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DISCRETE STRUCTURES IN FINITE TYPE CLUSTER ALGEBRAS

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by

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ABSTRACT OF DISSERTATION

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Abstract

Due to their recursive definition, manipulating cluster algebras in an efficient way can be hard. Several combinatorial models have been developed in order to overcame this difficulty; here we investigate some of them in the finite tipe case.

In the first part of this thesis, using the parametrization of cluster variables by their g-vectors explicitly computed by S.-W. Yang and A. Zelevinsky, we extend the original construction of generalized associahedra by F. Chapoton, S. Fomin and A. Zelevinsky to any choice of acyclic initial cluster, and compare it to the one given by C. Hohlweg, C. Lange, and H. Thomas in the setup of Cambrian fans developed by N. Reading and D. Speyer.

In the second part we provide an explicit Dynkin diagrammatic description of the c-vectors and the d-vectors (the denominator vectors) of any cluster algebra of finite type with principal coefficients and any initial exchange matrix.

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CHAPTER 1

Introduction

Cluster algebras are a class of commutative rings equipped with a distinguished set of generators (*cluster variables*) grouped into overlapping subsets (*clusters*) of the same cardinality. Both the generators and the algebraic relations among them (*exchange relations*) are constructed recursively by a process called *mutation* modeled by a skew-symmetrizable integer matrix.

They were introduced by Fomin and Zelevinsky [**FZ02**] as an algebraic framework to understand total positivity and dual canonical bases in semisimple algebraic groups. The definition is, at a first glance, extremely technical. Nevertheless they form a natural class of objects to study as testified by the abundance of different fields in which they occur: Poisson geometry, discrete integrable systems, quiver representations, Calabi-Yau categories, and Teichmüller theory just to name a few.

What makes cluster algebras interesting from the point of view of algebraic combinatorics is the fact that, in many different ways, they are intimately related to Lie theory. In particular the language of root systems applies naturally to cluster algebras yielding several beautiful results.

In this spirit, it turns out that *finite type* cluster algebras (those having only a finite number of clusters) are classified by finite type Dynkin diagrams. Moreover both the cluster variables and the exchange relations of such an algebra can be associated to certain subsets of an appropriate root (or weight) lattice.

Because of their recursive definition, manipulating cluster algebras in an efficient way can be complicated. Our aim is to circumvent this difficulty by investigating the properties of these subsets extracting from them valuable information on the corresponding algebras.

1.1. Generalities on cluster algebras

In order to formalize the concepts mentioned before we need to introduce some terminology.

To begin, recall that a *semifield* $\mathbb{P} = (\mathbb{P}, \cdot, \oplus)$ is a abelian multiplicative group endowed with a binary operation, $\oplus$, which is commutative, associative and distributive with respect to the multiplication in $\mathbb{P}$. As noted in [**FZ02**, Section 5] the group ring $\mathbb{ZP}$ is a domain since any semifield $\mathbb{P}$ is torsion-free. Examples of semifields include the set of positive real numbers $\mathbb{R}_+ = (\mathbb{R}_+, \cdot, +)$ with the usual multiplication and addition, and the set of *subtraction-free* rational expressions in n variables again with the usual product and sum.

Fix a semifield $\mathbb{P}$ and let I be a finite set of cardinality n. Let $\mathbb{QP}(\mathbf{u})$ be the field of rational functions in the indeterminates $\mathbf{u} = \{u_i\}_{i\in I}$ with coefficients in $\mathbb{QP}$.

Definition 1.1.1. A *seed* is a triple

$$(B, \mathbf{y}, \mathbf{x})$$

where B is a skew-symmetrizable $n \times n$ integer matrix (the *exchange matrix* of the seed), $\mathbf{y} = \{y_i\}_{i\in I}$ is a tuple of elements of $\mathbb{P}$ (the *coefficients* of the seed), and $\mathbf{x} = \{x_i\}_{i\in I}$ is a tuple of algebraically independent elements of $\mathbb{QP}(\mathbf{u})$ (the *cluster* of the seed).

Given a seed, for each $k \in I$, one can construct a different seed by a *mutation* in direction k:

$$\mu_k (B, \mathbf{y}, \mathbf{x}) := (\mu_k(B), \mu_k(\mathbf{y}), \mu_k(\mathbf{x}))\,.$$

Apart from the chosen k, the new matrix $\mu_k(B)$ depends only on B; $\mu_k(\mathbf{y})$ depends both on B and $\mathbf{y}$, while $\mu_k(\mathbf{x})$ depends on the entire seed $(B, \mathbf{x}, \mathbf{y})$.

More precisely the entries of $B' = \mu_k(B)$ are given by

$$b'_{ij} := \begin{cases} -b_{ij} & \text{if } k \in \{i,j\} \\ b_{ij} + b_{ik}\,[b_{kj}]_+ + [b_{ik}]_+\, b_{kj} & \text{otherwise} \end{cases} \tag{1.1.1}$$

where $[b]_+$ denotes $\max(b,0)$. The new coefficient tuple $\mathbf{y}' = \mu_k(\mathbf{y})$ is defined by

$$(1.1.2) \qquad y'_i := \begin{cases} y_i^{-1} & \text{if } i = k \\ y_i y_k^{[b_{ki}]_+} (y_k \oplus 1)^{-b_{ki}} & \text{if } i \neq k \end{cases}$$

and the cluster $x' = \mu_k(\mathbf{x})$ is obtained from $\mathbf{x}$ replacing x_k with the cluster variable x'_k defined by the *exchange relation*

$$(1.1.3) \qquad x'_k := \frac{y_k \prod x_i^{[b_{ik}]_+} + \prod x_i^{[-b_{ik}]_+}}{(y_k \oplus 1)\, x_k}.$$

By direct inspection it is easy to check that every mutation is an involution i.e. for every $k \in I$ and for any seed $(B, \mathbf{y}, \mathbf{x})$

$$\mu_k \circ \mu_k (B, \mathbf{y}, \mathbf{x}) = (B, \mathbf{y}, \mathbf{x}).$$

Definition 1.1.2. Fix an *initial seed* $(B_0, \mathbf{y}_0, \mathbf{x}_0)$. The *cluster algebra* $\mathcal{A}(B_0, \mathbf{x}_0, \mathbf{y}_0)$ is the subring of $\mathbb{QP}(\mathbf{u})$ generated over $\mathbb{ZP}$ by all the cluster variables obtained by sequences of mutations from the initial seed.

Note that, up to an isomorphism of $\mathbb{QP}(\mathbf{u})$, we can always assume that the initial cluster is $\mathbf{u}$; we will therefore omit to specify the initial cluster when this does not generate confusion.

Note also that the choice of initial seed is not relevant for the cluster algebra itself; indeed the same cluster algebra can be obtained from any seed *mutationally equivalent* to $(B_0, \mathbf{y}_0, \mathbf{x}_0)$. However the models we are going to discuss here depend on this choice.

A cluster algebra is said to be of *finite type* if it contains only finitely many clusters otherwise it is of *infinite type*. Cluster algebras of finite type form one of the most basic and better understood classes of cluster algebras; they were classified in one of the first papers on the topic ([**FZ03a**]). It turns out that the property of being of finite type depends only on the *mutation class* of the initial exchange matrix B_0 (the set of all matrices obtainable from B_0 by a sequence of mutations) and not on the choice of coefficients.

An extra piece of terminology is needed to state the result. The *Cartan counterpart* $A(B)$ of a skew-symmetrizable matrix B is the matrix defined by

$$a_{ij} := \begin{cases} -|b_{ij}| & \text{if } i \neq j \\ 2 & \text{if } i = j. \end{cases}$$

It is a symmetrizable (generalized) Cartan matrix in the sense of Kac [**Kac90**].

THEOREM 1.1.3 ([**FZ03a**, Theorem 1.4]). *All cluster algebras $\mathcal{A}(B_0, \bullet, \bullet)$ are simultaneously of finite or infinite type. They are of finite type if and only if there exist a matrix B in the mutation class of B_0 such that $A(B)$ is a finite type Cartan matrix. Moreover the Dynkin type of $A(B)$ is uniquely determined by the mutation class of B_0.*

By extension we say that a skew-symmetrizable matrix B_0 is of *(cluster) finite type*, and more specifically, of *(cluster) type Z*, according to the type Z of $A(B)$ above. From now on, unless otherwise specified, all cluster algebras are assumed to be of finite type.

1.2. g-vectors, cluster fans, and generalized associahedra

Much information on the structure of a cluster algebra $\mathcal{A}$ can be deduced directly from two purely combinatorial gadgets "dual" to one another: its cluster complex and its exchange graph.

The *exchange graph* is the graph whose vertices are the clusters of $\mathcal{A}$. Its edges are given by exchange relations: two clusters are connected by an edge if and only if they can be obtained from one another by a single mutation.

The *cluster complex* is an abstract simplicial complex with cluster variables as vertices and clusters as maximal simplices.

When a cluster algebra is of finite type both its cluster complex and its exchange graph are finite. The goal of Chapter 2 is to better understand these two objects.

This problem had been already addressed in [**FZ03b, CFZ02**] in the special case of a *bipartite* initial seed (i.e. when the rows of the initial exchange matrix have a definite sign). Under this assumption, in [**FZ03a**], Fomin and Zelevinsky provided an explicit combinatorial description of the cluster complex obtained by labeling its vertices with *almost-positive roots* in the corresponding root system.

They constructed a function on ordered pairs of labels, called *compatibility degree*, encoding whether the corresponding cluster variables are *compatible* (i.e. they belong to the same cluster), *exchangeable* (i.e. related by an exchange relation) or neither. Its definition is purely combinatorial and does not refer to the cluster algebra but just to the labels. The description of the cluster complex they presented is in terms of this function: compatible pairs of almost positive roots form its 1-skeleton; higher dimension simplices are given by the cliques of the 1-skeleton.

In [**FZ03b**] the authors improved on this combinatorial model explaining how almost positive roots give a geometric realization of the cluster complex. They showed that the positive real span of the labels in any simplex of the cluster complex is a cone in a complete simplicial fan: the *cluster fan*. Among the applications of this realization there is a parametrization of cluster monomials in $\mathcal{A}$ with points of the root lattice Q.

Further study ([**CFZ02**]) of the cluster fan showed that it is the normal fan of a distinguished polytope: the *generalized associahedron* of the given type. Its description is completely explicit: the authors discussed all the constrains that its *support function* must satisfy and then provided a concrete function that meets them. The exchange graph of $\mathcal{A}$ is the 1-skeleton of its generalized associahedron. As a bonus from the construction they also get an explicit formula for all the exchange relations in the coefficient-free case in terms of the roots labeling the corresponding cluster variables.

As noted above the construction in [**FZ03b**] and [**CFZ02**] depends on the labeling of cluster variables of $\mathcal{A}$ by almost positive roots; such a parametrization is provided by their *denominator vectors* with respect to a bipartite initial cluster.

This is not the only possible choice of labels; another option is to use *g-vectors.* To introduce them we need some more terminology.

Definition 1.2.1. The *tropical* semifield on n elements, $\mathrm{Trop}(\mathbf{z})$, is the abelian multiplicative group of Laurent monomials in the indeterminates $\mathbf{z} = \{z_i\}_{i\in I}$ endowed with the auxiliary operation $\oplus$ defined by

$$\prod_{i\in I} z_i^{a_i} \oplus \prod_{i\in I} z_i^{b_i} := \prod_{i\in I} z_i^{\min(a_i,b_i)}.$$

Definition 1.2.2. A cluster algebra $\mathcal{A}(B_0, \mathbf{y}_0, \mathbf{x}_0) \subset \mathbb{QP}(\mathbf{x}_0)$ is said to have *principal coefficients* (at the initial seed) if $\mathbb{P} = \mathrm{Trop}(\mathbf{y}_0)$. We will denote such algebras by $\mathcal{A}_\bullet(B_0)$.

Cluster algebras with principal coefficients can be equipped with a $\mathbb{Z}^n$-grading in such a way that every cluster variable is an homogeneous element of $\mathbb{QP}(\mathbf{x}_0)$.

Proposition 1.2.3. [**FZ07**, Proposition 6.1] *Every cluster variable in* $\mathcal{A}_\bullet(B_0)$ *is homogeneous with respect to the* $\mathbb{Z}^n$*-grading given by*

$$\deg(x_i) := \mathbf{e}_i \qquad \deg(y_i) := -\mathbf{b}_i$$

where, as usual, $\mathbf{e}_i$ *is the* i*-th unit vector while* $\mathbf{b}_i$ *is the* i*-th column of* B_0*.*

Definition 1.2.4. The *g-vector* of a cluster variable in $\mathcal{A}_\bullet(B_0)$ is its homogeneous degree.

The definition can be extended, in view of the "separation of additions" formula [**FZ07**, Theorem 3.7], to all other choices of coefficients; we will therefore refer to the g-vectors for the cluster variables in any cluster algebra.

An explicit description of all the g-vectors of any finite type cluster algebra with *acyclic* initial seed (i.e. any cluster algebra $\mathcal{A}(B_0, \mathbf{y}_0, \mathbf{x}_0)$ such that $A(B_0)$ is a finite type Cartan matrix) was given explicitly in [**YZ08**] in terms of weights. Using a *Coxeter element* c in the Weyl group of $A(B_0)$ to encode B_0 (see Eqs. (2.1.1) and (2.1.2) for the details) they were able to identify the g-vectors of $\mathcal{A}(B_0, \mathbf{y}_0, \mathbf{x}_0)$ with a subset $\Pi(c)$ of the corresponding weight lattice P.

The first goal of Chapter 2 is to extend the results from [**FZ03b**] and [**CFZ02**] to each of these new parametrizations of cluster variables. Retracing the steps in those papers, for any choice of acyclic initial seed, we will construct a complete simplicial fan realizing the cluster complex and we will show that it is the normal fan to a geometric realization of a generalized associahedron. We can summarize our claims as follows:

THEOREM 1.2.5. *Let $\mathcal{A} = \mathcal{A}(B_0, \mathbf{y}_0, \mathbf{x}_0)$ be a cluster algebra of finite type with an acyclic initial seed and let c be the Coxeter element encoding B_0. Let $\Pi(c)$ be the labeling set and $(\bullet||\bullet)_c$ its compatibility degree function both constructed in* [**YZ08**]. *Then*

(1) *every c-cluster in $\Pi(c)$ (i.e. every maximal subset of $\Pi(c)$ consisting of pairwise compatible weights) is a $\mathbb{Z}$-basis of the weight lattice P.*
(2) *The positive linear spans of the simplices in the clique complex induced by $(\bullet||\bullet)_c$ on $\Pi(c)$ form a complete simplicial fan $\mathcal{F}_c^\Pi$ realizing the cluster complex. Cluster monomials of $\mathcal{A}$ are in bijection with points of P.*
(3) *$\mathcal{F}_c^\Pi$ is the normal fan to a simple polytope: a geometric realization of the associated generalized associahedron.*
(4) *If $\mathcal{A}$ is coefficient-free then all its exchange relations are explicitly determined by the labels of exchangeable cluster variables.*

The proof will be split into sub-statements, namely Theorems 2.1.2, 2.1.4, 2.1.7, and 2.1.10. Some of these results were already proved in less generality or were already conjectured; we will provide explicit references in Section 2.1.

It turns out that our polytopes are the same as those studied in [**HLT11**] in the setup of *Cambrian fans* developed by Reading and Speyer. The construction we propose, however, is different from the one by Hohlweg, Lange, and Thomas. This provides us with an alternative prospective on c-cluster combinatorics that allows us to recover all the exchange relations of the associated coefficient-free cluster algebra and to answer positively to Problem 4.1 posed in [**Hoh12**].

To explain what we mean by "different" recall that the definition of Cambrian fans is given in terms of its maximal cones as opposed to the definition of cluster fans that builds up from the 1-skeleton. Indeed to each Coxeter element c of a finite type Weyl group W one can associate a lattice congruence on the group itself (seen as a lattice for the right weak order). This produces a coarsening of the associated *Coxeter fan* obtained by glueing together cones corresponding to elements in the same class (recall that the Coxeter fan is the complete simplicial fan in the weight space of W whose maximal cones are the images of the fundamental Weyl chamber under the action of the group). The approach used in [**HLT11**] to show that the Cambrian fans are polytopal follows the same philosophy: they begin from the *generalized permutahedron* associated to W seen as intersection of half-spaces and, again using the lattice congruence induced by c, they remove a certain subset of them to make it into a generalized associahedron.

The second goal of Chapter 2 is to show that the generalizations of the cluster fans we propose coincide with the Cambrian fans of Reading and Speyer. To do so it suffices to show that the polyhedral models for the generalized associahedra we build are the same as the realizations given in [**HLT11**]. Note that, in type A, the interaction between the geometric realizations of the associahedron by Hohlweg, Lange, and Thomas and the original realization by Chapoton, Fomin, and Zelevinsky has been already investigated in [**CSZ11**].

Chapter 2 is structured as follows: in Section 2.1, after having recalled the required terminology and having set up some notations, we discuss in more details our generalizations of the results in [**FZ03b**] and [**CFZ02**] and we provide an idea of the strategy we adopt to prove them. We then recall some more terminology and explain how our construction relates to Cambrian fans and to the polytopes from [**HLT11**].

In Section 2.2 we introduce our main tool: the set of *c-almost-positive roots* $\Phi_{\text{ap}}(c)$. Many arguments from [**FZ03b**] and [**CFZ02**] require to perform an induction on the rank of the cluster algebra; the labeling of cluster variables by almost-positive roots is ideal for such a purpose. In our case, however, we are given a set of weights to parametrize the vertices of the cluster complex therefore we can not generalize those proofs directly. The solution we adopt is to identify the weight lattice with the root lattice in such a way that the restriction to a smaller rank cluster sub-algebra can be expressed easily in terms of the labels in a new set $\Phi_{\text{ap}}(c)$ (the image of $\Pi(c)$ under this identification).

