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Statics and dynamics of dislocations: A variational approach

Dottorato in Matematica, Roma «La Sapienza» (2014).

<http://www.bdim.eu/item?id=tesi_2014_DeLucaLucia_1>

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Statics and dynamics of dislocations: A variational approach

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Anno Accademico 2013–2014

Introduction

In Materials Science, dislocations are line imperfections in the crystalline structure of materials and their presence (together with their motion) plays a fundamental role in a large variety of phenomena, such as plastic deformations in metals, phase transitions, crystal growth, crack propagation, ductile-brittle behaviour. The theory of dislocations is relatively young: Although Volterra's "distorsioni" were introduced for the first time in the early 1900th ([69]), a systematic theory of dislocations has been developed only thirty years later by Orowan ([54]), Polanyi ([55]) and Taylor ([66, 67]) in order to explain some experimental results relative to the plastic properties of crystals. After that, several phenomenological models for plasticity that account for the presence of dislocations at different scales have been proposed. In the last decades, both the mathematical and the mechanical engineering communities have shown an increasing interest and effort in the derivation and improvement of these models starting from fundamental microscopic models, describing single dislocation lines and their collective behaviour (see for instance [26, 36, 37, 38, 9, 39, 50]). In this direction, new mathematical methods have been developed in order to study the formation of dislocations and their dynamics. The purpose of this thesis is to contribute to the mathematical research in this field in the context of the variational models.

Loosely speaking, a dislocation can be represented by a measure concentrate on the dislocation line, to which it is associated a vector, called *Burgers vector*. In all the thesis, we focus on the case of straight dislocations. The Burgers vector, allows to classify these dislocations in two main types: *edge* if the Burgers vector is orthogonal to the dislocation line, and *screw* if it is parallel. Roughly speaking, the former are obtained adding an "extra half plane" of atoms into the crystal whereas the latter are produced by skewing a crystal so that the atomic plane produces a spiral ramp around the dislocation. In this thesis, we study the asymptotic behaviour of the elastic energy, stored in a crystal, induced by a configuration of screw (respectively edge) dislocations as the atomic scale goes to zero; moreover, we propose a purely variational approach to the study of the dynamics of screw dislocations. Our analysis is based on Γ-convergence (see [18, 29]).

We consider an elastic body with cylindrical simmetry, so that the mathematical formulation involves only problems set on the cross section Ω of the crystal. Assuming that dislocations are straight lines orthogonal to Ω, each of them is completely identified by its intersection x_i with Ω and by a vector $\xi_i \in \mathbb{R}^3$ (representing the Burgers vector). If we are in presence of a system of screw dislocations, then ξ_i are vertical vectors whose (suitably rescaled) modulus is an integer $d_i = |\xi_i| \in \mathbb{Z}$, representing the multiplicity of the dislocation. In the case of edge dislocations, ξ_i are horizontal vectors and hence with a little abuse of notation we can identify them with vectors in $\mathbb{R}^2$. More precisely, $\xi_i \in \mathbb{S} \subset \mathbb{R}^2$ where $\mathbb{S}$ is a discrete lattice representing the class of all the horizontal translations under which the crystal is invariant.

We pass now to the description of the energies we consider.

As for the screw dislocations, we study a *purely discrete* model. We consider the illustrative case of a square lattice of size ε with nearest neighbors interactions, following along the lines of the more general theory introduced in [11]. In this framework, a vertical *displacement* is a scalar function u defined on the nodes of the lattice in Ω, i.e., $u : \Omega \cap \varepsilon\mathbb{Z}^2 \to \mathbb{R}$ and the (isotropic) elastic energy associated with u, in absence of dislocations, is given by

$$E_\varepsilon^{el}(u) := \frac{1}{2} \sum_{i,j\in\Omega\cap\varepsilon\mathbb{Z}^2, |i-j|=\varepsilon} |\,\mathrm{d}u(i,j)|^2,$$

where $\mathrm{d}u$ is the *discrete* gradient of u (namely $\mathrm{d}u(i,j) = u(i) - u(j)$ with $|i-j| = \varepsilon$). To introduce the dislocations in this framework, we adopt the formalism of the discrete pre-existing strains as in [11] and [3]. More precisely, a pre-existing strain is a function β^p representing the plastic part of the strain defined on pairs of nearest neighbors and valued in $\mathbb{Z}$. The idea is that the plastic strain does not store elastic energy and hence it has to be subtracted to the discrete gradient in order to obtain the elastic strain β^e. In view of this additive decomposition $\mathrm{d}u = \beta^p + \beta^e$, we have that the elastic strain is not curl-free (in a discrete suitable sense) and that the discrete dislocation measure $\mu(u) := \mathrm{curl}\,\beta^e = -\mathrm{curl}\,\beta^p$ is nothing but the incompatibilty maesure of β^e, namely $\mu(u)$ maesures how far is β^e from being a gradient. Summarizing, the elastic energy of u is obtained by minimizing the functional

$$E_\varepsilon^{el}(u) := \frac{1}{2} \sum_{i,j\in\Omega\cap\varepsilon\mathbb{Z}^2, |i-j|=\varepsilon} |\beta^e(i,j)|^2 = \frac{1}{2} \sum_{i,j\in\Omega\cap\varepsilon\mathbb{Z}^2, |i-j|=\varepsilon} |\,\mathrm{d}u(i,j) - \beta^p(i,j)|^2$$

with respect to the plastic strain β^p. Since by our kinematic assumption the plastic strain takes values in $\mathbb{Z}$, it is clear that the optimal β^p_u is obtained projecting $\mathrm{d}u$ on $\mathbb{Z}$. Therefore, the elastic energy of u is given by

$$SD_\varepsilon(u) := \frac{1}{2} \sum_{i,j\in\Omega\cap\varepsilon\mathbb{Z}^2, |i-j|=\varepsilon} \mathrm{dist}^2(u(i) - u(j), \mathbb{Z}). \tag{0.1}$$

Since in the case of edge dislocations, a purely discrete model is not well-established, we consider a *semi-discrete* (mesoscopic) model in which the dislocations are modeled individually, while the underlying atomic lattice is averaged out. It is clear that in this case only the horizontal components of the strain are relevant and hence the natural setting is given by the *plane* elasticity. In plane linear elasticity, a planar displacement is a regular vector field $u : \Omega \to \mathbb{R}^2$. The equilibrium equations have the form $\mathrm{Div}\,\mathbb{C}[e(u)] = 0$, where $e(u) := \frac{1}{2}(\nabla u + (\nabla u)^{\mathrm{T}})$ is the infinitesimal strain tensor, and $\mathbb{C}$ is a linear operator from $\mathbb{M}^{2\times 2}$ into itself usually referred to as the *elasticity tensor*, incorporating the material properties of the crystal. It satisfies

$$c_1|\xi^{\mathrm{sym}}|^2 \le \frac{1}{2}\mathbb{C}\,\xi : \xi \le c_2|\xi^{\mathrm{sym}}|^2 \qquad \text{for any } \xi \in \mathbb{M}^{2\times 2}, \tag{0.2}$$

where c_1 and c_2 are two given positive constants and $\xi^{\rm sym} := \frac{1}{2}(\xi + \xi^{\rm T})$. The corresponding elastic energy, in absence of dislocations, is given by

$$\int_\Omega W(\beta)\,\mathrm{d}x, \tag{0.3}$$

where $\beta = \nabla u$ is the displacement gradient field and $W(\xi) = \frac{1}{2}\mathbb{C}\,\xi : \xi = \frac{1}{2}\mathbb{C}\,\xi^{\rm sym} : \xi^{\rm sym}$ is the elastic energy density. If a configuration of edge dislocations $\mu = \sum_{i=1}^N \xi_i \delta_{x_i}$ is present, we introduce the class of admissible fields β associated with μ as the matrix valued fields whose circulation around the dislocation x_i is equal to ξ_i (once again "$\mu = \mathrm{Curl}\beta$"). These fields by definition have a singularity at each x_i and are not in $L^2(\Omega;\mathbb{M}^{2\times 2})$. To set up a variational formulation we then follow the so called *core radius approach*. More precisely, we introduce a scale parameter ε, proportional to the lattice spacing of the crystal, and we compute the energy outside the core region $\cup_i B_\varepsilon(x_i)$. As in the scalar case of screw dislocations, the elastic energy stored in the core region is negligible and the elastic distortion decays as the inverse of the distance from the dislocations, therefore it is commonly accepted in literature that the linearized elasticity provides a good approximation of the elastic energy stored outside the core region (see [62] for a justification of these arguments in terms of Γ-convergence).

We introduce the elastic energy induced by an arbitrary configuration of dislocations μ and an admissible field β

$$E_\varepsilon^{\rm el}(\mu,\beta) := \int_{\Omega_\varepsilon(\mu)} W(\beta)\,\mathrm{d}x, \qquad (\Omega_\varepsilon(\mu) := \Omega \setminus \bigcup_i B_\varepsilon(x_i)). \tag{0.4}$$

By minimizing the elastic energy (0.4) among all admissible fields, we obtain the elastic energy $\mathcal{E}_\varepsilon^{\rm el}(\mu)$ induced by μ.

Summarizing, from a mathematical point of view, dislocations are nothing but the topological singularities of the strain field and hence they can be studied in analogy with other better understood singularities. In particular, dislocations exhibits many similarities with vortices in superconductors, studied within the Ginzburg-Landau model. We recall that, for a given $\varepsilon > 0$, the Ginzburg-Landau energy $GL_\varepsilon : H^1(\Omega;\mathbb{R}^2) \to \mathbb{R}$ is defined by

$$GL_\varepsilon(w) = \frac{1}{2}\int_\Omega |\nabla w|^2\,\mathrm{d}x + \frac{1}{2\varepsilon^2}\int_\Omega (1-|w|^2)^2\,\mathrm{d}x. \tag{0.5}$$

Whereas the variational approach to dislocations is relatively young, the variational analysis as $\varepsilon \to 0$ of GL_ε has been the subject of a vast literature starting from the pioneering book [15]. The analysis in [15] shows that, as ε tends to zero, vortex-like singularities appear by energy minimization (induced for instance by the boundary conditions), and each singularity carries a quantum of energy of order $|\log \varepsilon|$. Removing this leading term, the so-called *self-energy*, from the energy, a finite quantity remains, called *renormalized energy*, depending on the positions of the singularities. This asymptotic analysis has been also developed through the solid formalism of Γ-convergence (see [45, 46, 59, 61, 6]). For the convenience of the reader, we briefly recall it in Chapter 1 of this thesis. It turns out that the relevant object to deal with is the distributional Jacobian Jw, which, in the continuous setting, plays the role of the distribution of dislocations; loosely speaking, as $\varepsilon \to 0$, the "double-well" potential in (0.5) forces the Jacobian Jw to concentrate on points, the vortices, which are topological singularities of the gradient of w as well as the dislocations are topological singularities of the admissible field β. A remarkable fact is that these Γ-convergence results also contain a compactness statement. Indeed, for sequences with bounded energy the vorticity measure is not in general bounded in mass; this is due to the fact that many dipoles are compatible with a logarithmic energy bound. Therefore, the compactness of the vorticity measures fails in the usual sense of weak star convergence. Nevertheless, compactness holds in the flat topology, i.e., in the dual of Lipschitz continuous functions with compact support. An essential tool in the proof of the Γ-convergence result is given by the so-called *ball construction*, introduced independently by Sandier [58] and Jerrard [45]: It consists in providing suitable pairwise disjoint annuli, where much of the energy is stored, and estimating from below the energy on each of such annuli, using the following easy lower bound

$$\frac{1}{2}\int_{B_R\setminus B_r} |\nabla w|^2 \, \mathrm{d}x \geq \pi |\deg(w, \partial B_R)| \log \frac{R}{r}, \quad w \in H^1(B_R \setminus B_r; \mathcal{S}^1). \tag{0.6}$$

Recently, part of this Γ-convergence analysis has been exported to two-dimensional discrete systems. In [56], it has been proved that the screw dislocations functionals $\frac{SD_\varepsilon}{|\log \varepsilon|}$ Γ-converge to $|\mu|(\Omega)$, where μ is the limiting vorticity measure and is given by a finite sum of Dirac masses. In [3], it has been shown that the energies $\frac{SD_\varepsilon}{|\log \varepsilon|^h}$ and $\frac{GL_\varepsilon}{|\log \varepsilon|^h}$ (with $h \geq 1$) are *variationally equivalent*, which means, roughly speaking, that they have the same Γ-limit (up to a factor) with respect to the same convergence. The Γ-limit $|\mu|(\Omega)$ is not affected by the position of the singularities and hence does not account for their interaction, which is an essential ingredient in order to study the dynamics.

The purpose of this thesis is two-fold: on one hand, we derive the renormalized energy for SD_ε (see Chapter 2) and introduce a purely variational approach to the dynamics of screw dislocations (see Chapter 4), on the other hand, we extend the Ginzburg-Landau analysis in the self-energy regime to the vectorial case of edge dislocations (see Chapter 5). Moreover in Chapter 3 we derive the renormalized energy also in the case of anisotropic and long range interaction screw dislocations energies.

Now we delineate the main features of the Γ-convergence analysis for the statics and the dynamics of screw dislocations contained in Chapters 2, 3 and 4 (see also [**4**, **32**]). We first notice that the functional SD_ε can be regarded as a specific example of scalar system governed by a 1-periodic potential f acting on nearest neighbors, whose energy is of the type

$$F_\varepsilon(u) := \sum_{i,j \in \Omega \cap \varepsilon\mathbb{Z}^2,\, |i-j|=\varepsilon} f(u(i) - u(j)). \tag{0.7}$$

The functionals in (0.7) have the advantage to include not only the screw dislocations systems, but also another celebrated model which allows to describe the formation of topological singularities. This is the so-called XY model ([**14**, **48**, **49**]): Here the order parameter is a vectorial spin field

$v : \Omega \cap \varepsilon\mathbb{Z}^2 \to \mathcal{S}^1$ and the energy is given by

$$XY_\varepsilon(v) := \frac{1}{2} \sum_{i,j \in \Omega\cap\varepsilon\mathbb{Z}^2,\, |i-j|=\varepsilon} |v(i) - v(j)|^2. \tag{0.8}$$

It is easy to see that $XY_\varepsilon(v)$ can be written in terms of a representative of the phase of v, defined as a scalar field u such that $v = e^{2\pi i u}$ and that $XY_\varepsilon(u)$ has the form in (0.7).

As mentioned above, the first step is given by the derivation of the renormalized energy for the functionals F_ε by Γ-convergence, using the notion of Γ-convergence expansion introduced in [10] (see also [21]). Precisely, in Theorem 2.6 we prove that, given $M \in \mathbb{N}$, the functionals $F_\varepsilon(u) - M\pi|\log\varepsilon|$ Γ-converge to $\mathbb{W}(\mu) + M\gamma$, where μ is a sum of M singularities x_i with degrees $d_i = \pm 1$. Here $\mathbb{W}$ is the renormalized energy as in the Ginzburg-Landau setting, defined by

$$\mathbb{W}(\mu) := -\pi \sum_{i\neq j} d_i d_j \log|x_i - x_j| - \pi \sum_i d_i R_0(x_i),$$

where R_0 is a suitable harmonic function (see (1.5)) which rules the interaction among the singularities and the boundary of Ω, and γ can be viewed as a core energy, depending on the specific discrete interaction energy (see (2.39)).

An intermediate step to prove Theorem 2.6 is Theorem 2.2 (ii), which establishes a localized lower bound of the energy around the limiting vortices. This result is obtained using the ball construction, that we have to slightly revise in order to include our discrete energies. Indeed, in Proposition 3.4 we prove a lower bound for F_ε similar to (0.6), but with R/r replaced by $R/(r + C\varepsilon|\log\varepsilon|)$, the error being due to the discrete structure of our energies. Nevertheless, this weaker estimate, inserted in the ball construction machinery, is refined enough to prove the lower bound in Theorem 2.2 (ii). In Chapter 3 we use analogue tools in order to extend this Γ-convergence analysis to anisotropic energies with nearest neighbors interaction in the triangular lattice and to the case of isotropic long range interactions (see also [32]).

Chapter 4 is devoted to the analysis of metastable configurations for F_ε and to our variational approach to the dynamics of discrete topological singularities. The dynamics is driven by a *discrete parabolic flow* of the renormalized energy and it is based on the minimizing movements approach.

We now draw a parallel between the continuous Ginzburg-Landau model and our discrete systems, stressing out the peculiarities of our framework.

In [51], [44], [60], it has been proved that the parabolic flow of GL_ε can be described, as $\varepsilon \to 0$, by the gradient flow of the renormalized energy $\mathbb{W}(\mu)$. Precisely the limiting flow is a measure $\mu(t) = \sum_{i=1}^M d_{i,0}\delta_{x_i(t)}$, where $x(t) = (x_1(t), \dots, x_M(t))$ solves

$$\begin{cases} \dot{x}(t) = -\dfrac{1}{\pi}\nabla W(x(t)) \\ x(0) = x_0\,, \end{cases} \tag{0.9}$$

with $W(x(t)) = \mathbb{W}(\mu(t))$. The advantage of this description is that the effective dynamics is described by an ODE involving only the positions of the singularities. This result has been derived through a purely variational approach in [60], based on the idea that the gradient flow structure is consistent with Γ-convergence, under some assumptions which imply that the slope of the approximating functionals converges to the slope of their Γ-limit. The gradient flow approach to dynamics used in the Ginzburg-Landau context fails for our discrete systems. In fact, the free energy of discrete systems is often characterized by the presence of many energy barriers, which affect the dynamics and are responsible for pinning effects (for a variational description of pinning effects in discrete systems see [19], [20] and the references therein). As a consequence of our Γ-convergence analysis, we show that F_ε has many local minimizers. Precisely, in Theorem 4.5 and Theorem 4.6 we show that, under suitable assumptions on the potential f, given any configuration of singularities $x \in \Omega^{2M}$, there exists a stable configuration $\tilde{x}$ at a distance of order ε from x. Starting from these configurations, the gradient flow of F_ε is clearly stuck. Moreover, these stable configurations are somehow attractive wells for the dynamics. These results are proved for a general class of energies, including SD_ε, while the case of the XY_ε energy, to our knowledge, is still open. A similar analysis of stable configurations in the triangular lattice has been recently carried on in [42], combining PDEs techniques with variational arguments, while our approach is purely variational and based on Γ-convergence.

On one hand, our analysis is consistent with the well-known pinning effects due to energy barriers in discrete systems; on the other hand, it is also well understood that dislocations are able to overcome the energetic barriers to minimize their interaction energy (see [22, 35, 43, 57]). The mechanism governing these phenomena is still matter of intense research. Certainly, thermal effects and statistical fluctuations play a fundamental role. Such analysis is beyond the purposes of this thesis. Instead, we raise the question whether there is a simple variational mechanism allowing singularities to overcome the barriers, and then which would be the effective dynamics. We face these questions, following the minimizing movements approach à la De Giorgi ([7, 8, 14]). More precisely, we discretize time by introducing a time scale $\tau > 0$, and at each time step we minimize a total energy, which is given by the sum of the free energy plus a dissipation. For any fixed τ, we refer to this process as discrete gradient flow. This terminology is due to the fact that, as τ tends to zero, the discrete gradient flow is nothing but the Euler implicit approximation of the continuous gradient flow of F_ε. Therefore, as $\tau \to 0$ it inherits the degeneracy of F_ε, and pinning effects are dominant. The scenario changes completely if instead we keep τ fixed, and send $\varepsilon \to 0$. In this case, it turns out that, during the step by step energy minimization, the singularities are able to overcome the energy barriers, that are of order ε. Finally, sending $\tau \to 0$ the solutions of the discrete gradient flows converge to a solution of (0.9). In our opinion, this purely variational approach based on minimizing movements, mimics in a realistic way more complex mechanisms, providing an efficient and simple view point on the dynamics of discrete topological singularities in two dimensions.

Summarizing, in order to observe an effective dynamics of the vortices we are naturally led to let $\varepsilon \to 0$ for a fixed time step τ, obtaining a discrete gradient flow of the renormalized energy. A technical issue is that the renormalized energy is not bounded from below, and therefore, in the step by step minimization we are led to consider local rather than global minimizers. Precisely, we minimize the energy in a δ neighborhood of the minimizer at the previous step. Without this care, already at the first step we would have the trivial solution $\mu = 0$, corresponding to the fact that dipoles annihilate and the remaining singularities reach the boundary of the domain. Nevertheless, for τ small the minimizers do not touch the constraint, so that they are in fact true local minimizers.

In order to discuss some mathematical aspects of the asymptotic analysis of discrete gradient flows as ε, $\tau \to 0$, we need to clarify the specific choice of the dissipations we deal with. The canonical dissipation corresponding to continuous parabolic flow is clearly the L^2 dissipation. On the other hand, once ε is sent to zero, we have a finite dimensional gradient flow of the renormalized energy, for which it is more natural to consider as dissipation the Euclidean distance between the singularities. This, for $\varepsilon > 0$, corresponds to the introduction of a 2-Wasserstein type dissipation, D_2, between the vorticity measures. For two Dirac deltas D_2 is nothing but the square of the Euclidean distance of the masses (for the definition of D_2 see (4.20)). We are then led to consider also the discrete gradient flow with this dissipation. By its very definition D_2 is continuous with respect to the flat norm and this makes the analysis as $\varepsilon \to 0$ rather simple and somehow instructive in order to face the more complex case of L^2 dissipation.

We first discuss in details the discrete gradient flows with flat dissipation. To this purpose, it is convenient to introduce the functional $\mathcal{F}_\varepsilon(\mu)$, defined as the minimum of $F_\varepsilon(u)$ among all u whose vorticity measure $\mu(u)$ is equal to μ. We fix an initial condition $\mu_0 := \sum_{i=1}^M d_{i,0}\delta_{x_{i,0}}$ with $|d_{i,0}| = 1$ and a sequence $\mu_{\varepsilon,0} \in X_\varepsilon$ satisfying

$$\mu_{\varepsilon,0} \overset{\text{flat}}{\to} \mu_0, \qquad \lim_{\varepsilon\to 0} \frac{\mathcal{F}_\varepsilon(\mu_{\varepsilon,0})}{|\log\varepsilon|} = \pi|\mu_0|(\Omega).$$

Then given $\delta > 0$ and let $\varepsilon, \tau > 0$ we define $\mu^\tau_{\varepsilon,k}$ by the following minimization problem

$$\mu^\tau_{\varepsilon,k} \in \operatorname{argmin}\left\{\mathcal{F}_\varepsilon(\mu) + \frac{\pi D_2(\mu,\mu^\tau_{\varepsilon,k-1})}{2\tau} : \mu \in X_\varepsilon, \right. \\ \left. \|\mu - \mu^\tau_{\varepsilon,k-1}\|_{\text{flat}} \le \delta\right\} \tag{0.10}$$

with $\mu^\tau_{\varepsilon,0} = \mu_{\varepsilon,0}$. As a direct consequence of our Γ-convergence analysis, in Theorem 4.14 we will show that, as $\varepsilon \to 0$, $\mu^\tau_{\varepsilon,k}$ converges, up to a subsequence, to a solution $\mu^\tau_k \in X$ to

$$\mu^\tau_k \in \operatorname{argmin}\left\{\mathbb{W}(\mu) + \frac{\pi D_2(\mu,\mu^\tau_{k-1})}{2\tau} \ : \ \mu = \sum_{i=1}^M d_{i,0}\delta_{x_i},\ \left\|\mu - \mu^\tau_{k-1}\right\|_{\text{flat}} \le \delta\right\}.$$

After identifying the vorticity measure with the positions of its singularities, we get that the vortices x^τ_k of μ^τ_k satisfy the following finite-dimensional problem

$$x_k^\tau \in \operatorname{argmin}\left\{ W(x) + \frac{\pi|x - x_{k-1}^\tau|^2}{2\tau} :\ x \in \Omega^M,\ \sum_{i=1}^M |x_i - x_{i,k-1}^\tau| \le \delta \right\}.$$

In Theorem 4.13 we show that this constrained scheme converges, as $\tau \to 0$, to the gradient flow of the renormalized energy (0.9), until a maximal time $\tilde{T}_\delta$. The proof of this fact follows the standard Euler implicit method, with some care to handle local rather than global minimization. Moreover, as $\delta \to 0$, $\tilde{T}_\delta$ converges to the critical time T^* (see Definition 4.10), at which either a vortex touches the boundary or two vortices collapse.

We now discuss the discrete gradient flow with the L^2 dissipation. Once again, we consider a step by step minimization problem as in (0.10), with $\|\mu - \mu_{\varepsilon,k-1}^\tau\|_{\text{flat}}^2$ replaced by $\|v - v_{\varepsilon,k-1}^\tau\|_{L^2}^2 / |\log\tau|$. More precisely,

$$u_{\varepsilon,k}^\tau \in \operatorname{argmin}\left\{ F_\varepsilon(u) + \frac{\|e^{2\pi i u} - e^{2\pi i u_{\varepsilon,k-1}}\|_{L^2}^2}{2\tau|\log\tau|} : \|\mu(u) - \mu(u_{\varepsilon,k-1}^\tau)\|_{\text{flat}} \le \delta \right\}.$$

The prefactor $1/|\log\tau|$ in front of the dissipation can be viewed as a time reparametrization, on which we will comment later.

The asymptotics of these discrete gradient flow as $\varepsilon \to 0$ relies again on a Γ-convergence analysis, which keeps memory also of the L^2 limit v of the variable $e^{2\pi i u_\varepsilon}$. Under suitable assumptions on the initial data, in Theorem 4.27 we show that, as $\varepsilon \to 0$, the solutions $u_{\varepsilon,k}^\tau$ converge, up to a subsequence, to a solution to

$$\begin{aligned} v_k^\tau \in \operatorname{argmin}\Bigg\{ \mathcal{W}(v) + \frac{\|v - v_{k-1}^\tau\|_{L^2}^2}{2\tau|\log\tau|} \ :\ v \in H^1_{\text{loc}}(\Omega \setminus \cup_{i=1}^M \{y_{i,k}\}; \mathcal{S}^1), \\ Jv = \sum_{i=1}^M d_{i,0}\delta_{y_{i,k}},\ \|Jv - Jv_{k-1}^\tau\|_{\text{flat}} \le \delta \Bigg\}, \end{aligned}$$

where Jv is the distributional Jacobian of v and $\mathcal{W}$ is the renormalized energy in terms of v (see Theorem 2.9); namely, $\min_{Jv=\mu} \mathcal{W}(v) = \mathbb{W}(\mu)$. Now, we wish to send τ to zero. This step is much more delicate than in the case of flat dissipation. Indeed, it is at this stage that we adopt the abstract method introduced in [60], and exploit it in the context of minimizing movements instead of gradient flows. This method relies on the proof of two energetic inequalities; the first relates the slope of the approximating functionals with the slope of the renormalized energy; the second one relates the scaled L^2 norm underlying the parabolic flow of GL_ε with the Euclidean norm of the time derivative of the limit singularities. In our discrete in time framework, we adapt the arguments in [60] by replacing derivatives by finite differences. A heuristic argument to justify the prefactor $1/|\log\tau|$ is that it is the correct scaling for the canonical vortex $x/|x|$. Indeed, given $V \in \mathbb{R}^2$ representing the vortex velocity, a direct computation shows that

$$\lim_{\tau\to 0} \frac{1}{\tau|\log\tau|} \left\| \frac{x}{|x|} - \frac{x - \tau V}{|x - \tau V|} \right\|_2^2 = \pi|V|^2. \tag{0.11}$$

As a matter of fact, in order to get a non trivial dynamics in the limit, we have to accelerate the time scale as τ tends to zero. This feature is well known also in the parabolic flow of Ginzburg-Landau functionals. In this respect, we observe that our time scaling is expressed directly as a function of the time step τ, while for the functionals GL_ε it depends on the only scale parameter of the problem, which is the length scale ε. The explicit computation in (0.11) has not an easy counterpart for general solutions v_k^τ, and (0.11) has to be replaced by more sophisticated estimates (see (4.55) and (4.94)). This point is indeed quite technical, and makes use of a lot of analysis developed in [**59**], [**60**].

Summarizing, we have noticed that the analogies between screw dislocations and vortices extend to the complete static analysis and, somehow, to their dynamics. As for the edge dislocations, the precise relation between the two frameworks appears less clear. The main difference is that, in the last case, the framework is that of plane elasticity and hence the problem is vectorial and the energy is not coercive, depending only on the symmetric part of the deformation gradient.

We develop the Γ-convergence analysis for edge dislocations in Chapter 5 (see also [**31**]). As mentioned above, the elastic energy induced by a finite distribution of edge dislocations μ is given by

$$\mathcal{E}_\varepsilon^{\mathrm{el}}(\mu) = \min_{\mathrm{Curl}\beta=\mu} \int_{\Omega_\varepsilon(\mu)} W(\beta)\,\mathrm{d}x.$$

This variational formulation has been considered in [**24**] by Cermelli and Leoni who study the limit of the elastic energy induced by a fixed configuration of edge dislocations (and of its minimizers) as the atomic scale ε tends to zero.

In order to perform a meaningful analysis in terms of Γ-convergence i.e., allowing also the distribution of dislocations to be optimized, we consider the functional

$$\mathcal{E}_\varepsilon(\mu) = \mathcal{E}_\varepsilon^{\mathrm{el}}(\mu) + |\mu|(\Omega), \tag{0.12}$$

where the total variation of μ in Ω, $|\mu|(\Omega)$, represents the energy stored in the region surrounding the dislocations. We remark that this term is essential; indeed, without the core energy any configuration μ such that $\Omega_\varepsilon(\mu) = \emptyset$ would induce no energy. On the other hand, its specific choice does not affect the Γ-limit (see [**56**]). In this respect, it can be seen as the discrete counterpart of the double-well potential in GL_ε.

The Γ-convergence analysis for the functionals $\mathcal{E}_\varepsilon$ has been studied by Garroni, Leoni and Ponsiglione in [**36**] under the assumption that the dislocations are well separated. They perform a complete analysis, which includes also different energetic regimes. It is well-known that, also in this vectorial case, as in the scalar case of screw dislocations, a finite number of edge dislocations has an elastic energy of order $|\log\varepsilon|$. Note that in the regimes $|\log\varepsilon|^h$, $h > 1$, the number of defects N_ε increases, tending to infinity as $\varepsilon \to 0$, and the interaction between singularities becomes relevant; in particular they show in terms of Γ-convergence that in the critical $|\log\varepsilon|^2$ energy regime (that corresponds to $N_\varepsilon \approx |\log\varepsilon|$), the two effects of interaction energy and self-energy are balanced. The limit energy is of the form

$$\int_\Omega W(\beta)\,\mathrm{d}x + \int_\Omega \varphi\left(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}\right)\mathrm{d}|\mu|, \tag{0.13}$$

where φ is a positively 1-homogeneous density function defined by a suitable cell problem formula, determined only by the elasticity tensor $\mathbb{C}$ and the geometric structure of the crystal. This structure of the limit energy is set in the framework of so called *strain gradient theories* for plasticity (see [34, 40, 27]). Very recently, the analysis in [36] has been exported to the case of *nonlinear* semi-discrete dislocation energy (see [53]).

We perform the Γ-convergence analysis for the energy induced by a finite system of edge dislocations, without assuming the dislocations to be fixed, uniformly bounded in mass or well-separated. More precisely, in Theorem 5.4 we prove that the Γ-limit of the functionals $\frac{\mathcal{E}_\varepsilon}{|\log\varepsilon|}$ is given by

$$\mathcal{F}(\mu) := \int_\Omega \varphi\left(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}\right) \mathrm{d}|\mu|,$$

where φ is obtained throught a cell problem formula as in (0.13). This Γ-convergence result is obtained with respect to the flat convergence of the dislocations measures and exploits once again the strong analogy with the Ginzburg-Landau setting. As mentioned above, the parallel between edge dislocations and vortices is more delicate. Indeed, in the asymptotics of edge dislocations, relaxation effects, that are encoded in the definition of φ, take place. Moreover, the fact that the energy is not coercive, depending only on the symmetric part of the strain, introduces specific difficulties in the analysis, which make the proofs of compactness and Γ-liminf a challenging task.

The idea of the proof of Theorem 5.4 relies on the ball construction technique but the lower bound of the energy on annuli in this case is given by

$$\int_{B_R\setminus B_r} W(\beta)\,\mathrm{d}x \geq \frac{c_1|\xi|^2}{2\pi K(R/r)} \log\frac{R}{r}, \quad \beta \in H^1(B_R\setminus B_r;\mathbb{R}^2),\ \mathrm{Curl}\beta = \xi \text{ in } B_R;$$

here $K(\rho)$ is the Korn's constant and is such that $K(\rho) \to +\infty$ as $\rho \to 1$ (see Section 5.6 for more details). As a consequence, we have to perform the ball construction avoiding too thin annuli (where the Korn's constant blows up). This will be done in Section 5.2 where we construct an *ad hoc* discrete version of the ball construction. Once this ball construction is done, we deduce a lower bound with a pre-factor error due to the use of Korn's inequality. Then, compactness is easily deduced in Section 5.3 using arguments similar to [59]. In view of this analysis, we can easily find the required annuli where the energy concentrates, providing the optimal lower bound (see Section 5.4). Finally, in Section 5.5 we provide the upper bound, concluding the proof of our Γ-convergence result.

In conclusion, this thesis is organized as follows:

> In Chapter 1 we introduce the notions of Γ-convergence and Γ-expansion following the approach in [18] and [10]. Here we collect some results in [58, 59, 60, 61] relative to the statics and dynamics of vortices in the Ginzburg-Landau framework.
> In Chapter 2 we give the Γ-expansion of the functionals F_ε. All the results proved in this chapter are contained in [4].

In Chapter 3 we perform the Γ-convergence analysis of the anisotropic and long range interaction energies. These results are contained in [**32**].
In Chapter 4 we use the Γ-convergence analysis of Chapter 2 to prove the existence of many metastable configurations for the systems we consider. Then we introduce our purely variational approach to the study of their dynamics. All the results of this chapter are contained in [**4**].
In Chapter 5 we prove our Γ-convergence result for $\frac{\mathcal{E}_\varepsilon}{|\log\varepsilon|}$. The results in this chapter are proved in [**31**].

Contents

Acknowledgements

My deep gratitude goes at first to my advisor Adriana Garroni for the proposed problems, her continuous advice and for many interesting discussions about this thesis (and many other matters); it would not have been possible to accomplish this dissertation without her wise guidance and her patience. I would like to express my sincere gratitude to Roberto Alicandro and Marcello Ponsiglione, for the precious help and support.

I wish to thank the people I met in Rome, Adriano, Marta, Federico and all other PhD colleagues and students.

A special mention goes to Matteo for his intelligence and his friendness. Tell Angela I love her very much, she knows.

Finally I wish to thank Pino, Carmela and Giorgia for their constant support and encouragement.

CHAPTER 1

Ginzburg-Landau functionals

In this chapter we recall some results of [58, 59, 60, 61] relative to the Γ-convergence analysis of the Ginzburg-Landau functionals, which will be useful in the following of this thesis.

Ginzburg-Landau functionals arise in condensed-matter physics; they have been originally introduced as a phenomenological phase-field type free-energy of a superconductor, near the superconducting transition, in absence of an external magnetic field. They involve a complex-valued order parameter, which we denote by w, that describes the local state of the material, $|w| \leq 1$ being a local density. The crucial set is the zero-set of w: since w is complex-valued, it can have a non-zero degree around its zeroes, which are then called *vortices*, i.e. *topological defects* in dimension 2. We recall that $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary. The Ginzburg-Landau energy (without magnetic field) is given by

$$GL_\varepsilon(w) := \frac{1}{2}\int_\Omega |\nabla w|^2 + \frac{1}{2\varepsilon^2}(1-|w|^2)^2 \,\mathrm{d}x,$$

and it is defined over $H^1(\Omega;\mathbb{C})$; here ε is a length-scale parameter, usually referred to as the *coherence length* while GL_ε is the corresponding free energy of the system.

In the first section we introduce the basic notion of Γ-convergence, then we present the results relative to the static Γ-convergence analysis of GL_ε and in the last sections we state some results about the limiting dynamics of the vortices which are proved in [60] and [59].

1.1. Γ-convergence

Here we introduce the fundamental notion of Γ-convergence following the notation in [18] (see also [29]). Γ-convergence is designed to express the convergence of minimum problems: Given a family of functionals F_ε defined on a metric space (X, d), it may be convenient to study the asymptotic behaviour of a family of problems

$$m_\varepsilon = \min\{F_\varepsilon(x) \,:\, x \in X\} \tag{1.1}$$

not through the direct study of the properties of the solutions x_ε but defining a suitable limit energy $F^{(0)}$ such that, as $\varepsilon \to 0$, the problem

$$m_0 = \min\{F^{(0)}(x) \,:\, x \in X_0\} \tag{1.2}$$

is a "good approximation" of (1.1), i.e., $m_\varepsilon \to m_0$ and $x_\varepsilon \to x_0$, where x_0 is itself a solution of m_0. This latter requirement must be thought upon extraction of a subsequence if the 'target' minimum problem admits more than a solution. Of course, in order for this procedure to make sense we must require a *equi-coerciveness* (or *compactness*) property for the energies F_ε.

Definition 1.1. Let (X, d) be a metric space. For any $\varepsilon > 0$, let $F_\varepsilon : X \to [-\infty, +\infty]$. We say that the sequence $\{F_\varepsilon\}$ is equi-coercive if for any sequence $\{x_\varepsilon\} \subset X$ with $\sup_\varepsilon F_\varepsilon(x_\varepsilon) \le C$ for some $C \in \mathbb{R}$, there exists $x \in X$ such that, up to a subsequence, $d(x_\varepsilon, x) \to 0$.

Definition 1.2. Let (X, d) be a metric space. For any $\varepsilon > 0$, let $F_\varepsilon, F^{(0)} : X \to [-\infty, +\infty]$. We say that the sequence $\{F_\varepsilon\}$ Γ-converges to F_0 at x if the following inequalities hold true

(i) (Γ-liminf inequality) for every $\{x_\varepsilon\}$ with $d(x_\varepsilon, x) \to 0$, it holds $F^{(0)}(x) \le \liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon)$.

(ii) (Γ-limsup inequality) there exists a (recovery) sequence $\{\bar{x}_\varepsilon\}$, such that $d(\bar{x}_\varepsilon, x) \to 0$ and $F^{(0)}(x) \ge \limsup_{\varepsilon\to 0} F_\varepsilon(\bar{x}_\varepsilon)$.

We say that $\{F_\varepsilon\}$ Γ-converges to $F^{(0)}$ in X ($F_\varepsilon \xrightarrow{\Gamma} F^{(0)}$) if $\{F_\varepsilon\}$ Γ-converges to $F^{(0)}$ at x for every $x \in X$.

The inequality (i) means that $F^{(0)}$ is a *lower bound* for the sequence $\{F_\varepsilon\}$ in the sense that $F^{(0)}(x) \le F_\varepsilon(x_\varepsilon) + \mathrm{o}(1)$ whenever $d(x_\varepsilon, x) \to 0$ and (ii) implies that $F^{(0)}$ is an *upper bound* for $\{F_\varepsilon\}$. We remark that the lower bound uniquely involves minimization and optimization procedures and is totally *ansatz*-free, whereas the computation of the upper bound is usually referred to an *ansatz*, suggested by the structure of the minimizing sequences, on the construction of the recovery sequence $\{\bar{x}_\varepsilon\}$.

We are now in a position to state the fundamental theorem of Γ-convergence, which ensures the convergence of the infima m_ε in (1.1) to the minimum m_0 in (1.2) and that every cluster point of a minimizing sequence is a minimum point for $F^{(0)}$.

Theorem 1.3. *Let (X, d) be a metric space. Let $\{F_\varepsilon\}$ be a equi-coercive sequence of functions on X and let $F^{(0)}$ be such that $F_\varepsilon \xrightarrow{\Gamma} F^{(0)}$. Then*

$$\exists \min F^{(0)} = \lim_{\varepsilon\to 0} \inf_X F_\varepsilon.$$

Moreover if $\{x_\varepsilon\}$ is a precompact sequence such that

$$\lim_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) = \lim_{\varepsilon\to 0} \inf_X F_\varepsilon,$$

then every limit of a subsequence of $\{x_\varepsilon\}$ is a minimum point for $F^{(0)}$.

Proof. Let $\{x_\varepsilon\}$ be a minimizing sequence for F_ε, that is

$$\liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) = \liminf_{\varepsilon\to 0} \inf_X F_\varepsilon.$$

Since $\{F_\varepsilon\}$ is equi-coercive, there exists $x_0 \in X$ such that, up to a subsequence, $d(x_\varepsilon, x) \to 0$. By (i) in Definition 1.2, we have immediately

$$\inf_X F^{(0)} \le F^{(0)}(x_0) \le \liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) = \liminf_{\varepsilon\to 0} \inf_X F_\varepsilon. \tag{1.3}$$

Moreover for any $y \in X$ let $\{\bar{x}_{\varepsilon,y}\}$ be a recovery sequence for x according with Definition 1.2(ii). Then, by (1.3), we get for any y

$$F^{(0)}(x_0) \le \liminf_{\varepsilon\to 0} \inf_X F_\varepsilon \le \limsup_{\varepsilon\to 0} F_\varepsilon(\bar{x}_{\varepsilon,y}) \le F^{(0)}(y).$$

It follows that $F^{(0)}(x_0) = \min_X F^{(0)}$ and that

$$F^{(0)}(x_0) = \liminf_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon).$$

$\square$

It is clear that the hidden element in the procedure of the computation of the Γ-limit is the choice of the right metric d. This is actually one of the main issues in the problem: a convergence is not given beforehand and should be chosen in such a way that it implies the equi-coercivenss of the sequence $\{F_\varepsilon\}$. In fact there are two terms in competition: on one hand, a very weak convergence, with many converging sequences makes the equi-coerciveness property easier to fulfill but at the same time it makes the Γ-liminf inequality more difficult to hold. A related issue is that of the *correct energy scaling*. In fact, in many cases, the functionals F_ε have to be suitably scaled in order to give rise to an equi-coercive sequence with respect to a meaningful convergence.

We give now the notion of asymptotic development of a sequence of functionals by Γ-convergence (or Γ-expansion) as it has been introuduced in [10]. The idea is that of introducing an *asymptotic expansion*

$$F_\varepsilon = F^{(0)} + \varepsilon F^{(1)} + \ldots + \varepsilon^k F^{(k)} + \mathrm{o}(\varepsilon^k)$$

of the sequence F_ε in such a way that the knowledge of the functionals $F^{(k)}$ gives an additional information on the limit points of minimizers. Precisely: Any limit point of a sequence of minimizers x_ε will also be a minimizer of each of the functionals $F^{(k)}$ appearing in the development above. We focus on the first-order expansion by Γ-convergence.

Definition 1.4. [10, Definition 1.3] With the notation above, we say that the first-order asymptotic development

$$F_\varepsilon = F^{(0)} + \varepsilon F^{(1)} + \mathrm{o}(\varepsilon) \tag{1.4}$$

holds, if we have

$$\lim_{\varepsilon \to 0} \frac{F_\varepsilon - m_0}{\varepsilon} = F^{(1)}.$$

For $k = 0, 1$, we denote by U_k the set of minimizers of $F^{(k)}$.

Theorem 1.5. [10, Theorem 1.2] *With the notation above, assume that the first-order Γ-expansion in (1.4) holds true. Let $\{x_\varepsilon\} \subset X$ be such that $d(x_\varepsilon, \bar{x}) \to 0$ for some $\bar{x} \in X$ and $F_\varepsilon(x_\varepsilon) = \min\{F_\varepsilon(x) : x \in X\}$. Then $\bar{x}$ is a minimizer of both $F^{(0)}$ and $F^{(1)}$ in U^0. Moreover , if m_1 denotes the infimum of $F^{(1)}$ on U^0, then*

$$m_\varepsilon = m_0 + \varepsilon m_1 + \mathrm{o}(\varepsilon).$$

1.2. Γ-expansion of GL_ε

We now introduce the basic functional spaces we use in this thesis. We recall that $W^{1,\infty}(\Omega)$ is the space of Lipschitz continuous functions in Ω and $W_0^{1,\infty}(\Omega)$ is the subspace of functions with compact support.

Given $w \in H^1(\Omega;\mathbb{C})$, we recall that the Jacobian Jw of w is the L^1 function defined by

$$Jw := \det \nabla w.$$

Moreover, we can consider Jw as an element of the dual of $W^{1,\infty}$ by setting

$$\langle Jw, \varphi\rangle := \int_\Omega Jw\, \varphi \,\mathrm{d}x \qquad \text{for every of } \varphi \in W^{1,\infty}(\Omega).$$

Notice that Jw can be written in a divergence form as
$Jw = \operatorname{div}(w_1\partial_{x_2}w_2, -w_1\partial_{x_1}w_2)$, i.e., for any $\varphi \in W_0^{1,\infty}(\Omega)$,

$$\langle Jw, \varphi\rangle = -\int_\Omega w_1\partial_{x_2}w_2\partial_{x_1}\varphi - w_1\partial_{x_1}w_2\partial_{x_2}\varphi \,\mathrm{d}x.$$

Equivalently, we have that $Jw = \operatorname{curl}(w_1\nabla w_2)$ and $Jw = \operatorname{curl} j(w)$, where

$$j(w) := w_1\nabla w_2 - w_2\nabla w_1,$$

is the so-called *current*.

Notice that if $w \in L^\infty(\Omega;\mathbb{C})$, then Jw is in the dual of $H^1(\Omega)$. Let $A \subset \Omega$ with Lipschitz boundary. Then we have

$$\int_A Jw \,\mathrm{d}x = \frac{1}{2}\int_A \operatorname{curl} j(w)\,\mathrm{d}x := \frac{1}{2}\int_{\partial A} j(w)\cdot t\,\mathrm{d}s,$$

where t is the tangent field to ∂A and the last integral is meant in the sense of $H^{-\frac{1}{2}}$.

Let $h \in H^{\frac{1}{2}}(\partial A;\mathbb{C})$ with $|h| \geq \alpha > 0$. The *degree* of h is defined as follows

$$\deg(h,\partial A) := \frac{1}{2\pi}\int_{\partial A} j(h/|h|)\cdot t\,\mathrm{d}s.$$

It is well-known that the definition above is well-posed and that $\deg(h,\partial A) \in \mathbb{Z}$; moreover, whenever $w \in H^1(A;\mathbb{C})$, $|w| \geq \beta > 0$ in A, $\deg(w,\partial A) = 0$, where in the notation of degree we identify w with its trace. Finally, $\deg(w,\partial A)$ is stable with respect to the strong convergence in $H^1(A;\mathbb{C})$. Notice that w can be written in polar coordinates as $w(x) = \rho(x)e^{i\theta(x)}$ on ∂A with $|\rho| \geq \alpha$, where θ is the so-called *lifting* of w. In particular, if $\rho(x) \equiv 1$ on ∂A, then the current $j(w)$ is nothing but the gradient of θ and the degree of w coincides with the circulation of $\nabla\theta$ on ∂A.

Moreover, by [17, Theorem 1] (see also [17, Remark 3]), if A is simply connected and $\deg(w,\partial A) = 0$, then the lifting can be selected in $H^{\frac{1}{2}}(\partial A)$ with the map $u \mapsto \theta$ continuous. If the degree d is not zero, then the lifting can be locally selected in $H^{\frac{1}{2}}(\partial A)$ with a "jump" of order $2\pi d$.

We state here the Γ-convergence theorem for $\frac{GL_\varepsilon}{|\log\varepsilon|}$ that collects result proved in [45, 46, 1].

Theorem 1.6. *The following Γ-convergence result holds.*

(i) *(Compactness) Let $\{w_\varepsilon\} \subset H^1(\Omega;\mathbb{C})$ be such that $GL_\varepsilon(w_\varepsilon) \leq C|\log\varepsilon|$ for some positive C. Then, up to a subsequence, $Jw_\varepsilon \overset{\mathrm{flat}}{\to} \pi\mu$, where $\mu := \sum_{i=1}^N d_i\delta_{x_i}$ for some $x_i \in \Omega$, $d_i \in \mathbb{Z}$.*

(ii) *(Γ-liminf inequality) Let $\{w_\varepsilon\} \subset H^1(\Omega;\mathbb{C})$ be such that $Jw_\varepsilon \overset{\text{flat}}{\to} \pi\mu := \pi\sum_{i=1}^N d_i\delta_{x_i}$. Then, there exists $C \in \mathbb{R}$ such that, for any $i = 1,\dots,N$ and for every $\sigma < \frac{1}{2}\mathrm{dist}(x_i, \partial\Omega \cup \bigcup_{j\neq i} x_j)$, we have*

$$\liminf_{\varepsilon\to 0} GL_\varepsilon(w_\varepsilon, B_\sigma(x_i)) - \pi|d_i|\log\frac{\sigma}{\varepsilon} \geq C.$$

In particular

$$\liminf_{\varepsilon\to 0} GL_\varepsilon(w_\varepsilon) - \pi|\mu|(\Omega)\log\frac{\sigma}{\varepsilon} \geq C.$$

(iii) *(Γ-limsup inequality) For every $\mu := \sum_{i=1}^N d_i\delta_{x_i}$, there exists $\{w_\varepsilon\} \subset H^1(\Omega;\mathbb{C})$ such that $Jw_\varepsilon \overset{\text{flat}}{\to} \pi\mu$ and*

$$\pi|\mu|(\Omega) \geq \limsup_{\varepsilon\to 0}\frac{GL_\varepsilon}{|\log\varepsilon|}.$$

Before stating the first order Γ-convergence of $GL_\varepsilon - \pi|\mu|(\Omega)|\log\varepsilon|$ to the renormalized energy introduced in [15], we recall the main definitions and results of [15] we need.

Let $\mu = \sum_{i=1}^M d_i\delta_{x_i}$ with $d_i \in \{-1,+1\}$ and $x_i \in \Omega$. In order to define the renormalized energy, consider the following problem

$$\begin{cases} \Delta\Phi = 2\pi\mu & \text{in } \Omega \\ \Phi = 0 & \text{on } \partial\Omega, \end{cases}$$

and let $R_0(x) = \Phi(x) - \sum_{i=1}^M d_i \log|x - x_i|$. Notice that R_0 is harmonic in Ω and $R_0(x) = -\sum_{i=1}^M d_i\log|x-x_i|$ for any $x \in \partial\Omega$. The *renormalized energy* corresponding to the configuration μ is then defined by

$$\mathbb{W}(\mu) := -\pi\sum_{i\neq j} d_i d_j \log|x_i - x_j| - \pi\sum_i d_i R_0(x_i). \tag{1.5}$$

Let $\sigma > 0$ be such that the balls $B_\sigma(x_i)$ are pairwise disjoint and contained in Ω and set $\Omega^\sigma := \Omega \setminus \bigcup_{i=1}^M B_\sigma(x_i)$. A straightforward computation shows that

$$\mathbb{W}(\mu) = \lim_{\sigma\to 0}\frac{1}{2}\int_{\Omega^\sigma}|\nabla\Phi|^2\,\mathrm{d}x - M\pi|\log\sigma|\,, \tag{1.6}$$

In this respect the renormalized energy represents the finite energy induced by μ once the leading logarithmic term has been removed.

Consider the following minimization problems

$$\begin{aligned} m(\sigma,\mu) &:= \min_{w\in H^1(\Omega^\sigma;\mathcal{S}^1)}\left\{\frac{1}{2}\int_{\Omega^\sigma}|\nabla w|^2\,\mathrm{d}x \;:\; \deg(w,\partial B_\sigma(x_i)) = d_i\right\}, \\ \tilde{m}(\sigma,\mu) &:= \min_{w\in H^1(\Omega^\sigma;\mathcal{S}^1)}\left\{\frac{1}{2}\int_{\Omega^\sigma}|\nabla w|^2\,\mathrm{d}x \;: \right. \qquad (1.7) \\ &\qquad\qquad \left. w(\cdot) = \frac{\alpha_i}{\sigma^{d_i}}(\cdot - x_i)^{d_i}\text{on } \partial B_\sigma(x_i),\ |\alpha_i| = 1\right\}, \end{aligned}$$

$$\gamma_{GL}(\varepsilon,\sigma) := \min_{w\in H^1(B_\sigma;\mathbb{C})}\left\{GL_\varepsilon(w,B_\sigma) \;:\; w(x)\llcorner\partial B_\sigma = \frac{x}{|x|}\right\}. \tag{1.8}$$

Theorem 1.7. [15] *It holds*

$$\lim_{\sigma\to 0} m(\sigma,\mu) - \pi|\mu|(\Omega)|\log\sigma| = \lim_{\sigma\to 0} \tilde{m}(\sigma,\mu) - \pi|\mu|(\Omega)|\log\sigma| = \mathbb{W}(\mu). \quad (1.9)$$

Moreover, for any fixed $\sigma > 0$, the following limit exists

$$\lim_{\varepsilon\to 0}(\gamma_{GL}(\varepsilon,\sigma) - \pi|\log\frac{\varepsilon}{\sigma}|) =: \gamma \in \mathbb{R}. \quad (1.10)$$

We are now in a position to state the Γ-expansion of GL_ε.

