ and we define

$$w_\varepsilon = \phi \hat{v} + (1 - \phi)\hat{v}_\varepsilon.$$

Let remark that we can choose R such that

$$\lim_{\mu \longrightarrow 0} \lim_{\varepsilon \longrightarrow 0} \int_{Q_R \backslash Q_{R-\mu}} |\nabla w_\varepsilon|^2 dx = 0 \tag{3.5.37}$$

$$\lim_{\mu \longrightarrow 0} \lim_{\varepsilon \longrightarrow 0} \int_{Q_R} |\nabla w_\varepsilon - \nabla \hat{v}_\varepsilon|^2 dx = 0 \tag{3.5.38}$$

indeed by definition of w_ε we only need to prove the first one and we have

$$\int_{Q_R \backslash Q_{R-\mu}} |\nabla w_\varepsilon|^2 dx \leq \frac{1}{\mu^2} \int_{Q_R \backslash Q_{R-\mu}} |\hat{v}_\varepsilon - \hat{v}|^2 dx + \int_{Q_R \backslash Q_{R-\mu}} |\nabla \hat{v}|^2 dx + \int_{Q_R \backslash Q_{R-\mu}} |\nabla \hat{v}_\varepsilon|^2 dx.$$

The first term on the right hand side of the inequality goes to zero when ε goes to zero by the L^2 strong convergence of $\hat{v}_\varepsilon$ to $\hat{v}$, while the second term goes to zero when μ goes to zero. To show that also the last term goes to zero we just need to prove that there exists $R \in (0, |\bar{Q}|^{1/2})$ and a constant $C > 0$ such that

$$|\nabla \hat{v}_\varepsilon|^2 \leq C \quad \text{in} \quad Q_R \backslash Q_{R-\mu}$$

but this is immediate since $\hat{v}_\varepsilon$ is bounded in $H^1(\bar{Q})$ (choosing μ suitable small). By the (3.5.37) and the (3.5.38) follows that the energy given by $(w_\varepsilon, \hat{D}_\varepsilon)$ is arbitrarily close to that given by $(\hat{v}_\varepsilon, \hat{D}_\varepsilon)$ which is (with a simple computation)

$$\frac{1}{2}\beta\lambda(\lambda + \delta)|\bar{Q}| + \frac{1}{2}\beta\lambda\delta|\bar{Q}|$$

The second term is the product of k with the scalar d where k is the dissipation linked to the threshold λ by the relation in the statement. With this choice of λ we have that the energy given by $\hat{v}(x)$ (with no damage but exceeding the threshold) can be decreased putting damaged in a way to bring the strain outside the damage region down to the threshold λ (see Remark 1 in [43]). So, we conclude that

$$\inf\left\{\frac{1}{2}\int_{\bar{Q}}\sigma_{D'}|\nabla w|^2dx+k|D'| : (w-\hat{v})\in H^1_0(\bar{Q}), D'\subset\bar{Q}\right\}\leq\frac{1}{2}\beta\lambda(\lambda+\delta)|\bar{Q}|+\frac{1}{2}\beta\lambda\delta|\bar{Q}| \tag{3.5.39}$$

$$=E^{wf}_{tot}(\hat{v},\emptyset,\bar{Q})-\frac{1}{2}\beta\delta^2|\bar{Q}| \tag{3.5.40}$$

where the equality comes from the fact that

$$E^{wf}_{tot}(\hat{v},\emptyset,\bar{Q})=\frac{1}{2}\beta(\lambda+\delta)^2|\bar{Q}|$$

Step 2: we start showing that by the properties (3) of $\bar{Q}$ we have for each $t\in[0,T]$

$$|E^{wf}_{tot}(\tilde{v}_0(t),\tilde{D}_0(t),\bar{Q})-E^{wf}_{tot}(\hat{v},\emptyset,\bar{Q})|\leq o_\varepsilon(1)|\bar{Q}| \tag{3.5.41}$$

indeed we have

$$E^{wf}_{tot}(\tilde{v}_0(t),\tilde{D}_0(t),\bar{Q})-E^{wf}_{tot}(\hat{v},\emptyset,\bar{Q})=\int_{\bar{Q}}\sigma_{\tilde{D}_0(t)}|\nabla\tilde{v}_0(t)|^2dx+k|\tilde{D}_0(t)\cap\bar{Q}|-\beta\int_{\bar{Q}}|\nabla\hat{v}|^2dx \tag{3.5.42}$$

$$\leq\beta\int_{\bar{Q}}(|\nabla\tilde{v}_0(t)|^2-|\nabla\hat{v}|^2)dx+k|\tilde{D}_0(t)\cap\bar{Q}| \tag{3.5.43}$$

and using the property for numbers $|a|^2-|b|^2\leq|a-b|^2+2|a-b||b|$ we obtain (using also Holder inequality) that

$$E^{wf}_{tot}(\tilde{v}_0(t),\tilde{D}_0(t),\bar{Q})-E^{wf}_{tot}(\hat{v},\emptyset,\bar{Q})\leq C\int_{\bar{Q}}|\nabla\tilde{v}_0(t)-\nabla\hat{v}|^2dx+k|\tilde{D}_0(t)\cap\bar{Q}| \tag{3.5.44}$$

with $C>0$, and using the properties (3) we have (3.5.41).
Now considering the *inf problem* in (3.5.34) we note that we can use as test function $h(x):=w(x)+(\tilde{v}_0(x)-\hat{v}(x))$ where $w(x)$ is an arbitrary admissible test function in the *inf problem* in (3.5.32), so, using (3.5.41) we obtain the inequality (3.5.34).
Step 3.
We come back to the square $Q^1(0)$ and consider the problem

$$\inf_{w,D'}\left\{\frac{1}{2}\int_{Q^1(0)}\sigma_{D'}|\nabla w|^2dx+k|D'| : (w-\tilde{v}_0(t))\in H^1_0(Q^1(0)), D'\subset Q^1(0), D'\supseteq\tilde{D}_0\right\}.$$

Using the previous steps we can construct a pair (w,D') with $D'\supseteq\tilde{D}_0$ and $w=\tilde{v}_0(t)$ outside $\bar{Q}$ such that

$$E_{tot}(w,D',Q^1(0))\leq E_{tot}(\tilde{v}_0(t),\tilde{D}_0(t),Q^1(0))+(-\frac{1}{2}\beta\delta^2+o_\varepsilon(1))|\bar{Q}|$$

So for $\varepsilon<<1$ we have a contradiction since with an admissible competitor the total energy decreases more than the one given by the pait $(\tilde{v}_0(t),\tilde{D}_0(t))$. $\square$

3.5.2. Proof of the main Theorem.

In this subsection we prove the main result of the chapter stated in Theorem 3.6 that is considering the quasi-static evolution $(u^\varepsilon(t),\chi_{D^\varepsilon(t)})$ we have that, up to subsequences, the G-limit $A(t)$ of $\sigma_{D^\varepsilon(t)}$, the weak limit $u(t)$ of $u^\varepsilon(t)$ and the weak* limit $\theta(t)$ of $\chi_{D^\varepsilon(t)}$ define a quasi-static evolution as in Definition 3.1.
To do it we divide the proof in two steps. In the first one we will prove that it holds the threshold condition of the Definition 3.4 for $(u^\varepsilon(t),\chi_{D^\varepsilon(t)})$ and $\sigma_{D^\varepsilon(t)}$, then using this fact,

in the second step we will prove that $A(t)$, $u(t)$ and $\theta(t)$ satisfy the minimality condition introduce by Garroni and Larsen (see Definition 3.1). In this way since these limits satisfy the energy balance and the damage irreversibility for the energy (3.2.1) (by Theorem 3.7) we have that they define a quasi-static evolution in the sense of Definition 3.1.

PROOF. **Step 1.**

Given $f(t) \in H^{-1}(\Omega)$ we note that by the minimality condition $u^\varepsilon(t)$ is the unique solution of

$$-div(\sigma_{D^\varepsilon(t)} \nabla u^\varepsilon(t)) = f(t).$$

Now defined

$$A_\varepsilon^\delta := \{x \in (D^\varepsilon(t))^c : |\nabla u^\varepsilon(t)| > \lambda + \delta\} \tag{3.5.45}$$

with $\delta > 0$, we want to prove that

$$|A_\varepsilon^\delta| \longrightarrow 0 \tag{3.5.46}$$

when ε goes to 0.
We proceed by contradiction: suppose there exists $\gamma > 0$ s.t. $\limsup_\varepsilon |A_\varepsilon^\delta| = \gamma$. So considering ε small enough we have $|A_\varepsilon^\delta| > \frac{\gamma}{2}$ and, considering the set of indeces Λ_ε^ν (see Lemma 3.5.1),

$$\frac{\gamma}{2} < |A_\varepsilon^\delta| = \sum_{i \in \Lambda_\varepsilon^\nu} |A_\varepsilon^\delta \cap Q_i^\varepsilon| + \sum_{i \notin \Lambda_\varepsilon^\nu} |A_\varepsilon^\delta \cap Q_i^\varepsilon|$$

using (3.5.28) we obtain

$$\sum_{i \in \Lambda_\varepsilon^\delta} |A_\varepsilon^\delta \cap Q_i^\varepsilon| > \frac{\gamma}{2} - \nu$$

which implies that there exists an index $j \in \Lambda_\varepsilon^\nu$ such that

$$|A_\varepsilon^\delta \cap Q_j^\varepsilon| > \left(\frac{\gamma}{2} - \nu\right)\frac{|Q_j^\varepsilon|}{|\Omega|}. \tag{3.5.47}$$

Considering this index j we can make in Q_j^ε the blow-up argument of previous section, so defining (neglecting the dependence on t)

$$\tilde{v}_j^\varepsilon(x) := \frac{1}{\rho_\varepsilon}(u_j^\varepsilon(\rho_\varepsilon x + x_j^\varepsilon) - \bar{u}_j^\varepsilon)$$

$$\tilde{D}_j^\varepsilon := \left((D^\varepsilon \cap Q_j^\varepsilon) - x_j^\varepsilon\right)\frac{1}{\rho_\varepsilon}$$

we can apply Lemma 3.5.2 and call their limits (up to subsequences) $\tilde{D}_0$ and $\tilde{v}_0$.
In this way (3.5.47) becomes

$$|\{x \in Q^1(0) \backslash \tilde{D}_j^\varepsilon(t) : |\nabla \tilde{v}_j^\varepsilon| > \lambda + \delta\}| > \left(\frac{\gamma}{2} - \nu\right)\frac{|Q^1(0)|}{|\Omega|}. \tag{3.5.48}$$

Now we want to pass to the limit in ε to have a contradiction, and to this aim the weak-convergence of $\nabla \tilde{v}_\varepsilon(t)$ is not enough, so we prove that (at fixed t) the convergence is strong in H^1.
We have for each $t \in [0, T]$ (and up to subsequence) $\tilde{v}_j^\varepsilon(t) \rightharpoonup \tilde{v}_0(t)$ in $H^1(Q^1(0))$ so, since $H^1(Q^1(0))$ is an Hilbert space we just need to prove the convergence of the norms. By compactness we have strong convergence for each $t \in [0, T]$ (and up to subsequence) of $\tilde{v}_j^\varepsilon(t)$ to $\tilde{v}_0(t)$ in $L^2(Q^1(0))$, that means we need just to show the L^2 strong convergence of the gradient. To this aim we note that $\tilde{v}_j^\varepsilon(t)$ satisfy the Euler-Lagrange equation

$$-div(\sigma_{\tilde{D}_j^\varepsilon} \nabla \tilde{v}_j^\varepsilon) = g_j^\varepsilon$$

with $g_j^\varepsilon(x) = \rho_\varepsilon f(\rho_\varepsilon x + x_j^\varepsilon)$, and using as test function $v = \tilde{v}_j^\varepsilon(t) - \tilde{v}_0(t)$ in the weak formulation we obtain

$$\int_{Q^1(0)} \sigma_{\tilde{D}_j^\varepsilon(t)} |\nabla \tilde{v}_j^\varepsilon(t) - \nabla \tilde{v}_0(t)|^2 dx = \langle g_j^\varepsilon, \tilde{v}_j^\varepsilon(t) - \tilde{v}_0(t) \rangle - \int_{Q^1(0)} \sigma_{\tilde{D}_j^\varepsilon(t)} \nabla \tilde{v}_0(t)(\nabla \tilde{v}_j^\varepsilon(t) - \nabla \tilde{v}_0(t))$$

and since for each $t \in [0,T]$ (and up to subsequences) $\tilde{v}_j^\varepsilon(t) - \tilde{v}_0(t) \rightharpoonup 0$ in $H^1(\Omega)$ and $g_j^\varepsilon \longrightarrow 0$ in H^{-1} the first term on the right hand side converges to zero. At the same time by the L^2-strong convergence of $\sigma_{\tilde{D}_j^\varepsilon}(t)$ to $\sigma_{\tilde{D}_0}(t)$ we have that also the last term in the right hand-side goes to zero, and since it holds

$$\alpha \int_{Q^1(0)} |\nabla \tilde{v}_j^\varepsilon(t) - \nabla \tilde{v}_0(t)|^2 dx \leq \int_{Q^1(0)} \sigma_{\tilde{D}_j^\varepsilon}(t) |\nabla \tilde{v}_j^\varepsilon(t) - \nabla \tilde{v}_0(t)|^2 dx$$

we obtain the strong L^2-convergence of the gradient (up to subsequences). Now passing to the limit in the subsequences in (3.5.48) we have

$$|\{x \in Q^1(0) \backslash \tilde{D}_0(t) : |\nabla \tilde{v}_0(t)| \geq \lambda + \delta\}| \geq \left(\frac{\gamma}{2} - \nu\right) \frac{|Q^1(0)|}{|\Omega|} > 0$$

since ν can be chosen arbitrarily small. Otherwise, since $j \in \Lambda_\varepsilon^\nu$ it holds by Proposition 3.10 that

$$|\{x \in Q^1(0) \backslash \tilde{D}_0(t) : |\nabla \tilde{v}_0(t)| > \lambda\}| = 0$$

and so a contradiction.

Step 2.