Section 2.3 deals with bipartite orientations. We show that, in this case, our results follow directly from their analogues from [**FZ03b**] and [**CFZ02**].

Section 2.4 contains the proofs of some technical results we need in Section 2.5 where we complete the proofs of the main results of the first part of Chapter 2.

The chapter is concluded by Section 2.6 where we show that our realizations of the generalized associahedra coincide with those constructed by Hohlweg, Lange, and Thomas and therefore that our generalization of cluster fans is a different presentation of Cambrian fans.

1.3. c-vectors and d-vectors

g-vectors are not the only example of integer vectors naturally associated to cluster algebras; other two distinguished families are given by *c-vectors* and *d-vectors*; they correspond respectively to exchange relations and cluster variables.

The definition of d-vectors can be given independently from the choice of coefficients. In view of the Laurent phenomenon [**FZ02**, Theorem 3.1], every cluster variable x in a cluster algebra $\mathcal{A}(B_0, \mathbf{y}_0, \mathbf{x}_0)$ can be expressed as a ratio

$$x = \frac{N(\mathbf{x}_0)}{\prod_{i \in I} x_i^{d_i}}$$

where N is a polynomial in $\mathbb{QP}[\mathbf{x}_0]$ not divisible by any of the initial cluster variables $\mathbf{x}_0 = \{x_i\}$. The *d-vector* of x is the vector $(d_i)_{i \in I}$. It is important to remark that this definition does not imply that d-vectors are sign-coherent (i.e. with entries either all non-negative or all non-positive).

To introduce c-vectors, instead, we need principal coefficients. The *c-vectors* of a cluster algebra $\mathcal{A}_\bullet(B_0)$ are the exponent vectors of the coefficients appearing in its seeds.

We will refer to the map associating to each cluster variable (resp. coefficient) its d-vector (resp. c-vector) as the *tropicalization map*. The origin of the name can be made explicit introducing universal semifields and semifield homomorphisms but for our purposes we do not need to do so.

It has been partially recognized and proved that, the c- and d-vectors of $\mathcal{A}_\bullet(B_0)$ are roots in the root system of the (generalized) Cartan matrix $A(B_0)$. When B_0 is skew-symmetric, thanks to Kac's theorem [**Kac80**], it is enough to prove that they can be identified with the *dimension vectors* of some indecomposable modules of the path algebra $kQ(B_0)$ for the quiver $Q(B_0)$ corresponding to B_0. In fact, this is a common method of proving many known cases. We are going to discuss this subject in more detail in Section 3.1.

Among finite type cluster algebras the most studied are those of *simply-laced types* (i.e. types A_n, D_n, E_6, E_7, E_8 according to the classification of [**FZ03a**]). In these cases the *cluster-tilted algebra* $\Lambda(B_0)$, introduced in [**BMR07**] as a certain quotient of the path algebra $kQ(B_0)$, plays a key role in the study of $\mathcal{A}_\bullet(B_0)$ [**CCS05, CCS06, BMR$^+$06a, BMR08, BMR06b**].

A c-vector is said to be *positive* if it is a nonzero vector and its components are all nonnegative. A d-vector is *non-initial* if it is the d-vector of a cluster variable not in the initial cluster. It was proved by [**CCS06, BMR07**] that the set of all the non-initial d-vectors of $\mathcal{A}_\bullet(B_0)$ coincides with the set of the dimensions vectors of all the indecomposable $\Lambda(B_0)$-modules. Moreover, it was recently proved by [**NC12, NC**] that the set of all the positive c-vectors of $\mathcal{A}_\bullet(B_0)$ also coincides with the same set. See Theorems 3.1.5 and 3.1.6.

In spite of this beautiful and complete, representation-theoretic description of c- and d-vectors for finite type, little is known about their *explicit* form, except for type A_n [**CCS05, Par11, Tra11**]. The purpose of Chapter 3 is to fill this gap and to provide an *explicit Dynkin diagrammatic description* of the c- and d-vectors for cluster algebras of any finite type with any initial exchange matrix. All the results contained there and that we will now summarize were obtained jointly with T. Nakanishi.

For any skew-symmetrizable matrix B_0 of cluster finite type, we present the Cartan matrix $A(B_0)$ as a Dynkin diagram $X(B_0)$ in the usual way following [**Kac90**]. Note that, in general, $X(B_0)$ is not a finite type Dynkin diagram.

In Section 3.2, for each finite type Z, we provide the following two lists explicitly:

• the list $\mathcal{X}(Z)$ of the Dynkin diagrams $X(B_0)$ of all the skew-symmetrizable matrices B_0 of cluster type Z (for each B_0 the vertices of $X(B_0)$ are naturally identified with the elements of I),

• the list $\mathcal{W}(Z)$ of the "candidates" of positive c-vectors and non-initial d-vectors of $\mathcal{A}_\bullet(B_0)$ for any $X(B_0) \in \mathcal{X}(Z)$ in the form of *weighted Dynkin diagrams*, namely, Dynkin diagrams with a positive integer attached to each vertex.

For a pair $X(B_0) \in \mathcal{X}(Z)$ and $W \in \mathcal{W}(Z)$, an embedding of the diagram part of W into $X(B_0)$ as a full sub-diagram is denoted by $W \subset X(B_0)$. Such an embedding is not necessarily unique if it exists; we distinguish them up to isomorphism of W. An embedding $W \subset X(B_0)$ is identified with an integer vector $v = (v_i)_{i\in I}$ such that the support of v is the diagram part of W and the nonzero integer v_i is the weight of W at i. For each skew-symmetrizable matrix B_0 of cluster type Z, let us introduce the sets

$$
\begin{aligned}
\mathcal{V}(B_0) &:= \{\, W \subset X(B_0) \mid W \in \mathcal{W}(Z) \,\}, \\
\mathcal{C}(B_0) &:= \{\, \text{all } c\text{-vectors of } \mathcal{A}_\bullet(B_0) \,\}, \\
\mathcal{C}_+(B_0) &:= \{\, \text{all positive } c\text{-vectors of } \mathcal{A}_\bullet(B_0) \,\}, \\
\mathcal{D}(B_0) &:= \{\, \text{all non-initial } d\text{-vectors of } \mathcal{A}_\bullet(B_0) \,\}.
\end{aligned} \tag{1.3.1}
$$

For finite type cluster algebras, it turns out that

$$\mathcal{C}(B_0) = \mathcal{C}_+(B_0) \sqcup (-\mathcal{C}_+(B_0)), \tag{1.3.2}$$

therefore, we can concentrate on $\mathcal{C}_+(B_0)$. Our main result is stated as follows.

THEOREM 1.3.1. *Let B_0 be any skew-symmetrizable matrix of cluster finite type. Then, the sets $\mathcal{C}_+(B_0)$, $\mathcal{D}(B_0)$, and $\mathcal{V}(B_0)$ coincide.*

As an immediate and important corollary, for simply laced types the set $\mathcal{V}(B_0)$ also coincides with the set of the dimension vectors of all the indecomposable modules of the cluster-tilted algebra $\Lambda(B_0)$, thereby yielding a representation-theoretic result.

To prove Theorem 1.3.1 we use the surface realization of cluster algebras [**FG07, FST08, FT12**] for types A_n and D_n. The case A_n is easy, but the case D_n is (much) more involved. Then we apply the folding method [**Dup08, Dem11**] to types D_{n+1} and A_{2n-1} to obtain types B_n and C_n, respectively. Exceptional types are studied by direct inspection with the help of the software by Keller [**Kel**] and the cluster algebra package [**MS12**] of Sage [**S$^+$12**] written by Musiker and Stump; we rely on Corollaries 3.1.7 and 3.1.11 to simplify computations in type E_8. In classical types our derivation is purely combinatorial and does not refer to any results from representation theory. On the one side, this may be unsatisfactory due to the lack of a direct representation-theoretic explanation; on the other side, this is the reason why we get the result easily. In particular, we obtain an alternative proof of the known equality $\mathcal{C}_+(B_0) = \mathcal{D}(B_0)$ for types A_n and D_n, and also several results on non-simply laced types, for which the representation-theoretic method is not yet fully available.

From the explicit list of positive c-vectors and non-initial d-vectors provided by Theorem 1.3.1 we deduce the following result. The statements (1) and (3) generalize to all finite types properties known only for simply-laced types (cf. Corollaries 3.1.8 and 3.1.11).

THEOREM 1.3.2. *Let B_0 be any skew-symmetrizable matrix of cluster finite type.*

(1) *All c-vectors and d-vectors of $\mathcal{A}_\bullet(B_0)$ are roots of the root system of $A(B_0)$. For simply-laced types they are Schur roots.*
(2) *A c-vector (d-vector) of $\mathcal{A}_\bullet(B_0)$ is a real root if and only if its support in $X(B_0)$ is a tree.*
(3) *The cardinality $|\mathcal{C}_+(B_0)| = |\mathcal{D}(B_0)|$ depends only on the cluster type Z of B_0 and it is equal to the number of positive roots in the root system of type Z. Explicitly it is equal to $nh/2$, where n and h are the rank and the Coxeter number of type Z (see Table 1.3.1).*
(4) *The set $\mathcal{C}_+(B_0) = \mathcal{D}(B_0)$ only depends on $A(B_0)$, the Cartan counterpart of B_0.*

TABLE 1.3.1. Coxeter numbers and numbers of positive roots.

Type	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
h	$n+1$	$2n$	$2n$	$2n-2$	12	18	30	12	6
$nh/2$	$n(n+1)/2$	n^2	n^2	$n(n-1)$	36	63	120	24	6

While proving Theorem 1.3.1 we also obtain the following interesting result.

THEOREM 1.3.3. *Let B_0 be any skew-symmetrizable matrix of cluster finite type. Any c-vector (d-vector) of $\mathcal{A}_\bullet(B_0)$ occurs in some bipartite seed.*

Chapter 3 is structured as follows. In Section 3.1 we give more background and a short survey of the known results on c- and d-vectors and their consequences in order to connect our result to representation theory of quivers. In Section 3.2 we describe the sets $\mathcal{X}(Z)$ and $\mathcal{W}(Z)$ for any finite type Z.

The proofs of Theorems 1.3.1 and 1.3.3 for classical types are split into several Propositions and use different techniques. In Section 3.3 we use the surface realization ([**FG07, FST08, FT12**]) of cluster algebras to prove the results for types A_n and D_n. In Section 3.4 we extend the folding construction of [**Dup08**] to deal with types B_n and C_n.

The chapter is concluded by Section 3.5 where we prove Theorem 1.3.2. In appendix we add the complete analysis needed in the proof of Propositions 3.3.10 and 3.3.11.

CHAPTER 2

Generalized associahedra

2.1. Preliminaries

We start by setting up notation and recalling some terminology and results from [**YZ08**]. Let I be a finite type Dynkin diagram; with a small abuse of notation denote by I also its vertex set. Let W be the associated Weyl group with simple reflections $\{s_i\}_{i \in I}$ and let $A = (a_{ij})_{i,j \in I}$ be the corresponding Cartan matrix.

Recall that an element c of W is said to be *Coxeter* if every simple reflection appears in a reduced expression of c exactly once. To each Coxeter element c associate a skew-symmetrizable matrix $B(c) = (b_{ij})_{i,j \in I}$ as follows. For i and j in I, write $i \prec_c j$ if i and j are connected by an edge and s_i precedes s_j in a reduced expression of c. Set then

$$b_{ij} := \begin{cases} -a_{ij} & \text{if } i \prec_c j \\ a_{ij} & \text{if } j \prec_c i \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.1.1}$$

Note also that Coxeter elements are in bijection with orientation of I under the convention

$$j \to i \qquad \Leftrightarrow \qquad i \prec_c j\,. \tag{2.1.2}$$

Remark 2.1.1. In each Weyl group there is a distinguished class of Coxeter elements (call them *bipartite*) corresponding to orientations of I in which each node is either a source or a sink. Following the notation of [**FZ03b**], we denote bipartite Coxeter elements by t.

For a given Coxeter element c denote by $\mathcal{A}_0(c)$ the coefficient-free cluster algebra with the initial B-matrix $B(c)$. Let $\{\omega_i\}_{i\in I}$ be the set of fundamental weights associated to I and w_0 the longest element in W. Set $h(i;c)$ to be the minimum positive integer such that

$$c^{h(i;c)}\omega_i = -\omega_{i^*}$$

where $\omega_{i^*} := -w_0\omega_i$ (Cf. Proposition 1.3 in [**YZ08**]).

By theorem 1.4 in [**YZ08**] the set of weights

$$\Pi(c) := \{c^m\omega_i : i \in I, 0 \leq m \leq h(i;c)\}$$

parametrize the cluster variables in $\mathcal{A}_0(c)$. The correspondence is given associating to each cluster variable its g-vector as defined in [**FZ07**]; in particular cluster variables in the initial cluster correspond to fundamental weights.

The set $\Pi(c)$ can be made into an abstract simplicial complex of pure dimension $n-1$ (the *c-cluster complex*) as follows. The cluster algebra structure induces a permutation on $\Pi(c)$

$$\tau_c^{\Pi}(\lambda) := \begin{cases} \omega_i & \text{if } \lambda = -\omega_i \\ c\lambda & \text{otherwise} \end{cases}$$

and a (unique) τ_c^{Π}-invariant *c-compatibility degree* function defined by the initial conditions

$$(\omega_i||\lambda)_c^{\Pi} := \left[(c^{-1}-1)\lambda;\alpha_i\right]_+$$

where $[\bullet;\alpha_i]$ is the coefficient of α_i in $\bullet$ expressed in the basis of simple roots and $[\bullet]_+$ denotes $\max\{\bullet, 0\}$ (Cf. Proposition 5.1 in [**YZ08**]).

Note that the action of τ_c^{Π} on $\Pi(c)$ is, by construction, compatible with the action of w_0 on I; that is any τ_c^{Π}-orbit contains a unique pair $\{\omega_i, \omega_{i^*}\}$ (or a single fundamental weight ω_i if $i = i^*$).

Call two weights λ and μ in $\Pi(c)$ *c-compatible* if

$$(\lambda||\mu)_c^{\Pi} = 0\,.$$

This definition makes sense since the c-compatibility degree satisfies

$$(\lambda||\mu)_c^\Pi = 0 \Leftrightarrow (\mu||\lambda)_c^\Pi = 0\,.$$

The c-cluster complex Δ_c^Π is defined to be the abstract simplicial complex on the vertex set $\Pi(c)$ whose 1-skeleton is given by c-compatible pairs of weights and whose higher dimensional simplex are given by the cliques of its 1-skeleton. We refer to its maximal simplices as *c-clusters*; this name already appeared in the work of Reading and Speyer in a different setup, we will discuss later on how the two notions are related.

The first step in order to construct a complete simplicial fan realizing the c-cluster complex is to show that we can associate an n-dimensional cone to each c-cluster.

THEOREM 2.1.2. *Each c-cluster in Δ_c^Π is a $\mathbb{Z}$-basis of the weight lattice P.*

Remark 2.1.3. Theorem 2.1.2 was conjectured in [**FZ07**] (Conjecture 7.10(2)) and then proved in [**DWZ10b**] (Theorem 1.7) under the assumption that the initial exchange matrix is skew-symmetric.

Let $\mathcal{F}_c^\Pi$ be the collection of all the cones in $P_\mathbb{R}$ that are positive linear span of simplices in the c-cluster complex.

THEOREM 2.1.4. *$\mathcal{F}_c^\Pi$ is a complete simplicial fan.*

Remark 2.1.5. This is a generalization of Theorem 1.10 in [**FZ03b**], and our proof is inspired by the one in that paper. In particular we will deduce the result from the following proposition (mimicking Theorem 3.11 in there).

Proposition 2.1.6. *Every point μ in the weight lattice P can be uniquely be written as*

$$\mu = \sum_{\lambda \in \Pi(c)} m_\lambda \lambda \tag{2.1.3}$$

where all the coefficients m_λ *are non-negative integers and* $m_\lambda m_\nu = 0$ *whenever* $(\lambda||\nu)_c^\Pi \neq 0$

The expression (2.1.3) is called the *c-cluster expansion* of μ.

A simplicial fan is said to be *polytopal* if it is the normal fan to a simple polytope. Recall that, given a simple full-dimensional polytope T in a vector space V its *support function* F is the piecewise linear function on V^* defined by

$$\begin{array}{rccl} F: & V^* & \longrightarrow & \mathbb{R} \\ & \varphi & \longmapsto & \max\{\varphi(x)|x \in T\} \end{array}$$

and its *normal fan* is the complete simplicial fan in V^* whose maximal cones are the domains of linearity of F. Note that, in dimension greater than 2, not every simplicial fan needs to be the normal fan of a polytope (see for example section 1.5 in [**Ful93**]).

Our next goal is to show that the c-cluster fans we constructed so far are polytopal. In view of Theorem 2.1.4, each function defined on $\Pi(c)$ extends uniquely to a continuous, piecewise linear function on $P_\mathbb{R}$ linear on the maximal cones of $\mathcal{F}_c^\Pi$. In particular, every function

$$f : I \longrightarrow \mathbb{R}$$

satisfying $f(i) = f(i^*)$ gives rise to a continuous, τ_c^Π-invariant, piecewise-linear function $F_c = F_{c;f}$, by setting

$$F_c(c^m\omega_i) := f(i)$$

for all $c^m\omega_i \in \Pi(c)$, and then extending it to $P_\mathbb{R}$ as above.