Theorem 1.8. *The following Γ-convergence result holds.*

(i) *(Compactness) Let $M \in \mathbb{N}$ and let $\{w_\varepsilon\} \subset H^1(\Omega;\mathbb{C})$ be a sequence satisfying $GL_\varepsilon(w_\varepsilon) - M\pi|\log\varepsilon| \leq C$. Then, up to a subsequence, $Jw_\varepsilon \overset{\text{flat}}{\to} \pi\mu$ for some $\mu = \sum_{i=1}^N d_i\delta_{x_i}$ with $d_i \in \mathbb{Z}\setminus\{0\}$, $x_i \in \Omega$ and $\sum_i |d_i| \leq M$. Moreover, if $\sum_i |d_i| = M$, then $\sum_i |d_i| = N = M$, namely $|d_i| = 1$ for any i.*

(ii) *(Γ-liminf inequality) Let $\{w_\varepsilon\} \subset H^1(\Omega;\mathbb{C})$ be such that $Jw_\varepsilon \overset{\text{flat}}{\to} \mu$, with $\mu = \sum_{i=1}^M d_i\delta_{x_i}$ with $|d_i| = 1$ and $x_i \in \Omega$ for every i. Then,*

$$\liminf_{\varepsilon\to 0} GL_\varepsilon(w_\varepsilon) - M\pi|\log\varepsilon| \geq \mathbb{W}(\mu) + M\gamma_{GL}. \quad (1.11)$$

(iii) *(Γ-limsup inequality) Given $\mu = \sum_{i=1}^M d_i\delta_{x_i}$ with $|d_i| = 1$ and $x_i \in \Omega$ for every i, there exists $\{w_\varepsilon\} \subset H^1(\Omega;\mathbb{C})$ with $Jw_\varepsilon \overset{\text{flat}}{\to} \mu$ such that*

$$GL_\varepsilon(w_\varepsilon) - M\pi|\log\varepsilon| \to \mathbb{W}(\mu) + M\gamma_{GL}.$$

The basic tool in order to prove Theorem 1.6 and 1.8 is given by the *ball construction* introduced by Sandier in [58]: It consists in an efficient way of selecting balls where the energy concentrates. In the next subsection we briefly delineate the main features of this powerful machinery that we will use with some modifications in order to prove the main results of this thesis.

1.2.1. Ball construction for GL_ε. Let $\mathcal{B} = \{B_{R_1}(x_1), \ldots, B_{R_N}(x_N)\}$ be a finite family of pairwise disjoint balls in $\mathbb{R}^2$. The ball construction consists in letting the balls alternatively expand and merge each other. The expansion phase consists in letting all the balls expand without changing their centers in such a way that at each (artificial) time the ratio $\theta(t) := r_i(t)/r_i$ is independent of i. The expansion phase stops at the first time T when two balls $B_{r_i(t)}(x_i)$ and $B_{r_j(t)}(x_j)$ bump into each other. Then the merging phase begins. It consists in collecting the balls $B_{R_i(T)}(x_i)$ in subclasses and merging all the balls of each subclass in a larger ball $B_{R_j}(y_j)$ with the following properties:

(a) R_j is not larger than the sum of all the radii of the balls $B_{R_i(T)}(x_i)$ contained in $B_{R_j}(y_j)$;
(b) the balls $B_{R_j}(y_j)$ are pairwise disjoint.

After the merging, we define in each ball $B_{R_j}(y_j)$ a *seed size* s_j by $R_j/s_j = \theta(T) = 1 + T$ (we set $s_j = r_j$ for $t = 0$). Then another expansion phase begins, during the seed sizes are left constant and $\theta(t) := R_j(t)/s_j = 1 + t$ for $t \geq T$. The procedure consists in alternating merging to expansion phase until a last phase where only a ball expands. Notice that, by property (a), the sum of the seed sizes $\sum_j s_j$ does not increase during the merging. In particular

$$\sum_j R_j(t) = \sum_j (1+t)s_j \le (1+t)\sum_i r_i.$$

Now we assume that to each ball $B_{r_i}(x_i)$ of the original family $\mathcal{B}$ corresponds some integer multiplicity $d_i \in \mathbb{Z}$ and set $\mu := \sum_{i=1}^N d_i \delta_{x_i}$. Let $F(\mathcal{B}, \mu, U)$ be defined as follows: If $A_{r,R}(y) := B_R(y) \setminus B_r(y)$ is an annulus which does not intersect any $B_{r_i}(x_i)$, we set

$$F(\mathcal{B}, \mu, A_{r,R}(y)) := \pi |\mu(B_r(y))| \log \frac{R}{r};$$

then, for any open set $U \subset \mathbb{R}^2$, we set

$$F(\mathcal{B}, \mu, U) := \sup \sum_i F(A_i),$$

where the sup is over all finite families of pairwise disjoint annuli $A_i \subset U$ which do not intersect any $B_{r_i}(x_i)$.

Remark 1.9. The definition of F is justified by the following fact. Let $w \in H^1(\Omega \setminus \cup_{B\in\mathcal{B}} B; \mathcal{S}^1)$ and let $\mu := \sum_{B\in\mathcal{C}} \deg(w, \partial B)\delta_{x_B}$, where $\mathcal{C}$ denotes the family of balls in $\mathcal{B}$ which are contained in Ω and x_B is the center of B. Then, by Jensen inequality it follows that

$$\begin{aligned} \frac{1}{2}\int_{B_R \setminus B_r} |\nabla w|^2 \, \mathrm{d}x &\ge \frac{1}{2}\int_r^R \int_{\partial B_\rho} |\,(w \times \nabla w) \cdot \tau|^2 ds\, d\rho \\ &\ge \int_r^R \frac{1}{\rho}\pi d_i^2 d\rho \ge \pi |d_i| \log \frac{R}{r}. \end{aligned} \tag{1.12}$$

for any annulus $A_{r,R}(y)$ such that $B_{r_i}(x_i) \subset B_r(x)$ and $B_R(x) \cap \cup_{j\neq i} B_{r_j}(x_j) = \emptyset$. As an easy consequence, one has

$$\frac{1}{2}\int_{U \setminus \cup_{i=1}^N B_{r_i}(x_i)} |\nabla w|^2 \, \mathrm{d}x \ge F(\mathcal{B}, \mu, U). \tag{1.13}$$

Let $\mathcal{B}(t)$ be the family of balls at time t (with the convention $\mathcal{B}(t) = \mathcal{B}(t^-)$ if t is a merging time). By the construction above it easily follows that for any $B \in \mathcal{B}(t)$

$$F(\mathcal{B}, \mu, B) \ge \pi |\mu(B)| \log(1+t). \tag{1.14}$$

We refer to [**58**, **59**, **61**, **6**, **45**] for the proofs of Theorems 1.6 and 1.8. We remark only that one of the most challenging tasks is the proof of the compactness property. The main idea is to use the potential term in (0.5) in order to show that the zeroes of $|w_\varepsilon|$ concentrate on a family $\mathcal{B}$ of balls such that the sum of their radii $Rad(\mathcal{B})$ satisfies $Rad(\mathcal{B}) \le C\varepsilon^{\frac{1}{3}} |\log \varepsilon|$. Plugging a Dirac mass, corresponding to the degree of w_ε, into each of these balls, and using the Ball Construction above, one can obtain a sequence of measures $\tilde{\mu}_\varepsilon$ that approximates Jw_ε, carrying all the topological information and bringing the required compactness.

1.3. Γ-convergence of gradient flows

In this section we present the abstract approach introduced in [60] to study the convergence of heat flows of families of energies which Γ-converge to a limiting energy. In Section 4.3 we will revisit this result in order to extend it to our discrete gradient flow.

The general framework is the following: Let $\mathcal{M}$ be an open subset of an affine space associated to a Banach space $\mathcal{B}$ and let $\mathcal{N}$ be an open subset of a finite-dimensional vector space $\mathcal{B}'$. We assume that $\mathcal{B}$ embeds continuously into a Hilbert space X_ε, $\mathcal{B}'$ into Y. Let $\{E_\varepsilon\}$ be a family of C^1 functionals defined over $\mathcal{M}$, which Γ-converges to a C^1 functional F defined over $\mathcal{N}$.

We first give the definitions we need.

Definition 1.10. [60, Definition 1.1] Let $T > 0$ and let $\{u_\varepsilon(t,\cdot)\} \subset \mathcal{M}$ for $t \in [0,T)$. We say that $\{E_\varepsilon\}$ Γ-converges along the trajectory $u_\varepsilon(t)$ in the sense S to F if there exists $u : [0,T) \to \mathcal{N}$, such that, up to a subsequence, $u_\varepsilon(t) \xrightarrow{S} u(t)$ for any $t \in [0,T)$ and

$$F(u(t)) \leq \liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon(t)) \qquad \forall t \in [0,T).$$

Definition 1.11. [60, Definition 1.2] If the differential $\mathrm{d}E_\varepsilon(u)$ of E_ε at u is also linear continuous on X_ε, we denote by $\nabla_{X_\varepsilon} E_\varepsilon(u)$ the gradient at $u \in \mathcal{M}$ for the structure X_ε, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} E_\varepsilon(u + t\phi) = \mathrm{d}E_\varepsilon(u)\cdot\phi = \langle \nabla_{X_\varepsilon} E_\varepsilon(u), \phi\rangle_{X_\varepsilon}.$$

If this gradient does not exist, we use the convention $\|\nabla_{X_\varepsilon} E_\varepsilon(u)\|_{X_\varepsilon} = +\infty$.

Definition 1.12. [60, Definition 1.3] A solution to the gradient flow of E_ε (with respect to the structure X_ε) on $[0,T)$ is a map $u_\varepsilon \in H^1((0,T);X_\varepsilon)$ such that

$$\partial_t u_\varepsilon(t) = -\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon) \qquad \forall t \in [0,T). \tag{1.15}$$

Such a solution is conservative if

$$E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) = \int_0^t \|\partial_t u_\varepsilon(t)\|^2_{X_\varepsilon}\,\mathrm{d}s \qquad \forall t \in [0,T).$$

If $\{u_\varepsilon\}$ is a family of solutions on $[0,T)$ to the gradient flow of E_ε along which $\{E_\varepsilon\}$ Γ-converges to F and $u_\varepsilon(t) \xrightarrow{S} u(t)$ for any $t \in [0,T)$, we define the energy excess $D(t)$ as

$$D(t) := \limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon(t)) - F(u(t)).$$

We notice that $D(t) \geq 0$. If $D(0) = 0$, we say that the family of solutions $\{u_\varepsilon\}$ is well-prepared initially.

We are now in a postion to state the abstract result in [60] which makes a rigorous connection between the convergence of gradient flows and the Γ-convergence structure.

Theorem 1.13. [60, Theorem 1.4] *Let $\{E_\varepsilon\}$ and F be C^1 functionals over $\mathcal{M}$ and $\mathcal{N}$ respectively and let $\{u_\varepsilon\}$ be a family of conservative solutions to the flow for E_ε (in the sense of (1.15)) on $[0,T)$, with $u_\varepsilon(0) \xrightarrow{S} u_0$,* *along which $\{E_\varepsilon\}$ Γ-converges to F in the sense of Definition 1.10. Assume moreover that conditions (i) and (ii) below are satisfied.*

(i) *For a subsequence $\{u_\varepsilon\}$ such that $u_\varepsilon(t) \xrightarrow{S} u(t)$ for any $t \in [0,T)$, it holds that $u \in H^1((0,T);Y)$ and there exists $f \in L^1((0,T))$ such that for every $s \in [0,T)$*

$$\liminf_{\varepsilon\to 0} \int_0^s \|\partial_t u_\varepsilon(t)\|_{X_\varepsilon}^2 \,\mathrm{d}t \geq \int_0^s \Big(\|\partial_t u(t)\|_Y^2 - f(t)D(t)\Big)\,\mathrm{d}t.$$

(ii) *If $u_\varepsilon(t) \xrightarrow{S} u(t)$ for any $t \in [0,T)$, there exists a locally bounded function g on $[0,T)$ such that for any $t_0 \in [0,T)$ and any v defined in a neighborhood of t_0 satisfying*

$$\begin{cases} \partial_t v(t_0) = -\nabla_Y F(u(t_0)) \\ v(t_0) = u(t_0), \end{cases}$$

there exists $v_\varepsilon(t)$ such that $v_\varepsilon(t_0) = u_\varepsilon(t_0)$ and

$$\limsup_{\varepsilon\to 0} \|\partial_t v_\varepsilon(t_0)\|_{X_\varepsilon}^2 \leq \|\partial_t v(t_0)\|_Y^2 + g(t_0)D(t_0),$$

$$\liminf_{\varepsilon\to 0} -\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0} E_\varepsilon(v_\varepsilon(t)) \geq -\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0} F(v) - g(t_0)D(t_0).$$

Then, if $\{u_\varepsilon\}$ is well-prepared initially ($D(0) = 0$), then $\{u_\varepsilon(t)\}$ is well-prepared for every $t \in [0,T)$, all the inequalities above are equalities, $u_\varepsilon(t) \xrightarrow{S} u(t)$ for any $t \in [0,T)$ where u is the solution to

$$\begin{cases} \partial_t u = -\nabla_Y F(u) \\ u(0) = u_0. \end{cases}$$

We omit the proof of Theorem 1.13. We remark only that the convergence of gradient flows does not follow from the Γ-convergence only, since slight perturbations of the energy landscape of E_ε may add local minima which disappear in the limit. The assumptions (i) and (ii) we need, guarantee that the C^1 structure of the energy landscape also converges.

In [60], this abstract scheme is used in order to prove the convergence result for the heat flow of the Ginzburg-Landau equation.

In this case, the space X_ε is nothing but the $L^2(\Omega;\mathbb{C})$ endowed with the standard scalar product of L^2 scaled by $\frac{1}{|\log\varepsilon|}$ and hence

$$\|\cdot\|_{X_\varepsilon} = \frac{1}{\sqrt{|\log\varepsilon|}}\|\cdot\|_{L^2(\Omega;\mathbb{C})}.$$

Fixed $M \in \mathbb{N}$, then $E_\varepsilon(w) = GL_\varepsilon(w) - M\pi|\log\varepsilon| - M\gamma_{GL}$ and (1.15) is given by

$$\frac{1}{|\log\varepsilon|}\partial_t w = \Delta w + \frac{w}{\varepsilon^2}(1-|w|^2). \tag{1.16}$$

Moreover $Y = \mathbb{R}^{2M}$ and its norm is given by $\|\cdot\|_Y = \frac{1}{\sqrt{\pi}}\|\cdot\|_{\mathbb{R}^{2M}}$. Finally, we say that $w_\varepsilon(t) \xrightarrow{S} x(t) = (x_1(t),\ldots,x_M(t)) \in Y$ for some $t \geq 0$ if $Jw_\varepsilon(t) \xrightarrow{\text{flat}} \sum_{i=1}^M d_{i,0}\delta_{x_i(t)}$ where $d_{1,0},\ldots,d_{M,0} \in \{+1,-1\}$ do not depend on t.

We remark that with this notation, the family $\{w_\varepsilon(t)\}$ of solutions to (1.16) is well prepared initially if $\{w_\varepsilon(0)\}$ is a recovery sequence in the sense of Theorem 1.8 (iii).

Using Theorem 1.13 and Proposition 1.17, the authors prove the following result.

Theorem 1.14. [60, Theorem 1.6] *Let $\{w_\varepsilon\}$ be a family of solutions to (1.16) with homogeneous Neumann boundary conditions such that $Jw_\varepsilon(0) \stackrel{\text{flat}}{\to} \pi\sum_{i=1}^M d_{i,0}\delta_{x_{i,0}}$, where $M \in \mathbb{N}$, $d_{i,0} \in \{+1,-1\}$ and $x_{i,0}$ are distinct points of $\mathbb{R}^2$. Assume moreover that $\{w_\varepsilon(t)\}$ is well-prepared initially, that is,*

$$\lim_{\varepsilon\to 0} GL_\varepsilon(w_\varepsilon(0)) - M\pi|\log\varepsilon| - M\gamma_{GL} = \mathbb{W}(\sum_{i=1}^M d_{i,0}\delta_{x_{i,0}}).$$

Then, there exists a time $T^* > 0$ such that $Jw_\varepsilon(t) \stackrel{\text{flat}}{\to} \sum_{i=1}^M d_{i,0}\delta_{x_i(t)}$ and

$$\lim_{\varepsilon\to 0} GL_\varepsilon(w_\varepsilon(t)) - M\pi|\log\varepsilon| - M\gamma_{GL} = \mathbb{W}(\sum_{i=1}^M d_{i,0}\delta_{x_i(t)})$$

for any $t \in [0, T^*)$, where $x(t) = (x_1(t), \dots, x_M(t))$ is the solution to

$$\begin{cases} \dot{x}_i(t) = -\frac{1}{\pi}\partial_{x_i}\mathbb{W}(\sum_{i=1}^M d_{i,0}\delta_{x_i(t)}) & t \in [0, T^*) \\ x_i(0) = x_{i,0}, \end{cases}$$

and T^* is the minimum among the collision time and the exit time from Ω for this law of motion.

The main difficulty of proving this result consists in showing that the assumptions of Theorem 1.13 are satisfied by the energies GL_ε and $\mathbb{W}$. The proof of assumption (i) is very smart and it is obtained exploiting an important result of [59], that we recall in the following section since it will be useful in order to prove our result on the dynamics of screw dislocations.

1.4. Product-Estimate

In this section we collect some results in [59] that are used in the proofs of Section 4.3.

We first introduce some notation. Let B be an open bounded subset of $\mathbb{R}^3$. Given $w = (w_1, w_2) \in H^1(B;\mathbb{C})$, its Jacobian Jw can be regarded as a 2-form in $\mathbb{R}^3$ given by

$$Jw = \mathrm{d}w_1 \wedge \mathrm{d}w_2 = \sum_{j<k}(\partial_j w_1 \partial_k w_2 - \partial_j w_2 \partial_k w_1)\,\mathrm{d}x_j \wedge \mathrm{d}x_k. \tag{1.17}$$

Thus Jw acts on vector fields $X, Y \in C(B;\mathbb{R}^3)$ with the standard rule that

$$\mathrm{d}x_j \wedge \mathrm{d}x_k(X,Y) = \frac{1}{2}\left(X_jY_k - X_kY_j\right).$$

The Jacobian Jw can be also identified with a 1-dimensional current $\star Jw$ which acts on 1-forms $\omega = \omega_1\,\mathrm{d}x_1 + \omega_2\,\mathrm{d}x_2 + \omega_3\,\mathrm{d}x_3$ as

$$\langle \star Jw, \omega\rangle = \int_B Jw \wedge \omega\,.9$$

In particular, for any $X, Y \in C(B;\mathbb{R}^3)$

$$\langle \star Jw, X \wedge Y\rangle = \int_B Jw(X,Y)\,\mathrm{d}x,$$

where, with a little abuse of notation, we identify 1-forms with vector fields.

Let $T>0$. For a given $w\in H^1([0,T]\times\Omega;\mathbb{R}^2)$, we denote by μ, V^1, V^2 the L^1 functions such that

$$Jw=\mu\,\mathrm{d}x_1\wedge\mathrm{d}x_2+V^1\,\mathrm{d}x_1\wedge\mathrm{d}t+V^2\,\mathrm{d}x_2\wedge\mathrm{d}t\,. \tag{1.18}$$

The theorem below collects the results of Theorem 1 and Theorem 3 in [59].

Theorem 1.15. *Let $w_\varepsilon\in H^1([0,T]\times\Omega;\mathbb{R}^2)$ be such that*

$$\int_0^T GL_\varepsilon(w_\varepsilon(t,\cdot))\,\mathrm{d}t+\int_0^T\int_\Omega|\partial_t w_\varepsilon(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t\le C|\log\varepsilon|\,. \tag{1.19}$$

Then, there exists a rectifiable integer 1-current J such that, up to a subsequence,

$$\frac{\star Ju_\varepsilon}{\pi}\to J\quad in\ (C_c^{0,\gamma}([0,T]\times\Omega;\mathbb{R}^3))',\ \forall\gamma\in(0,1].$$

Moreover, for any $X,Y\in C^0_c([0,T]\times\Omega;\mathbb{R}^3)$

$$\liminf_{\varepsilon\to0}\frac{1}{\pi|\log\varepsilon|}\left(\int_{[0,T]\times\Omega}|X\cdot\nabla w_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\int_{[0,T]\times\Omega}|Y\cdot\nabla w_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac12}\ge|\langle J,X\wedge Y\rangle|\,.$$

If in addition we assume that

$$\sup_{t\in[0,T]}GL_\varepsilon(w_\varepsilon(t,\cdot))\le C|\log\varepsilon|, \tag{1.20}$$

then, J can be written as in (1.18) with $\mu\in C^{0,\frac12}([0,T];(C_c^{0,\gamma}(\Omega))')$ for every $\gamma\in(0,1]$ and V^1, $V^2\in L^2([0,T];\mathcal{M}(\Omega))$.

Finally, up to a subsequence,

$$\mu_\varepsilon(t)\overset{\text{flat}}{\to}\mu(t)\qquad for\ all\ t\in[0,T].$$

We now state a variant of Corollary 4 in [59] which is a direct consequence of Theorem 1.15.

Corollary 1.16. *Let $0\le t_1<t_2$ and let $w_\varepsilon\in H^1([t_1,t_2]\times\Omega;\mathbb{R}^2)$ be such that (1.19) holds true with $[0,T]$ replaced by $[t_1,t_2]$, and such that for all $t\in[t_1,t_2]$*

$$\frac12\int_\Omega|\nabla w_\varepsilon(t,x)|^2\,\mathrm{d}x\le M\pi|\log\varepsilon|+C$$

for some $M\in\mathbb{N}$ and $C\in\mathbb{R}$. Assume moreover that

$$\mu_\varepsilon(t)\overset{\text{flat}}{\to}\mu(t):=\pi\sum_{i=1}^M d_i\delta_{x_i(t)},$$

with $|d_i|=1$ and $x_i(t)\in C([t_1,t_2];\Omega)$ for every i with $x_i(t)\neq x_j(t)$ for $i\neq j$.

Then, for any $X,Y\in C^0_c(\Omega;\mathbb{R}^3)$

$$\liminf_{\varepsilon\to0}\frac{1}{|\log\varepsilon|}\int_{[t_1,t_2]\times\Omega}\langle X\cdot\nabla w_\varepsilon,Y\cdot\nabla w_\varepsilon\rangle\,\mathrm{d}x\,\mathrm{d}t=\pi\int_{t_1}^{t_2}\sum_{i=1}^M\langle X(x_i(t)),Y(x_i(t))\rangle\,\mathrm{d}t.$$

Here we state a result analogous to Corollary 7 in [59].

Proposition 1.17. *Let $\tilde{T} > 0$ and let $w_\varepsilon \in H^1([0,\tilde{T}] \times \Omega; \mathbb{R}^2)$ be such that (1.19) holds true, and such that for all $t \in [0,\tilde{T}]$*

$$\frac{1}{2}\int_\Omega |\nabla w_\varepsilon(t,x)|^2 \,\mathrm{d}x \leq M\pi|\log\varepsilon| + C$$

for some $M \in \mathbb{N}$ and $C \in \mathbb{R}$. Assume moreover that

$$\mu_\varepsilon(t) \overset{\text{flat}}{\to} \mu(t) := \pi\sum_{i=1}^M d_i \delta_{x_i(t)},$$

with $|d_i| = 1$ and $x_i(t) \in C([0,\tilde{T}];\Omega)$ for any i with $x_i(t) \neq x_j(t)$ for $i \neq j$. Then

$$\liminf_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|}\int_{[0,\tilde{T}]\times\Omega} |\partial_t w_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \geq \pi\sum_{i=1}^M \int_0^{\tilde{T}} |\dot{x}_i|^2 \,\mathrm{d}t. \tag{1.21}$$

Proof. The proof of this result coincides with the one of Corollary 7 in [59], the only difference being that [59] assumes that for every $t \in [0,\hat{T}]$

$$\frac{1}{\varepsilon^2}\int_\Omega (1-|w_\varepsilon(t,x)|^2)^2 \,\mathrm{d}x \leq C|\log\varepsilon|.$$

Here this assumption is replaced by (1.19), which is enough to apply Corollary 1.16. Once the statement of Corollary 1.16 holds true, the rest of the proof follows exactly as in in [59]. $\square$

CHAPTER 2

Γ-convergence expansion for systems of screw dislocations

In this chapter we perform the Γ-convergence expansion of the energy SD_ε defined in (0.1) and for the functional XY_ε in (0.8). The results proved in this chapter are contained in [4].

As mentioned in the Introduction, both SD_ε and XY_ε can be regarded as specific examples of scalar systems F_ε, whose energy is governed by periodic potentials acting on first neighbors. In the next section we introduce this general class of energies and the discrete formalism used in the analysis of the problem we deal with. We will follow the approach of [11]; specifically, we will use the formalism and the notation in [3] (see also [56]).

In the following of this thesis, all the results are proved for this general class of functionals.

2.1. The discrete model for topological singularities

The discrete lattice. For every $\varepsilon > 0$, we define $\Omega_\varepsilon \subset \Omega$ as follows

$$\Omega_\varepsilon := \bigcup_{i\in\varepsilon\mathbb{Z}^2 :\, i+\varepsilon Q\subset\overline{\Omega}} (i+\varepsilon Q),$$

where $Q = [0,1]^2$ is the unit square. Moreover we set $\Omega^0_\varepsilon := \varepsilon\mathbb{Z}^2 \cap \Omega_\varepsilon$, and $\Omega^1_\varepsilon := \left\{(i,j) \in \Omega^0_\varepsilon \times \Omega^0_\varepsilon :\ |i-j| = \varepsilon,\, i \le j\right\}$ (where $i \le j$ means that $i_l \le j_l$ for $l \in \{1,2\}$). These objects represent the reference lattice and the class of nearest neighbors, respectively. The cells contained in Ω_ε are labeled by the set of indices $\Omega^2_\varepsilon = \left\{i \in \Omega^0_\varepsilon :\ i+\varepsilon Q \subset \Omega_\varepsilon\right\}$. Finally, we define the *discrete boundary* of Ω as

$$\partial_\varepsilon\Omega := \partial\Omega_\varepsilon \cap \varepsilon\mathbb{Z}^2. \tag{2.1}$$

In the following, we will extend the use of these notations to any given open subset A of $\mathbb{R}^2$.

2.1.1. Discrete functions and discrete topological singularities.

As mentioned in the Introduction the dislocations can be seen as topological singularities of the deformation gradient. Here we introduce the notion of topological singularity in our discrete setting. To this purpose, we first set

$$\mathcal{AF}_\varepsilon(\Omega) := \left\{u : \Omega^0_\varepsilon \to \mathbb{R}\right\},$$

which represents the class of admissible scalar functions on Ω^0_ε.

Moreover, we introduce the class of admissible fields from Ω^0_ε to the set $\mathcal{S}^1$ of unit vectors in $\mathbb{R}^2$

$$\mathcal{AXY}_\varepsilon(\Omega) := \left\{v : \Omega^0_\varepsilon \to \mathcal{S}^1\right\}, \tag{2.2}$$

Notice that, to any function $u \in \mathcal{AF}_\varepsilon(\Omega)$, we can associate a function $v \in \mathcal{AXY}_\varepsilon(\Omega)$ setting

$$v = v(u) := e^{2\pi i u}.$$

With a little abuse of notation for every $v : \Omega^0_\varepsilon \to \mathbb{R}^2$ we denote

$$\|v\|^2_{L^2} = \sum_{j\in\Omega^0_\varepsilon} \varepsilon^2 |v(j)|^2 . \tag{2.3}$$

Now we can introduce a notion of discrete vorticity corresponding to both scalar and $\mathcal{S}^1$ valued functions. To this purpose, let $P : \mathbb{R} \to \mathbb{Z}$ be defined as follows

$$P(t) = \operatorname{argmin} \{|t - s| : s \in \mathbb{Z}\} , \tag{2.4}$$

with the convention that, if the argmin is not unique, then we choose the minimal one.

Let $u \in \mathcal{AF}_\varepsilon(\Omega)$ be fixed. For every $i \in \Omega^2_\varepsilon$ we introduce the vorticity

$$\begin{aligned} \alpha_u(i) := & P(u(i+\varepsilon e_1) - u(i)) + P(u(i+\varepsilon e_1+\varepsilon e_2) - u(i+\varepsilon e_1)) \\ & - P(u(i+\varepsilon e_1+\varepsilon e_2) - u(i+\varepsilon e_2)) - P(u(i+\varepsilon e_2) - u(i)). \end{aligned} \tag{2.5}$$

One can easily see that the vorticity α_u takes values in $\{-1, 0, 1\}$. Finally, we define the vorticity measure $\mu(u)$ as follows

$$\mu(u) := \sum_{i\in\Omega^2_\varepsilon} \alpha_u(i)\delta_{i+\frac{\varepsilon}{2}(e_1+e_2)}. \tag{2.6}$$

This definition of vorticity extends to $\mathcal{S}^1$ valued fields in the obvious way, by setting $\mu(v) = \mu(u)$ where u is any function in $\mathcal{AF}_\varepsilon(\Omega)$ such that $v(u) = v$.

Let $\mathcal{M}(\Omega)$ be the space of Radon measures in Ω and set

$$\begin{aligned} X &:= \left\{ \mu \in \mathcal{M}(\Omega) \,:\, \mu = \sum_{i=1}^N d_i \delta_{x_i} \,,\, N \in \mathbb{N}, d_i \in \mathbb{Z} \setminus \{0\} \,, x_i \in \Omega \right\}, \\ X_\varepsilon &:= \left\{ \mu \in X : \mu = \sum_{i\in\Omega^2_\varepsilon} \alpha(i)\delta_{i+\frac{\varepsilon}{2}(e_1+e_2)} \,,\ \alpha(i) \in \{-1,0,1\} \right\}. \end{aligned} \tag{2.7}$$

We will denote by $\mu_n \overset{\text{flat}}{\to} \mu$ the flat convergence of μ_n to μ, i.e., in the dual $W^{-1,1}$ of $W^{1,\infty}_0$.

2.1.2. The discrete energy. Here we introduce a class of energy functionals defined on $\mathcal{AF}_\varepsilon(\Omega)$. We will consider periodic potentials $f : \mathbb{R} \to \mathbb{R}$ which satisfy the following assumptions: For any $a \in \mathbb{R}$

(1) $f(a + z) = f(a)$ for any $z \in \mathbb{Z}$,
(2) $f(a) \geq \frac{1}{2}|e^{2\pi i a} - 1|^2 = 1 - \cos 2\pi a$,
(3) $f(a) = 2\pi^2(a - z)^2 + \mathrm{O}(|a - z|^3)$ for any $z \in \mathbb{Z}$.

For any $u \in \mathcal{AF}_\varepsilon(\Omega)$, we define

$$F_\varepsilon(u) := \sum_{(i,j)\in\Omega^1_\varepsilon} f(u(i) - u(j)). \tag{2.8}$$

As explained in the Introduction, the main motivation for our analysis comes from the study discrete screw dislocations in crystals and XY spin systems. We introduce the basic energies for these two models as in [3].

We recall that the energy functional for the screw dislocations model is defined by

$$SD_\varepsilon(u) := \frac{1}{2} \sum_{(i,j)\in\Omega^1_\varepsilon} \operatorname{dist}^2(u(i)-u(j),\mathbb{Z}), \tag{2.9}$$

where $u \in \mathcal{AF}_\varepsilon(\Omega)$. It is easy to see that this potential fits (up to the prefactor $4\pi^2$) with our general assumptions.

As for the XY model, for any $v:\Omega^0_\varepsilon \to \mathcal{S}^1$, we define

$$XY_\varepsilon(v) := \frac{1}{2} \sum_{(i,j)\in\Omega^1_\varepsilon} |v(i)-v(j)|^2. \tag{2.10}$$

Also this potential fits our framework, once we rewrite it in terms of the phase u of v. Indeed, setting $f(a) = 1-\cos(2\pi a)$, we have

$$XY_\varepsilon(v) = \sum_{(i,j)\in\Omega^1_\varepsilon} f(u(i)-u(j)) \qquad \text{with} \;\; v = e^{2\pi i u}. \tag{2.11}$$

We notice that assumption (2) on F_ε reads as

$$F_\varepsilon(u) \geq XY_\varepsilon(e^{2\pi i u}). \tag{2.12}$$

Let $\{T_i^\pm\}$ be the family of the ε-simplices of $\mathbb{R}^2$ whose vertices are of the form $\{i, i\pm\varepsilon e_1, i\pm\varepsilon e_2\}$, with $i\in\varepsilon\mathbb{Z}^2$. For any $v:\Omega^0_\varepsilon\to\mathcal{S}^1$, we denote by $\tilde v:\Omega_\varepsilon\to\mathcal{S}^1$ the piecewise affine interpolation of v, according with the triangulation $\{T_i^\pm\}$. It is easy to see that, up to boundary terms, $XY_\varepsilon(v)$ corresponds to the Dirichlet energy of $\tilde v$ in Ω_ε; more precisely

$$\frac{1}{2}\int_{\Omega_\varepsilon} |\nabla\tilde v|^2\,\mathrm{d}x + C\varepsilon \geq XY_\varepsilon(v) \geq \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla\tilde v|^2\,\mathrm{d}x, \tag{2.13}$$

where C depends only on Ω.

Remark 2.1. Let $v:\Omega^0_\varepsilon\to\mathcal{S}^1$. One can easily verify that if A is an open subset of Ω and if $|\tilde v| > c > 0$ on ∂A_ε, then

$$\mu(v)(A_\varepsilon) = \deg(\tilde v, \partial A_\varepsilon), \tag{2.14}$$

where the *degree* of a function $w\in H^{\frac{1}{2}}(\partial A;\mathbb{R}^2)$ with $|w|\geq c>0$, is defined by

$$\deg(w,\partial A) := \frac{1}{2\pi}\int_{\partial A}\left(\frac{w}{|w|}\times\nabla\frac{w}{|w|}\right)\cdot\tau\,\mathrm{d}s\,, \tag{2.15}$$

with $v\times\nabla v := v_1\nabla v_2 - v_2\nabla v_1$, for $v\in H^1(A;\mathbb{R}^2)$.
In particular, whenever $|\tilde v| > 0$ on $i+\varepsilon Q$ we have $\mu(v)(i+\varepsilon Q) = 0$.

2.2. Localized lower bounds

In this section we will prove a lower bound for the energies F_ε localized on open subsets $A\subset\Omega$. We will use the standard notation $F_\varepsilon(\cdot, A)$ (and as well $XY_\varepsilon(\cdot, A)$) to denote the functional F_ε defined in (2.8) with Ω replaced by A.

To this purpose, thanks to assumption (2) on the energy density f, it will be enough to prove a lower bound for the XY_ε energy. As a consequence of this lower bound, we obtain a sharp zero-order Γ-convergence result for the functionals F_ε.

2.2.1. The zero-order Γ-convergence. Here we prove the Γ-convergence result for F_ε, which can be seen as the discrete version of Theorem [**?**]. We recall that the space of finite sums of weighted Dirac masses has been denoted in (2.7) by X.

Theorem 2.2. *The following Γ-convergence result holds.*

(i) *(Compactness) Let $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ be such that $F_\varepsilon(u_\varepsilon) \le C|\log\varepsilon|$ for some positive C. Then, up to a subsequence, $\mu(u_\varepsilon) \stackrel{\text{flat}}{\to} \mu$, for some $\mu \in X$.*

(ii) *(Localized Γ-liminf inequality) Let $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ be such that $\mu(u_\varepsilon) \stackrel{\text{flat}}{\to} \mu = \sum_{i=1}^M d_i \delta_{x_i}$ with $d_i \in \mathbb{Z} \setminus \{0\}$ and $x_i \in \Omega$. Then, there exists a constant $C \in \mathbb{R}$ such that, for any $i = 1, \dots, M$ and for every $\sigma < \frac{1}{2}\mathrm{dist}(x_i, \partial\Omega \cup \bigcup_{j\neq i} x_j)$, we have*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, B_\sigma(x_i)) - \pi|d_i|\log\frac{\sigma}{\varepsilon} \ge C. \tag{2.16}$$

In particular

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) - \pi|\mu|(\Omega)\log\frac{\sigma}{\varepsilon} \ge C. \tag{2.17}$$

(iii) *(Γ-limsup inequality) For every $\mu \in X$, there exists a sequence $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ such that $\mu(u_\varepsilon) \stackrel{\text{flat}}{\to} \mu$ and*

$$\pi|\mu|(\Omega) \ge \limsup_{\varepsilon\to 0} \frac{F_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}.$$

The above theorem has been proved in [**56**] for $F_\varepsilon = SD_\varepsilon$ and in [**3**] for $F_\varepsilon = XY_\varepsilon$, with (ii) replaced by the standard global Γ-liminf inequality

$$\pi|\mu|(\Omega) \le \liminf_{\varepsilon\to 0} \frac{F_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}, \tag{2.18}$$

which is clearly implied by (2.17).

By (2.12), the compactness property (i) follows directly from the zero-order Γ-convergence result for the XY_ε energies, while the proof of (ii) requires a specific analysis. For the convenience of the reader we will give a self contained proof of both (i) and (ii) of Theorem 2.2. We will omit the proof of the Γ-lim sup inequality (iii) which is standard and identical to the XY_ε case.

Before giving the proof of Theorem 2.2, we need to revisit the ball construction introduced in Subsection 1.2.1 in order to include our discrete case.

2.2.2. Lower bound on annuli. As noticed in Remark 1.9, the key estimate in order to prove compactness in the continuous Ginzburg-Landau is given by a sharp lower bound on annuli (see (1.12)). In the following we will prove an analogous lower bound for the energy $XY_\varepsilon(v,\cdot)$ in an annulus in which the piecewise affine interpolation $\tilde v$ satisfies $|\tilde v| \ge \frac{1}{2}$. In view of (2.12) such a lower bound will hold also for the energy F_ε.

Proposition 2.3. *Fix $\varepsilon > 0$ and let $\sqrt{2}\varepsilon < r < R - 2\sqrt{2}\varepsilon$. For any function $v : (B_R \setminus B_r) \cap \varepsilon\mathbb{Z}^2 \to \mathcal{S}^1$ with $|\tilde{v}| \geq \frac{1}{2}$ in $B_{R-\sqrt{2}\varepsilon} \setminus B_{r+\sqrt{2}\varepsilon}$, it holds*

$$XY_\varepsilon(v, B_R \setminus B_r) \geq \pi|\mu(v)(B_r)| \log \frac{R}{r + \varepsilon\big(\alpha|\mu(v)(B_r)| + \sqrt{2}\big)}, \tag{2.19}$$

where $\alpha > 0$ is a universal constant.

Proof. By (2.13), using Fubini's theorem, we have that

$$XY_\varepsilon(v, B_R \setminus B_r) \geq \frac{1}{2} \int_{r+\sqrt{2}\varepsilon}^{R-\sqrt{2}\varepsilon} \int_{\partial B_\rho} |\nabla \tilde{v}|^2 \, \mathrm{d}s \, \mathrm{d}\rho. \tag{2.20}$$

Fix $r + \sqrt{2}\varepsilon < \rho < R - \sqrt{2}\varepsilon$ and let T be a simplex of the triangulation of the ε-lattice. Set $\gamma_T(\rho) := \partial B_\rho \cap T$, let $\bar{\gamma}_T(\rho)$ be the segment joining the two extreme points of $\gamma_T(\rho)$ and let $\bar{\gamma}(\rho) = \bigcup_T \bar{\gamma}_T(\rho)$; then

$$\begin{aligned} \frac{1}{2} \int_{\partial B_\rho} |\nabla \tilde{v}|^2 \, \mathrm{d}s &= \frac{1}{2} \int_{\cup_T \gamma_T(\rho)} |\nabla \tilde{v}|^2 \, \mathrm{d}s = \frac{1}{2} \sum_T |\nabla \tilde{v}_{|_T}|^2 \mathcal{H}^1(\gamma_T(\rho)) \qquad (2.21) \\ &\geq \frac{1}{2} \sum_T |\nabla \tilde{v}_{|_T}|^2 \mathcal{H}^1(\bar{\gamma}_T(\rho)) = \frac{1}{2} \int_{\bar{\gamma}(\rho)} |\nabla \tilde{v}|^2 \, \mathrm{d}s. \end{aligned}$$

Set $m(\rho) := \min_{\bar{\gamma}(\rho)} |\tilde{v}|$; using Jensen's inequality and the fact that $\mathcal{H}^1(\bar{\gamma}(\rho)) \leq \mathcal{H}^1(\partial B_\rho)$ we get

$$\begin{aligned} \frac{1}{2} \int_{\bar{\gamma}(\rho)} |\nabla \tilde{v}|^2 \, \mathrm{d}s &\geq \frac{1}{2} \int_{\bar{\gamma}(\rho)} m^2(\rho) \left| \left(\frac{\tilde{v}}{|\tilde{v}|} \times \nabla \frac{\tilde{v}}{|\tilde{v}|} \right) \cdot \tau \right|^2 \mathrm{d}s \qquad (2.22) \\ &\geq \frac{1}{2} \frac{m^2(\rho)}{\mathcal{H}^1(\bar{\gamma}(\rho))} \left| \int_{\bar{\gamma}(\rho)} \left(\frac{\tilde{v}}{|\tilde{v}|} \times \nabla \frac{\tilde{v}}{|\tilde{v}|} \right) \cdot \tau \, \mathrm{d}s \right|^2 \\ &\geq \frac{m^2(\rho)}{\rho} \pi |d|^2 \qquad (2.23) \end{aligned}$$

where we have set $d := \deg(\tilde{v}, \partial B_\rho) = \mu(v)(B_r)$, which does not depend on ρ since $|\tilde{v}| \geq 1/2$.

Now, let $T(\rho)$ be the simplex in which the minimum $m(\rho)$ is attained. Without loss of generality we assume that $T(\rho) = T^-_{\bar{\imath}}$ for some $\bar{\imath} \in \varepsilon\mathbb{Z}^2$. Let P one of the points of $\bar{\gamma}(\rho)$ for which $|\tilde{v}(P)| = m(\rho)$. By elementary geometric arguments, one can show that

$$\frac{1}{2} \int_{\partial B_\rho} |\nabla \tilde{v}|^2 \, \mathrm{d}s \geq \tilde{\alpha} \frac{1 - m^2(\rho)}{\varepsilon}, \tag{2.24}$$

for some universal positive constant $\tilde{\alpha}$.

In view of (3.14), (3.18) and (3.19), for any $r + \sqrt{2}\varepsilon < \rho < R - \sqrt{2}\varepsilon$ we have

$$\frac{1}{2} \int_{\partial B_\rho} |\nabla \tilde{v}|^2 \, \mathrm{d}s \geq \frac{m^2(\rho)}{\rho} \pi |d| \vee \tilde{\alpha} \frac{1 - m^2(\rho)}{\varepsilon} \geq \frac{\pi |d| \tilde{\alpha}}{\varepsilon \pi |d| + \tilde{\alpha} \rho}.$$

By this last estimate and (3.13) we get

$$XY_\varepsilon(v, B_R \setminus B_r) \geq \pi|\mu(v)(B_r)| \log \frac{\varepsilon(\frac{\pi}{\tilde{\alpha}}|\mu(v)(B_r)| - \sqrt{2}) + R}{\varepsilon(\frac{\pi}{\tilde{\alpha}}|\mu(v)(B_r)| + \sqrt{2}) + r}. \tag{2.25}$$

Assuming, without loss of generality, $\tilde{\alpha} < 1$, we immediately get (3.12) for $\alpha = \frac{\pi}{\tilde{\alpha}}$. □

2.2.3. Ball Construction for F_ε. Here we introduce the ball construction for the functionals F_ε. Let $\mathcal{B} = \{B_{R_1}(x_1), \dots, B_{R_N}(x_N)\}$ be a finite family of pairwise disjoint balls in $\mathbb{R}^2$ and let $\mu = \sum_{i=1}^N d_i \delta_{x_i}$ with $d_i \in \mathbb{Z} \setminus \{0\}$. Let F be a positive superadditive set function on the open subsets of $\mathbb{R}^2$, i.e., such that $F(A \cup B) \geq F(A) + F(B)$, whenever A and B are open and disjoint. We will assume that there exists $c > 0$ such that

$$F(A_{r,R}(x)) \geq \pi |\mu(B_r(x))| \log \frac{R}{c + r}, \tag{2.26}$$

for any annulus $A_{r,R}(x) = B_R(x) \setminus B_r(x)$, with $A_{r,R}(x) \subset \Omega \setminus \bigcup_i B_{R_i}(x_i)$.

The purpose of this construction is to select a family of larger and larger annuli in which the main part of the energy F concentrates. Let t be a parameter which represents an artificial time. For any $t > 0$ we want to construct a finite family of balls $\mathcal{B}(t)$ which satisfies the following properties

(1) $\bigcup_{i=1}^N B_{R_i}(x_i) \subset \bigcup_{B \in \mathcal{B}(t)} B$,
(2) the balls in $\mathcal{B}(t)$ are pairwise disjoint,
(3) $F(B) \geq \pi |\mu(B)| \log(1 + t)$ for any $B \in \mathcal{B}(t)$,
(4) $\sum_{B \in \mathcal{B}(t)} R(B) \leq (1 + t) \sum_i R_i + (1 + t) c N (N^2 + N + 1)$, where $R(B)$ denotes the radius of the ball B.

We construct the family $\mathcal{B}(t)$, closely following the strategy of the ball construction introduced in Subsection 1.2.1 with slight modifications that include our case: The only difference in our discrete setting is the appearance of the error term $c > 0$ in (3.21) and in (4), while in the continuous setting $c = 0$.

In this case, the ball construction starts with an expansion phase if $\mathrm{dist}(B_{R_i}(x_i), B_{R_j}(x_j)) > 2c$ for all $i \neq j$, and with a merging phase otherwise. Assume that the first phase is an expansion. It consists in letting the balls expand, without changing theirs centers, in such a way that, at each (artificial) time, the ratio $\theta(t) := \frac{R_i(t)}{c + R_i}$ is independent of i. We will parametrize the time enforcing $\theta(t) = 1 + t$. Note that with this choice $R_i(0) = R_i + c$ so that the balls $\{B_{R_i(0)}(x_i)\}$ are pairwise disjoint. The first expansion phase stops at the first time T_1 when two balls bump into each other. Then the merging phase begins. It consists in identifying a suitable partition $\{S_j^1\}_{j=1,\dots,N_n}$ of the family $\left\{B_{R_i(T_1)}(x_i)\right\}$, and, for each subclass S_j^1, in finding a ball $B_{R_j^1}(x_j^1)$ which contains all the balls in S_j^1 such that the following properties hold:

i) for every $j \neq k$, $\mathrm{dist}(B_{R_j^1}(x_j^1), B_{R_k^1}(x_k^1)) > 2c$;
ii) $R_j^1 - Nc$ is not larger than the sum of all the radii of the balls $B_{R_i(T_1)}(x_i) \in S_j^1$, i.e., contained in $B_{R_j^1}(x_j^1)$.

After the merging, another expansion phase begins, during which we let the balls $\left\{B_{R_j^1}(x_j^1)\right\}$ expand in such a way that, for $t \geq T_1$, for every j we have

$$\frac{R_j^1(t)}{c + R_j^1} = \frac{1 + t}{1 + T_1}. \tag{2.27}$$

Again note that $R_j^1(T_1) = R_j^1 + c$. We iterate this process obtaining a set of merging times $\{T_1, \dots, T_n\}$, and a family $\mathcal{B}(t) = \{B_{R_j^k(t)}(x_j^k)\}_j$ for $t \in [T_k, T_{k+1})$, for all $k = 1, \dots, n-1$. Notice that $n \le N$. If the condition $\mathrm{dist}(B_{R_i}(x_i), B_{R_j}(x_j)) > 2c$ for all $i \ne j$, is not satisfied we clearly can start this process with a merging phase (in this case $T_1 = 0$).

By construction, we clearly have (1) and (2). We now prove (4). Set $N(t) = \sharp\{B \in \mathcal{B}(t)\}$ and $I(t) = \{1, \dots, N(t)\}$. Moreover, for every merging time T_k and $1 \le j \le N(T_k)$, set

$$I_j(T_k) := \left\{ i \in I(T_{k-1}) : B_{R_i^{k-1}}(x_i^{k-1}) \subset B_{R_j^k}(x_j^k) \right\}.$$

By ii) and (2.27) it follows that for any $1 \le k \le n$

$$\begin{aligned}
\sum_{j=1}^{N(T_k)} (R_j^k - Nc) &\le \sum_{j=1}^{N(T_k)} \sum_{l \in I_j(T_k)} R_l^{k-1}(T_k) \\
&= \sum_{j=1}^{N(T_{k-1})} \left(\frac{1+T_k}{1+T_{k-1}} R_j^{k-1} + \frac{1+T_k}{1+T_{k-1}} c \right) \\
&= \frac{1+T_k}{1+T_{k-1}} \sum_{j=1}^{N(T_{k-1})} R_j^{k-1} + \frac{1+T_k}{1+T_{k-1}} c N(T_{k-1}) \qquad (2.28) \\
&\le \frac{1+T_k}{1+T_{k-1}} \sum_{j=1}^{N(T_{k-1})} R_j^{k-1} + (1+T_k) c N.
\end{aligned}$$

Let $T_k \le t < T_{k+1}$ for some $1 \le k \le n$; by (2.27) and iterating (2.28) we get

$$\begin{aligned}
\sum_{j=1}^{N(T_k)} R_j^k(t) &= \frac{1+t}{1+T_k} \sum_{j=1}^{N(T_k)} R_j^k + \frac{1+t}{1+T_k} c\, N(T_k) \\
&\le (1+t) \sum_{i=1}^{N} R_i + (1+t) c N (N^2 + N + 1),
\end{aligned} \qquad (2.29)$$

and this concludes the proof of (4).

It remains to prove (3). For $t = 0$ it is trivially satisfied. We will show that it is preserved during the merging and the expansion times. Let T_k be a merging time and assume that (3) holds for all $t < T_k$. Then for every $j \in I(T_k)$

$$\begin{aligned}
F(B_{R_j^k}(x_j^k)) &\ge \sum_{l \in I_j(T_k)} F(B_{R_l^{k-1}(T_k)}(x_l^{k-1})) \\
&\ge \pi \log(1+T_k) \sum_{l=1}^{j} |\mu(B_{R_l^{k-1}(T_k)}(x_l^{k-1}))| \\
&\ge \pi \log(1+T_k) |\mu(B_{R_j^k}(x_j^k))|.
\end{aligned}$$

Finally, for a given $t \in [T_k, T_{k+1})$ and for any ball $B_{R_i^k(t)}(x_i^k(t)) \in \mathcal{B}(t)$ we have

$$F(B_{R_i^k(t)}(x_i^k)) \geq F(B_{R_i^k(t)}(x_i^k) \setminus B_{R_i^k}(x_i^k)) + F(B_{R_i^k}(x_i^k))$$
$$\geq \pi|\mu(B_{R_i^k(t)}(x_i^k))| \log \frac{1+t}{1+T_k} + \pi|\mu(B_{R_i^k(t)}(x_i^k))| \log(1+T_k)$$
$$= \pi|\mu(B_{R_i^k(t)}(x_i^k))| \log(1+t),$$

where we have used that $\frac{R_i^k(t)}{c+R_i^k} = \frac{1+t}{1+T_k}$.

2.2.4. Proof of Theorem 2.2. First, we give an elementary lower bound of the energy localized on a single square of the lattice, whose proof is immediate.

Proposition 2.4. *There exists a positive constant β such that for any $\varepsilon > 0$, for any function $v \in \mathcal{AXY}_\varepsilon(\Omega)$ and for any $i \in \Omega^2_\varepsilon$ such that the piecewise affine interpolation $\tilde{v}$ of v satisfies $\min_{i+\varepsilon Q} |\tilde{v}| < \frac{1}{2}$, it holds $XY_\varepsilon(v, i+\varepsilon Q) \geq \beta$.*

Proof of Theorem 2.2. By (2.12), it is enough to prove (i) and (ii) for $F_\varepsilon = XY_\varepsilon$, using as a variable $v_\varepsilon = e^{2\pi i u_\varepsilon}$. The proof of (iii) is standard and left to the reader.