Now to prove that $A(t)$, $u(t)$ and $\theta(t)$ satisfy the minimality condition as in Definition 3.1, we first show that, for every $t \in [0,T]$, $(u^\varepsilon(t), D^\varepsilon(t))$ is an almost minimizers for (0.0.11) in the following sense:

$$E_{el}(u^\varepsilon(t), D^\varepsilon(t)) + k|D^\varepsilon(t)| \tag{3.5.49}$$

$$\leq \min_{v \in H_0^1(\Omega)} \left\{ E_{el}(v, D) + k|D| \right\} + o_\varepsilon(1) \tag{3.5.50}$$

where

$$E_{el}(v, D) := \int_\Omega \sigma_D |\nabla u|^2 dx - \langle f, v \rangle,$$

for all $D \supseteq D^\varepsilon(t)$. For each ε and t we fix D and we call the related minimum point in (3.5.49) $v = \hat{u}^\varepsilon(t)$. Considering test set $D = \hat{D}^\varepsilon(t)$ given by

$$\hat{D}^\varepsilon(t) = D^\varepsilon(t) \cup E^\varepsilon$$

with E^ε such that

$$E^\varepsilon \cap D^\varepsilon(t) = \emptyset$$

and using Lemma 12 in [43] we have

$$E_{el}(u^\varepsilon(t), D^\varepsilon(t)) - E_{el}(\hat{u}^\varepsilon(t), \hat{D}^\varepsilon(t)) \leq \frac{(\beta - \alpha)\beta}{2\alpha} \|\nabla u^\varepsilon(t)\|^2_{L^2(\hat{D}^\varepsilon(t) \backslash D^\varepsilon(t))}. \tag{3.5.51}$$

Introducing the set

$$A_\varepsilon^\delta := \{x \notin D^\varepsilon(t) : |\nabla u^\varepsilon(t)| > \lambda + \delta\} \quad \text{and} \quad B_\varepsilon^\delta := \{x \notin D^\varepsilon(t) : |\nabla u^\varepsilon(t)| \leq \lambda + \delta\}$$

which divide the undamaged domain (at time t) in two regions, one in which the gradient of $\nabla u^\varepsilon(t)$ exceeds the threshold $\lambda+\delta$ and the complementary one the right hand-side term in (3.5.51) can be written as

$$\frac{(\beta-\alpha)\beta}{2\alpha}\int_{A_{\varepsilon,\delta}\cap E^\varepsilon}|\nabla u^\varepsilon(t)|^2dx+\frac{(\beta-\alpha)\beta}{2\alpha}\int_{B_{\varepsilon,\delta}\cap E^\varepsilon}|\nabla u^\varepsilon(t)|^2dx.$$

It can be proved that $|\nabla u^\varepsilon(t)|^2$ can be seen as sum of a term which is equiintegrable and one which goes to zero in measure (see [42] for details) and since by Step 1 $u^\varepsilon(t)$ satisfies the threshold property, we have $|A^\delta_\varepsilon|$ goes to zero when ε goes to zero, which implies that the first term goes to zero in the limit in ε. The second term can be estimated as

$$\frac{(\beta-\alpha)\beta}{2\alpha}\int_{B_{\varepsilon,\delta}\cap E^\varepsilon}|\nabla u^\varepsilon(t)|^2\le\frac{(\beta-\alpha)\beta}{2\alpha}|E^\varepsilon|(\lambda+\delta)^2=\left(k+\delta^2\frac{(\beta-\alpha)\beta}{2\alpha}+\delta\frac{(\beta-\alpha)\lambda\beta}{\alpha}\right)|E^\varepsilon|$$

where the last equality comes from the relation between λ and k. So we have

$$E_{el}(u^\varepsilon(t),D^\varepsilon(t))-E_{el}(\hat{u}^\varepsilon(t),\hat{D}^\varepsilon(t))\le o_\varepsilon(1)+k|E^\varepsilon|+o_\delta(1)|E^\varepsilon|$$

By this estimates using that $E^\varepsilon\cap D^\varepsilon(t)=0$ we have

$$E_{el}(u^\varepsilon(t),D^\varepsilon(t))+k|D^\varepsilon(t)|\le E_{el}(\hat{u}^\varepsilon,\hat{D}^\varepsilon(t))+k|\hat{D}(t)|+o_\varepsilon(1)+o_\delta(1)$$

and passing to the limit in δ we obtain (3.5.49). Now given $\hat{\theta}(t)=\hat{\theta}(t,x)\in[0,1]$ and $\hat{A}(t)=\hat{A}(t,x)\in G_{\hat{\theta}(t,x)}(\alpha I,\beta I)$ we can consider $\hat{D}^\varepsilon(t)\supseteq D^\varepsilon(t)$ such that $\hat{D}^\varepsilon(t)\overset{*}{\rightharpoonup}\theta(t)$ and $\sigma_{\hat{D}^\varepsilon(t)}$ G-converges to $\hat{A}(t)$ and passing to the limit in ε we obtain the minimality condition as in Definition 3.1. $\square$

CHAPTER 4

Dynamic of the damage: Energy and threshold approach

4.1. Main results

The main original result of this chapter is given in Theorem 4.1. We prove that a minimizing sequence $(u_n(t), D_n(t))$ for minimum problems for the time-discrete energy in (0.0.11) with a discrete version of kinetic term converges to a pair $(u(t), \theta(t))$ satisfying a (weak) homogenized version of elasto-dynamic equation of (0.0.21), an energy inequality, irreversibility of the damage and a threshold property. In the second part, using a different approach (dealing directly with the momentum equation), we show that the solution of the problem (0.0.21) for discrete time considering the damage updated at each time step through a threshold criterion, converges (as the time step goes to zero) to a couple $(u(t), \theta(t))$ satisfying, also with this approach, a (weak) homogenized version of elasto-dynamic equation of (0.0.21) (see Theorem 4.7). In the final section we suggest a definition of Threshold solution.

4.2. Energy approach

To show the existence of an evolution (in terms of displacement and damage) for the (weak) homogenized version of Euler-Lagrange equation of (0.0.21) we will apply explicitly the well-established method for showing existence for rate-independent processes ([**59, 61, 66, 67**]) adjusted to the coupling with the inertial term. As explained in the Introduction of this thesis the strategy consists in a discretization of the time and to solve minimum problems at next time steps. Then, from such solutions, after having defined piecewise and affine functions the idea is to pass in the limit in the partition of the time through some compactness property and to verify that the limit satisfies an Euler-Lagrange equation of the motion. We start defining the formulation of the incremental problem.

4.2.1. The formulation of the incremental problem.

For every $n \in \mathbb{N}$, we fix a time scale $\Delta t = \frac{1}{n^p}$ with $0 < p < 1$ (this restriction will be clear at the end of the proof of the Lemma 4.2.2) and we consider a partition of the time interval $[0, T]$ given by points $t_0^n = 0$ and t_i^n with $i \geq 1$ such that $t_i^n - t_{i-1}^n = \Delta t$. To avoid heavy notation, sometimes we write t_i instead of t_i^n.
Given $f \in W^{1,1}([0, T]; H^{-1}(\Omega))$, $\Omega \subset \mathbb{R}^n$, with $n \geq 1$, we consider the energy functional

$$E(t, u, D) := \frac{1}{2} \int_\Omega \sigma_D |\nabla u|^2 dx + k|D| - \langle f(t), u \rangle \tag{4.2.1}$$

and starting from initial conditions $D_0 := \emptyset$ and $(u_0, v_0) \in H^1(\Omega) \times L^2(\Omega)$ we define $(u_0^n, D_0^n) := (u_0, D_0)$ and iteratively $(u_i^n, D_i^n) \in H_0^1(\Omega) \times \mathcal{P}(\Omega)$ as follows:

- We choose $(\tilde{u}_1^n, D_1^n)$ such that

$$E(t_1^n, \tilde{u}_1^n, D_1^n) + \frac{1}{2} \left\| \frac{\tilde{u}_i^n - u_0}{\Delta t} - v_0 \right\|_{L^2}^2 \leq \inf_{u \in H_0^1;\ D \supseteq D_0} E(t_1^n, u, D) + \frac{1}{2} \left\| \frac{u - u_0}{\Delta t} - v_0 \right\|_{L^2}^2 + \frac{1}{2^i n} \tag{4.2.2}$$

and, fixed D_1^n, we define u_1^n be the minimizer of

$$E(t_1^n, u, D_i^n) + \frac{1}{2}\left\|\frac{u-u_0}{\Delta t} - v_0\right\|_{L^2}^2. \tag{4.2.3}$$

- Analogously for $i \geq 1$ we choose $(\tilde{u}_{i+1}^n, D_{i+1}^n)$ such that

$$\begin{aligned} &E(t_{i+1}^n, \tilde{u}_{i+1}^n, D_{i+1}^n) + \frac{1}{2}\left\|\frac{\tilde{u}_{i+1}^n - u_i^n}{\Delta t} - \frac{u_i^n - u_{i-1}^n}{\Delta t}\right\|_{L^2}^2 \leq \\ &\inf_{u\in H_0^1;\ D\supseteq D_i^n} E(t_{i+1}^n, u, D) + \frac{1}{2}\left\|\frac{u-u_i^n}{\Delta t} - \frac{u_i^n - u_{i-1}^n}{\Delta t}\right\|_{L^2}^2 + \frac{1}{2^i n} \end{aligned} \tag{4.2.4}$$

  and fixed D_{i+1}^n we define u_{i+1}^n be the minimizer of

$$E(u, D_{i+1}^n, t_{i+1}^n) + \frac{1}{2}\left\|\frac{u-u_i^n}{\Delta t} - \frac{u_i^n - u_{i-1}^n}{\Delta t}\right\|_{L^2}^2. \tag{4.2.5}$$

We define $D_n(0) := D_0$ and for every $t \in (t_i^n, t_{i+1}^n]$ we define the following piecewise constant and affine functions

$$\tilde{u}_n(t) = u_{i+1}^n \qquad u_n(t) = u_i^n + (t - t_i^n)\frac{u_{i+1}^n - u_i^n}{\Delta t} \qquad D_n(t) = D_{i+1}^n \tag{4.2.6}$$

$$v_n(t) = \frac{u_i^n - u_{i-1}^n}{\Delta t} + \frac{(t-t_i^n)}{\Delta t}\left(\frac{u_{i+1}^n - u_i^n}{\Delta t} - \frac{u_i^n - u_{i-1}^n}{\Delta t}\right) \qquad f_{i+1}^n := f(t_{i+1}^n). \tag{4.2.7}$$

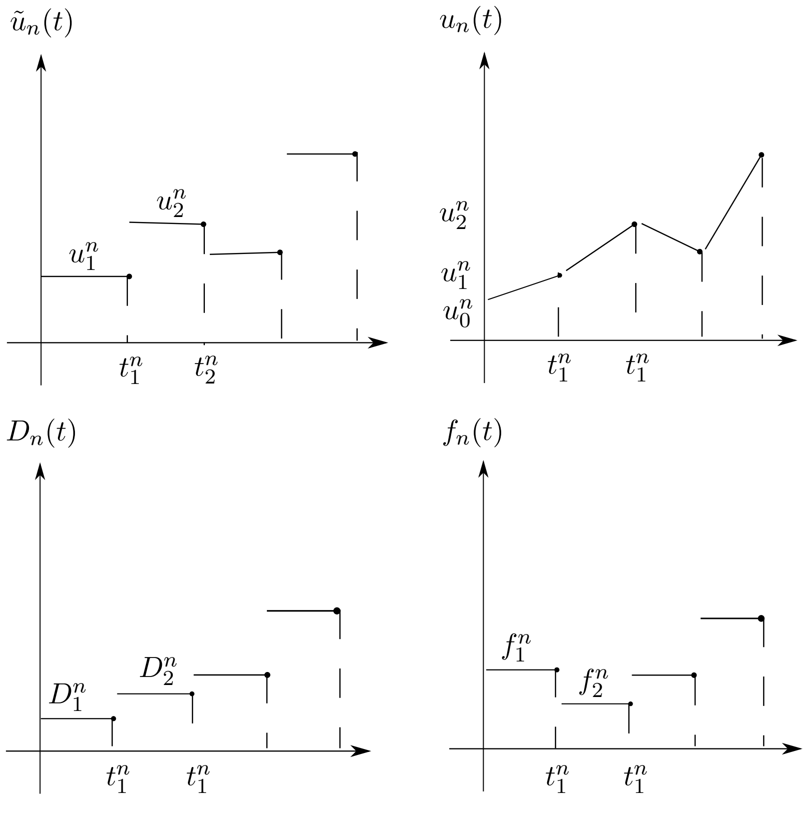

FIGURE 1.

In this energetic framework the main result that we will prove is the following:

THEOREM 4.1. *Let* $(u_n(t), D_n(t))$ *as in* (4.2.6) *and* (4.2.7).
Then there exists $u \in W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1_0(\Omega))$, $\theta(t) \in L^\infty(\Omega;[0,1])$ *and* $A(t) \in L^\infty(\Omega;\mathcal{F}(\alpha,\beta))$ *such that, up to subsequences,*

$$u_n(t) \overset{H^1}{\rightharpoonup} u(t), \qquad \chi_{D_n(t)} \overset{*}{\rightharpoonup} \theta(t), \qquad \sigma_{D_n(t)} \xrightarrow{G} A(t) \tag{4.2.8}$$

with $\theta(t)$ *increasing and* $A(t)$ *decreasing in time.*
Moreover $(u_n(t), D_n(t))$ *and the limit* $(u(t), \theta(t), A(t))$ *are such that:*

- ***Euler-Lagrange equation***: *for every* $\phi \in L^2(0,T;H^1(\Omega)) \cap W^{1,1}(0,T;L^2(\Omega))$ *it holds*

$$-\int_0^T\int_\Omega \dot{u}(t)\dot{\phi}\,dx\,dt + \int_\Omega \dot{u}(T)\phi(T)dx - \int_\Omega \dot{u}(0)\phi(0)dx + \int_0^T\int_\Omega A(t)\nabla u(t)\nabla\phi\,dx\,dt = \int_0^T \langle f(t),\phi\rangle\,dt \tag{4.2.9}$$

- ***Energy Balance inequality***: *given*

$$E_{tot}(t) := \frac{1}{2}\|\dot{u}(t)\|^2_{L^2} + \frac{1}{2}\int_\Omega A(t)\nabla u(t)\nabla u(t)dx + k\int_\Omega \theta(t)dx - \langle f(t), u(t)\rangle,$$

  it holds

$$E_{tot}(t) \le E_{tot}(0) - \int_0^t \langle \dot{f}(s), u(s)\rangle ds, \tag{4.2.10}$$

- ***Threshold condition***: *for each* $\delta > 0$ *it holds*

$$\lim_{n\longrightarrow\infty} |\{x \notin D_n(t) : |\nabla u_n(t)| > \lambda + \delta\}| = 0 \tag{4.2.11}$$

  with $\lambda := \sqrt{\dfrac{2k}{\beta(\beta-\alpha)}}$

This result will be consequence of propositions and lemmas proved in next subsections.

4.2.2. Convergence result and threshold condition. In this subsection we want to prove (4.2.8) using compactness properties and an a priori estimate following [27] . We will also prove the threshold condition in (4.2.11) following the blow-up argument proposed in [43] and used in Chapter 3.