Let $\mathrm{Asso}_c^f(W)$ be the subset of $P_\mathbb{R}^*$ defined by

$$\mathrm{Asso}_c^f(W) := \{\varphi \in P_\mathbb{R}^* \,|\, \varphi(\lambda) \leq F_c(\lambda), \forall \lambda \in \Pi(c)\}\ . \tag{2.1.4}$$

Theorem 2.1.7. *If* $f : I \to \mathbb{R}$ *is such that*

(1) *for any $i \in I$*

$$f(i) = f(i^*)$$

(2) *for any $j \in I$*

$$\sum_{i \in I} a_{ij} f(i) > 0$$

then $\mathrm{Asso}_c^f(W)$ is a simple n-dimensional polytope with support function F_c. Furthermore, the domains of linearity of F_c are exactly the maximal cones of $\mathcal{F}_c^{\Pi}$, hence the normal fan of $\mathrm{Asso}_c^f(W)$ is $\mathcal{F}_c^{\Pi}$.

Remark 2.1.8. Theorem 2.1.7 is a generalization of Theorem 1.5 in [**CFZ02**]. Its proof uses the result by Chapoton, Fomin, and Zelevinsky as base case.

The following examples illustrate the above results. We represent a point $\varphi \in P_{\mathbb{R}}^*$ by a tuple $(z_i := \varphi(\omega_i))_{i \in I}$. We also use the standard numeration of simple roots and fundamental weights from [**Bou68**].

The construction carried on in this paper, as it will be explained in details in Section 2.3, coincides with the one in [**FZ03b**] and [**CFZ02**] when c is a bipartite Coxeter element. Therefore the first example in which something interesting arises is $c = s_1 s_2 s_3$ in type A_3. In this case $\Pi(c)$ consists of two τ_c^{Π}-orbits:

$$\begin{tikzcd} \omega_1 \arrow[r, "\tau_c"] & -\omega_1 + \omega_2 \arrow[r, "\tau_c"] & -\omega_2 + \omega_3 \arrow[r, "\tau_c"] & -\omega_3 \arrow[r, "\tau_c"] & \omega_3 \arrow[r, "\tau_c"] & -\omega_1 \arrow[lllll, bend left, "\tau_c"] \end{tikzcd}$$

and

$$\begin{tikzcd} \omega_2 \arrow[r, "\tau_c"] & -\omega_1 + \omega_3 \arrow[r, "\tau_c"] & -\omega_2 \arrow[ll, bend left, "\tau_c"] \end{tikzcd}$$

It is not surprising that the number of orbits and their lengths are the same as the A_3 example in [**CFZ02**]: they depend only on the type of the cluster algebra and not on the choice of a Coxeter element. Since in this case $w_0 \omega_1 = -\omega_3$ we have $1^* = 3$ therefore we need to impose $f(1) = f(3)$; condition (2) in Theorem 2.1.7

becomes

$$0 < f(1) < f(2) < 2f(1)$$

and the corresponding polytope $\mathrm{Asso}_c^f(W)$ is defined by the inequalities

$$\max\{z_1, -z_1 + z_2, -z_2 + z_3, -z_3, z_3, -z_1\} \leq f(1)$$

$$\max\{z_2, -z_1 + z_3, -z_2\} \leq f(2)\,.$$

This polytope is shown in Figure 2.1.1. Note that, to make pictures easier to plot and view, the angles between fundamental weights are not drawn to scale, and each facet is labeled by the weight it is orthogonal to.

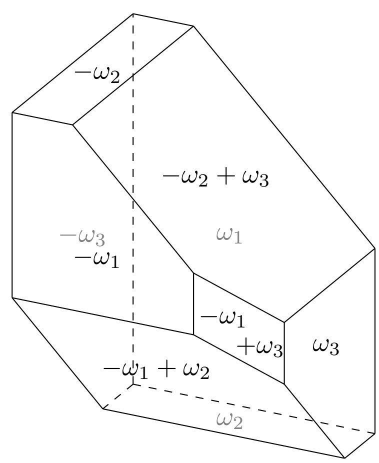


FIGURE 2.1.1. $\mathrm{Asso}_c^f(W)$ in type A_3 for $c = s_1 s_2 s_3$

Now let $c = s_1 s_2 s_3$ in type C_3. Then the set $\Pi(c)$ consists of three orbits:

$$\begin{tikzcd} \omega_1 \arrow[r, "\tau_c"] & -\omega_1 + \omega_2 \arrow[r, "\tau_c"] & -\omega_2 + \omega_3 \arrow[r, "\tau_c"] & -\omega_1 \arrow[lll, bend left, "\tau_c"] \end{tikzcd}$$

$$\begin{tikzcd} \omega_2 \arrow[r, "\tau_c"] & -\omega_1 + \omega_3 \arrow[r, "\tau_c"] & -\omega_1 - \omega_2 + \omega_3 \arrow[r, "\tau_c"] & -\omega_2 \arrow[lll, bend left, "\tau_c"] \end{tikzcd}$$

$$\begin{tikzcd} \omega_3 \arrow[r, "\tau_c"] & -2\omega_1 + \omega_3 \arrow[r, "\tau_c"] & -2\omega_2 + \omega_3 \arrow[r, "\tau_c"] & -\omega_3 \arrow[lll, bend left, "\tau_c"] \end{tikzcd}$$

Condition (2) in Theorem 2.1.7 reads

$$f(2) < 2f(1)$$

$$f(1) + f(3) < 2f(2)$$

$$f(2) < f(3)$$

as in the corresponding example in [**CFZ02**]. The polytope is given by the inequalities

$$\max\{z_1, -z_1 + z_2, -z_2 + z_3, -z_1\} < f(1)$$

$$\max\{z_2, -z_1 + z_3, -z_1 - z_2 + z_3, -z_2\} < f(2)$$

$$\max\{z_3, -2z_1 + z_3, -2z_2 + z_3, -z_3\} < f(3)$$

and it is shown in Figure 2.1.2 using the same conventions of Figure 2.1.1.

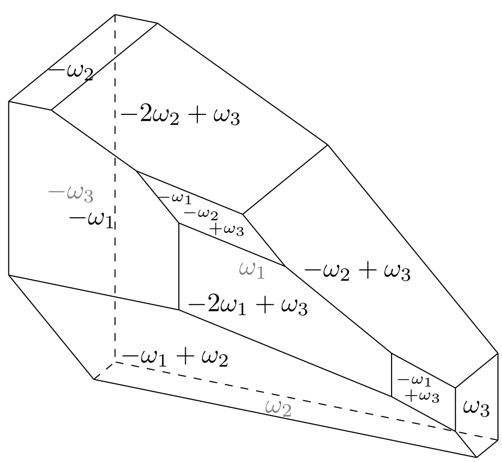


FIGURE 2.1.2. $\mathrm{Asso}_c^f(W)$ in type C_3 for $c = s_1s_2s_3$

To prove the results we discussed so far will use two types of argument. The first one is induction on the rank of I. Unfortunately the set $\Pi(c)$, and in general the whole weight lattice P, does not behave nicely when considering sub-diagrams of I. It is then convenient to introduce an auxiliary set of labels: the *c-almost positive roots*:

$$\Phi_{\mathrm{ap}}(c) := (c^{-1} - 1)\Pi(c)$$

whose behaviour is more manageable. On the one hand the new set is related to the old one by a linear transformation therefore any property proved for $\Phi_{\mathrm{ap}}(c)$ can be transported back to $\Pi(c)$.

On the other hand $\Phi_{\mathrm{ap}}(c)$ is modeled after the set $\Phi_{\geq -1}$ introduced in [**FZ03b**]. It differs from the latter in several respects: first it still consists of g-vectors (in an odd-looking basis) and not denominator vectors; second it contains all the positive roots (as $\Phi_{\geq -1}$ does) but the negative simples are replaced by other negative roots depending on the choice of the Coxeter element c. However, contrary to what happens for $\Pi(c)$, it retains a notion of subset corresponding to a Dynkin sub-diagram. In order to use induction on $|I|$ it will then suffice to show that the c-compatibility degree on $\Phi_{\mathrm{ap}}(c)$ is preserved when restricting to a sub-diagram of I (this is the content of Proposition 2.2.12).

To explain the second type of argument we need an observation on Coxeter elements. For a given Coxeter element c, we call a simple reflection s_i *initial* (resp. *final*) if c admits a reduced expression of the form $c = s_i v$ (resp. $c = v s_i$). Conjugating any Coxeter element by an initial or final reflection produces another Coxeter element; call such a conjugation an *elementary move* and call two Coxeter elements related by a single elementary move *adjacent*. The following is a well known fact; a proof can be found in [**GP00**] Theorem 3.1.4.

Lemma 2.1.9. *Any Coxeter element can be reached from any other via a sequence of elementary moves.*

We will construct maps $\sigma_i^{\pm 1}$ relating sets of c-almost positive roots for adjacent Coxeter elements. These maps will not be linear so, a priori, they might not preserve all the properties we are interested into. Our strategy will be to show that, for any Coxeter element c, there exists a bipartite Coxeter element t and a sequence of elementary moves relating the two, such that all the corresponding maps $\sigma_i^{\pm 1}$ preserve the desired properties. This will reduce our statements to the bipartite case. Our results will then follow from another important property of the set of c-almost positive roots: when the Coxeter element is bipartite, there exists bijection

$$t_- : \Phi_{\geq -1} \to \Phi_{\mathrm{ap}}(t)$$

which is induced by a linear map. This will allow us, in this particular case, to deduce our results from their analogs from [**FZ03b**] and [**YZ08**].

As a byproduct of the construction we get an explicit description of the exchange relations of $\mathcal{A}_0(c)$. Two cluster variables $x_{\lambda,c}$ and $x_{\mu,c}$ in it are exchangeable if and only if $(\lambda||\mu)_c^\Pi = (\mu||\lambda)_c^\Pi = 1$. Denote by T the cyclic group generated by τ_c^Π. The proof of Theorem 2.1.7 relies on the fact that, except in some degenerate cases, for any pair of weights λ and μ in $\Pi(c)$ corresponding to a pair of exchangeable cluster variables, the set

$$\left\{\tau\left(\tau^{-1}(\lambda) + \tau^{-1}(\mu)\right)\right\}_{\tau \in T}$$

consists of two vectors: $\lambda + \mu$ and another one denoted by $\lambda \uplus_c \mu$.

Use Theorem 2.1.4 to label all cluster monomials in $\mathcal{A}_0(c)$ by points of P:

$$x_{\sum m_\lambda \lambda, c} := \prod x_{\lambda,c}^{m_\lambda}.$$

THEOREM 2.1.10. *All the exchange relations in $\mathcal{A}_0(c)$ are of the form*

$$x_{\lambda,c}x_{\mu,c} = x_{\lambda+\mu,c} + x_{\lambda \uplus_c \mu, c}$$

We now discuss the connection of $\mathcal{F}_c^\Pi$ with the Cambrian fan defined in [**Rea06**]. First recall some definitions and results from [**HLT11**].

Let D be the fundamental Weyl chamber, i.e., the $\mathbb{R}_+$-span of the fundamental weights. The *Coxeter fan* $\mathcal{F}$ is the complete simplicial fan in $P_\mathbb{R}$ whose maximal cones are the images of D under the action of W. It is well known that the correspondence

$$w \mapsto w(D)$$

is a bijection between W and the set of maximal cones of $\mathcal{F}$; moreover $\mathcal{F}$ is the normal fan to a distinguished polytope: the permutahedron (see e.g. [**Pos09**]).

Using the (right) weak order, W can be regarded as a lattice with minimal and maximal element e and w_0 respectively. To each lattice congruence on W corresponds a fan that coarsens $\mathcal{F}$ as shown in [**Rea05**]; maximal cones in the new fan are obtained glueing together cones of $\mathcal{F}$ corresponding to elements of W belonging to the same equivalence class.

Fix a Coxeter element c and one of its reduced expressions. For any subset $J \subset I$, denote by c_J the sub-word of c obtained omitting the simple reflections $\{s_i\}_{i\in I\setminus J}$. Let c^∞ be the formal word obtained concatenating infinitely many copies of c. Every reduced expression of $w \in W$ can be seen as a sub-word of c^∞; call the *c-sorting word* of w the lexicographically first sub-word of c^∞ realizing it. The c-sorting word of w can be encoded by a sequence of subsets $I_1, I_2, \dots I_k$ of I (the *c-factorization* of w) so that

$$w = c_{I_1} c_{I_2} \cdots c_{I_k} .$$

Note that the c-factorization of w is independent on the reduced expression chosen for c: it depends only on the Coxeter element itself.

Definition 2.1.11. An element w in W is

- *c-sortable* if its c-factorization is such that

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k$$

- *c-antisortable* if ww_0 is c^{-1}-sortable

As an example pick $c = s_1s_2s_3$ in type A_3, then $s_2s_3s_2$ is c-sortable with the c-factorization $\{2,3\}, \{2\}$, the element $s_2s_3s_1s_2s_1$ is c-antisortable while $s_2s_3s_2s_1$ is neither.

For any element w in W, again in the weak order, there exist a unique minimal c-antisortable element above it and a unique maximal c-sortable below it; denote them by $\pi_c^{\uparrow}(w)$ and $\pi_{\downarrow}^c(w)$ respectively.

Proposition 2.1.12 (cf. [**Rea07**]). *For any $w \in W$ the sets*

$$\left(\pi_{\downarrow}^c\right)^{-1}\left(\pi_{\downarrow}^c(w)\right)$$

and

$$\left(\pi_c^{\uparrow}\right)^{-1}\left(\pi_c^{\uparrow}(w)\right)$$

coincide; they are intervals in the lattice W with minimal element $\pi_{\downarrow}^c(w)$ and maximal element $\pi_c^{\uparrow}(w)$.

Define a lattice congruence on W by setting

$$v \sim w \quad \Leftrightarrow \quad \pi_{\downarrow}^c(v) = \pi_{\downarrow}^c(w)\,. \tag{2.1.5}$$

The c-Cambrian fan $\mathcal{F}_c^C$ (defined in [**RS09**]) is the complete simplicial fan obtained from $\mathcal{F}$ by coarsening with respect to the lattice congruence (2.1.5); its maximal cones are parametrized by c-sortable elements.

In [**HLT11**] it was shown that, for any point a in the fundamental Weyl chamber, there is a unique simple polytope $\mathrm{Asso}_c^a(W)$ with normal fan $\mathcal{F}_c^C$ and such that a is a vertex of $\mathrm{Asso}_c^a(W)$.

We have now all the required notations to state our last result.

Theorem 2.1.13. *For every $f : I \to \mathbb{R}$ satisfying the hypothesis of Theorem 2.1.7 there exists a point $a \in D$ such that the polytopes $\mathrm{Asso}_c^a(W)$ and $\mathrm{Asso}_c^f(W)$ coincide.*

As a direct consequence we get

Corollary 2.1.14. *The c-Cambrian fan $\mathcal{F}_c^C$ and the c-cluster fan $\mathcal{F}_c^{\Pi}$ coincide.*

2.2. The set $\Phi_{\mathbf{ap}}(c)$

Fix a Dynkin diagram I and let $\Phi = \Phi_+ \sqcup \Phi_-$ be the corresponding root system. For convenience we identify I with $\{1, \dots n\}$ so that a chosen Coxeter element is $c = s_1 \cdots s_n$.

For $i \in I$ let

$$\beta_i^c := s_n \cdots s_{i+1} \alpha_i \,. \tag{2.2.1}$$

Remark 2.2.1. It is known that the roots (2.2.1) are exactly the positive roots that are mapped into negative roots by c; moreover they form a $\mathbb{Z}$-basis of the root lattice Q since the linear map sending each α_i to β_i^c is unitriangular.

We call $\Phi_{\mathrm{ap}}(c) := \Phi_+ \cup \{-\beta_i^c\}_{i \in I}$ the set of the *c-almost-positive roots* and define a bijection $\tau_c^\Phi : \Phi_{\mathrm{ap}}(c) \to \Phi_{\mathrm{ap}}(c)$ by setting, for $\alpha \in \Phi_{\mathrm{ap}}(c)$,

$$\tau_c^\Phi(\alpha) := \begin{cases} -\beta_i^c & \text{if } \alpha = \beta_i^c \\ c\alpha & \text{otherwise}\,. \end{cases}$$

Definition 2.2.2. The *c-compatibility degree* on $\Phi_{\mathrm{ap}}(c)$ is the unique τ_c^Φ-invariant function

$$(\bullet||\bullet)_c^\Phi : \Phi_{\mathrm{ap}}(c) \times \Phi_{\mathrm{ap}}(c) \longrightarrow \mathbb{N}$$

defined by the initial conditions

$$(-\beta_i^c||\alpha)_c^\Phi := [\alpha; \alpha_i]_+ \ .$$

These definitions are justified by the following proposition.

Proposition 2.2.3. *The linear map*

$$\phi_c := (c^{-1} - 1) : P_{\mathbb{R}} \longrightarrow Q_{\mathbb{R}}$$

is invertible and restricts to an isomorphism of the weight lattice P with the root lattice Q sending $\Pi(c)$ to $\Phi_{ap}(c)$. Moreover ϕ_c intertwines τ_c^Π and τ_c^Φ and transform *the compatibility degree* $(\bullet||\bullet)^{\Pi}_{c}$ *on* $\Pi(c)$ *into the compatibility degree* $(\bullet||\bullet)^{\Phi}_{c}$ *on* $\Phi_{ap}(c)$.