Proof of (i). For every $\varepsilon > 0$, set $I_\varepsilon := \left\{ i \in \Omega^2_\varepsilon \, : \, \min_{i+\varepsilon Q} |\tilde{v}_\varepsilon| \leq \frac{1}{2} \right\}$. Notice that, in view of Remark 2.1, $\mu(v_\varepsilon)$ is supported in $I_\varepsilon + \frac{\varepsilon}{2}(e_1 + e_2)$.

Starting from the family of balls $B_{\frac{\sqrt{2}\varepsilon}{2}}(i + \frac{\varepsilon}{2}(e_1 + e_2)))$, and eventually passing through a merging procedure we can construct a family of pairwise disjoint balls

$$\mathcal{B}_\varepsilon := \left\{ B_{R_{i,\varepsilon}}(x_{i,\varepsilon}) \right\}_{i=1,\dots,N_\varepsilon},$$

with $\sum_{i=1}^{N_\varepsilon} R_{i,\varepsilon} \leq \varepsilon \sharp I_\varepsilon$. Then, by Proposition 2.4 and by the energy bound, we immediately have that $\sharp I_\varepsilon \leq C |\log \varepsilon|$ and hence

$$\sum_{i=1}^{N_\varepsilon} R_{i,\varepsilon} \leq \varepsilon C |\log \varepsilon|. \tag{2.30}$$

We define the sequence of measures

$$\mu_\varepsilon := \sum_{i=1}^{N_\varepsilon} \mu(v_\varepsilon)(B_{R_{i,\varepsilon}}(x_{i,\varepsilon})) \delta_{x_{i,\varepsilon}}.$$

Since $|\mu_\varepsilon(B)| \leq \sharp I_\varepsilon$ for each ball $B \in \mathcal{B}_\varepsilon$, by (3.12) we deduce that (3.21) holds with $F(\cdot) = XY_\varepsilon(v_\varepsilon, \cdot \setminus \cup_{B \in \mathcal{B}_\varepsilon} B)$ and $c = \varepsilon(\alpha \sharp I_\varepsilon + 2\sqrt{2})$.

We let the balls in the families $\mathcal{B}_\varepsilon$ grow and merge as described in Subsection 3.2.3, and let $\mathcal{B}_\varepsilon(t) := \left\{ B_{R_{i,\varepsilon}(t)}(x_{i,\varepsilon}(t)) \right\}$ be the corresponding family of balls at time t. Set moreover $t_\varepsilon := \frac{1}{\sqrt{\varepsilon}} - 1$, $N_\varepsilon(t_\varepsilon) := \sharp \mathcal{B}_\varepsilon(t_\varepsilon)$ and define

$$\nu_\varepsilon := \sum_{\substack{i=1,\dots,N_\varepsilon(t_\varepsilon) \\ B_{R_{i,\varepsilon}(t_\varepsilon)}(x_{i,\varepsilon}(t_\varepsilon)) \subset \Omega}} \mu_\varepsilon(B_{R_{i,\varepsilon}(t_\varepsilon)}(x_{i,\varepsilon}(t_\varepsilon))) \delta_{x_{i,\varepsilon}(t_\varepsilon)}. \tag{2.31}$$

By (3) in Subsection 3.2.3, for any $B \in \mathcal{B}_\varepsilon(t_\varepsilon)$, with $B \subseteq \Omega$, we have

$$XY_\varepsilon(v_\varepsilon, B) \geq \pi|\mu_\varepsilon(B)| \log(1+t_\varepsilon) = \pi|\nu_\varepsilon(B)|\tfrac{1}{2}|\log\varepsilon|;$$

by the energy bound, we have immediately that $|\nu_\varepsilon|(\Omega) \leq M$ and hence $\{\nu_\varepsilon\}$ is precompact in the weak* topology. By (4) in Subsection 3.2.3, it follows that

$$\sum_{j=1}^{N_\varepsilon(t_\varepsilon)} R_j(t_\varepsilon) \leq C\sqrt{\varepsilon}\ (\sharp I_\varepsilon)^4,$$

which easily implies that $\|\nu_\varepsilon - \mu_\varepsilon\|_{\text{flat}} \to 0$; moreover, using (3.22), it is easy to show that $\|\mu_\varepsilon - \mu(v_\varepsilon)\|_{\text{flat}} \to 0$ as $\varepsilon \to 0$ (see [6] for more details). We conclude that also $\mu(v_\varepsilon)$ is precompact in the flat topology.

Proof of (ii). Fix $i \in \{1, \dots, M\}$. Without loss of generality, and possibly extracting a subsequence, we can assume that

$$\begin{aligned}\liminf_{\varepsilon\to 0} XY_\varepsilon(v_\varepsilon, B_\sigma(x_i)) - \pi|d_i||\log\varepsilon| \\ = \lim_{\varepsilon\to 0} XY_\varepsilon(v_\varepsilon, B_\sigma(x_i)) - \pi|d_i||\log\varepsilon| < +\infty. \quad (2.32)\end{aligned}$$

We consider the restriction $\bar v_\varepsilon \in \mathcal{AXY}_\varepsilon(B_\sigma(x_i))$ of v_ε to $B_\sigma(x_i)$. Notice that $\operatorname{supp}(\mu(\bar v_\varepsilon) - \mu(v_\varepsilon) \llcorner B_\sigma(x_i)) \subseteq B_\sigma(x_i) \setminus B_{\sigma-\varepsilon}(x_i)$. On the other hand, by (3.24) and Proposition 2.4 it follows that

$$|\mu(v_\varepsilon)|(B_\sigma(x_i) \setminus B_{\sigma-\varepsilon}(x_i)) \leq C|\log\varepsilon|. \quad (2.33)$$

Then, using (2.33) one can easily get

$$\|\mu(\bar v_\varepsilon) - \mu(v_\varepsilon)\llcorner B_\sigma(x_i)\|_{\text{flat}} \to 0, \quad (2.34)$$

and hence

$$\|\mu(\bar v_\varepsilon) - d_i\delta_{x_i}\|_{\text{flat}} \to 0. \quad (2.35)$$

We repeat the ball construction procedure used in the proof of (i) with Ω replaced by $B_\sigma(x_i)$, v_ε by $\bar v_\varepsilon$ and I_ε by

$$I_{i,\varepsilon} := \left\{ j \in (B_\sigma(x_i))^2_\varepsilon \ :\ \min_{j+\varepsilon Q} |\tilde v_\varepsilon| \leq \frac{1}{2} \right\}.$$

We denote by $\mathcal{B}_{i,\varepsilon}$ the corresponding family of balls and by $\mathcal{B}_{i,\varepsilon}(t)$ the family of balls constructed at time t.

Fix $0 < \gamma < 1$ such that

$$(1-\gamma)(|d_i|+1) > |d_i|\,. \quad (2.36)$$

Let $t_{\varepsilon,\gamma} = \varepsilon^{\gamma-1} - 1$ and let $\nu_{\varepsilon,\gamma}$ be the measure defined as in (3.23) with Ω replaced by $B_\sigma(x_i)$ and t_ε replaced by $t_{\varepsilon,\gamma}$. As in the previous step, since $\gamma > 0$ we deduce that $\|\nu_{\varepsilon,\gamma} - d_i\delta_{x_i}\|_{\text{flat}} \to 0$; moreover, for any $B \in \mathcal{B}_{i,\varepsilon}(t_{\varepsilon,\gamma})$ we have

$$XY_\varepsilon(v_\varepsilon, B) \geq \pi|\nu_{\varepsilon,\gamma}(B)|(1-\gamma)|\log\varepsilon|. \quad (2.37)$$

Now, if $\liminf_{\varepsilon\to 0} |\nu_{\varepsilon,\gamma}|(B_\sigma(x_i)) > |d_i|$, then, thanks to (3.26), (3.10) holds true. Otherwise we can assume that $|\nu_{\varepsilon,\gamma}|(B_\sigma(x_i)) = |d_i|$ for ε small enough. Then $\nu_{\varepsilon,\gamma}$ is a sum of Dirac masses concentrated on points which converge to x_i, with weights all having the same sign and summing to d_i. Let $C_1 > 0$

be given and set $\bar{t}_\varepsilon := \frac{\sigma}{C_1(\sharp I_{i,\varepsilon})^4\varepsilon} - 1$. By (2.29), we have that any ball $B \in \mathcal{B}_{i,\varepsilon}(\bar{t}_\varepsilon)$ satisfies

$$\mathrm{diam}(B) \leq \frac{C_2}{C_1}\sigma,$$

where $C_2 > 0$ is a universal constant. We fix $C_1 > 2C_2$ so that $\mathrm{diam}(B) < \frac{\sigma}{2}$. Recall that, for ε small enough, $\mathrm{supp}(\nu_{\varepsilon,\gamma}) \subseteq B_{\sigma/2}(x_i)$; hence if $B \in \mathcal{B}_{i,\varepsilon}(\bar{t}_\varepsilon)$ with $\mathrm{supp}(\nu_{\varepsilon,\gamma}) \cap B \neq \emptyset$, then $B \subseteq B_\sigma(x_i)$ and one can easily show that

$$\mu(\bar{v}_\varepsilon)\Big(\bigcup_{\substack{B\in\mathcal{B}_{i,\varepsilon}(\bar{t}_\varepsilon)\\ B\subset B_\sigma(x_i)}} B\Big) = d_i.$$

We have immediately that

$$XY_\varepsilon(\bar{v}_\varepsilon, B_\sigma(x_i)\backslash\cup_{B\in\mathcal{B}_{i,\varepsilon}}B) \geq \pi\sum_{\substack{B\in\mathcal{B}_{i,\varepsilon}(\bar{t}_\varepsilon)\\ B\subset B_\sigma(x_i)}} |\mu(\bar{v}_\varepsilon)(B)|\log(1+\bar{t}_\varepsilon) \geq \pi|d_i|\log\frac{\sigma}{C_1(\sharp I_{i,\varepsilon})^4\varepsilon}.$$

On the other hand, by Proposition 2.4 there exists a positive constant β such that

$$XY_\varepsilon(\bar{v}_\varepsilon, j+\varepsilon Q) \geq \beta \quad \text{for any } j \in I_{i,\varepsilon};$$

therefore, $XY_\varepsilon(\bar{v}_\varepsilon, \bigcup_{B\in\mathcal{B}_{i,\varepsilon}} B) \geq \beta\sharp I_{i,\varepsilon}$. Finally, we get

$$\begin{aligned} XY_\varepsilon(\bar{v}_\varepsilon, B_\sigma(x_i)) &\geq XY_\varepsilon(\bar{v}_\varepsilon, B_\sigma(x_i)\setminus\cup_{B\in\mathcal{B}_{i,\varepsilon}}B) + XY_\varepsilon(\bar{v}_\varepsilon, \cup_{B\in\mathcal{B}_{i,\varepsilon}}B) \\ &\geq \pi|d_i|\log\frac{\sigma}{\varepsilon} - \log\left(C_1(\sharp I_{i,\varepsilon})^4\right) + \sharp I_{i,\varepsilon}\beta \geq \pi|d_i|\log\frac{\sigma}{\varepsilon} + C \end{aligned}$$

and (3.10) follows sending $\varepsilon \to 0$. □

2.3. Γ-expansion for F_ε

In this section we will derive the Γ-expansion of the functionals F_ε, analogously to what we state in the continuous Ginzburg-Landau framework (see Theorem 1.8). More precisely, we show that the Γ-limit of $F_\varepsilon - \pi|\mu|(\Omega)|\log\varepsilon|$ is given by the renormalized energy in (1.5) plus a term which accounts for the discrete core energy to this order. Indeed, whereas the renormalized energy is the same as in the continuous Ginzburg-Landau framework, the core energy depends on the potential f of the functional F_ε. To this purpose, instead of (1.8), we consider the following discrete minimum problem

$$\gamma_F(\varepsilon,\sigma) := \min_{u\in\mathcal{AF}_\varepsilon(B_\sigma)} \left\{F_\varepsilon(u, B_\sigma) \,:\, 2\pi u(\cdot) = \theta(\cdot) \text{ on } \partial_\varepsilon B_\sigma\right\}, \tag{2.38}$$

where the discrete boundary ∂_ε is defined in (2.1) and $\theta(x)$ is the polar coordinate $\arctan x_2/x_1$, also referred to as the lifting of the function $\frac{x}{|x|}$.

Proposition 2.5. *For any fixed $\sigma > 0$, the following limit exists finite*

$$\lim_{\varepsilon\to 0}(\gamma_F(\varepsilon,\sigma) - \pi|\log\frac{\varepsilon}{\sigma}|) =: \gamma_F \in \mathbb{R}. \tag{2.39}$$

Proof of (2.39). First, by scaling, it is easy to see that $\gamma_F(\varepsilon,\sigma) = I(\frac{\varepsilon}{\sigma})$ where $I(t)$ is defined by

$$I(t) := \min\left\{F_1(u, B_{\frac{1}{t}}) \mid 2\pi u = \theta \text{ on } \partial_1 B_{\frac{1}{t}}\right\}.$$

We aim to prove that

$$0 < t_1 \le t_2 \;\Rightarrow\; I(t_1) \le \pi \log \frac{t_2}{t_1} + I(t_2) + O(t_2). \tag{2.40}$$

Notice that by (2.40) it easily follows that $\lim_{t\to 0^+}(I(t) - \pi|\log t|)$ exists and is not $+\infty$. Moreover, by Theorem 2.2, there exists a universal constant C such that

$$I(t) \ge \pi|\log t| + C \qquad \forall t \in (0,1].$$

We conclude that $\lim_{t\to 0^+}(I(t) - \pi|\log t|)$ is not $-\infty$.

In order to complete the proof we have to show that (2.40) holds. To this end, set $A_{r,R} = B_R \setminus B_r$, and let θ be the lifting of the function $\frac{x}{|x|}$. Since $|\nabla\theta(x)| \le c/r$ for every $x \in A_{r,R}$, by standard interpolation estimates (see for instance [25]) and using assumption (3) on f, we have that, as $r < R \to \infty$,

$$F_1(\theta/2\pi, A_{r,R}) \le \pi \log \frac{R}{r} + O(1/r). \tag{2.41}$$

Let u_2 be a minimizer for $I(t_2)$ and for any $i \in \mathbb{Z}^2$ define

$$u_1(i) := \begin{cases} u_2(i) & \text{if } |i| \le \frac{1}{t_2} \\ \frac{\theta(i)}{2\pi} & \text{if } \frac{1}{t_2} \le |i| \le \frac{1}{t_1}, \end{cases}$$

By (2.41) we have

$$\begin{aligned} I(1/R) \le \sum_{\substack{(i,j)\in(B_r)^1_1 \\ i,j\in(B_r)_1}} f(u_1(i) - u_1(j)) + \sum_{\substack{(i,j)\in(A_{r-\sqrt{2},R})^1_1 \\ i,j\in(A_{r-\sqrt{2},R})_1}} f(u_1(i) - u_1(j)) \\ \le I(1/r) + \pi \log \frac{r}{R} + O(1/r), \end{aligned}$$

which yields (2.40) for $r = \frac{1}{t_2}$ and $R = \frac{1}{t_1}$. $\square$

2.3.1. The main Γ-convergence result. We are now in a position to state the Γ-expansion result for the functionals F_ε.

Theorem 2.6. *The following Γ-convergence result holds.*

(i) *(Compactness) Let $M \in \mathbb{N}$ and let $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ be a sequence satisfying $F_\varepsilon(u_\varepsilon) - M\pi|\log\varepsilon| \le C$. Then, up to a subsequence, $\mu(u_\varepsilon) \stackrel{\text{flat}}{\to} \mu$ for some $\mu = \sum_{i=1}^N d_i\delta_{x_i}$ with $d_i \in \mathbb{Z}\setminus\{0\}$, $x_i \in \Omega$ and $\sum_i |d_i| \le M$. Moreover, if $\sum_i |d_i| = M$, then $\sum_i |d_i| = N = M$, namely $|d_i| = 1$ for any i.*

(ii) *(Γ-lim inf inequality) Let $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ be such that $\mu(u_\varepsilon) \stackrel{\text{flat}}{\to} \mu$, with $\mu = \sum_{i=1}^M d_i\delta_{x_i}$ with $|d_i| = 1$ and $x_i \in \Omega$ for every i. Then,*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) - M\pi|\log\varepsilon| \ge \mathbb{W}(\mu) + M\gamma. \tag{2.42}$$

(iii) *(Γ-lim sup inequality) Given $\mu = \sum_{i=1}^M d_i\delta_{x_i}$ with $|d_i| = 1$ and $x_i \in \Omega$ for every i, there exists $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ with $\mu(u_\varepsilon) \stackrel{\text{flat}}{\to} \mu$ such that*

$$F_\varepsilon(u_\varepsilon) - M\pi|\log\varepsilon| \to \mathbb{W}(\mu) + M\gamma.$$

In our analysis it will be convenient to introduce the energy functionals F_ε in term of the variable μ, i.e., by minimizing F_ε with respect to all $u \in \mathcal{AF}_\varepsilon(\Omega)$ with $\mu(u) = \mu$. Precisely, let $\mathcal{F}_\varepsilon : X \to [0, +\infty]$ be defined by

$$\mathcal{F}_\varepsilon(\mu) := \inf \{F_\varepsilon(u) \ : \ u \in \mathcal{AF}_\varepsilon(\Omega), \mu(u) = \mu\} . \tag{2.43}$$

Theorem 2.6 can be rewritten in terms of $\mathcal{F}_\varepsilon$ as follows.

Theorem 2.7. *The following Γ-convergence result holds.*

(i) *(Compactness) Let $M \in \mathbb{N}$ and let $\{\mu_\varepsilon\} \subset X$ be a sequence satisfying $\mathcal{F}_\varepsilon(\mu_\varepsilon) - M\pi|\log \varepsilon| \le C$. Then, up to a subsequence, $\mu_\varepsilon \stackrel{\text{flat}}{\to} \mu = \sum_{i=1}^N d_i \delta_{x_i}$ with $d_i \in \mathbb{Z} \setminus \{0\}$, $x_i \in \Omega$ and $\sum_i |d_i| \le M$. Moreover, if $\sum_i |d_i| = M$, then $\sum_i |d_i| = N = M$, namely $|d_i| = 1$ for every i.*

(ii) *(Γ-$\liminf$ inequality) Let $\{\mu_\varepsilon\} \subset X$ be such that $\mu_\varepsilon \stackrel{\text{flat}}{\to} \mu = \sum_{i=1}^M d_i \delta_{x_i}$ with $|d_i| = 1$ and $x_i \in \Omega$ for every i. Then,*

$$\liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(\mu_\varepsilon) - M\pi|\log \varepsilon| \ge \mathbb{W}(\mu) + M\gamma. \tag{2.44}$$

(iii) *(Γ-$\limsup$ inequality) Given $\mu = \sum_{i=1}^M d_i \delta_{x_i}$ with $|d_i| = 1$ and $x_i \in \Omega$ for every i, there exists $\{\mu_\varepsilon\} \subset X$ with $\mu_\varepsilon \stackrel{\text{flat}}{\to} \mu$ such that*

$$\mathcal{F}_\varepsilon(\mu_\varepsilon) - M\pi|\log \varepsilon| \to \mathbb{W}(\mu) + M\gamma. \tag{2.45}$$

2.3.2. The proof of Theorem 2.6. Recalling that $F_\varepsilon(u) \ge XY_\varepsilon(e^{2\pi i u})$, the proof of the compactness property (i) will be done for $F_\varepsilon = XY_\varepsilon$, and will be deduced by Theorem 2.2. On the other hand, the constant γ in the definition of the Γ-limit depends on the details of the discrete energy F_ε, and its derivation requires a specific proof.

Proof of (i): Compactness. The fact that, up to a subsequence, $\mu(u_\varepsilon) \stackrel{\text{flat}}{\to} \mu = \sum_{i=1}^N d_i \delta_{x_i}$ with $\sum_{i=1}^N |d_i| \le M$ is a direct consequence of the zero order Γ-convergence result stated in Theorem 2.2 (i). Assume now $\sum_{i=1}^N |d_i| = M$ and let us prove that $|d_i| = 1$. Let $0 < \sigma_1 < \sigma_2$ be such that $B_{\sigma_2}(x_i)$ are pairwise disjoint and contained in Ω and let ε be small enough so that $B_{\sigma_2}(x_i)$ are contained in Ω_ε. For any $0 < r < R$ and $x \in \mathbb{R}^2$, set $A_{r,R}(x) := B_R(x) \setminus B_r(x)$. Since $F_\varepsilon(u_\varepsilon) \ge XY_\varepsilon(e^{2\pi i u_\varepsilon})$,

$$F_\varepsilon(u_\varepsilon) \ge \sum_{i=1}^N XY_\varepsilon(e^{2\pi i u_\varepsilon}, B_{\sigma_1}(x_i)) + \sum_{i=1}^N XY_\varepsilon(e^{2\pi i u_\varepsilon}, A_{\sigma_1,\sigma_2}(x_i)). \tag{2.46}$$

To ease notation we set $v_\varepsilon = e^{2\pi i u_\varepsilon}$ and we indicate with $\tilde{v}_\varepsilon$ the piecewise affine interpolation of v_ε. Moreover let t be a positive number and let ε be small enough so that $t > \sqrt{2}\varepsilon$. Then, by (3.10) and (2.13), we get

$$F_\varepsilon(u_\varepsilon) \ge \pi \sum_{i=1}^N |d_i| \log \frac{\sigma_1}{\varepsilon} + \frac{1}{2} \sum_{i=1}^N \int_{A_{\sigma_1+t,\sigma_2-t}(x_i)} |\nabla \tilde{v}_\varepsilon|^2 \, \mathrm{d}x + C. \tag{2.47}$$

By the energy bound, we deduce that $\int_{A_{\sigma_1+t,\sigma_2-t}(x_i)} |\nabla \tilde{v}_\varepsilon|^2 \, \mathrm{d}x \le C$ and hence, up to a subsequence, $\tilde{v}_\varepsilon \rightharpoonup v_i$ in $H^1(A_{\sigma_1+t,\sigma_2-t}(x_i); \mathbb{R}^2)$ for some field v_i. Moreover, since

$$\frac{1}{\varepsilon^2}\int_{A_{\sigma_1+t,\sigma_2-t}(x_i)}(1-|\tilde v_\varepsilon|^2)^2\,\mathrm{d}x\le CXY_\varepsilon(v_\varepsilon)\le C\log\frac{1}{\varepsilon},$$

(see Lemma 2 in [2] for more details), we deduce that $|v_i|=1$ a.e.

Furthermore, by standard Fubini's arguments, for a.e. $\sigma_1+t<\sigma<\sigma_2-t$, up to a subsequence the trace of $\tilde v_\varepsilon$ is bounded in $H^1(\partial B_\sigma(x_i);\mathbb{R}^2)$, and hence it converges uniformly to the trace of v_i. By the very definition of degree it follows that $\deg(v_i,\partial B_\sigma(x_i))=d_i$.

Hence, by (1.12), for every i we have

$$\frac{1}{2}\int_{A_{\sigma_1+t,\sigma_2-t}(x_i)}|\nabla v_i|^2\,\mathrm{d}x\ge|d_i|^2\pi\log\frac{\sigma_2-t}{\sigma_1+t}. \tag{2.48}$$

By (3.34) and (3.35), we conclude that for ε sufficiently small

$$\begin{aligned}F_\varepsilon(u_\varepsilon)&\ge\pi\sum_{i=1}^N\left(|d_i|\log\frac{\sigma_1}{\varepsilon}+|d_i|^2\log\frac{\sigma_2-t}{\sigma_1+t}\right)+C\\&\ge M\pi|\log\varepsilon|+\pi\sum_{i=1}^N(|d_i|^2-|d_i|)\log\frac{\sigma_2}{\sigma_1}+\pi\sum_{i=1}^N|d_i|^2\log\frac{\sigma_1(\sigma_2-t)}{\sigma_2(\sigma_1+t)}+C.\end{aligned}$$

The energy bound yields that the sum of the last two terms is bounded; letting $t\to0$ and $\sigma_1\to0$, we conclude $|d_i|=1$.

Proof of (ii): Γ-liminf inequality. Fix $r>0$ so that the balls $B_r(x_i)$ are pairwise disjoint and compactly contained in Ω. Let moreover $\{\Omega^h\}$ be an increasing sequence of open smooth sets compactly contained in Ω such that $\cup_{h\in\mathbb{N}}\Omega^h=\Omega$. Without loss of generality we can assume that $F_\varepsilon(u_\varepsilon)\le M\pi|\log\varepsilon|+C$, which together with Theorem 2.2 yields

$$F_\varepsilon(u_\varepsilon,\Omega\setminus\bigcup_{i=1}^M B_r(x_i))\le C. \tag{2.49}$$

We set $v_\varepsilon:=e^{2\pi i u_\varepsilon}$ and we denote by $\tilde v_\varepsilon$ the piecewise affine interpolation of v_ε. For every $r>0$, by (3.36) and by (2.12) we deduce $XY_\varepsilon(v_\varepsilon\setminus\bigcup_{i=1}^N B_r(x_i))\le C$. Fix $h\in\mathbb{N}$ and let ε be small enough so that $\Omega^h\subset\Omega_\varepsilon$. Then,

$$\frac{1}{2}\int_{\Omega^h\setminus\bigcup_{i=1}^N B_{2r}(x_i)}|\nabla\tilde v_\varepsilon|^2\,\mathrm{d}x\le C;$$

therefore, by a diagonalization argument, there exists a unitary field v with $v\in H^1(\Omega\setminus\cup_{i=1}^M B_\rho(x_i);\mathcal{S}^1)$ for any $\rho>0$ and a subsequence $\{\tilde v_\varepsilon\}$ such that $\tilde v_\varepsilon\rightharpoonup v$ in $H^1_{\mathrm{loc}}(\Omega\setminus\cup_{i=1}^M\{x_i\};\mathbb{R}^2)$.

Let $\sigma>0$ be such that $B_\sigma(x_i)$ are pairwise disjoint and contained in Ω^h. For any $0<r<R<+\infty$ and for any $x\in\mathbb{R}^2$, set $A_{r,R}(x):=B_R(x)\setminus B_r(x)$, $A_{r,R}:=A_{r,R}(0)$. Let $t\le\sigma$, and consider the minimization problem

$$\min_{w\in H^1(A_{t/2,t};\mathcal{S}^1)}\left\{\frac{1}{2}\int_{A_{t/2,t}}|\nabla w|^2\,\mathrm{d}x\,:\,\deg(w,\partial B_{\frac{t}{2}})=1\right\}.$$

It is easy to see that the minimum is $\pi \log 2$ and that the set of minimizers is given by (the restriction at $A_{t/2,t}$ of the functions in)

$$\mathcal{K} := \left\{ \alpha \frac{z}{|z|} \,:\, \alpha \in \mathbb{C}, |\alpha| = 1 \right\}. \tag{2.50}$$

Set

$$\mathrm{d}_t(w, \mathcal{K}) := \min \left\{ \|\nabla w - \nabla v\|_{L^2(A_{t/2,t};\mathbb{R}^2)} \,:\, v \in \mathcal{K} \right\}. \tag{2.51}$$

It is easy to see that for any given $\delta > 0$ there exists a positive $\omega(\delta)$ (independent of t) such that if $\mathrm{d}_t(\tilde{v}_\varepsilon(\cdot + x_i), \mathcal{K}) \geq \delta$, then

$$\liminf_{\varepsilon \to 0} \frac{1}{2} \int_{A_{\frac{t}{2}+\sqrt{2}\varepsilon, t-\sqrt{2}\varepsilon}(x_i)} |\nabla \tilde{v}_\varepsilon|^2 \,\mathrm{d}x \geq \pi \log 2 + \omega(\delta). \tag{2.52}$$

By a scaling argument we can assume $t = 1$. Then, arguing by contradiction, if there exists a subsequence $\{\tilde{v}_\varepsilon\}$ such that

$$\lim_{\varepsilon \to 0} \frac{1}{2} \int_{A_{\frac{1}{2}+\sqrt{2}\varepsilon, 1-\sqrt{2}\varepsilon}(x_i)} |\nabla \tilde{v}_\varepsilon|^2 \,\mathrm{d}x = \pi \log 2,$$

then, by the lower semicontinuity of the L^2 norm, we get

$$\pi \log 2 \leq \frac{1}{2} \int_{A_{1/2,1}(x_i)} |\nabla v|^2 \,\mathrm{d}x \leq \lim_{\varepsilon \to 0} \frac{1}{2} \int_{A_{\frac{1}{2}+\sqrt{2}\varepsilon, 1-\sqrt{2}\varepsilon}(x_i)} |\nabla \tilde{v}_\varepsilon|^2 \,\mathrm{d}x = \pi \log 2. \tag{2.53}$$

It follows that $v(\cdot + x_i) \in \mathcal{K}$, and that $\tilde{v}_\varepsilon \to v$ strongly in $H^1(A_{1/2,1}(x_i); \mathbb{R}^2)$, which yields the contradiction $\mathrm{dist}(v(\cdot + x_i), \mathcal{K}) \geq \delta$.

Let $L \in \mathbb{N}$ be such that $L\,\omega(\delta) \geq \mathbb{W}(\mu) + M(\gamma - \pi \log \sigma - C)$ where C is the constant in (3.10). For $l = 1, \ldots, L$, set $C_l(x_i) := B_{2^{1-l}\sigma}(x_i) \setminus B_{2^{-l}\sigma}(x_i)$.

We distinguish among two cases.

First case: for ε small enough and for every fixed $1 \leq l \leq L$, there exists at least one i such that $\mathrm{d}_{2^{1-l}\sigma}(\tilde{v}_\varepsilon(\cdot + x_i), \mathcal{K}) \geq \delta$. Then, by (3.10), (3.39) and the lower semicontinuity of the L^2 norm, we conclude

$$\begin{aligned} F_\varepsilon(u_\varepsilon, \Omega^h) &\geq \sum_{i=1}^{M} XY_\varepsilon(v_\varepsilon, B_{2^{-L}\sigma}(x_i)) + \sum_{l=1}^{L} \sum_{i=1}^{M} XY_\varepsilon(v_\varepsilon, C_l(x_i)) \\ &\geq M(\pi \log \frac{\sigma}{2^L} + \pi |\log \varepsilon| + C) + L(M\pi \log 2 + \omega(\delta)) + \mathrm{o}(\varepsilon) \\ &\geq M\pi |\log \varepsilon| + M\gamma + \mathbb{W}(\mu) + \mathrm{o}(\varepsilon). \end{aligned}$$

Second case: Up to a subsequence, there exists $1 \leq \bar{l} \leq L$ such that for every i we have $\mathrm{d}_{\bar{\sigma}}(\tilde{v}_\varepsilon(\cdot + x_i), \mathcal{K}) \leq \delta$, where $\bar{\sigma} := 2^{1-\bar{l}}\sigma$. Let $\alpha_{\varepsilon,i}$ be the unitary vector such that $\|\tilde{v}_\varepsilon - \alpha_{\varepsilon,i} \frac{x-x_i}{|x-x_i|}\|_{H^1(C_{\bar{l}}(x_i);\mathbb{R}^2)} = \mathrm{d}_{\bar{\sigma}}(\tilde{v}_\varepsilon(\cdot + x_i), \mathcal{K})$.

One can construct a function $\bar{u}_\varepsilon \in \mathcal{AF}_\varepsilon(\Omega)$ such that

(i) $\bar{u}_\varepsilon = u_\varepsilon$ on $\partial_\varepsilon(\mathbb{R}^2 \setminus B_{2^{-\bar{l}}\sigma}(x_i))$;
(ii) $e^{2\pi i \bar{u}_\varepsilon} = \alpha_\varepsilon e^{i\theta}$ on $\partial_\varepsilon B_{2^{1-\bar{l}}\sigma}(x_i)$
(iii) $F_\varepsilon(u_\varepsilon, B_{\bar{\sigma}}(x_i)) \geq F_\varepsilon(\bar{u}_\varepsilon, B_{\bar{\sigma}}(x_i)) + r(\varepsilon, \delta)$ with $\lim_{\delta \to 0} \lim_{\varepsilon \to 0} r(\varepsilon, \delta) = 0$.

The proof of (i)-(iii) is quite technical, and consists in adapting standard cut-off arguments to our discrete setting. For the reader convenience we skip the details of the proof, and assuming (i)-(iii) we conclude the proof of the lower bound.

By Theorem (1.7), we have that

$$
\begin{aligned}
F_\varepsilon(u_\varepsilon) \geq XY_\varepsilon(v_\varepsilon, \Omega^h \setminus \bigcup_{i=1}^M B_{\bar\sigma}(x_i)) + \sum_{i=1}^M F_\varepsilon(u_\varepsilon, B_{\bar\sigma}(x_i))\\
\geq \frac{1}{2}\int_{\Omega^h \setminus \bigcup_{i=1}^M B_{\bar\sigma}(x_i)} |\nabla \tilde v_\varepsilon|^2 \,\mathrm{d}x + \sum_{i=1}^M F_\varepsilon(\bar u_\varepsilon, B_{\bar\sigma}(x_i)) + r(\varepsilon,\delta) + \mathrm{o}(\varepsilon)\\
\geq \frac{1}{2}\int_{\Omega^h \setminus \bigcup_{i=1}^M B_{\bar\sigma}(x_i)} |\nabla \tilde v_\varepsilon|^2 \,\mathrm{d}x + M(\gamma - \pi \log\frac{\varepsilon}{\bar\sigma}) + r(\varepsilon,\delta) + \mathrm{o}(\varepsilon)\\
\geq \frac{1}{2}\int_{\Omega \setminus \bigcup_{i=1}^M B_{\bar\sigma}(x_i)} |\nabla v|^2 \,\mathrm{d}x + M(\gamma - \pi \log\frac{\varepsilon}{\bar\sigma}) + r(\varepsilon,\delta) + \mathrm{o}(\varepsilon) + \mathrm{o}(1/h)\\
\geq M\pi|\log\varepsilon| + M\gamma + \mathbb{W}(\mu) + r(\varepsilon,\delta) + \mathrm{o}(\varepsilon) + \mathrm{o}(\bar\sigma) + \mathrm{o}(1/h).
\end{aligned}
$$

The proof follows sending $\varepsilon \to 0$, $\delta \to 0$, $\sigma \to 0$ and $h \to \infty$.

Proof of (iii): Γ*-limsup inequality.* This proof is standard in the continuous case, and we only sketch its discrete counterpart. Let w_σ be a function that agrees with a minimizer of (3.28) in $\Omega \setminus \bigcup_{i=1}^M B_\sigma(x_i) =: \Omega^\sigma$. Then, $w_\sigma = \alpha_i \frac{x - x_i}{\sigma}$ on $\partial B_\sigma(x_i)$ for some $|\alpha_i| = 1$.

For every $\rho > 0$ we can always find a function $w_{\sigma,\rho} \in C^\infty(\overline{\Omega^\sigma}; \mathcal{S}^1)$ such that $w_{\sigma,\rho} = \alpha_i \frac{x - x_i}{\sigma}$ on $\partial B_\sigma(x_i)$, and

$$
\frac{1}{2}\int_{\Omega^\sigma} |\nabla w_{\sigma,\rho}|^2 \,\mathrm{d}x - \frac{1}{2}\int_{\Omega^\sigma} |\nabla w_\sigma|^2 \,\mathrm{d}x \leq \rho.
$$

Moreover, for every i let $w_i \in \mathcal{AXY}_\varepsilon(B_\sigma(x_i))$ be a function which agrees with $\alpha_i \frac{x - x_i}{|x - x_i|}$ on $\partial_\varepsilon B_\sigma(x_i)$ and such that its phase minimizes problem (3.29). If necessary, we extend w_i to $(\overline{B_\sigma(x_i)} \cap \varepsilon\mathbb{Z}^2) \setminus (B_\sigma(x_i))^0_\varepsilon$ to be equal to $\alpha_i \frac{x - x_i}{|x - x_i|}$. Finally, define the function $w_{\varepsilon,\sigma,\rho} \in \mathcal{AXY}_\varepsilon(\Omega)$ which coincides $w_{\sigma,\rho}$ on $\Omega^\sigma \cap \varepsilon\mathbb{Z}^2$ and with w_i on $\overline{B_\sigma(x_i)} \cap \varepsilon\mathbb{Z}^2$. Then, in view of assumption (3) on f, a straightforward computation shows that any phase $u_{\varepsilon,\sigma,\rho}$ of $w_{\varepsilon,\sigma,\rho}$ is a recovery sequence, i.e.,

$$
\lim_{\varepsilon \to 0} F_\varepsilon(u_{\varepsilon,\sigma,\rho}) - M\pi|\log\varepsilon| = M\gamma + \mathbb{W}(\mu) + o(\rho,\sigma),
$$

with $\lim_{\sigma \to 0} \lim_{\rho \to 0} o(\rho,\sigma) = 0$.

2.3.3. Γ-convergence analysis in the L^2 topology. Here we prove a Γ-convergence result for $F_\varepsilon(u_\varepsilon) - M\pi|\log\varepsilon|$, where M is fixed positive integer, with respect to the flat convergence of $\mu(u_\varepsilon)$ and the L^2-convergence of $\tilde v_\varepsilon$, where $\tilde v_\varepsilon : \Omega_\varepsilon \to \mathbb{R}^2$ is the piecewise affine interpolation of $e^{2\pi i u_\varepsilon}$.

To this purpose, for $N \in \mathbb{N}$ let us first introduce the set

$$
\begin{aligned}
\mathcal{D}_N := \{v \in L^2(\Omega; \mathcal{S}^1) : Jv = \pi \sum_{i=1}^N d_i \delta_{x_i} \text{ with } |d_i| = 1,\, x_i \in \Omega,\\
v \in H^1_{\mathrm{loc}}(\Omega \setminus \mathrm{supp}(Jv); \mathcal{S}^1)\}.
\end{aligned}
\tag{2.54}
$$

Notice that, if $v \in \mathcal{D}_M$, then the function

$$\frac{1}{2}\int_{\Omega\setminus\bigcup_{i=1}^M B_\sigma(x_i)} |\nabla v|^2 \,\mathrm{d}x - M\pi|\log\sigma|,$$

is monotonically decreasing with respect to σ. Therefore, it is well defined the functional $\mathcal{W} : L^2(\Omega;\mathcal{S}^1) \to \bar{\mathbb{R}}$ given by

$$\mathcal{W}(v) = \begin{cases} \lim_{\sigma\to 0}\frac{1}{2}\int_{\Omega\setminus\bigcup_{i=1}^M B_\sigma(x_i)} |\nabla v|^2 \,\mathrm{d}x - M\pi|\log\sigma| & \text{if } v \in \mathcal{D}_M; \\ -\infty & \text{if } v \in \mathcal{D}_N \text{ for some } N < M; \\ +\infty & \text{otherwise} \end{cases} \tag{2.55}$$

Notice that, by (1.9) we have that, for every $\mu = \sum_{i=1}^M d_i\delta_{x_i}$ with $|d_i| = 1$

$$\mathbb{W}(\mu) = \min_{\substack{v\in H^1_{\mathrm{loc}}(\Omega\setminus\mathrm{supp}(\mu);\mathcal{S}^1)\\ Jv=\mu}} \mathcal{W}(v). \tag{2.56}$$

Remark 2.8. We can rewrite $\mathcal{W}(v)$ as follows

$$\mathcal{W}(v) = \frac{1}{2}\int_{\Omega\setminus\cup_i B_\rho(x_i)} |\nabla v|^2 \,\mathrm{d}x + M\pi\log\rho + \sum_{i=1}^M\sum_{j=0}^{+\infty}\left(\frac{1}{2}\int_{C_{i,j}} |\nabla v|^2 \,\mathrm{d}x - \pi\log 2\right),$$

where $C_{i,j}$ denotes the annulus $B_{2^{-j}\rho}(x_i) \setminus B_{2^{-(j+1)}\rho}(x_i)$. In particular, for the lower bound (1.12) we deduce that

$$\sup_{i,j}\frac{1}{2}\int_{C_{i,j}} |\nabla v|^2 \,\mathrm{d}x \le \pi\log 2 + \mathcal{W}(v) - M\pi\log\rho. \tag{2.57}$$

Theorem 2.9. *Let $M \in \mathbb{N}$ be fixed. The following Γ-convergence result holds.*

(i) (Compactness) Let $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ be such that $F_\varepsilon(u_\varepsilon) \le M\pi|\log\varepsilon| + C$. Then, up to a subsequence, $\mu(u_\varepsilon) \stackrel{\mathrm{flat}}{\to} \mu = \sum_{i=1}^N d_i\delta_{x_i}$ with $d_i \in \mathbb{Z}\setminus\{0\}$, $x_i \in \Omega$ and $\sum_{i=1}^N |d_i| \le M$. Moreover, if $\sum_{i=1}^N |d_i| = M$, then $|d_i| = 1$ and up to a further subsequence $\tilde{v}_\varepsilon \rightharpoonup v$ in $H^1_{\mathrm{loc}}(\Omega\setminus\mathrm{supp}(\mu);\mathbb{R}^2)$ for some $v \in \mathcal{D}_M$.

(ii) (Γ-liminf inequality) Let $v \in \mathcal{D}_M$ and let $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ be such that $\mu(u_\varepsilon) \stackrel{\mathrm{flat}}{\to} Jv$ and $\tilde{v}_\varepsilon \to v$ in $L^2(\Omega;\mathbb{R}^2)$. Then,

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) - M\pi|\log\varepsilon| \ge \mathcal{W}(v) + M\gamma. \tag{2.58}$$

(iii) (Γ-limsup inequality) Given $v \in \mathcal{D}_M$, there exists $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ such that $\mu(u_\varepsilon) \stackrel{\mathrm{flat}}{\to} Jv$, $\tilde{v}_\varepsilon \rightharpoonup v$ in $H^1_{\mathrm{loc}}(\Omega\setminus\mathrm{supp}(Jv);\mathbb{R}^2)$ and

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) - M\pi|\log\varepsilon| = \mathcal{W}(v) + M\gamma. \tag{2.59}$$

Proof. *Proof of (i).* The compactness properties concerning the sequence $\{\mu(u_\varepsilon)\}$ are given in Theorem 2.6 (i) while the weak convergence up to a subsequence of $\{\tilde{v}_\varepsilon\}$ to a unitary field v such that $v \in \mathcal{D}_M$ has been shown in the first lines of the proof of Theorem 2.6 (ii).

Proof of (ii). The proof of Γ-liminf inequality follows strictly the one of Theorem 2.6 (ii) and we leave it to the reader.

Proof of (iii). Let $Jv = \pi\sum_{i=1}^M d_i\delta_{x_i}$, with $x_i \in \Omega$, $|d_i| = 1$. Fix $\sigma > 0$ and $\Omega^\sigma := \Omega \setminus \cup_{i=1}^M B_\sigma(x_i)$. Without loss of generality we can assume that $\mathcal{W}(v) < +\infty$ and hence for some fixed constant $C > 0$ and for every σ

$$\frac{1}{2}\int_{\Omega^\sigma} |\nabla v|^2 \, \mathrm{d}x \le M\pi|\log\sigma| + C.$$

Now, fix $\sigma > 0$, and let $C_{i,j}$ denote the annulus $B_{2^{-j}\sigma}(x_i) \setminus B_{2^{-(j+1)}\sigma}(x_i)$. By Remark 2.8, it follows that for every $i = 1, \dots, M$

$$\lim_{j\to\infty} \frac{1}{2}\int_{C_{i,j}} |\nabla v|^2 \, \mathrm{d}x = \pi\log 2. \tag{2.60}$$

Recall that $\pi\log 2$ is the minimal possible energy in each annulus, and that the class of minimizers is given by the set $\mathcal{K}$ defined in (3.37). Using standard scaling arguments and (2.60), one can show (see (2.53)) that for any $j \in \mathbb{N}$ there exists a unitary vector $\alpha_{i,j}$ such that

$$\frac{1}{2}\int_{C_{i,j}} \left|\nabla\left(v - \alpha_{i,j}\frac{x - x_i}{|x - x_i|}\right)\right|^2 \mathrm{d}x = r(i,j), \tag{2.61}$$

with $\lim_{j\to\infty} r(i,j) = 0$. Moreover, we can find a function $w_j \in C^\infty(\overline{\Omega^{2^{-j}\sigma}}; \mathcal{S}^1)$ such that

$$\frac{1}{2}\int_{\Omega^{2^{-j}\sigma}} |\nabla w_j - \nabla v|^2 \, \mathrm{d}x \le \frac{1}{j}. \tag{2.62}$$

Let $\varphi \in C^1([\frac{1}{2}, 1]; [0,1])$ be such that $\varphi(\frac{1}{2}) = 1$ and $\varphi(1) = 0$, and let define the function $v_{i,j}$ in $C_{i,j}$, with

$$v_{i,j}(x) := \varphi(2^j\sigma^{-1}|x - x_i|)\alpha_{i,j}\frac{x - x_i}{|x - x_i|} + (1 - \varphi(2^j\sigma^{-1}|x - x_i|))w_j(x).$$

Then define the function v_j as follows

$$v_j = \begin{cases} w_j & \text{in } \Omega^{2^{-j}\sigma} \\ v_{i,j} & \text{in } C_{i,j}\,. \end{cases} \tag{2.63}$$

Finally for every i we denote by $\bar{v}^\varepsilon_{i,j} \in \mathcal{AXY}_\varepsilon(B_{2^{-j-1}\sigma}(x_i))$ a function which agrees with $\alpha_{i,j}\frac{x-x_i}{|x-x_i|}$ on $\partial_\varepsilon B_{2^{-j-1}\sigma}(x_i)$ and such that its phase (up to an additive constant) minimizes problem (3.29). If necessary, we extend $\bar{v}_{i,j}$ to $(\overline{B_{2^{-j-1}\sigma}(x_i)} \cap \varepsilon\mathbb{Z}^2) \setminus (B_{2^{-j-1}\sigma}(x_i))^0_\varepsilon$ to be equal to $\alpha_{i,j}\frac{x-x_i}{|x-x_i|}$. Finally, consider the field the $v_{\varepsilon,j}$ which coincides with v_j on the nodes of $\Omega^{2^{-j-1}\sigma}$ and with $\bar{v}^\varepsilon_{i,j}$ on $\overline{B_{2^{-j}\sigma}(x_i)} \cap \varepsilon\mathbb{Z}^2$. In view of assumption (3) on f, a straightforward computation shows that any phase $u_{\varepsilon,j}$ of $v_{\varepsilon,j}$ satisfies

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_{\varepsilon,j}) - M\pi|\log\varepsilon| = M\gamma + \frac{1}{2}\int_{\Omega^{2^{-j}\sigma}} |\nabla v|^2 \, \mathrm{d}x - M\pi|\log(2^{-j}\sigma)| + \mathrm{o}(j),$$

with $\lim_{j\to\infty} \mathrm{o}(j) = 0$. A standard diagonal argument yields that there exists $j(\varepsilon) \to 0$ such that $u_{\varepsilon,j(\varepsilon)}$ is a recovery sequence in the sense of (2.59). □

CHAPTER 3

Γ-convergence expansion for anisotropic and long range interaction energies

In this chapter we develop the Γ-convergence analysis for anisotropic energies defined in the triangular lattice. Moreover we will show that a similar expansion holds if we consider isotropic long range interactions which satisfy a decay assumption. The results of this chapter are contained in the preprint [32]. We first introduce the notation we need.

3.1. The discrete model

We recall that the properties of the potential f and the definition of the functional spaces $\mathcal{AF}_\varepsilon(\Omega)$ and $\mathcal{AXY}_\varepsilon(\Omega)$ are given in Section 2.1.

The discrete energies. Let $c_{e_1}, c_{e_2} > 0$ and let $c_{e_1+e_2} \geq 0$. For any $u \in \mathcal{AF}_\varepsilon(\Omega)$ we define the anisotropic energy in the triangular lattice as

$$\begin{aligned} F^T_\varepsilon(u) = \frac{1}{2}\sum_{i\in\Omega^2_\varepsilon} & c_{e_1} f(u(i+\varepsilon e_1) - u(i)) + c_{e_2} f(u(i+\varepsilon e_2) - u(i)) \\ & + c_{e_1+e_2} f(u(i+\varepsilon e_1+\varepsilon e_2) - u(i)). \end{aligned} \tag{3.1}$$

We now introduce the isotropic long range interaction energy. Set $\mathbb{Z}^2_+ := \{\xi \in \mathbb{Z}^2 \,:\, \xi\cdot e_2 \geq 0\}$, where $\cdot$ denote the usual scalar product in $\mathbb{R}^2$. Let $\{c_\xi\}_\xi$ be a family of non-negative constants labeled with $\xi \in \mathbb{Z}^2_+$ such that $c_\xi = c_{\xi^\perp}$, $c_{e_1} = c_{e_2} > 0$ and $\sum_{\xi\in\mathbb{Z}^2_+} c_\xi|\xi|^2 < +\infty$. We define the energy

$$F^{lr}_\varepsilon(u) := \frac{1}{2}\sum_{\xi\in\mathbb{Z}^2_+} c_\xi \sum_{\substack{i\in\Omega^0_\varepsilon \\ [i,i+\varepsilon\xi]\subset\Omega_\varepsilon}} f(u(i+\varepsilon\xi)) - u(i)). \tag{3.2}$$

Analogously to the isotropic case, we set

$$\begin{aligned} SD^T_\varepsilon(u) &:= \frac{1}{2}\sum_{i\in\Omega^2_\varepsilon} c_{e_1}\mathrm{dist}^2(u(i+\varepsilon e_1) - u(i), \mathbb{Z}) \\ c_{\varepsilon_2}\mathrm{dist}^2(u(i+\varepsilon e_2) - u(i), \mathbb{Z}) &+ c_{e_1+e_2}\mathrm{dist}^2(u(i+\varepsilon e_1+\varepsilon e_2) - u(i), \mathbb{Z}) \\ SD^{lr}_\varepsilon(u) &:= \frac{1}{2}\sum_{\xi\in\mathbb{Z}^2_+} c_\xi \sum_{\substack{i\in\Omega^0_\varepsilon \\ [i,i+\varepsilon\xi]\subset\Omega_\varepsilon}} \mathrm{dist}^2(u(i+\varepsilon\xi) - u(i), \mathbb{Z}). \end{aligned}$$

and

$$XY^T_\varepsilon(v) := \frac{1}{2}\sum_{i\in\Omega^2_\varepsilon} c_{e_1}|v(i+\varepsilon e_1)-v(i)|^2 + c_{\varepsilon_2}|v(i+\varepsilon e_2)-v(i)|$$
$$+c_{e_1+e_2}|v(i+\varepsilon e_1+\varepsilon e_2)-v(i)|^2,$$
$$XY^{lr}_\varepsilon(v) := \frac{1}{2}\sum_{\xi\in\mathbb{Z}^2_+} c_\xi \sum_{\substack{i\in\Omega^0_\varepsilon\\ [i,i+\varepsilon\xi]\subset\Omega_\varepsilon}} |v(i+\varepsilon\xi)-v(i)|^2$$

Once again, we notice that assumption (2) on f implies

$$F^T_\varepsilon(u) \geq XY^T_\varepsilon(e^{2\pi i u}) \quad . \tag{3.3}$$

as well as $F^{lr}_\varepsilon(u) \geq XY^{lr}_\varepsilon(e^{2\pi i u})$.

For any $v:\Omega^0_\varepsilon\to\mathcal{S}^1$, we recall that $\tilde v:\Omega_\varepsilon\to\mathcal{S}^1$ is the piecewise affine interpolation of v, according with the triangulation $\{i+\varepsilon T^\pm\}$ defined in Section 2.1.

For any $A\subset\Omega$ and for any $w\in H^1(A;\mathbb{R}^2)$ we set

$$\mathcal{F}^T(w,A) := \frac{1}{2}\int_A c_{e_1}|\partial_{e_1}w|^2 + c_{e_2}|\partial_{e_2}w|^2 + 2c_{e_1+e_2}|\partial_{\frac{e_1+e_2}{\sqrt{2}}}w|^2\,\mathrm{d}x. \tag{3.4}$$

It is easy to see that there exists a positive constant C depending only on Ω such that

$$\mathcal{F}^T(\tilde v,\Omega_\varepsilon)+C\varepsilon \geq XY^T_\varepsilon(v) \geq \mathcal{F}^T(\tilde v,\Omega_\varepsilon), \tag{3.5}$$

where $\partial_{\frac{e_1+e_2}{\sqrt{2}}}\tilde v(x) := (\partial_{e_1}\tilde v(i),\partial_{e_2}\tilde v(i))\cdot\frac{e_1+e_2}{\sqrt{2}}$ for any $x\in i+\varepsilon Q = i+\varepsilon[0,1]^2$.

3.2. Γ-expansion for F^T_ε

In this section we develop the Γ-convergence analysis of the functionals F^T_ε as $\varepsilon\to 0$. This analysis is closely related to the one given in the isotropic case in Chapter 2, but it requires some cares due to the presence of the anisotropy coefficients and of the interaction in the direction $\frac{e_1+e_2}{\sqrt{2}}$.