LEMMA 4.2.1. For each $j = 0,..\lfloor T/\Delta t\rfloor$ it holds

$$\begin{aligned}&\|\dot{u}_n(t)\|^2_{L^2} + \int_\Omega \sigma_{D_n(t)}|\nabla u_n(t^n_{j+1})|^2dx + \Delta t\int_0^{t^n_{j+1}} \|\dot{v}_n(s)\|^2_{L^2}ds + \Delta t\int_0^{t^n_{j+1}}\int_\Omega \sigma_{D_n(t)}|\nabla \dot{u}_n(s)|^2dxds \\ &= 2\int_0^{t^n_{j+1}} \langle f_n(s), \dot{u}_n(s)\rangle ds + \int_\Omega \sigma_{D_n(0)}|\nabla u_n(0)|^2dx + \|\dot{u}(0)\|^2_{L^2} - (\beta-\alpha)\sum_{i=0}^j \int_{D^n_{i+1}\setminus D^n_i} |\nabla u^n_i|^2dx\end{aligned} \tag{4.2.12}$$

PROOF. Since u^n_{i+1} is the minimum point for the functional in (4.2.5) it satisfies the following weak Euler-Lagrange equation

$$\int_\Omega \sigma_{D^n_{i+1}}\nabla u^n_{i+1}\nabla\varphi dx + \int_\Omega \left(\frac{u^n_{i+1}-u^n_i}{\Delta t} - \frac{u^n_i - u^n_{i-1}}{\Delta t}\right)\frac{\varphi}{\Delta t}dx - \langle f^{i+1}_n, \varphi\rangle = 0. \tag{4.2.13}$$

for each $\varphi \in H^1_0(\Omega)$.
Choosing $\varphi = u^n_{i+1} - u^n_i$ we have

$$
\begin{aligned}
&\int_\Omega \sigma_{D^n_{i+1}}|\nabla u^n_{i+1}|^2 - \int_\Omega \sigma_{D^n_{i+1}}\nabla u^n_{i+1}\nabla u^n_i dx + \int_\Omega \left|\frac{u^n_{i+1}-u^n_i}{\Delta t}\right|^2 dx \\
&- \int_\Omega \frac{u^n_{i+1}-u^n_i}{\Delta t}\frac{u^n_i-u^n_{i-1}}{\Delta t}dx - \langle f^{i+1}_n, (u^n_{i+1}-u^n_i)\rangle = 0
\end{aligned}
\tag{4.2.14}
$$

and using the identity $\frac{1}{2}\|g\|^2_{L^2} - \int_\Omega g\cdot h dx = \frac{1}{2}\|g-h\|^2_{L^2} - \frac{1}{2}\|h\|^2_{L^2}$ with $g = \nabla u^n_{i+1}\sqrt{\sigma_{D^n_{i+1}}}$ and $h = \nabla u^n_i\sqrt{\sigma_{D^n_{i+1}}}$ for the first two terms on the left side of the previous equality and $g = \frac{u^n_{i+1}-u^n_i}{\Delta t}$ and $h = \frac{u^n_i - u^n_{i-1}}{\Delta t}$ for the second two terms we easily obtain

$$
\begin{aligned}
&\left\|\frac{u^n_{i+1}-u^n_i}{\Delta t}\right\|^2_{L^2} + \left\|\frac{u^n_{i+1}-u^n_i}{\Delta t} - \frac{u^n_i-u^n_{i-1}}{\Delta t}\right\|^2_{L^2} \\
&+ \int_\Omega \sigma_{D^n_{i+1}}|\nabla u^n_{i+1}|^2 dx + \int_\Omega \sigma_{D^{i+1}_n}|\nabla u^n_{i+1} - \nabla u^n_i|^2 dx \\
&= 2\langle f^{i+1}_n, u^n_{i+1}-u^n_i\rangle + \int_\Omega \sigma_{D^n_{i+1}}|\nabla u^n_i|^2 dx + \left\|\frac{u^n_i-u^n_{i-1}}{\Delta t}\right\|^2_{L^2}.
\end{aligned}
$$

Summing over $i = 0, .., j$ and using the identity $\sigma_{D^n_{i+1}} = \sigma_{D^n_i} - (\beta-\alpha)\chi_{D^n_{i+1}\setminus D^n_i}$ we have for $t \in (t_j, t_{j+1}]$

$$
\begin{aligned}
&\left\|\frac{u^n_{j+1}-u^n_j}{\Delta t}\right\|^2_{L^2} + \sum_{i=0}^{j}\|\Delta t\dot{v}_n(t)\|^2_{L^2} + \int_\Omega \sigma_{D^n_{j+1}}|\nabla u^n_{j+1}|^2 dx + \sum_{i=0}^{j}\int_\Omega \sigma_{D^n_{i+1}}|\nabla u^n_{i+1} - \nabla u^n_i|^2 dx \\
&= 2\sum_{i=0}^{j}\langle f^{i+1}_n(u^n_{i+1}-u^n_i)\rangle + \int_\Omega \sigma_{D_n(0)}|\nabla u_n(0)|^2 dx + \|\dot{u}(0)\|^2_{L^2} - (\beta-\alpha)\sum_{i=1}^{j}\int_{D^n_{i+1}\setminus D^n_i}|\nabla u^n_i|^2 dx
\end{aligned}
$$

From which follows immediately (4.2.12) by definitions in (4.2.6) and (4.2.7). $\square$

Let note that from the previous lemma we have immediately that for each $i \geq 0$

$$
\begin{aligned}
&\|\dot{u}_n(t)\|^2_{L^2} + \alpha\|\nabla u_n(t^n_{i+1})\|^2_{L^2} + \Delta t\int_0^{t^n_{i+1}}\|\dot{v}_n(t)\|^2_{L^2}dt + \Delta t\int_0^{t^n_{i+1}}\|\nabla\dot{u}_n(t)\|^2_{L^2}dt \\
&\qquad \leq \beta\|\nabla u_n(0)\|_{L^2} + \|\dot{u}(0)\|^2_{L^2} + 2\|f\|_{L^2(0,T;H^{-1}(\Omega))}T^{1/2}\max_{t\in[0,T]}\|\dot{u}_n(t)\|_{L^2}.
\end{aligned}
\tag{4.2.15}
$$

in which the right-hand side above is bounded as long as $M_n := \max_{t\in[0,T]}\|\dot{u}_n(t)\|_{L^2}$ is bounded. From (4.2.15) we immediately have that

$$
M^2_n \leq \beta\|\nabla u(0)\|^2_{L^2(\Omega)} + \|\dot{u}(0)\|^2_{L^2(\Omega)} + 2\|f\|_{L^2(0,T;H^{-1}(\Omega))}T^{1/2}M_n. \tag{4.2.16}
$$

This implies that M_n is bounded, and so is the right-hand side of (4.2.15). From this estimates we obtain also that

i) u_n is bounded in $W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1_0)$,
ii) $\tilde{u}_n$ is bounded in $L^\infty(0,T;H^1(\Omega))$.

By the definition of the sequences u_n and v_n and the minimality of u^n_{i+1} we deduce that for a.e. $t \in [0,T]$ we have

$$
\int_\Omega \dot{v}_n(t)\phi\, dx + \int_\Omega \sigma_{D_n(t)}\nabla\tilde{u}_n(t)\nabla\phi\, dx = \langle f_n(t), \phi\rangle \tag{4.2.17}
$$

for every $\phi \in H^1_0(\Omega)$ which implies that

$$\left| \int_\Omega \dot{v}_n(t)\phi dx \right| \leq (\beta \|\tilde{u}_n(t)\|_{H^1} + \|f_n(t)\|_{H^{-1}})\|\phi\|_{H^1}$$

for every $\phi \in H^1_0(\Omega)$ and hence by i) we have that

iii) v_n is bounded in $W^{1,\infty}(0,T;H^{-1}(\Omega))$.

As in [27] is easy to have that up to subsequence $u_n \rightharpoonup u$ in $H^1(0,T;L^2(\Omega))$ and $v_n \rightharpoonup v$ in $H^1(0,T;H^{-1}(\Omega))$. We also have that $\dot{u}(t) = v(t)$ in $L^2(\Omega)$ a.e. $t \in (0,T)$. Indeed it holds

$$\|\dot{u}_n(t) - v_n(t)\|_{H^{-1}} = \|v_n(t^n_{i+1}) - v_n(t)\|_{H^{-1}} \leq \int_{t^n_i}^{t^n_{i+1}} \|\dot{v}_n(s)\|_{H^{-1}} ds \leq c\Delta t$$

which goes to zero when $n \longrightarrow \infty$ (remind $\Delta t = 1/n^p$ with $p > 0$). By this we conclude that $v(t) = \dot{u}(t)$ in H^{-1} , and since by (4.2.15) and (4.2.16) we also have that v_n is bounded in $W^{1,\infty}(0,T;L^2(\Omega))$ we obtain by a density argument that

$$\langle v(t), \varphi\rangle_{H^{-1}\times H^1_0} = \langle v(t), \varphi\rangle_{L^2\times L^2} = \langle \dot{u}(t), \varphi\rangle_{H^{-1}\times H^1_0} = \langle \dot{u}(t), \varphi\rangle_{L^2\times L^2}$$

that is $v(t) = \dot{u}(t)$ in $L^2(\Omega)$.

Moreover using Helly's Theorem and properties of G-convergence and arguing as in Lemma 3.3.1 (Chapter 3) it is easy to obtain that there exists a subsequence of $(u_n(t), D_n(t), \sigma_{D_n(t)})$ and there exists $u(t) \in H^1_0(\Omega)$, $\theta(t) \in L^\infty(\Omega;[0,1])$ and $A(t) \in L^\infty(\Omega;\mathcal{F}(\alpha,\beta))$ such that (for such subsequence)

$$u_n(t) \overset{H^1}{\rightharpoonup} u(t), \qquad \chi_{D_n(t)} \overset{*}{\rightharpoonup} \theta(t), \qquad \sigma_{D_n(t)} \overset{G}{\longrightarrow} A(t), \tag{4.2.18}$$

with $\theta(t)$ increasing and $A(t)$ decreasing in time, for each $t \in [0,T]$. This conclude the first part of Theorem 4.1. We now prove the threshold property as stated in (4.2.11).

Proposition 4.2. *Given $(u_n(t), D_n(t))$ as in (4.2.6) it holds the following threshold condition:*

$$\lim_{n\longrightarrow\infty} |\{x \notin D_n(t) : |\nabla u_n(t)| > \lambda + \delta\}| = 0 \tag{4.2.19}$$

for each $\delta > 0$ and $\lambda := \sqrt{\frac{2k}{\beta(\beta-\alpha)}}$,

Proof. We first prove the result for (u^n_i, D^n_i) then by a convexity argument we will easily obtain the claim. The first part of the proof is strictly similar to the one in [43] and we will follow the same steps as done in Chapter 3.

Given a set $Q \subseteq \Omega$ we define

$$E(u, D, Q) := \frac{1}{2}\int_Q \sigma_{D\cap Q}|\nabla u| + k|D \cap Q|.$$

We define

$$E^n := E^n_{i,\delta} := \{x \notin D^n_i : |\nabla u^n_i| > \lambda + \delta\}$$

and we suppose by contradiction that there exists $\delta > 0$ such that

$$\limsup_{n\longrightarrow\infty} |E^n_{i,\delta}| = 2\eta$$

with $\eta > 0$, which implies that (up to subsequences)

$$|E^n| > \eta \tag{4.2.20}$$

for $n \geq \bar{n}$ for a fixed $\bar{n} >> 1$.

In the first part of the proof we show that for $n >> 1$ (so using (4.2.20)) there exists an explicit constant $c > 0$ and $(\tilde{w}^n_i, \tilde{D}^n_i)$ admissible for the minimum problem (4.2.4) such that

$$E(\tilde{w}^n_i, \tilde{D}^n_i, \Omega) \leq E(u^n_i, D^n_i, \Omega) - c \tag{4.2.21}$$

i.e. decreasing the elastic part of the energy in the whole Ω.
In the second part we will consider also the kinetic part of the problem obtaining competitors which total energy is less the one given by (u_i^n, D_i^n) and so a contradiction because of the minimality property of (u_i^n, D_i^n).

Part 1.

Now for each $n \geq \bar{n}$ we consider a covering of E^n made of squares Q such that

1) the center $\bar{x}$ of the square is in E_n and it is a Lebesgue point for u_i^n and ∇u_i^n, i.e. it holds
$$\lim_{r \longrightarrow 0^+} \frac{1}{|B_r(\bar{x})|} \int_{B_r(\bar{x})} |u_i^n(\bar{x}) - u_i^n(y)|^p dy = 0$$
(and the same for ∇u_i^n) for all $p \geq 1$, where $B_r(\bar{x})$ is the ball with center $\bar{x}$ and radius r.
2) two sides of Q are orthogonal to $\nabla u_i^n(\bar{x})$;
3) defined $\bar{u}_i^n(x) := u_i^n(\bar{x}) + \nabla u_i^n(\bar{x}) \cdot (x - \bar{x})$ we have
$$\|u_i^n - \bar{u}_i^n\|^2_{H^1(Q)} \leq \varepsilon |Q|$$
$$|D_i^n \cap Q| \leq \varepsilon |Q|$$

Let note that since Q is a square of a covering of E^n it depends on n, δ and i and by definition it depends also on ε and its measure goes to zero when ε goes to zero. Moreover for each ε it is a fine covering of E^n so we can choose a finite number of disjoint square to cover E^n.

We can divide the proof of the first part in 3 steps.
As first step we will show that considering test functions in each Q with the same boundary condition of $\bar{u}_i^n$ in ∂Q (instead of u_i^n) we can decrease in this square the elastic energy given by the pair $(\bar{u}_i^n, \emptyset)$ using a process of lamination, in particular we will show that for each $\sigma > 0$ there exist v_i^n and $\hat{D}_i^n$ such that
$$E(v_i^n, \hat{D}_i^n, Q) \leq E(\bar{u}_i^n, \emptyset, Q) - \frac{1}{2}\beta\delta^2|Q| + \sigma|Q| \tag{4.2.22}$$
with $v_i^n = \bar{u}_i^n$ on ∂Q.