Proof. To show that ϕ_c is a lattice isomorphism, in view of Remark 2.2.1, it suffices to establish that

$$\phi_c(\omega_i) = -\beta_i^c\,.$$

Using the well-known property

$$s_i\omega_j = \begin{cases} \omega_i - \alpha_i & \text{if } i = j \\ \omega_j & \text{otherwise,} \end{cases}$$

we have

$$\phi_c(\omega_i) = s_n \cdots s_1\omega_i - \omega_i = s_n \cdots s_{i+1}(s_i\omega_i - \omega_i) = s_n \cdots s_{i+1}(-\alpha_i) = -\beta_i^c.$$

The sets $\Pi(c)$ and $\Phi_{\mathrm{ap}}(c)$ have the same cardinality. Indeed Proposition 1.7 in [**YZ08**] states that, for every i, the sum $h(i,c) + h(i^*,c)$ is equal to the Coxeter number h, hence

$$|\Pi(c)| = \sum_{i\in I}(h(i,c)+1) = \frac{1}{2}\sum_{i\in I}(h(i,c)+h(i^*,c)+2) = \frac{1}{2}\sum_{i\in I}(h+2) = |\Phi_{\mathrm{ap}}(c)|\,.$$

To conclude the proof of the first part it suffices to check that any weight in $\Pi(c)\setminus\{\omega_i\}_{i\in I}$ is mapped to a positive root. This was already showed in [**YZ08**] during the proof of the inequalities (1.8) in it.

To show that, for any $\alpha \in \Phi_{\mathrm{ap}}(c)$,

$$\phi_c^{-1}\left(\tau_c^{\Phi}(\alpha)\right) = \tau_c^{\Pi}\left(\phi_c^{-1}(\alpha)\right)$$

there are two cases to consider:

(1) if $\alpha = \beta_i^c$ then

$$\phi_c^{-1}\left(\tau_c^{\Phi}(\beta_i^c)\right) = \phi_c^{-1}\left(-\beta_i^c\right) = \omega_i = \tau_c^{\Pi}(-\omega_i) = \tau_c^{\Pi}\left(\phi_c^{-1}\left(\beta_i^c\right)\right)$$

(2) if $\alpha \neq \beta_i^c$ for any i then

$$\phi_c^{-1}\left(\tau_c^{\Phi}(\alpha)\right) = \phi_c^{-1}(c\alpha) = (c^{-1}-1)^{-1}c\alpha = c(c^{-1}-1)^{-1}\alpha = \tau_c^{\Pi}\left(\phi_c^{-1}(\alpha)\right) .$$

To conclude the proof it is sufficient to show that both compatibility degrees satisfy the same initial conditions. On the one hand we have

$$(-\beta_i^c||\alpha)_c^{\Phi} = [\alpha;\alpha_i]_+$$

and on the other

$$\left(\phi_c^{-1}(-\beta_i^c)\,||\phi_c^{-1}(\alpha)\right)_c^{\Pi} = \left(\omega_i||\phi_c^{-1}(\alpha)\right)_c^{\Pi} = \left[(c^{-1}-1)(c^{-1}-1)^{-1}\alpha;\alpha_i\right]_+ .$$

□

Remark 2.2.4. As in the case of $\Pi(c)$ the action of τ_c^{Φ} on $\Phi_{\mathrm{ap}}(c)$ and the action of w_0 on I are compatible, i.e. there exist $m \in \mathbb{Z}$ such that

$$\left(\tau_c^{\Phi}\right)^m(-\beta_i^c) = -\beta_j^c$$

if and only if $j = i$ or $j = i^*$.

We can now rephrase Theorems 2.1.2, 2.1.4, 2.1.7, 2.1.10, and Proposition 2.1.6 in this new setup.

Let Δ_c^{Φ} be the abstract simplicial complex having elements of $\Phi_{\mathrm{ap}}(c)$ as vertices and with subsets of pairwise compatible roots as simplices; similarly to the case of $\Pi(c)$, we call *c-clusters* the maximal (by inclusion) simplices.

In view of Proposition 2.2.3, Theorem 2.1.2 is equivalent to the following.

THEOREM 2.2.5. *Each c-cluster in Δ_c^{Φ} is a $\mathbb{Z}$-basis of the root lattice Q.*

Definition 2.2.6. For any γ in Q we call a *c-cluster expansion* of γ an expression

$$\gamma = \sum_{\alpha \in \Phi_{\mathrm{ap}}(c)} m_\alpha \alpha$$

where all the coefficients m_α are nonnegative integers such that $m_\alpha m_\delta = 0$ whenever $(\alpha||\delta)^\Phi_c \neq 0$.

The counterpart of Proposition 2.1.6 is the following:

Proposition 2.2.7. *Any γ in the root lattice Q admits a unique c-cluster expansion.*

Remark 2.2.8. Our proof of Proposition 2.2.7 will mimic, step by step, the proof of Theorem 3.11 in [**CFZ02**]. A sketch of a different proof, more similar to the others in this paper, will be also given.

Let $\mathcal{F}^\Phi_c$ be the set of all the cones in the space $Q_\mathbb{R}$ that are the positive linear span of simplices of the complex Δ^Φ_c. A direct consequence of Proposition 2.2.7 is the following counterpart of Theorem 2.1.4.

THEOREM 2.2.9. *$\mathcal{F}^\Phi_c$ is a complete simplicial fan.*

As for the case of $\Pi(c)$, once Theorem 2.2.9 is established, any function defined on $\Phi_{\mathrm{ap}}(c)$ can be extended to a continuous, piecewise linear function on $Q_\mathbb{R}$ that is linear on the maximal cones of $\mathcal{F}^\Phi_c$. In particular, any function

$$f : I \longrightarrow \mathbb{R}$$

such that $f(i) = f(i^*)$ gives rise to a τ^Φ_c-invariant, continuous, piecewise-linear function

$$F_c = F_{c;f} : Q_\mathbb{R} \longrightarrow \mathbb{R}$$

by setting

$$F_c(-\beta^c_i) := f(i)$$

and extending, first to $\Phi_{\mathrm{ap}}(c)$ and then to $Q_\mathbb{R}$, as prescribed.

Let $\mathrm{Asso}^{f,\Phi}_c(W)$ be the subset of $Q^*_\mathbb{R}$ defined by

$$\mathrm{Asso}^{f,\Phi}_c(W) := \{\varphi \in Q^*_\mathbb{R} \mid \varphi(\alpha) \leq F_c(\alpha) \quad \forall \alpha \in \Phi_{\mathrm{ap}}(c)\} \ . \tag{2.2.2}$$

Theorem 2.2.10. *If $f : I \to \mathbb{R}$ is such that*

(1) *for any $i \in I$*

$$f(i) = f(i^*)$$

(2) *for any $j \in J$*

$$\sum_{i \in I} a_{ij} f(i) > 0$$

then $\mathrm{Asso}_c^{f,\Phi}(W)$ is a simple n-dimensional polytope with support function F_c. Furthermore, the domains of linearity of F_c are exactly the maximal cones of $\mathcal{F}_c^\Phi$, hence the normal fan of $\mathrm{Asso}_c^{f,\Phi}(W)$ is $\mathcal{F}_c^\Phi$.

Again by Proposition 2.2.3, Theorem 2.2.10 implies Theorem 2.1.7.

The proof of Theorem 2.2.10 is based on an explicit characterization of the roots in $\Phi_{\mathrm{ap}}(c)$ belonging to adjacent maximal cones of $\mathcal{F}_c^\Phi$. Namely there exist two c-cluster C_α and C_γ such that $C_\alpha \setminus \{\alpha\} = C_\gamma \setminus \{\gamma\}$ if and only if

$$(\alpha||\gamma)_c^\Phi = 1 = (\gamma||\alpha)_c^\Phi$$

(Cf. Lemma 2.5.10). For all such pairs of roots the set

$$\left\{ \left(\tau_c^\Phi\right)^{-m} \left(\left(\tau_c^\Phi\right)^m (\alpha) + \left(\tau_c^\Phi\right)^m (\beta) \right) \right\}_{m \in \mathbb{Z}}$$

consists (when I has no connected component with only one node) of precisely two vectors, $\alpha + \gamma$ and $\alpha \uplus_c \gamma$; their c-cluster expansion are supported on $C_\alpha \cup C_\gamma$ and they are disjoint (Cf. Proposition 2.5.7 and Corollary 2.5.9).

Let $\mathcal{A}_0(c)$ the coefficient-free cluster algebra with initial orientation given by c; label its cluster variables by roots in $\Phi_{\mathrm{ap}}(c)$ and, in view of Proposition 2.2.7, its cluster monomials by points in the root lattice. Using this notation Theorem 2.1.10 can be restated as follows.

Theorem 2.2.11. *All the exchange relations in $\mathcal{A}_0(c)$ are of the form*

$$x_{\alpha,c} x_{\gamma,c} = x_{\alpha+\gamma,c} + x_{\alpha \uplus_c \gamma,c}$$

for suitable c-almost positive roots such that

$$(\alpha||\gamma)_c^\Phi = 1 = (\gamma||\alpha)_c^\Phi.$$

As mentioned before the main advantage of the labels $\Phi_{\mathrm{ap}}(c)$ over $\Pi(c)$ is that it is easier to set up inductions on $|I|$. Let $J \subset I$ be a sub diagram of I. Fix a Coxeter element c for I and denote by c_J the sub-word of c obtained omitting all the simple reflections $\{s_i\}_{i \in I \setminus J}$. By construction c_J is a Coxeter element in the Weyl group W_J (we denote by W_J the standard parabolic subgroup of W generated by $\{s_j\}_{j \in J}$). Let

$$\iota : \Phi_{\mathrm{ap}}^J(c_J) \longrightarrow \Phi_{\mathrm{ap}}(c)$$

be the "twisted" inclusion map given by

$$\iota(\alpha) := \begin{cases} -\beta_i^c & \text{if } \alpha = -\beta_i^{c_J},\, i \in J \\ \alpha & \text{otherwise.} \end{cases} \tag{2.2.3}$$

From this moment on, unless it is not clear from the context, superscripts Φ and Π will be omitted in order to make notation less heavy.

Denote by $(\bullet||\bullet)_{c_J}^J$ the c_J-compatibility degree on $\Phi_{\mathrm{ap}}^J(c_J)$. The key property is this:

Proposition 2.2.12. *Let α and γ be roots in $\Phi_{ap}^J(c_J)$. Then*

$$(\iota(\alpha)||\iota(\gamma))_c = (\alpha||\gamma)_{c_J}^J \ .$$

Remark 2.2.13. In the setup of almost positive roots the analog of this statement is point 3 of Proposition 3.3 in [**FZ03b**]; there the map ι is the ordinary inclusion. A proof of Proposition 2.2.12 will be given in Section 2.4.

The original construction in [**FZ03b**] does not distinguish among the possible bipartite orientations of I. With this motivation in mind consider the map $\alpha \mapsto \overline{\alpha}$

between $\Phi_{\mathrm{ap}}(c)$ and $\Phi_{\mathrm{ap}}(c^{-1})$ defined by

$$\overline{\alpha} := \begin{cases} -\beta_i^{c^{-1}} & \text{if } \alpha = -\beta_i^c,\, i \in I \\ \alpha & \text{otherwise}\,. \end{cases} \tag{2.2.4}$$

Proposition 2.2.14. *For any α and γ in $\Phi_{ap}(c)$*

$$(\alpha||\gamma)_c = (\overline{\alpha}||\overline{\gamma})_{c^{-1}}\ .$$

Proof. Initial conditions agree:

$$(-\beta_i^c||\alpha)_c = [\alpha;\alpha_i]_+ = [\overline{\alpha};\alpha_i]_+ = \left(-\beta_i^{c^{-1}}||\overline{\alpha}\right)_{c^{-1}} = \left(\overline{-\beta_i^c}||\overline{\alpha}\right)_{c^{-1}}\ .$$

It suffices then to show that, for any $\alpha \in \Phi_{\mathrm{ap}}(c)$

$$\overline{\tau_c(\alpha)} = \tau_{c^{-1}}^{-1}(\overline{\alpha})\,.$$

There are three cases to be considered.

(1) If $\alpha = -\beta_i^c$ for some $i \in I$ then on the one hand

$$\overline{\tau_c(-\beta_i^c)} = \overline{-c\beta_i^c} = \overline{-s_1\cdots s_n(s_n\cdots s_{i+1}\alpha_i)} = \overline{s_1\cdots s_{i-1}\alpha_i} = \beta_i^{c^{-1}};$$

on the other hand

$$\tau_{c^{-1}}^{-1}\left(\overline{-\beta_i^c}\right) = \tau_{c^{-1}}^{-1}\left(-\beta_i^{c^{-1}}\right) = \beta_i^{c^{-1}}\ .$$

(2) When $\alpha = \beta_i^c$

$$\overline{\tau_c(\beta_i^c)} = \overline{-\beta_i^c} = -\beta_i^{c^{-1}} = s_1\cdots s_i\alpha_i$$

multiplying and dividing by $s_{i+1}\cdots s_n$ we get

$$s_1\cdots s_i(s_{i+1}\cdots s_n s_n\cdots s_{i+1})\alpha_i = \left(c^{-1}\right)^{-1}\beta_i^c = \tau_{c^{-1}}^{-1}\left(\beta_i^c\right) = \tau_{c^{-1}}^{-1}\left(\overline{\beta_i^c}\right)\ .$$

(3) Finally for $\alpha \neq \pm\beta_i^c$

$$\overline{\tau_c(\alpha)} = \overline{c\alpha} = c\alpha = (c^{-1})^{-1}\alpha = \tau_{c^{-1}}^{-1}(\overline{\alpha})\,.$$

□

2.3. The bipartite case

In this section we assume that the Dynkin diagram I is connected; the statements in the general case are easily reduced to this. Since any connected Dynkin diagram I is a tree, we can split I into two disjoint subsets I_+ and I_- such that every edge in I has one endpoint in I_+ and one in I_-. Up to relabeling, this can be done in a unique way. A bipartite Coxeter can thus be written as

$$t = t_\varepsilon t_{-\varepsilon} \tag{2.3.1}$$

where ε denotes a sign and

$$t_\varepsilon := \prod_{i\in I_\varepsilon} s_i$$

(the expression makes sense since the factors commute with each other). By our assumption there are precisely two bipartite Coxeter elements in W: $t = t_+t_-$ and $t^{-1} = t_-t_+$.

Let $\Phi_{\geq -1}$ be the set of *almost positive roots*, i.e.

$$\Phi_{\geq -1} := \Phi_+ \cup \{-\alpha_i\}_{i\in I}$$

introduced in [**FZ03b**] to parametrize cluster variables in the special case of bipartite initial cluster. On it there are two involutions τ_+ and τ_- defined by

$$\tau_\varepsilon(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_i \text{ and } i \in I_{-\varepsilon} \\ t_\varepsilon\alpha & \text{otherwise} \end{cases}$$

and a unique $\{\tau_+, \tau_-\}$-invariant compatibility degree function $(\bullet||\bullet)_{\geq -1}$ satisfying

$$(-\alpha_i||\gamma)_{\geq -1} = [\gamma;\alpha_i]_+\,.$$

Call two almost positive roots α and γ *compatible* if

$$(\alpha||\gamma)_{\geq -1} = 0 = (\gamma||\alpha)_{\geq -1} \ . \tag{2.3.2}$$

The *cluster complex* $\Delta_{\geq -1}$ is the abstract simplicial complex induced on $\Phi_{\geq -1}$ by the compatibility degree function; its simplices are subsets of pairwise compatible almost positive roots. As before call the maximal simplices *clusters* and consider the set $\mathcal{F}_{\geq -1}$ of all simplicial cones generated by simplices.

As we mentioned in the introduction our construction is based on the results for the bipartite case given in [**FZ03b**] and [**CFZ02**]. From the first paper we will need the following.

Proposition 2.3.1.

(1) [Proposition 3.3 (2)] *For any pair of almost positive roots α and γ, we have*

$$(\alpha||\gamma)_{\geq -1} = 0$$

if and only if $(\gamma||\alpha)_{\geq -1} = 0$.

(2) [Proposition 3.3 (3)] *Let J be a subset of I and denote by $\Phi^J_{\geq -1}$ the corresponding set of almost positive roots. Let α and γ be roots in $\Phi^J_{\geq -1}$, then*

$$(\alpha||\gamma)_{\geq -1} = (\alpha||\gamma)^J_{\geq -1}$$

where $(\bullet||\bullet)^J_{\geq -1}$ denotes the compatibility degree function on $\Phi^J_{\geq -1}$.

(3) [Theorem 1.8] *Each cluster in the cluster complex is a $\mathbb{Z}$-basis of the root lattice Q.*

(4) [Theorem 3.11] *Any $\gamma \in Q$ admits a unique* cluster expansion. *In other words γ can uniquely be written as*

$$\gamma = \sum_{\alpha \in \Phi_{\geq -1}} m_\alpha \alpha$$

so that all the coefficients m_α are negative integers and $m_\alpha m_{\alpha'} = 0$ if $(\alpha||\alpha')_{\geq -1} \neq 0$.

(5) [Theorem 1.10] $\mathcal{F}_{\geq -1}$ *is a complete simplicial fan in* $Q_{\mathbb{R}}$.

The results we will need form [**CFZ02**] can be summarized as follows.

Proposition 2.3.2. *Suppose that I has at least 2 vertices. Let α and γ be almost positive roots such that $(\alpha||\gamma)_{\geq -1} = 1 = (\gamma||\alpha)_{\geq -1}$. Then we have:*

(1) [Theorem 1.14] *The set*

$$\left\{\tau\left(\tau^{-1}(\alpha) + \tau^{-1}(\gamma)\right)\right\}_{\tau \in T}$$

where T denotes the group generated by τ_+ and τ_-, consists of exactly two elements $\alpha + \gamma$ and $\alpha \uplus \gamma$.