3.2.1. The zero-order Γ-convergence. Here we prove the Γ-converegence result for the energies $\frac{F^T_\varepsilon}{|\log\varepsilon|}$ as $\varepsilon\to 0$. By (3.3), it is enough to prove the compactness property and the Γ-liminf inequality for the functional XY^T_ε whereas the construction of the recovery sequence is standard.

We start with a simple lemma which allows to write the energy in (3.4) as the (multiple of the) Dirichlet energy of a suitably modified field. To simplify the notation, we set $\lambda := \sqrt{c_{e_1}c_{e_2}+c_{e_1}c_{e_1+e_2}+c_{e_2}c_{e_1+e_2}}$.

Lemma 3.1. *Let A be an open subset of Ω and let $w\in H^1(A;\mathbb{R}^2)$. There exist two positive numbers $\lambda_1,\lambda_2>0$ and two orthonormal vectors ν_1,ν_2 depending only on $c_{e_1},c_{e_2},c_{e_1+e_2}$, such that*

$$\mathcal{F}^T(w,A) = \frac{\lambda}{2}\int_A \frac{\lambda_1^2}{\lambda_2^2}|\partial_{\nu_1}w(x)|^2 + \frac{\lambda_2^2}{\lambda_1^2}|\partial_{\nu_2}w(x)|^2\,\mathrm{d}x. \tag{3.6}$$

Proof. It is easy to see that

$$\mathcal{F}^T(w,A) = \frac{1}{2}\int_A (c_{e_1}+c_{e_1+e_2})|\partial_{e_1}w|^2+(c_{e_2}+c_{e_1+e_2})|\partial_{e_2}w|^2+2c_{e_1+e_2}\partial_{e_1}w\partial_{e_2}w\,\mathrm{d}x.$$

Let consider the quadratic form on $\mathbb{R}$ defined by

$$Q(X,Y) := (c_{e_1} + c_{e_1+e_2})X^2 + (c_{e_2} + c_{e_1+e_2})Y^2 + 2c_{e_1+e_2}XY.$$

We first notice that if $c_{e_1+e_2} = 0$ than the matrix associated with Q is diagonal and we set

$$\lambda_1 := \sqrt[4]{c_{e_1}}, \quad \lambda_2 := \sqrt[4]{c_{e_2}}, \quad \nu_1 := e_1, \quad \nu_2 := e_2.$$

From now on, we will focus on the case $c_{e_1+e_2} > 0$.

Set $D := \sqrt{(c_{e_1} - c_{e_2})^2 + 4c^2_{e_1+e_2}}$, a straightforward computation shows that the eigenvalues of Q are given by

$$\eta_1 := \frac{c_{e_1} + c_{e_2} + 2c_{e_1+e_2} - D}{2} \qquad \eta_2 := \frac{c_{e_1} + c_{e_2} + 2c_{e_1+e_2} + D}{2}.$$

Since, by assumption, $c_{e_1}, c_{e_2} > 0$ and $c_{e_1+e_2} \geq 0$, η_1 and η_2 are both strictly positive and hence there exists $\lambda_1, \lambda_2 > 0$ such that $\eta_1 = \lambda_1^4$ and $\eta_1 = \lambda_2^4$. We notice that $\eta_1 \neq \eta_2$ since $c_{e_1+e_2} > 0$. Let consider an orthonormal basis of eigenvectors $\{\nu_1, \nu_2\}$ relative to η_1 and η_2 by setting

$$\nu_1 := \frac{1}{\sqrt{(c_{e_1} - c_{e_2} + D)^2 + 4c^2_{e_1+e_2}}} \left(-2c_{e_1+e_2}, c_{e_1} - c_{e_2} + D\right),$$

$$\nu_2 := \frac{1}{\sqrt{(c_{e_1} - c_{e_2} - D)^2 + 4c^2_{e_1+e_2}}} \left(2c_{e_1+e_2}, -(c_{e_1} - c_{e_2} - D)\right).$$

Since

$$\begin{aligned} \mathcal{F}^T(w, A) &= \frac{1}{2}\int_A \lambda_1^4 |\partial_{\nu_1} w|^2 + \lambda_2^4 |\partial_{\nu_2} w|^2 \,\mathrm{d}x \\ &= \frac{\lambda_1^2\lambda_2^2}{2}\int_A \frac{\lambda_1^2}{\lambda_2^2}|\partial_{\nu_1} w|^2 + \frac{\lambda_2^2}{\lambda_1^2}|\partial_{\nu_2} w|^2 \,\mathrm{d}x, \end{aligned}$$

the conclusion easily follows noticing that $\lambda^2 = \eta_1\eta_2 = \lambda_1^4\lambda_2^4$. □

By Lemma 3.1, a straightforward computation yields the following result.

Corollary 3.2. *Let A be an open subset of Ω, let $w \in H^1(\Omega; \mathbb{R}^2)$ and let $\lambda_1, \lambda_2, \eta_1, \eta_2$ as in Lemma 3.1. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by*

$$L : \nu_1 \mapsto \frac{\lambda_2}{\lambda_1}\nu_1,\ \nu_2 \mapsto \frac{\lambda_1}{\lambda_2}\nu_2, \tag{3.7}$$

the field $w_L(y) := w(L^{-1}y)$ satisfies

$$\mathcal{F}^T(w, A) = \frac{\lambda}{2}\int_{L(A)} |\nabla w_L(y)|^2 \,\mathrm{d}y. \tag{3.8}$$

From now on, for any $\rho > 0$, we set

$$E_\rho(x) := L^{-1}(B_\rho(Lx)). \tag{3.9}$$

We are in a position to prove the zero-order Γ-convergence result.

Theorem 3.3. *The following Γ-convergence result holds.*

(i) *(Compactness) Let $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ be such that $F_\varepsilon^T(u_\varepsilon) \leq C|\log \varepsilon|$ for some positive C. Then, up to a subsequence, $\mu(u_\varepsilon) \stackrel{\text{flat}}{\to} \mu$, for some $\mu \in X$.*

(ii) *(Localized Γ-liminf inequality) Let $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ be such that $\mu(u_\varepsilon) \stackrel{\text{flat}}{\to} \mu = \sum_{i=1}^M d_i \delta_{x_i}$, with $d_i \in \mathbb{Z} \setminus 0$ and $x_i \in \Omega$. Then, there exists a constant $C \in \mathbb{R}$ such that, for any $i = 1, \dots, M$ and for every $\sigma < \frac{1}{2}\operatorname{dist}(L(x_i), L(\partial\Omega) \cup \bigcup_{j \neq i} L(x_j))$, we have*

$$\liminf_{\varepsilon \to 0} F_\varepsilon^T(u_\varepsilon, E_\sigma(x_i)) - \pi\lambda|d_i| \log \frac{\sigma}{\varepsilon} \geq C, \tag{3.10}$$

where L is defined in (3.7). In particular

$$\liminf_{\varepsilon \to 0} F_\varepsilon^T(u_\varepsilon) - \pi\lambda|\mu|(\Omega) \log \frac{\sigma}{\varepsilon} \geq C.$$

(iii) *(Γ-limsup inequality) For every $\mu \in X$, there exists a sequence $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ such that $\mu(u_\varepsilon) \stackrel{\text{flat}}{\to} \mu$ and*

$$\pi\lambda|\mu|(\Omega) \geq \limsup_{\varepsilon \to 0} \frac{F_\varepsilon^T(u_\varepsilon)}{|\log \varepsilon|}.$$

Before giving the proof of Theorem 3.3, we revisit in our anisotropic case the ball construction in Section 3.2.3.

3.2.2. Lower bound on elliptic annuli. Let $0 < r < R$ and let $w \in H^1(E_R \setminus E_r; \mathcal{S}^1)$ with $\deg(w, \partial E_R) = d$. Set $w_L(y) := w(L^{-1}y)$ where L is the change of variable in (3.7), by (3.8) and Jensen's inequality, we get

$$\begin{aligned} \mathcal{F}^T(w, E_R \setminus E_r) &= \frac{\lambda}{2} \int_{B_R \setminus B_r} |\nabla w_L(y)|^2 \, \mathrm{d}y \\ &\geq \frac{\lambda}{2} \int_r^R \int_{\partial B_\rho} |(w_L \times \nabla w_L) \cdot \tau|^2 \, \mathrm{d}s \, \mathrm{d}\rho \geq \lambda \int_r^R \frac{1}{\rho} \pi d^2 \, \mathrm{d}\rho \geq \lambda\pi|d| \log \frac{R}{r}, \end{aligned} \tag{3.11}$$

where we have used that $\deg(w_L, \partial B_r) = d$ since $\det \nabla L = 1$.

Set $m := \min\left\{\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_1}\right\}$.

Proposition 3.4. *Fix $\varepsilon > 0$ and let $m2\sqrt{2}\varepsilon < r < R - m2\sqrt{2}\varepsilon$. For any function $v : (E_R \setminus E_r) \cap \varepsilon\mathbb{Z}^2 \to \mathcal{S}^1$ with $|\tilde{v}| \geq \frac{1}{2}$ in $E_{R-\sqrt{2}\varepsilon} \setminus E_{r+\sqrt{2}\varepsilon}$, it holds*

$$XY_\varepsilon^T(v, E_R \setminus E_r) \geq \lambda\pi|\mu(v)(E_r)| \log \frac{R}{r + \varepsilon\big(\alpha|\mu(v)(E_r)| + m^{-1}\sqrt{2}\big)}, \tag{3.12}$$

where $\alpha > 0$ is a universal constant.

Proof. By (3.5), using Fubini's theorem, we have that

$$XY_\varepsilon^T(v, E_R \setminus E_r) \geq \int_{r+m^{-1}\sqrt{2}\varepsilon}^{R-m^{-1}\sqrt{2}\varepsilon} \mathcal{F}^T(\tilde{v}, \partial E_\rho) \, \mathrm{d}\rho. \tag{3.13}$$

Fix $r + m^{-1}\sqrt{2}\varepsilon < \rho < R - m^{-1}\sqrt{2}\varepsilon$ and let T be a simplex of the triangulation of the ε-lattice. Set $\gamma_T(\rho) := \partial E_\rho \cap T$, let $\bar{\gamma}_T(\rho)$ be the segment joining the two extreme points of $\gamma_T(\rho)$ and let $\bar\gamma(\rho) = \bigcup_T \bar\gamma_T(\rho)$; then

$$\mathcal{F}^T(\tilde v, \partial E_\rho) = \frac{1}{2}\int_{\cup_T \gamma_T(\rho)} c_{e_1}|\partial_{e_1}\tilde v|^2 + c_{e_2}|\partial_{e_2}\tilde v|^2 + c_{e_1+e_2}|\partial_{\frac{e_1+e_2}{\sqrt2}}\tilde v|^2\,\mathrm{d}s \tag{3.14}$$

$$= \frac{1}{2}\sum_T (c_{e_1}|\partial_{e_1}\tilde v_{|T}|^2 + c_{e_2}|\partial_{e_2}\tilde v_{|T}|^2 + 2c_{e_1+e_2}|\partial_{\frac{e_1+e_2}{\sqrt2}}\tilde v_{|T}|^2)\mathcal{H}^1(\gamma_T(\rho))$$

$$\geq \tfrac{1}{2}\textstyle\sum_T (c_{e_1}|\partial_{e_1}\tilde v_{|T}|^2 + c_{e_2}|\partial_{e_2}\tilde v_{|T}|^2 + 2c_{e_1+e_2}|\partial_{\frac{e_1+e_2}{\sqrt2}}\tilde v_{|T}|^2)\mathcal{H}^1(\bar\gamma_T(\rho))$$

$$= \mathcal{F}^T(\tilde v, \bar\gamma_\rho). \tag{3.15}$$

Set $m(\rho) := \min_{\bar\gamma(\rho)}|\tilde v|$. Recalling the definition of the change of variable L in (3.7), let $\tilde v_L(y) := \tilde v(L^{-1}y)$. By Corollary 3.2 and in particular by (3.6), we have

$$\mathcal{F}^T(\tilde v, \bar\gamma(\rho)) = \frac{\lambda}{2}\int_{L(\bar\gamma(\rho))} |\nabla \tilde v_L(y)|^2\,\mathrm{d}y \tag{3.16}$$

using Jensen's inequality and the fact that $\mathcal{H}^1(\bar\gamma(\rho)) \leq \mathcal{H}^1(\partial E_\rho) = \mathcal{H}^1(L^{-1}(\partial B_\rho))$, we get

$$\frac12\int_{\bar\gamma(\rho)}|\nabla\tilde v_L|^2\,\mathrm{d}s \geq \frac12\int_{L(\bar\gamma(\rho))} m^2(\rho)\left|\left(\frac{\tilde v_L}{|\tilde v_L|}\times\nabla\frac{\tilde v_L}{|\tilde v_L|}\right)\cdot\tau\right|^2\mathrm{d}s \tag{3.17}$$

$$\geq \frac12\frac{m^2(\rho)}{\mathcal{H}^1(L(\bar\gamma(\rho)))}\left|\int_{L(\bar\gamma(\rho))}\left(\frac{\tilde v_L}{|\tilde v_L|}\times\nabla\frac{\tilde v_L}{|\tilde v_L|}\right)\cdot\tau\,\mathrm{d}s\right|^2$$

$$\geq \frac{m^2(\rho)}{\rho}\pi|d|^2 \tag{3.18}$$

where we have set $d := \deg(\tilde v, \partial E_\rho) = \mu(v)(E_r)$, which does not depend on ρ since $|\tilde v| \geq 1/2$ and coincide with $\deg(\tilde v_L, \partial B_\rho)$.

Now, let $T(\rho)$ be the simplex in which the minimum $m(\rho)$ is attained. Without loss of generality we assume that $T(\rho) = T^-_{\bar\imath}$ for some $\bar\imath \in \varepsilon\mathbb{Z}^2$. Let P one of the points of $\bar\gamma(\rho)$ for which $|\tilde v(P)| = m(\rho)$. By elementary geometric arguments, one can show that

$$\mathcal{F}^T(\tilde v, \partial E_\rho) \geq \tilde\alpha\frac{1-m^2(\rho)}{\varepsilon}, \tag{3.19}$$

for some universal positive constant $\tilde\alpha$.

In view of (3.16), (3.14), (3.18) and (3.19), for any $r + m^{-1}\sqrt2\varepsilon < \rho < R - m^{-1}\sqrt2\varepsilon$ we have

$$\mathcal{F}^T(\tilde v, \partial E_\rho) \geq \lambda\frac{m^2(\rho)}{\rho}\pi|d| \vee \tilde\alpha\frac{1-m^2(\rho)}{\varepsilon} \geq \frac{\lambda\pi|d|\tilde\alpha}{\varepsilon\lambda\pi|d| + \tilde\alpha\rho}.$$

By this last estimate and (3.13) we get

$$XY^T_\varepsilon(v, E_R\setminus E_r) \geq \lambda\pi|\mu(v)(E_r)|\log\frac{\varepsilon(\frac{\lambda\pi}{\tilde\alpha}|\mu(v)(E_r)| - m^{-1}\sqrt2) + R}{\varepsilon(\frac{\lambda\pi}{\tilde\alpha}|\mu(v)(E_r)| + m^{-1}\sqrt2) + r}. \tag{3.20}$$

Assuming, without loss of generality, $\tilde\alpha < \dfrac{\lambda m\pi}{\sqrt2}$, we immediately get (3.12) for $\alpha = \frac{\pi}{\tilde\alpha}$. □

3.2.3. Ellipse Construction. Here we introduce a slide modification of the *ball construction* in Section 3.2.3. The main difference is that, in order to deal with our anisotropic energies, we have to consider ellipses in place of balls.

Let $\mathcal{E} = \{E_{R_1}(x_1), \ldots, E_{R_N}(x_N)\}$ be a finite family of pairwise disjoint ellipses in $\mathbb{R}^2$ of the type in (3.9) and let $\mu = \sum_{i=1}^N d_i \delta_{x_i}$ with $d_i \in \mathbb{Z} \setminus \{0\}$. Let F be a positive superadditive set function on the open subsets of $\mathbb{R}^2$, i.e., such that $F(A \cup B) \geq F(A) + F(B)$, whenever A and B are open and disjoint. We will assume that there exists $c, C > 0$ such that

$$F(A_{r,R}(x)) \geq C\pi|\mu(E_r(x))| \log \frac{R}{c+r}, \tag{3.21}$$

for any annulus $A_{r,R}(x) = E_R(x) \setminus E_r(x)$, with $A_{r,R}(x) \subset \Omega \setminus \bigcup_i E_{R_i}(x_i)$.

Let t be a parameter which represents an artificial time. Using the same strategy in Section 3.2.3 we can construct, for any $t > 0$, a finite family of ellipses $\mathcal{E}(t)$ which satisfies the following properties

(1) $\bigcup_{i=1}^N E_{R_i}(x_i) \subset \bigcup_{E \in \mathcal{E}(t)} E$,
(2) the ellipses in $\mathcal{E}(t)$ are pairwise disjoint,
(3) $F(E) \geq C\pi|\mu(E)| \log(1+t)$ for any $E \in \mathcal{E}(t)$,
(4) $\sum_{E \in \mathcal{E}(t)} R(E) \leq (1+t)\sum_i R_i + (1+t)cN(N^2+N+1)$, where $R(E)$ denotes the radius of the ball $L(E)$.

3.2.4. Proof of Theorem 3.3. Using the fact that $XY_\varepsilon^T \geq \min\{c_{e_1}, c_{e_2}\} XY_\varepsilon$ and Proposition 2.4, it is immediate tho prove the following lemma.

Lemma 3.5. *There exists a positive constant β such that for any $\varepsilon > 0$, for any function $v \in \mathcal{AXY}_\varepsilon(\Omega)$ and for any $i \in \Omega_\varepsilon^2$ such that the piecewise affine interpolation $\tilde{v}$ of v satisfies $\min_{i+\varepsilon Q} |\tilde{v}| < \frac{1}{2}$, it holds $XY_\varepsilon^T(v, i + \varepsilon Q) \geq \beta$.*

By (2.12), it is enough to prove (i) and (ii) for $F_\varepsilon^T = XY_\varepsilon^T$, using as a variable $v_\varepsilon = e^{2\pi i u_\varepsilon}$.

Proof of (i). For every $\varepsilon > 0$, set $I_\varepsilon := \left\{i \in \Omega_\varepsilon^2 : \min_{i+\varepsilon Q} |\tilde{v}_\varepsilon| \leq \frac{1}{2}\right\}$. Notice that, in view of Remark 2.1, $\mu(v_\varepsilon)$ is supported in $I_\varepsilon + \frac{\varepsilon}{2}(e_1 + e_2)$.

Starting from the family of ellipses $E_{m\frac{\sqrt{2}\varepsilon}{2}}(i + \frac{\varepsilon}{2}(e_1+e_2)))$, and eventually passing through a merging procedure we can construct a family of pairwise disjoint ellipses

$$\mathcal{E}_\varepsilon := \left\{E_{R_{i,\varepsilon}}(x_{i,\varepsilon})\right\}_{i=1,\ldots,N_\varepsilon},$$

with $\sum_{i=1}^{N_\varepsilon} R_{i,\varepsilon} \leq m\varepsilon \sharp I_\varepsilon$. Then, by Lemma 2.4 and by the energy bound, we immediately have that $\sharp I_\varepsilon \leq C|\log \varepsilon|$ and hence

$$\sum_{i=1}^{N_\varepsilon} R_{i,\varepsilon} \leq \varepsilon C|\log \varepsilon|. \tag{3.22}$$

We define the sequence of measures

$$\mu_\varepsilon := \sum_{i=1}^{N_\varepsilon} \mu(v_\varepsilon)(E_{R_{i,\varepsilon}}(x_{i,\varepsilon}))\delta_{x_{i,\varepsilon}}.$$

Since $|\mu_\varepsilon(E)| \leq \sharp I_\varepsilon$ for each ellipse $E \in \mathcal{E}_\varepsilon$, by (3.12) we deduce that (3.21) holds with $F(\cdot) = XY_\varepsilon^T(v_\varepsilon, \cdot \setminus \cup_{E \in \mathcal{E}_\varepsilon} E)$, $C = \lambda$ and $c = \varepsilon(\alpha \sharp I_\varepsilon + \sqrt{2}m^{-1})$.

We let the ellipses in the families $\mathcal{E}_\varepsilon$ grow and merge as described in Subsection 3.2.3, and let $\mathcal{E}_\varepsilon(t) := \left\{E_{R_{i,\varepsilon}(t)}(x_{i,\varepsilon}(t))\right\}$ be the corresponding family of balls at time t. Set moreover $t_\varepsilon := \frac{1}{\sqrt{\varepsilon}} - 1$, $N_\varepsilon(t_\varepsilon) := \sharp\mathcal{E}_\varepsilon(t_\varepsilon)$ and define

$$\nu_\varepsilon := \sum_{\substack{i=1,\dots,N_\varepsilon(t_\varepsilon)\\ B_{R_{i,\varepsilon}(t_\varepsilon)}(x_{i,\varepsilon}(t_\varepsilon))\subset\Omega}} \mu_\varepsilon(E_{R_{i,\varepsilon}(t_\varepsilon)}(x_{i,\varepsilon}(t_\varepsilon)))\delta_{x_{i,\varepsilon}(t_\varepsilon)}. \tag{3.23}$$

By (3) in Subsection 3.2.3, for any $E \in \mathcal{E}_\varepsilon(t_\varepsilon)$, with $E \subseteq \Omega$, we have

$$XY_\varepsilon^T(v_\varepsilon, E) \geq \lambda\pi|\mu_\varepsilon(E)|\log(1+t_\varepsilon) = \pi|\nu_\varepsilon(E)|\tfrac{1}{2}|\log\varepsilon|;$$

by the energy bound, we have immediately that $|\nu_\varepsilon|(\Omega) \leq M$ and hence $\{\nu_\varepsilon\}$ is precompact in the weak* topology. By (4) in Subsection 3.2.3, it follows that

$$\sum_{j=1}^{N_\varepsilon(t_\varepsilon)} R_j(t_\varepsilon) \leq C\sqrt{\varepsilon}\ (\sharp I_\varepsilon)^4,$$

which easily implies that $\|\nu_\varepsilon - \mu_\varepsilon\|_{\mathrm{flat}} \to 0$; moreover, using (3.22), it is easy to show that $\|\mu_\varepsilon - \mu(v_\varepsilon)\|_{\mathrm{flat}} \to 0$ as $\varepsilon \to 0$ (see [6] for more details). We conclude that also $\mu(v_\varepsilon)$ is precompact in the flat topology.

Proof of (ii). The proof of (ii) coincides with the one of (ii) in Theorem 2.6 using the ellipse construction in place of the ball construction. We briefly sketch it.

Fix $i \in \{1,\dots,M\}$. Without loss of generality, and possibly extracting a subsequence, we can assume that

$$\liminf_{\varepsilon\to0} XY_\varepsilon^T(v_\varepsilon, E_\sigma(x_i)) - \lambda\pi|d_i||\log\varepsilon| = \lim_{\varepsilon\to0} XY_\varepsilon^T(v_\varepsilon, E_\sigma(x_i)) - \lambda\pi|d_i||\log\varepsilon| < +\infty. \tag{3.24}$$

We consider the restriction $\bar v_\varepsilon \in \mathcal{AXY}_\varepsilon(E_\sigma(x_i))$ of v_ε to $E_\sigma(x_i)$, it is easy to see that

$$\|\mu(\bar v_\varepsilon) - d_i\delta_{x_i}\|_{\mathrm{flat}} \to 0. \tag{3.25}$$

We repeat the ellipse construction procedure used in the proof of (i) with Ω replaced by $E_\sigma(x_i)$, v_ε by $\bar v_\varepsilon$ and I_ε by

$$I_{i,\varepsilon} := \left\{ j \in (E_\sigma(x_i))^2_\varepsilon \ :\ \min_{j+\varepsilon Q} |\tilde v_\varepsilon| \leq \frac{1}{2} \right\}.$$

We denote by $\mathcal{E}_{i,\varepsilon}$ the corresponding family of balls and by $\mathcal{E}_{i,\varepsilon}(t)$ the family of balls constructed at time t.

Fix $0 < \gamma < 1$ such that

$$(1-\gamma)(|d_i|+1) > |d_i|\,. \tag{3.26}$$

Let $t_{\varepsilon,\gamma} = \varepsilon^{\gamma-1} - 1$ and let $\nu_{\varepsilon,\gamma}$ be the measure defined as in (3.23) with Ω replaced by $E_\sigma(x_i)$ and t_ε replaced by $t_{\varepsilon,\gamma}$. As in the previous step, since $\gamma > 0$ we deduce that $\|\nu_{\varepsilon,\gamma} - d_i\delta_{x_i}\|_{\mathrm{flat}} \to 0$; moreover, for any $E \in \mathcal{E}_{i,\varepsilon}(t_{\varepsilon,\gamma})$ we have

$$XY_\varepsilon^T(v_\varepsilon, E) \geq \lambda\pi|\nu_{\varepsilon,\gamma}(E)|(1-\gamma)|\log\varepsilon|. \tag{3.27}$$

Now, if $\liminf_{\varepsilon\to0}|\nu_{\varepsilon,\gamma}|(E_\sigma(x_i)) > |d_i|$, then, thanks to (3.26), (3.10) holds true. Otherwise we can assume that $|\nu_{\varepsilon,\gamma}|(E_\sigma(x_i)) = |d_i|$ for ε small enough.

Then $\nu_{\varepsilon,\gamma}$ is a sum of Dirac masses concentrated on points which converge to x_i, with weights all having the same sign and summing to d_i. Using the properties of ellipse construction, one can easily show that

$$\mu(\bar v_\varepsilon)\Big(\bigcup_{\substack{E\in\mathcal{E}_{i,\varepsilon}(\bar t_\varepsilon)\\ E\subset E_\sigma(x_i)}} E\Big)=d_i.$$

We have immediately that

$$\begin{aligned}XY_\varepsilon^T(\bar v_\varepsilon, E_\sigma(x_i)\setminus\cup_{E\in\mathcal{E}_{i,\varepsilon}}E)&\geq \pi\lambda\sum_{\substack{E\in\mathcal{E}_{i,\varepsilon}(\bar t_\varepsilon)\\ E\subset E_\sigma(x_i)}}|\mu(\bar v_\varepsilon)(E)|\log(1+\bar t_\varepsilon)\\ &\geq \lambda\pi|d_i|\log\frac{\sigma}{C_1(\sharp I_{i,\varepsilon})^4\varepsilon}.\end{aligned}$$

On the other hand, by Proposition 2.4 there exists a positive constant β such that

$$XY_\varepsilon^T(\bar v_\varepsilon, j+\varepsilon Q)\geq\beta\quad\text{for any } j\in I_{i,\varepsilon};$$

therefore, $XY_\varepsilon^T(\bar v_\varepsilon,\bigcup_{E\in\mathcal{E}_{i,\varepsilon}}E)\geq\beta\sharp I_{i,\varepsilon}$. Finally, we get

$$\begin{aligned}XY_\varepsilon^T(\bar v_\varepsilon,E_\sigma(x_i))&\geq XY_\varepsilon^T(\bar v_\varepsilon,E_\sigma(x_i)\setminus\cup_{E\in\mathcal{E}_{i,\varepsilon}}E)+XY_\varepsilon^T(\bar v_\varepsilon,\cup_{E\in\mathcal{E}_{i,\varepsilon}}E)\\ &\geq\pi\lambda|d_i|\log\frac{\sigma}{\varepsilon}-\log\big(C_1(\sharp I_{i,\varepsilon})^4\big)+\sharp I_{i,\varepsilon}\beta\geq\lambda\pi|d_i|\log\frac{\sigma}{\varepsilon}+C\end{aligned}$$

and (3.10) follows sending $\varepsilon\to0$.

Proof of (iii) By a standard density argument we can assume $d_i=\pm1$. Let $u_{\varepsilon,i}(x):=\pm\theta_i^L(x)$, where $\theta_i^L(x):=\theta(Lx-Lx_i)$ and $\theta(y)$ is the polar coordinate $\arctan y_2/y_1$. Then a recovery sequence is given by $u_\varepsilon=\sum_{i=1}^M u_{\varepsilon,i}$. The straightforwanrd computations are left to the reader.

In the following Subsections we will prove the first order Γ-convergence of the functionals F_ε^T to the renormalized energy, introduced in the continuous framework of Ginzburg-Landau energies in [15]. To this purpose we begin by extending the many definitions and results of [15] in order to include our anisotropic case.

3.2.5. The anisotropic renormalized energy. Fix $\mu=\sum_{i=1}^M d_i\delta_{x_i}$ with $d_i\in\{-1,+1\}$ and $x_i\in\Omega$. We recall the definition of L given in (3.7). With a little abuse of notation we set $L\mu:=\sum_{i=1}^M d_i\delta_{Lx_i}$ and we define

$$\mathbb{W}^T(\mu):=\lambda\mathbb{W}(L\mu),$$

where $\mathbb{W}$ is defined in (1.5) and $\lambda=\sqrt{c_{e_1}c_{e_2}+c_{e_1}c_{e_1+e_2}+c_{e_2}c_{e_1+e_2}}$.

For any $y\in\mathbb{R}^2\setminus\{0\}$, we define $\theta(y)$ as the polar coordinate $\arctan y_2/y_1$ and let $\theta^L(x):=\theta(Lx)$. As done above we define $\theta_i^L(x):=\theta(Lx-Lx_i)$ for any $i=1,\dots,M$.

We consider the following auxiliary minimum problems.

$$m^T(\sigma,\mu):=\min_{w\in H^1(\Omega_\sigma^T;\mathcal{S}^1)}\left\{\mathcal{F}^T(w)\,:\,\deg(w,\partial E_\sigma(x_i))=d_i\right\},$$

$$\tilde m^T(\sigma,\mu):=\min_{w\in H^1(\Omega_\sigma^T;\mathcal{S}^1)}\left\{\mathcal{F}^T(w)\,:w(\cdot)=\alpha_i e^{id_i\theta_i^L(\cdot)}\text{on }\partial E_\sigma(x_i),\,|\alpha_i|=1\right\}.\tag{3.28}$$

Given $\varepsilon > 0$, we introduce the discrete minimization problem in the ellipse E_σ

$$\gamma^T(\varepsilon,\sigma) \quad := \quad \min_{u\in\mathcal{AF}_\varepsilon(E_\sigma)} \left\{F^T_\varepsilon(u,E_\sigma)\,:\, 2\pi\, u(\cdot) = \theta^L(\cdot) \text{ on } \partial_\varepsilon E_\sigma\right\} \tag{3.29}$$

By Theorem 1.7, using the change of variable L in (3.7), one can obtain the following result whose proof is left to the reader.

Theorem 3.6. *It holds*

$$\lim_{\sigma\to 0} m^T(\sigma,\mu) - \lambda\pi|\mu|(\Omega)|\log\sigma| = \lim_{\sigma\to 0}\tilde m^T(\sigma,\mu) - \lambda\pi|\mu|(\Omega)|\log\sigma| = \mathbb{W}^T(\mu). \tag{3.30}$$

Moreover, for any fixed $\sigma > 0$, the following limit exists finite

$$\lim_{\varepsilon\to 0}(\gamma^T(\varepsilon,\sigma) - \lambda\pi|\log\frac{\varepsilon}{\sigma}|) =: \gamma^T \in \mathbb{R} \tag{3.31}$$

3.2.6. The first-order Γ-convergence result. We are now in a position to state the first-order Γ-convergence theorems for the functionals F^T_ε.

Theorem 3.7. *The following Γ-convergence result holds.*

(i) *(Compactness) Let $M \in \mathbb{N}$ and let $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ be a sequence satisfying $F^T_\varepsilon(u_\varepsilon) - M\lambda\pi|\log\varepsilon| \le C$. Then, up to a subsequence, $\mu(u_\varepsilon) \overset{\text{flat}}{\to} \mu$ for some $\mu = \sum_{i=1}^N d_i\delta_{x_i}$ with $d_i \in \mathbb{Z}\setminus\{0\}$, $x_i \in \Omega$ and $\sum_i |d_i| \le M$. Moreover, if $\sum_i |d_i| = M$, then $\sum_i |d_i| = N = M$, namely $|d_i| = 1$ for any i.*

(ii) *(Γ-$\liminf$ inequality) Let $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ be such that $\mu(u_\varepsilon) \overset{\text{flat}}{\to} \mu$, with $\mu = \sum_{i=1}^M d^i\delta_{x_i}$ with $|d_i| = 1$ and $x_i \in \Omega$ for every i. Then,*
$$\liminf_{\varepsilon\to 0} F\varepsilon^T(u_\varepsilon) - M\lambda\pi|\log\varepsilon| \ge \mathbb{W}^T(\mu) + M\gamma^T. \tag{3.32}$$

(iii) *(Γ-$\limsup$ inequality) Given $\mu = \sum_{i=1}^M d_i\delta_{x_i}$ with $|d_i| = 1$ and $x_i \in \Omega$ for every i, there exists $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ with $\mu(u_\varepsilon) \overset{\text{flat}}{\to} \mu$ such that*
$$F^T_\varepsilon(u_\varepsilon) - M\lambda\pi|\log\varepsilon| \to \mathbb{W}^T(\mu) + M\gamma^T.$$

PROOF. The proof of Theorem 3.7 closely follows the proof of Theorem 2.6. Recalling that $F^T_\varepsilon(u) \ge XY^T_\varepsilon(e^{2\pi iu})$, the proof of the compactness property (i) will be done for $F^T_\varepsilon = XY^T_\varepsilon$. On the other hand, the constant γ^T depends on the potential f and on the constants $c_{e_1}, c_{e_2}, c_{e_1+e_2}$, so its derivation requires a specific proof.

Proof of (i): Compactness. The fact that, up to a subsequence, $\mu(u_\varepsilon) \overset{\text{flat}}{\to} \mu = \sum_{i=1}^N d_i\delta_{x_i}$ with $\sum_{i=1}^N |d_i| \le M$ is a direct consequence of the zero order Γ-convergence result stated in Theorem 3.3 (i). Assume now $\sum_{i=1}^N |d_i| = M$ and let us prove that $|d_i| = 1$. Let $0 < \sigma_1 < \sigma_2$ be such that $E_{\sigma_2}(x_i)$ are pairwise disjoint and contained in Ω and let ε be small enough so that $E_{\sigma_2}(x_i)$ are contained in Ω_ε. For any $0 < r < R$ and $x \in \mathbb{R}^2$, set $A_{r,R}(x) := E_R(x) \setminus E_r(x)$. Since $F^T_\varepsilon(u_\varepsilon) \ge XY^T_\varepsilon(e^{2\pi iu_\varepsilon})$,

$$F^T_\varepsilon(u_\varepsilon) \ge \sum_{i=1}^N XY^T_\varepsilon(e^{2\pi iu_\varepsilon}, E_{\sigma_1}(x_i)) + \sum_{i=1}^N XY^T_\varepsilon(e^{2\pi iu_\varepsilon}, E_{\sigma_1,\sigma_2}(x_i)). \tag{3.33}$$

To ease notation we set $v_\varepsilon = e^{2\pi i u_\varepsilon}$ and we indicate with $\tilde v_\varepsilon$ the piecewise affine interpolation of v_ε. Moreover let t be a positive number and let ε be small enough so that $t > m\sqrt{2}\varepsilon$. Then, by (3.10) and (3.5), we get

$$F^T_\varepsilon(u_\varepsilon) \geq \lambda\pi \sum_{i=1}^N |d_i| \log\frac{\sigma_1}{\varepsilon} + \mathcal{F}^T(\tilde v_\varepsilon, A_{\sigma_1+t,\sigma_2-t}(x_i)) + C. \tag{3.34}$$

By the energy bound and by the definition of $\mathcal{F}^T$, we deduce that

$$\int_{A_{\sigma_1+t,\sigma_2-t}(x_i)} |\nabla \tilde v_\varepsilon|^2 \,\mathrm{d}x \leq \frac{2}{\min\{c_{e_1}, c_{e_2}\}}\mathcal{F}^T(\tilde v_\varepsilon, A_{\sigma_1+t,\sigma_2-t}(x_i)) \leq C$$

and hence, up to a subsequence, $\tilde v_\varepsilon \rightharpoonup v_i$ in $H^1(A_{\sigma_1+t,\sigma_2-t}(x_i);\mathbb{R}^2)$ for some field v_i. Moreover, since

$$\frac{1}{\varepsilon^2}\int_{A_{\sigma_1+t,\sigma_2-t}(x_i)} (1-|\tilde v_\varepsilon|^2)^2 \,\mathrm{d}x \leq C XY^T_\varepsilon(v_\varepsilon) \leq C|\log\varepsilon|,$$

(see Lemma 2 in [2] for more details), we deduce that $|v_i| = 1$ a.e..

Furthermore, by standard Fubini's arguments, for a.e. $\sigma_1 + t < \sigma < \sigma_2 - t$, up to a subsequence the trace of $\tilde v_\varepsilon$ is bounded in $H^1(\partial E_\sigma(x_i);\mathbb{R}^2)$, and hence it converges uniformly to the trace of v_i. By the very definition of degree it follows that $\deg(v_i, \partial E_\sigma(x_i)) = d_i$.

Hence, by (3.11), for every i we have

$$\mathcal{F}^T(v_i, A_{\sigma_1+t,\sigma_2-t}(x_i)) \geq \lambda|d_i|^2\pi\log\frac{\sigma_2 - t}{\sigma_1 + t}. \tag{3.35}$$

By (3.34) and (3.35), we conclude that for ε smal enoughl

$$F^T_\varepsilon(u_\varepsilon) \geq \lambda\pi\sum_{i=1}^N\left(|d_i|\log\frac{\sigma_1}{\varepsilon} + |d_i|^2\log\frac{\sigma_2 - t}{\sigma_1 + t}\right) + C$$

$$\geq \lambda\pi\left(M|\log\varepsilon| + \sum_{i=1}^N(|d_i|^2 - |d_i|)\log\frac{\sigma_2}{\sigma_1} + \sum_{i=1}^N|d_i|^2\log\frac{\sigma_1(\sigma_2 - t)}{\sigma_2(\sigma_1 + t)}\right) + C.$$

The energy bound yields that the sum of the last two terms is bounded; letting $t \to 0$ and $\sigma_1 \to 0$, we conclude $|d_i| = 1$.

Proof of (ii): Γ-liminf inequality. Fix $r > 0$ so that the ellipses $E_r(x_i)$ are pairwise disjoint and compactly contained in Ω. Let moreover $\{\Omega^h\}$ be an increasing sequence of open smooth sets compactly contained in Ω such that $\cup_{h\in\mathbb{N}}\Omega^h = \Omega$. Without loss of generality we can assume that $F^T_\varepsilon(u_\varepsilon) \leq \lambda M\pi|\log\varepsilon| + C$, which together with Theorem 3.3 yields

$$F^T_\varepsilon(u_\varepsilon, \Omega\setminus\bigcup_{i=1}^M E_r(x_i)) \leq C. \tag{3.36}$$

We set $v_\varepsilon := e^{2\pi i u_\varepsilon}$ and we denote by $\tilde v_\varepsilon$ the piecewise affine interpolation of v_ε. For every $r > 0$, by (3.36) and (2.12) we deduce $XY^T_\varepsilon(v_\varepsilon, \Omega\setminus\bigcup_{i=1}^N E_r(x_i)) \leq C$. Fix $h \in \mathbb{N}$ and let ε be small enough so that $\Omega^h \subset \Omega_\varepsilon$. Since

$$\int_{\Omega^h\setminus\bigcup_{i=1}^N E_r(x_i)} |\nabla\tilde v_\varepsilon|^2 \leq \frac{2}{\min\{c_{e_1}, c_{e_2}\}} XY^T_\varepsilon(v_\varepsilon, \Omega\setminus\bigcup_{i=1}^M E_r(x_i)) \leq C,$$

by a diagonalization argument, there exists a unitary field $v \in H^1(\Omega \setminus E_r(x_i); \mathcal{S}^1)$ such that, up to a subsequence, $\tilde{v}_\varepsilon \rightharpoonup v$ in $H^1_{\mathrm{loc}}(\Omega \setminus \cup_{i=1}^M \{x_i\}; \mathbb{R}^2)$.

Let $\sigma > 0$ be such that $E_\sigma(x_i)$ are pairwise disjoint and contained in Ω^h. For any $0 < r < R < +\infty$ and for any $x \in \mathbb{R}^2$, set $A_{r,R}(x) := E_R(x) \setminus E_r(x)$, $A_{r,R} := A_{r,R}(0)$. Let $t \leq \sigma$, and consider the minimization problem

$$\min_{w \in H^1(A_{t/2,t}; \mathcal{S}^1)} \left\{ \mathcal{F}^T(w, A_{t/2,t}) : \deg(w, \partial E_{\frac{t}{2}}) = 1 \right\}.$$

It is easy to see that the minimum is $\lambda\pi \log 2$ and that the set of minimizers is given by (the restriction at $A_{t/2,t}$ of the functions in)

$$\mathcal{K} := \left\{ \alpha \frac{Lz}{|Lz|} \, : \, \alpha \in \mathbb{C}, |\alpha| = 1 \right\}. \tag{3.37}$$

Set

$$\mathrm{d}_t(w, \mathcal{K}) := \min \left\{ \mathcal{F}^T(w - v, A_{t/2,t}) \, : \, v \in \mathcal{K} \right\}. \tag{3.38}$$

For any $v \in \mathcal{K}$ and $w \in H^1(A_{t/2,t}; \mathbb{R}^2)$, using the change of variable L in (3.7), we have

$$\mathcal{F}^T(w - v, A_{t/2,t}) = \lambda \int_{B_t \setminus B_{t/2}} |\nabla w_L - \nabla v_L|^2 \, \mathrm{d}y,$$

where we have set $w_L(y) := w(L^{-1}y)$ and $v_L(y) := v(L^{-1}y)$. By this fact, it follows that (see [4] for further details) for any given $\delta > 0$ there exists a positive $\omega(\delta)$ (independent of t) such that if $\mathrm{d}_t(\tilde{v}_\varepsilon(\cdot), \mathcal{K}_i) \geq \delta$, then

$$\mathcal{F}^T(\tilde{v}_\varepsilon; A_{\frac{t}{2}+m^{-1}\sqrt{2}\varepsilon, t+m^{-1}\sqrt{2}\varepsilon}) \geq \lambda\pi \log 2 + \omega(\delta), \tag{3.39}$$

where $\mathcal{K}_i := \left\{ \alpha \frac{Lz - Lx_i}{|Lx - Lx_i|} \, : \, \alpha \in \mathbb{C}, |\alpha| = 1 \right\}$.

Let $P \in \mathbb{N}$ be such that $P\,\omega(\delta) \geq \mathbb{W}^T(\mu) + M(\gamma^T - \lambda\pi \log \sigma - C)$ where C is the constant in (3.10). For $p = 1, \ldots, P$, set $C_p(Lx_i) := E_{2^{1-p}\sigma}(x_i) \setminus E_{2^{-p}\sigma}(x_i)$.

Now, if for ε small enough and for every fixed $1 \leq p \leq P$, there exists at least one i such that $\mathrm{d}_{2^{1-p}\sigma}(\tilde{v}_\varepsilon, \mathcal{K}_i) \geq \delta$, then by (3.10), (3.39) and the lower semicontinuity of the functional $\mathcal{F}^T$, we conclude

$$\begin{aligned} F^T_\varepsilon(u_\varepsilon, \Omega^h) &\geq \sum_{i=1}^M XY^T_\varepsilon(v_\varepsilon, E_{2^{-P}\sigma}(x_i)) + \sum_{p=1}^P \sum_{i=1}^M XY^T_\varepsilon(v_\varepsilon, C_p(x_i)) \\ &\geq \lambda M(\pi \log \frac{\sigma}{2^P} + \pi|\log \varepsilon| + C) + L(M\lambda\pi \log 2 + \omega(\delta)) + \mathrm{o}(\varepsilon) \\ &\geq M\lambda\pi|\log \varepsilon| + M\gamma^T + \mathbb{W}^T(\mu) + \mathrm{o}(\varepsilon). \end{aligned}$$

Assume that, up to a subsequence, there exists $1 \leq \bar{p} \leq P$ such that for every i we have $\mathrm{d}_{\bar{\sigma}}(\tilde{v}_\varepsilon, \mathcal{K}_i) \leq \delta$, where $\bar{\sigma} := 2^{1-\bar{p}}\sigma$. Let $\alpha_{\varepsilon,i}$ be the unitary vector such that $\mathcal{F}^T(\tilde{v}_\varepsilon - \alpha_{\varepsilon,i} \frac{Lx - Lx_i}{|Lx - Lx_i|}, C_l(x_i)); \mathbb{R}^2) = \mathrm{d}_{\bar{\sigma}}(\tilde{v}_\varepsilon, \mathcal{K}_i)$.

One can construct a function $\bar{u}_\varepsilon \in \mathcal{AF}_\varepsilon(\Omega)$ such that

(i) $\bar{u}_\varepsilon = u_\varepsilon$ on $\partial_\varepsilon(\mathbb{R}^2 \setminus E_{2^{-\bar{l}}\sigma}(x_i))$;

(ii) $e^{2\pi i \bar{u}_\varepsilon(i)} = \alpha_{\varepsilon,i} \frac{Li - x_i}{|Li - Lx_i|}$ on $\partial_\varepsilon E_{2^{1-\bar{l}}\sigma}(x_i)$

(iii) $F^T_\varepsilon(u_\varepsilon, E_{\bar{\sigma}}(x_i)) \geq F^T_\varepsilon(\bar{u}_\varepsilon, E_{\bar{\sigma}}(x_i)) + r(\varepsilon, \delta)$ with $\lim_{\delta \to 0} \lim_{\varepsilon \to 0} r(\varepsilon, \delta) = 0$.

The proof of (i)-(iii) is quite technical, and consists in adapting standard cut-off arguments to our discrete setting. For the reader convenience we skip the details of the proof, and assuming (i)-(iii) we conclude the proof of the lower bound.

By Theorem (1.7), we have that

$$
\begin{aligned}
F_\varepsilon^T(u_\varepsilon) &\geq XY_\varepsilon^T(v_\varepsilon, \Omega^h \setminus \bigcup_{i=1}^M E_{\bar\sigma}(x_i)) + \sum_{i=1}^M F_\varepsilon^T(u_\varepsilon, E_{\bar\sigma}(x_i)) \\
&\geq \mathcal{F}^T(\tilde v_\varepsilon, \Omega \setminus \cup_{i=1}^M E_{\bar\sigma}(x_i)) + \sum_{i=1}^M F_\varepsilon^T(\bar u_\varepsilon, E_{\bar\sigma}(x_i)) + r(\varepsilon,\delta) + \mathrm{o}(\varepsilon) \\
&\geq \mathcal{F}^T(\tilde v_\varepsilon, \Omega \setminus \cup_{i=1}^M E_{\bar\sigma}(x_i)) + M(\gamma^T - \lambda\pi \log\frac{\varepsilon}{\bar\sigma}) + r(\varepsilon,\delta) + \mathrm{o}(\varepsilon) \\
&\geq \mathcal{F}^T(v, \Omega \setminus \cup_{i=1}^M E_{\bar\sigma}(x_i)) + M(\gamma^T - \lambda\pi \log\frac{\varepsilon}{\bar\sigma}) + r(\varepsilon,\delta) + \mathrm{o}(\varepsilon) + \mathrm{o}(1/h) \\
&\geq M\lambda\pi|\log\varepsilon| + M\gamma^T + \mathbb{W}^T(\mu) + r(\varepsilon,\delta) + \mathrm{o}(\varepsilon) + \mathrm{o}(\bar\sigma) + \mathrm{o}(1/h).
\end{aligned}
$$

The proof follows sending $\varepsilon \to 0$, $\delta \to 0$, $\sigma \to 0$ and $h \to \infty$.

Proof of (iii): Γ-limsup inequality. This proof in analogue to the one given in (iii) of Theorem 2.6 for the isotropic case. We only sketch its anisotropic counterpart in our triangular lattice. Let w_σ be a function that agrees with a minimizer of (3.28) in $\Omega \setminus \bigcup_{i=1}^M E_\sigma(x_i) = \Omega_\sigma^T$. Then, $w_\sigma = \alpha_i e^{i\theta_i^\lambda}$ on $\partial E_\sigma(x_i)$ for some $|\alpha_i| = 1$.

For every $\rho > 0$ we can always find a function $w_{\sigma,\rho} \in C^\infty(\overline{\Omega_\sigma^T}; \mathcal{S}^1)$ such that $w_{\sigma,\rho} = \alpha_i e^{i\theta_i^L}$ on $\partial E_\sigma(x_i)$, and

$$
\mathcal{F}^T(w_{\sigma,\rho}, \Omega_\sigma^T) - \mathcal{F}^T(w_\sigma, \Omega_\sigma^T) \leq \rho.
$$

Moreover, for every i let $w_i \in \mathcal{AXY}_\varepsilon(E_\sigma(x_i))$ be a function which agrees with $\alpha_i e^{i\theta_i^L}$ on $\partial_\varepsilon E_\sigma(x_i)$ and such that its phase minimizes problem (3.29). If necessary, we extend w_i to $(\overline{E_\sigma(x_i)} \cap \varepsilon\mathbb{Z}^2) \setminus (E_\sigma(x_i))_\varepsilon^0$ to be equal to $\alpha_i e^{i\theta_i^L}$. Finally, define the function $w_{\varepsilon,\sigma,\rho} \in \mathcal{AXY}_\varepsilon(\Omega)$ which coincides $w_{\sigma,\rho}$ on $\Omega_\sigma \cap \varepsilon\mathbb{Z}^2$ and with w_i on $\overline{E_\sigma(x_i)} \cap \varepsilon\mathbb{Z}^2$. In view of assumption (3) on f, a straightforward computation shows that any phase $u_{\varepsilon,\sigma,\rho}$ of $w_{\varepsilon,\sigma,\rho}$ is a recovery sequence, i.e.,

$$
\lim_{\varepsilon\to 0} F_\varepsilon^T(u_{\varepsilon,\sigma,\rho}) - M\lambda\pi|\log\varepsilon| = M\gamma^T + \mathbb{W}^T(\mu) + o(\rho,\sigma),
$$

with $\lim_{\sigma\to 0}\lim_{\rho\to 0} o(\rho,\sigma) = 0$. □

3.3. Γ-expansion for F_ε^{lr}

Here we give the asymptotic expansion by Γ-convergence of the functional F_ε^{lr}. The idea is to decompose the energy F_ε^{lr} in the sum of isotropic F_ε energies and to use for each of these energies the Γ-convergence analysis developed in Chapter 2. To this aim, using that $c_\xi = c_{\xi^\perp}$ for any $\xi \in \mathbb{Z}_+^2$, we have that for any $u \in \mathcal{AF}_\varepsilon(\Omega)$

$$
F_\varepsilon^{lr}(u) = \frac{1}{2} \sum_{\substack{\xi\in\mathbb{Z}_+^2 \\ \xi\cdot e_1 \geq 0}} c_\xi \sum_{\substack{i\in\Omega_\varepsilon^0 \\ i+\varepsilon\xi\in\Omega_\varepsilon^0}} f(u(i+\varepsilon\xi) - u(i))
$$

More precisely, using an idea in [2], given $\xi \in \mathbb{Z}^2$ with $\xi \cdot e_1 \geq 0$, we may partition $\mathbb{Z}^2$ as follows

$$\mathbb{Z}^2 = \bigcup_{h=1}^{|\xi|^2} (z_h + \mathbb{Z}\xi \oplus \mathbb{Z}\xi^\perp)$$

where $\{z_h\}_{h=1,\dots,|\xi|^2} = \{z \in \mathbb{Z}^2 \,:\, 0 \leq z \cdot \xi < |\xi|,\, 0 \leq z \cdot \xi^\perp < |\xi|\}$. Then we may write

$$F_\varepsilon^{lr}(u) = \sum_{\substack{\xi \in \mathbb{Z}^2_+ \\ \xi \cdot e_1 \geq 0}} c_\xi \sum_{h=1}^{|\xi|^2} F_\varepsilon^{\xi,h}(u),$$

where

$$F_\varepsilon^{\xi,h}(u) := \frac{1}{2} \sum_{i \in \mathbb{Z}_\varepsilon^{\xi,h}(\Omega)} f(u(i+\varepsilon\xi) - u(i)) + f(u(i+\varepsilon\xi^\perp) - u(i)) \tag{3.40}$$

and $\mathbb{Z}_\varepsilon^{\xi,h}(\Omega) := \Omega_\varepsilon \cap \varepsilon(z_h + \mathbb{Z}\xi \oplus \mathbb{Z}\xi^\perp)$. In the following we will extend the definition of $F_\varepsilon^{\xi,h}$ to any open $A \subset \mathbb{R}^2$, by using the standard notation $F_\varepsilon^{\xi,h}(\cdot, A)$.