To do it we recall a technical result (used also in Chapter 3) to match the boundary conditions of special (almost) test functions (which will be piecewise linear functions) with the boundary conditions of $\bar{u}_i^n$.
In the second step, using the previous one, we will show that in each square Q we can lower the energy given by (u_i^n, D_i^n) using a test function with the same boundary conditions of u_i^n and choosing ε sufficiently small, i.e. we will show that for each $s > 0$ we can choose ε small in such a way that there exists w_i^n and $\hat{D}_n^i \subset Q$ such that
$$E(\hat{w}_i^n, \hat{D}_n^i, Q) \leq E(u_i^n, D_i^n, Q) - \frac{1}{2}\beta\delta^2|Q| + \sigma|Q| + s|Q| \tag{4.2.23}$$
with $\hat{w}_i^n = u_i^n$ on ∂Q.
Finally in the third step we will use the previous steps to construct an admissible pair for the problem (4.2.4) that has in Ω the elastic energy lower than the one given by (u_i^n, D_i^n).
Step 1.
We consider an arbitrary square of the covering of E^n. To avoid heavy notation we can assume that $\bar{x} = 0$ and $u_i^n(\bar{x}) = 0$ and so we have $\bar{u}_i^n(x) = \nabla u_i^n(0) \cdot x$
We consider the continuous periodic function $z(y)$ such that $z(0) = 0$, $z(1) = \lambda + \delta$ such that
$$z'(y) = \begin{cases} \frac{\beta}{\alpha}\lambda, & \text{if } y \in (0, d) \\ \lambda, & \text{if } y \in (d, 1) \end{cases} \tag{4.2.24}$$

where d is univocally determined and it holds $d = \frac{(\nabla u_i^n(0)-\lambda)\alpha}{\lambda(\beta-\alpha)}$ and we define

$$\bar{v}_{i,h}^n(x) := z\left(\frac{x}{h}\cdot\frac{\nabla u_i^n(0)}{|\nabla u_i^n(0)|}\right)$$

$$\hat{D}_{i,h}^n := \left\{x \in Q : z'(\frac{x}{h}\cdot\frac{\nabla u_i^n(0)}{|\nabla u_i^n(0)|}) = \frac{\beta}{\alpha}\lambda\right\}$$

It is easy to see that $\bar{v}_{i,h}^n$ converges strongly to $\bar{u}_i^n$ in $L^2(Q)$ when $h \longrightarrow 0$ and that it is bounded in $H^1(Q)$. Now we match the boundary conditions of $\bar{v}_{i,h}^n$ with the ones of $\bar{u}_i^n$ using the cut-off function (cfr. details see [43], Remark 12)

$$\phi(y) = \begin{cases} 1, & \text{if } y \in Q\backslash Q_R \\ 0, & \text{if } y \in Q_{R-\mu} \end{cases} \tag{4.2.25}$$

and such that $|\nabla\phi| = \frac{1}{\mu}$ in $Q_R\backslash Q_{R-\mu}$ where $R \in (0, |Q|^{1/2})$ will be suitable chosen and $\mu \in (0, R)$ and we define

$$v_{i,h}^n = \phi\bar{u}_i^n + (1-\phi)\bar{v}_{i,h}^n.$$

Let remark that we can choose R such that

$$\lim_{\mu\longrightarrow 0}\lim_{h\longrightarrow 0}\int_{Q_R\backslash Q_{R-\mu}} |\nabla v_{i,h}^n|^2 = 0 \tag{4.2.26}$$

$$\lim_{\mu\longrightarrow 0}\lim_{h\longrightarrow 0}\int_{Q_R} |\nabla v_{i,h}^n - \nabla\bar{v}_{i,h}^n|^2 = 0 \tag{4.2.27}$$

By the (4.2.26) and the (4.2.27) follows that the energy in $|Q|$ given by $(v_{i,h}^n, \hat{D}_{i,h}^n)$ is arbitrarily close to that given by $(\bar{v}_{i,h}^n, \hat{D}_{i,h}^n)$ which is (with a simple computation)

$$\frac{1}{2}\beta\lambda(\nabla u_i^n(0))|Q| + \frac{1}{2}\beta\lambda\delta|Q|$$

So, we conclude that for each $\sigma > 0$ there exists $v_i^n := v_{i,h}^n$ and $\hat{D}_i^n := \hat{D}_{i,h}^n$ (with $h << 1$) such that

$$E(v_i^n, \hat{D}_i^n, Q) \le E(\bar{v}_{i,h}^n, \hat{D}_{i,h}^n, Q) + \sigma = E(\bar{u}_i^n, \emptyset, Q) - \frac{1}{2}\beta(|\nabla u_i^n(0)| - \lambda)^2|Q| + \sigma|Q| \tag{4.2.28}$$

$$< E(\bar{u}_i^n, \emptyset, Q) - \frac{1}{2}\beta\delta^2|Q| + \sigma|Q| \tag{4.2.29}$$

where the last inequality comes from the fact that $(|\nabla u_i^n(0)| - \lambda) > \delta$ and so the (4.2.22).

Step 2.

We start showing that by the properties (3) of Q we have

$$|E(u_i^n, D_i^n, Q) - E(\bar{u}_i^n, \emptyset, Q)| \le o_\varepsilon(1)|Q| \tag{4.2.30}$$

indeed we have

$$E(u_i^n, D_i^n, Q) - E(\bar{u}_i^n, \emptyset, Q) = \int_Q \sigma_{D_i^n}|\nabla u_i^n|^2dx + k|D_i^n \cap Q| - \beta\int_Q |\nabla\bar{u}_i^n|^2dx \tag{4.2.31}$$

$$\le \beta\int_Q (|\nabla u_i^n|^2 - |\nabla\bar{u}_i^n|^2)dx + k|D_i^n \cap Q| \tag{4.2.32}$$

and using the property for numbers $|a|^2 - |b|^2 \le |a-b|^2 + 2|a-b||b|$ we obtain (using also Holder inequality) that

$$E(u_i^n, D_i^n, Q) - E(\bar{u}_i^n, \emptyset, Q) \le C\int_Q (|\nabla u_i^n - \nabla\bar{u}_i^n|^2)dx + k|D_i^n \cap Q| \tag{4.2.33}$$

with $C > 0$, and using the properties (3) we have (4.2.30).
Now considering $\hat{w}_i^n := v_i^n + (u_i^n - \bar{u}_i^n)$, with v_i^n as in (4.2.28), it has the same boundary condition in Q of u_i^n. So by (4.2.22) and (4.2.30) we have that for each $s > 0$ we can take ε sufficiently small in such a way that

$$E(\hat{w}_i^n, \hat{D}_n^i, Q) \leq E(u_i^n, D_i^n, Q) - \frac{1}{2}\beta\delta^2|Q| + \sigma|Q| + s|Q|, \tag{4.2.34}$$

i.e. the inequality (4.2.23).
Step 3.
We come back to whole Ω and consider the problem

$$\inf_{w,D'} \left\{ \frac{1}{2}\int_\Omega \sigma_{D'}|\nabla w|^2 dx + k|D'| : (w - u_i^n) \in H_0^1(\Omega), D' \supseteq D_i^n \right\}.$$

Iterating the previous steps for each square of the covering of E^n we can construct for each $s > 0$ and $\sigma > 0$ a pair $(\tilde{w}_i^n, \tilde{D}_i^n)$ with $\tilde{D}_i^n \supseteq D_i^n$ and $\tilde{D}_i^n = D_i^n$ outside E^n, and $w_i^n = u_i^n$ outside E_n such that

$$E(\tilde{w}_i^n, \tilde{D}_i^n, \Omega) \leq E(u_i^n, D_i^n, \Omega) + (-\frac{1}{2}\beta\delta^2 + s)|E^n| + \sigma|E^n| \tag{4.2.35}$$

And since for $n \geq \bar{n}$, $\varepsilon << 1$ and $\delta > 0$ we can have σ and s small as we want we obtain that there exist $c > 0$ such that

$$E(\tilde{w}_i^n, \tilde{D}_i^n, \Omega) \leq E(u_i^n, D_i^n, \Omega) - c$$

i.e. the inequality (4.2.21).
Part 2.
Now we link the result for the elastic part and dissipation of the energy of (u_i^n, D_i^n) with the (almost) minimality property of (u_i^n, D_i^n) for the *total* energy (elastic+kinetic) in such a way to obtain a contradiction and so the validity of (4.2.19). By the (almost) minimality property of (u_i^n, D_i^n) we have

$$\begin{aligned} &E(u_i^n, D_i^n) + \frac{1}{2}\left\| \frac{u_i^n - u_{i-1}^n}{\Delta t} - \frac{u_{i-1}^n - u_{i-2}^n}{\Delta t} \right\|_{L^2(\Omega)}^2 \leq \\ &E(\tilde{w}_i^n, \tilde{D}_i^n) + \frac{1}{2}\left\| \frac{\tilde{w}_i^n - u_{i-1}^n}{\Delta t} - \frac{u_{i-1}^n - u_{i-2}^n}{\Delta t} \right\|_{L^2(\Omega)}^2 + \frac{1}{2^i n} \end{aligned} \tag{4.2.36}$$

with $(\tilde{w}_i^n, \tilde{D}_i^n)$ as in (4.2.35).
Now we focus on the second term of the right side of the inequality, and to avoid heavy notation we call

$$A := \frac{\tilde{w}_i^n - u_{i-1}^n}{\Delta t}, \qquad B := \frac{u_{i-1}^n - u_{i-2}^n}{\Delta t}, \qquad \text{and} \qquad C := \frac{u_i^n - u_{i-1}^n}{\Delta t}.$$

We have

$$\frac{1}{2}\|A - B\|_{L^2}^2 = \frac{1}{2}\|C - B\|_{L^2}^2 \tag{4.2.37}$$

$$+ \frac{1}{2}\left[\Big(\|A - B\|_{L^2} - \|C - B\|_{L^2} \Big) \Big(\|A - B\|_{L^2} + \|C - B\|_{L^2} \Big) \right] \tag{4.2.38}$$

$$\leq \frac{1}{2}\|C - B\|_{L^2}^2 + \left[\|A - C\|_{L^2} \Big(\|A - C\|_{L^2} + 4C_{n,i} \Big) \right] \tag{4.2.39}$$

where the constant $C_{n,i}$ is an upper bound for the kinetic part of the discrete solution i.e. it is such that:

$$\frac{1}{2}\left\|\frac{u_i^n-u_{i-1}^n}{\Delta t}-\frac{u_{i-1}^n-u_{i-2}^n}{\Delta t}\right\|_{L^2(\Omega)}\le C_{n,i}.$$

Explicitly we have (in the notation of Part 1)

$$\|A-C\|_{L^2(\Omega)}=\left\|\frac{\tilde{w}_i^n-u_{i-1}^n}{\Delta t}-\frac{u_i^n-u_{i-1}^n}{\Delta t}\right\|_{L^2(\Omega)}=\left\|\frac{\tilde{w}_i^n-u_i^n}{\Delta t}\right\|_{L^2(\Omega)} \tag{4.2.40}$$

$$=\sum_{Q\in E^n}\left\|\frac{\hat{w}_i^n-u_i^n}{\Delta t}\right\|_{L^2(Q)}=\sum_{Q\in E^n}\left\|\frac{v_i^n-\bar{u}_i^n}{\Delta t}\right\|_{L^2(Q)} \tag{4.2.41}$$

$$=\frac{1}{\Delta t}\sum_{Q\in E^n}\left\|(1-\phi)(\bar{v}_{i,h}^n-\bar{u}_i^n)\right\|_{L^2(Q)} \tag{4.2.42}$$

By the convergence of $\bar{v}_{i,h}^n$ to $\bar{u}_i^n$, when h goes to zero, we obtain finally the estimates

$$\|A-C\|_{L^2(\Omega)}\le\frac{|E^n|}{|Q|}\frac{o_h(1)}{\Delta t} \tag{4.2.43}$$

By this estimates we have that the second term of the right side of (4.2.36) satisfies

$$\begin{aligned}\frac{1}{2}\left\|\frac{\tilde{w}_i^n-u_{i-1}^n}{\Delta t}-\frac{u_{i-1}^n-u_{i-2}^n}{\Delta t}\right\|^2_{L^2(\Omega)}\le&\frac{1}{2}\left\|\frac{u_i^n-u_{i-1}^n}{\Delta t}-\frac{u_{i-1}^n-u_{i-2}^n}{\Delta t}\right\|^2_{L^2(\Omega)}\\&+\left(2C_{n,i}+\frac{|E^n|}{|Q|}\frac{o_h(1)}{2\Delta t}\right)\frac{|E^n|}{|Q|}\frac{o_h(1)}{\Delta t}\end{aligned} \tag{4.2.44}$$

and combining (4.2.35), (4.2.36) and (4.2.44) we obtain that for each n

$$|E^n|\left(-\frac{1}{2}\beta\delta+s+\sigma|Q|+\left(2C_{n,i}+\frac{|E^n|}{|Q|}\frac{o_h(1)}{2\Delta t}\right)\frac{o_h(1)}{|Q|\Delta t}\right)+\frac{1}{2^i n}\ge 0$$

but, for fixed $n\ge\bar{n}$ and $\varepsilon<<1$, we can have (choosing appropriate competitors) h, s and σ small as we want (as we noticed in step 3 of part 1), so such that

$$|E_n|\left(-\frac{1}{2}\beta\delta+s+\sigma+\left(2C_{n,i}+\frac{|E^n|}{|Q|}\frac{o_h(1)}{2\Delta t}\right)\frac{o_h(1)}{|Q|\Delta t}\right)+\frac{1}{2^i n}<0$$

and so a contradiction.
This proves that for each $i\ge 1$

$$\lim_{n\longrightarrow\infty}|\{x\notin D_i^n:|\nabla u_i^n|>\lambda+\delta\}|=0. \tag{4.2.45}$$

Finally by a convexity argument we obtain the claim in (4.2.19). Indeed, for each $t\in[0,T]$ there exists $i\ge 0$ such that $t\in(t_i^n,t_{i+1}^n]$ and so by definition of $u_n(t)$ and $D_n(t)$ we have

$$\begin{aligned}&|\{x\notin D_n(t):|\nabla u_n(t)|>\lambda+\delta\}|=\\&|\{x\notin D_{i+1}^n:\left|\nabla u_i^n\left(1-\frac{t-t_i^n}{\Delta t}\right)+\nabla u_{i+1}^n\frac{t-t_i^n}{\Delta t}\right|>\lambda+\delta\}|.\end{aligned} \tag{4.2.46}$$

Obviously we have

$$\left|\nabla u_i^n\left(1-\frac{t-t_i^n}{\Delta t}\right)+\nabla u_{i+1}^n\frac{t-t_i^n}{\Delta t}\right|\le|\nabla u_i^n|\left(1-\frac{t-t_i^n}{\Delta t}\right)+|\nabla u_{i+1}^n|\frac{t-t_i^n}{\Delta t} \tag{4.2.47}$$

and since $x\notin D_{i+1}^n$ we also have that $x\notin D_i^n$ and by (4.2.45) we obtain that for each $\varepsilon>0$ there exists $\bar{n}>>1$ such that for each $n>\bar{n}$

$$|\nabla u^n_{i+1}| \le \lambda + \delta + \varepsilon \qquad \text{and} \qquad |\nabla u^n_i| \le \lambda + \delta + \varepsilon \tag{4.2.48}$$

for $x \notin D^n_{i+1}$. Combining (4.2.46), (4.2.47) and (4.2.48) we conclude the proof. $\square$

4.2.3. Euler-Lagrange equation and Energy Inequality. In this subsection following the strategy in [30] we prove that the limit $(u(t), \theta(t), A(t))$ satisfies the energy inequality (4.2.10) while following the strategy in [17] we prove that it satisfies a weak form of the following Euler-Lagrange equation

$$\ddot{u} - \operatorname{div}(A(t)\nabla u) = f \qquad \text{in } \Omega \tag{4.2.49}$$

with zero boundary condition.