(2) [Lemma 2.3] *Any root appearing with a positive coefficient in the cluster expansion of $\alpha + \gamma$ or $\alpha \uplus \gamma$ is compatible with both α, γ, and with any other root compatible with both α and γ.*

(3) [Lemma 2.4] *Let $f : I \to \mathbb{R}_+$ be any function such that, for any $i \in I$,*

$$f(i) = f(i^*)$$

and

$$\sum_{i \in I} a_{ij} f(i) > 0$$

for any $j \in I$. Let $F_{\geq -1} : Q_{\mathbb{R}} \to \mathbb{R}$ be a continuous piecewise-linear function on $Q_{\mathbb{R}}$ that is linear on the maximal cones of $\mathcal{F}_{\geq -1}$, invariant under the action of T, and such that

$$F_{\geq -1}(-\alpha_i) = f(i)\,.$$

Then,

$$F_{\geq -1}(\alpha) + F_{\geq -1}(\gamma) > \max\left\{F_{\geq -1}(\alpha + \gamma), F_{\geq -1}(\alpha \uplus \gamma)\right\}\,.$$

The statements regarding $\alpha \uplus \gamma$ are not expressed explicitly in [**CFZ02**] but can be recovered immediately from the corresponding statements about $\alpha + \gamma$. Indeed, let τ be such that

$$\alpha \uplus \gamma = \tau^{-1}(\tau(\alpha) + \tau(\gamma)) .$$

Any root appearing with positive coefficient in the cluster expansion of $\tau(\alpha \uplus \gamma)$ is compatible with $\tau(\alpha)$, $\tau(\gamma)$ and with any root compatible with both. Since τ preserves the compatibility degree we get 2. Similarly for 3:

$$F_{\geq -1}(\tau(\alpha)) + F_{\geq -1}(\tau(\gamma)) > F_{\geq -1}\left(\tau(\alpha) + \tau(\gamma)\right) = F_{\geq -1}(\tau(\alpha \uplus \gamma))$$

since $F_{\geq -1}$ is invariant under the action of τ we can conclude

$$F_{\geq -1}(\alpha) + F_{\geq -1}(\gamma) > F_{\geq -1}(\alpha \uplus \gamma) .$$

We need to translate the above results to $\Phi_{\text{ap}}(t)$; in order to do so we need a bijection between $\Phi_{\geq -1}$ and $\Phi_{\text{ap}}(t)$ induced by a linear map. Note that Proposition 2.2.3 together with Lemma 5.2 in [**YZ08**] already provide a bijection but it is not induced by a linear map.

Note also that, for a bipartite Coxeter element $t = t_+ t_-$, the negative roots in $\Phi_{\text{ap}}(t)$ are the roots $-\beta_i^t$ given by

$$-\beta_i^t = \begin{cases} -\alpha_i & i \in I_- \\ -t_-\alpha_i & i \in I_+ . \end{cases}$$

Proposition 2.3.3. *The linear involution*

$$t_- : Q_{\mathbb{R}} \longrightarrow Q_{\mathbb{R}}$$

restricts to an automorphism of Q and to a bijection

$$t_- : \Phi_{\geq -1} \longrightarrow \Phi_{ap}(t) .$$

Proof. It suffices to show that, for any root α in $\Phi_{\geq -1}$, we have $t_-(\alpha) \in \Phi_{\mathrm{ap}}(t)$. Let us first deal with roots whose image is negative.

- If $i \in I_+$, then
$$t_-(-\alpha_i) = -t_-\alpha_i = -\beta_i^t\,.$$
- If $i \in I_-$, then
$$t_-(\alpha_i) = s_i\alpha_i = -\alpha_i = -\beta_i^t\,.$$

This also shows that $t_-(-\alpha_i) = \beta_i^t$ if $i \in I_-$. For any other root α in $\Phi_{\geq -1}$, that is, for any positive root not in $\{\alpha_i\}_{i\in I_-}$, the image $t_-(\alpha)$ is positive since the support of any positive root is a connected sub-diagram of the Dynkin diagram and t_- sends any root to itself plus a linear combination of simple roots indexed by I_-.

□

Proposition 2.3.4. *The map t_- intertwines τ_t with $\tau_-\tau_+$ and preserves compatibility degree. In other words, for any almost positive roots α and γ, we have*
$$\tau_t(t_-\alpha) = t_-\tau_-\tau_+(\alpha)$$
and
$$(\alpha||\gamma)_{\geq -1} = (t_-\alpha||t_-\gamma)_t$$

Proof. Proceed by direct inspection;

- if $t_-\alpha = \alpha_i = \beta_i^t$ for $i \in I_-$, that is if $\alpha = -\alpha_i$ with $i \in I_-$, then
$$\tau_t(t_-(-\alpha_i)) = \tau_t(\beta_i^t) = -\beta_i^t = -\alpha_i = t_-\alpha_i = t_-\tau_-\tau_+(-\alpha_i)$$
- if $t_-\alpha = \beta_i^t$ for $i \in I_+$, i.e. if $\alpha = \alpha_i$ with $i \in I_+$, then
$$\tau_t(t_-\alpha_i) = \tau_t(\beta_i^t) = -\beta_i^t = -t_-\alpha_i = t_-(-\alpha_i) = t_-\tau_-\tau_+(\alpha_i)$$
- in any other case
$$\tau_t(t_-\alpha) = tt_-\alpha = t_+\alpha = t_-t_-t_+\alpha = t_-\tau_-\tau_+(\alpha)\,.$$

To conclude the proof it is enough to show that

$$(-\alpha_i||\gamma)_{\geq -1} = (-t_-\alpha_i||t_-\gamma)_t$$

for any $\gamma \in \Phi_{\geq -1}$ and any $i \in I$. If i is in I_+ then

$$(-\alpha_i||\gamma)_{\geq -1} = [\gamma; \alpha_i]_+ = [t_-\gamma; \alpha_i]_+ = \left(-\beta_i^t||t_-\gamma\right)_t = (t_-(-\alpha_i)||t_-\gamma)_t$$

where the second equality holds because t_- does not contain s_i. If $i \in I_-$ then, on the one hand we have $(-\alpha_i||\gamma)_{\geq -1} = [\gamma; \alpha_i]_+$ on the other

$$(t_-(-\alpha_i)||t_-\gamma)_t = (\alpha_i||t_-\gamma)_t = (\tau_t\alpha_i||\tau_t t_-\gamma)_t = \left(-\beta_i^t||\tau_t t_-\gamma\right)_t = [\tau_t t_-\gamma; \alpha_i]_+ .$$

Now there are three cases:

(1) if γ is α_j with $j \in I_+$ then $\tau_t t_-\gamma = \tau_t(\beta_j^t) = -\beta_j^t$ and

$$[\alpha_j; \alpha_i]_+ = 0 = \left[-\beta_j^t; \alpha_i\right]_+ .$$

(2) If γ is $-\alpha_j$ with $j \in I_-$ then

$$\tau_t t_-\gamma = \tau_t(\beta_j^t) = -\beta_j^t = -\alpha_j = \gamma .$$

(3) For any other γ we have $\tau_t t_-\gamma = tt_-\gamma = t_+\gamma$ and

$$[t_+\gamma; \alpha_i]_+ = [\gamma; \alpha_i]_+$$

since s_i does not appear in t_+.

□

Since the map t_- is linear, all the properties of $\Phi_{\geq -1}$ translate to $\Phi_{\mathrm{ap}}(t)$:

Corollary 2.3.5.

(1) *For any α and γ in $\Phi_{ap}(t)$ we have $(\alpha||\gamma)_t = 0$ if and only if $(\gamma||\alpha)_t = 0$.*

(2) *For any $J \subset I$ and any pair of roots α and γ in $\Phi_{ap}^J(t_J)$*

$$(\iota\alpha||\iota\gamma)_t = (\alpha||\gamma)_{t_J}^J .$$

(3) *Each t-cluster in the simplicial complex Δ_t^Φ is a $\mathbb{Z}$-basis of the root lattice Q.*

(4) *Any $\gamma \in Q$ admits a unique t-cluster expansion. That is, γ can be uniquely written as*

$$\gamma = \sum_{\alpha \in \Phi_{ap}(t)} m_\alpha \alpha$$

so that all the coefficients m_α are non negative integers and $m_\alpha m_{\alpha'} = 0$ whenever $(\alpha||\alpha')_t \neq 0$.

(5) *The set $\mathcal{F}_t$ is a complete simplicial fan in $Q_\mathbb{R}$.*

Proof. The only non trivial claim is (2); it is enough to show that it holds when $I \setminus J = \{j\}$. Since t is bipartite, s_j is either initial or final (cf. (2.3.1)). Using Proposition 2.2.14 we can assume it is initial, i.e. $j \in I_+$. We have then

$$(\iota\alpha||\iota\gamma)_t = (t_-\iota\alpha||t_-\iota\gamma)_{\geq -1} = (t_-\alpha||t_-\gamma)^J_{\geq -1} = (\alpha||\gamma)^J_{t_J}$$

where the second equality holds since, for any root α in $\Phi^J_{\mathrm{ap}}(t_J)$,

$$[t_-\iota(\alpha); \alpha_j] = 0$$

hence $t_-\iota(\alpha)$ is in Φ^J. Indeed if α is positive then $t_-\alpha$ contains α_j only if α does since s_j does not appear in t_-; if $\alpha = -\beta_i^{t_J}$ with $i \in I_-$ then $t_-\iota(-\beta_i^{t_J}) = \alpha_i$ and finally if $\alpha = -\beta_i^{t_J}$ with $i \in I_+ \setminus \{j\}$ then $t_-\iota(-\beta_i^{t_J}) = -\alpha_i$.

□

Corollary 2.3.6. *Suppose that I has at least 2 vertices. If α and γ in $\Phi_{ap}(t)$ are such that $(\alpha||\gamma)_t = 1 = (\gamma||\alpha)_t$, then*

(1) *The set*

$$\left\{\tau_t^m\left(\tau_t^{-m}(\alpha) + \tau_t^{-m}(\gamma)\right)\right\}_{m\in\mathbb{Z}}$$

consists of exactly two elements $\alpha + \gamma$ and $\alpha \uplus_t \gamma$.

(2) *Any root appearing with a positive coefficient in the cluster expansion of* $\alpha + \gamma$ *or* $\alpha \uplus_t \gamma$ *is compatible with both* α, γ, *and with any other root compatible with both* α *and* γ.

(3) *Let* $f : I \to \mathbb{R}_+$ *be any function such that, for any* $i \in I$,

$$f(i) = f(i^*)$$

and

$$\sum_{i \in I} a_{ij} f(i) > 0$$

for any $j \in I$. *Let* $F_t : Q_{\mathbb{R}} \to \mathbb{R}$ *be a continuous piecewise-linear function on* $Q_{\mathbb{R}}$ *that is linear on the maximal cones of* $\mathcal{F}_t$, *invariant under the action of* τ_t, *and such that*

$$F_t(-\beta_i^t) = f(i)\,.$$

Then,

$$F_t(\alpha) + F_t(\gamma) > \max\left\{F_t(\alpha+\gamma), F_t(\alpha \uplus \gamma)\right\}\,.$$

2.4. Some technical results

As anticipated we need to lift elementary moves to the level of $\Phi_{\mathrm{ap}}(c)$. We concentrate first on conjugation by initial simple reflections. Fix the Coxeter element $c = s_1 \cdots s_n$ and consider the bijection

$$\sigma_1 : \Phi_{\mathrm{ap}}(c) \longrightarrow \Phi_{\mathrm{ap}}(s_1 c s_1)$$

defined by

$$\sigma_1(\alpha) := \begin{cases} \alpha_1 (= \beta_1^{s_1 c s_1}) & \text{if } \alpha = -\beta_1^c \\ s_1 \alpha & \text{otherwise.} \end{cases} \tag{2.4.1}$$

Note that σ_1 sends $-\beta_i^c$ to $-\beta_i^{s_1 c s_1}$ for any $i \neq 1$.

Proposition 2.4.1. *The map σ_1 intertwines τ_c and $\tau_{s_1cs_1}$, i.e., for any α in $\Phi_{ap}(c)$, we have*

$$\tau_{s_1cs_1}\left(\sigma_1(\alpha)\right) = \sigma_1\left(\tau_c(\alpha)\right) .$$

Moreover it preserves the compatibility degree, i.e. for any α and γ in $\Phi_{ap}(c)$

$$(\alpha||\gamma)_c = (\sigma_1\alpha||\sigma_1\gamma)_{s_1cs_1} .$$

PROOF. It suffices to notice that σ_1 is the composition

$$\Phi_{\mathrm{ap}}(c) \xrightarrow{\phi_c^{-1}} \Pi(c) \xrightarrow{\psi^{-1}_{s_1cs_1,c}} \Pi(s_1cs_1) \xrightarrow{\phi_{s_1cs_1}} \Phi_{\mathrm{ap}}(s_1cs_1)$$

where $\psi^{-1}_{s_1cs_1,c}$ is the bijection

$$\psi^{-1}_{s_1cs_1,c}(\lambda) := \begin{cases} -\omega_1 & \text{if } \lambda = \omega_1 \\ s_1\lambda & \text{otherwise} \end{cases}$$

defined by Lemma 5.3 in [**YZ08**] and ϕ_c is the map of Proposition 2.2.3. Indeed, if $\alpha \neq \alpha_1$ then

$$\phi_{s_1cs_1} \circ \psi^{-1}_{s_1cs_1,c} \circ \phi_c^{-1}\left(\alpha\right) = (s_1c^{-1}s_1 - 1)s_1(c^{-1}-1)^{-1}\alpha = s_1\alpha$$

and

$$\phi_{s_1cs_1} \circ \psi^{-1}_{s_1cs_1,c} \circ \phi_c^{-1}\left(\alpha_1\right) = (s_1c^{-1}s_1 - 1)(-\omega_1) = \alpha_1 .$$

The bijection σ_1 satisfies the desired property because all the maps that define it do.

$\square$

To use simultaneously induction on the rank of I and elementary moves we need to prove some type of compatibility between σ_1 and ι. It suffices to inspect their interaction in the case when ι is induced removing only one node (say i) from I.

Proposition 2.4.2. *For $J = I \setminus \{i\}$ and $i \neq 1$ let c_J be the Coxeter element of W_J obtained by deleting s_i from c and let σ_1^J be the map corresponding to the conjugation by s_1 in W_J. For any root α in $\Phi_{ap}^J(c_J)$ we have*

$$\sigma_1(\iota_c(\alpha)) = \iota_{s_1 c s_1}(\sigma_1^J(\alpha)) .$$

Proof. There are three cases to be considered.

(1) If $\alpha = -\beta_1^{c_J}$ then

$$\sigma_1\left(\iota_c(-\beta_1^{c_J})\right) = \sigma_1\left(-\beta_1^c\right) = \alpha_1 = \iota_{s_1 c s_1}(\alpha_1) = \iota_{s_1 c s_1}\left(\sigma_1^J(-\beta_1^{c_J})\right) .$$

(2) If $\alpha = -\beta_j^{c_J}$ and $j \neq 1$ then $\iota_c(-\beta_j^{c_J}) = -\beta_j^c$ therefore

$$\sigma_1\left(\iota_c(-\beta_j^{c_J})\right) = -\beta_j^{s_1 c s_1} = \iota_{s_1 c s_1}\left(-\beta_j^{s_1 c_J s_1}\right) = \iota_{s_1 c s_1}\left(\sigma_1^J(-\beta_j^{c_J})\right) .$$

(3) If α is positive

$$\sigma_1\left(\iota_c(\alpha)\right) = \sigma_1(\alpha) = s_1\alpha = \iota_{s_1 c s_1}\left(s_1\alpha\right) = \iota_{s_1 c s_1}\left(\sigma_1^J(\alpha)\right) .$$

The last equality holds since, α being positive, $s_1\alpha \neq \alpha_1$ and the third because if $s_1\alpha$ is not positive then $\alpha = \alpha_1$ and $-\beta_1^{s_1 c s_1} = -\beta_1^{s_1 c_J s_1} = -\alpha_1$.

□

Remark 2.4.3. The definition of σ_1 can be replicated to get the maps σ_i corresponding to conjugation by any initial simple reflection s_i. It is clear that, to get the maps corresponding to elementary moves that conjugate c by a final simple reflection, it suffices to consider the inverses σ_i^{-1}.

As a first application of the elementary moves let us show that the definition of c-compatible pair of roots make sense.

Lemma 2.4.4. *For any α and γ in $\Phi_{ap}(c)$*

$$(\alpha||\gamma)_c = 0 \Leftrightarrow (\gamma||\alpha)_c = 0 .$$

Proof. If c is bipartite the statement is true by point 1 in Corollary 2.3.5. It is then enough to show that the property is preserved under $\sigma_i^{\pm 1}$. Suppose that it holds for $c = s_1 \cdots s_n$. If $(\alpha||\gamma)_{s_1 c s_1} = 0$ then

$$\left(\sigma_1^{-1}\alpha||\sigma_1^{-1}\gamma\right)_c = 0 = \left(\sigma_1^{-1}\gamma||\sigma_1^{-1}\alpha\right)_c .$$

Therefore $(\gamma||\alpha)_{s_1 c s_1} = 0$.

□

Our next goal is to show that "distant" roots are compatible. We need to introduce some terminology. For any positive root α define its support to be the set

$$\text{Supp}(\alpha) := \left\{ i \in I \,\middle|\, [\alpha;\alpha_i] \neq 0 \right\} \tag{2.4.2}$$

and extend the definition to $\Phi_{\text{ap}}(c)$ declaring

$$\text{Supp}(-\beta_i^c) := \{i\} . \tag{2.4.3}$$

Remark 2.4.5. If α and γ are roots with supports contained in two different connected components of I then $(\alpha||\gamma)_c = 0$ since τ_c preserves connected components.