Finally, for every $\xi \in \mathbb{Z}^2_+$ with $\xi \cdot e_1 \geq 0$ and for $i \in \varepsilon(z_h + \mathbb{Z}\xi \oplus \mathbb{Z}\xi^\perp)$, we set

$$\begin{aligned} \alpha_u^{\xi,h} := {} & P(u(i+\varepsilon\xi) - u(i)) + P(u(i+\varepsilon\xi + i + \varepsilon\xi^\perp) - u(i+\varepsilon\xi)) \\ & - P(u(i+\varepsilon\xi+\varepsilon\xi^\perp) - u(i+\varepsilon\xi^\perp)) - P(u(i+\varepsilon\xi^\perp) - u(i)), \end{aligned}$$

and define the ξ-discrete vorticity $\mu^{\xi,h}(u)$ as

$$\mu^{\xi,h}(u) := \sum_{\substack{i \in \Omega \cap \varepsilon(z_h + \mathbb{Z}\xi \oplus \mathbb{Z}\xi^\perp) \\ i + \varepsilon Q_\xi \subset \Omega}} \alpha_u(i) \delta_{i + \frac{\varepsilon}{2}(\xi + \xi^\perp)}.$$

We remark that if $\xi = e_1$, then the index h above is necessarily equal to 1 and hence $\mu^{e_1,1}(u)$ coincides with $\mu(u)$ defined in (2.6).

Here we state the zero-order Γ-convergence result for the functionals F_ε^{lr}. We remark that this result has been proved in [2] for the XY_ε^{lr}. Once again, we notice that assumption (2) on f implies $F_\varepsilon^{lr}(u) \geq XY_\varepsilon^{lr}(e^{2\pi i u})$.

Theorem 3.8. *The following* Γ*-convergence result holds.*

(i) *(Compactness) Let* $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ *be such that* $F_\varepsilon^{lr}(u_\varepsilon) \leq C|\log \varepsilon|$ *for some positive* C*. Then, up to a subsequence,* $\mu(u_\varepsilon) \overset{\text{flat}}{\to} \mu$*, for some* $\mu \in X$*.*

(ii) *(Localized* Γ*-liminf inequality) Let* $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ *be such that* $\mu(u_\varepsilon) \overset{\text{flat}}{\to} \mu = \sum_{i=1}^M d_i \delta_{x_i}$ *with* $d_i \in \mathbb{Z} \setminus \{0\}$ *and* $x_i \in \Omega$*. Then, there exists a constant* $C \in \mathbb{R}$ *such that, for any* $i = 1, \dots, M$ *and for every* $\sigma < \frac{1}{2}\mathrm{dist}(x_i, \partial\Omega \cup \bigcup_{j \neq i} x_j)$*, we have*

$$\liminf_{\varepsilon \to 0} F_\varepsilon^{lr}(u_\varepsilon, B_\sigma(x_i)) - \pi \sum_{\substack{\xi \in \mathbb{Z}^2_+ \\ \xi \cdot e_1 \geq 0}} c_\xi |\xi|^2 |d_i| \log \frac{\sigma}{\varepsilon} \geq C. \tag{3.41}$$

In particular

$$\liminf_{\varepsilon\to 0} F_\varepsilon^{lr}(u_\varepsilon) - \pi \sum_{\substack{\xi\in\mathbb{Z}^2_+\\ \xi\cdot e_1\geq 0}} c_\xi|\xi|^2|\mu|(\Omega)\log\frac{\sigma}{\varepsilon} \geq C.$$

(iii) *(Γ-limsup inequality) For every $\mu \in X$, there exists a sequence $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ such that $\mu(u_\varepsilon) \stackrel{\text{flat}}{\to} \mu$ and*

$$\sum_{\substack{\xi\in\mathbb{Z}^2_+\\ \xi\cdot e_1\geq 0}} c_\xi|\xi|^2\pi|\mu|(\Omega) \geq \limsup_{\varepsilon\to 0}\frac{F_\varepsilon^{lr}(u_\varepsilon)}{|\log\varepsilon|}.$$

Since the proof is based essentially on Theorem 2.2 and on the proof of Theorem 4.8 in [2] we briefly sketch it.

Sketch of the proof. Since $c_{e_1} > 0$ the compactness property is a direct consequence of Theorem 2.2(i).

As for the Γ-liminf inequality, fix $i \in \{1,\dots,M\}$, without loss of generality, we can assume that

$$\liminf_{\varepsilon\to 0} F_\varepsilon^{lr}(u_\varepsilon, B_\sigma(x_i)) - \pi \sum_{\substack{\xi\in\mathbb{Z}^2_+\\ \xi\cdot e_1\geq 0}} c_\xi|\xi|^2|d_i|\log\frac{\sigma}{\varepsilon}$$
$$= \lim_{\varepsilon\to 0} F_\varepsilon^{lr}(u_\varepsilon, B_\sigma(x_i)) - \pi \sum_{\substack{\xi\in\mathbb{Z}^2_+\\ \xi\cdot e_1\geq 0}} c_\xi|\xi|^2|d_i|\log\frac{\sigma}{\varepsilon} < +\infty.$$

Since each of the functionals $F_\varepsilon^{\xi,h}$ (defined in (3.40)) is bounded from above by $C|\log\varepsilon|$, using the same strategy in the proof of Theorem 4.8 in [2], one can show that for any ξ and for any $h = 1,\dots,|\xi|^2$

$$\|\mu^{\xi,h}(u_\varepsilon) - \mu(u_\varepsilon)\|_{\text{flat}} \to 0$$

and hence $\mu^{\xi,h}(u_\varepsilon) \stackrel{\text{flat}}{\to} \mu$. So we can apply Theorem 2.2 (ii) to each of the functionals $F_\varepsilon^{\xi,h}$ and hence there exists a universal constant C such that

$$\liminf_{\varepsilon\to 0} F_\varepsilon^{\xi,h}(u_\varepsilon) - \pi|d_i|\log\frac{\sigma}{\varepsilon} \geq C.$$

By summing over $h = 1,\dots,|\xi|^2$ and over ξ we get (3.41).

The proof of the Γ-limsup inequality is standard and left to the reader. □

Finally, we state the first order Γ-convergence result for F_ε^{lr}. To this aim we have to introduce the following discrete minimum problem

$$\gamma_F^{lr}(\varepsilon,\sigma) := \min_{u\in\mathcal{AF}_\varepsilon(B_\sigma)}\{F_\varepsilon^{lr}(u, B_\sigma) \,:\, 2\pi u(\cdot) = \theta(\cdot) \text{ on } \partial_\varepsilon B_\sigma\},$$

where the discrete boundary ∂_ε in (2.1) and $\theta(x)$ is the polar coordinate $\arctan x_2/x_1$, also referred to as the lifting of the function $\frac{x}{|x|}$.

The following Proposition is the analogous of Proposition 2.39 in the long range interaction case.

Proposition 3.9. *For any fixed $\sigma>0$, the following limit exists finite*
$$\lim_{\varepsilon\to 0}(\gamma_F^{lr}(\varepsilon,\sigma)-\pi\sum_{\substack{\xi\in\mathbb{Z}^2_+\\ \xi\cdot e_1\geq 0}} c_\xi|\xi|^2|\log\frac{\varepsilon}{\sigma}|)=:\gamma_F^{lr}\in\mathbb{R}. \tag{3.42}$$

Proof of (3.42). First, by scaling, it is easy to see that $\gamma_F(\varepsilon,\sigma)^{lr}=I^{lr}(\frac{\varepsilon}{\sigma})$ where $I^{lr}(t)$ is defined by
$$I^{lr}(t):=\min\left\{F_1^{lr}(u,B_{\frac{1}{t}})\,|\,2\pi u=\theta \text{ on } \partial_1 B_{\frac{1}{t}}\right\}.$$
We aim to prove that
$$0<t_1\leq t_2\ \Rightarrow\ I^{lr}(t_1)\leq\pi\sum_{\substack{\xi\in\mathbb{Z}^2_+\\ \xi\cdot e_1\geq 0}} c_\xi|\xi|^2\log\frac{t_2}{t_1}+I^{lr}(t_2)+O(t_2). \tag{3.43}$$
By (3.43) and by Theorem 3.8(ii), it follows that
$$\exists\lim_{t\to 0^+}(I^{lr}(t)-\pi\sum_{\substack{\xi\in\mathbb{Z}^2_+\\ \xi\cdot e_1\geq 0}} c_\xi|\xi|^2|\log t|)>-\infty.$$
We prove now that (3.43) holds true. To this end, set $A_{r,R}:=B_R\setminus B_r$, and let θ be the lifting of the function $\frac{x}{|x|}$. Since
$$|\nabla\theta(x)|=\sqrt{\left|\partial_{\frac{\xi}{|\xi|}}\theta(x)\right|^2+\left|\partial_{\frac{\xi^\perp}{|\xi|}}\theta(x)\right|^2}\leq\frac{c}{r}$$
for every $\xi\in\mathbb{Z}^2$ and for every $x\in A_{r,R}$, by standard interpolation estimates (see for instance [25] and [2]) and using assumption (3) on f, we have that, as $0<r<R\to\infty$,

$$\begin{aligned}
F_1^{lr}(\theta/2\pi,A_{r,R})&=\frac{1}{2}\sum_{\substack{\xi\in\mathbb{Z}^2_+\\ \xi\cdot e_1\geq 0}} c_\xi\sum_{h=1}^{|\xi|^2}F_1^{\xi,h}(\frac{\theta}{2\pi},A_{r,R})\\
&=\frac{1}{2}\sum_{\substack{\xi\in\mathbb{Z}^2_+\\ \xi\cdot e_1\geq 0}} c_\xi\sum_{h=1}^{|\xi|^2}\sum_{i\in\mathbb{Z}_1^{\xi,h}(A_{r,R})}\left|\frac{\theta(i+\xi)-\theta(i)}{|\xi|}\right|^2+\left|\frac{\theta(i+\xi^\perp)-\theta(i)}{|\xi|}\right|^2+\mathrm{O}(1/r)\\
&\leq\pi\sum_{\substack{\xi\in\mathbb{Z}^2_+\\ \xi\cdot e_1\geq 0}} c_\xi|\xi|^2\log\frac{R}{r}+\mathrm{O}(1/r).
\end{aligned} \tag{3.44}$$

Let u_2 be a minimizer for $I(t_2)$ and for any $i\in\mathbb{Z}^2$ define
$$u_1(i):=\begin{cases} u_2(i) & \text{if } |i|\leq\frac{1}{t_2}\\ \frac{\theta(i)}{2\pi} & \text{if } \frac{1}{t_2}\leq|i|\leq\frac{1}{t_1},\end{cases}$$
By (3.44) we have
$$I(1/R)\leq I(1/r)+\pi\sum_{\substack{\xi\in\mathbb{Z}^2_+\\ \xi\cdot e_1\geq 0}} c_\xi|\xi|^2\log\frac{r}{R}+O(1/r),$$

which yields (3.43) for $r = \frac{1}{t_2}$ and $R = \frac{1}{t_1}$. $\square$

To ease the notation, for any $\mu = \sum_{i=1}^{M} d_i \delta_{x_i}$ with $|d_i| = 1$ and $x_i \in \Omega$, we set

$$\mathbb{W}^{lr}(\mu) := \sum_{\substack{\xi \in \mathbb{Z}^2_+ \\ \xi \cdot e_1 \geq 0}} c_\xi |\xi|^2 \mathbb{W}(\mu),$$

where $\mathbb{W}$ is defined in (1.5).

Theorem 3.10. *The following* Γ*-convergence result holds.*

(i) *(Compactness) Let* $M \in \mathbb{N}$ *and let* $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ *be a sequence satisfying* $F^{lr}_\varepsilon(u_\varepsilon) - M\pi \sum_{\substack{\xi \in \mathbb{Z}^2_+ \\ \xi \cdot e_1 \geq 0}} c_\xi |\xi|^2 |\log \varepsilon| \leq C$. *Then, up to a subsequence,* $\mu(u_\varepsilon) \overset{\text{flat}}{\to} \mu$ *for some* $\mu = \sum_{i=1}^{N} d_i \delta_{x_i}$ *with* $d_i \in \mathbb{Z} \setminus \{0\}$, $x_i \in \Omega$ *and* $\sum_i |d_i| \leq M$. *Moreover, if* $\sum_i |d_i| = M$, *then* $\sum_i |d_i| = N = M$, *namely* $|d_i| = 1$ *for any* i.

(ii) *(*Γ*-*$\liminf$ *inequality) Let* $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ *be such that* $\mu(u_\varepsilon) \overset{\text{flat}}{\to} \mu$, *with* $\mu = \sum_{i=1}^{M} d_i \delta_{x_i}$ *with* $|d_i| = 1$ *and* $x_i \in \Omega$ *for every* i. *Then,*

$$\liminf_{\varepsilon \to 0} F^{lr}_\varepsilon(u_\varepsilon) - M\pi \sum_{\substack{\xi \in \mathbb{Z}^2_+ \\ \xi \cdot e_1 \geq 0}} c_\xi |\xi|^2 |\log \varepsilon| \geq \mathbb{W}^{lr}(\mu) + M\gamma^{lr}_F.$$

(iii) *(*Γ*-*$\limsup$ *inequality) Given* $\mu = \sum_{i=1}^{M} d_i \delta_{x_i}$ *with* $|d_i| = 1$ *and* $x_i \in \Omega$ *for every* i, *there exists* $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$ *with* $\mu(u_\varepsilon) \overset{\text{flat}}{\to} \mu$ *such that*

$$F^{lr}_\varepsilon(u_\varepsilon) - M\pi \sum_{\substack{\xi \in \mathbb{Z}^2_+ \\ \xi \cdot e_1 \geq 0}} c_\xi |\xi|^2 |\log \varepsilon| \to \mathbb{W}^{lr}(\mu) + M\gamma^{lr}_F.$$

The proof of the theorem above is the same of Theorem 2.6 and it is omitted.

CHAPTER 4

Metastability and dynamics of screw dislocations

In this chapter we apply the Γ-convergence analysis in Chapter 2 in order to study the dynamics of the discrete topological singularities. All the results proved here are in [4].

In Section 4.1 below we will prove the existence of many local minimizers for a large class of discrete energy functionals F_ε, which includes the screw dislocations energy but not the XY functionals. As a consequence we have that any solution to the gradient flow of F_ε starting from these minimizers is still. In Section 4.2 and 4.3 we introduce a purely variational approach in order to analyze an effective dynamics of the singularities which overcome the pinning effect due to the presence of local minima.

4.1. Analysis of local minimizers

In this section we will assume some further hypotheses for the energy density f in addition to (1), (2) and (3):

(4) $f \in C^0([-\frac{1}{2}, \frac{1}{2}]) \cap C^2((-\frac{1}{2}, \frac{1}{2}))$;

(5) There exists a neighborhood I of $\frac{1}{2}$ such that for every $x \in I$ we have $C_1(\frac{1}{2} - x)^2 < f(\frac{1}{2}) - f(x)$ for some $C_1 > 0$ and $\sup_{t\in(-\frac{1}{2},\frac{1}{2})} f''(t) < \frac{1}{9}C_1$;

(6) f is increasing in $[0, \frac{1}{2}]$ and even.

Notice that these conditions are satisfied by the energy density of the screw dislocations functionals, $f(a) = \mathrm{dist}^2(a, \mathbb{Z})$, while they are not satisfied by the energy density of the XY model.

4.1.1. Antipodal configurations and energy barriers. When a discrete singularity of $\mu(v)$ moves to a neighboring cell, then v has to pass through an antipodal configuration $v(i) = -v(j)$ (i.e., such that the corresponding phase u satisfies $\mathrm{dist}(u(i) - u(j), \mathbb{Z}) = \frac{1}{2}$). We will show that such configurations are energy barriers.

Lemma 4.1. *There exist $\alpha > 0$ and $E > 0$ such that the following holds: Let $u \in \mathcal{AF}_\varepsilon(\Omega)$ such that $\mathrm{dist}(u(i)-u(j), \mathbb{Z}) > \frac{1}{2}-\alpha$ for some $(i,j) \in \Omega^1_\varepsilon$. Then there exists a function w, with $w = u$ in $\Omega^0_\varepsilon\setminus\{i\}$ such that $F_\varepsilon(w) \le F_\varepsilon(u)-E$.*

Proof. As a consequence of assumption (5), it is easy to see that there exists $\gamma > 0$ and a positive constant C_2 such that

$$f(\tfrac{1}{2}) - f(\gamma) - f(\tfrac{1}{2} - \gamma) > C_2\,. \tag{4.1}$$

First, we prove the statement assuming $f \in C^2(\mathbb{R})$. In this case, assumption (5) implies that $f'(\frac{1}{2}) = 0$ and $|f''(\frac{1}{2})| > C_1$.

Without loss of generality we can assume that $u(i) = 0$. For sake of notation we set

$$E^i(u) = \sum_{|l-i|=\varepsilon} f(u(l))\,. \tag{4.2}$$

We will assume that $i \notin \partial_\varepsilon\Omega$, so that i has exactly four nearest neighbors, denoted by j, k_1, k_2 and k_3. The case $i \in \partial_\varepsilon\Omega$ is fully analogous (some explicit computations are indeed shorter), and left to the reader. By assumption

$$E^i(u) \geq f(\tfrac{1}{2} + \alpha) + \sum_{l=1}^{3} f(u(k_l))\,. \tag{4.3}$$

We will distinguish two cases.

Case 1: There exists at least a nearest neighbor, say k_1, such that $\mathrm{dist}(u(k_1), \mathbb{Z}) \geq \frac{1}{2} - \alpha$. In this case we have that

$$E^i(u) \geq 2f(\tfrac{1}{2} + \alpha) + f(u(k_2)) + f(u(k_3)). \tag{4.4}$$

Now there are two possibilities. In fact we may have either that $\mathrm{dist}(u(k_2), \mathbb{Z}) \vee \mathrm{dist}(u(k_3), \mathbb{Z}) < 3\alpha$, or that $\mathrm{dist}(u(k_2), \mathbb{Z}) \vee \mathrm{dist}(u(k_3), \mathbb{Z}) \geq 3\alpha$.

In the first case, set $w(i) = \gamma$ with γ as in (4.1). Then, by continuity we have

$$E^i(w) = 2f(\tfrac{1}{2} - \gamma) + 2f(\gamma) + \mathrm{o}(1),$$

where $\mathrm{o}(1) \to 0$ as $\alpha \to 0$. From (4.4) we have $E^i(u) \geq 2f(\frac{1}{2} + \alpha)$, which together with (4.1) yields

$$E^i(u) - E^i(w) \geq 2(f(\tfrac{1}{2} + \alpha) - f(\tfrac{1}{2})) + C_2 + \mathrm{o}(1) = C_2 + \mathrm{o}(1) \tag{4.5}$$

as $\alpha \to 0$. Suppose now that $\mathrm{dist}(u(k_2), \mathbb{Z}) \vee \mathrm{dist}(u(k_3), \mathbb{Z}) \geq 3\alpha$. Then we define $w(i) = \frac{1}{2}$ and we get

$$E^i(w) \leq 2f(\alpha) + f(\tfrac{1}{2}) + f(\tfrac{1}{2} + 3\alpha)\,.$$

Moreover, thanks to assumption (6) of f we have $E^i(u) \geq 2f(\frac{1}{2}+\alpha)+f(3\alpha)$. We conclude that

$$E^i(u) - E^i(w) \geq \frac{7}{2}\alpha^2(f''(0) - f''(\tfrac{1}{2})) \geq \frac{7}{2}\alpha^2 C_1 \tag{4.6}$$

Case 2: For every i it holds $\mathrm{dist}(u(k_i), \mathbb{Z}) < \frac{1}{2} - \alpha$. Set $w(i) = \eta$ with $|\eta| = 3\alpha$ and $\eta \sum_{l=1}^{3} f'(u(k_l)) \geq 0$. Then

$$\begin{aligned}
E^i(u) - E^i(w) &\geq f(\tfrac{1}{2} + \alpha) - f(\tfrac{1}{2} + \alpha - |\eta|) + \sum_{l=1}^{3} f(u(k_l)) - f(u(k_l) - \eta) \\
&= \frac{1}{2}|f''(\tfrac{1}{2})||\eta|(|\eta| - 2\alpha) + \eta\sum_{l=1}^{3} f'(u(k_l)) - \frac{1}{2}\eta^2\sum_{l=1}^{3} f''(u(k_l)) + o(\eta^2) \\
&\geq \frac{1}{2}|f''(\tfrac{1}{2})|3\alpha^2 - \frac{9}{2}\alpha^2\sum_{l=1}^{3} f''(u(k_l)) + o(\alpha^2) \geq \frac{3}{2}(C_1 - 9\sup_t f''(t))\alpha^2 + o(\alpha^2)
\end{aligned} \tag{4.7}$$

The combination of Step 1 and Step 2 allows to conclude the proof in the case of $f \in C^2(\mathbb{R})$, by choosing α small enough and $E = (7C_1 \wedge 3(C_1 -$

$9\sup_t f''(t)))\alpha^2/2$. The general case can be recover by approximating f in a neighborhood of $\frac{1}{2}$ with C^2 functions still satisfying assumptions (4)-(6). $\square$

Note that in the case of $f(a) = \mathrm{dist}^2(a, \mathbb{Z})$ the proof of the above Lemma can be obtained by a direct computation without the regularization.

Remark 4.2. Note that the function w constructed in Lemma 4.1 has a discrete vorticity that can be different from the one of u only in the four ε-squares sharing i as a vertex, and hence $\|\mu(u) - \mu(w)\|_{\mathrm{flat}} \le 2\varepsilon$.

Definition 4.3. We say that a function $u \in \mathcal{AF}_\varepsilon(\Omega)$ satisfies the α-cone condition if

$$\mathrm{dist}(u(i) - u(j), \mathbb{Z}) \le \frac{1}{2} - \alpha \qquad \text{for every} \quad (i,j) \in \Omega^1_\varepsilon \,.$$

Remark 4.4. Note that if $u \in \mathcal{AF}_\varepsilon(\Omega)$ satisfies the α-cone condition for some $\alpha > 0$, then for every $w \in \mathcal{AF}_\varepsilon(\Omega)$ such that $\sum_{i \in \Omega^0_\varepsilon} |w(i) - u(i)|^2 < \frac{\alpha^2}{16}$ we have $\mu(w) = \mu(u)$. In other words, the vorticity measure $\mu(u)$ is stable with respect to small variations of u.

4.1.2. Metastable configurations and pinning. As a consequence of Lemma 4.1 we prove the existence of a minimizer for the energy F_ε, under assumptions (1)-(6) with singularities close to prescribed positions.

Theorem 4.5. *Given $\mu_0 = \sum_{i=1}^M d_i \delta_{x_i}$ with $x_i \in \Omega$ and $d_i \in \{1, -1\}$ for $i = 1, \dots, M$, there exists a constant $K \in \mathbb{N}$ such that, for ε small enough, there exists $k_\varepsilon \in \{1, \dots, K\}$ such that the following minimum problem is well-posed*

$$\min\{F_\varepsilon(u) \, : \, \|\mu(u) - \mu_0\|_{\mathrm{flat}} \le k_\varepsilon \varepsilon\}. \tag{4.8}$$

Moreover, let α be given by Lemma 4.1; any minimizer u_ε of the problem in (4.8) satisfies the α-cone condition and it is a local minimizer for F_ε.

Proof. For any $k \in \mathbb{N} \cup \{0\}$, we set

$$I^k_\varepsilon := \inf\{F_\varepsilon(u) \, : \, \|\mu(u) - \mu_0\|_{\mathrm{flat}} \le (M + 2k)\varepsilon\}, \tag{4.9}$$

By constructing explicit competitors one can show that

$$I^0_\varepsilon \le M\pi |\log \varepsilon| + C. \tag{4.10}$$

Then, we consider a minimizing sequence $\{u^{k,n}_\varepsilon\}$ for I^k_ε. It is not restrictive to assume that $0 \le u^{k,n}_\varepsilon(i) \le 1$ for any $i \in \Omega^0_\varepsilon$; therefore, up to a subsequence, $u^{k,n}_\varepsilon \to u^k_\varepsilon$ as $n \to \infty$ for some $u^k_\varepsilon \in \mathcal{AF}_\varepsilon(\Omega)$. Note that if u^k_ε satisfies the α-cone condition, then it is a minimizer for I^k_ε.

Set $\bar{k} := \lceil \frac{C - \mathbb{W}(\mu_0) - M\gamma}{E} \rceil + 1$ and assume by contradiction that there exists a subsequence, still labelled with ε, such that for every $k \in \{0, 1, \dots, \bar{k}\}$, there exists a bond $(i_\varepsilon, j_\varepsilon) \in \Omega^1_\varepsilon$, with $\mathrm{dist}(u^k_\varepsilon(i_\varepsilon) - u^k_\varepsilon(j_\varepsilon), \mathbb{Z}) > \frac{1}{2} - \alpha$. Thus, for n large enough, we have

$$\mathrm{dist}(u^{k,n}_\varepsilon(i_\varepsilon) - u^{k,n}_\varepsilon(j_\varepsilon), \mathbb{Z}) > \frac{1}{2} - \alpha.$$

By Lemma 4.1, there exists a function $w^{k,n}_\varepsilon \in \mathcal{AF}_\varepsilon(\Omega)$ such that $w^{k,n}_\varepsilon \equiv u^{k,n}_\varepsilon$ in $\Omega^0_\varepsilon \setminus \{i\}$ and $F_\varepsilon(w^{k,n}_\varepsilon) \le F_\varepsilon(u^{k,n}_\varepsilon) - E$ for some $E > 0$. By construction (see Remark 4.2) we have that $\|\mu(w_\varepsilon^{k,n}) - \mu(u_\varepsilon^{k,n})\|_{\mathrm{flat}} \le 2\varepsilon$. It follows that

$$I_\varepsilon^{k+1} \le F_\varepsilon(w_\varepsilon^{k,n}) \le I_\varepsilon^k - E.$$

By an easy induction argument on k and by (4.10), we have immediately that

$$I_\varepsilon^k \le I_\varepsilon^0 - kE \le M\pi|\log\varepsilon| + C - kE. \tag{4.11}$$

By the lower bound (2.42) in Theorem 2.6, (4.11), and the definition of $\bar{k}$ we get

$$\mathbb{W}(\mu_0) + M\gamma \le \liminf_{\varepsilon\to 0} I_\varepsilon^{\bar{k}} - M\pi|\log\varepsilon| \le C - \bar{k}E \le \mathbb{W}(\mu_0) + M\gamma - E,$$

and so the contradiction. Then the statement holds for $K = M + 2\bar{k}$.

□

Let $\varepsilon > 0$ and let $u_\varepsilon^0 \in \mathcal{AF}_\varepsilon(\Omega)$. We say that $u_\varepsilon = u_\varepsilon(t)$ is a gradient flow of F_ε from u_ε^0 if u_ε is a solution of

$$\begin{cases} \frac{1}{|\log\varepsilon|}\dot{u}_\varepsilon = -\nabla F_\varepsilon(u_\varepsilon) & \text{in } (0,+\infty)\times\Omega_\varepsilon^0 \\ u_\varepsilon(0) = u_\varepsilon^0 & \text{in } \Omega_\varepsilon^0. \end{cases}$$

Clearly $u_\varepsilon(t) \in \mathcal{AF}_\varepsilon(\Omega)$, and we will write $u_\varepsilon(t,i)$ in place of $u_\varepsilon(t)(i)$.

Theorem 4.6. *Let $\mu_0 = \sum_{i=1}^M d_i\delta_{x_i}$ with $x_i \in \Omega$ and $d_i \in \{1,-1\}$ for $i = 1,\dots,M$. Let $\{u_\varepsilon^0\} \subset \mathcal{AF}_\varepsilon(\Omega)$ be such that*

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon^0) - M\pi|\log\varepsilon| = \mathbb{W}(\mu_0) + M\gamma. \tag{4.12}$$

Let α be given by Lemma 4.1. Then, for ε small enough, the following facts hold.

(i) *u_ε^0 satisfy the α-cone condition.*
(ii) *The gradient flow $u_\varepsilon(t)$ of F_ε from u_ε^0 satisfies $\mu(u_\varepsilon(t)) = \mu(u_\varepsilon^0)$ for every $t > 0$.*
(iii) *There exists $\bar{u}_\varepsilon^0$ such that $\bar{u}_\varepsilon^0 \in \operatorname{argmin}\{F_\varepsilon(u) : \mu(u) = \mu(u_\varepsilon^0)\}$. Moreover $\bar{u}_\varepsilon^0$ satisfies the α-cone condition and it is a local minimizer for F_ε.*

Proof. *Proof of (i).* Assume, by contradiction, that there exists a sequence $\varepsilon_k \to 0$ such that $u_{\varepsilon_k}^0$ does not satisfy the α-cone condition, namely for every $k \in \mathbb{N}$ there exists a bond $(i_k, j_k) \in \Omega_{\varepsilon_k}^1$ with

$$\operatorname{dist}(u_{\varepsilon_k}^0(i_k) - u_{\varepsilon_k}^0(j_k), \mathbb{Z}) > \frac{1}{2} - \alpha.$$

By Lemma 4.1, for any k there exists a function $w_{\varepsilon_k} \in \mathcal{AF}_{\varepsilon_k}(\Omega)$ such that $w_{\varepsilon_k} \equiv u_{\varepsilon_k}^0$ in $\Omega_{\varepsilon_k}^0 \setminus \{i_k\}$ and

$$F_{\varepsilon_k}(w_{\varepsilon_k}) \le F_\varepsilon(u_{\varepsilon_k}^0) - E \le F_{\varepsilon_k}(u_{\varepsilon_k}^0) - E. \tag{4.13}$$

Moreover, by construction (see Remark 4.2) we have that $\|\mu(w_{\varepsilon_k}) - \mu(u_{\varepsilon_k}^0)\|_{\mathrm{flat}} \le 2\varepsilon_k$ and so $\mu(w_{\varepsilon_k}) \stackrel{\mathrm{flat}}{\to} \mu_0$. By the lower bound (2.42) in Theorem 2.6, we get

$$\mathbb{W}(\mu_0) + M\gamma \le \liminf_{\varepsilon_k\to 0} F_{\varepsilon_k}(w_{\varepsilon_k}) - M\pi|\log\varepsilon_k| \tag{4.14}$$

$$\le \lim_{\varepsilon_k\to 0} F_{\varepsilon_k}(u_{\varepsilon_k}^0) - M\pi|\log\varepsilon_k| - E = \mathbb{W}(\mu_0) + M\gamma - E,$$

and so the contradiction.

Proof of (ii). Assume, by contradiction, that there exists a sequence $\varepsilon_k \to 0$ such that the gradient flows $u_{\varepsilon_k}(t)$ of F_{ε_k} from $u^0_{\varepsilon_k}$ does not satisfy (ii). Let t_k be the first time (in fact, the infimum) for which $\mu(u_{\varepsilon_k}(t_k)) \neq \mu(u^0_{\varepsilon_k})$; then, there exists $(i_k, j_k) \in \Omega^1_{\varepsilon_k}$ such that $\mathrm{dist}(u_{\varepsilon_k}(t_k, i_k) - u_{\varepsilon_k}(t_k, j_k), \mathbb{Z}) > \frac{1}{2} - \alpha$. By Lemma 4.1 there exists $w_{\varepsilon_k}(t_k) \in \mathcal{AF}_{\varepsilon_k}(\Omega)$ such that $w_{\varepsilon_k}(t_k) \equiv u_{\varepsilon_k}(t_k)$ in $\Omega^0_{\varepsilon_k} \setminus \{i_k\}$ and $F_{\varepsilon_k}(w_{\varepsilon_k}(t_k)) \leq F_{\varepsilon_k}(u_{\varepsilon_k}(t_k)) - E$, for some positive constant E independent of k. Moreover, by (4.2), we have that

$$\|\mu(u^0_{\varepsilon_k}) - \mu(w_{\varepsilon_k}(t_k))\|_{\mathrm{flat}} = \|\mu(u_{\varepsilon_k}(t_k)) - \mu(w_{\varepsilon_k}(t_k))\|_{\mathrm{flat}} \leq 2\varepsilon_k;$$

Therefore, by the lower bound (2.42) in Theorem 2.6, arguing as in (4.14), we get a contradiction.

Proof of (iii). Let $\{u^n_\varepsilon\}$ be a minimizing sequence for the minimum problem in (iii). We can always assume that $0 \leq u^n_\varepsilon(i) \leq 1$ for any $i \in \Omega^0_\varepsilon$; therefore, up to a subsequence, $u^n_\varepsilon \to \bar{u}^0_\varepsilon$ as $n \to \infty$ for some $\bar{u}^0_\varepsilon \in \mathcal{AF}_\varepsilon(\Omega)$. To prove that $\bar{u}^0_\varepsilon$ (for ε small enough) is a minimizer, it is enough to show that $\mu(\bar{u}^0_\varepsilon) = \mu(u_\varepsilon)$; this follows once we have proved that $\bar{u}^0_\varepsilon$ satisfies the α-cone condition (see Remark 4.4). Assume by contradiction that there exists a sequence $\varepsilon_k \to 0$ such that $\mathrm{dist}(\bar{u}^0_\varepsilon(i_k) - \bar{u}^0_\varepsilon(j_k), \mathbb{Z}) > \frac{1}{2} - \alpha$ for some bond $(i_k, j_k) \in \Omega^1_{\varepsilon_k}$. Then, for n large enough, we have

$$F_{\varepsilon_k}(u^n_{\varepsilon_k}) \leq F_{\varepsilon_k}(u^0_{\varepsilon_k}) + \varepsilon_k, \qquad \mathrm{dist}(u^n_{\varepsilon_k}(i) - u^n_{\varepsilon_k}(j), \mathbb{Z}) > \frac{1}{2} - \alpha\,. \tag{4.15}$$

Let $\bar{n}$ so that (4.15) holds. By Lemma 4.1, there exists a function $w_{\varepsilon_k} \in \mathcal{AF}_{\varepsilon_k}(\Omega)$ such that $w_{\varepsilon_k} \equiv u^{\bar{n}}_{\varepsilon_k}$ in $\Omega^0_{\varepsilon_k} \setminus \{i\}$ and

$$F_{\varepsilon_k}(w_{\varepsilon_k}) \leq F_\varepsilon(u^{\bar{n}}_{\varepsilon_k}) - E \leq F_{\varepsilon_k}(u^0_{\varepsilon_k}) - E + \varepsilon_k. \tag{4.16}$$

By construction (see Remark 4.2), we have that $\|\mu(w_{\varepsilon_k}) - \mu(u^0_{\varepsilon_k})\|_{\mathrm{flat}} = \|\mu(w_{\varepsilon_k}) - \mu(u^{\bar{n}}_{\varepsilon_k})\|_{\mathrm{flat}} \leq 2\varepsilon_k$. Therefore, by the lower bound (2.42) in Theorem 2.6, arguing as in (4.14), we get a contradiction.

Finally, by the α-cone condition and Remark 4.4, we have immediately that $F_\varepsilon(\bar{u}^0_\varepsilon) \leq F_\varepsilon(w)$ for any function $w \in \mathcal{AF}_\varepsilon(\Omega)$ with $\|w - u\|_{L^2} \leq \frac{\alpha}{4}$, and hence $\bar{u}^0_\varepsilon$ is a local minimizer of F_ε .

□

Remark 4.7. By Theorem 4.6 it easily follows that there exists $t_n \to \infty$ such that $u^\infty_\varepsilon := \lim_{t_n \to \infty} u_\varepsilon(t_n)$ is a critical point of F_ε.

4.2. Discrete gradient flow of $\mathcal{F}_\varepsilon$ with flat dissipation

In Section 4.1 we have seen that the energy $\mathcal{F}_\varepsilon$ has many local minimizers. In particular, Theorem 4.5 shows that the length-scale of metastable configurations of singularities is of order ε. In this section we consider a discrete in time gradient flow of the energy F_ε which allows to overcome the pinning effect due to the presence of local minima and then to study an effective dynamics of the vortices. This is done following the minimizing movements method.

It turns out that, for ε smaller than the time step τ, the vortices overcome the energetic barriers and the dynamics is described (as $\varepsilon, \tau \to 0$) by the gradient flow of the renormalized energy (see Definition 4.10). This process requires the introduction of a suitable dissipation. In this section we consider a dissipation which is continuous with respect to the flat norm. To this purpose, we notice that, identifying each $\mu = \sum_{i=1}^{N} d_i\delta_{x_i}$ with a 0-current, it can be shown that

$$\|\mu\|_{\text{flat}} = \min\{|S|\,,\, S \text{ 1-current},\, \partial S\llcorner\Omega = \mu\} \tag{4.17}$$

(see [33, Section 4.1.12]). Moreover, it is an established result in the optimal transport theory (see for instance [68, Theorem 5.30]) that the minimization in (4.17) can be restricted to the family

$$\mathcal{S}(\mu) := \Bigg\{ S = \sum_{l=1}^{L} m_l[p_l, q_l] \,:\, L \in \mathbb{N}, m_l \in \mathbb{Z}, \,,\; p_l\,, q_l \in \operatorname{supp}(\mu) \cup \partial\Omega\,,$$
$$\partial S\llcorner\Omega = \sum_{l=1}^{L} m_l(\delta_{q_l} - \delta_{p_l})\llcorner\Omega = \mu \Bigg\}\,,$$

where $m[p, q]$ denotes the 1-rectifiable current supported on the oriented segment of vertices p and q, and with multiplicity m (for a self-contained proof of this fact we refer also to [52, Proposition 4.4]). Notice that, given $S \in \mathcal{S}(\mu)$, $|S| = \sum_{l=1}^{L} |m_l||q_l - p_l|$.

We define our dissipation in two steps.

First assume that $\nu_1 = \sum_{i=1}^{N_1} d_i^1\delta_{x_i^1}$ and $\nu_2 = \sum_{j=1}^{N_2} d_j^2\delta_{x_j^2}$ with $d_i^1, d_j^2 \in \mathbb{N}$ for every $i = 1, \dots, N_1$ and $j = 1, \dots, N_2$ and set

$$\tilde{D}_2(\nu_1, \nu_2) := \min \Bigg\{ \sum_{l=1}^{L} |q_l - p_l|^2 \,:\, L \in \mathbb{N},\, q_l \in \operatorname{supp}(\nu_1) \cup \partial\Omega,\, p_l \in \operatorname{supp}(\nu_2) \cup \partial\Omega,$$
$$\sum_{l=1}^{L} \delta_{q_l}\llcorner\Omega = \nu_1, \sum_{l=1}^{L} \delta_{p_l}\llcorner\Omega = \nu_2 \Bigg\}.$$

It is easy to see that $\tilde{D}_2^{\frac{1}{2}}$ is a distance. Actually, $\|\nu_1 - \nu_2\|_{\text{flat}}$ and $D_2(\nu_1, \nu_2)$ can be rewritten as

$$\|\nu_1 - \nu_2\|_{\text{flat}} = \min_{\lambda} \int_{\bar{\Omega}\times\bar{\Omega}} |x - y| d\lambda(x, y)\,,$$
$$\tilde{D}_2(\nu_1, \nu_2) = \min_{\lambda} \int_{\bar{\Omega}\times\bar{\Omega}} |x - y|^2 d\lambda(x, y)\,,$$

where the minimum is taken over all measures λ which are sums of Dirac deltas in $\bar{\Omega} \times \bar{\Omega}$ with integer coefficients, and have marginals restricted to Ω given by ν_1 and ν_2. This clarifies the connection of the flat distance and of our dissipation with the Wasserstein distances W_1 and W_2, defined on pairs of probability measures in $\mathbb{R}^2$, respectivelly (see for instance [68]).

From the very definition of $\tilde{D}_2$ one can easily check that

$$\tilde{D}_2(\nu_1 + \rho_1, \nu_2 + \rho_2) \leq \tilde{D}_2(\nu_1, \nu_2) + \tilde{D}_2(\rho_1, \rho_2) \tag{4.18}$$

for any ρ_1 and ρ_2 sums of positive Dirac masses, and

$$\tilde{D}_2(\nu_1, \nu_2) \leq \operatorname{diam}(\Omega)\|\nu_1 - \nu_2\|_{\text{flat}}\,. \tag{4.19}$$

For the general case of $\mu_1 = \sum_{i=1}^{N_1} d_i^1 \delta_{x_i^1}$ and $\mu_2 = \sum_{i=1}^{N_2} d_i^2 \delta_{x_i^2}$ with $d_i^1, d_i^2 \in \mathbb{Z}$ we set

$$D_2(\mu_1, \mu_2) := \tilde{D}_2(\mu_1^+ + \mu_2^-, \mu_2^+ + \mu_1^-), \tag{4.20}$$

where μ_j^+ and μ_j^- are the positive and the negative part of μ_j. As a consequence of (4.18) and (4.19) we have that D_2 is continuous with respect to the flat norm.

We are now in a position to introduce the discrete gradient flow of F_ε with respect to the dissipation D_2.

Definition 4.8. Fix $\delta > 0$ and let $\varepsilon, \tau > 0$. Given $\mu_{\varepsilon,0} \in X_\varepsilon$, we say that $\{\mu_{\varepsilon,k}^\tau\}$, with $k \in \mathbb{N} \cup \{0\}$, is a solution of the flat discrete gradient flow of $\mathcal{F}_\varepsilon$ from $\mu_{\varepsilon,0}$ if $\mu_{\varepsilon,0}^\tau = \mu_{\varepsilon,0}$, and for any $k \in \mathbb{N}$, $\mu_{\varepsilon,k}^\tau$ satisfies

$$\mu_{\varepsilon,k}^\tau \in \operatorname{argmin} \left\{ \mathcal{F}_\varepsilon(\mu) + \frac{\pi \mathrm{D}_2(\mu, \mu_{\varepsilon,k-1}^\tau)}{2\tau} : \mu \in X_\varepsilon, \, \|\mu - \mu_{\varepsilon,k-1}^\tau\|_{\mathrm{flat}} \leq \delta \right\}. \tag{4.21}$$

Notice that the existence of a minimizer is obvious, since μ lies in X_ε which is a finite set.

We want to analyze the limit as $\varepsilon \to 0$ of the flat discrete gradient flow. To this purpose, let $\mu_0 := \sum_{i=1}^M d_{i,0} \delta_{x_{i,0}}$ with $|d_{i,0}| = 1$, and let $\mu_{\varepsilon,0} \in X_\varepsilon$ be such that

$$\mu_{\varepsilon,0} \overset{\text{flat}}{\to} \mu_0, \qquad \lim_{\varepsilon \to 0} \frac{\mathcal{F}_\varepsilon(\mu_{\varepsilon,0})}{|\log \varepsilon|} = \pi |\mu_0|(\Omega).$$

In Theorem 4.14 we will show that, as $\varepsilon \to 0$, the sequence $\mu_{\varepsilon,k}^\tau$ converges to some $\mu_k^\tau \in X$, whose singularities have the same degrees of those of the initial datum. Therefore, it is convenient to regard the renormalized energy as a function only of the positions of M singularities. To this end we introduce the following notation

$$W(x) := \mathbb{W}(\mu) \qquad \text{where } \mu = \sum_{i=1}^M d_{i,0} \delta_{x_i} \qquad \text{for every } x \in \Omega^M.$$

The right notion for the limit as $\varepsilon \to 0$ of flat discrete gradient flows of $\mathcal{F}_\varepsilon$ is given by the following definition of discrete gradient flow of the renormalized energy.

Definition 4.9. Let $\delta > 0$, $K \in \mathbb{N} \cup \{0\}$, and $\tau > 0$. Fix $x_0 \in \Omega^M$. We say that $\{x_k^\tau\}$ with $k = 0, 1, \ldots, K$, is a solution of the discrete gradient flow of W from x_0 if $x_0^\tau = x_0$ and, for any $k = 1, \ldots, K$, $x_k^\tau \in \Omega^M$ satisfies

$$x_k^\tau \in \operatorname{argmin} \left\{ W(x) + \frac{\pi |x_k^\tau - x_{k-1}^\tau|^2}{2\tau} \, : \, x \in \Omega^M, \, \sum_{i=1}^M |x_i - x_{i,k-1}^\tau| \leq \delta \right\}, \tag{4.22}$$

where $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^k$ for any $k \in \mathbb{N}$.

In Theorem 4.13 we show that, as $\tau \to 0$, this discrete time evolution converges, until a maximal time $\tilde{T}_\delta$, to the gradient flow of the renormalized energy given by the following definition.

Definition 4.10. Let $M \in \mathbb{N}$ and $x_0 \in \Omega^M$. The gradient flow of the renormalized energy from x_0 is given by

$$\begin{cases} \dot{x}(t) = -\frac{1}{\pi}\nabla W(x(t)) \\ x(0) = x_0. \end{cases} \tag{4.23}$$

We denote by T^* the maximal time of existence of the solution, and we notice that until the time T^* the solution is unique, and that T^* is the minimal critical time among the first collision time and the exit time from Ω.

As $\delta \to 0$, $\tilde{T}_\delta$ converges to the critical time T^*. Notice that the renormalized energy is not bounded from below and it blows up to $-\infty$ whenever one of these critical events occur. This justifies the introduction of the parameter δ, in order to explore local minima. Nevertheless, the solutions of flat discrete gradient flows defined above do not touch the constraint and hence, they satisfy the corresponding unconstrained Euler-Lagrange equations.

4.2.1. Flat discrete gradient flow of W. Fix initial conditions

$$x_0 = (x_{1,0}, \dots, x_{M,0}) \in \Omega^M, \qquad d_{1,0}, \dots, d_{M,0} \in \{-1, 1\},$$

and fix $\delta > 0$ such that

$$\min\{\frac{1}{2}\mathrm{dist}_{i\neq j}(x_{i,0}, x_{j,0}), \mathrm{dist}(x_{i,0}, \partial\Omega)\} - 2\delta =: c_\delta > 0. \tag{4.24}$$

Definition 4.11. We say that a solution of the discrete gradient flow $\{x_k^\tau\}$ of W from x_0 is maximal if the minimum problem in (4.22) does not admit a solution for $k = K + 1$.

Let $\{x_k^\tau\}$ be a maximal solution of the flat discrete gradient flow of W from x_0, according with Definitions 4.9, 4.11; we set

$$k_\delta^\tau = k_\delta^\tau(\{x_k^\tau\}) := \min\{k \in \{1, \dots, K\} \ : \\ \min\{\frac{1}{2}\mathrm{dist}_{i\neq j}(x_{i,k}^\tau, x_{j,k}^\tau), \mathrm{dist}(x_{i,k}^\tau, \partial\Omega)\} \le 2\delta\,\}. \tag{4.25}$$

We notice that, since $|x_{k_\delta^\tau}^\tau - x_{k_\delta^\tau - 1}^\tau| \le \delta$ and

$$\min\{\frac{1}{2}\mathrm{dist}_{i\neq j}(x_{i,k_\delta^\tau-1}^\tau, x_{j,k_\delta^\tau-1}^\tau), \mathrm{dist}(x_{i,k_\delta^\tau-1}^\tau, \partial\Omega)\} > 2\delta,$$

then

$$\min\{\frac{1}{2}\mathrm{dist}_{i\neq j}(x_{i,k_\delta^\tau}^\tau, x_{j,k_\delta^\tau}^\tau), \mathrm{dist}(x_{i,k_\delta^\tau}^\tau, \partial\Omega)\} > \delta,$$

i.e., $k_\delta^\tau < K$. It follows that, for any $k = 0, 1, \dots, k_\delta^\tau$, we have

$$x_k^\tau \in K_\delta, \tag{4.26}$$

where K_δ is the compact set given by

$$K_\delta := \left\{ x \in \Omega^M \ : \ \min\{\frac{1}{2}\mathrm{dist}_{i\neq j}(x_i, x_j), \mathrm{dist}(x_i, \partial\Omega)\} \ge \delta \right\}. \tag{4.27}$$

Notice that W is smooth on K_δ. In particular, we can set

$$C_\delta := \max_{x \in K_\delta}(W(x_0) - W(x)). \tag{4.28}$$

Proposition 4.12. *For τ small enough the following holds. For every $k = 1, \dots, k_\delta^\tau$, we have that $\sum_{i=1}^M |x_{i,k}^\tau - x_{i,k-1}^\tau| < \delta$ and*

$$\partial_{x_i} W(x_k^\tau) + \pi \frac{x_{i,k}^\tau - x_{i,k-1}^\tau}{\tau} = 0 \qquad \text{for } i = 1, \dots, M. \tag{4.29}$$

In particular, for every $k = 1, \dots, k_\delta^\tau$

$$|x_k^\tau - x_{k-1}^\tau| \le \max_{x \in K_\delta} |\nabla W(x)| \tau. \tag{4.30}$$

Proof. Since the energy W is clearly decreasing in k, for every $k = 1, \dots, k_\delta^\tau$ we have

$$\frac{|x_k^\tau - x_{k-1}^\tau|^2}{2\tau} \le \frac{1}{\pi}(W(x_{k-1}^\tau) - W(x_k^\tau)) \le W(x_0) - W(x_k^\tau) \le C_\delta. \tag{4.31}$$

It follows that for τ small enough $\sum_{i=1}^M |x_{i,k}^\tau - x_{i,k-1}^\tau| < \delta$. Therefore, the minimality of x_k^τ clearly implies (4.29), as well as (4.30). $\square$

Let $x(t)$ be the solution of the gradient flow of W with initial datum x_0 (see (4.23)) and let T^* be its maximal existence time. We set

$$T_\delta := \inf \left\{ t \in [0, T^*] : \ \min\{\frac{1}{2} \mathrm{dist}_{i \neq j}(x_i(t), x_j(t)), \mathrm{dist}(x_i(t), \partial\Omega)\} \le 2\delta \right\}. \tag{4.32}$$

Notice that by definition we have

$$\lim_{\delta \to 0} T_\delta = T^*. \tag{4.33}$$

For $0 \le t \le k_\delta^\tau \tau$, we denote by $x^\tau(t) = (x_1^\tau(t), \dots, x_M^\tau(t))$ the piecewise affine in time interpolation of $\{x_k^\tau\}$.

Theorem 4.13. *Let $\{x_k^\tau\}_{\tau > 0}$ be a family of maximal solutions of the flat discrete gradient flow of W from x_0. Then,*

$$\tilde{T}_\delta := \liminf_{\tau \to 0} k_\delta^\tau \tau \ge T_\delta, \tag{4.34}$$

where k_δ^τ is defined in (4.25) and T_δ is defined in (4.32).

Moreover, for every $0 < T < \tilde{T}_\delta$, $x^\tau \to x$ uniformly on $[0, T]$. Finally, $\tilde{T}_\delta \to T^*$ as $\delta \to 0$.

Proof. By the very definition of k_δ^τ, it is easy to prove that

$$|x_{k_\delta^\tau}^\tau - x_0^\tau| > c_\delta\,,$$

where c_δ is defined in (4.24). Moreover, by (4.30), for τ small enough we get

$$|x_{k_\delta^\tau}^\tau - x_0^\tau| \le \sum_{k=1}^{k_\delta^\tau} |x_k^\tau - x_{k-1}^\tau| \le \max_{x \in K_\delta} |\nabla W(x)| k_\delta^\tau \tau,$$

and hence

$$k_\delta^\tau \tau \ge \frac{c_\delta}{\max_{x \in K_\delta} |\nabla W(x)|} > 0.$$

From (4.30) it is easy to see that x^τ are equibounded and equicontinuous in $[0, \tau k_\delta^\tau]$, and hence by Ascoli Arzelà Theorem, they uniformly converge, up to a subsequence, to a function x on $[0,T]$, for every $T<\tilde T_\delta$. Let $t\in(0,\tilde T_\delta)$ and let $h>0$, by (4.29) we get

$$x^\tau(\tau\lfloor(t+h)/\tau\rfloor)-x^\tau(\tau\lfloor t/\tau\rfloor)=\sum_{k=\lfloor t/\tau\rfloor}^{\lfloor(t+h)/\tau\rfloor-1}x^\tau_{k+1}-x^\tau_k=-\frac{\tau}{\pi}\sum_{k=\lfloor t/\tau\rfloor}^{\lfloor(t+h)/\tau\rfloor-1}\nabla W(x^\tau_k). \tag{4.35}$$

Taking the limit as $\tau\to0$, and then $h\to0$, we obtain that the limit x is the unique solution of (4.23).

Moreover, it is easy to see that $x^\tau(\tau k^\tau_\delta)\to x(\tilde T_\delta)$ and hence by definition of k^τ_δ, it is easy to see that (4.34) holds true and $\tilde T_\delta<T^*$. Since $T_\delta\to T^*$ (see (4.33)) we conclude that $\tilde T_\delta\to T^*$ as $\delta\to0$.

□

4.2.2. Flat discrete gradient flow of $\mathcal F_\varepsilon$. We are now in a position to state and prove the convergence of discrete gradient flows as $\varepsilon\to0$.