PROPOSITION 4.3. *Given $(u(t), \theta(t), A(t))$ limit of $(u_n(t), D_n(t), \sigma_{D_n(t)})$ it holds*

$$-\int_0^T \int_\Omega \dot{u}(t)\dot{\phi}\, dx\, dt + \int_\Omega \dot{u}(t)\phi(t) dx - \int_\Omega u(0)\phi(0) dx + \int_0^T \int_\Omega A(t)\nabla u(t)\nabla\phi\, dx\, dt = \int_0^T \langle f(t), \phi\rangle\, dt \tag{4.2.50}$$

for every $\phi \in L^2(0,T; H^1_0(\Omega)) \cap W^{1,1}(0,T; L^2(\Omega))$

PROOF. Integrating in time (4.2.17), with $\phi \in L^2(0,T; H^1_0(\Omega)) \cap W^{1,1}(0,T; L^2(\Omega))$, taking the limit as $n \to +\infty$ and denoting by $\sigma(t)$ the weak limit in $L^2(\Omega)$ of $\sigma_{D_n(t)}\nabla\tilde{u}_n(t)$ we obtain

$$-\int_0^T \int_\Omega \dot{u}(t)\dot{\phi}\, dx\, dt + \int_\Omega \dot{u}(t)\phi(t) dx - \int_\Omega u(0)\phi(0) dx + \int_0^T \int_\Omega \sigma\nabla\phi\, dx\, dt = \int_0^T \langle f(t), \phi\rangle\, dt \tag{4.2.51}$$

We only need to show that $\sigma(t) = A(t)\nabla u(t)$ a.e. $t \in (0,T)$. Let us denote by $\theta(t)$ the weak-* limit in $L^\infty(\Omega)$ of $\chi_{D_n(t)}$. By the monotonicity in t of the damage sets $D_n(t)$ we deduce that $\Theta(t) = \int_\Omega \theta(t)$ is increasing and hence continuous up to a countable set of points in $(0,T)$. Let us fix $\tau \in (0,T)$ be a point of continuity of $\Theta(t)$ and $h > 0$. Integrating in time (4.2.17) from $\tau - h$ to τ (and multiplying by $1/h$) we get

$$\begin{aligned}&\int_\Omega \sigma_{D_n(\tau)} \left(\fint_{\tau-h}^{\tau} \nabla\tilde{u}_n(t)\, dt \right) \nabla\phi\, dx = \\ &-\int_\Omega \frac{v_n(\tau) - v_n(\tau-h)}{h}\phi\, dx + \int_\Omega \left(\fint_{\tau-h}^{\tau} (\sigma_{D_n(\tau)} - \sigma_{D_n(t)})\nabla\tilde{u}_n(t)\, dt \right) \nabla\phi\, dx + \fint_{\tau-h}^{\tau} \langle f_n(t), \phi\rangle\end{aligned} \tag{4.2.52}$$

with $\phi \in H^1_0(\Omega)$. We define

$$\bar{u}_n = \fint_{\tau-h}^{\tau} \tilde{u}_n(t)\, dt \qquad \bar{u} = \fint_{\tau-h}^{\tau} u(t)\, dt$$

and we denote by $\hat{u}_n$ the unique solution of the following elliptic problem

$$\begin{cases} -\operatorname{div}(\sigma_{D_n(\tau)}\nabla\hat{u}_n) = -\operatorname{div}(A(\tau)\nabla\bar{u}) & \text{in } \Omega \\ \hat{u}_n(\tau) = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{4.2.53}$$

As a consequence of the G-convergence of $\sigma_{D_n(\tau)}$ to $A(\tau)$ we deduce that $\hat{u}_n$ converges to $\bar{u}$ weakly in $H^1_0(\Omega)$ and hence that $\hat{u}_n - \bar{u}_n \rightharpoonup 0$ weakly in $H^1_0(\Omega)$. Using $\hat{u}_n - \bar{u}_n$ as test function in (4.2.52) and (4.2.53) we get

$$\begin{aligned}\int_\Omega \sigma_{D_n(\tau)}|\nabla(\hat{u}_n-\bar{u}_n)|^2dx = &-\int_\Omega \frac{v_n(\tau)-v_n(\tau-h)}{h}(\hat{u}_n-\bar{u}_n)\,dx\\ &-\int_\Omega\left(\fint_{\tau-h}^{\tau}(\beta-\alpha)\chi_{D_n(\tau)\setminus D_n(t)}\nabla\tilde{u}_n(t)\,dt\right)\nabla(\hat{u}_n-\bar{u}_n)\,dx\\ &+\fint_{\tau-h}^{\tau}\langle f_n(t),\hat{u}_n-\bar{u}_n\rangle dt\\ &-\int_\Omega A(\tau)\nabla\bar{u}\nabla(\hat{u}_n-\bar{u}_n)\end{aligned}$$

From this, using the boundness of v_n and the convergence to zero of $\hat{u}_n-\bar{u}_n$ we get

$$\begin{aligned}\int_\Omega \sigma_{D_n(\tau)}|\nabla(\hat{u}_n-\bar{u}_n)|^2dx &\le \int_\Omega\left(\fint_{\tau-h}^{\tau}(\beta-\alpha)\chi_{D_n(\tau)\setminus D_n(t)}|\nabla\tilde{u}_n(t)|\,dt\right)|\nabla(\hat{u}_n-\bar{u}_n)|\,dx+o(1)\\ &\le\left[\int_\Omega\left(\fint_{\tau-h}^{\tau}(\beta-\alpha)\chi_{D_n(\tau)\setminus D_n(t)}|\nabla\tilde{u}_n(t)|\,dt\right)^2dx\right]^{\frac{1}{2}}\|\nabla(\hat{u}_n-\bar{u}_n)\|_{L^2}+o(1)\end{aligned}\tag{4.2.54}$$

where we applied Hölder inequality. Now by Jensen inequality and Remark 1 and (4.2.19), we get

$$\begin{aligned}\int_\Omega\left(\fint_{\tau-h}^{\tau}\chi_{D_n(\tau)\setminus D_n(t)}|\nabla\tilde{u}_n(t)|\,dt\right)^2dx &\le\int_\Omega\fint_{\tau-h}^{\tau}\chi_{D_n(\tau)\setminus D_n(t)}|\nabla\tilde{u}_n(t)|^2\,dt\,dx\\ &\le\fint_{\tau-h}^{\tau}\int_{D_n(\tau)\setminus D_n(t)}|\nabla\tilde{u}_n(t)|^2\,dx\,dt\\ &\le M\fint_{\tau-h}^{\tau}|D_n(\tau)\setminus D_n(t)|\,dt+o(1)\\ &\le M|D_n(\tau)\setminus D_n(\tau-h)|+o(1)\,.\end{aligned}\tag{4.2.55}$$

Applying Young's inequality from (4.2.54) and (4.2.55) we obtain that there exists a constant $C>0$ such that

$$\int_\Omega|\nabla(\hat{u}_n-\bar{u}_n)|^2dx\le C|D_n(\tau)\setminus D_n(\tau-h)|+o(1)\,,\tag{4.2.56}$$

i.e.

$$\limsup_{n\to\infty}\int_\Omega|\nabla(\hat{u}_n-\bar{u}_n)|^2dx\le C(\Theta(\tau)-\Theta(\tau-h))\,.\tag{4.2.57}$$

From this we get

$$\begin{aligned}\int_\Omega\left(\fint_{\tau-h}^{\tau}\sigma_{D_n(t)}\nabla\tilde{u}_n(t)\,dt-\sigma_{D_n(\tau)}\nabla\hat{u}_n\right)^2dx &\le 2\int_\Omega\left(\fint_{\tau-h}^{\tau}(\beta-\alpha)\chi_{D_n(\tau)\setminus D_n(t)}|\nabla\tilde{u}_n(t)|\,dt\right)^2dx\\ &+2\beta^2\int_\Omega|\nabla(\hat{u}_n-\bar{u}_n)|^2dx\\ &\le C(\Theta(\tau)-\Theta(\tau-h))+o(1)\,.\end{aligned}$$

Now by the definition of $\hat{u}_n$, taking the limit as $n\to+\infty$ we get

$$\int_\Omega\left(\fint_{\tau-h}^{\tau}\sigma(\tau)\,dt-A(\tau)\nabla\bar{u}\right)^2dx\le C(\Theta(\tau)-\Theta(\tau-h))\,.\tag{4.2.58}$$

Using the fact that a.e. $\tau \in (0,T)$ is a Lebesgue point of $\sigma(\tau)$ and $u(\tau)$ and a continuity point for $\Theta(\tau)$, taking the limit as $h \to 0$ we get

$$\int_\Omega |\sigma(\tau) - A(\tau)\nabla u(\tau)|^2 dx \le \lim_{h\to 0} C(\Theta(\tau) - \Theta(\tau - h)) = 0\,, \tag{4.2.59}$$

which concludes the proof. □

Finally remains to prove the energy inequality (4.2.10). This is done in the next Lemma in which we use the same technique as in [30] We define

$$E_{tot}(t,u,\theta,A) := \frac{1}{2}\|\dot u\|^2_{L^2} + \frac{1}{2}\int_\Omega A\nabla u \nabla u dx + k\int_\Omega \theta dx - \langle f(t), u\rangle$$

LEMMA 4.2.2.
Given $(u(t),\theta(t),A(t))$ the limit of $(u_n(t), \chi_{D_n}(t), \sigma_{D_n}(t))$ and

$$E_{tot}(t) := E_{tot}(t,u(t),\theta(t),A(t))$$

it holds

$$E_{tot}(t) \le E_{tot}(0) - \int_0^t \langle \dot f(s), u(s)\rangle ds$$

PROOF. Considering (u^n_{i+1}, D^n_{i+1}) we have by almost minimality condition (4.2.5)

$$\begin{aligned}&\frac{1}{2}\int_\Omega \sigma_{D^n_{i+1}}|\nabla u^n_{i+1}|^2 dx + k|D^n_{i+1}| + \frac{1}{2}\left\|\frac{u^n_{i+1}-u^n_i}{\Delta t} - \frac{u^n_i - u^n_{i-1}}{\Delta t}\right\|^2_{L^2} - \langle f^n_{i+1}, u^n_{i+1}\rangle\\ &\le \frac{1}{2}\int_\Omega \sigma_D|\nabla u|^2 dx + k|D| + \frac{1}{2}\left\|\frac{u-u^n_i}{\Delta t} - \frac{u^n_i-u^n_{i-1}}{\Delta t}\right\|^2_{L^2} - \langle f^n_{i+1}, u\rangle + \frac{1}{2^i n}\end{aligned} \tag{4.2.60}$$

with $D \supseteq D^n_i$ and $u \in H^1(\Omega)$.
We consider as test set $D = D^n_{i+1}\backslash E_\lambda$ with E_λ such that

- $D^n_{i+1} \supseteq E_\lambda$,
- $E_\lambda \cap D^n_i = \emptyset$
- $|E_\lambda| = \lambda(|D^n_{i+1}| - |D^n_i|)$

with $\lambda \in (0,1)$.
It is easy to see that it implies

$$|D| = \lambda(|D^n_i| - |D^n_{i+1}|) + |D^n_{i+1}|, \qquad \text{and} \qquad \sigma_D = \sigma_{D^n_{i+1}} + (\beta-\alpha)\chi_{E_\lambda}.$$

We use this test set D in the right hand-side of (4.2.60) which becomes

$$\begin{aligned}&\frac{1}{2}\int_\Omega \sigma_{D^n_{i+1}}|\nabla u|^2 dx + \frac{(\beta-\alpha)}{2}\int_{E_\lambda}|\nabla u|^2 dx\\ &+ k\lambda(|D^n_i| - |D^n_{i+1}|) + k|D^n_{i+1}| + \frac{1}{2}\left\|\frac{u-u^n_i}{\Delta t} - \frac{u^n_i-u^n_{i-1}}{\Delta t}\right\|^2_{L^2} - \langle f^n_{i+1}, u\rangle + \frac{1}{2^i n}.\end{aligned} \tag{4.2.61}$$

Now we consider as test function $\bar u := u^n_{i+1} - \lambda(u^n_{i+1} - u^n_i)$, and since

$$\begin{aligned}\frac{1}{2}\int_\Omega \sigma_{D^n_{i+1}}|\nabla \bar u|^2 dx - \langle f^n_{i+1}, \bar u\rangle &= \frac{1}{2}\int_\Omega \sigma_{D^n_{i+1}}|\nabla u^n_{i+1}|^2 dx + \lambda\left(\frac{\lambda-2}{2}\right)\int_\Omega \sigma_{D^n_{i+1}}|\nabla u^n_{i+1}|^2 dx\\ &+ \frac{\lambda^2}{2}\int_\Omega \sigma_{D^n_{i+1}}|\nabla u^n_i|^2 dx + \lambda(1-\lambda)\int_\Omega \sigma_{D^n_{i+1}}\nabla u^n_{i+1}\nabla u^n_i dx - \langle f^n_{i+1}, u^n_{i+1}\rangle + \lambda\langle f^n_{i+1}, u^n_{i+1} - u^n_i\rangle\end{aligned} \tag{4.2.62}$$

and

$$\frac{1}{2}\left\|\frac{\bar{u}-u_i^n}{\Delta t}-\frac{u_i^n-u_{i-1}^n}{\Delta t}\right\|_{L^2}^2=\frac{1}{2}\left\|\frac{u_{i+1}^n-u_i^n}{\Delta t}-\frac{u_i^n-u_{i-1}^n}{\Delta t}\right\|_{L^2}^2+\frac{\lambda^2}{2}\left\|\frac{u_{i+1}^n-u_i^n}{\Delta t}\right\|_{L^2}^2 \\ -\lambda\int_\Omega\left(\frac{u_{i+1}^n-u_i^n}{\Delta t}-\frac{u_i^n-u_{i-1}^n}{\Delta t}\right)\left(\frac{u_{i+1}^n-u_i^n}{\Delta t}\right)dx. \tag{4.2.63}$$

Considering (4.2.60), (4.2.61), (4.2.62), (4.2.63) and dividing by λ we have

$$\begin{aligned}\left(\frac{2-\lambda}{2}\right)\int_\Omega\sigma_{D_{i+1}^n}|\nabla u_{i+1}^n|^2dx+\int_\Omega\left(\frac{u_{i+1}^n-u_i^n}{\Delta t}-\frac{u_i^n-u_{i-1}^n}{\Delta t}\right)(\frac{u_{i+1}^n-u_i^n}{\Delta t})&\\ -(1-\lambda)\int_\Omega\sigma_{D_{i+1}}\nabla u_i^n\nabla u_{i+1}^n dx&\\ \leq\frac{\lambda}{2}\int_\Omega\sigma_{D_{i+1}^n}|\nabla u_i^n|^2dx+\frac{\lambda}{2}\left\|\frac{u_{i+1}^n-u_i^n}{\Delta t}\right\|^2&\\ +k(|D_i^n|-|D_{i+1}^n|)+\frac{(\beta-\alpha)}{2}\int_{E_\lambda}|\nabla\bar{u}|^2dx&\\ +\Delta t\langle f_{i+1}^n,\frac{u_{i+1}^n-u_i^n}{\Delta t}\rangle+\frac{1}{\lambda 2^i n}&\end{aligned} \tag{4.2.64}$$

We note that the first and third term of the left hand-side satisfy

$$\left(\frac{2-\lambda}{2}\right)\int_\Omega\sigma_{D_{i+1}^n}|\nabla u_{i+1}^n|^2dx-(1-\lambda)\int_\Omega\sigma_{D_{i+1}}\nabla u_i^n\nabla u_{i+1}^n dx \\ \geq(1-\lambda)\int_\Omega\sigma_{D_{i+1}^n}\nabla u_{i+1}^n\nabla(u_{i+1}^n-u_i^n)dx. \tag{4.2.65}$$