We can improve on Remark 2.4.5.

Definition 2.4.6. Call two roots α and γ *spaced* if, for any $i \in \text{Supp}(\alpha)$ and for any $j \in \text{Supp}(\gamma)$, $a_{ij} = 0$.

Remark 2.4.7. Note that if $(-\beta_i^c||\alpha)_c \neq 0$ then α and $-\beta_i^c$ are not spaced.

Proposition 2.4.8. *Let α and γ be roots in $\Phi_{ap}(c)$. If α and γ are spaced then*

$$(\alpha||\gamma)_c = 0.$$

Proof. Using Remark 2.4.5 we can assume that I is connected. If any of α and γ is a negative root we are done by Lemma 2.4.4 and Remark 2.4.7. Let then both α and γ be positive roots.

Supports of positive roots are connected subgraphs of the Dynkin diagram I. Since α and γ are spaced, there must exist at least one vertex on the shortest path connecting $\mathrm{Supp}(\alpha)$ and $\mathrm{Supp}(\gamma)$ not belonging to either of the supports. Let i be the nearest to $\mathrm{Supp}(\alpha)$ of such vertices. Let I' be a connected component of $I \setminus \{i\}$ of type A and containing one of the two support; there exist such a component because we are in finite type. Assume α is the root whose support is contained in I' (the other case is identical). We will proceed by induction on the cardinality of I'.

Let j be the only vertex in $\mathrm{Supp}(\alpha)$ connected to i. Without loss of generality we can assume $j \prec_c i$, i.e., s_j precedes s_i in any reduced expression of c. If this is not the case we can use Proposition 2.2.14 since two roots are spaced if and only if their images under the involution $\delta \mapsto \overline{\delta}$ are spaced.

Apply τ_c^{-1} to both α and γ; they are positive so τ_c^{-1} acts as c^{-1} on them. By construction we have

$$\mathrm{Supp}\left(\tau_c^{-1}\alpha\right) \subseteq I' \setminus \{j\}$$

and

$$\mathrm{Supp}\left(\tau_c^{-1}\gamma\right) \subseteq (I \setminus I') \cup \{i\}$$

where both relations hold since s_j is applied before s_i and α belongs to a type A component of I . If one among $\tau_c^{-1}\alpha$ and $\tau_c^{-1}\gamma$ is negative we are done (again using Lemma 2.4.4 if needed) otherwise the statement follows by induction on $|I'|$.

□

To complete the proof of Proposition 2.2.12 we need to sharpen Lemma 2.1.9. From this moment on we will denote a word on the alphabet $\{s_i\}_{i \in I}$ (up to commutations) by $\mathbf{w}$; the corresponding element in W will be denoted by w. For convenience we will record a sequence of elementary moves by the corresponding word. As an example in type A_4 (again using the standard numeration of simple roots from [**Bou68**]) the sequence of elementary moves

$$s_1s_2s_3s_4 \to s_2s_3s_4s_1 \to s_1s_3s_4s_2$$

will be encoded by $\mathbf{w} = s_2s_1$; indeed

$$(s_2s_1)(s_1s_2s_3s_4)(s_2s_1)^{-1} = s_1s_3s_4s_2 \,.$$

The key observation is given by the following Lemma.

Lemma 2.4.9. *For any pair of Coxeter elements c and c' and for any $i \in I$, there exist a sequence of elementary moves connecting c and c' that does not contain s_i.*

Proof. The result is obvious once we notice that both the sequences of simple moves $\mathbf{w} = s_1$ and $\mathbf{w}' = s_2 \cdots s_n$ acts in the same way on $c = s_1 \cdots s_n$.

□

The Dynkin diagram I is in general a forest. For any leaf i in I, i.e., for any node belonging to a single edge, denote by $i_\#$ the only other node of I connected to i.

Lemma 2.4.10. *For any leaf $i \in I$ and for any Coxeter element c there exist a bipartite Coxeter element t and a sequence of elementary moves $\mathbf{w}$ such that*

(1) $c = \mathbf{w}t\mathbf{w}^{-1}$
(2) $\mathbf{w}$ *contains neither s_i, nor $s_{i_\#}$.*

Proof. According to Lemma 2.4.9 we can find a sequence of elementary moves $\mathbf{w}$ not containing $s_{i_\#}$ that transform c into either of the bipartite Coxeter elements. By construction s_i commutes with all reflections appearing in $\mathbf{w}$ since it commutes with all the simple reflections except $s_{i_\#}$. Therefore, using commutations relations, we can always find such a $\mathbf{w}$ containing at most one copy of s_i. Choosing now a bipartite Coxeter element t in which s_i and $s_{i_\#}$ appear in the same order in which they appear in c we have that $\mathbf{w}$ does not contain s_i.

□

Let i be a leaf of I and c be any Coxeter element. Let t and $\mathbf{w}$ be respectively the bipartite Coxeter element and the sequence of elementary moves constructed in Lemma 2.4.10. To fix ideas suppose that s_i appears on the left of $s_{i_\#}$ in c (and in t); the other case can be dealt with exactly in the same way but multiplying on the right instead of on the left. Denote by c_J and t_J the corresponding Coxeter elements for the Dynkin sub diagram $J = I \setminus \{i\}$. By our assumption $c_J = s_i c$ and $t_J = s_i t$.

Lemma 2.4.11. *In the notation just established* $\mathbf{w}$ *is a sequence of elementary moves in* W_J *conjugating* t_J *and* c_J*.*

Proof.

$$c_J = s_i c = s_i \mathbf{w} t \mathbf{w}^{-1} = \mathbf{w} s_i t \mathbf{w}^{-1} = \mathbf{w} t_J \mathbf{w}^{-1}.$$

□

We now have all the required tools to prove Proposition 2.2.12.

Proof of Proposition 2.2.12. We can assume, without loss of generality, I to be irreducible. It suffices to show that the result holds when J is obtained from I removing one node i. Let α and γ be roots in $\Phi_{\mathrm{ap}}(c_J)$. There are two cases to consider depending on the relative position of $\mathrm{Supp}(\alpha)$, $\mathrm{Supp}(\gamma)$ and i.

(1) If $\mathrm{Supp}(\alpha)$ and $\mathrm{Supp}(\gamma)$ belong to different connected components of J then

$$(\alpha||\gamma)^J_{c_J} = 0 = (\iota(\alpha)||\iota(\gamma))_c\,.$$

The first equality holds because of Remark 2.4.5 and the second one is an instance of Proposition 2.4.8.

(2) If $\mathrm{Supp}(\alpha)$ and $\mathrm{Supp}(\gamma)$ belong to the same connected component of the Dynkin diagram J then we can assume i to be a leaf of I. Let $i_\#$ be the only vertex in I connected to i. By Lemmata 2.4.10 and 2.4.11, there

exist a sequence of elementary moves $\mathbf{w}$ and a bipartite Coxeter element t such that

$$c = \mathbf{w} t \mathbf{w}^{-1}$$

$$c_J = \mathbf{w} t_J \mathbf{w}^{-1}$$

and $\mathbf{w}$ contains neither s_i, nor $s_{i_\#}$. Denote by $\sigma_{\mathbf{w}}$ the composition of the maps σ_i corresponding to $\mathbf{w}$. By construction neither of $\sigma_{\mathbf{w}}\alpha$ and $\sigma_{\mathbf{w}}\gamma$ contains i in its support. Using point 2 of Corollary 2.3.5 we can conclude

$$(\alpha||\gamma)_c = (\sigma_{\mathbf{w}}\alpha||\sigma_{\mathbf{w}}\gamma)_t = (\sigma_{\mathbf{w}}\alpha||\sigma_{\mathbf{w}}\gamma)^J_{t_J} = (\alpha||\gamma)^J_{c_J}.$$

□

2.5. Proof of the main results

To prove Theorem 2.2.5 we will use the following easy observation.

Lemma 2.5.1. *Let $J \subset I$ be a Dynkin sub diagram. There is a bijection between c-clusters in $\Phi_{ap}(c)$ containing $\{-\beta_i^c\}_{i \in I\setminus J}$ and c_J-clusters in $\Phi^J_{ap}(c_J)$.*

The existence of such a bijection is a direct consequence of the fact that, if $-\beta_i^c$ is in a c-cluster C, then for any other root γ in that c-cluster $i \notin \mathrm{Supp}(\gamma)$. In particular any positive root in C is contained in a complement of the space generated by

$$\{-\beta_i^c\}_{-\beta_i^c \in C \cap \Phi_-}.$$

Definition 2.5.2. Call a c-cluster C in $\Phi_{\mathrm{ap}}(c)$ *positive* if $C \subset \Phi_+$.

Proof of Theorem 2.2.5. As already mentioned in Remark 2.2.1, the set

$$\{-\beta_i^c\}_{i\in I}$$

is a $\mathbb{Z}$-basis of Q. By Lemma 2.5.1 it suffices to show that the theorem holds for any given positive cluster C. Apply τ_c^{-1} to C; since it is positive τ_c^{-1} acts on all the roots in it as c^{-1}. Since c^{-1} is a product of reflections, C is a $\mathbb{Z}$-basis if and only if

$\tau_c^{-1}C$ is a $\mathbb{Z}$-basis. Continue to apply τ_c^{-1} until one of the roots is sent to a negative one. This will happen because, similarly to the case of $\Pi(c)$, any τ_c-orbit contains precisely two negative roots $\{-\beta_i^c, -\beta_{i^*}^c\}$ or a single negative root $-\beta_i^c$ if $i = i^*$. Remove the negative root just obtained again using Lemma 2.5.1 and conclude by induction on the rank of the root system.

□

Remark 2.5.3. The proof just proposed is a straightforward adaptation of the proof of Theorem 1.8 in [**FZ03b**] to the new setup of c-almost positive roots.

PROOF OF PROPOSITION 2.2.7. Let $c = s_1 \dots s_n$; for $\gamma \in Q$ write

$$\gamma = -\sum_{i \in I} m_{-\beta_i^c} \beta_i^c + \gamma_+$$

where the coefficients $m_{-\beta_i^c}$ are the non-negative integers uniquely defined (since the change of basis $\alpha_i \mapsto \beta_i^c$ is triangular) by the recursive formula

$$m_{-\beta_i^c} := \left[-\gamma - \sum_{j=1}^{i-1} m_{-\beta_j^c} \beta_j^c ; \alpha_n \right]_+$$

(we use the convention that the empty sum is 0). By construction γ_+, the *positive part* of γ, is in the positive cone of the sub-root lattice generated by

$$\left\{ \alpha_i | m_{-\beta_i^c} = 0 \right\}.$$

Clearly $\gamma \in Q$ has a unique c-cluster expansion if and only if γ_+ does. Without loss of generality we can thus assume that γ is in the positive cone Q_+.

The root $-\beta_i^c$ can appear with a positive coefficient in a c-cluster expansion of γ only if the coefficient $[\gamma; \alpha_i]$ is negative therefore the result holds for $\gamma = 0$ and we can assume $\gamma \neq 0$.

If $\sum_{\alpha \in \Phi_+} m_\alpha \alpha$ is a c-cluster expansion of γ then

$$\tau_c^{-1}\gamma = c^{-1}\gamma = c^{-1}\left(\sum_{\alpha \in \Phi_+} m_\alpha \alpha \right) = \sum_{\alpha \in \Phi_+} m_\alpha c^{-1} \alpha = \sum_{\alpha \in \Phi_+} m_\alpha \tau_{c^{-1}} \alpha$$

is a c-cluster expansion of $\tau_c^{-1}\gamma$. In other words γ has a unique c-cluster expansion if and only if $\tau_c^{-1}\gamma$ does. Applying τ_c^{-1} a sufficient number of times γ can be moved outside of the positive cone. We can then take its positive part and conclude by induction on the rank of the root system.

$\square$

Remark 2.5.4. This proof, as its analog in [**FZ03b**], has the advantage of considering one Coxeter element at a time. An alternative strategy could have been the following. The claim holds for bipartite Coxeter elements by point 4 in Corollary 2.3.5. Using the fact that the maps σ_i preserve compatibility degree one can then transfer the property to other sets of c-almost positive roots.

As in [**FZ03b**], Theorem 2.2.9 follows from Proposition 2.2.7. For the sake of completeness we replicate the proof here.

Proof of Theorem 2.2.9. It suffices to show that

(1) no two cones of $\mathcal{F}_c^\Phi$ have a common interior point
(2) the union of all cones is $Q_\mathbb{R}$.

Assume by contradiction that there exist a point in the common interior of two cones. Since $Q_\mathbb{Q}$ is dense in $Q_\mathbb{R}$ we may assume that such point is in $Q_\mathbb{Q}$. Clearing the denominators there is then a common point in Q which contradicts the uniqueness of the c-cluster expansion. Therefore the interiors of any two cones are disjoint.

Since for any $\gamma \in Q$ there exist a c-cluster expansion the union of all the cones $\mathbb{R}_+ C$ contains Q; since this union is closed in $Q_\mathbb{R}$ and stable under the action of $\mathbb{R}_+$ it must contain all of $Q_\mathbb{R}$ and we are done.

$\square$

To prove Theorem 2.2.10 we will apply the criterion provided by Lemma 2.1 in [**CFZ02**]. Let us restate it in the particular case we need.

Lemma 2.5.5. *Let F_c be a continuous piecewise-linear function*

$$F_c : Q_{\mathbb{R}} \longrightarrow \mathbb{R}$$

linear on the maximal cones of the fan $\mathcal{F}_c^{\Phi}$ (as such F_c is uniquely determined by its values on $\Phi_{ap}(c)$). Then $\mathcal{F}_c^{\Phi}$ is the normal fan to a unique full-dimensional polytope with support function F_c if and only if F_c satisfy the following system of inequalities. For any pair of adjacent c-clusters C_α and C_γ let α be the only root in $C_\alpha \setminus C_\gamma$ and γ the only root in $C_\gamma \setminus C_\alpha$. Let

$$m_\alpha \alpha + m_\gamma \gamma = \sum_{\delta \in C_\alpha \cap C_\gamma} m_\delta \delta \tag{2.5.1}$$

be the unique (up to non-zero scalar multiple) linear dependence on the elements of $C_\alpha \cup C_\gamma$ with m_α and m_γ positive. Then

$$m_\alpha F_c(\alpha) + m_\gamma F_c(\gamma) > \sum_{\delta \in C_\alpha \cap C_\gamma} m_\delta F_c(\delta)\,. \tag{2.5.2}$$

In particular the domains of linearity of F_c are exactly the maximal cones of $\mathcal{F}_c^{\Phi}$.

To apply Lemma 2.5.5 we make the relations (2.5.1) more explicit by exploring the interaction of σ_1 and τ_c. Note that, having established Theorem 2.2.9, any vector-valued function on $\Phi_{\mathrm{ap}}(c)$ can be extended to a continuous piecewise-linear map on $Q_{\mathbb{R}}$; in particular this is the case for τ_c and σ_i.

To avoid degenerate cases, from now on, assume that every connected component of I contains at least 2 vertices. As before, let c be $s_1 \cdots s_n$.