Theorem 4.14. *Let $\mu_0:=\sum_{i=1}^M d_{i,0}\delta_{x_{i,0}}$ with $|d_{i,0}|=1$. Let $\mu_{\varepsilon,0}\in X_\varepsilon$ be such that*

$$\mu_{\varepsilon,0}\overset{\text{flat}}{\to}\mu_0,\qquad \lim_{\varepsilon\to0}\frac{\mathcal F_\varepsilon(\mu_{\varepsilon,0})}{|\log\varepsilon|}=\pi|\mu_0|(\Omega).$$

Let $\delta>0$ be fixed such that $\min\left\{\frac12\mathrm{dist}_{i\neq j}(x_{i,0},x_{j,0}),\mathrm{dist}(x_{i,0},\partial\Omega)\right\}>2\delta$. Given $\tau>0$, let $\mu^\tau_{\varepsilon,k}$ be a solution of the flat discrete gradient flow of $\mathcal F_\varepsilon$ from $\mu_{\varepsilon,0}$.

Then, up to a subsequence, for any $k\in\mathbb N$ we have $\mu^\tau_{\varepsilon,k}\overset{\text{flat}}{\to}\mu^\tau_k$, for some $\mu^\tau_k\in X$ with $|\mu^\tau_k|(\Omega)\le M$.

Moreover there exists a maximal solution of the discrete gradient flow, $x^\tau_k=(x^\tau_{1,k},\dots,x^\tau_{M,k})$, of W from $x_0=(x_{1,0},\dots,x_{M,0})$, according with Definition 4.9, such that

$$\mu^\tau_k=\sum_{i=1}^M d_{i,0}\delta_{x^\tau_{i,k}}\qquad \textit{for every } k=1,\dots,k^\tau_\delta,$$

where k^τ_δ is defined in (4.25).

Proof. Since $\mathcal F_\varepsilon(\mu^\tau_{\varepsilon,k})$ is not increasing in k, we have

$$\mathcal F_\varepsilon(\mu^\tau_{\varepsilon,k})\le\mathcal F_\varepsilon(\mu_{\varepsilon,0})\le M\pi|\log\varepsilon|+\mathrm o(|\log\varepsilon|).$$

By Theorem 2.7(i), we have that, up to a subsequence, $\mu^\tau_{\varepsilon,k}\overset{\text{flat}}{\to}\mu^\tau_k\in X$, with $|\mu^\tau_k|(\Omega)\le M$ and $\|\mu^\tau_k-\mu^\tau_{k-1}\|_{\text{flat}}\le\delta$. Let $\tilde k^\tau_\delta$ be defined by

$$\begin{aligned}\tilde k^\tau_\delta:=\sup\{k\in\mathbb N:\ \mu^\tau_l=\sum_{i=1}^M d_{i,0}\delta_{x^\tau_{i,l}},\\ \min\{\frac12\mathrm{dist}_{i\neq j}(x^\tau_{i,l},x^\tau_{j,l}),\mathrm{dist}(x^\tau_{i,l},\partial\Omega)\}>2\delta,\ l=0,\dots,k\}.\end{aligned}\tag{4.36}$$

Since $|\mu^\tau_{\tilde{k}^\tau_\delta+1}|(\Omega)\le M$ and $\|\mu^\tau_{\tilde{k}^\tau_\delta+1}-\mu^\tau_{\tilde{k}^\tau_\delta}\|_{\mathrm{flat}}\le\delta$, we deduce that $\mu_{\tilde{k}^\tau_\delta+1}=\sum_{i=1}^M d_{i,0}\delta_{x^\tau_{i,\tilde{k}^\tau_\delta+1}}$, while

$$\min\{\frac{1}{2}\mathrm{dist}_{i\neq j}(x^\tau_{i,\tilde{k}^\tau_\delta+1},x^\tau_{j,\tilde{k}^\tau_\delta+1}),\mathrm{dist}(x^\tau_{i,\tilde{k}^\tau_\delta+1},\partial\Omega)\}\le 2\delta\,. \tag{4.37}$$

Moreover, since $\|\mu^\tau_k-\mu^\tau_{k-1}\|_{\mathrm{flat}}\le\delta$, it is easy to see that at each step $k=1,\dots,\tilde{k}^\tau_\delta+1$ and for every singularity $x^\tau_{i,k-1}$ of μ^τ_{k-1}, there is exactly one singularity of μ^τ_k at distance at most δ from $x^\tau_{i,k-1}$; we relabel it $x^\tau_{i,k}$. Therefore, by definition of D_2 , we have that for $k=1,\dots,\tilde{k}^\tau_\delta+1$

$$D_2(\mu^\tau_k,\mu^\tau_{k-1})=|x^\tau_k-x^\tau_{k-1}|^2. \tag{4.38}$$

We now show that for $k=0,1,\dots,\tilde{k}^\tau_\delta+1$, x^τ_k satisfies (4.22). For any measure $\mu=\sum_{i=1}^M d_{i,0}\delta_{y_i}$ with $\|\mu-\mu^\tau_{k-1}\|_{\mathrm{flat}}\le\delta$, by Theorem 2.7 (iii) there exists a recovery sequence $\{\mu_\varepsilon\}$ such that $\mathcal{F}_\varepsilon(\mu_\varepsilon)-M\pi|\log\varepsilon|\to\mathbb{W}(\mu)+M\gamma$ as $\varepsilon\to0$. By a standard density argument we can assume that $\|\mu_\varepsilon-\mu^\tau_{\varepsilon,k-1}\|_{\mathrm{flat}}\le\delta$. Therefore by (ii) of Theorem 2.7, using the fact that $\mu^\tau_{\varepsilon,k}$ satisfies (4.21) and the continuity of D_2 with respect to the flat norm, we get

$$\begin{aligned}\mathbb{W}(\mu^\tau_k)+M\gamma\quad &+\quad\frac{\pi D_2(\mu^\tau_k,\mu^\tau_{k-1})}{2\tau}\\ &\le\quad\liminf_{\varepsilon\to0}\mathcal{F}_\varepsilon(\mu^\tau_{\varepsilon,k})-\pi M|\log\varepsilon|+\frac{\pi D_2(\mu^\tau_{\varepsilon,k},\mu^\tau_{\varepsilon,k-1})}{2\tau}\\ &\le\quad\lim_{\varepsilon\to0}\mathcal{F}_\varepsilon(\mu_\varepsilon)-\pi M|\log\varepsilon|+\frac{\pi D_2(\mu_\varepsilon,\mu^\tau_{\varepsilon,k-1})}{2\tau}\\ &=\quad\mathbb{W}(\mu)+M\gamma+\frac{\pi D_2(\mu,\mu^\tau_{k-1})}{2\tau},\end{aligned}$$

i.e., μ^τ_k satisfies

$$\mu^\tau_k\in\mathrm{argmin}\left\{\mathbb{W}(\mu)+\frac{\pi D_2(\mu,\mu^\tau_{k-1})}{2\tau}\ :\ \mu=\sum_{i=1}^M d_{i,0}\delta_{x_i},\ \|\mu,\mu^\tau_{k-1}\|_{\mathrm{flat}}\le\delta\right\}.$$

By (4.38) we have that x^τ_k is a solution of the discrete gradient flow of W from $x_0=(x_{1,0},\dots,x_{M,0})$ and by (4.37) that $\tilde{k}^\tau_\delta+1=k^\tau_\delta$.

□

4.3. Discrete gradient flow of F_ε with L^2 dissipation

In this section we introduce the discrete gradient flows of F_ε with L^2 dissipation (for the L^2 norm, we will use the notation introduced in (2.3)).

Definition 4.15. Fix $\delta>0$ and let $\varepsilon,\tau>0$. Given $u_{\varepsilon,0}\in\mathcal{AF}_\varepsilon(\Omega)$, we say that $\{u^\tau_{\varepsilon,k}\}$, with $k\in\mathbb{N}\cup\{0\}$, is a solution of the L^2 discrete gradient flow of F_ε from $u_{\varepsilon,0}$ if $u^\tau_{\varepsilon,0}=u_{\varepsilon,0}$, and for any $k\in\mathbb{N}$, $u^\tau_{\varepsilon,k}$ satisfies

$$\begin{aligned}u^\tau_{\varepsilon,k}\in\mathrm{argmin}\Bigg\{&F_\varepsilon(u)+\frac{\|e^{2\pi iu}-e^{2\pi iu^\tau_{\varepsilon,k-1}}\|^2_{L^2}}{2\tau|\log\tau|}\ :\ u\in\mathcal{AF}_\varepsilon(\Omega),\\ &\|\mu(u)-\mu(u^\tau_{\varepsilon,k-1})\|_{\mathrm{flat}}\le\delta\Big\}\,.\end{aligned} \tag{4.39}$$

The constraint $\|\mu(u)-\mu(u^\tau_{\varepsilon,k-1})\|_{\mathrm{flat}} \le \delta$ is not closed in the L^2 topology. Nevertheless, in Subsection 4.3.2 we prove an existence result for such a discrete gradient flow.

In the parabolic flow of Ginzburg-Landau functionals it is well known that, as $\varepsilon \to 0$, the dynamics becomes slower and slower, and in order to capture a non trivial dynamics it is needed to scale the time by $|\log\varepsilon|$ (see for instance [60]). In our discrete in time evolution, with $\tau \gg \varepsilon$, it turns out that the natural scaling involves the time step τ instead of the length scale ε. Such a time-scaling is plugged into the discrete dynamics through the $1/|\log\tau|$ pre-factor in front of the L^2 dissipation.

As in Section 4.2, we want to consider the limit as $\varepsilon \to 0$ of such a discrete gradient flow. To this purpose, we will exploit the Γ-convergence analysis developed in Section 2.3.3. The limit dynamics will be described by a discrete gradient flow (that we shall define in the following) of the functional $\mathcal{W}$ (defined in (2.55)).

Let $v_0 \in \mathcal{D}_M$ (see (2.54)) be an initial condition with $\mathcal{W}(v_0) < +\infty$, and let $u_{\varepsilon,0}$ be a recovery sequence for v_0 in the sense of (2.59). We will show that the solutions $u^\tau_{\varepsilon,k}$ of the L^2 discrete gradient flow of F_ε from $u_{\varepsilon,0}$ converge (according with the topology of our Γ-convergence analysis in Subsection 2.3.3) to some limit v^τ_k. Moreover, at each time step k, $v^\tau_k \in \mathcal{D}_M$, the Γ-limit $\mathcal{W}$ is finite, and the degrees of the singularities coincide with the degrees $d_{i,0}$ of the initial datum. Finally, $\{v^\tau_k\}$ is a solution of the L^2 discrete gradient flow according with the following definition.

Definition 4.16. Let $\delta, \tau > 0$ and $K \in \mathbb{N}$. We say that $\{v^\tau_k\}$, with $k = 0, 1, \dots, K$, is a solution of the L^2 discrete gradient flow of $\mathcal{W}$ from v_0 if $v^\tau_0 = v_0$ and, for any $k = 1, \dots, K$, v^τ_k satisfies

$$\begin{aligned} v^\tau_k \in \operatorname{argmin}\Big\{\mathcal{W}(v) + \frac{\|v - v^\tau_{k-1}\|^2_{L^2}}{2\tau|\log\tau|} \ :\ Jv = \sum_{i=1}^M d_{i,0}\delta_{y_{i,k}},\ y_{i,k} \in \Omega, \\ v \in H^1_{\mathrm{loc}}(\Omega \setminus \cup_{i=1}^M\{y_{i,k}\}; \mathcal{S}^1),\ \|Jv - Jv^\tau_{k-1}\|_{\mathrm{flat}} \le \delta\Big\}. \end{aligned} \tag{4.40}$$

As in Section 4.2, we first do the asymptotic analysis as $\tau \to 0$. In contrast with the flat gradient flow of W, such step will require a big effort, and will involve many ingredients developed in [60].

4.3.1. L^2 discrete gradient flow of $\mathcal{W}$.

Let $v_0 \in \mathcal{D}_M$ with $Jv_0 = \sum_{i=1}^M d_{i,0}\delta_{x_{i,0}}$, and fix $\delta > 0$ such that (4.24) holds true.

Definition 4.17. We say that a solution of the L^2 discrete gradient flow $\{v^\tau_k\}$ of $\mathcal{W}$ from v_0 is maximal if the minimum problem in (4.40) does not admit a solution for $k = K+1$.

Let $\{v^\tau_k\}$ be a maximal solution of the L^2 discrete gradient flow of $\mathcal{W}$ from v_0, let $Jv^\tau_k := \sum_{i=1}^M d_{i,0}\delta_{x^\tau_{i,k}}$, and let k^τ_δ be defined as in (4.25).

Remark 4.18. Since for any $i = 1, \dots, M$, we have that $|d_{i,0}| = 1$ and thanks to the constraint $\|Jv^\tau_k - Jv^\tau_{k-1}\|_{\mathrm{flat}} \le \delta$, we get that at each step $k = 1, \dots, k^\tau_\delta$ and for each singularity $x^\tau_{i,k-1}$ of Jv^τ_{k-1}, there is exactly one singularity of Jv^τ_k whose distance from $x^\tau_{i,k-1}$ is less than δ. We label this singularity $x^\tau_{i,k}$.

The above remark guarantees that the following definition is well posed.

Definition 4.19. We set $x^\tau_k := (x^\tau_{1,k}, \dots, x^\tau_{M,k})$, where $x^\tau_{i,k}$ are labeled according with Remark 4.18. Moreover, we define $x^\tau(t) := (x^\tau_1(t), \dots, x^\tau_M(t))$ as the piecewise affine in time interpolation of $\{x^\tau_k\}$.

As in Section 4.2 we have that $x^\tau_k \in K_\delta$, where K_δ is defined in (4.27). Moreover, the energy $\mathcal{W}$ is clearly decreasing in k. Since, for every $k = 1, \dots, k^\tau_\delta$ we have

$$\frac{\|v^\tau_k - v^\tau_{k-1}\|^2_{L^2}}{2\tau|\log\tau|} \le \mathcal{W}(v^\tau_{k-1}) - \mathcal{W}(v^\tau_k),$$

then

$$\begin{aligned}\sum_{k=1}^{k^\tau_\delta} \frac{\|v^\tau_k - v^\tau_{k-1}\|^2_{L^2}}{2\tau|\log\tau|} &\le \mathcal{W}(v_0) - \mathcal{W}(v^\tau_{k^\tau_\delta}) \\ &\le \mathcal{W}(v_0) - W(x^\tau_{k^\tau_\delta}) \le \mathcal{W}(v_0) - W(x_0) + C_\delta\,, \end{aligned} \tag{4.41}$$

where C_δ is defined in (4.28).

Proposition 4.20. *For every $k = 0, 1, \dots, k^\tau_\delta$ we have that $\|Jv^\tau_k - Jv^\tau_{k-1}\|_{\mathrm{flat}} < C\sqrt{\tau|\log\tau|}$, where $C > 0$ depends only on δ (and on the initial condition v_0).*

Proof. Fix $1 \le k \le k^\tau_\delta$ and $1 \le i \le M$. Set $\rho^\tau_{i,k} := \frac{1}{4}\mathrm{dist}(x^\tau_{i,k}, x^\tau_{i,k-1})$. Note that

$$\deg(v^\tau_k, \partial B_{\rho^\tau_{i,k}}(x^\tau_{i,k})) \ne 0 = \deg(v^\tau_{k-1}, \partial B_{\rho^\tau_{i,k}}(x^\tau_{i,k}))\,. \tag{4.42}$$

Moreover, since $\mathcal{W}(v^\tau_k) \le \mathcal{W}(v_0)$, from (2.57) we have that

$$\int_{B_{2\rho^\tau_{i,k}}(x^\tau_{i,k}) \setminus B_{\rho^\tau_{i,k}}(x^\tau_{i,k})} (|\nabla v^\tau_k|^2 + |\nabla v^\tau_{k-1}|^2)\,\mathrm{d}x \le 2\mathcal{W}(v_0) + C\,. \tag{4.43}$$

As a consequence of (4.42) and (4.43), we have that

$$(\mathrm{dist}(x^\tau_{i,k}, x^\tau_{i,k-1}))^2 \le C \int_{B_{2\rho^\tau_{i,k}}(x^\tau_{i,k}) \setminus B_{\rho^\tau_{i,k}}(x^\tau_{i,k})} |v^\tau_k - v^\tau_{k-1}|^2\,\mathrm{d}x\,. \tag{4.44}$$

Indeed, if by contradiction (4.44) does not hold, by a scaling argument we could find two sequences $\{w^n_1\}$ and $\{w^n_2\}$ of functions in $H^1(B_2 \setminus B_1; \mathcal{S}^1)$ such that

$$\int_{B_2 \setminus B_1} (|\nabla w^n_1|^2 + |\nabla w^n_2|^2)\,\mathrm{d}x \le 2\mathcal{W}(v_0) + C, \qquad \int_{B_2 \setminus B_1} |w^n_1 - w^n_2|^2\,\mathrm{d}x \to 0\,,$$

and such that $\deg(w^n_1, \partial B_\rho) \ne \deg(w^n_2, \partial B_\rho)$ for almost every $\rho \in [1, 2]$. This is impossible in view of the stability of the degree with respect to uniform convergence for continuous maps from $\mathcal{S}^1$ to $\mathcal{S}^1$.

Now, from (4.41) we have that

$$\int_{B_{2\rho^\tau_{i,k}}(x^\tau_{i,k}) \setminus B_{\rho^\tau_{i,k}}(x^\tau_{i,k})} |v^\tau_k - v^\tau_{k-1}|^2\,\mathrm{d}x \le C\tau|\log\tau|,$$

which together with (4.44) yields

$$\|Jv_k^\tau - Jv_{k-1}^\tau\|_{\mathrm{flat}} \le C\sqrt{\tau|\log\tau|}\,. \tag{4.45}$$

□

For every $k = 0, 1, \dots, k_\delta^\tau$ we set

$$D_k^\tau := \mathcal{W}(v_k^\tau) - W(x_k^\tau). \tag{4.46}$$

Moreover, set $\tilde{T}_\delta := \liminf_{\tau\to 0} k_\delta^\tau \tau$, and define for any $t \in [0, \tilde{T}_\delta)$, the *energy excess*

$$D(t) = \limsup_{\tau\to 0} D_{\lfloor t/\tau\rfloor}^\tau \ge 0. \tag{4.47}$$

Since $\mathcal{W}(v_k^\tau) \le \mathcal{W}(v_0)$, by (4.26) we have

$$D_k^\tau = \mathcal{W}(v_k^\tau) - W(x_k^\tau) \le \mathcal{W}(v_0) - W(x_k^\tau) \le D(0) + C_\delta\,, \tag{4.48}$$

where C_δ is defined in (4.28). From now on we will say that an initial condition v_0 is *well prepared* if $W(x_0) = \mathcal{W}(v_0)$, i.e., $D(0) = 0$.

We are in a position to state the main theorem of this section, which is the analogous of Theorem 1.14 in our case of discrete topological singularities.

Theorem 4.21. *Let v_0 be a well prepared initial condition. Let $\{v_k^\tau\}_{\tau>0}$ be a family of maximal solutions of the L^2 discrete gradient flow of $\mathcal{W}$ from v_0. Then,*

$$\tilde{T}_\delta := \liminf_{\tau\to 0} k_\delta^\tau \tau \ge T_\delta, \tag{4.49}$$

where k_δ^τ is defined in (4.25) and T_δ is defined in (4.32).

Moreover, for every $0 < T < \tilde{T}_\delta$, $x^\tau \to x$ uniformly on $[0, T]$, where x^τ is defined in Definition 4.19, and x is the solution of the gradient flow of W from x_0 according with Definition 4.10. Finally, $D(t) = 0$ for every $0 \le t < \tilde{T}_\delta$ and $\tilde{T}_\delta \to T^*$ as $\delta \to 0$.

Remark 4.22. As a consequence of the uniform convergence of x^τ and the estimate (4.45), one can prove that the 1-current associated to the polygonal x^τ (with the natural orientation and multiplicity given by the integers $d_{i,0}$), converges to the current associated to the limit x in the flat norm.

The proof of Theorem 4.21 is postponed at the end of the section, and will be obtained as a consequence of Theorem 4.23 below, which can be regarded as the discrete in time counterpart of Theorem 1.13.

Theorem 4.23. *Let v_0 be a well prepared initial datum, i.e., with $W(x_0) = \mathcal{W}(v_0)$. Let $\{v_k^\tau\}_{\tau>0}$ be solutions of the L^2 discrete gradient flow for $\mathcal{W}$ from v_0, let $T > 0$ be such that $k_\delta^\tau \ge \lfloor T/\tau\rfloor$ for every τ, and assume that $x^\tau \to x$ uniformly in $[0, T]$ for some $x(t) \in H^1([0, T]; \Omega^M)$. Moreover, assume that (i) and (ii) below are satisfied:*

(i) *(*Lower bound*) For any $s \in [0, T]$*

$$\liminf_{\tau\to 0} \frac{\tau}{|\log\tau|} \sum_{k=1}^{\lfloor \frac{s}{\tau}\rfloor} \left\| \frac{v_k^\tau - v_{k-1}^\tau}{\tau} \right\|_{L^2}^2 \ge \pi \int_0^s |\dot{x}(t)|^2 \, \mathrm{d}t\ .$$

(ii) *(*Construction*) For any* $k = 0, 1, \ldots, \lfloor T/\tau \rfloor - 1$, *there exists a field* $w^\tau_{k+1} \in H^1_{\mathrm{loc}}(\Omega \setminus \cup_{i=1}^M \{x^\tau_{i,k} - \frac{\tau}{\pi}\partial_{x_i} W(x^\tau_k)\}; \mathcal{S}^1)$ *and a constant* $M_\delta > 0$ *such that*

$$\mathcal{W}(v^\tau_k) - \mathcal{W}(w^\tau_{k+1}) \geq \frac{\tau}{\pi}|\nabla W(x^\tau_k)|^2 - \tau M_\delta D^\tau_k + \mathrm{o}(\tau),$$

$$\frac{1}{|\log \tau|}\left\|\frac{w^\tau_{k+1} - v^\tau_k}{\tau}\right\|^2_{L^2} \leq \frac{1}{\pi}|\nabla W(x^\tau_k)|^2 + \mathrm{o}(1).$$

Then, $D(t) = 0$ *for every* $t \in [0, T]$, *and* $x(t)$ *is a solution of the gradient flow* (4.23) *of* W *from* x_0 *on* $[0, T]$.

Proof. By (ii) and by the minimality of v^τ_{k+1}, we have

$$\mathcal{W}(v^\tau_k) - \mathcal{W}(v^\tau_{k+1}) = \mathcal{W}(v^\tau_k) - \mathcal{W}(w^\tau_{k+1}) + \mathcal{W}(w^\tau_{k+1}) - \mathcal{W}(v^\tau_{k+1})$$
$$\geq \frac{\tau}{\pi}|\nabla W(x^\tau_k)|^2 - \frac{\tau}{2|\log\tau|}\left\|\frac{w^\tau_{k+1} - v^\tau_k}{\tau}\right\|^2_{L^2} + \frac{\tau}{2|\log\tau|}\left\|\frac{v^\tau_{k+1} - v^\tau_k}{\tau}\right\|^2_{L^2} - \tau M_\delta D^\tau_k + \mathrm{o}(\tau)$$
$$\geq \frac{\tau}{2\pi}|\nabla W(x^\tau_k)|^2 + \frac{\tau}{2|\log\tau|}\left\|\frac{v^\tau_{k+1} - v^\tau_k}{\tau}\right\|^2_{L^2} - \tau M_\delta D^\tau_k + \mathrm{o}(\tau).$$

Now, let $s \in [0, T]$. Summing over $k = 0, 1, \ldots, \lfloor s/\tau \rfloor - 1$, we have

$$\mathcal{W}(v^\tau_0) - \mathcal{W}(v^\tau_{\lfloor s/\tau \rfloor}) \geq \frac{1}{2\pi}\int_0^{\tau\lfloor s/\tau\rfloor - \tau} |\nabla W(x^\tau_{\lfloor t/\tau \rfloor})|^2 \, \mathrm{d}t$$
$$+ \frac{\tau}{2|\log\tau|}\sum_{k=0}^{\lfloor s/\tau \rfloor - 1}\left\|\frac{v^\tau_{k+1} - v^\tau_k}{\tau}\right\|^2_{L^2} - M_\delta \int_0^{\tau\lfloor s/\tau\rfloor - \tau} D^\tau_{\lfloor t/\tau \rfloor}\, \mathrm{d}t + \mathrm{o}(1).$$

By the uniform convergence of x^τ to x in $[0, T]$ and the fact that $x \in H^1$, we have that also $x^\tau_{\lfloor \cdot/\tau \rfloor} \to x$ uniformly in $[0, T]$. Hence, passing to the lim inf as $\tau \to 0$, using (i) and (4.48), we get

$$\liminf_{\tau \to 0}(\mathcal{W}(v^\tau_0) - \mathcal{W}(v^\tau_{\lfloor s/\tau \rfloor})) \geq \frac{1}{2}\int_0^s \frac{1}{\pi}|\nabla W(x(t))|^2 + \pi|\dot{x}(t)|^2 \, \mathrm{d}t$$
$$-M_\delta \int_0^s D(t)\, \mathrm{d}t, \tag{4.50}$$

where $D(t)$ is defined in (4.47).

Since $\mathcal{W}(v^\tau_0) = \mathcal{W}(v_0) = W(x_0) = W(x(0))$, we have immediately that

$$\liminf_{\tau \to 0}(\mathcal{W}(v^\tau_0) - \mathcal{W}(v^\tau_{\lfloor s/\tau \rfloor})) = W(x(0)) - W(x(s)) - D(s). \tag{4.51}$$

Combining this with (4.50) yields

$$W(x(0)) - W(x(s)) - D(s) \geq \frac{1}{2}\int_0^s \frac{1}{\pi}|\nabla W(x(t))|^2 + \pi|\dot{x}(t)|^2 \, \mathrm{d}t$$
$$- M_\delta \int_0^s D(t)\, \mathrm{d}t. \tag{4.52}$$

Since

$$W(x(0)) - W(x(s)) = \int_0^s \langle -\nabla W(x(t)), \dot{x}(t)\rangle \, \mathrm{d}t$$
$$\leq \frac{1}{2}\int_0^s \frac{1}{\pi}|\nabla W(x(t))|^2 + \pi|\dot{x}(t)|^2 \, \mathrm{d}t, \tag{4.53}$$

then,

$$D(s) \leq M_\delta \int_0^s D(t)\,\mathrm{d}t.$$

Since $D(0) = 0$ by assumption, from Gronwall's lemma we find that $D(s) = 0$ for all $s \in [0, T]$.

Using that $D(s) = 0$, by (4.52) and (4.53) we obtain

$$\int_0^s |\frac{1}{\sqrt{\pi}} \nabla W(x(t)) + \sqrt{\pi}\,\dot{x}(t)|^2\,\mathrm{d}t \leq 0,$$

and hence $\dot{x}(t) = -\frac{1}{\pi}\nabla W(x(t))$ a.e. in $[0, T]$. □

The following propositions are devoted to shows that the hypothesis of Theorem 4.23 are satisfied by the L^2 discrete gradient flow defined in Definition 4.16.

Proposition 4.24. *Let $\{v_k^\tau\}_{\tau>0}$ be a family of maximal solutions of the L^2 discrete gradient flow of $\mathcal{W}$ from v_0, let k_δ^τ be as in (4.25), and let x^τ be defined as in Definition 4.19. Then*

$$\tilde{T}_\delta = \liminf_{\tau \to 0} k_\delta^\tau \tau \geq \pi \frac{c_\delta^2}{C_\delta}, \tag{4.54}$$

where C_δ and c_δ are defined in (4.28) and (4.24) respectively.

Moreover, there exists a map $x \in H^1([0, \tilde{T}_\delta]; \Omega^M)$ such that, up to a subsequence, $x^\tau \to x$ uniformly on $[0, T]$ for every $0 < T < \tilde{T}_\delta$ and

$$\liminf_{\tau \to 0} \frac{\tau}{|\log \tau|} \sum_{k=1}^{\lfloor \frac{T}{\tau} \rfloor} \left\| \frac{v_k^\tau - v_{k-1}^\tau}{\tau} \right\|_{L^2}^2 \geq \pi \int_0^T |\dot{x}(t)|^2\,\mathrm{d}t. \tag{4.55}$$

Proof. The starting point of the proof consists in applying Theorem 1.15 to piecewise affine interpolations in time of suitable regularizations of v_k^τ. Clearly, the Ginzburg-Landau energy of v_k^τ is not bounded. By the very definition of $\mathcal{W}$, we have

$$\frac{1}{2} \int_{\Omega \setminus \cup_i B_\tau(x_{i,k}^\tau)} |\nabla v_k^\tau|^2\,\mathrm{d}x - M\pi |\log \tau| \leq \mathcal{W}(v_k^\tau) \leq \mathcal{W}(v_0).$$

Moreover, the Dirichlet energy stored in $B_\tau(x_{i,k}^\tau) \setminus B_{\tau/2}(x_{i,k}^\tau)$ is bounded. Therefore, by standard cut off arguments, we can easily construct fields $\hat{v}_k^\tau$ which coincide with v_k^τ in $\Omega \setminus \cup_i B_\tau(x_{i,k}^\tau)$, are equal to zero in $B_{\tau/2}(x_{i,k}^\tau)$ and satisfy

$$\frac{1}{2} \int_\Omega |\nabla \hat{v}_k^\tau|^2\,\mathrm{d}x \leq M\pi |\log \tau| + C. \tag{4.56}$$

Then, we consider the piecewise affine in time interpolation $\hat{v}^\tau : [0, +\infty) \times \Omega \to \mathbb{R}^2$ of $\hat{v}_k^\tau$ defined by

$$\hat{v}^\tau(t, x) := \begin{cases} (1 - \dfrac{t - k\tau}{\tau})\hat{v}_k^\tau(x) + \dfrac{t - k\tau}{\tau}\hat{v}_{k+1}^\tau(x) & \text{if } k\tau \leq t \leq (k+1)\tau \leq k_\delta^\tau \tau, \\ \hat{v}_{k_\delta^\tau}^\tau(x) & \text{if } t > k_\delta^\tau \tau. \end{cases}$$

For every fixed $t > 0$, we denote by $\hat{\mu}^\tau(t)$ the (space) Jacobian of $\hat{v}^\tau$.
We will prove the theorem in several steps.

Step 1. There exists a map $x \in C^{0,\frac{1}{2}}([0,+\infty);\Omega^M)$ such that up to a subsequence, for every $T>0$ we have

$$\hat{\mu}^\tau(t) \stackrel{\text{flat}}{\to} \mu(t) := \pi \sum_{i=1}^{M} d_{i,0}\delta_{x_i(t)} \qquad \text{for every } t \in [0,T]\,. \tag{4.57}$$

Fix $T>0$. By the convexity of the Dirichlet energy and by (4.56), it follows that for any $t \in [0,T]$

$$\frac{1}{2}\int_\Omega |\nabla \hat{v}^\tau(t,x)|^2 \,\mathrm{d}x \le M\pi|\log\tau| + C. \tag{4.58}$$

Moreover, by the definition of $\hat{v}^\tau_k$, it follows that for any $k = 0,\dots,k^\tau_\delta - 1$

$$\left\|\hat{v}^\tau_{k+1} - \hat{v}^\tau_k\right\|^2_{L^2} \le \left\|v^\tau_{k+1} - v^\tau_k\right\|^2_{L^2} + C\tau^2;$$

therefore, by (4.41), we get

$$\int_{[0,T]\times\Omega} |\partial_t \hat{v}^\tau|^2 \,\mathrm{d}t\,\mathrm{d}x = \sum_{k=0}^{k^\tau_\delta-1} \tau \left\|\frac{\hat{v}^\tau_{k+1}-\hat{v}^\tau_k}{\tau}\right\|^2_{L^2} \le C|\log\tau|\,.$$

It is easy to see that for every $t \in [k\tau,(k+1)\tau]$

$$\frac{1}{\tau}\int_\Omega (1-|\hat{v}^\tau(t,x)|^2)^2 \,\mathrm{d}x \le \frac{C}{\tau}\left\|\hat{v}^\tau_{k+1} - \hat{v}^\tau_k\right\|^2_{L^2} \le C|\log\tau|. \tag{4.59}$$

In conclusion, for every $t \in [0,T]$ we have

$$\frac{1}{2}\int_\Omega |\nabla\hat{v}^\tau|^2 + \frac{1}{\tau}(1-|\hat{v}^\tau|^2)^2 \,\mathrm{d}x \le C|\log\tau|$$
$$\int_{[0,T]\times\Omega} |\partial_t\hat{v}^\tau|^2 \,\mathrm{d}t\,\mathrm{d}x \le C|\log\tau|.$$

By Theorem 1.15 applied with $\varepsilon = \sqrt{\tau}$ and recalling that $\mu(0) = \mu_0 = \sum_{i=1}^M d_{i,0}\delta_{x_{i,0}}$, we deduce that

$$\mu(t) = \pi\sum_{i=1}^{M(t)} d_i\delta_{x_i(t)}, \qquad \text{for all } t \in [0,T]$$

for some $x_i(t) \in C^{0,\frac{1}{2}}([0,T_i);\Omega)$ with $T_i \le T$. Here T_i represents the first time when $x_i(t)$ reaches $\partial\Omega$. Finally, by construction $x_i(t)$ are defined on $[0,T]$, distinct, and contained in Ω. The conclusion follows by a standard diagonalization argument.

Step 2. Set

$$\hat{T}_\delta := \inf\left\{t \in [0,+\infty) : \ \min\{\frac{1}{2}\mathrm{dist}_{i\neq j}(x_i(t),x_j(t)), \mathrm{dist}(x_i(t),\partial\Omega)\} \le 2\delta\right\}.$$

Then, $\tilde{T}_\delta \ge \hat{T}_\delta > 0$.

Since $x \in C^{0,\frac{1}{2}}$ and $x(0) = x_0$ satisfies

$$\min\{\frac{1}{2}\mathrm{dist}_{i\neq j}(x_{i,0},x_{j,0}), \mathrm{dist}(x_{i,0},\partial\Omega)\} > 2\delta,$$

we have $\hat{T}_\delta > 0$. Fixed $t > \tilde{T}_\delta$, by construction and Step 1 we have that

$$\hat{\mu}^\tau(t) = \hat{\mu}^\tau_{k^\tau_\delta \tau} \overset{\text{flat}}{\to} \pi \sum_{i=1}^{M} d_{i,0}\delta_{x_i(\tilde{T}_\delta)}. \tag{4.60}$$

Set $\mu^\tau(t) := \pi \sum_{i=1}^{M} d_{i,0}\delta_{x_i^\tau(t)}$ for $t \le k^\tau_\delta \tau$, where x_i^τ are defined in Definition 4.19. Let $0 \le k \le k^\tau_\delta$. Since $\mathrm{supp}(\hat{\mu}^\tau(k\tau)), \mathrm{supp}(\mu^\tau(k\tau)) \subseteq \cup_i B_\tau(x^\tau_{i,k})$ and $\hat{\mu}^\tau(k\tau)(B_\tau(x^\tau_{i,k})) = \mu^\tau(k\tau)(B_\tau(x^\tau_{i,k}))$, for any $\varphi \in C^{0,1}_c(\Omega)$ we have

$$\begin{aligned}\langle \hat{\mu}^\tau(k\tau) - \mu^\tau(k\tau), \varphi\rangle = &\sum_{i=1}^{M} \langle \hat{\mu}^\tau(k\tau) - \mu^\tau(k\tau), \varphi - \bar{\varphi}_i\rangle \\ &\le (|\hat{\mu}^\tau(k\tau)|(\Omega) + |\mu^\tau(k\tau)|(\Omega))\, \tau \|\nabla\varphi\|_{L^\infty}\,,\end{aligned}$$

where $\bar{\varphi}_i$ denotes the average of φ on $B_\tau(x^\tau_{i,k})$. Since, by Remark 2.8, we have

$$|\hat{\mu}^\tau(k\tau)|(\Omega) \le C \sum_{i=1}^{M} \int_{B_\tau(x^\tau_{i,k})\setminus B_{\frac{\tau}{2}}(x^\tau_{i,k})} |\nabla \hat{v}^\tau_k|^2 \, \mathrm{d}x \le C\,,$$

we deduce that

$$\max_{k=0,1,\dots,k^\tau_\delta} \|\hat{\mu}^\tau(k\tau) - \mu^\tau(k\tau)\|_{\mathrm{flat}} \le C\tau\,. \tag{4.61}$$

This fact together with (4.60) yields

$$\sum_{i=1}^{M_0} d_{i,0}\delta_{x^\tau_{i,k^\tau_\delta}} \overset{\text{flat}}{\to} \sum_{i=1}^{M} d_{i,0}\delta_{x_i(\tilde{T}_\delta)}.$$

Therefore, by the very definition of k^τ_δ, we have that for every $t > \tilde{T}_\delta$

$$\min\{\frac{1}{2}\mathrm{dist}_{i\neq j}(x_i(t), x_j(t)), \mathrm{dist}(x_i(t), \partial\Omega)\} \le 2\delta.$$

By continuity, the previous inequality holds also for $t = \tilde{T}_\delta$, so that we conclude that $\tilde{T}_\delta \ge \hat{T}_\delta > 0$.

Step 3. $x^\tau \to x$ uniformly on the compact subsets of $[0, \tilde{T}_\delta)$.

Let us show that

$$\max_{k=0,1,\dots,k^\tau_\delta} \|\hat{\mu}^\tau(k\tau) - \mu(k\tau)\|_{\mathrm{flat}} =: \|\hat{\mu}^\tau(\bar{k}^\tau\tau) - \mu(\bar{k}^\tau\tau)\|_{\mathrm{flat}} \to 0\,. \tag{4.62}$$

Up to a subsequence we can assume that $\bar{k}^\tau\tau$ converges to some $t_0 \in [0, \tilde{T}_\delta]$. The fields

$$\tilde{v}^\tau(t,x) := \begin{cases} \hat{v}^\tau(t,x) & \text{if } t \le \bar{k}^\tau\tau \\ \hat{v}^\tau(\bar{k}^\tau\tau, x) & \text{if } t > \bar{k}^\tau\tau \end{cases} \tag{4.63}$$

satisfy the assumptions of Theorem 1.15, applied with $\varepsilon = \sqrt{\tau}$; therefore, denoting by $\tilde{\mu}^\tau(t)$ the (space) Jacobian of $\tilde{v}^\tau$, we have that, up to a subsequence,

$$\tilde{\mu}^\tau(t) \overset{\text{flat}}{\to} \tilde{\mu}(t) := \begin{cases} \mu(t) & \text{if } t \le t_0 \\ \mu(t_0) & \text{if } t > t_0\,, \end{cases} \tag{4.64}$$

where the structure of $\tilde{\mu}$ is a consequence of the continuity guaranteed by Theorem 1.15. From (4.64) one can easily prove that $\hat{\mu}^\tau(\bar{k}^\tau\tau) - \mu(t_0)$ converges to zero in the flat norm and hence we get (4.62). Combining (4.61) with (4.62) we also deduce that

$$\max_{k=0,1,\dots,k_\delta^\tau} \|\mu^\tau(k\tau) - \mu(k\tau)\|_{\mathrm{flat}} \to 0. \tag{4.65}$$

Moreover, by the construction of μ^τ and (4.20), we have that

$$\max_{t\in[0,k_\delta^\tau\tau]} \|\mu^\tau(t) - \mu^\tau(\lfloor t/\tau\rfloor\tau)\|_{\mathrm{flat}} \to 0. \tag{4.66}$$

Using (4.66), (4.65) and that $\max_{t\in[0,k_\delta^\tau\tau]} \|\mu(\lfloor t/\tau\rfloor\tau) - \mu(t)\|_{\mathrm{flat}} \to 0$, by the triangular inequality we conclude that

$$\max_{t\in[0,k_\delta^\tau\tau]} |x^\tau(t) - x(t)| = \max_{t\in[0,k_\delta^\tau\tau]} \|\mu^\tau(t) - \mu(t)\|_{\mathrm{flat}} \to 0.$$

Step 4. The function x belongs to $H^1([0,\tilde{T}_\delta];\Omega^M)$, and, for any $T\in[0,\tilde{T}_\delta]$, (4.55) holds true. In particular, $\tilde{T}_\delta \ge \pi c_\delta^2/C_\delta$.

The proof of this step is obtained as a consequence of Proposition 1.17 applied to the fields $\hat{v}^\tau$, with $\varepsilon = \tau$ and $\tilde{T} = \tilde{T}_\delta$. By (4.41) and recalling (4.59), it easily follows that

$$\frac{1}{\tau^2}\int_0^{\tilde{T}_\delta}\int_\Omega (1-|\hat{v}^\tau|^2)^2\,\mathrm{d}x\,\mathrm{d}t \le C\sum_{k=1}^{k_\delta^\tau}\tau\left\|\frac{\hat{v}_{k+1}^\tau - \hat{v}_k^\tau}{\tau}\right\|_{L^2}^2 \le C|\log\tau|;$$

and hence (1.21) holds with $\varepsilon=\tau$ and $w_\varepsilon = v^\tau$. This fact together with (4.57) and (4.58) guarantees that the hypothesis of Proposition 1.21 are satisfied. Therefore, we deduce that (4.55) holds true with v_k^τ replaced by $\hat{v}_k^\tau$. Since $\|\hat{v}_k^\tau - v_k^\tau\|_{L^2} = \mathrm{O}(\tau^2)$, we deduce (4.55).

Finally, by (4.41) and recalling (4.28), we have

$$\begin{aligned}\pi\int_0^T |\dot{x}(t)|^2\,\mathrm{d}t &\le \liminf_{\tau\to0}\frac{\tau}{|\log\tau|}\sum_{k=1}^{\lfloor\frac{T}{\tau}\rfloor}\left\|\frac{v_k^\tau - v_{k-1}^\tau}{\tau}\right\|_{L^2}^2 \\ &\le \liminf_{\tau\to0}(\mathcal{W}(v_0) - \mathcal{W}(v^\tau_{\lfloor\frac{T}{\tau}\rfloor})) \le \liminf_{\tau\to0}(W(x_0) - W(x^\tau_{\lfloor\frac{T}{\tau}\rfloor})) \le C_\delta.\end{aligned} \tag{4.67}$$

By Hölder inequality, and recalling (4.24), we conclude

$$c_\delta \le |x(\tilde{T}_\delta) - x(0)| \le \int_0^{\tilde{T}_\delta}|\dot{x}|\,\mathrm{d}t \le \|\dot{x}\|_{L^2([0,\tilde{T}_\delta];\mathbb{R}^{2M})}\sqrt{\tilde{T}_\delta}. \tag{4.68}$$

By (4.67) and (4.68) we immediately get (4.54) □

Since we have proved assumption (i) in Theorem 4.24, it remains to prove only assumption (ii). To this aim, at each time step $k = 0,1,\dots,k_\delta^\tau$, we construct, a field w_{k+1}^τ whose vortices are obtained translating $x_{i,k}^\tau$ in the direction of the renormalized energy $\nabla W(x_k^\tau)$. The variation of the energy $\mathcal{W}$ associated to the fields v_k^τ and w_{k+1}^τ is proportional to the distance among the vortices of the two functions (i.e. $|\nabla W(x_k^\tau)|$) up to an error given by the energy excess D_k^τ defined in (4.48).

Proposition 4.25. *For any $k = 0, 1, \dots, k_\delta^\tau - 1$, there exists a field $w_{k+1}^\tau \in H^1_{\mathrm{loc}}(\Omega \setminus \bigcup_{i=1}^M \left\{ x_{i,k}^\tau - \frac{\tau}{\pi}\partial_{x_i} W(x_k^\tau) \right\}; \mathcal{S}^1)$ such that*

$$\mathcal{W}(v_k^\tau) - \mathcal{W}(w_{k+1}^\tau) \geq \frac{\tau}{\pi}|\nabla W(x_k^\tau)|^2 - M_\delta \tau D_k^\tau + \mathrm{o}(\tau) \tag{4.69}$$

$$\frac{\left\| w_{k+1}^\tau - v_k^\tau \right\|_{L^2}^2}{\tau^2 |\log \tau|} \leq \frac{1}{\pi}|\nabla W(x_k^\tau)|^2 + \mathrm{o}(1), \tag{4.70}$$

where M_δ is a positive constant depending only on δ.

Proof. Fix $k \in \{0, 1, \dots, k_\delta^\tau - 1\}$; to ease notation we set

$$V_i = (V_{i1}, V_{i2}) := -\frac{1}{\pi}\partial_{x_i} W(x_k^\tau), \qquad V := (V_1, \dots, V_M). \tag{4.71}$$

With a little abuse of notation, from now on we will set $x_i := x_{i,k}^\tau$ and $y_i = x_i + \tau V_i$ for every $i = 1, \dots, M$. By (4.25), the balls $B_{\delta/2}(x_i)$ are pairwise disjoint and contained in Ω.

In order to construct the field w_{k+1}^τ, we wish to "push" the vortices x_i along the direction V_i. For every $i = 1, \dots, M$, we can find smooth, compactly supported vector fields in Ω, X_{i1} and X_{i2} such that

$$\begin{aligned} X_{i1}(x) = (1,0) \quad X_{i2}(x) = (0,1) &\qquad \text{for } x \in B_{\delta/2}(x_i), \\ X_{i1}(x) = X_{i2}(x) = (0,0) &\qquad \text{for } x \in B_{\delta/2}(x_j), j \neq i \end{aligned}$$

and such that $\|\nabla X_{ij}\|_{L^\infty} \leq \frac{2}{\delta}$ for every $i, j = 1, \dots, M$. Then, we define $X_V = \sum_{i=1}^M \sum_{j=1,2} V_{ij} X_{ij}$. Since W is smooth in K_δ (see (4.27)), there exists a constant M_δ depending only on δ such that

$$\| \det \nabla X_V \|_{L^\infty} \leq \frac{1}{2} M_\delta \,. \tag{4.72}$$

For any $t \in [0, \tau]$, we define $\chi_t(x) := x + t X_V(x)$ for every $x \in \Omega$; notice that $\chi_t(x) = x + tV_i$ for $x \in B_{\delta/2}(x_i)$. For any $t \in [0, \tau]$ let Φ^t be the solution of

$$\begin{cases} \Delta \Phi^t = 2\pi \sum_{i=1}^M d_{i,0} \delta_{x_i + tV_i} & \text{in } \Omega \\ \Phi^t = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$R^t(x) := \Phi^t(x) - \sum_{i=1}^M d_{i,0} \log|x - x_i - tV_i|. \tag{4.73}$$

By definition R^t are smooth harmonic functions in Ω; we denote by $\tilde{R}^t$ theirs harmonic conjugates with zero average in Ω. Moreover, we denote by θ_i^t the polar coordinates centered at $x_i + tV_i$ and set $\tilde{\Phi}^t := \sum_{i=1}^M d_{i,0}\theta_i^t + \tilde{R}^t$. Notice that $\nabla \tilde{\Phi}^t$ is nothing but the $\pi/2$ rotation of $\nabla \Phi^t$. We define

$$\psi^t(\cdot) = \tilde{\Phi}^t(\chi_t(\cdot)) - \tilde{\Phi}^0(\cdot). \tag{4.74}$$

Notice that ψ^t is a smooth function in Ω, the singularities at x_i canceling out, and that it is smooth in space-time. In particular, using (4.26) one can show that, for τ small enough, there exists a constant C depending only on δ such that

$$\sup_{t \in [0,\tau]} \left(\|\nabla \psi^t\|_{L^\infty(\Omega)} + \|\frac{\mathrm{d}}{\mathrm{d}t}\psi^t\|_{L^\infty(\Omega)} \right) \leq C \,. \tag{4.75}$$

For any $0 < \sigma < \delta$, we define $\Omega^t_\sigma := \Omega \setminus \cup_{i=1}^M B_\sigma(x_i + tV_i)$. By definition of $\tilde{\Phi}^t$, the renormalized energy associated to the configuration $x^\tau_k + tV$ is given by

$$W(x^\tau_k + tV) = \lim_{\sigma\to 0} \frac{1}{2}\int_{\Omega^t_\sigma} |\nabla\tilde{\Phi}^t|^2 - M\pi|\log\sigma|. \tag{4.76}$$

Since $v^\tau_k \in H^1(\Omega^0_\sigma;\mathcal{S}^1)$, there exist a family $\{L_i\}_{i=1,\dots,M}$ of cuts of the domain Ω (L_i is a segment from x_i to $\partial\Omega$) and a function $\varphi^0 \in H^1(\Omega^0_\sigma \setminus \cup_{i=1}^M \{L_i\};\mathbb{R})$ such that $v^\tau_k = e^{i\varphi^0}$.

Recalling (4.74), we introduce the field w^τ_{k+1} defined by the following identity (notice that χ_τ is invertible for τ small enough)

$$w^\tau_{k+1}(\chi_\tau(x)) := v^\tau_k(x)e^{i\psi^\tau(x)} = e^{i(\varphi^0(x)+\psi^\tau(x))}. \tag{4.77}$$

By definition, $w^\tau_{k+1} \in H^1(\Omega^\tau_\sigma;\mathcal{S}^1)$ and $Jw^\tau_{k+1} = \sum_{i=1}^M d_{i,0}\delta_{y_i}$.

We notice that if $\varphi^0 = \tilde{\Phi}^0$, then by (4.76) we get

$$\begin{aligned}\mathcal{W}(v^\tau_k) - \mathcal{W}(w^\tau_{k+1}) &= \lim_{\sigma\to 0}\frac{1}{2}\int_{\Omega^0_\sigma}|\nabla v^\tau_k|^2\,\mathrm{d}x - \frac{1}{2}\int_{\Omega^\tau_\sigma}|\nabla w^\tau_{k+1}|^2\,\mathrm{d}y \qquad (4.78)\\ &= \lim_{\sigma\to 0}\frac{1}{2}\int_{\Omega^0_\sigma}|\nabla\tilde{\Phi}^0|^2\,\mathrm{d}x - \frac{1}{2}\int_{\Omega^\tau_\sigma}|\nabla\tilde{\Phi}^\tau|^2\,\mathrm{d}y\\ &= W(x^\tau_k) - W(x^\tau_k + \tau V). \qquad (4.79)\end{aligned}$$

Recalling (4.26) and (4.27), by Taylor expansion we conclude

$$W(x^\tau_k) - W(x^\tau_k - \frac{\tau}{\pi}\nabla W(x^\tau_k)) = \frac{\tau}{\pi}|\nabla W(x^\tau_k)|^2 + \mathrm{O}(\tau^2). \tag{4.80}$$

We show now that w^τ_{k+1} satisfies (4.69) even when v^τ_k is not optimal in energy. To this purpose, we show that the difference $\mathcal{W}(v^\tau_k) - \mathcal{W}(w^\tau_{k+1})$ can be bounded from below by the variation of the renormalized energy up to an error given by the defect D^τ_k defined in (4.46). More precisely, set

$$D^\tau_{\sigma,k} := \frac{1}{2}\int_{\Omega^0_\sigma}(|\nabla\varphi^0|^2 - |\nabla\tilde{\Phi}^0|^2)\,\mathrm{d}x, \tag{4.81}$$

so that $D^\tau_k = \lim_{\sigma\to 0} D^\tau_{\sigma,k}$. We want to prove that, for $0 < \sigma \ll \tau$,

$$\begin{aligned}\frac{1}{2}\int_{\Omega^0_\sigma}|\nabla v^\tau_k|^2\,\mathrm{d}x - \frac{1}{2}\int_{\Omega^\tau_\sigma}|\nabla w^\tau_{k+1}|^2\,\mathrm{d}y \geq \frac{1}{2}\int_{\Omega_\sigma}|\nabla\tilde{\Phi}^0|^2\,\mathrm{d}x\\ - \frac{1}{2}\int_{\Omega^\tau_\sigma}|\nabla\tilde{\Phi}^\tau|^2\,\mathrm{d}y - M_\delta\tau D^\tau_{\sigma,k} + \mathrm{O}(\sqrt{\sigma|\log\sigma|}).\end{aligned} \tag{4.82}$$

Notice that, taking the limit as $\sigma\to 0$ in (4.82), we get

$$\mathcal{W}(v^\tau_k) - \mathcal{W}(w^\tau_{k+1}) \geq W(x^\tau_k) - W(x^\tau_k - \tau\nabla W(x^\tau_k)) - M_\delta\tau D^\tau_k,$$

which, in view of (4.80), concludes the proof of (4.69).