Now considering the following identities (see [30] pag. 14,15-16):

- $\int_\Omega\sigma_{D_{i+1}^n}\nabla u_{i+1}^n(\nabla u_{i+1}^n-\nabla u_i^n)dx=$
$$\int_{t_i}^{t_{i+1}}\int_\Omega\sigma_{D_{i+1}^n}\nabla u_n(s)\nabla\dot{u}_n(s)ds+\frac{\Delta t}{2}\int_{t_i}^{t_{i+1}}\int_\Omega\sigma_{D_{i+1}^n}\nabla\dot{u}_n(s)\nabla\dot{u}_n(s)ds$$

- $$\int_\Omega\left(\frac{u_{i+1}^n-u_i^n}{\Delta t}-\frac{u_i^n-u_{i-1}^n}{\Delta t}\right)(\frac{u_{i+1}^n-u_i^n}{\Delta t})=\frac{1}{2}\left\|\frac{u_{i+1}^n-u_i^n}{\Delta t}\right\|_{L^2}^2 \\ -\frac{1}{2}\left\|\frac{u_i^n-u_{i-1}^n}{\Delta t}\right\|_{L^2}^2+\frac{\Delta t}{2}\int_{t_i}^{t_{i+1}}|\dot{v}_n(s)|^2ds$$

- $$\frac{\lambda}{2}\int_\Omega\sigma_{D_{i+1}^n}|\nabla u_i^n|^2dx=\frac{\lambda}{2\Delta t}\int_{t_i}^{t_{i+1}}\int_\Omega\sigma_{D_{i+1}^n}|\nabla u_n(s)|^2ds \\ -\frac{\lambda}{2}\int_{t_i}^{t_{i+1}}\int_\Omega\sigma_{D_{i+1}^n}\nabla u_n(s)\nabla\dot{u}(s)ds \\ +\frac{\lambda\Delta t}{12}\int_{t_i}^{t_{i+1}}\int_\Omega\sigma_{D_{i+1}^n}|\nabla\dot{u}_n(s)|^2ds$$

and using (4.2.65) the inequality (4.2.64) becomes

$$
\begin{aligned}
&(1-\lambda)\int_{t_i}^{t_{i+1}}\int_\Omega \sigma_{D^n_{i+1}}\nabla u_n(s)\nabla\dot u_n(s)ds+(1-\lambda)\frac{\Delta t}{2}\int_{t_i}^{t_{i+1}}\int_\Omega \sigma_{D^n_{i+1}}\nabla\dot u_n(s)\nabla\dot u_n(s)ds\\
&+\frac{1}{2}\left\|\frac{u^n_{i+1}-u^n_i}{\Delta t}\right\|^2_{L^2}-\frac{1}{2}\left\|\frac{u^n_{i}-u^n_{i-1}}{\Delta t}\right\|^2_{L^2}+\frac{\Delta t}{2}\int_{t_i}^{t_{i+1}}|\dot v_n(s)|^2ds\\
&\le\frac{\lambda}{2\Delta t}\int_{t_i}^{t_{i+1}}\int_\Omega \sigma_{D^n_{i+1}}|\nabla u_n(s)|^2ds-\frac{\lambda}{2}\int_{t_i}^{t_{i+1}}\int_\Omega \sigma_{D^n_{i+1}}\nabla u_n(s)\nabla\dot u(s)ds\\
&+\frac{\lambda\Delta t}{12}\int_{t_i}^{t_{i+1}}\int_\Omega \sigma_{D^n_{i+1}}|\nabla\dot u_n(s)|^2ds+\frac{\lambda}{2\Delta t}\int_{t_i}^{t_{i+1}}\|\dot u_n(s)\|ds\\
&+k(|D^n_i|-|D^n_{i+1}|)+(\beta-\alpha)\int_{E_\lambda}|\nabla\bar u|^2dx+\int_{t_i}^{t^{i+1}}\langle f^n_{i+1},\dot u_n(s)\rangle ds+\frac{1}{\lambda 2^i n}
\end{aligned}
\tag{4.2.66}
$$

which can be rewritten in the form

$$
\begin{aligned}
&\frac{2-\lambda}{2}\int_{t_i}^{t_{i+1}}\int_\Omega \sigma_{D^n_{i+1}}\nabla u_n(s)\nabla\dot u_n(s)ds+k(|D^n_{i+1}|-|D^n_i|)\\
&+\frac{1}{2}\left\|\frac{u^n_{i+1}-u^n_i}{\Delta t}\right\|^2_{L^2}-\frac{1}{2}\left\|\frac{u^n_{i}-u^n_{i-1}}{\Delta t}\right\|^2_{L^2}+\frac{\Delta t}{2}\int_{t_i}^{t_{i+1}}|\dot v_n(s)|^2ds\\
&\le\frac{(7\lambda-6)\Delta t}{12}\int_{t_i}^{t_{i+1}}\int_\Omega \sigma_{D^n_{i+1}}|\nabla\dot u_n(s)|^2ds+\frac{\lambda}{2\Delta t}\int_{t_i}^{t_{i+1}}\|\dot u_n(s)\|ds\\
&+\frac{\lambda}{2\Delta t}\int_{t_i}^{t_{i+1}}\int_\Omega \sigma_{D^n_{i+1}}|\nabla u_n(s)|^2ds++(\beta-\alpha)\int_{E_\lambda}|\nabla\bar u|^2dx-\int_{t_i}^{t^{i+1}}\langle\dot f(s),u_n(s)\rangle ds+\frac{1}{\lambda 2^i n}
\end{aligned}
\tag{4.2.67}
$$

Moreover let note that the first term on the left side of the inequality is (integrating by part)

$$
\frac{2-\lambda}{4}\left[\int_\Omega \sigma_{D^n_{i+1}}|\nabla u_n(t_{i+1})|^2dx-\int_\Omega\sigma_{D^n_i}|\nabla u_n(t_i)|^2dx+(\beta-\alpha)\int_{D^n_{i+1}\setminus D^n_i}|\nabla u_n(t_i)|^2dx\right]
\tag{4.2.68}
$$

with

$$
(\beta-\alpha)\int_{D^n_{i+1}\setminus D^n_i}|\nabla u_n(t_i)|^2dx\ge 0
\tag{4.2.69}
$$

Let remind that $\Delta t=\frac{1}{n^p}$ and let take $\lambda=\frac{1}{n^q}$ with $0<p<q<1$. In such a way, summing over $i=0,1,...j$ and sending n to infinity we have by properties of convergence of $u_n(t)$, $D_n(t)$ and $\sigma_{D_n(t)}$ that

$$
E_{tot}(u(t),\theta(t),A(t))\le E_{tot}(u(0),\theta(0),A(0))-\int_0^t\langle\dot f(s),u(s)ds\rangle
$$

which concludes the proof.

$\square$

REMARK 4.4. ***On the opposite energy inequality***.
Let note that Theorem 4.1, doesn't prove that an energy equality holds but only an inequality one. The inequality missing in the Theorem is

$$E_{tot}(t) \geq E_{tot}(0) - \int_0^t \langle \dot{f}(s), u(s) \rangle ds \tag{4.2.70}$$

It is interesting to note that in literature a lot of models of damage, plasticity, fracture in which the equality is obtained include a viscosity term, which gives a compactness property strongly used to prove the opposite inequality (see e.g. [27, 52, 75] and reference therein). In [30] it is studied a quasi-static evolution for a linearly-plasticity model and the missing inequality is obtained without the necessity of this compactness property (although it holds). However, since there's no damage evolution in such model the elasticity tensor remains constant in time and this helps to prove the opposite inequality. Such difficulty is also stressed in [76] (Remark 2.7) in which is proved only the same side of our energy inequality but in the framework of the "entropic" solution.
Here we want just to show where are the difficulties to prove the opposite inequality using standard methods. A first possibly approach should be to use minimality (and almost minimality) properties in (4.2.4) (and in (4.2.2)) to obtain and iteration formula, and then pass to the limit. But in so doing we arrive to obtain the following inequality

$$E(u_{i+1}^n, D_{i+1}^n) + F_i(u_{i+1}^n) + o(n) \geq E(u_i^n, D_i^n) + F_i(u_i^n)$$

where

$$E(v, D) = \frac{1}{2} \int_\Omega \sigma_D |\nabla v|^2 dx + k|D| - \langle f, v \rangle$$

and

$$F_i(v) := \frac{1}{2} \left\| \frac{v - u_i^n}{\Delta t} - \frac{u_i - u_{i-1}}{\Delta t} \right\|_{L^2(\Omega)}$$

which results to be not easily iterable because of the form of the kinetic part.
Another approach could be to start from equality (4.2.12) and pass to the limit using the fact that

$$-\lim_{n \longrightarrow \infty} (\beta - \alpha) \sum_{i=0}^{j} \int_{D_{i+1}^n \setminus D_i^n} |\nabla u_i^n|^2 dx \geq -k \int_\Omega \theta(t) dx + k \int_\Omega \theta(0) dx.$$

which can be easily proved. But the very big problem in the equality (4.2.12) is the limit

$$\lim_{n \longrightarrow \infty} \Delta t \int_0^{t_{j+1}^n} \int_\Omega \sigma_{D_n(t)} |\nabla \dot{u}_n(s)| ds dx$$

Indeed since we have not compactness property for the sequence $\nabla \dot{u}_n(s)$ we can not prove that such limit goes to zero. A way to have this compactness property could be to have in the problem a damping term as in [27], which is not our case.

4.3. A Threshold approach

In this section, to study the damage evolution of our model defined in (0.0.21) we use an alternative approach, based on tackle directly equation in (0.0.21) instead of consider the energy associated. Such idea comes from the fact that both in the energetic approach for dynamic both in the model with the perimeter (studied in Chapter 3), the solution satisfies a threshold condition in the undamaged region (see respectively (4.2.19) and (3.5.45)-(3.5.46)). So this leads to investigate if it's possible to construct a solution of the problem imposing directly some threshold condition.

4.3.1. The formulation of the incremental problem.

We fix $n \in \mathbb{N}$ and we start solving problem (0.0.21) with no damage and until the gradient of solution u has not exceeded a threshold $\lambda > 0$ on a set of measure greater than $1/n$. Then we damage this set and we solve again (0.0.21) with the new damage and updated initial conditions with the same criterion as in step 1. Iteratively we construct a sequence of function $(v^n(t), D^n(t))$ that we show (as consequence of results in [17]) to converge to a pair satisfying, also using this approach, a weak relaxed version of the momentum equation in (0.0.21).

We also take care to construct the partition of $[0, T]$ in such a way that the intervals have size less (or equal) of $1/n$.

Starting with fixed initial condition $p \in H_0^1(\Omega)$ and $q \in L^2(\Omega)$ and external loading $f(t) \in H^{-1}$ we solve step by step the equation (0.0.21), defining at each step the damage set (according to the threshold criterion).

Step 1.

We start from $t = \tau_0 := 0$, no damage, and fixed threshold $\lambda > 0$. We consider $v^{n,0}(x, t)$ the solution of the problem

$$\begin{cases} \ddot{v} - div(\beta \nabla v) = f(t); \\ v(t, x) = 0 \qquad \text{in } \partial\Omega \\ v(0, x) = p(x); \\ \dot{v}(0, x) = q(x) \end{cases} \tag{4.3.71}$$

we define

$$\bar{D}^1(t) := \{x \in \Omega : \exists s \in [0, t) : |\nabla v^{n,0}(x, s)| > \lambda\}$$

and

$$\tau^1_{1/n} := inf\{t > 0 : |\bar{D}^1(t)| > \frac{1}{n}\}.$$

The first time step of the partition is given by

$$\tau_1 := \min\{\tau_0 + \frac{1}{n}, \tau^1_{1/n}, T\}.$$

Let note that by definition the characteristic function $\chi_{\bar{D}^1(t)}(x)$ is monotone (increasing) in time for each $x \in \Omega$, which allows to define the "updated" damage D_1^n as

$$D_1^n := \bar{D}^1(\tau_1^+)$$

where $\bar{D}^1(\tau_1^+)$ is characterized by

$$\chi_{\bar{D}^1(\tau_1^+)}(x) := \lim_{s \longrightarrow \tau_1^+} \chi_{\bar{D}^1(s)}(x).$$

Step 2.
Analogously we consider $v^{n,1}(x,t)$ the solution of the problem

$$\begin{cases} \ddot{v} - div(\sigma_{D_1^n} \nabla v) = f(t); \\ v(t,x) = 0 \qquad \text{in } \partial\Omega \\ v(\tau_1, x) = v^{n,0}(\tau_1); \\ \dot{v}(\tau_1, x) = \dot{v}^{n,0}(\tau_1) \end{cases} \tag{4.3.72}$$

and defining as before

$$\bar{D}^2(t) := \{x \in \Omega \backslash D_1^n : \exists s \in [\tau_1, t) : |\nabla v^{n,1}(x,s)| > \lambda\}$$

and

$$\tau_{1/n}^2 := inf\{t > \tau_1 : |\bar{D}^2(t)| > \frac{1}{n}\}.$$

$$\tau_2 := \min\{\tau_1 + \frac{1}{n}, \tau_{1/n}^2, T\}.$$

we can define the "updated" damage D_2^n as

$$D_2^n := D_1^n \cup \bar{D}^2(\tau_2^+)$$

where $\bar{D}^2(\tau_2^+)$ is characterized by

$$\chi_{\bar{D}^2(\tau_2^+)}(x) := \lim_{s \longrightarrow \tau_2^+} \chi_{\bar{D}^2(s)}(x).$$

Step k + 1.
Iterating the process we arrive to consider $v^{n,k}(x,t)$ the solution of the problem

$$\begin{cases} \ddot{v} - div(\sigma_{D_k^n} \nabla v) = f(t); \\ v(t,x) = 0 \qquad \text{in } \partial\Omega \\ v(\tau_k, x) = v^{n,k-1}(\tau_k); \\ \dot{v}(\tau_k, x) = \dot{v}^{n,k-1}(\tau_k) \end{cases} \tag{4.3.73}$$

and again defining

$$\bar{D}^{k+1}(t) := \{x \in \Omega \backslash D_k^n : \exists s \in [\tau_k, t) : |\nabla v^{n,k}(x,s)| > \lambda\} \tag{4.3.74}$$

and

$$\tau_{1/n}^{k+1} := inf\{t > \tau_k : |\bar{D}^{k+1}(t)| > \frac{1}{n}\}. \tag{4.3.75}$$

$$\tau_{k+1} := \min\{\tau_k + \frac{1}{n}, \tau_{1/n}^{k+1}, T\}. \tag{4.3.76}$$

we obtain the damage D_{k+1}^n as

$$D_{k+1}^n := D_k^n \cup \bar{D}^{k+1}(\tau_{k+1}^+) \tag{4.3.77}$$

where $\bar{D}^{k+1}(\tau_{k+1}^+)$ is characterized by

$$\chi_{\bar{D}^{k+1}(\tau_{k+1}^+)}(x) := \lim_{s \longrightarrow \tau_{k+1}^+} \chi_{\bar{D}^{k+1}(s)}(x).$$

At this point we define $D^n(t) := \emptyset$ if $t \in [0, \tau_1)$ and

$$D^n(t) := D_k^n \qquad \text{and} \qquad v^n(t) := v^{n,k}(t) \tag{4.3.78}$$

if $t \in [\tau_k, \tau_{k+1})$, we have by construction that $v^n(t)$ satisfies (at least formally)

$$\begin{cases} \ddot{v} - div(\sigma_{D^n(t)} \nabla v) = f(t); \\ v(0,x) = p(x); \\ \dot{v}(0,x) = q(x) \end{cases} \tag{4.3.79}$$

for each $t \in [0, T]$.