Lemma 2.5.6. *Let α and γ be roots in $\Phi_{ap}(c)$ such that*

$$(\alpha||\gamma)_c = 1 = (\gamma||\alpha)_c\ .$$

Then

$$\sigma_1^{-1}\left(\sigma_1(\alpha) + \sigma_1(\gamma)\right)$$

is either $\alpha+\gamma$ *or*

$$\tau_c\left(\tau_c^{-1}(\alpha)+\tau_c^{-1}(\gamma)\right)$$

and it is different from $\alpha+\gamma$ *only if one of the two roots (say,* α*) is* $-\beta_1^c$*; in this case*

$$\sigma_1^{-1}\left(\sigma_1(-\beta_1^c)+\sigma_1(\gamma)\right)=\gamma-\alpha_1\in Q_+\,.$$

PROOF. If both α and γ are positive roots then

$$\sigma_1^{-1}\left(\sigma_1(\alpha)+\sigma_1(\gamma)\right)=\sigma_1^{-1}\left(s_1\alpha+s_1\gamma\right)=\sigma_1^{-1}\left(s_1\left(\alpha+\gamma\right)\right).$$

Let

$$\alpha+\gamma=\sum_{\delta\in\Phi_{\mathrm{ap}}(c)}m_\delta\delta$$

be the c-cluster expansion of $\alpha+\gamma$; all the roots δ such that $m_\delta\neq 0$ are positive; therefore

$$\sigma_1^{-1}\left(s_1\left(\alpha+\gamma\right)\right)=\sigma_1^{-1}\left(s_1\left(\sum m_\delta\delta\right)\right)=\sigma_1^{-1}\left(\sigma_1\left(\alpha+\gamma\right)\right)=\alpha+\gamma.$$

It remains to consider the case in which one of the two roots is negative (they cannot be both negative); we can, by symmetry, assume α to be the negative one. If $\alpha=-\beta_i^c$ with $i\neq 1$ then

$$\sigma_1^{-1}\left(\sigma_1(-\beta_i^c)+\sigma_1(\gamma)\right)=\sigma_1^{-1}\left(-s_1\beta_i^c+s_1\gamma\right)=\sigma_1^{-1}\left(s_1\left(\gamma-\beta_i^c\right)\right)\,.$$

Let

$$\gamma-\beta_i^c=\sum_{\delta\in\Phi_{\mathrm{ap}}(c)}m_\delta\delta$$

be the c-cluster expansion of $\gamma-\beta_i^c$. None of the roots δ appearing with a positive coefficient is $-\beta_1^c$ since

$$[\gamma-\beta_i^c;\alpha_1]\geq 0$$

($-\beta_1^c$ is the only negative root having α_1 with non-zero coefficient). Therefore on $\gamma - \beta_i^c$ the actions of s_1 and of σ_1 are the same. We get

$$\sigma_1^{-1}\left(\sigma_1(-\beta_i^c) + \sigma_1(\gamma)\right) = \sigma_1^{-1}\left(s_1\left(\gamma - \beta_i^c\right)\right) = \sigma_1^{-1}\left(\sigma_1\left(\gamma - \beta_i^c\right)\right) = \gamma - \beta_i^c\,.$$

Finally if $\alpha = -\beta_1^c$ then on the one hand we have

$$\sigma_1^{-1}\left(\sigma_1(-\beta_1^c) + \sigma(\gamma)\right) = \sigma_1^{-1}\left(\alpha_1 + s_1\gamma\right) = \sigma_1^{-1}\left(s_1(\gamma - \alpha_1)\right)$$

$\gamma - \alpha_1$ is in Q_+ therefore

$$\sigma_1^{-1}\left(s_1(\gamma - \alpha_1)\right) = \sigma_1^{-1}\left(\sigma_1(\gamma - \alpha_1)\right) = \gamma - \alpha_1$$

On the other hand

$$\tau_c\left(\tau_c^{-1}(-\beta_1^c) + \tau_c^{-1}(\gamma)\right) = \tau_c\left(\beta_1^c + c^{-1}\gamma\right) = \tau_c\left(c^{-1}(\gamma - \alpha_1)\right)$$

we can interchange c^{-1} and τ_c^{-1} because $\gamma - \alpha_1$ is in Q_+ and conclude

$$\tau_c\left(c^{-1}(\gamma - \alpha_1)\right) = \tau_c\left(\tau_c^{-1}(\gamma - \alpha_1)\right) = \gamma - \alpha_1.$$

$\square$

Proposition 2.5.7. *Let α and γ be roots in $\Phi_{ap}(c)$ such that*

$$(\alpha||\gamma)_c = 1 = (\gamma||\alpha)_c\,.$$

Then

$$\left\{\tau_c^m\left(\tau_c^{-m}(\alpha) + \tau_c^{-m}(\gamma)\right)\right\}_{m\in\mathbb{Z}}$$

consist of exactly two elements, one is $\alpha+\gamma$; denote the other by $\alpha \uplus_c \gamma$.

PROOF. If c is bipartite there is nothing to prove by point 1 in Corollary 2.3.6. In view of Lemma 2.5.6, for any other Coxeter element $c = s_1 \cdots s_n$ and any integer

m we have

$$\tau_{s_1cs_1}^{m}\left(\tau_{s_1cs_1}^{-m}(\sigma_1(\alpha))+\tau_{s_1cs_1}^{-m}(\sigma_1(\gamma))\right)=\sigma_1\tau_c^m\sigma_1^{-1}\left(\sigma_1\tau_c^{-m}(\alpha)+\sigma_1\tau_c^{-m}(\gamma)\right).$$

For a suitable m' in $\{m, m+1\}$ we get

$$\sigma_1\tau_c^m\sigma_1^{-1}\left(\sigma_1\tau_c^{-m}(\alpha)+\sigma_1\tau_c^{-m}(\gamma)\right)=\sigma_1\tau_c^{m'}\left(\tau_c^{-m'}(\alpha)+\tau_c^{-m'}(\gamma)\right).$$

Therefore

$$\{\sigma_1(\alpha)+\sigma_1(\gamma), \sigma_1(\alpha)\uplus_{s_1cs_1}\sigma_1(\gamma)\}\subseteq\sigma_1\{\alpha+\gamma, \alpha\uplus_c\gamma\}$$

and the claim follows reversing the role of c and s_1cs_1.

□

Remark 2.5.8. Following Remark 1.15 in [**CFZ02**], if I contains a component of type A_1 let α_1 and $-\alpha_1$ be the corresponding roots. In view of Remark 2.4.5 they are both compatible with any root in $\Phi_{\mathrm{ap}}(c)\setminus\{\alpha_1, -\alpha_1\}$. By direct inspection, we have

$$(-\alpha_1||\alpha_1)_c = 1 = (\alpha_1||-\alpha_1)_c$$

In this case their sum is 0 and it is natural to declare $-\alpha_1\uplus_c\alpha_1$ to be 0 too.

Corollary 2.5.9. *If*

$$(\alpha||\gamma)_c = 1 = (\gamma||\alpha)_c$$

then every root appearing with positive coefficient in the cluster expansion of either $\alpha+\gamma$ or $\alpha\uplus_c\gamma$ is compatible with α, γ and with any other root compatible with both α and γ.

Proof. The statement is true in the bipartite case by point 2 in Corollary 2.3.6. For an arbitrary Coxeter element $c = s_1\cdots s_n$ the result can be deduced using elementary moves: from the previous proof we have

$$\{\sigma_1(\alpha)+\sigma_1(\gamma), \sigma_1(\alpha)\uplus_{s_1cs_1}\alpha_1(\gamma)\}=\sigma_1\left(\{\alpha+\gamma, \alpha\uplus_c\gamma\}\right) \tag{2.5.3}$$

and the claim follows since σ_1 preserves compatibility degrees.

$\square$

Lemma 2.5.10. *In every dependence relation (2.5.1) we have*

$$(\alpha||\gamma)_c = 1 = (\gamma||\alpha)_c \,. \tag{2.5.4}$$

Furthermore, after normalization the relation (2.5.1) is just the c-cluster expansion of $\alpha+\gamma$:

$$\alpha+\gamma = \sum_{\delta\in\Phi_{ap}(c)} m_\delta \delta \,.$$

Proof. Normalize (2.5.1) so that coefficients are coprime integers. By Theorem 2.2.5 all the coefficients in

$$\alpha = -\frac{m_\gamma}{m_\alpha}\gamma + \sum_{\delta\in C_\alpha\cap C_\gamma} \frac{m_\delta}{m_\alpha}\delta$$

are integers forcing $m_\alpha = 1$ (it is positive by hypothesis). In a similar fashion $m_\gamma = 1$.

To show (2.5.4), using the τ_c-invariance of the compatibility degree, it suffices to consider the case $\alpha = -\beta_i^c$. We have

$$\gamma = \beta_i^c + \sum_{\delta\in C_{-\beta_i^c}\cap C_\gamma} m_\delta \delta$$

and thus

$$(-\beta_i^c||\gamma)_c = [\gamma;\alpha_i]_+ = 1$$

since i is not in $\mathrm{Supp}(\delta)$ for any δ in $C_{-\beta_i^c}\cap C_\gamma$.

The fact that, after the normalization, the dependence (2.5.1) is the c-cluster expansion of $\alpha+\gamma$ is a direct application of Corollary 2.5.9. Any root appearing with non zero coefficient in the cluster expansion of $\alpha+\gamma$ is compatible with α, γ, and with any other root compatible with both α and γ, therefore it is a root in $C_\alpha\cap C_\gamma$.

□

Proposition 2.5.7 together with Corollary 2.5.9 and Lemma 2.5.10 allow us to compute exchange relations. Let $\mathcal{A}_0(c)$ be the coefficient-free cluster algebra with initial exchange matrix $B(c)$ and denote by $\{x_{\alpha,c}\}_{\alpha\in\Phi_{\mathrm{ap}}(c)}$ its cluster variables. Due to Proposition 2.2.7 all the cluster monomials are in bijection with points of Q. Namely we can write

$$x_{\gamma,c} := \prod_{\delta\in\Phi_{\mathrm{ap}}(c)} x_{\delta,c}^{m_\delta}$$

where

$$\gamma = \sum_{\delta\in\Phi_{\mathrm{ap}}(c)} m_\delta \delta$$

is the cluster expansion of $\gamma \in Q$.

Proof of Theorem 2.2.11. The statement is true when c is a bipartite Coxeter element (cf. (5.1) in [**FZ03a**]). Let $c = s_1 \cdots s_n$. We have

$$\begin{aligned}
x_{\alpha,s_1cs_1}x_{\gamma,s_1cs_1} &= x_{\sigma_1^{-1}(\alpha),c}x_{\sigma_1^{-1}(\gamma),c}\\
&= x_{\sigma_1^{-1}(\alpha)+\sigma_1^{-1}(\gamma),c} + x_{\sigma_1^{-1}(\alpha)\uplus_c\sigma_1^{-1}(\gamma),c}\\
&= x_{\sigma_1(\sigma_1^{-1}(\alpha)+\sigma_1^{-1}(\gamma)),s_1cs_1} + x_{\sigma_1(\sigma_1^{-1}(\alpha)\uplus_c\sigma_1^{-1}(\gamma)),s_1cs_1}\\
&= x_{\alpha+\gamma,s_1cs_1} + x_{\alpha\uplus_{s_1cs_1}\gamma,s_1cs_1}.
\end{aligned}$$

□

Recall Remark 2.2.4: by construction of the map τ_c, there is one τ_c-orbit in $\Phi_{\mathrm{ap}}(c)$ for each w_0-orbit in I, i.e., there exist $-\beta_j^c$ such that

$$\tau_c^m(-\beta_i^c) = -\beta_j^c$$

if and only if $j \in \{i, i^*\}$.

Since σ_j sends $-\beta_i^c$ to $\left\{\pm\beta_i^{s_jcs_j}\right\}$ the τ_c-orbit of $-\beta_i^c$ gets mapped to the $\tau_{s_jcs_j}$-orbit of $-\beta_i^{s_jcs_j}$. In particular, for any function

$$f : I \longrightarrow \mathbb{R}$$

such that

$$f(i) = f(i^*)$$

we get a family of maps, one for each Coxeter element c,

$$F_c = F_{c;f} : \Phi_{\mathrm{ap}}(c) \longrightarrow \mathbb{R}$$

defined setting $F_c(-\beta_i^c) := f(i)$ and extending by τ_c-invariance. These maps are invariant under the action of σ_i, that is

$$F_{s_i c s_i}(\sigma_i(\alpha)) = F_c(\alpha) \tag{2.5.5}$$

for any c, any i initial in c, and any α in $\Phi_{\mathrm{ap}}(c)$. From now on assume that F_c has been defined in this way and extend it to a continuous, piecewise-linear function

$$F_c : Q_{\mathbb{R}} \longrightarrow \mathbb{R}$$

linear on maximal cones of $\mathcal{F}_c$.

Proposition 2.5.11. *Fix any function*

$$f : I \longrightarrow \mathbb{R}$$

such that

(1) *for any* $i \in I$
$$f(i) = f(i^*)$$

(2) *for any* $j \in I$
$$\sum_{i \in I} a_{ij} f(i) > 0\,.$$

Then for any pair of roots α *and* γ *in* $\Phi_{ap}(c)$ *such that*

$$(\alpha||\gamma)_c = 1 = (\gamma||\alpha)_c$$

the following inequality holds

$$F_c(\alpha) + F_c(\gamma) > \max\left\{F_c(\alpha+\gamma), F_c(\alpha \uplus_c \gamma)\right\} .$$

Proof. The bipartite case was taken care of by point 3 in Corollary 2.3.6. Let $c = s_1 \cdots s_n$ be any Coxeter element. Using elementary moves, (2.5.3) and (2.5.5) we get

$$\begin{aligned}
F_{s_1 c s_1}(\sigma_1(\alpha)) + F_{s_1 c s_1}(\sigma_1(\gamma)) &= F_c(\alpha) + F_c(\gamma) \\
&> \max\left\{F_c(\alpha+\gamma), F_c(\alpha \uplus_c \gamma)\right\} \\
&= \max\left\{F_{s_1 c s_1}(\sigma_1(\alpha+\gamma)), F_{s_1 c s_1}(\sigma_1(\alpha \uplus_c \gamma))\right\} \\
&= \max\left\{F_{s_1 c s_1}(\sigma_1(\alpha)+\sigma_1(\gamma)), F_{s_1 c s_1}(\sigma_1(\alpha) \uplus_{s_1 c s_1} \sigma_1(\gamma))\right\}
\end{aligned}$$

as desired.

□

Proof of Theorem 2.2.10. It is enough to note that the Proposition 2.5.11 together with Lemma 2.5.10 satisfy the requirements of Lemma 2.5.5.

□

2.6. Relation between $\mathcal{F}_c^{\Pi}$ and the c-Cambrian fan $\mathcal{F}_c^{C}$

We start by recalling some results and terminology from [**HLT11**].

Definition 2.6.1. (cf. Proposition 1.1 in [**HLT11**]) Fix a Coxeter element c and call an element $w \in W$ a *c-singleton* if w is both c-sortable and c-antisortable.

Note that both w_0 and the identity element of W are c-singletons for any choice of c. Denote by $\mathbf{w}_0$ the c-sorting word of w_0.

Theorem 2.6.2. *(cf. Theorem 1.2 in* [**HLT11**]*) An element $w \in W$ is a c-singleton if and only if it has a reduced expression which is a prefix of $\mathbf{w}_0$ up to commutations.*

THEOREM 2.6.3. *(cf. Theorem 2.6 in* [**HLT11**]*) For any ray ρ of $\mathcal{F}_c^C$, there exist a unique fundamental weight ω_i and a (non unique) c-singleton w such that*

$$\rho = \mathbb{R}_+ \cdot w\omega_i \,.$$

Conversely for any c-singleton w and any fundamental weight ω_i, the weight $w\omega_i$, lies on a ray of $\mathcal{F}_c^C$.

We will use Theorems 2.6.2 and 2.6.3 to relate the rays of $\mathcal{F}_c^C$ to the elements of the set $\Pi(c)$.

Definition 2.6.4. Given a Coxeter element $c \in W$, we call a reduced expression $c = s_1 \cdots s_n$ *greedy* if

$$h(i,c) \geq h(j,c)$$

whenever $i < j$.

Lemma 2.6.5. *Any Coxeter element c admits a greedy reduced expression.*

PROOF. Consider a reduced expression $s_1 \cdots s_n$ for c and suppose $h(i,c) < h(i+1,c)$ for some i. Let i be the minimal index with this property. Using Proposition 1.6 in [**YZ08**] we can deduce that i and $i+1$ are not connected in the Coxeter graph. Indeed if they were connected then we would have $i \prec_c i+1$ and thus $h(i,c) \geq h(i+1,c)$ which is in contradiction with our assumption. Therefore s_i and s_{i+1} commute and $s_1 \cdots s_{i+1}s_i \cdots s_n$ is another reduced expression for c; we can now conclude by induction.

$\square$

Remark 2.6.6. Greedy reduced expressions, in general, are not unique; for example $s_2s_4s_1s_3$ and $s_4s_2s_1s_3$ are both greedy reduced expression of the same Coxeter element in type A_4 (again we used the standard numeration of roots from [**Bou68**]).

Lemma 2.6.7. *For any vertices i and j of the Dynkin diagram at distance d from each other the difference $h(i,c) - h(j,c)$ is at most d.*

PROOF. It is enough to observe that if i and j are adjacent then either $i \prec_c j$ or $j \prec_c i$ so $|h(i,c) - h(j,c)| < 1$ by Proposition 1.6 in [**YZ08**]. Therefore each step on the minimal path in I connecting i and j contribute at most 1 to the difference $h(i,c) - h(j,c)$.

$\square$

Fix a greedy reduced expression for c. With some abuse of notation, we denote this expression also as c. Denote by $\mathbf{w}_m$ the sub-word of c^m obtained by omitting in the l-th copy of c all the transpositions s_i such that $h(i,c) < l$. Observe that having taken a greedy reduced expression for c, if we write $I_1, \dots I_m$ for the c-factorization of $\mathbf{w}_m$, then

$$I_1 \supseteq I_2 \cdots \supseteq I_m.$$

In particular if $\mathbf{w}_m$ is a reduced word then w_m, the corresponding element of W, is c-sortable. Let $m_c = \max_{i \in I}\{h(i,c)\}$, our goal is to show that the word $\mathbf{w}_{m_c}$ is a reduced expression for w_0.