We now prove (4.82). By the change of variable $y = \chi_\tau(x)$ and by definition of w^τ_{k+1} in (4.77), we get

$$\begin{aligned}\frac{1}{2}\int_{\Omega^\tau_\sigma}|\nabla w^\tau_{k+1}|^2\,\mathrm{d}y &= \frac{1}{2}\int_{\Omega^0_\sigma}|\nabla w^\tau_{k+1}(\chi_\tau)|^2|J\chi_\tau|\,\mathrm{d}x\\ &= \frac{1}{2}\int_{\Omega^0_\sigma}|\nabla\varphi^0 + \nabla\psi^\tau|^2|J\chi_\tau|\,\mathrm{d}x \qquad (4.83)\end{aligned}$$

We claim that the following two estimates hold.

$$\frac{1}{2}\int_{\Omega_\sigma^0} |\nabla\varphi^0|^2|J\chi_\tau|\,\mathrm{d}x \le \frac{1}{2}\int_{\Omega_\sigma^0} |\nabla\tilde{\Phi}^0|^2|J\chi_\tau|\,\mathrm{d}x + (1+M_\delta\tau)D^\tau_{\sigma,k}, \tag{4.84}$$

$$\int_{\Omega_\sigma^0} \langle\nabla\psi^\tau,\nabla\varphi^0\rangle|J\chi_\tau|\,\mathrm{d}x = \int_{\Omega_\sigma^0} \langle\nabla\psi^\tau,\nabla\tilde{\Phi}^0\rangle|J\chi_\tau|\,\mathrm{d}x + \mathrm{O}(\sqrt{\sigma|\log\sigma|}). \tag{4.85}$$

By (4.84) and (4.85), we conclude the proof of (4.82) as follows: Using (4.74) and the change of variables $y = \chi_\tau(x)$, by (4.83) we get

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_\sigma^\tau} |\nabla w^\tau_{k+1}|^2\,\mathrm{d}y \\
&\quad\le \frac{1}{2}\int_{\Omega_\sigma^0} |\nabla\tilde{\Phi}^0 + \nabla\psi^\tau|^2|J\chi_\tau|\,\mathrm{d}x + (1+M_\delta\tau)D^\tau_{\sigma,k} + \mathrm{O}(\sqrt{\sigma|\log\sigma|}) \\
&\qquad\qquad = \frac{1}{2}\int_{\Omega_\sigma^\tau} |\nabla\tilde{\Phi}^\tau|^2\,\mathrm{d}y + (1+M_\delta\tau)D^\tau_{\sigma,k} + \mathrm{O}(\sqrt{\sigma|\log\sigma|})\,.
\end{aligned} \tag{4.86}$$

By (4.81) and straightforward algebraic manipulations we obtain (4.82).

Now, we will prove the claims (4.84) and (4.85). Claim (4.84) follows by

$$\begin{aligned}
\frac{1}{2}\int_{\Omega_\sigma^0} (|\nabla\varphi^0|^2 - |\nabla\tilde{\Phi}^0|^2)|J\chi_\tau|\,\mathrm{d}x &\le \|J\chi_\tau\|_{L^\infty} D^\tau_{\sigma,k} \\
&\le (1+\frac{1}{2}M_\delta\tau + O(\tau^2))D^\tau_{\sigma,k} \le (1+M_\delta\tau)D^\tau_{\sigma,k}.
\end{aligned}$$

We pass to the proof of (4.85). We have

$$\begin{aligned}
&\int_{\Omega_\sigma^0} \langle\nabla\psi^\tau,\nabla\varphi^0\rangle|J\chi_\tau|\,\mathrm{d}x \\
&\quad = \int_{\Omega_\sigma^0} \langle\nabla\psi^\tau,\nabla\tilde{\Phi}^0\rangle|J\chi_\tau|\,\mathrm{d}x + \int_{\Omega_\sigma^0} \langle\nabla\psi^\tau,\nabla\varphi^0 - \nabla\tilde{\Phi}^0\rangle|J\chi_\tau|\,\mathrm{d}x.
\end{aligned} \tag{4.87}$$

Using again that $\|J\chi_\tau\|_{L^\infty} \le 1 + M_\delta\tau$ and Hölder inequality, we get

$$\begin{aligned}
&\int_{\Omega_\sigma^0} \langle\nabla\psi^\tau,\nabla\varphi^0 - \nabla\tilde{\Phi}^0\rangle|J\chi_\tau|\,\mathrm{d}x \\
&\qquad \le (1+M_\delta\tau)\left(\int_{\Omega_\sigma^0}|\nabla\psi^\tau|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\Omega_\sigma^0}|\nabla\varphi^0 - \nabla\tilde{\Phi}^0|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.88}$$

Moreover, since $\mathcal{W}(v^\tau_k) \le \mathcal{W}(v_0)$, we have

$$\begin{aligned}
\int_{\Omega_\sigma^0} |\nabla\varphi^0 - \nabla\tilde{\Phi}^0|^2\,\mathrm{d}x &\le 2\int_{\Omega_\sigma^0} (|\nabla\varphi^0|^2 + |\nabla\tilde{\Phi}^0|^2)\,\mathrm{d}x \\
&\le 4\mathcal{W}(v_0) + 4M\pi|\log\sigma| + o_\sigma(1).
\end{aligned}$$

By (4.74), since X_V has compact support in Ω we have

$$\frac{\partial\psi^t}{\partial\nu} = \frac{\partial\Phi^t}{\partial\nu^\perp} - \frac{\partial\Phi}{\partial\nu^\perp} = 0 \qquad \text{on } \partial\Omega\,.$$

Therefore, in view of (4.75),

$$\int_{\Omega^0_\sigma} |\nabla\psi^\tau|^2 \,\mathrm{d}x = \int_{\partial\Omega} \psi^\tau \frac{\partial\psi^\tau}{\partial\nu} \,\mathrm{d}s - \sum_{i=1}^{M} \int_{B_\sigma(x_i)} |\nabla\psi^\tau|^2 \,\mathrm{d}s \le C\sigma^2.$$

Combining the above estimates with (4.87) and (4.88) we get (4.85).

To complete the proof it remains to show that (4.70) holds. By definition of $w^\tau_{k+1}(x)$ (see (4.77)), we have immediately

$$\|v^\tau_k - w^\tau_{k+1}\|^2_{L^2} = \int_\Omega \left| v^\tau_k - v^\tau_k(\chi^{-1}_\tau) e^{i\psi^\tau(\chi^{-1}_\tau)} \right|^2 \,\mathrm{d}y$$

$$= \int_\Omega \left| v^\tau_k - v^\tau_k(\chi^{-1}_\tau) \right|^2 \,\mathrm{d}y \tag{4.89}$$

$$+ \int_\Omega \left| v^\tau_k(\chi^{-1}_\tau) - v^\tau_k(\chi^{-1}_\tau) e^{i\psi^\tau(\chi^{-1}_\tau)} \right|^2 \,\mathrm{d}y \tag{4.90}$$

$$+ \ 2\int_\Omega \langle v^\tau_k - v^\tau_k(\chi^{-1}_\tau), v^\tau_k(\chi^{-1}_\tau) - v^\tau_k(\chi^{-1}_\tau) e^{i\psi^\tau(\chi^{-1}_\tau)} \rangle \,\mathrm{d}y. \tag{4.91}$$

In order to prove (4.70) it is enough to show that

$$\textstyle\int_\Omega \left| v^\tau_k - v^\tau_k(\chi^{-1}_\tau) \right|^2 \,\mathrm{d}y \le \pi\tau^2 |\log\tau| \ |V|^2 + \mathrm{o}(\tau^2|\log\tau|) \ , \tag{4.92}$$

$$\textstyle\int_\Omega \left| v^\tau_k(\chi^{-1}_\tau) - v^\tau_k(\chi^{-1}_\tau) e^{i\psi^\tau(\chi^{-1}_\tau)} \right|^2 \,\mathrm{d}y \le C\tau^2; \tag{4.93}$$

indeed, once we got (4.92) and (4.93), by Hölder inequality, we have immediately that the integral in (4.91) is $\mathrm{O}(\tau^2\sqrt{|\log\tau|})$.

First, we prove (4.93). By the change of variable $y = \chi_\tau(x)$ and the fact that $\psi^0 = 0$, in view of (4.75), we obtain

$$\int_\Omega \left| v^\tau_k(\chi^{-1}_\tau) - v^\tau_k(\chi^{-1}_\tau) e^{i\psi^\tau(\chi^{-1}_\tau)} \right|^2 \,\mathrm{d}y = \int_\Omega \left| 1 - e^{i\psi^\tau} \right|^2 |J\chi_\tau| \,\mathrm{d}x$$
$$\le (1 + M_\delta\tau) \| \frac{\mathrm{d}}{\mathrm{d}t}\psi^t \|^2_{L^\infty(\Omega)} \tau^2 |\Omega| \le C\tau^2 \,.$$

Finally, to complete the proof of the Theorem it remains to show that (4.92) holds. By Hölder inequality, we have

$$\int_{\Omega\setminus\cup_i B_{\delta/2}(y_i)} |v^\tau_k - v^\tau_k(\chi^{-1}_\tau)|^2 \,\mathrm{d}y$$
$$\le \int_{\Omega\setminus\cup_i B_{\delta/2}(y_i)} \tau \int_0^\tau |\nabla v^\tau_k(\chi^{-1}_t)|^2 \| \frac{\mathrm{d}}{\mathrm{d}t}\chi^{-1}_t \|^2_{L^\infty} \,\mathrm{d}y \,\mathrm{d}t$$
$$\le C(1 + M_\delta\tau)\tau^2 (\mathcal{W}(v^\tau_k) + M\pi |\log\frac{\delta}{2}|) \le C\tau^2 \,,$$

where C depends only on δ and we have used that $\mathcal{W}(v^\tau_k) \le \mathcal{W}(v_0)$.

In order to complete the proof of(4.92), it is enough to show that

$$\int_{B_{\delta/2}(y_i)} |v^\tau_k - v^\tau_k(\chi^{-1}_\tau)|^2 \,\mathrm{d}y \le \pi\tau^2 |\log\tau| |V_i|^2 + \mathrm{o}(\tau^2|\log\tau|). \tag{4.94}$$

Let $N > 0$ be given; then, for any $i = 1, \dots, M$,

$$\int_{B_{\delta/2}(y_i)} |v_k^\tau(\chi_\tau^{-1}) - v_k^\tau|^2 \,\mathrm{d}y \le \int_{B_{\delta/2}(y_i)\setminus B_{N\tau}(y_i)} |v_k^\tau(\chi_\tau^{-1}) - v_k^\tau|^2 \,\mathrm{d}y + 4N^2\tau^2\pi. \tag{4.95}$$

Without loss of generality we can assume $d_{i,0} = \deg(v_k^\tau, \partial B_{\delta/2}(x_i)) = 1$.

We first show the estimate (4.94) in the case $v_k^\tau = \frac{x-x_i}{|x-x_i|}$ in $B_{\delta/2}(x_i)$. Let (r,θ) be the polar coordinates with respect to y_i; denoting by $\alpha = \alpha(r,\theta)$ the angle between the vectors $\frac{y-y_i}{|y-y_i|}$ and $v_k^\tau(y) = \frac{y-y_i+\tau V_i}{|y-y_i+\tau V_i|}$, we have

$$\int_{B_{\delta/2}(y_i)\setminus B_{N\tau}(y_i)} \left| \frac{y-y_i}{|y-y_i|} - \frac{y-y_i+\tau V_i}{|y-y_i+\tau V_i|} \right|^2 \mathrm{d}y = \int_{N\tau}^{\delta/2} r\,\mathrm{d}r \int_0^{2\pi} 4\sin^2\frac{\alpha}{2}\,\mathrm{d}\theta. \tag{4.96}$$

Using elementary geometry identities and Taylor expansion, for $N\tau \le r \le \delta/2$ we get

$$\sin\alpha = \frac{\tau|V_i|\sin\theta}{r} \frac{1}{\sqrt{1+\frac{\tau^2|V_i|^2}{r^2} - 2\frac{\tau|V_i|\cos\theta}{r}}} = \frac{\tau|V_i|\sin\theta}{r}(1+\mathrm{O}(1/N)),$$

so that $\sin^2\frac{\alpha}{2} = \frac{\tau^2|V_i|^2\sin^2\theta}{4r^2} + \mathrm{O}(1/N)$. Therefore, by (4.96) we get

$$\begin{aligned}\int_{B_{\delta/2}(y_i)\setminus B_{N\tau}(y_i)} &\left| \frac{y-y_i}{|y-y_i|} - \frac{y-y_i+\tau V_i}{|y-y_i+\tau V_i|} \right|^2 \mathrm{d}y \\ &= \tau^2|V_i|^2 \int_{N\tau}^{\delta/2} \frac{1}{r}\,\mathrm{d}r \int_0^{2\pi} \sin^2\theta\,\mathrm{d}\theta + \mathrm{O}(1/N) \\ &= \pi\tau^2|\log\tau||V_i|^2 + \pi\tau^2\log\frac{\delta}{2N}|V_i|^2 + \mathrm{O}(1/N)). \end{aligned} \tag{4.97}$$

Then, (4.94) follows (in the case $v_k^\tau = \frac{x-x_i}{|x-x_i|}$) by choosing $N = |\log\tau|$.

Now, we prove (4.94) in the general case, i.e., without assuming $v_k^\tau = \frac{x-x_i}{|x-x_i|}$. Set $L := \lfloor \log_2 \frac{\delta}{2N\tau} \rfloor$ and let θ_i be the angle in polar coordinates with center in y_i, i.e., the phase of the function $\frac{y-y_i}{|y-y_i|}$. For every $l = 1, \dots, L$, we set

$$C_l(y_i) := B_{2^{-l}\delta}(y_i) \setminus B_{2^{-l-1}\delta}(y_i), \quad \tilde{C}_l(y_i) := B_{2^{-l+1}\delta}(y_i) \setminus B_{2^{-l-2}\delta}(y_i).$$

Set $\tilde{\varphi}^0_{i,l} = \frac{1}{|\tilde{C}_l(y_i)|}\int_{\tilde{C}_l(y_i)} \varphi^0(x)\,\mathrm{d}y$ and notice that the average of θ_i is equal to π. We have

$$\int_{B_{\delta/2}(y_i)\setminus B_{N\tau}(y_i)} |v^\tau_k(\chi_\tau^{-1}) - v^\tau_k|^2\,\mathrm{d}y = \sum_{l=1}^L \int_{C_l(y_i)} |v^\tau_k(\chi_\tau^{-1}) - v^\tau_k|^2\,\mathrm{d}y$$
$$= \sum_{l=1}^L \int_{C_l(y_i)} |e^{i(\varphi^0(\chi_\tau^{-1}) - \tilde{\varphi}^0_{i,l}+\pi)} - e^{i(\varphi^0 - \tilde{\varphi}^0_{i,l}+\pi)}|^2\,\mathrm{d}y$$
$$= \sum_{l=1}^L \int_{C_l(y_i)} |e^{i\theta_i(\chi_\tau^{-1})} - e^{i\theta_i}|^2\,\mathrm{d}y$$
$$+ \int_{C_l(y_i)} |e^{i(\varphi^0(\chi_\tau^{-1}) - \tilde{\varphi}^0_{i,l}+\pi)} - e^{i(\varphi^0 - \tilde{\varphi}^0_{i,l}+\pi)} - (e^{i\theta_i(\chi_\tau^{-1})} - e^{i\theta_i})|^2\,\mathrm{d}y$$
$$+2\int_{C_l(y_i)} \langle e^{i\theta_i(\chi_\tau^{-1})} - e^{i\theta_i}, e^{i(\varphi^0(\chi_\tau^{-1}) - \tilde{\varphi}^0_{i,l}+\pi)} - e^{i(\varphi^0 - \tilde{\varphi}^0_{i,l}+\pi)} - (e^{i\theta_i(\chi_\tau^{-1})} - e^{i\theta_i})\rangle\,\mathrm{d}y\,.$$

Estimating the last term of the right hand side of the above formula by Hölder's inequality and recalling (4.97), in order to prove (4.94) it is enough to show the following estimate

$$\sum_{l=1}^L \int_{C_l(y_i)} \left| e^{i(\varphi^0(\chi_\tau^{-1}) - \tilde{\varphi}^0_{i,l}+\pi)} - e^{i(\varphi^0 - \tilde{\varphi}^0_{i,l}+\pi)} - (e^{i\theta_i(\chi_\tau^{-1})} - e^{i\theta_i})\right|^2\,\mathrm{d}y \le C\tau^2\,.$$

By definition of χ_τ, for any $y \in C_l(y_i)$, $\chi_\tau^{-1}(y) = y - \tau V_i$ and then

$$\begin{aligned} e^{i(\varphi^0(\chi_\tau^{-1}(y)) - \tilde{\varphi}^0_{i,l}+\pi)} - e^{i(\varphi^0(y) - \tilde{\varphi}^0_{i,l}+\pi)} &= -\int_0^\tau \nabla e^{i(\varphi^0(y-tV_i) - \tilde{\varphi}^0_{i,l}+\pi)}\cdot V_i\,\mathrm{d}t\,, \\ e^{i\theta_i(\chi_\tau^{-1}(y))} - e^{i\theta_i(y)} &= -\int_0^\tau \nabla e^{i\theta_i(y-tV_i)}\cdot V_i\,\mathrm{d}t\,; \end{aligned}$$

then, by Jensen and Cauchy inequalities,

$$\int_{C_l(y_i)} \left| e^{i(\varphi^0(\chi_\tau^{-1}) - \tilde{\varphi}^0_{i,l}+\pi)} - e^{i(\varphi^\tau_k - \tilde{\varphi}^0_{i,l}+\pi)} - (e^{i\theta_i(\chi_\tau^{-1})} - e^{i\theta_i})\right|^2\,\mathrm{d}y$$
$$= \int_{C_l(y_i)} \left| \int_0^\tau (\nabla e^{i\theta_i(y-tV_i)} - \nabla e^{i(\varphi^0(y-tV_i) - \tilde{\varphi}^0_{i,l}+\pi)})\cdot V_i\,\mathrm{d}t\right|^2\,\mathrm{d}y$$
$$\le \tau|V_i|^2 \int_{C_l(y_i)}\mathrm{d}y \int_0^\tau \left|\nabla e^{i\theta_i(y-tV_i)} - \nabla e^{i(\varphi^0(y-tV_i) - \tilde{\varphi}^0_{i,l}+\pi)}\right|^2\,\mathrm{d}t$$
$$\le \tau^2|V_i|^2 \int_{\tilde{C}_l(y_i)} \left|\nabla e^{i\theta_i} - \nabla e^{i(\varphi^0 - \tilde{\varphi}^0_{i,l}+\pi)}\right|^2\,\mathrm{d}y. \quad (4.98)$$

Furthermore

$$
\begin{aligned}
&\sum_{l=1}^{L}\int_{\tilde C_l(y_i)}|\nabla e^{i\theta_i}-\nabla e^{i(\varphi^0-\tilde\varphi^0_{i,l}+\pi)}|^2\,\mathrm{d}y\\
&\le 2\sum_{l=1}^{L}\int_{\tilde C_l(y_i)}|\nabla e^{i\theta_i}|^2|1-e^{i(\varphi^0-\tilde\varphi^0_{i,l}+\pi-\theta_i)}|^2\,\mathrm{d}y+\int_{\tilde C_l(y_i)}|\nabla(\theta_i-\varphi^0)|^2\,\mathrm{d}y\\
&\le 2\sum_{l=1}^{L}\int_{\tilde C_l(y_i)}2^{2l+4}\delta^{-2}|e^{i(\theta_i-\varphi^0)}-e^{i(-\tilde\varphi^0_{i,l}+\pi)}|^2\,\mathrm{d}y+\int_{\tilde C_l(y_i)}|\nabla(\theta_i-\varphi^0)|^2\,\mathrm{d}y,
\end{aligned}
\tag{4.99}
$$

where the last inequality follows from the fact that $|\nabla e^{i\theta_i}(y)|^2=\frac{1}{|y-y_i|^2}$ and that $2^{-l-2}\delta\le|y-y_i|\le 2^{-l+1}\delta$ for $y\in\tilde C_l(x_i)$.

Finally, by Poincaré inequality, it follows that

$$
\begin{aligned}
&\int_{\tilde C_l(y_i)}|e^{i(\theta_i-\varphi^0)}-e^{i(-\tilde\varphi^0_{i,l}+\pi)}|^2\,\mathrm{d}x\\
&\le\int_{\tilde C_l(x_i)}|\theta_i-\varphi^0-(\pi-\tilde\varphi^0_{i,l})|^2\,\mathrm{d}x\le C2^{-2l}\delta^2\int_{\tilde C_l(y_i)}|\nabla(\theta^i-\varphi^0)|^2\,\mathrm{d}y\ ,
\end{aligned}
\tag{4.100}
$$

where C is a positive constant. By the minimality of θ_i, we have

$$
\int_{\tilde C_l(y_i)}|\nabla(\varphi^0-\theta_i)|^2\,\mathrm{d}y=\int_{\tilde C_l(y_i)}|\nabla\varphi^0|^2-\int_{\tilde C_l(y_i)}|\nabla\theta_i|^2\,\mathrm{d}y\ .
$$

By (4.99) and Remark 2.8, we obtain

$$
\begin{aligned}
\sum_{l=1}^{L}\int_{\tilde C_l(y_i)}&|\nabla e^{i(\varphi^0-\tilde\varphi^\tau_{i,l}+\pi)}-\nabla e^{i\theta_i}|^2\,\mathrm{d}y\\
&\le C\sum_{l=1}^{L}\int_{\tilde C_l(y_i)}(|\nabla\varphi^0|^2-|\nabla\theta_i|^2)\,\mathrm{d}y\\
=C\sum_{l=1}^{L}&\left(\int_{\tilde C_l(y_i)}|\nabla\varphi^0|^2\,\mathrm{d}x-6\pi\log 2\right)\le C\,.
\end{aligned}
$$

This together with (4.98) concludes the proof. □

We are now in a position to prove Theorem 4.21.

Proof of Theorem 4.21. By Theorems 4.24, 4.25, we can apply Theorem 4.23 for any $T<\pi\frac{c_\delta^2}{C_\delta}$, and in view of (4.54) we obtain that

1) x^τ converges to the solution x of (4.23), uniformly on $[0,T]$;
2) $D(T)=0$.

Let $T^{max}\le\liminf_{\tau\to0}k^\tau_\delta\tau$ be the maximal time such that 1) and 2) holds true on $[0,T)$ for every $T<T^{max}$. Recalling (see (4.33)) that $T_\delta\to T^*$ as $\delta\to0$, it remains only to prove that $T^{max}\ge T_\delta$. This follows by a standard

continuation argument: Assume by contradiction that $T^{max} < T_\delta$, and let $T < T^{max}$. Then we have

$$\min_{t\in[0,T^{max}]} \min\{\frac{1}{2}\mathrm{dist}_{i\neq j}(x_i(t),x_j(t)), \mathrm{dist}(x_i(t),\partial\Omega)\} - 2\delta = c'_\delta > 0.$$

Consider now $x^\tau_{\lfloor T/\tau\rfloor}$, $v^\tau_{\lfloor T/\tau\rfloor}$ as the initial condition of a new L^2 discrete gradient flow. Notice that, in view of 2), these initial conditions are well prepared; the fact that the initial time is not zero is not relevant, since all the equations are autonomous. Moreover, even if the initial conditions depend on τ, they converge as $\tau \to 0$. Therefore, Theorems 4.24, 4.25, and Theorem 4.23 still hold true with the obvious modifications, and we easily deduce that 1) and 2) holds true as long as $0 \le t - T \le (c'_\delta)^2/C_\delta$. This time interval in which we can extend the solution is independent of $T < T^{max}$, which contradicts the maximality of T^{max}. □

4.3.2. L^2 discrete gradient flow of F_ε. We conclude this section by analyzing the existence of the L^2 discrete gradient flow of F_ε and studying its asymptotic behaviour as $\varepsilon \to 0$. The existence will be obtained for ε small enough by making use of the auxiliary problem studied in the previous section. To this aim it is convenient to introduce a relaxed version of such discrete evolution.

Definition 4.26. Fix $\delta > 0$ and let $\varepsilon, \tau > 0$. Given $u_{\varepsilon,0} \in \mathcal{AF}_\varepsilon(\Omega)$, we say that $\{\bar u^\tau_{\varepsilon,k} : k \in \mathbb{N}\}$, is a relaxed L^2 discrete gradient flow of F_ε from $u_{\varepsilon,0}$ if $\bar u^\tau_{\varepsilon,0} = u_{\varepsilon,0}$ and, for any $k \in \mathbb{N}$, there exists a sequence $\{u^\tau_{\varepsilon,k,n}\}_n$ such that

$$\begin{aligned} \lim_{n\to\infty} \|e^{2\pi i u^\tau_{\varepsilon,k,n}} - e^{2\pi i \bar u^\tau_{\varepsilon,k}}\|_{L^2} = 0,\\ \|\mu(u^\tau_{\varepsilon,k,n}) - \mu(\bar u^\tau_{\varepsilon,k-1})\|_{\mathrm{flat}} \le \delta \quad \text{for every } n\in\mathbb{N},\\ \lim_{n\to\infty} F_\varepsilon(u^\tau_{\varepsilon,k,n}) + \frac{\|e^{2\pi i u^\tau_{\varepsilon,k,n}} - e^{2\pi i \bar u^\tau_{\varepsilon,k-1}}\|^2_{L^2}}{2\tau|\log\tau|} = I^\tau_{\varepsilon,k}\,, \end{aligned} \tag{4.101}$$

where

$$I^\tau_{\varepsilon,k} = \inf_{u\in\mathcal{AF}_\varepsilon(\Omega)} \left\{ F_\varepsilon(u) + \frac{\|e^{2\pi i u} - e^{2\pi i \bar u^\tau_{\varepsilon,k-1}}\|^2_{L^2}}{2\tau|\log\tau|} \ :\ \|\mu(u) - \mu(\bar u^\tau_{\varepsilon,k-1})\|_{\mathrm{flat}} \le \delta \right\}.$$

The existence of such relaxed discrete gradient flow is obvious. To show that it is actually a strong L^2 discrete gradient flow it is enough to show that $\|\mu(\bar u^\tau_{\varepsilon,k}) - \mu(\bar u^\tau_{\varepsilon,k-1})\|_{\mathrm{flat}} \le \delta$. A key argument is given by the following estimate that one can easly check by contradiction

$$\limsup_{n\to+\infty} \|\mu(u^\tau_{\varepsilon,k,n}) - \mu(\bar u^\tau_{\varepsilon,k})\|_{\mathrm{flat}} \le C\varepsilon\sharp\{(i,j)\in\Omega^1_\varepsilon : \mathrm{dist}(\bar u^\tau_{\varepsilon,k}(i) - \bar u^\tau_{\varepsilon,k}(j), \mathbb{Z}) = \tfrac{1}{2}\} \tag{4.102}$$

Theorem 4.27. *Let v_0 be such that $\mathcal{W}(v_0) < +\infty$ and let $Jv_0 = \sum_{i=1}^M d_{i,0}\delta_{x_{i,0}} =: \mu_0$ with $|d_{i,0}| = 1$. Let $u_{\varepsilon,0} \in \mathcal{AF}_\varepsilon(\Omega)$ such that*

$$\mu(u_{\varepsilon,0}) \overset{\mathrm{flat}}{\to} \mu_0, \qquad F_\varepsilon(u_{\varepsilon,0}) \le \pi|\mu_0|(\Omega)\log\varepsilon + C.$$

Let $\delta > 0$ be fixed such that $\min\left\{\frac{1}{2}\mathrm{dist}_{i\neq j}(x_{i,0},x_{j,0}), \mathrm{dist}(x_{i,0},\partial\Omega)\right\} > 2\delta$. Given $\tau > 0$, let $\bar u^\tau_{\varepsilon,k}$ be a relaxed L^2 discrete gradient flow of F_ε from $u_{\varepsilon,0}$.

Then, up to a subsequence, for any $k \in \mathbb{N}$ we have $\mu(\bar{u}^\tau_{\varepsilon,k}) \stackrel{\text{flat}}{\to} \mu^\tau_k$, for some $\mu^\tau_k \in X$ with $|\mu^\tau_k|(\Omega) \le M$ and there exists a maximal L^2 discrete gradient flow, v^τ_k, of $\mathcal{W}$ from v_0, according with Definition 4.16, such that

$$\mu^\tau_k = Jv^\tau_k = \sum_{i=1}^M d_{i,0}\delta_{x^\tau_{i,k}}\,, \text{ for every } k = 1, \dots, k^\tau_\delta\,, \tag{4.103}$$

with k^τ_δ as defined in (4.25).

Moreover denoting by $\tilde{v}^\tau_{\varepsilon,k}$ the piecewise affine interpolation of $e^{2\pi i\bar{u}^\tau_{\varepsilon,k}}$, we have

$$\tilde{v}^\tau_{\varepsilon,k} \rightharpoonup v^\tau_k \quad \text{in } H^1_{\text{loc}}(\Omega \setminus \cup_{i=1}^M\{x^\tau_{i,k}\};\mathbb{R}^2)\,,\ \text{for every } k = 1, \dots, k^\tau_\delta\,. \tag{4.104}$$

Finally for τ and ε small enough such $\bar{u}^\tau_{\varepsilon,k}$ is indeed a minimizer of problem (4.39) and hence it is a (strong) L^2 discrete gradient flow.

Proof. The proof of this result uses the first order Γ-convergence result (Theorem 2.9) and follows closely the proof of the analogous statement in Section 4.2 (see Theorem 4.14). Indeed by the definition of relaxed L^2 discrete gradient flow we have that for any $k \in \mathbb{N}$

$$F_\varepsilon(\bar{u}^\tau_{\varepsilon,k}) + \frac{\left\| e^{2\pi i\bar{u}^\tau_{\varepsilon,k}} - e^{2\pi i\bar{u}^\tau_{\varepsilon,k-1}}\right\|^2_{L^2}}{2\pi\tau|\log\tau|} \le F_\varepsilon(\bar{u}^\tau_{\varepsilon,k-1}).$$

By induction on k, one can show that

$$F_\varepsilon(\bar{u}^\tau_{\varepsilon,k}) \le F_\varepsilon(u_{\varepsilon,0},\Omega) \le M\pi|\log\varepsilon| + C\,.$$

This estimate together with (4.102) we obtain that $\|\mu(\bar{u}^\tau_{\varepsilon,k}) - \mu(\bar{u}^\tau_{\varepsilon,k-1})\|_{\text{flat}} \le \delta + C\varepsilon|\log\varepsilon|$. Then using the Compactness result stated in Theorem 2.6(i), and arguing as in the proof of Theorem 4.14 we deduce (4.103) and (4.104).

In order to show that, for ε small enough, $\bar{u}^\tau_{\varepsilon,k}$ is an L^2 discrete gradient flow according with Definition 4.15, it is enough to recall that thanks to Proposition 4.20 we have that $\|\mu^\tau_k - \mu^\tau_{k-1}\|_{\text{flat}} \le C\sqrt{\tau|\log\tau|}$. Then the conclusion follows by the convergence in the flat norm of $\mu(\bar{u}^\tau_{\varepsilon,k})$ to μ^τ_k. □

CHAPTER 5

Γ-convergence analysis of systems of edge dislocations

Here we develop the static Γ-convergence analysis for the energy dislocations. All the results in this chapter are proved in [31].

5.1. The main result

In this section we state the main result of this chapter and introduce the required preliminaries and notation. We recall that Ω is a bounded open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary. Let $\mathbb{S}$ be the class of horizontal slips (translations) under which the crystal is invariant. It is generated by a set $S := \{v_1, v_2\} \subset \mathbb{R}^2$, where v_i are called *primitive vectors*, i.e., $\mathbb{S} = \mathrm{Span}_{\mathbb{Z}} S$ (we are implicitly assuming that Ω lies on a slip plane of the crystal). For instance, in the case of cubic crystals we would choose $S = \{e_1, e_2\}$, while for fcc crystals S can be chosen as $S = \{e_1, \frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2\}$.

The space of finite distributions of edge dislocations X_{edge} is given by

$$X_{\mathrm{edge}} := \{\mu \in \mathcal{M}(\Omega, \mathbb{R}^2) : \mu = \sum_{i=1}^{N} \xi_i \delta_{x_i},\, N \in \mathbb{N},\, x_i \in \Omega,\, \xi_i \in \mathbb{S}\},$$

where $\mathcal{M}(\Omega, \mathbb{R}^2)$ denotes the set of vector valued Radon measures on Ω. We endow X_{edge} with the *flat norm* $\|\mu\|_{\mathrm{flat}}$ defined by

$$\|\mu\|_{\mathrm{flat}} := \sup_{\|\phi\|_{W_0^{1,\infty}(\Omega)} \le 1} \int_\Omega \phi \, \mathrm{d}\mu;$$

in particular, we can consider X_{edge} as a subspace of $W^{-1,1}(\Omega)$. We will denote by $\mu_h \overset{\mathrm{flat}}{\to} \mu$ the flat convergence of μ_h to μ.

Fix $\varepsilon > 0$. Given $\mu \in X$, we denote by

$$\Omega_\varepsilon(\mu) := \Omega \setminus \bigcup_{x_i \in \mathrm{supp}(\mu)} B_\varepsilon(x_i).$$

With a little abuse of terminology we will call admissible strain associated with μ any field $\beta \in \mathcal{AS}_\varepsilon(\mu)$, where

$$\begin{aligned} \mathcal{AS}_\varepsilon(\mu) \quad := \quad & \Big\{\beta \in L^2(\Omega_\varepsilon(\mu); \mathbb{M}^{2\times 2}) : \ \mathrm{Curl}\beta = 0 \text{ in } \Omega_\varepsilon(\mu), \\ & \int_{\partial A} \beta(s) \cdot t(s)\, ds = \mu(A) \text{ for every open set } A \subset \Omega \\ & \quad \text{with } \partial A \text{ smooth: } \partial A \subset \Omega_\varepsilon(\mu), \text{ and } \int_{\Omega_\varepsilon(\mu)} (\beta - \beta^{\mathrm{T}})\, \mathrm{d}x = 0\Big\}. \end{aligned} \tag{5.1}$$

Here t denotes the tangent vector to ∂A and the integrand $\beta \cdot t$ is intended in the sense of traces (see Theorem 2 page 204 in [30]). Note that if the balls $B_\varepsilon(x_i)$, with $x_i \in \mathrm{supp}(\mu)$, are pairwise disjoint and contained in Ω, then the circulation condition in (5.1) reads as

$$\int_{\partial B_\varepsilon(x_i)} \beta(s) \cdot t(s)\, ds = \xi_i \,.$$

The elastic energy associated with a strain $\beta \in \mathcal{AS}_\varepsilon(\mu)$ is defined by

$$E_\varepsilon(\mu, \beta) := \int_{\Omega_\varepsilon(\mu)} W(\beta)\, \mathrm{d}x,$$

where $W(\beta) = \frac{1}{2}\mathbb{C}\beta : \beta$. The energy $\mathcal{E}_\varepsilon : X_{\mathrm{edge}} \to \mathbb{R}$ induced by the distribution of dislocations μ is given by

$$\mathcal{E}_\varepsilon(\mu) := \min_{\beta \in \mathcal{AS}_\varepsilon(\mu)} E_\varepsilon(\mu, \beta) + |\mu|(\Omega).$$

The rescaled energy functionals $\mathcal{F}_\varepsilon : X_{\mathrm{edge}} \to \mathbb{R}$ are defined by

$$\mathcal{F}_\varepsilon(\mu) := \frac{1}{|\log \varepsilon|} \mathcal{E}_\varepsilon(\mu). \tag{5.2}$$

The main result of this Chapter is the study in terms of Γ-convergence with respect to the flat topology of the functionals $\mathcal{F}_\varepsilon(\mu)$. We show in Theorem 5.4 that the Γ-limit is obtained by a suitable relaxation of the so-called prelogarithmic factor ψ, that we define as follows: Given $\xi \in \mathbb{R}^2$, we set, in agreement with [36]

$$\psi(\xi) := \min \Big\{ \int_{\partial B_1} W(\Gamma(\theta))\, \mathrm{d}\theta \;:\; \Gamma \in L^2(\partial B_1, \mathbb{M}^{2\times 2}),\ \mathrm{Curl} \frac{1}{\rho}\Gamma(\theta) = 0, \\ \int_{\partial B_1} \Gamma(\theta) \cdot t(\theta)\, \mathrm{d}\theta = \xi \Big\}, \tag{5.3}$$

where (ρ, θ) are polar coordinates in $\mathbb{R}^2$, $t(\theta)$ denotes the tangent vector to ∂B_1, and the equation $\mathrm{Curl}\frac{1}{\rho}\Gamma(\theta) = 0$ has to be understood in the sense of distributions in $\mathbb{R}^2 \setminus \{0\}$. The minimum in (5.3) is attained by a function denoted by Γ_ξ which is unique up to additive skew matrices.

The displacement $u_{\mathbb{R}^2}(\xi)$ induced on the whole plane by a straight infinite dislocation centered at 0 with multiplicity ξ is computed explicitly in the literature (see e.g.,[12, formula (4.1.25)]) and it is of the form

$$u^\xi_{\mathbb{R}^2}(\rho, \theta) = F_\xi(\theta) + g_\xi \log \rho \,,$$

where $g_\xi \in \mathbb{R}^2$ and the function F_ξ is given by $F_\xi(\theta) = \int_0^\theta f_\xi(\omega)\, \mathrm{d}\omega$ for a suitable function $f_\xi \in C^0(\partial B_1; \mathbb{R}^2)$, with $\int_0^{2\pi} f_\xi(\omega)\, \mathrm{d}\omega = \xi$. The corresponding strain field is given by

$$\beta^\xi_{\mathbb{R}^2}(\rho, \theta) := \frac{1}{\rho} \left(f_\xi(\theta) \otimes (-\sin\theta, \cos\theta) + g_\xi \otimes (\cos\theta, \sin\theta) \right). \tag{5.4}$$

The equations satisfied by $\beta^\xi_{\mathbb{R}^2}$ are

$$\begin{cases} \mathrm{Curl}\beta^\xi_{\mathbb{R}^2} = \xi\delta_0 & \text{in } \mathbb{R}^2; \\ \mathrm{Div}\mathbb{C}\beta^\xi_{\mathbb{R}^2} = 0 & \text{in } \mathbb{R}^2. \end{cases} \tag{5.5}$$

It can be proved that a field satisfying (5.4) and (5.5) is unique, so that the fields $f_\xi(\cdot)$ and g_ξ are determined by the vector ξ and the elasticity tensor $\mathbb{C}$. More precisely $\beta^\xi_{\mathbb{R}^2}$ is given by

$$\beta^\xi_{\mathbb{R}^2}(\rho,\theta) = \frac{1}{\rho}\Gamma_\xi(\theta)\,, \tag{5.6}$$

where Γ_ξ is a minimizer of (5.3) (see [36]). In particular

$$\psi(\xi) = \int_{\partial B_1} W(\Gamma_\xi(\theta))\,\mathrm{d}\theta = \lim_{\varepsilon\to 0}\frac{1}{\log\frac{1}{\varepsilon}}\int_{B_1\setminus B_\varepsilon} W(\beta^\xi_{\mathbb{R}^2})\,\mathrm{d}x, \tag{5.7}$$

where B_ρ denotes the ball of radius ρ and center 0.

Let us introduce for any given $\xi\in\mathbb{R}^2$ and for $0<r<R$, the space

$$\begin{aligned}\mathcal{AS}_{r,R}(\xi) := \{\beta\in L^2(B_R\setminus B_r;\mathbb{M}^{2\times 2}) : \mathrm{Curl}\beta = 0, \int_{\partial B_r}\beta\cdot t\,\mathrm{d}s = \xi,\\ \int_{B_R\setminus B_r}(\beta-\beta^{\mathrm{T}})\,\mathrm{d}x = 0\}.\end{aligned} \tag{5.8}$$

The relation between the prelogarithmic factor defined in (5.3) and our energy is clarified by the following proposition (proved in [36], Corollary 6).

Proposition 5.1. *There exists a constant $C_0>0$ such that*

$$|\psi(\xi)-\psi_\varepsilon(\xi)| \le C_0\frac{|\xi|^2}{|\log\varepsilon|}\,,$$

where

$$\psi_\varepsilon(\xi) := \frac{1}{|\log\varepsilon|}\min_{\beta\in\mathcal{AS}_{\varepsilon,1}(\xi)}\int_{B_1\setminus B_\varepsilon} W(\beta)\,\mathrm{d}x\,.$$

Remark 5.2. In our analysis it will be convenient to introduce the following notation for the elastic energy of a dislocation in the annulus $B_R\setminus B_r$

$$\psi_{r,R}(\xi) := \frac{1}{\log R-\log r}\min_{\beta\in\mathcal{AS}_{r,R}(\xi)}\int_{B_R\setminus B_r} W(\beta)\,\mathrm{d}x\,. \tag{5.9}$$

Using a change of variables we clearly have $\psi_{r,R}(\xi) = \psi_{\frac{r}{R}}(\xi)$, and hence

$$|\psi(\xi)-\psi_{r,R}(\xi)| \le C_0\frac{|\xi|^2}{\log R-\log r}.$$

In particular

$$\lim_{\frac{r}{R}\to 0}\psi_{r,R}(\xi) = \psi(\xi)\,.$$

We introduce the density function $\varphi:\mathbb{S}\mapsto[0,+\infty)$ of the energy $\mathcal{F}$ through the following relaxation procedure

$$\varphi(\xi) := \inf\left\{\sum_{k=1}^N|\lambda_k|\psi(\xi_k) : \ \sum_{k=1}^N\lambda_k\xi_k = \xi,\, N\in\mathbb{N}, \lambda_k\in\mathbb{Z}, \xi_k\in\mathbb{S}\right\}. \tag{5.10}$$

It can be easily proved (see [36]) that the infimum in (5.10) is in fact a minimum.

Definition 5.3. We say that $b\in\mathbb{S}$ is a *Burgers vector* if $\varphi(b)=\psi(b)$, and denote by $\mathfrak{B}$ the class of such vectors.

It is easy to see that $\mathbb{S} = \mathrm{Span}_{\mathbb{Z}}\mathfrak{B}$ and that in the relaxation in (5.10) we can replace $\mathbb{S}$ with $\mathfrak{B}$, namely for every $\xi \in \mathbb{S}$ we have

$$\varphi(\xi) = \min\left\{\sum_{i=1}^{k} |\lambda_i|\psi(b_i) : \xi = \sum_{i=1}^{k} \lambda_i b_i,\ \lambda_i \in \mathbb{Z},\ b_i \in \mathfrak{B}\right\}. \tag{5.11}$$

The limit energy induced by a configuration μ is the functional

$$\mathcal{F}(\mu) := \sum_{i=1}^{N} \varphi(\xi_i) \qquad \text{for any } \mu = \sum_{i=1}^{N} \xi_i \delta_{x_i} \in X\,, \tag{5.12}$$

The following Γ-convergence result holds.

Theorem 5.4. *Let $\mathcal{F}_\varepsilon$ and $\mathcal{F}$ be defined by (5.2) and (5.12).*

(i) *(Compactness) Let $\varepsilon_h \to 0$ and let $\{\mu_h\}$ be a sequence in X_{edge} such that $\mathcal{F}_{\varepsilon_h}(\mu_h) \le M$ for some positive constant M independent of h. Then, (up to a subsequence) $\mu_h \stackrel{\mathrm{flat}}{\to} \mu \in X_{\mathrm{edge}}$.*

(ii) *(Γ-liminf inequality) Let $\{\mu_h\} \subset X_{\mathrm{edge}}$ be such that $\mu_h \stackrel{\mathrm{flat}}{\to} \mu$. Then*

$$\mathcal{F}(\mu) \le \liminf_{h\to+\infty} \mathcal{F}_{\varepsilon_h}(\mu_h).$$

iii *(Γ-limsup inequality) For every $\mu \in X$, there exists $\{\mu_h\} \subset X_{\mathrm{edge}}$, such that $\mu_h \stackrel{\mathrm{flat}}{\to} \mu$ and*

$$\limsup_{h\to+\infty} \mathcal{F}_{\varepsilon_h}(\mu_h) \le \mathcal{F}(\mu).$$

The proofs of the compactness and the Γ-liminf inequality are quite technical and are based on the "ball construction" technique. As explained in the Introduction, a specific difficulty of our context of plane elasticity is due to the fact that the energy depends only on the symmetric part of the field β. Moreover, the optimal Korn's inequality constant blows up on thin annuli, and the function $\psi_{r,R}$ defined in (5.9) vanishes as $R/r \to 1$ (see Example 5.13). It is then not clear how to estimate the energy from below on thin annuli. For this reason, in the implementation of the ball construction technique, we will work only with annuli whose ratio of the radii is given by a constant $c > 1$. To this purpose we have to revisit the standard ball construction in Subsection 1.2.1. We will introduce the needed discrete ball construction in the next section.

5.2. Revised Ball Construction

As mentioned above, the main goal of the ball construction is to provide the key lower bounds (see Proposition 5.6) on annular sets, needed in the proof of the Γ-liminf inequality and of the compactness. First, we give a lower bound for the energy on a single annulus $B_R \setminus B_r$.

Lemma 5.5. *Given $0 < r < R$ and $\xi \in \mathbb{R}^2$, for any admissible configuration $\beta \in \mathcal{AS}_{r,R}(\xi)$ (defined in (5.8)) we have*

$$\int_{B_R\setminus B_r} |\beta^{\mathrm{sym}}|^2 \,\mathrm{d}x \ge \frac{|\xi|^2}{2\pi}\frac{1}{K(R/r)}\log\frac{R}{r},$$

where $K(R/r)$ is the Korn's constant defined according with (5.35).

Proof. We introduce a cut L on the annulus $B_R \backslash B_r$ so that $(B_R \backslash B_r) \backslash L$ is simply connected, and exploit the fact that β is a curl free field in $B_R \setminus B_r$. More precisely, there exists a function $u \in H^1((B_R \backslash B_r) \backslash L; \mathbb{R}^2)$ with $\nabla u = \beta$ in $(B_R \setminus B_r) \setminus L$. From the circulation condition in (5.8), applying Jensen inequality, it is easy to see that

$$\int_{B_R \backslash B_r} |\nabla u|^2 \, \mathrm{d}x \geq \int_r^R \frac{1}{2\pi\rho} \left| \int_0^{2\pi} \nabla u \cdot t \, \mathrm{d}\theta \right|^2 \mathrm{d}\rho = \frac{|\xi|^2}{2\pi} \log \frac{R}{r};$$

the thesis follows directly by applying classical Korn's inequality (Theorem 5.35). □

For any given $C > 0$, let $f : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$f(r, R, t) := Ct \log \frac{R}{r}. \tag{5.13}$$

Clearly f satisfies the following properties

i) $f(r, \rho, t) + f(\rho, R, t) = f(r, R, t)$ for every $t > 0$ and $0 < r < \rho < R$;

ii) if $f(r_i, R_i, 1) = \alpha$ for every $i = 1, \ldots, m$, for some $\alpha \in \mathbb{R}^+$, then

$$\alpha = f(\sum_{i=1}^m r_i, \sum_{i=1}^m R_i, 1).$$

Fix $\mu = \sum_{i=1}^N \xi_i \delta_{x_i} \in X$, and set

$$\omega_\varepsilon := \bigcup_{i=1}^N B_\varepsilon(x_i). \tag{5.14}$$

Proposition 5.6. *Let $c > 1$ be fixed and let f be defined as in (5.13). Let F be a positive additive set function on the open subsets of Ω that satisfies*

$$F(B_R(x)) \geq f(r, R, |\mu(B_R(x))|) + F(B_r(x)), \tag{5.15}$$

for every $x \in \mathbb{R}^2$ and every r, $R \in \mathbb{R}^+$ with $\frac{R}{r} = c$ such that $B_R(x) \setminus B_r(x) \subset \Omega \setminus \omega_\varepsilon$. Finally, let $\rho > 0$ and let A be an open subset of Ω such that $\operatorname{dist}(x_i, \partial A) \geq \rho$ for all $x_i \in A$. Then,

$$F(A) \geq |\mu(A)| f(c^N \varepsilon N, \frac{\rho}{2c}, 1). \tag{5.16}$$

The statement of Proposition 5.6 is proved by computing a lower bound for the energy on a sequence of larger and larger annuli in which the main part of the energy is stored. We follow closely the strategy of the ball construction introduced by Sandier in 1.2.1. The main difference is that we need to construct annular sets with radii satisfying $R/r = c$. To this purpose, our ball construction consists in a discrete rather than continuous process in which at each step either all the balls expand or some of them merge together. We proceed by introducing our discrete ball construction.

Discrete Ball Construction

Let $\{x_i\}_{i=1,\ldots,N}$ be a set of points in $\mathbb{R}^2$, $c > 1$, and $\varepsilon > 0$. We set $N^0 := N$, $x_i^0 = x_i$, $R_i^0 = r_i^0 = \varepsilon$, for every $1 \leq i \leq N^0$ and $\mathcal{B}_0 = \{B_{R_i^0}(x_i^0)\}_{i=1,\ldots,N^0}$. Given x_i^{n-1}, R_i^{n-1}, r_i^{n-1} for $i = 1, \ldots, N^{n-1}$, we construct recursively x_i^n, R_i^n, r_i^n, for $i = 1, \ldots, N^n$, as follows. First, we consider the family of balls

$\{B_{cR_i^{n-1}}(x_i^{n-1})\}$. If these balls are pairwise disjoint, we say that n is an *expansion time*. In this case, we set $N^n = N^{n-1}$, and

$$x_i^n = x_i^{n-1}, \quad R_i^n = cR_i^{n-1}, \quad r_i^n = r_i^{n-1} \qquad \text{for all } i = 1, \dots, N^n.$$

If, otherwise, the balls in $\{B_{cR_i^{n-1}}(x_i^{n-1})\}$ are not pairwise disjoint, we say that n is a *merging time*. The merging consists in identifying a suitable partition $\{S_j\}_{j=1,\dots,N^n}$ of the family $\{B_{cR_i^{n-1}}(x_i^{n-1})\}$ and, for each subclass S_j, in finding a ball $B_{R_j^n}(x_j^n)$ which contains all the balls in S_j with the following properties:

i) the balls $B_{R_j^n}(x_j^n)$ of the new family are pairwise disjoint;

ii) R_j^n is not larger than the sum of all the radii of the balls $B_{cR_i^{n-1}}(x_i^{n-1}) \in S_j$, i.e., contained in $B_{R_j^n}(x_j^n)$.

Such a construction can be always done by an induction argument, for more details we refer to [58]. After the merging, we reset all the quantities introduced above as follows: x_j^n and R_j^n for $j = 1, \dots, N^n$ are determined by the merging construction, while the parameters r_j^n, referred to as the *seed sizes*, are defined so that, for all $1 \le i \le N^{n-1}$ and $1 \le j \le N^n$, we have

$$\frac{R_j^n}{r_j^n} = \frac{R_i^{n-1}}{r_i^{n-1}},$$

and hence

$$f(r_j^n, R_j^n, 1) = f(r_i^{n-1}, R_i^{n-1}, 1). \tag{5.17}$$

Furthermore, at any step n, we define a parameter τ_n that counts the number of merging occurred until the n-th step. More precisely, if n is an expansion time $\tau_n = \tau_{n-1}$ whereas if it is a merging time $\tau_n = \tau_{n-1} + 1$. In this way, at time n we have made $n - \tau_n$ expansions and τ_n merging.

Definition 5.7. We refer to the construction above as the *Discrete Ball Construction* associated with the points $\{x_i\}_{i=1,\dots,N}$. In particular, for every $n \in \mathbb{N}$ we have defined a family of balls

$$\mathcal{B}_n = \{B_{R_i^n}(x_i^n)\}_{i=1,\dots,N^n},$$

a family of seed sizes $\{r_i^n\}_{i=1,\dots,N^n}$ and the merging counter τ^n.

We are now in a position to prove Proposition 5.6.