Remark 4.5. *By definition of the partition given by* $\{\tau_k\}_{k\geq 0}$ *we have that*

$$\tau_k - \tau_{k-1} \leq \frac{1}{n}$$

and that there exists $M = M(n)$ *such that* $\tau_M = T$

Remark 4.6. *The amount of damage added at each time step depends by how* τ_{k+1} *is chosen in (4.3.76) and by the property of (dis)continuity of* $\bar{D}^k(t)$.
In case that $\tau_{k+1} \neq \tau^{k+1}_{1/n}$ *(and* $\tau_{k+1} \neq T$*) we have that*

$$|D^n_{k+1} \backslash D^n_k| = |\bar{D}^{k+1}(\tau^+_{k+1})| \leq \frac{1}{n} \tag{4.3.80}$$

In case that $\tau_{k+1} = \tau^{k+1}_{1/n}$ *(and* $\tau_{k+1} \neq T$*) we have two possibilities: if* τ_k *is a continuity point for* $|\bar{D}^{k+1}(t)|$ *then*

$$|D^n_{k+1} \backslash D^n_k| = |\bar{D}^{k+1}(\tau_{k+1})| = \frac{1}{n} \tag{4.3.81}$$

otherwise if τ_{k+1} *is a discontinuity point for* $|\bar{D}^{k+1}(t)|$ *then it holds*

$$|D^n_{k+1} \backslash D^n_k| = |\bar{D}^{k+1}(\tau^+_{k+1})| > \frac{1}{n} \tag{4.3.82}$$

4.3.2. Convergence result and weak momentum equation. In this subsection, through a more general result due to Casado et al. in [17], we will precise in which sense the equation in (4.3.79) is solved and we stress a result of homogenization.

Theorem 4.7. *Let* $f \in W^{1,1}(0,T;H^{-1}(\Omega))$. *The pair* $(v^n(t), D^n(t))$ *constructed as before satisfies the following weak momentum equation:*

$$\int_0^T \int_\Omega \dot{v}^n(t)\dot{\varphi}(t)dxdt - \int_0^T \sigma_{D^n(t)}\nabla v^n(t)\varphi(t)dxdt = \int_0^t \int_\Omega f(t)\varphi(t)dxdt \tag{4.3.83}$$

for each $\varphi \in \mathcal{D}([0,T]) \times H^1_0(\Omega)$, *with initial boundary conditions* $(p(x), q(x))$. *Moreover it holds a.e. in* $[0,T]$

$$\|\dot{v}^n(t)\|^2_{L^2} + \|v^n(t)\|^2_{H^1} \leq C(\|q\|^2_{H^1} + \|p\|^2_{L^2} + \|f(t)\|^2_{H^{-1}}) \tag{4.3.84}$$

with $C > 0$.
Furthermore there exists $v \in L^\infty(0,T;H^1_0(\Omega))$, $\theta(t) \in L^\infty(0,T;L^\infty(\Omega))$ *and* $A(t) \in L^\infty(0,T;\mathcal{F}(\alpha,\beta))$ *such that (up to subsequences)*

$$v^n \rightharpoonup v \quad in \quad L^\infty(0,T;H^1_0(\Omega)) \tag{4.3.85}$$

$$\chi_{D_n} \rightharpoonup \theta \quad in \quad L^\infty(0,T;L^\infty(\Omega)) \tag{4.3.86}$$

$$\sigma_{D^n(t)} \xrightarrow{G} A(t) \tag{4.3.87}$$

with $(u(t), A(t))$ *satisfying the following (weak) momentum equation*

$$\int_0^T \int_\Omega \dot{v}(t)\dot{\varphi}(t)dxdt - \int_0^T A(t)\nabla v^n(t)\varphi(t)dxdt = \int_0^t \int_\Omega f(t)\varphi(t)dxdt \tag{4.3.88}$$

for each $\varphi \in \mathcal{D}([0,T]) \times H^1_0(\Omega)$, *with initial boundary conditions* $(p(x), q(x))$.

This theorem is consequence of two results proved in [17], the first one is the following and the second one is given in Theorem 4.9 (we write them in the more abstract form).

THEOREM 4.8. *Let $V \subset H \subset V'$ Hilbert space, $(\cdot,\cdot)$ the scalar product in H and $\langle\cdot,\cdot\rangle$ the dual product between V and V'. Given $A \in BV(0,T;\mathcal{L}(V,V'))$ with $A(t)$ symmetric and hyperbolic operator, $f \in \mathcal{M}(0,T,H)$, $g \in BV(0,T;V)$, $p(x) \in V$, $q(x) \in H$, then there exists unique $u \in L^\infty(0,T;V)$ with $u_t \in L^\infty(0,T;H)$ solution of*

$$\begin{cases} \ddot{u} + A(t)u(t) = f(t) + g(t); \\ u(0) = p(x); \\ \dot{u}(0) = q(x) \end{cases} \tag{4.3.89}$$

in the following weak sense

$$\begin{cases} (\ddot{u}(t),v) + \langle A(t)u(t),v\rangle = (f(t),v) + \langle g(t),v\rangle \ \text{ in } \ D'(0,T), \ \text{for each} \ \ v \in V; \\ u(0) = p(x) \\ \dot{u}(0^+) = q(x) \end{cases} \tag{4.3.90}$$

Moreover it holds a.e. $[0,T]$

$$\|\dot{u}(t)\|_H^2 + \|u(t)\|_V^2 \le C(\|q\|_V^2 + \|p\|_V^2 + \|f\|^2_{\mathcal{M}(0,T;H)} + \|g\|^2_{BV(0,T;V')}) \tag{4.3.91}$$

a.e. in t.

Applying this result with $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, $g = 0$, $f \in W^{1,1}(0,T;H^{-1})$, A such that $A(t)\cdot = -div(\sigma_{D^n(t)}\nabla\cdot)$ with $D^n(t) \subset \Omega$ defined as before we have that there exists unique $v^n \in L^\infty(0,T;H_0^1(\Omega))$ with $\dot{v}^n \in L^\infty(0,T;L^2(\Omega))$ satisfying (4.3.83) and (4.3.84). The only thing that we must prove to apply this theorem is that

$$A \in BV(0,T;\mathcal{L}(H_0^1,H^{-1}))$$

i.e. that

$$\|A\|_{BV(0,T;\mathcal{L}(H_0^1,H^{-1}))} := \sup_{[0=t_0<t_1<\ldots<t_m=T]} \sum_{i=1}^m \|A(t_i) - A(t_{i-1})\|_{\mathcal{L}(H_0^1,H^{-1})} < \infty$$

We have

$$\begin{aligned} \|A(t_i) - A(t_{i-1})\|_{\mathcal{L}(H_0^1,H^{-1})} &:= \sup_{\varphi\in H_0^1, \|\varphi\|_{H^1}=1} \|(A(t_i) - A(t_{i-1}))\varphi\|_{H^{-1}} && (4.3.92) \\ &= \sup_{\varphi\in H_0^1, \|\varphi\|_{H^1}=1} \{ \sup_{\phi\in H_0^1, \|\phi\|_{H^1}=1} \langle (A(t_i) - A(t_{i-1})\varphi), \phi\rangle \} && (4.3.93) \\ &= (\beta-\alpha) \sup_{\varphi,\phi} \int_{D^n(t_i)\setminus D^n(t_{i-1})} \nabla\varphi\nabla\phi dx && (4.3.94) \\ &\le (\beta-\alpha)|D^n(t_i)\setminus D^n(t_{i-1})| && (4.3.95) \end{aligned}$$

which implies that

$$\begin{aligned} \|A\|_{BV(0,T;\mathcal{L}(H_0^1,H^{-1}))} &:= \sup_{[0=t_0<t_1<\ldots<t_m=T]} \sum_{i=1}^m (\beta-\alpha)|D^n(t_i)\setminus D^n(t_{i-1})| \\ &= (\beta-\alpha)|D^n(T)\setminus D^n(0)| < \infty && (4.3.96) \end{aligned}$$

so we can apply Theorem 4.8 to our case
The second part of Theorem 4.7 is a direct consequence of the following theorem proved in [17] that guarantees the possibility to pass in the limit in the problem (4.3.79) having that the limit of v_n satisfies the homogenized version of the hyperbolic equation in (0.0.21).

Theorem 4.9. *Given $A_n \in BV(0,T;\mathcal{L}(V,V'))$ symmetric and hyperbolic operator, $p_n \in H_0^1(\Omega)$, $q_n \in L^2(\Omega)$, $f \in W^{1,1}(0,T;H^{-1}(\Omega))$ such that for almost every t*

$$A_n(t) \xrightarrow{G} A(t) \tag{4.3.97}$$

$$p_n \overset{H^1}{\rightharpoonup} p$$

$$q_n \overset{L^2}{\rightharpoonup} q$$

then the unique solution $u^n(t)$ (in the weak sense of (4.3.90)) of

$$\begin{cases} \ddot{u} - div(A_n(t)\nabla u) = f(t); \\ u(0) = p(x) \\ \dot{u}(0) = q(x) \end{cases} \tag{4.3.98}$$

is such that (up to subsequence)

$$u^n \overset{\star}{\rightharpoonup} u \quad in \quad L^\infty(0,T;H_0^1(\Omega))$$

$$\dot{u}^n \overset{\star}{\rightharpoonup} \dot{u} \quad in \quad L^\infty(0,T;L^2(\Omega))$$

with $u(t)$ that solves (in the weak sense on (4.3.90)

$$\begin{cases} \ddot{u} - div(A(t)\nabla u) = f(t); \\ u(0) = p(x) \\ \dot{u}(0) = q(x) \end{cases} \tag{4.3.99}$$

Applying this theorem to the problem (4.3.79) we immediately obtain the second part of the Theorem 4.7.

Remark 4.10. ***Threshold property.***
The evolution $(v^n(t), D^n(t))$ given in (4.3.78), in case of continuity of the measure $|\bar{D}^{k+1}(t)|$ (see definition (4.3.74)) at the point $t = \tau_{k+1}$, satisfies the following threshold property:

$$|\{x \in (D^n(t))^c : |\nabla v^n(x,t)| > \lambda\}| \longrightarrow 0 \tag{4.3.100}$$

when n goes to infinity. Indeed fixed $t \in [0,T]$ we have that there exists $k \geq 0$ such that $t \in [\tau_k, \tau_{k+1})$ and by definition of $v^n(t)$ and $D^n(t)$ we have that (4.3.100) reduces to prove that

$$|\{x \in (D_{k+1}^n \backslash D_k^n) : |\nabla v_k^n| > \lambda\}| \longrightarrow 0 \tag{4.3.101}$$

since by construction $|\nabla v_k^n| \leq \lambda$ in $(D_{k+1}^n)^c$. Now, since we are supposing that (for each $k \geq 0$) $|\bar{D}^{k+1}(t)|$ is continuous at the point $t = \tau_{k+1}$ we have that (see Remark 4.6 equations (4.3.80) and (4.3.81))

$$|D_{k+1}^n \backslash D_k^n| = |\bar{D}^{k+1}(\tau_{k+1})| \leq \frac{1}{n}$$

from which follows the validity of (4.3.101) and so the threshold property (4.3.100).

The previous remark and the threshold property proved for the solution obtained through an energetic approach (see (4.2.19)) suggests to give a definition of evolution in the hyperbolic case using a threshold formulation as done in the elliptic case as in [43]. This is presented in the final remarks of the present thesis.

Conclusions and Perspectives

We have presented a damage model for which the study of the relations between quasi-static evolutions (q.s.e.), Γ-convergence and homogenization effects turn to be very efficient, showing moreover interesting threshold properties.
In this framework we proved, in the 1-D case and for an oscillating energy, a commutative result between q.s.e. and Γ-convergence which is not covered by the well known theory presented in [**64**]. To do it we considered *approximable* q.s.e. which allowed to characterize the measure for the damage sets. A first open question is whether it is possible to obtain the same result without require explicitly the approximability property, or in other words, whether each q.s.e. for the model presented is also an *approximable* q.s.e.. A second open question is whether this results can be obtained, and in which sense, in the n-D case. We still believe that the mechanism will be similar (i.e. the one-dimensional problem captures the main features of this process). Nevertheless, since in more than one dimension many different microstructures are possible for a composite material, the precise statement for the corresponding damage evolution must be more abstract or involve very fine properties of the G-closure.
For the perimeter-regularized energy we proved the convergence of the related q.s.e. to a q.s.e. for the relaxed energy in the sense of the evolution proposed in [**43**], which implies threshold properties for the limit. This result confirms the validity of the homogenized q.s.e. defined in [**43**] instead of the previous definition proposed in [**33**]. We believe that the perimeter penalization term goes to zero for the optimal damage set (as ε goes to zero), as we assumed, but till now the proof of it is an open problem (except for the time $t = 0$).
In the dynamic framework we presented two different approaches to study the damage evolution that brought to consider homogenization effects for the evolution. We proved that both limit evolutions can be approximated (by construction), they satisfy a monotonicity property for the approximating damage sets and a threshold property (in case of continuity of damage for the threshold approach). By these we propose here a definition of a *Threshold evolution* which strictly follows the definition given in [**43**] for a non-dynamic problem. Such definition turns to be more intuitive than an energetic one and it could be easier to study through numerical computation methods:

Definition 4.11. Given the equation

$$\ddot{u} - div(A\nabla u) = f(t) \tag{4.3.102}$$

we say that $(A(t), u(t))$, with $A(t,x) \in G(\alpha,\beta)$ and $u(t,x) \in H^1_0(\Omega)$, is a threshold evolution if there exist $\lambda > 0$ and $(D_n(t), u_n(t))$ such that

$$\ddot{u}_n - div(\sigma_{D_n(t)}\nabla u_n) = f(t)$$

$$\sigma_{D_n} \xrightarrow{G} A(t), \qquad u_n(t) \rightharpoonup u(t)$$

and

1) $D_n(t)$ is increasing in time,
2) For each $t > 0$

$$|\{x \in D_n^c(t) : |\nabla u_n(x,t)| > \lambda + \delta\}| \longrightarrow 0$$

for each $\delta > 0$ when $n \longrightarrow \infty$

3) For each $T > 0$ and for each $E_n \subseteq D_n(T)$: $\liminf |E_n| > 0$ defining

$$\tau := \inf\{t > 0 : |E_n \cap D_n(t)| > 0\}$$

and considering v_n solution of

$$\begin{cases} \ddot{v}_n - div(\sigma_{D_n(t)\setminus E_n} \nabla v_n) = f(t) \\ v_n(0,x) = u_n(0,x) \\ \dot{v}_n(0,x) = \dot{u}_n(0,x) \end{cases} \tag{4.3.103}$$

then, for each $\delta > 0$, and $\Delta t << 1$, it holds

$$\liminf |\{x \in D_n(t) \cap E_n : |\nabla v_n(t,x)| > \lambda - \delta, t \in [\tau, \tau + \Delta t]\}| > 0.$$

As remarked in Chapter 3, through a very similar definition, it was proved in [**43**], that for elliptic problems, a particular solution obtained using an energetic approach is also a threshold evolution. A first question is whether the same result is still valid in the hyperbolic case. Moreover is not clear which relation there is between the *limit solution* found through the two different approaches and if (at least) one of them is a Threshold evolution according to Definition 4.11. Finally we think that a threshold criterion could be useful in some way for the uniqueness of the problem, selecting a solution among all the possible evolutions, but, till now, this direction is not investigated yet.

Bibliography

[1] ALLAIRE G., AUBRY S., JOUVE F., Simulation numrique de l'endommagement l'aide du modle Francfort-Marigo, *Actes du 29me congrs d'analyse numrique,* ESAIM, proceedings, **3** (1998) 1-9.

[2] ALLAIRE G., JOUVE F., VAN GOETHEM N., A level set method for numerical simulation of damage evolution, *Proceedings of ICIAM 2007 Zurich*, R.J eltsch and G. Wanner eds., EMS, Zurich, 2009, 3-22.

[3] AMBROSIO L., BUTTAZZO G., An optimal design problem with perimiter penalization, *Calc. Var.* ,**1** (1993) 55-69.

[4] AMBROSIO L., GIGLI N., A user's guide to optimal transport. *Modelling and Optimisation of Flows on Networks*, B. Piccoli and M. Rascle eds. Lecture Notes in Mathematics 2062, pp. Springer, Berlin, 2013, 1-155.

[5] AURICCHIO F., PETRINI L., Improvements and algorithmical considerations on a recent three-dimensionale model describing strees-induced solid phas transformations, *Int. J. Numer. Meth Engng.*, **55** (2002) 1255-1284.

[6] AURICCHIO F., STEFANELLI U., Well-Posedness and approximation for a one-dimentional model for shape memory alloys, *M3AS: Math. Models Meth. Appl. Sci.* **15** (2005) 1301-1327.

[7] BONETTI E., BONFANTI G., Well-posedness results for a model of damage in thermoviscoelastic materials, *Ann. I. H. Poincar* **25** (2008) 11871208.

[8] BONETTI E., SCHIMPERNA G., SEGATTI A., On a doubly nonlinear model for evolution of damaging in viscoelastic materials, *Jou. Diff. Eq.* **218** 1 (2005) 91116.

[9] BONETTI E., SCHIMPERNA G., Local existence for Frmond's model of damage in elastic materials, *Cont. Mech. Therm.* **16** 4 (2004).

[10] BABADJIAN J-F., A quasi-static evolution model for the interaction between fracture and damage *Arch. Rat. Mech. and Anal.* **200** (2001) 945-1002.

[11] BENSOUSSAN A., LIONS J.L., PAPANICOLAU G., Asymptotic Analysis for Periodic Structures, *North-Holland, Amsterdam* (1978).

[12] BOURDIN B., FRANCFORT, G., MARIGO J., The variational approach to fracture. *J. Elasticity* **91** (2008) 5-148.

[13] BRAIDES A., Γ-convergence for Beginners *Oxford University Press*, Oxford, 2002.

[14] BRAIDES A., Local minimization, variational evolution and Γ-convergence, *Lecture Notes in Math., Springer, Berlin,* **2094** 2013.

[15] BRAIDES A., CASSANO B., GARRONI A., SARROCCO D., Quasi-static damage evolution and homogenization: a case study of non-commutability *To appear in Ann. Insti. Henri Poincar*

[16] BRAIDES A., LARSEN C.J., Γ-convergence for stable states and local minimizers, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, **10** (2011) 193-206.

[17] CASADO-DIAZ, COUCE-CALVO J., MAESTRE F., MARTIN-GOMEZ J., Homogenization and correctos for the wave equation with discontinuous coefficients in time, *Jour. Matha. Anal. and Appl.* **379** (2011), 664-681.

[18] CHISTYAKOV V.V:, On Mappings of bounded variation with values in a metric spaces, *Uspekhi Mat. Nauk* **54** (1999) 189-190.

[19] COLOMBINI F.,SPAGNOLO F., Some examples of hyperbolic equations without local solvability, *Ann. Sci. Ec. Norm. Sup.* **22** (1989) 109-125.

[20] COLOMBINI F.,SPAGNOLO F., On the convergence of solutions of hyperbolic equations, *Comm. Partial Differential Equations* **3** (1978) 73-103.

[21] COLOMBINI F.,SPAGNOLO F., Hyperbolic equations with coefficients rapidly oscillating in time: a result of nonstability, *J. differential Equations* **52** (1984) 24-38.

[22] CONTI S., THEIL F., Single-slip elasto-plastic microstructures, *Arch. Rat. Mech. Anal.* **178** (2005) 125-148.

[23] DACOROGNA B., Direct Methods in the calculus of Variations. *Applied Mathematical Sciences, Springer-Verlag* **78** 1989.

[24] Dal Maso G., An introduction to Γ-Convergence, *Progress in Nonlinear Differential Equations and Their Applications, Springer* **8** 1993.
[25] Dal Maso G., De Simone A., Mora G., Quasi-static evolution problems for linearly elastic-perfectly plastic materials., *Arch. Rat. Mech. Anal.* **180** (2006) 237-291.
[26] Dal Maso G., Francfort G., Toader R., Quasi-static crack growth in nonlinear elasticity, *Arch. Rat. Mech. Anal.* **176** (2005) 165-225.
[27] Dal Maso G., Larsen C.J., Existence for wave equations on domains with arbitrary growing cracks. *Atti Accad. Naz. Lincei. Mat. Appl.* **22** (2011) 387-408.
[28] Dal Maso G., Toader R., A model for the quasi-static growth of brittle fractures: existence and approximation results, *Arch. Rat. Mech. Anal.* **162** (2002) 101-135.
[29] Dal Maso G., Toader R., A model for the quasi-static growth in elastoplastic materials: the two dimensional case, *Arch. Rat. Mech. Anal.* **962** (2010) 897-906.
[30] Dal Maso G., Scala R., Quasi-static Evolution in perfect plasticity as limit of dynamic process, *To Appear.*
[31] De Giorgi E., Franzoni T., Su un tipo di convergenza variazionale, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Mat.* **8** 58 (1975) 842-850.
[32] De Giorgi E., Spagnolo S., Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine, *Boll. Un. Mat. Ital.* **4** (1973) 391-411.
[33] Francfort G., Garroni A., A Variational View of Brittle Damage Evolution, *Arch. Rat. Mech. Anal.* **182** (1) (2006) 125-152.
[34] Francfort G., Larsen C.J., Existence and convergence for quasi-static evolution in brittle fracture. *Comm. Pure Appl. Math.* **56** 10 (2003) 1465-1500.
[35] Francfort G., Marigo J., Stable damage evolution in a brittle continuous media, *Eur. J. Mech. A/Solids* **12** 2 (1993) 149-189.
[36] Francfort G., Marigo J., Revisiting brittle fracture as an energy minimization problem, *J. Mech. Phys. Solids* **46** (1998) 1319-1342.
[37] Francfort G., Mielke A., Existence results for a class of rate independent materials with non-convex elastic energies, *Journal fr die reine und angewandte Mathematik* **595** (2006) 55-91.
[38] Francfort G., Murat F., Homogenization and optimal bounds in linear elasticity, *Archive for Rational Mechanics and Analysis* **94** (1986) 307-334.
[39] Fremond M., Non-smooth thermomechanics, *Springer-Verlag, Berlin* 2002.
[40] Fremond M., Kuttler K., Shillor M., Existence and Uniqueness of Solutions for a Dynamic One-Dimensional Damage Model *Jou. Math. An. Appl.* **229** 1 (1999), 271294.
[41] Fremond M., Nedjar B., Damage, gradient of damage and principle of virtual power, *Internat. J. Solids Structures* **33** (1996) 1083-1103.
[42] Fonseca I., Muller S., Pedregal P., Analysis of concentration and oscillation effects generated by gradients, *Siam J. Math. Anal.* **29** 3 (1998) 736-756.
[43] Garroni A., Larsen C.J., Threshold-based quasi-static Brittle Damage Evolution, *Arch. Rat. Mech. Anal.* **194** (2009) 585–609.
[44] Germain P., Cours de Mcanique des Milieux Continus. *Masson, Paris* 1973.
[45] Heinemann C., Rocca E., Damage processes in thermoviscoelastic materials with damage-dependent thermal expansion coefficients , *To appear.*
[46] Hurd A.E., Sattinger D.H., Questions of existence and uniqueness for hyperbolic equations with disconinuous coefficients, *Trans. Amer. Math. Soc.* **132** (1968) 159-174.
[47] Ireman P., Nguyen Q., Using the gradient of temperature and internal parameters in continuum thermodynamics. *C.R. Mecanique* **332** (2004) 249-255.
[48] Kachanov L.M., Time of the rupture process under creep conditions. *Izv. Akad. Nauk. S.S.S.R., Otd. Tech. Nauk.* **8** (1958) 26-31.
[49] Kruzik M., Mielke A., Roubicek T., Modelling of microstructure and its evolution in shape-memory-alloy single-crystal, in particular in CuAlNi, *Meccanica* **40** (2005) 389-418.
[50] Larsen C.J., Regularity of components in optimal design problems with perimiter penalization. *Calc. of Var. adn Part. Diff. Eq.* **16** (2003) 17-29.
[51] Larsen C.J., Ortner C., Suli E,, Existence of solutions to a regularized model of dynamic fracture. *Math. Mod. Meth. Appl. Sci.* **20**, 07, (2010) 1021-1048.
[52] Lazzaroni G., Rossi R., Thomas M., Toader R., Rate-independent damage in thermo-viscoelastic materials with inertia, *To Appear*
[53] Lorentz E., Andireux S., A variational formulation for nonlocal damage models, *Int. J. Plasticity* **15** (1999) 119-138.

[54] Lemaitre J., Thorie mcanique de l'endommagement isotrope applique la fatigue des mtaux, *Structures sous chargements cycliques, ENPC, Paris* 1978.

[55] Lemaitre J., Chaboche J.L., Aspect phnomnologique de la rupture par endommagement, *J. Mc. Appl* **2** (1978) 317-365.

[56] LIN F.H., Variational problems with free interfaces, *Calc. Var.* **1** (1993) 149-168.

[57] Mainik A., Mielke A., Existence results for energetic models for rate-independent systems. *Calc. Var. Partial Differential Equations*, 2005.

[58] Meyers N.G., An Lp estimate for the gradient of solutions of second order elliptic divergence equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **3** (1963) 189206.

[59] Mielke A., Existence of minimzers in incremental elasto-plasticity with finite strain, *SIAM J. Math Analysis* **36** (2004) 384-404.

[60] Mielke A., Energetic formulation of multiplicative elasto-plasticity with finite strain. *NSIAM J. Math Analysis* 2003.

[61] Mielke A., Differential, energetic, and metric formulations for rate-independent processes, *Nonlinear PDE's and applications, Lecture Notes in Math. Springer, Berlin* (2011), 87-170.

[62] Mielke A., Rindler F., Reverse approximation of energetic solutions to rate-independent processes, *NoDEA Nonlinear Differential Equations Appl.* **16** (1) (2009) 7-40.

[63] Mielke A., Rossi R., Savaré G., BV solutions and viscosity approximations of rate-independent systems, *ESAIM Control Optim. Calc. Var.* **18** (2012) 36-80.

[64] Mielke A., Roubichek T., Stefanelli U., Γ-limits and relaxations for rate-independent evolutionary problems, *Calc. Var. Part. Diff. Equ.* **31** (2008) 387-416.

[65] Mielke A., Roubichek T., Rate-independent damage processes in nonlinear elasticity, M^3AS *Math. Models Methods Appl. Sci.* **16** (2006) 177-209.

[66] Mielke A, Theil F., A mathematical model for rate-independent phase tranformations with hysteresis. *In: H.D. Alber, R. Balean and R. Farwig, (eds.), Proceedings of the Workshop on "Model of Continuum Mechanics in Analysis and Engineering". Shaker Verlag, Aachen* (1999) 117-129.

[67] Mielke A, Theil F., On rate-independent hysteresis models. *NoDEA Nonlinear Differential Equations Appl.* **11** (2) (2004) 151-189.

[68] Mielke A, Theil F., Levitas V., A variational formulation of rate-independent phase trasformations using an extreme principle. *Arch. Rational Mech. Anal.* **162** (2002) 137-177.

[69] Murat F., Tartar L.. H-convergence, *Topics in the mathematical modelling of composite materials, 21-43, Birkhäuser, A. Cherkaev* and *R.V. Kohn, editors, Progress in Nonlinear Differential Equations and Their Applications* **31** (1997).

[70] Murat F., Tartar L., Calcul de variations et homognisation, *Les mthodes de l'homognisation: thorie et applications en physique, Collection Etudes et Rechercher EDF* Yrolles (1997) 319-369.

[71] Ortiz M., Repetto E., Nonconvex energy minimization and dislocation structures in ductile single crystals, *J.Mech. Phys. Solids* **47** (2) (1999) 397-462.

[72] Papanicolau G., Bensoussan B., Lions L., Asymptotic Analysis for Periodic Structures, Elsevier, 1978.

[73] Raitums U., On the local representation of G-closure, *Arch. Ration. Mech. Anal.* **158** 3 (2001) 213-234.

[74] Rindler F., Approximation of rate-independent optimal control problems. *SIAM J. Numer. Anal.* **47** 5 (2009) 3884-3909.

[75] Rocca E., Rossi R., A degenerating PDE system for phase transitions and damage, *Math. Mod. Meth, Appl. Sci.,* **24** 7 (2014) 1-9.

[76] Rocca E., Rossi R., "Entropic" solutions to a thermodynamically consistent PDE system for phase transitions and damage , *To appear.*

[77] Roubicek T., Rate-independent processes in viscous solids at small strains, *Math. Meth. Appl. Sci.* **32** 7 (2009) 825862.

[78] Roubicek T., Thermodynamics of rate independent processes in viscous solids at small strains, *SIAM J. Math. Anal.* **40** (2010) 256-297.

[79] Spagnolo S., Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche. *Ann. Scuola Norm. Sup. Pisa CI. Sci.* **22** (III) (1968) 571-597.

[80] Thomas M., Quasi-static damage evolution with spatial BV-regularization, *Disc. Cont. Dyn. Sys., Series S* **6** (1) (2013) 235-255.