Proof of Proposition 5.6. Consider the Discrete Ball Construction associated to the points $x_i \in A$. The balls in $\mathcal{B}_n$ satisfy

$$R_j^n \le c^n \varepsilon \,\sharp\{i : B_\varepsilon(x_i) \subset B_{R_j^n}(x_j^n)\} \tag{5.18}$$

$$r_j^n \le c^{\tau^n} \varepsilon \,\sharp\{i : B_\varepsilon(x_i) \subset B_{R_j^n}(x_j^n)\}. \tag{5.19}$$

We first prove (5.18) by induction arguing as follows. If n is an expansion time, then we clearly have $R_j^n = c\, R_j^{n-1}$. While if n is a merging time, by construction (namely, by property ii)) we have

$$R_j^n \le c \sum_{i:\, B_{R_i^{n-1}}(x_i^{n-1}) \subset B_{R_j^n}(x_j^n)} R_i^{n-1}.$$

As for the proof of (5.19), notice that

$$\begin{aligned}\frac{R_j^n}{r_j^n} &= c\frac{R_i^{n-1}}{r_i^{n-1}} \quad \text{if } n \text{ is an expansion step, for any } j \in N^n = N^{n-1},\ i \in N^{n-1},\\ \frac{R_n^j}{r_n^j} &= \frac{R_i^{n-1}}{r_i^{n-1}} \quad \text{if } n \text{ is a merging step, for any } j \in N^n,\ i \in N^{n-1}\ .\end{aligned}$$

We deduce that $\frac{R_j^n}{r_j^n} = c^{n-\tau^n}$. Therefore, (5.19) follows by (5.18) since

$$r_j^n = \frac{R_j^n}{c^{n-\tau^n}} \le \frac{c^n}{c^{n-\tau^n}}\varepsilon\,\sharp\{i:\ B_\varepsilon(x_i)\subset B_{R_j^n}(x_j^n)\} = c^{\tau^n}\,\varepsilon\,\sharp\{i:\ B_\varepsilon(x_i)\subset B_{R_j^n}(x_j^n)\}.$$

The main point of this construction is that it provides the following lower bound: for every $n\in\mathbb{N}$ and for every $j=1,\dots,N_n$

$$F(B_j^n) \ge |\mu(B_j^n)| f(r_j^n, R_j^n, 1), \tag{5.20}$$

where, for sake of simplicity, we have set $B_j^n := B_{R_j^n}(x_j^n)$.

We prove (5.20) by an induction argument. For $n=0$ there is nothing to prove. Suppose that the inequality is true at time $n-1$. If n is an expansion time, then

$$\begin{aligned}&F(B_j^n) = F(B_j^n\setminus B_j^{n-1}) + F(B_j^{n-1}) \ge f(R_j^{n-1}, R_j^n, |\mu(B_j^n)|)\\ &+ f(r_j^{n-1}, R_j^{n-1}, |\mu(B_j^{n-1})|)|\mu(B_j^n)|f(r_j^{n-1}, R_j^n, 1) = |\mu(B_j^n)| f(r_j^n, R_j^n, 1),\end{aligned}$$

where we have used (5.15), the induction hypothesis, the fact that the quantity $|\mu(B_j^{n-1})|$ does not vary during the expansion times and that, since n is an expansion time, $r_j^{n-1} = r_j^n$.

It remains to prove that inequality (5.20) is preserved during a merging time. Let n be a merging time and let $\{B_i^{n-1}\}_{i\in I}\subset B_j^n$. Since $\mu(B_n^j) = \sum_{i\in I}\mu(B_i^{n-1})$, we have $|\mu(B_j^n)| \le \sum_{i\in I}|\mu(B_i^{n-1})|$. Then, using (5.17), we conclude

$$\begin{aligned}F(B_j^n) &\ge \sum_{i\in I} F(B_i^{n-1})\\ &\ge \sum_{i\in I}|\mu(B_i^{n-1})| f(r_i^{n-1}, R_i^{n-1}, 1) \ge |\mu(B_j^n)| f(r_j^n, R_j^n, 1).\end{aligned}$$

Finally, let $\bar n\in\mathbb{N}$ be the first integer such that at least one ball in $\mathcal{B}^{\bar n}$ intersects ∂A. Clearly $\sum_{i=1}^{N^{\bar n}} R_i^{\bar n} \ge \rho/2$; moreover, by (5.19), we immediately deduce $\sum_{i=1}^{N^{\bar n}} r_i^{\bar n} \le c^N\varepsilon N$. Now we distinguish two cases. If $\bar n$ is an expansion time, then using (5.20) and property ii) of f, we get

$$\begin{aligned}F(A) &\ge \sum_{i=1}^{N^{\bar n-1}} F(B_i^{\bar n-1}) \ge \sum_{i=1}^{N^{\bar n-1}} |\mu(B_i^{\bar n-1})| f(r_i^{\bar n-1}, R_i^{\bar n-1}, 1)\\ &= \sum_{i=1}^{N^{\bar n}} |\mu(B_i^{\bar n})| f(r_i^{\bar n}, \frac{R_i^{\bar n}}{c}, 1) = \sum_{i=1}^{N^{\bar n}} |\mu(B_i^{\bar n})| f(\sum_{k=1}^{N^{\bar n}} r_k^{\bar n}, \frac{1}{c}\sum_{k=1}^{N^{\bar n}} R_k^{\bar n}, 1)\\ &\qquad\qquad \ge |\mu(A)| f(c^N\varepsilon\, N, \frac{\rho}{2c}, 1).\end{aligned}$$

If otherwise n is a merging time, then we conclude

$$\begin{aligned} F(A) &\geq \sum_{i=1}^{N^{\bar{n}-1}} F(B_i^{\bar{n}-1}) \geq \sum_{i=1}^{N^{\bar{n}-1}} |\mu(B_i^{\bar{n}-1})| f(r_i^{\bar{n}-1}, R_i^{\bar{n}-1}, 1) \geq \sum_{j=1}^{N_{\bar{n}}} |\mu(B_j^{\bar{n}})| f(r_j^{\bar{n}}, R_j^{\bar{n}}, 1) \\ &= \sum_{j=1}^{N^{\bar{n}}} |\mu(B_j^{\bar{n}})| f(\sum_{k=1}^{N^{\bar{n}}} r_k^{\bar{n}}, \sum_{k=1}^{N^{\bar{n}}} R_k^{\bar{n}}, 1) \geq |\mu(A)| f(c^N \varepsilon\, N, \frac{\rho}{2}, 1). \end{aligned}$$

Since $c > 1$, the conclusion follows. $\square$

Remark 5.8. Notice that, in order to prove (5.16), we gained indeed the following stronger estimate: for every $n \in \mathbb{N}$, we have

$$F(A) \geq \sum_{\substack{B_i^n \in \mathcal{B}^n \\ B_i^n \subset A}} |\mu(B_i^n)| f(c^N \varepsilon\, N, \sum_{k=1}^{N^n} R_k^n, 1)\,.$$

5.3. Compactness

The first step in order to prove the compactness and the Γ-liminf inequality is to show a lower bound for the elastic energy of a "cluster" of dislocations. Let $\mu := \sum_{i=1}^N \xi_i \delta_{x_i} \in X$ and $\varepsilon > 0$. We recall that ω_ε is defined in (5.14) and that $K(c)$ is the Korn's constant for an annulus with a cut, whose ratio of the radii is c (see (5.35)). Finally, we recall that c_1 is the constant in (0.2).

Lemma 5.9. *Fix $\varepsilon > 0$, let $\mu := \sum_{i=1}^N \xi_i \delta_{x_i} \in X$ for some $x_i \in \Omega$ and $\xi_i \in \mathbb{S}$, and let $\beta \in \mathcal{AS}_\varepsilon(\mu)$. Finally, let $0 < \delta < 1$ and $A \subset \Omega$ be open. If* $\operatorname{dist}(x_i, \partial A) \geq \varepsilon^\delta$ *for all $x_i \in A$, then, for every constant $c > 1$, we have*

$$\int_{A \setminus \omega_\varepsilon} W(\beta)\, \mathrm{d}x \geq c_1 \frac{|\mu(A)|}{2\pi K(c)} ((1-\delta)|\log \varepsilon| - (N+1)\log c - \log 2N). \quad (5.21)$$

Proof. We apply Proposition 5.6 for f defined as in (5.13) with $C = \frac{c_1}{2\pi K(c)}$ and

$$F(U) = E_\varepsilon(\mu, \beta, U) := \int_{U \setminus \omega_\varepsilon} W(\beta)\, \mathrm{d}x, \quad (5.22)$$

for all open subsets U of Ω. By Lemma 5.5 and (0.2) we deduce that (5.15) holds. Setting $\rho = \varepsilon^\delta$, from (5.16) we conclude

$$\begin{aligned} \int_{A \setminus \omega \varepsilon} W(\beta)\, \mathrm{d}x &\geq |\mu(A)| f(c^N \varepsilon\, N, \frac{\varepsilon^\delta}{2c}, 1) = c_1 \frac{|\mu(A)|}{2\pi K(c)} \log \frac{\varepsilon^\delta}{2c^{N+1} \varepsilon\, N} \\ &= c_1 \frac{|\mu(A)|}{2\pi K(c)} ((1-\delta)|\log \varepsilon| - (N+1)\log c - \log 2N). \end{aligned}$$

$\square$

We are now in a position to prove the compactness result. The strategy of the proof is the one for compactness of Jacobians in the context of Ginzburg-Landau energies (see Proof of Theorem 1.6(i) above). The idea is to modify a sequence of measures $\{\mu_h\}$ with equi-bounded energy by identifying clusters of dislocations with Dirac masses whose multiplicity is given by the effective Burgers vector of the cluster, i.e. the total mass of the cluster. Applying our lower bound, we show that the modified sequence $\{\tilde\mu_h\}$ is bounded in variation and then weakly* converges, up to a subsequence, to some $\mu \in X$. We deduce the convergence of μ_h to μ with respect to the flat norm by the fact that $\mu_h - \tilde\mu_h$ has vanishing flat norm.

Proof of Theorem 5.4(i). Let $\varepsilon_h \to 0$ as $h \to +\infty$ and let $\mu_h = \sum_{i=1}^{N_h} \xi_{i,h}\delta_{x_{i,h}}$ be a sequence such that $\sup_h \mathcal{F}_{\varepsilon_h}(\mu_h) \le M$ for some positive constant M. We have to prove that (up to a subsequence) $\mu_h \overset{\text{flat}}{\to} \mu$ for some $\mu \in X_{\text{edge}}$.

Fix $0 < \delta < 1$ and let

$$A_{\varepsilon_h^\delta}(\mu_h) = \bigcup_{x_{i,h}\in\operatorname{supp}(\mu_h)} B_{\varepsilon_h^\delta}(x_{i,h})\,.$$

Notice in particular that $\operatorname{dist}(x_{i,h}, \partial A_{\varepsilon_h^\delta}) \ge \varepsilon_h^\delta$. Let $\{C_{l,h}^\delta\}_{l=1}^{L_h}$ be the family of the connected components of $A_{\varepsilon_h^\delta}(\mu_h)$ which are contained in Ω and satisfy $|\mu_h(C_{l,h}^\delta)| > 0$. By Lemma 5.9 we deduce that for every $l = 1, \dots, L_h$ and $\beta_h \in \mathcal{AS}_{\varepsilon_h}(\mu_h)$

$$\int_{C_{l,h}^\delta l\setminus\omega_{\varepsilon_h}} W(\beta_h)\,\mathrm{d}x \ge c_1 \frac{|\mu_h(C_{l,h}^\delta)|}{2\pi K(c)}((1-\delta)|\log\varepsilon_h| - (N_h+1)\log c - \log 2N_h).$$

Since $N_h \le |\mu_h|(\Omega) \le \mathcal{E}_{\varepsilon_h}(\mu_h) \le M|\log\varepsilon_h|$, we deduce

$$\mathcal{E}_{\varepsilon_h}(\mu_h) \ge c_1 \sum_{l=1}^{L_h} \frac{|\mu_h(C_{l,h}^\delta)|}{2\pi K(c)}\Big((1-\delta - M\log c)|\log\varepsilon_h| - \log(2cM|\log\varepsilon_h|)\Big) \tag{5.23}$$

If $c-1$ is small enough we deduce that $L_h \le \tilde L$ for some $\tilde L$ independent of h, so that, up to a subsequence, we have $L_h \equiv L \in \mathbb{N}$. For any $l = 1, \dots, L$, let $\tilde x_{\delta,h}^l \in C_{l,h}^\delta$ be fixed and set

$$\tilde\mu_h = \sum_{l=1}^{L} \mu_h(C_{l,h}^\delta)\delta_{\tilde x_{l,h}^\delta}.$$

From (5.23) we easily see that $|\tilde\mu_h|(\Omega)$ is uniformly bounded; hence the sequence $\{\tilde\mu_h\}$ is precompact in X with respect to the weak* topology, and therefore also with respect to the flat topology. It remains to prove that $\|\mu_h - \tilde\mu_h\|_{\text{flat}} \to 0$ as $h \to +\infty$. Fix $\phi \in W_0^{1,\infty}(\Omega)$ with $\|\phi\|_{W_0^{1,\infty}(\Omega)} \le 1$. Let $D_{l,h}^\delta$, $l = 1, \dots \tilde N_h$ be the connected components of $A_{\varepsilon_h^\delta}$ which are not contained in Ω, and let $E_{l,h}^\delta$, $l = 1, \dots \hat N_h$ be the remaining ones, i.e., contained in Ω. Since $\phi = 0$ on $\partial\Omega$ and $\|\phi\|_{W_0^{1,\infty}(\Omega)} \le 1$ we have

$$|\phi(x)| \le \operatorname{diam}(D_{l,h}^\delta) \le 2N_h\varepsilon_h^\delta \le 2M\varepsilon_h^\delta|\log\varepsilon_h| \qquad \text{for all } x \in D_{l,h}^\delta, \tag{5.24}$$

and so

$$\int_{D_{l,h}^\delta} \phi\,\mathrm{d}(\mu_h - \tilde\mu_h) \le \sup_{D_{\delta,h}^l} |\phi| \int_{D_{l,h}^\delta} \mathrm{d}(|\mu_h| + |\tilde\mu_h|) \le (|\mu_h| + |\tilde\mu_h|)(D_{l,h}^\delta)\, 2M\varepsilon_h^\delta|\log\varepsilon_h|.$$

Set $\bar{\phi}_l = \frac{1}{|E^\delta_{l,h}|}\int_{E^\delta_{l,h}} \phi \,\mathrm{d}x$. As in (5.24), we deduce $|\phi - \bar{\phi}_l| \le 2M\varepsilon^\delta_h |\log \varepsilon_h|$ for all $x \in E^\delta_{l,h}$. Therefore, for every $l = 1, \dots \hat{N}_h$ we have

$$\int_{E^\delta_{l,h}} \phi \,\mathrm{d}(\mu_h - \tilde{\mu}_h) = \int_{E^\delta_{l,h}} (\phi - \bar{\phi}_l)\,\mathrm{d}(\mu_h - \tilde{\mu}_h) + \int_{E^\delta_{l,h}} \bar{\phi}_l \,\mathrm{d}(\mu_h - \tilde{\mu}_h)$$
$$\le (|\mu_h| + |\tilde{\mu}_h|)(E^\delta_{l,h})\,\mathrm{diam}\,(E^\delta_{l,h}) \le (|\mu_h| + |\tilde{\mu}_h|)(E^\delta_{l,h})\, 2M\varepsilon^\delta_h |\log \varepsilon_h|.$$

It follows that

$$\int_\Omega \phi \,\mathrm{d}(\mu_h - \tilde{\mu}_h) = \sum_{l=1}^{\tilde{N}_h} \int_{D^\delta_{l,h}} \phi \,\mathrm{d}(\mu_h - \tilde{\mu}_h) + \sum_{l=1}^{\hat{N}_h} \int_{E^\delta_{l,h}} \phi \,\mathrm{d}(\mu_h - \tilde{\mu}_h)$$
$$\le (|\mu_h| + |\tilde{\mu}_h|)(\Omega)\Big(4M\varepsilon^\delta_h |\log \varepsilon_h|\Big) \le C|\log \varepsilon_h|^2 \varepsilon^\delta_h, \quad (5.25)$$

which tends to zero as $\varepsilon_h \to 0$. By the very definition of the flat norm it follows that $\|\mu_h - \tilde{\mu}_h\|_{\mathrm{flat}} \to 0$ as h tends to infinity. □

5.4. Lower bound

In the proof of the Γ-liminf inequality we will first suitably remove the clusters of dislocations with zero multiplicity. To this purpose we need a lemma providing upper bounds for the energy on suitable annuli surrounding such clusters. We will use the notation of the discrete ball construction (see Definition 5.7).

Lemma 5.10. *For any given $\varepsilon > 0$, let $\mu \in X_{\mathrm{edge}}$ and $\beta \in \mathcal{AS}_\varepsilon(\mu)$ be fixed. Let $0 < \gamma < \alpha < 1$ and let $c > 1$ be such that $\log c < \frac{|\log \varepsilon|(\alpha-\gamma)}{|\mu|(\Omega)+1}$.*

Then there exists $\bar{n} \in \mathbb{N}$ such that

(i) $\varepsilon^\alpha \le \sum_{i=1}^{N_{\bar{n}}} R^{\bar{n}}_i \le \varepsilon^\gamma$*;*

(ii) $\bar{n}$ *is not a merging time;*

(iii) $\displaystyle\int_{\Omega \cap \cup_i B_{c R^{\bar{n}}_i}(x^{\bar{n}}_i) \setminus B_{R^{\bar{n}}_i}(x^{\bar{n}}_i)} W(\beta)\,\mathrm{d}x \le \frac{\log c\, E_\varepsilon(\mu,\beta)}{|\log \varepsilon|(\alpha-\gamma) - \log c(|\mu|(\Omega)+1)}$.

Proof. We denote by n_α the first step n in the ball construction such that $\sum_{i=1}^{N_n} R^n_i \ge \varepsilon^\alpha$ and similarly we set n_γ, so that for every $n_\alpha \le n \le n_\gamma - 1$ (i) holds true. Notice that in the ball construction

$$\sum_{i=1}^{N^n} R^n_i \le c \sum_{i=1}^{N^{n-1}} R^{n-1}_i.$$

By a straightforward computation we get

$$\varepsilon^\gamma \le c^{n_\gamma - n_\alpha + 1}\varepsilon^\alpha,$$

and so $n_\gamma - n_\alpha \ge \frac{(\alpha-\gamma)\log\frac{1}{\varepsilon}}{\log c} - 1$. Recalling that the total number of merging is smaller than $|\mu|(\Omega)$, we deduce that

$$n_\gamma - 1 - \tau^{n_\gamma - 1} - (n_\alpha - 1 - \tau^{n_\alpha - 1}) \ge \frac{(\alpha-\gamma)|\log \varepsilon|}{\log c} - 1 - |\mu|(\Omega),$$

where the left hand side represents the number of expansion times between n_α and $n_\gamma - 1$. The thesis follows by the mean value theorem since

$$E_\varepsilon(\mu,\beta) \geq \sum_{\substack{n_\alpha \leq n \leq n_\gamma - 1 \\ n \text{ is an expansion time}}} \int_{\Omega \cap \cup_i B_{c R_i^n}(x_i^n) \setminus B_{R_i^n}(x_i^n)} W(\beta)\, \mathrm{d}x.$$

$\square$

Proof of Theorem 5.4(ii). Let $\varepsilon_h \to 0$ as $h \to +\infty$. For any $h \in \mathbb{N}$, let $\mu_h \in X_{\mathrm{edge}}$, with $\mu_h = \sum_{i=1}^{N_h} \xi_{i,h} \delta_{x_{i,h}}$, such that $\mu_h \overset{\text{flat}}{\to} \mu$ for some $\mu = \sum_{i=1}^{N} \xi_i \delta_{x_i} \in X$. We have to prove that

$$\mathcal{F}(\mu) \leq \liminf_{h \to +\infty} \mathcal{F}_{\varepsilon_h}(\mu_h).$$

By a standard localization argument we can assume $\mu = \xi_0 \delta_{x_0}$ for some $\xi_0 \in \mathbb{S}$, $x_0 \in \Omega$. Moreover, we can assume that $\liminf_{h\to+\infty} \mathcal{F}_{\varepsilon_h}(\mu_h) = \lim_{h\to+\infty} \mathcal{F}_{\varepsilon_h}(\mu_h) \leq M$, for some positive constant M.

Let $\beta_h \in \mathcal{AS}_{\varepsilon_h}(\mu_h)$ be the strain that realizes the minimum in (0.4), namely $E_{\varepsilon_h}(\mu_h, \beta_h) = \min_{\beta \in \mathcal{AS}_{\varepsilon_h}(\mu_h)} E_{\varepsilon_h}(\mu_h, \beta)$. The idea is to give a lower bound for the energy on a finite number of shrinking balls where both the energy and the flat norm concentrate. To this purpose fix $0 < \gamma < \alpha < 1$, $c > 1$ such that

$$\log c < \min\left\{ \frac{\alpha - \gamma}{M+1}, \frac{1-\alpha}{M} \right\}. \tag{5.26}$$

Since

$$N_h = |\mu_h|(\Omega) \leq M |\log \varepsilon_h|, \tag{5.27}$$

we can apply Lemma 5.10; in particular, let $\bar n$ be such that $\varepsilon_h^\alpha \leq \sum_{i=1}^{N^{\bar n}} R_i^{\bar n} \leq \varepsilon_h^\gamma$. Consider the family of balls $B_i^{\bar n} := B_{R_i^{\bar n}}(x_i^{\bar n})$ in $\mathcal{B}_{\bar n}$ such that $B_{cR_i^{\bar n}}(x_i^{\bar n}) \subset \Omega$. We denote by $J_h \subset \{1, \ldots, N_{\bar n}\}$ the set of indices i such that $B_{cR_i^{\bar n}}(x_i^{\bar n}) \subset \Omega$ and $\mu_h(B_i^{\bar n}) = 0$, and by $I_h \subset \{1, \ldots, N_{\bar n}\}$ the set of indices i such that $B_{cR_i^{\bar n}}(x_i^{\bar n}) \subset \Omega$ and $\mu_h(B_i^{\bar n}) \neq 0$.

We prove that I_h is finite. Recalling the definition of E_{ε_h} in (5.22) and in view of Remark 5.8 applied with $f(r, R, t) = \frac{c_1}{2\pi K(c)} t \log \frac{R}{r}$ we obtain

$$\begin{aligned} E_{\varepsilon_h}(\mu_h, \beta_h, \cup_{i\in I_h} B_i^{\bar n}) &\geq \sum_{i \in I_h} |\mu_h(B_i^{\bar n})| f(c^{N_h} \varepsilon_h N_h, \sum_{i=1}^{N_{\bar n}} R_i^{\bar n}, 1) \\ &\geq \sum_{i\in I_h} c_1 \frac{|\mu_h(B_i^{\bar n})|}{2\pi K(c)} \Big((1 - \alpha - M \log c) |\log \varepsilon_h| - \log(M |\log \varepsilon_h|) \Big), \end{aligned}$$

where we have used $\sum_{i=1}^{N^{\bar n}} R_i^{\bar n} \geq \varepsilon_h^\alpha$ and (5.27). Notice that, since $\mathbb{S}$ is a discrete set, $|\mu_h(B_i^{\bar n})| > c > 0$ for every $i \in I_h$. Since $E_{\varepsilon_h}(\mu_h, \beta_h, \cup_{i \in I_h} B_i^{\bar n}) \leq M |\log \varepsilon_h|$, and $1 - \alpha - M \log c > 0$ (see (5.26)), we conclude that $\sharp I_h$ is uniformly bounded. Up to a subsequence, we have $\sharp I_h = L$ for every $h \in \mathbb{N}$, for some $L \in \mathbb{N}$.

Consider now $i \in J_h$. Recalling that $\mathrm{Curl}\beta_h = 0$ in the annulus $C_i^{\bar n} := B_{c R_i^{\bar n}}(x_i^{\bar n}) \setminus B_{R_i^{\bar n}}(x_i^{\bar n})$ and $\mu_h(B_{c R_i^{\bar n}}(x_i^{\bar n})) = 0$, we get that $\beta_h = \nabla v_{i,h}^{\bar n}$ for some

$v_{i,h}^{\bar n} \in H^1(C_i^{\bar n};\mathbb{R}^2)$. Thus, applying Korn's inequality (Theorem 5.12) to $v_{i,h}^{\bar n}$, we deduce that

$$\int_{C_i^{\bar n}} |\nabla v_{i,h}^{\bar n} - A_{i,h}^{\bar n}|^2 \,\mathrm{d}x \le K(c) \int_{C_i^{\bar n}} |(\nabla v_{i,h}^{\bar n})^{\mathrm{sym}}|^2 \,\mathrm{d}x = K(c) \int_{C_i^{\bar n}} |\beta_h^{\mathrm{sym}}|^2 \,\mathrm{d}x,$$

where $A_{i,h}^{\bar n}$ is a suitable skew-symmetric matrix. By a standard extension argument, there exists a function $u_{i,h}^{\bar n} \in H^1(B_{c\,R_i^{\bar n}}(x_i^{\bar n});\mathbb{R}^2)$ such that $\nabla u_{i,h}^{\bar n} = \nabla v_{i,h}^{\bar n} - A_{i,h}^{\bar n}$ in $C_i^{\bar n}$ and

$$\begin{aligned}\int_{B_{c\,R_i^{\bar n}}(x_i^{\bar n})} |\nabla u_{i,h}^{\bar n}|^2 \,\mathrm{d}x &\le C_1 \int_{C_i^{\bar n}} |\nabla v_{i,h}^{\bar n} - A_{i,h}^{\bar n}|^2 \,\mathrm{d}x \\ &\le C_1 K(c) \int_{C_i^{\bar n}} |\beta_h^{\mathrm{sym}}|^2 \,\mathrm{d}x, \qquad (5.28)\end{aligned}$$

for some positive constant C_1. Consider the field $\tilde\beta_h : \Omega \to \mathbb{M}^{2\times 2}$ defined by

$$\tilde\beta_h(x) := \begin{cases} \nabla u_{i,h}^{\bar n}(x) + A_{i,h}^{\bar n} & \text{if } x \in B_i^{\bar n} \text{ with } i \in J_h, \\ \beta_h(x) & \text{otherwise in } \Omega_{\varepsilon_h}(\mu_h). \end{cases}$$

It follows, by the definition of $\tilde\beta_h$, by the fact that the matrices $A_{i,h}^{\bar n}$ are skew symmetric, and by (5.28), that for every $i \in J_h$ the following inequalities hold

$$\begin{aligned}\int_{B_{cR_i^{\bar n}}(x_i^{\bar n})} W(\tilde\beta_h)\,\mathrm{d}x &\le c_2 \int_{B_{cR_i^{\bar n}}(x_i^{\bar n})} |\tilde\beta_h^{\mathrm{sym}}|^2 \,\mathrm{d}x \\ &\le c_2 \int_{C_{\bar n}^i} |\beta_h^{\mathrm{sym}}|^2 \,\mathrm{d}x + c_2 \int_{B_i^{\bar n}} |\tilde\beta_h^{\mathrm{sym}}|^2 \,\mathrm{d}x \\ &\le \frac{c_2}{c_1}(1 + C_1 K(c)) \int_{C_i^{\bar n}} W(\beta_h)\,\mathrm{d}x\,,\end{aligned}$$

where c_1 and c_2 are the constants in (0.2). Applying Lemma 5.10, we deduce

$$\begin{aligned}\frac{1}{|\log\varepsilon_h|} &\int_{\bigcup_{i\in J_h} B_{cR_i^{\bar n}}(x_i^{\bar n})} W(\tilde\beta_h)\,\mathrm{d}x \\ &\le \frac{c_2}{c_1} \frac{(1 + C_1 K(c)) M \log c}{|\log\varepsilon_h|(\alpha - \gamma - M\log c) - \log c}, \qquad (5.29)\end{aligned}$$

which vanishes as $\varepsilon_h \to 0$.

Let us introduce the modified measure

$$\hat\mu_h = \sum_{i\in I_h} \mu_h(B_i^{\bar n})\delta_{x_i^{\bar n}}.$$

Arguing as in the proof of the compactness property, and more precisely of estimate (5.25), we deduce that $\hat\mu_h - \mu_h \xrightarrow{\text{flat}} 0$, and hence, up to a subsequence, $\hat\mu_h \overset{*}{\rightharpoonup} \xi^0\delta_{x^0}$.

The points $x_i^{\bar n}$, $i \in I_h$ converge, up to a subsequence, to some point in a finite set of points $\{y_0 = x_0, y_1, \dots, y_{L'}\}$ contained in $\bar\Omega$. Let $\rho > 0$ be such that $B_{2\rho}(x_0) \subset\subset \Omega$ and $B_{2\rho}(y_j) \cap B_{2\rho}(y_k) = \emptyset$ for all $j \neq k$. Then,

$$x_i^{\bar n} \in B_\rho(y_j) \qquad \text{for some } j \text{ and for } h \text{ large enough}\,.$$

Thus, using the convergence of $\hat{\mu}_h$ to $\xi_0\delta_{x_0}$, one can show that for h large enough

$$\sum_{x_i^{\bar n}\in B_\rho(x_0)} \mu_h(B_i^{\bar n}) = \xi_0\,. \tag{5.30}$$

We finally introduce the measure

$$\tilde\mu_h = \mu_h \llcorner \cup_{i\in I_h(\rho)} B_i^{\bar n}\,,$$

where we have introduced the notation $I_h(\rho) = \{i \in I_h \,:\, x_i^{\bar n} \in B_\rho(x_0)\}$; by (5.29), it follows that

$$\mathcal{E}_{\varepsilon_h}(\mu_h) \geq \int_{\Omega_{\varepsilon_h}(\mu_h)\setminus \cup_{i\in J_h} B_{cR_i^{\bar n}}} W(\beta_h)\,\mathrm{d}x \geq \int_{\Omega_{\varepsilon_h}(\tilde\mu_h)\cap B_{2\rho}(x_0)} W(\tilde\beta_h)\,\mathrm{d}x + \mathrm{o}(1). \tag{5.31}$$

It remains to prove the lower bound for the right hand side of (5.31). Fix $0<\eta<\gamma$. Let us denote by $g_h : [\eta,\gamma]\to\{1,\ldots,L\}$ the function which associates with any $\delta\in(\eta,\gamma)$ the number $g_h(\delta)$ of the connected components of $\cup_{i\in I_h(\rho)} B_{\varepsilon_h^\delta}(x_i^{\bar n})$. For every $h\in\mathbb{N}$, the function g_h is monotone so that it can have at most L discontinuities. Let us denote by $\delta_{i,h}$ for $i=1,\ldots,\hat L\leq L$ such points of discontinuity, with

$$\eta \leq \delta_{1,h} < \ldots < \delta_{\hat L,h} \leq \gamma.$$

It is easy to see that there exists a finite set $\Delta = \{\delta_0,\delta_1,\ldots,\delta_{\tilde L}\}$ with $\delta_i<\delta_{i+1}$, such that, up to a subsequence $\{\delta_{i,h}\}_{h\in\mathbb{N}}$ converges to some point in Δ, as $h\to+\infty$, for every $i=1,\ldots,\hat L$. We may always assume $\delta_0=\eta$, $\delta_{\tilde L}=\gamma$ and $\tilde L\leq \hat L+2$.

Now, for any fixed $\sigma>0$ small enough and for h large enough (i.e., such that for any $j=1,\ldots,\hat L$, $|\delta_{j,h}-\delta_i|<\sigma$ for some $\delta_i\in\Delta$) the function g_h is constant in the interval $[\delta_i+\sigma,\delta_{i+1}-\sigma]$. Thus for every $i=0,\ldots,\tilde L-1$ we can construct a finite family of $N_{i,h}$ annuli $C_{i,j,h} = B_{j,h,\varepsilon_h^{\delta_i+\sigma}} \setminus B_{j,h,\varepsilon_h^{\delta_{i+1}-\sigma}}$ with $j=1,\ldots,N_{i,h}$, such that $C_{i,j,h}$ are pairwise disjoint for all i and all j and

$$\bigcup_{k\in I_h(\rho)} B_k^{\bar n} \subseteq \bigcup_{j=1}^{N_{i,h}} B_{j,h,\varepsilon_h^{\delta_{i+1}-\sigma}} \tag{5.32}$$

for all $i=0,\ldots,\hat L$. Note that, for h large enough, $C_{i,j,h}\subset B_{2\rho}(x_0)$ for all i and j. Recalling (5.9) and in view of Remark 5.2, the following estimate holds

$$\begin{aligned}\int_{C_{i,j,h}} W(\tilde\beta_h)\,\mathrm{d}x &\geq |\log\varepsilon_h|(\delta_{i+1}-\delta_i-2\sigma)\psi_{\varepsilon_h^{\delta_{i+1}-\sigma},\varepsilon_h^{\delta_i+\sigma}}(\tilde\mu_h(B_{j,h,\varepsilon_h^{\delta_i+\sigma}}))\\ &\geq |\log\varepsilon_h|(\delta_{i+1}-\delta_i-2\sigma)\psi(\tilde\mu_h(B_{j,h,\varepsilon_h^{\delta_i+\sigma}})) - C_0|\tilde\mu_h(B_{j,h,\varepsilon_h^{\delta_i+\sigma}})|^2\,.\end{aligned}$$

Notice that in view of (5.32) and of the weak* convergence of $\{\hat\mu_h\}$, we have

$$|\tilde\mu_h(B_{j,h,\varepsilon_h^{\delta_i+\sigma}})| \leq \sum_{k\in I_h(\rho)} |\tilde\mu_h(B_k^{\bar n})| \leq |\hat\mu_h|(\Omega)\leq C_2\,,$$

for some $C_2 > 0$. Summing over $i = 0, \dots, \tilde{L} - 1$ and $j = 1, \dots, N_{h,i}$, we obtain the following chain of inequalities

$$\int_{\Omega_{\varepsilon_h}(\tilde{\mu}_h)\cap B_{2\rho}(x_0)} W(\tilde{\beta}_h)\,\mathrm{d}x \geq \sum_{i=0}^{\tilde{L}-1}\sum_{j=1}^{N_{i,h}} \int_{C_{i,j,h}} W(\tilde{\beta}_h)\,\mathrm{d}x$$

$$\geq \sum_{i=0}^{\tilde{L}-1}\sum_{j=1}^{N_{i,h}} \Big(|\log \varepsilon_h| (\delta_{i+1} - \delta_i - 2\sigma)\psi(\tilde{\mu}_h(B_{j,h,\varepsilon_h^{\delta_i+\sigma}})) - C_0|\tilde{\mu}_h(B_{j,h,\varepsilon_h^{\delta_i+\sigma}})|^2 \Big)$$

$$\geq |\log \varepsilon_h| \sum_{i=0}^{\tilde{L}-1} (\delta_{i+1} - \delta_i - 2\sigma)\varphi(\xi_0) - C_0 L^2 C_2^2\,,$$

where the last inequality is a consequence of (5.30), recalling the definition of φ (see (5.10)). Finally we get

$$\int_{\Omega_{\varepsilon_h}(\tilde{\mu}_h)\cap B_{2\rho}(x_0)} W(\tilde{\beta}_h)\,\mathrm{d}x \geq (\gamma - \eta - 2\sigma\tilde{L})|\log \varepsilon_h|\varphi(\xi_0) - C_0 L^2 C_2^2,$$

and hence using (5.31) we have

$$\liminf_{h\to+\infty} \mathcal{F}_{\varepsilon_h}(\mu_h) \geq (\gamma - \eta - 2\sigma\tilde{L})\varphi(\xi_0).$$

The Γ-liminf inequality follows by taking the limits $\sigma \to 0$, $\eta \to 0$ and $\gamma \to 1$. $\square$

5.5. Upper Bound

In this section we will prove the Γ-limsup inequality, namely we will show that for every $\mu \in X_{\mathrm{edge}}$ there exists a *recovery sequence* $\{\mu_h\} \subset X_{\mathrm{edge}}$ that converges to μ in the flat topology and satisfies

$$\limsup_{h\to+\infty} \mathcal{F}_{\varepsilon_h}(\mu_h) \leq \mathcal{F}(\mu).$$

We first assume that μ belongs to the subclass $\mathcal{D}$ of X_{edge} defined by

$$\mathcal{D} := \{\mu \in X \mid \mu = \sum_{i=1}^{N} b_i\delta_{x_i},\, b_i \in \mathfrak{B},\, x_i \neq x_j \text{ for } i \neq j\}.$$

where $\mathfrak{B}$ is the class of Burgers vectors defined in Definition 5.3. The general case is obtained by a standard diagonal argument.

Let $\mu = \sum_{i=1}^{N} b_i\delta_{x_i}$ in $\mathcal{D}$; then $\mathcal{F}(\mu) = \sum_{i=1}^{N}\varphi(b_i) = \sum_{i=1}^{N}\psi(b_i)$. In this case, the recovery sequence is given by the constant sequence $\mu_h \equiv \mu$ for every $h \in \mathbb{N}$. To show this, for every $i = 1, \dots, N$, let $\beta_{\mathbb{R}^2}^{b_i}$ be the planar strain field defined in the whole of $\mathbb{R}^2$ corresponding to the dislocation centered at x_i with Burgers vector b_i. Recalling (5.4), we set

$$\beta_i(x) := \beta_{\mathbb{R}^2}^{b_i}(x - x_i) = \frac{1}{|x - x_i|}\Gamma_{b_i}(\theta) \quad \text{where } \theta = \arctan\frac{x_2 - x_{i,2}}{x_1 - x_{i,1}}\,,$$

and define $\beta_\mu := \sum_{i=1}^N \beta_i$. Clearly $\beta_\mu \in \mathcal{AS}_{\varepsilon_h}(\mu_h)$ for every $h \in \mathbb{N}$. Then

$$\begin{aligned}
\mathcal{F}_{\varepsilon_h}(\mu_h) &= \frac{1}{|\log \varepsilon_h|} \min_{\beta \in \mathcal{AS}_{\varepsilon_h}(\mu_h)} \int_{\Omega_{\varepsilon_h}(\mu_h)} W(\beta)\,\mathrm{d}x \\
&\leq \frac{1}{|\log \varepsilon_h|} \int_{\Omega_{\varepsilon_h}(\mu_h)} W(\beta_\mu)\,\mathrm{d}x = \frac{1}{|\log \varepsilon_h|} \int_{\Omega_{\varepsilon_h}(\mu_h)} W\Big(\sum_{i=1}^N \beta_i\Big)\,\mathrm{d}x \\
&\leq \frac{1}{|\log \varepsilon_h|} \sum_{i=1}^N \int_{B_R(x_i)\setminus B_{\varepsilon_h}(x_i)} W(\beta_i)\,\mathrm{d}x \qquad (5.33) \\
&\quad + \frac{2}{|\log \varepsilon_h|} \sum_{i=1}^N \sum_{j=i+1}^N \int_{(\Omega\setminus B_{\varepsilon_h}(x_i))\setminus B_{\varepsilon_h}(x_j)} \mathbb{C}\beta_i : \beta_j\,\mathrm{d}x, \qquad (5.34)
\end{aligned}$$

where $R > \mathrm{diam}(\Omega)$. As for the integrals in (5.33), from (5.7) we have that for every $i = 1, \ldots, N$

$$\lim_{h\to+\infty} \frac{1}{|\log \varepsilon_h|} \int_{B_R(x_i)\setminus B_{\varepsilon_h}(x_i)} W(\beta_i)\,\mathrm{d}x = \psi(b_i).$$

In order to conclude, it suffices to prove that each term of the sum in (5.34) tends to 0 as $h \to +\infty$. To this purpose, for every $i, j = 1, \ldots, N$ with $i \neq j$ set $\rho_{ij} := \frac{|x_i - x_j|}{2}$. Then

$$\begin{aligned}
\int_{(\Omega\setminus B_{\varepsilon_h}(x_i))\setminus B_{\varepsilon_h}(x_j)} \mathbb{C}\beta_i : \beta_j\,\mathrm{d}x &= \int_{B_{\rho_{ij}}(x_i)\setminus B_{\varepsilon_h}(x_i)} \mathbb{C}\beta_i : \beta_j\,\mathrm{d}x \\
&\quad + \int_{B_{\rho_{ij}}(x_j)\setminus B_{\varepsilon_h}(x^j)} \mathbb{C}\beta_i : \beta_j\,\mathrm{d}x + \int_{(\Omega\setminus B_{\rho_{ij}}(x_i))\setminus B_{\rho_{ij}}(x_j)} \mathbb{C}\beta_i : \beta_j\,\mathrm{d}x.
\end{aligned}$$

Since $\beta_i \in L^2_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{x_i\})$ the last term in the right hand side is bounded. As for the first two integrals, it is enough to apply Hölder's inequality in order to obtain

$$\begin{aligned}
\int_{B_{\rho_{ij}}(x_i)\setminus B_{\varepsilon_h}(x_i)} \mathbb{C}\beta_i : \beta_j\,\mathrm{d}x &\leq \|\mathbb{C}\beta_i\|_{L^2(B_{\rho_{ij}}(x_i)\setminus B_{\varepsilon_h}(x_i))} \|\beta_j\|_{L^2(B_{\rho_{ij}}(x_i)\setminus B_{\varepsilon_h}(x_i))} \\
&\leq C\, \|\beta_i\|_{L^2(B_{\rho_{ij}}(x_i)\setminus B_{\varepsilon_h}(x_i))} \|\beta_j\|_{L^2(\Omega\setminus B_{\rho_{ij}}(x_j))}\,;
\end{aligned}$$

here and in the following lines C denotes a positive constant that may change from line to line. By (5.6) we get

$$\int_{B_{\rho_{ij}}(x_i)\setminus B_{\varepsilon_h}(x_i)} |\beta_i|^2\,\mathrm{d}x \leq C|\log \varepsilon_h|\,,$$

and hence

$$\int_{(\Omega\setminus B_{\varepsilon_h}(x_i))\setminus B_{\varepsilon_h}(x_j)} \mathbb{C}\beta_i : \beta_j\,\mathrm{d}x \leq C\sqrt{|\log \varepsilon_h|}$$

for every $i, j = 1, \ldots, N$ with $i \neq j$. Therefore,

$$\lim_{h\to+\infty} \frac{1}{|\log \varepsilon_h|} \int_{(\Omega\setminus B_{\varepsilon_h}(x_i))\setminus B_{\varepsilon_h}(x_j)} \mathbb{C}\beta_{i,h} : \beta_{j,h}\,\mathrm{d}x = 0,$$

and so

$$\limsup_{h\to+\infty} \mathcal{F}_{\varepsilon_h}(\mu_h) \leq \sum_{i=1}^{N} \psi(b_i) = \sum_{i=1}^{N} \varphi(b_i) = \mathcal{F}(\mu)\,.$$

We have proved that the Γ-limsup inequality holds for any $\mu \in \mathcal{D}$. Now we conclude noticing that $\mathcal{D}$ is dense in X with respect to the weak* topology, and hence with respect to the flat topology. More precisely, for any $\mu = \sum_{i=1}^{N} \xi_i \delta_{x_i}$, with $\xi_i \in \mathbb{S} = \mathrm{Span}_{\mathbb{Z}}\mathfrak{B}$ $(i = 1, \ldots, N)$, we can construct a sequence $\{\mu_k\} \subset \mathcal{D}$ such that $\mathcal{F}(\mu_k) = \mathcal{F}(\mu)$ and $\mu_k \stackrel{*}{\rightharpoonup} \mu$. Indeed, by (5.11), for every $i = 1, \ldots, N$ we can find a decomposition of $\xi_i = \sum_{j=1}^{s_i} \alpha_{ij} b_j$ such that $\varphi(\xi_i) = \sum_{j=1}^{s_i} |\alpha_{ij}|\psi(b_j)$. Now, for every $k \in \mathbb{N}$ we define

$$\mu_k = \sum_{i=1}^{N} \sum_{j=1}^{s_i} b_j \sum_{l=1}^{|\alpha_{ij}|} \delta_{x_{i,jl}(k)},$$

where for every k $x_{i,jl}(k)$ are distinct points in Ω, and $|x_{i,jl}(k) - x_i| \to 0$ as $k \to +\infty$. Clearly $\{\mu_k\} \subset \mathcal{D}$ and $\mu_k \stackrel{*}{\rightharpoonup} \mu$. Moreover

$$\mathcal{F}(\mu_k) = \sum_{i=1}^{N} \sum_{j=1}^{s_i} \sum_{l=1}^{|\alpha_{ij}|} \varphi(b_j) = \sum_{i=1}^{N} \sum_{j=1}^{s_i} |\alpha_{ij}|\psi(b_j) = \sum_{i=1}^{N} \varphi(\xi_i) = \mathcal{F}(\mu).$$

The thesis follows using a standard diagonal argument. Indeed, since for any measure in $\mathcal{D}$, the recovery sequence is given by the constant sequence, we have

$$\limsup_{h\to\infty} \mathcal{F}_{\varepsilon_h}(\mu_k) \leq \mathcal{F}(\mu_k) = \mathcal{F}(\mu).$$

Therefore, there exists a sequence $k_h \to \infty$ as $h \to \infty$ such that $\mu_h := \mu_{k_h}$ is a recovery sequence, i.e.,

$$\limsup_{h\to\infty} \mathcal{F}_{\varepsilon_h}(\mu_h) \leq \mathcal{F}(\mu).$$

□

Remark 5.11. In the proof of the Γ-limsup inequality we have shown that configurations of dislocations optimal in energy belong to the class $\mathcal{D}$. As a consequence, we get the same Γ-limit if we start from an energy for which the only admissible dislocations are those whose multiplicity belongs to $\mathfrak{B}$, i.e. to the set of Burgers vectors. Precisely, if we define

$$\mathcal{G}_\varepsilon(\mu) = \begin{cases} \mathcal{F}_\varepsilon(\mu) & \text{if } \mu \in \mathcal{D}, \\ +\infty & \text{otherwise}, \end{cases}$$

then $\mathcal{G}_\varepsilon$ still Γ-converge to the functional $\mathcal{F}$ defined in (5.12). In this respects, the class of Burgers vectors in $\mathfrak{B}$ are the building blocks to describe multiple dislocations in $\mathcal{S}$.

5.6. Korn's inequality in thin annuli

Here we revisit some results concerning the Korn's inequality in thin domains. First, we recall the Korn's inequality on annular sets with a cut.

Theorem 5.12 (Korn's inequality)**.** *Let $0 < r < R$, let $L := \{0\} \times (r, R)$, and let $u \in H^1((B_R \setminus B_r) \setminus L; \mathbb{R}^2)$ be such that $\int_{(B_R \setminus B_r)\setminus L} (\nabla u - \nabla u^{\mathrm{T}})\, \mathrm{d}x = 0$. Then, there exists a positive constant $K = K(R/r)$ such that*

$$\int_{(B_R \setminus B_r)\setminus L} |\nabla u|^2 \leq K(R/r) \int_{(B_R \setminus B_r)\setminus L} |(\nabla u)^{\mathrm{sym}}|^2 \, \mathrm{d}x, \tag{5.35}$$

where $(\nabla u)^{\mathrm{sym}} := \dfrac{\nabla u + \nabla u^{\mathrm{T}}}{2}$.

The proof of such theorem can be proved for instance covering the annulus $(B_R \setminus B_r) \setminus L$ with two open overlapping sets A_1, $A_2 \subset (B_R \setminus B_r)$ with Lipschitz boundary, and applying classical Korn's inequality on each A_i, see for instance [62].

The best constant K of the Korn's inequality on annular sets (without cuts) has been explicitly computed in [28]. In this context it's important to remark that such Korn's constant depends only on the ratio of the radii, and tends to infinity when this parameter tends to 1. In particular, we deduce that also $K(R/r) \to \infty$ as $R/r \to 1$.

A natural question is whether the best (i.e., the lower) Korn's costant blows up on thin annuli also in the class of our admissible strains $\mathcal{AS}_{r,R}(\xi)$. Let us show that, actually, this is the case. More precisely, let $\xi \in \mathbb{R}^2$ and let $r_n \to 1$. Then, there exists a sequence of strains $\beta_n \in \mathcal{AS}_{r_n,1}(\xi)$ such that

$$\int_{B_1 \setminus B_{r_n}} |\beta_n|^2 \, dx \geq c_n \int_{B_1 \setminus B_{r_n}} |\beta_n^{\mathrm{sym}}|^2 \, dx, \tag{5.36}$$

for some $c_n \to \infty$ as $n \to \infty$. Indeed, by [28] there exists a sequence $u_n \in H^1(B_1 \setminus B_{r_n}; \mathbb{R}^2)$ such that

$$\int_{B_1 \setminus B_{r_n}} |\nabla u_n|^2 \, dx \geq \tilde{c}_n \int_{B_1 \setminus B_{r_n}} |\nabla u_n^{\mathrm{sym}}|^2 \, dx$$

with $\tilde{c}_n \to \infty$ as $n \to \infty$. By homogeneity we may assume

$$\int_{B_1 \setminus B_{r_n}} |\nabla u_n^{\mathrm{sym}}|^2 \, dx = 1.$$

Let $\beta(\rho, \theta) := \frac{\xi}{2\pi\rho} \otimes (-\sin\theta, \cos\theta)$, and notice that $\beta \in \mathcal{AS}_{r_n,1}(\xi)$ for every n. Finally, set $\beta_n = \nabla u_n + \beta \in \mathcal{AS}_{r_n,1}(\xi)$; a straightforward computation shows that (5.36) holds.

The sequence β_n just constructed is such that its symmetric part is bounded in L^2, while its skew part blows up as $n \to \infty$. In particular, the linearized energy induced by β_n on the annuli $B_1 \setminus B_{r_n}$ is larger than $1 - r_n$. In the next example we construct a strain $\beta \in \mathcal{AS}_{r,1}(\xi)$ for every $0 < r < 1$ whose linearized energy density vanishes on thin annuli $B_1 \setminus B_r$ (as $r \to 1$), showing that the function $\psi_{r,R}$ defined in (5.9) vanishes as $R/r \to 1$.

Example 5.13. Let $S(x, y) : \mathbb{R}^2 \mapsto \mathbb{M}^{2\times 2}$ be defined by

$$S(x, y) := \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}.$$

We have Curl $S = (1, 0)$. Set

$$f(\rho, \theta) := \frac{\rho^2}{4} - \frac{1}{2} \log \rho.$$

Notice that $\Delta f = 1$, and hence curl $(-f_y, f_x) = 1$. Finally, set

$$\beta(x, y) := S(x, y) - \begin{pmatrix} -\frac{\partial f}{\partial y} & \frac{\partial f}{\partial x} \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that $\beta \in \mathcal{AS}_{r,1}((\pi, 0))$ for every $0 < r < 1$ and $|\beta^{\mathrm{sym}}|^2 \leq |\nabla f|^2$. Moreover, $|\nabla f| = 0$ on ∂B_1; a straightforward computation shows that

$$\lim_{r \to 1} \frac{1}{|\log r|} \int_{B_1 \setminus B_r} |\nabla f|^2 \, \mathrm{d}x = 0,$$

so that the density of the linearized elastic energy vanishes on thin annuli $B_1 \setminus B_r$ as $r \to 1$.

Conclusions and perspectives

We have obtained an asymptotic expansion by Γ-convergence for the functionals SD_ε that allows us to show existence of metastable configurations, and to introduce a discrete in time variational dynamics, which overcomes the energy barriers and mimics the effect of more complex mechanisms, as thermal effects. On the other hand, we have proved a Γ-convergence result for the edge dislocations energy without extra-assumptions on the position and on the number of singularities.

As for the screw dislocations, we believe that our Γ-convergence analysis for anisotropic energies defined in the triangular lattice can be used in order to prove, also in this case, the existence of metastable configurations and the convergence to a limit dynamics. A more delicate question is the Γ-expansion of the anisotropic long range (and finite range) interaction energy, for which a zero-order Γ-convergence result is not yet available.

For the SD_ε energies, we have described the dynamics up to the first collision time; it would be interesting to model the collision of discrete vortices, and study the dynamics after the critical time as in the Ginzburg-Landau setting (see [**16, 63, 64**]). Moreover, we have focused on Neumann boundary conditions, but we are confident that our analysis could be extended to the case of Dirichlet boundary conditions.

In the discrete dynamics we have analyzed two different dissipations. This is motivated also by applications. Indeed, the L^2 dissipation is a standard choice for parabolic flows and measures the variations in the spin variable. While, the dissipation D_2 is a natural choice in the study of screw dislocation dynamics, and can be viewed as a measure of the number of energy barriers to be overcome in order to move a dislocation. We note that, in the case of dislocations, one could also consider suitable variants of the D_2 dissipation account for the glide directions of the crystal (see [**5**]). This would lead to a different effective dynamics.

Having proved a pinning phenomenon, it remains open to characterize a critical ε-τ regime for the motion of dislocations, and an effective depinning threshold in this regime. This is a relevant issue and it might be worth facing it by using our variational approach.

The effective dynamics of our discrete systems agrees with the asymptotic parabolic flow of the Ginzburg-Landau functionals. In the latter, the time scaling needed to get a non-trivial effective dynamics depends on the space parameter ε. It is worth noticing that, in our discrete in time gradient flow with L^2 dissipation, the time scaling is expressed only in terms of the time step τ. In this respect, an analysis of critical ε-τ regimes would make an interesting bridge between these two approaches.

The case of edge dislocations is still open since a complete Γ-expansion of the energy is not yet available. The reason is that the lower bounds in Chapter 5 are sharp enough to guarantee the Γ-convergence result for the rescaled energies, but not so good to derive the renormalized energy. We hope that a refinement of these lower bounds could give on one hand the Γ-convergence expansion of the energy in the $|\log \varepsilon|$ regime and on the other hand to study the case of infinite singularities (corresponding to the $|\log \varepsilon|^2$ regime) without any a priori assumptions on their position.

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