Proposition 2.6.8. *For any $i \in I$ and any $m \leq h(i,c)$ we have $c^m \omega_i = \mathbf{w}_m \omega_i$.*

PROOF. Let $I_1, \dots I_m$ be the c-factorization of $\mathbf{w}_m$ with respect to the fixed greedy reduced expression of c. Observe that, for any j appearing in I_{l+1} and for any k missing from I_l

$$|h(k,c) - h(j,c)| \geq 2$$

and so, by Lemma 2.6.7, s_k and s_j commute. Consider now the element

$$\mathbf{w} = c_{I \setminus I_1} c_{I \setminus I_2} \cdots c_{I \setminus I_m}$$

Since $m \leq h(i,c)$, the reflection s_i will not appear in $\mathbf{w}$ and so $\mathbf{w}\omega_i = \omega_i$ hence $\mathbf{w}_m \mathbf{w} \omega_i = \mathbf{w}_m \omega_i$. Form the previous consideration we can move all the elements in the l-th copy of c in $\mathbf{w}$ up to the l-th block of $\mathbf{w}_m$ and obtain

$$c^m \omega_i = c_{I_1} c_{I \setminus I_1} \cdots c_{I_m} c_{I \setminus I_m} \omega_i = \mathbf{w}_m \mathbf{w} \omega_i = \mathbf{w}_m \omega_i\,.$$

$\square$

Proposition 2.6.9. $\mathbf{w}_{m_c}$ *is a reduced expression of* w_0*.*

Proof. To show that $\mathbf{w}_{m_c}$ is an expression of w_0 it is enough to show that both w_0 and $\mathbf{w}_{m_c}$ act in the same way on the weight space (the representation of W as reflection group of $P_{\mathbb{R}}$ is faithful). Fundamental weights form a basis of the weight space so it is enough to see how w_0 and $\mathbf{w}_{m_c}$ act on them. For any i we have $w_0\omega_i = -\omega_{i^*}$. On the other hand, using Proposition 2.6.8, we conclude that

$$\mathbf{w}_{m_c}\omega_i = \mathbf{w}_{h(i,c)}\omega_i = c^{h(i,c)}\omega_i = -\omega_{i^*}\,.$$

Therefore $\mathbf{w}_{m_c}$ is a word representing w_0. The fact that it is a reduced expression follows from considerations on its length; each reflection s_i appears exactly $h(i,c)$ times in it. Proposition 1.7 in [**YZ08**] states that, for every i, the sum $h(i,c)+h(i^*,c)$ is equal to the Coxeter number h, hence

$$\sum_{i\in I}\left(h(i,c)+h(i^*,c)\right) = |I|h = |\Phi|$$

but in this way we are counting the contribution of each i twice, i.e.

$$l(\mathbf{w}_{m_c}) \le \sum_{i\in I} h(i,c) = \frac{1}{2}\sum_{i\in I}\left(h(i,c)+h(i^*,c)\right) = \frac{1}{2}|\Phi| = |\Phi_+| = l(w_0)\,.$$

$\square$

Note that, in view of last Proposition, for any $m \le m_c$, $\mathbf{w}_m$ is a reduced expression in W (and w_m is c-sortable).

Proposition 2.6.10. *Fix a greedy reduced expression for* c*. Then* $\mathbf{w}_{m_c}$ *is the lexicographically first reduced expression of* w_0 *as a sub-word of* c^∞*. In other word* $\mathbf{w}_{m_c}$ *is the* c*-sorting word of* w_0*.*

Proof. It is enough to show that $w_m\alpha_i$ is a negative root for any i not in I_m. We have

$$0 < (\alpha_i, \omega_i) = (w_m\alpha_i, w_m\omega_i) = (w_m\alpha_i, w_0\omega_i) = (w_m\alpha_i, -\omega_{i^*})$$

thus $(w_m\alpha_i, \omega_{i^*}) < 0$ and so $w_m\alpha_i$ is a negative root.

□

Remark 2.6.11. Combining together Theorem 2.6.2 and Proposition 2.6.10 we get another characterization of c-singletons: they are all the prefixes of $\mathbf{w}_{m_c}$ up to commutations.

Proposition 2.6.12. *The sets of rays of $\mathcal{F}_c^{\Pi}$ and $\mathcal{F}_c^{C}$ coincide.*

Proof. Fix a greedy reduced expression for c. Let ρ be a ray of $\mathcal{F}_c^C$. By Theorem 2.6.3 there exist a c-singleton w and a fundamental weight ω_i such that

$$\rho = \mathbb{R}_+ w\omega_i.$$

Let m be the minimum integer such that w is a prefix of $\mathbf{w}_m$. By Proposition 2.6.8

$$w\omega_i = \mathbf{w}_m\omega_i = c^m\omega_i \in \Pi(c).$$

On the other hand, given any element $c^m\omega_i$ of $\Pi(c)$, let $\mathbf{w}_m$ be the corresponding sub-word of c^m as in Proposition 2.6.8; it is a c-singleton therefore $c^m\omega_i = \mathbf{w}_m\omega_i$ is a point on a ray of $\mathcal{F}_c^C$ by Theorem 2.6.3.

□

We can now define the polytope $\mathrm{Asso}_c^a(W)$. For any point a in $P_{\mathbb{R}}$ and for any ray ρ of $\mathcal{F}_c^C$ such that $\rho = \mathbb{R}_+ \cdot w\omega_j$, denote by $\mathcal{H}_\rho^a$ the half-space

$$\mathcal{H}_\rho^a := \left\{\varphi \in P_{\mathbb{R}}^* \mid \varphi(w\omega_j) \le (a, \omega_j)\right\} .$$

The main result in [**HLT11**] is that, if a lies in the interior of the fundamental Weyl chamber, the intersection of half-spaces

$$\mathrm{Asso}_c^a(W) := \bigcap \mathcal{H}_\rho^a$$

as ρ runs over all rays of $\mathcal{F}_c^C$ is a simple polytope and its normal fan is $\mathcal{F}_c^C$.

Proof of Theorem 2.1.13. In view of Proposition 2.6.12, the two polytopes become

$$\mathrm{Asso}_c^a(W) = \{\varphi \in P_{\mathbb{R}}^* \mid \varphi(c^m\omega_i) \leq (a, \omega_i)\, \forall i \in I, 0 \leq m \leq h(i,c)\}$$

and

$$\mathrm{Asso}_c^f(W) = \{\varphi \in P_{\mathbb{R}}^* \mid \varphi(c^m\omega_i) \leq f(i)\, \forall i \in I, 0 \leq m \leq h(i,c)\}\ .$$

For any function $f : I \longrightarrow \mathbb{R}$ let a be the point in $P_{\mathbb{R}}$ defined by the conditions

$$(a, \omega_i) := f(i)$$

for all $i \in I$. Imposing condition 2 of Theorem 2.1.7 on f is equivalent to ask for a to lie in the fundamental Weyl chamber; indeed a is in it if and only if the scalar product (α_j, a) is positive for every $j \in I$. Since $\alpha_j = \sum_{i \in I} a_{ij}\omega_i$ we have

$$(\alpha_j, a) = (\sum_{i \in I} a_{ij}\omega_i, a) = \sum_{i \in I} a_{ij}(a, \omega_i) = \sum_{i \in I} a_{ij} f(i) > 0\,.$$

We can thus conclude that, for any function $f : I \longrightarrow \mathbb{R}$ satisfying conditions 1 and 2 of Theorem 2.1.7, choosing a as above, we get

$$\mathrm{Asso}_c^a(W) = \mathrm{Asso}_c^f(W)\,.$$

$\square$

Remark 2.6.13. It is clear that, imposing condition 1 of Theorem 2.1.7, from our construction we get only the polytopes from [**HLT11**] obtained from points a invariant under the action of $-w_0$.

CHAPTER 3

c- and d-vectors

3.1. More background

We begin our analysis by observing that both c- and d-vectors satisfies some recurrence relation (a tropicalized version of (1.1.2) and (1.1.3) respectively).

Given a seed $(B, \mathbf{y}, \mathbf{x})$ in a cluster algebra with principal coefficients $\mathcal{A}_\bullet(B_0)$, the *D-matrix* associated to it is the integer matrix $D = (d_{ij})_{ij \in I}$ whose columns are the d-vectors of the cluster variables in $\mathbf{x}$. If $(B', \mathbf{y}', \mathbf{x}')$ is obtained from $(B, \mathbf{y}, \mathbf{x})$ mutating in direction k then its D-matrix $D' = (d'_{ij})_{ij \in I}$ is defined by the rule

$$d'_{ij} = \begin{cases} -d_{ik} + \max\left(\sum_{\ell \in I} d_{i\ell}[b_{\ell k}]_+, \sum_{\ell \in I} d_{i\ell}[-b_{\ell k}]_+\right) & j = k \\ d_{ij} & j \neq k \end{cases} \tag{3.1.1}$$

which can be deduced immediately from (1.1.3).

Similarly the *C-matrix* of $(B, \mathbf{y}, \mathbf{x})$ is the integer matrix $C = (c_{ij})_{ij \in I}$ having as columns the exponent vectors of the elements of $\mathbf{y}$. Using (1.1.2) it easy to see that the entries of the C-matrix $C' = (c'_{ij})_{ij \in I}$ of the mutated seed satisfy

$$c'_{ij} = \begin{cases} -c_{ij} & j = k \\ c_{ij} + c_{ik}[b_{kj}]_+ + [-c_{ik}]_+ b_{kj} & j \neq k. \end{cases} \tag{3.1.2}$$

3.1.1. Sign-coherence Conjecture.

Fomin and Zelevinsky made the following fundamental conjecture on c- and d-vectors, which plays an important role in the structure theory of cluster algebras (e.g., [**FZ07, NZ12**]).

Conjecture 3.1.1 (Sign-coherence Conjecture)**.** *Let B_0 be any skew-symmetrizable matrix.*

(i) [**FZ07**, Conjecture 5.5 & Proposition 5.6] *Any c-vector of $\mathcal{A}_\bullet(B_0)$ is a nonzero vector, and its components are either all non-negative or all non-positive.*

(ii) [**FZ07**, Conjectures 7.4 & 7.5] *Any non-initial d-vector of $\mathcal{A}_\bullet(B_0)$ is a nonzero vector, and its components are all nonnegative.*

The first part of the conjecture is equivalent to the fact that *the constant term of any F-polynomial of $\mathcal{A}_\bullet(B_0)$ is one* [**FZ07**], which is proved for any skew-symmetric matrix B_0 [**DWZ10a, Nag10, Pla11**], and also for a large class of skew-symmetrizable matrices [**Dem10**], in particular, for any skew-symmetrizable matrix which is mutation equivalent to an acyclic one.

The second part of the conjecture is proved, for example, for any skew-symmetric matrix B_0 arising from a surface [**FST08**], and more cases follow from the results in the rest of this subsection.

3.1.2. Root Conjecture. Recall that a skew-symmetric matrix $B = (b_{ij})_{i,j\in I}$ can be identified with a quiver $Q(B)$ without loops and 2-cycles by attaching b_{ij} arrows from vertex i to vertex j if $b_{ij} > 0$. This correspondence can be extended to the one between *skew-symmetrizable* matrices and *valued quivers* (see [**DR76**]).

Let $\Delta(A)$ be the *root system* associated with a symmetrizable Cartan matrix A, and let $\{\alpha_i\}_{i\in I}$ be its simple roots [**Kac90**]. A root $\alpha = \sum_{i\in I} c_i\alpha_i$ of $\Delta(A)$ is naturally identified with, either all nonnegative or all non-positive, nonzero integer vector $(c_i)_{i\in I}$. It is said to be *real* if there is an element w of the Weyl group of $\Delta(A)$ such that $w(\alpha)$ is a simple root; otherwise it is said to be *imaginary*. It is known that α is a real root if and only if $(\alpha,\alpha)_{TA} = {}^t\alpha TA\alpha > 0$, where T is any diagonal matrix with positive diagonal entries such that TA is symmetric. See [**Kac90**] for details.

In the study of cluster algebras, it becomes more and more apparent that there is some intimate interplay among three kinds of algebras, namely, cluster algebras, path algebras, and (quantized) Kac-Moody algebras. Naturally, root systems provide the common underlying structure. The starting point of the interplay is Kac's theorem, which generalizes celebrated Gabriel's theorem. Let k be an algebraically closed field below.

THEOREM 3.1.2 (**Kac's Theorem** [**Kac80, Kac82**]). *Let B_0 be any skew-symmetric matrix. Then, there exists an indecomposable module of the path algebra $kQ(B_0)$ with dimension vector α if and only if α is a positive root of $\Delta(A(B_0))$.*

In the above correspondence, if a positive root is the dimension vector of some indecomposable $kQ(B_0)$-module M such that $\mathrm{End}_{kQ(B_0)}(M) = k$, then it is called a *Schur* root. We use this notion later.

In view of cluster algebras, the extension of Theorem 3.1.2 to the valued quivers is desired and expected. Unfortunately, it is not fully achieved yet [**Hub05, DDPW08**]. Nevertheless, the perspective presented above guides us to the following natural refinement of Conjecture 3.1.1, jointly proposed with Andrei Zelevinsky.

Conjecture 3.1.3 (Root Conjecture)**.** *For any skew-symmetrizable matrix B_0 any c-vector of $\mathcal{A}_\bullet(B_0)$ is a root of $\Delta(A(B_0))$.*

As for d-vectors, they also satisfy the same root property in many known cases. However Marsh and Reiten recently found, in cluster affine type A, an example of a d-vector which is not a root of $\Delta(A(B_0))$ [**MR**]. We thank Robert Marsh and Idun Reiten for sharing with us this counterexample.

3.1.3. Results for finite type. Cluster algebras of finite type were studied in detail by various authors. Here we collect some of the known properties of their c- and d-vectors along with some consequences which are relevant to the present paper. For simplicity, we assume that a skew-symmetrizable matrix B_0 is *indecomposable* in this subsection.

The connection between the d-vectors and the root systems of finite type was first discovered by Fomin and Zelevinsky [**FZ03a**]. Recall that a skew-symmetrizable integer matrix B_0 is said to be *bipartite* if the corresponding valued quiver has only sinks and sources.

Theorem 3.1.4 ([**FZ03a**, Theorem 1.9]). *For any skew-symmetrizable bipartite matrix B_0 whose Cartan counterpart $A(B_0)$ is of finite type, the set $\mathcal{D}(B_0)$ coincides with the set of all the positive roots of $\Delta(A(B_0))$.*

The requirement of B_0 being bipartite was lifted later on in [**YZ08**]. In particular, in the skew-symmetric case, combining the above result with Gabriel's theorem, we get that the set $\mathcal{D}(B_0)$ also coincides with the set of all the dimension vectors of the path algebra $kQ(B_0)$. This result triggered the intensive representation-theoretic study of cluster algebras in the past decade.

For a skew-symmetric matrix B_0 of cluster finite type, let $\Lambda(B_0)$ be the corresponding cluster-tilted algebra, which is the path algebra of the quiver $Q(B_0)$ modulo the relations described by [**BMR06b**, Theorem 4.2]. Note that any indecomposable $\Lambda(B_0)$-module can also be regarded as an indecomposable $kQ(B_0)$-module. Let $\mathrm{Dim}(\Lambda(B_0))$ be the set of the dimension vectors of all the indecomposable $\Lambda(B_0)$-modules.

The following theorem by Caldero, Chapoton, and Schiffler [**CCS06**], and by Buan, Marsh, and Reiten [**BMR07**], extended Theorem 3.1.4 to any skew-symmetric matrix B_0 of cluster finite type.

Theorem 3.1.5 ([**CCS06**, Theorem 4.4 & Remark 4.5], [**BMR07**, Theorem 2.2]). *For any skew-symmetric matrix B_0 of cluster finite type, the sets $\mathcal{D}(B_0)$ and $\mathrm{Dim}(\Lambda(B_0))$ coincide.*

On the other hand, Nájera Chávez recently proved a parallel theorem for c-vectors.

Theorem 3.1.6 ([**NC12**, Theorem 4],[**NC**]). *For any skew-symmetric matrix B_0 of cluster finite type, the sets $\mathcal{C}_+(B_0)$ and $\mathrm{Dim}(\Lambda(B_0))$ coincide.*

The inclusion $\mathcal{C}_+(B_0) \subset \mathrm{Dim}(\Lambda(B_0))$ is a special case of [**NC12**, Theorem 4] (see Theorem 3.1.16), while the opposite inclusion is due to a yet unpublished result communicated to us by Alfredo Nájera Chávez [**NC**].

We have the following immediate corollary of Theorems 3.1.5 and 3.1.6.

Corollary 3.1.7. *For any skew-symmetric matrix B_0 of cluster finite type, the sets $\mathcal{C}_+(B_0)$ and $\mathcal{D}(B_0)$ coincide.*

It is known that, for any indecomposable $\Lambda(B_0)$-module M, $\mathrm{End}_{\Lambda(B_0)}(M) = k$ holds [**BMR+06a**, Section 8]. Thus, we have another corollary of Theorems 3.1.5 and 3.1.6.

Corollary 3.1.8. *For any skew-symmetric matrix B_0 of cluster finite type, all positive c-vectors and all non-initial d-vectors are Schur roots of $\Delta(A(B_0))$.*

For any skew-symmetric matrix B_0 of cluster finite type, let us introduce the set

$$\mathrm{Ind}(\Lambda(B_0)) = \{\,\text{all indecomposable } \Lambda(B_0)\text{-modules}\,\}. \tag{3.1.3}$$

The following remarkable fact holds.

Theorem 3.1.9 ([**BMR07**, Corollary 2.4]). *For any skew-symmetric matrix B_0 of cluster finite type, the cardinality $|\mathrm{Ind}(\Lambda(B_0))|$ only depends on the cluster type Z of B_0; it is equal to the number of positive roots of the root system of type Z.*

The dimension map

$$\underline{\dim} : \mathrm{Ind}(\Lambda(B_0)) \to \mathrm{Dim}(\Lambda(B_0)) \tag{3.1.4}$$

is surjective by definition. Actually, it is bijective by the following theorem.

Theorem 3.1.10. [**Rin11**, Theorem 1] *For any skew-symmetric matrix B_0 of cluster finite type, the map $\underline{\dim}$ in (3.1.4) is injective.*

We have an immediate corollary of Theorems 3.1.5, 3.1.6, 3.1.9, and 3.1.10.

Corollary 3.1.11. *For any skew-symmetric matrix B_0 of cluster finite type, the cardinality $|\mathcal{C}_+(B_0)| = |\mathcal{D}(B_0)|$ only depends on the cluster type Z of B_0, and it is equal to the number of positive roots of the root system of type Z.*

3.1.4. More general results. For completeness, we summarize some general results on c- and d-vectors beyond finite type and also give some examples, though we do not use them in the rest of the paper.

A skew-symmetrizable matrix B is *acyclic* if the corresponding valued quiver $Q(B)$ is acyclic, i.e., without oriented cycles. Let us first discuss the case of an acyclic skew-symmetric matrix B_0. Under this hypothesis, the cluster tilted algebra $\Lambda(B_0)$ is the path algebra $kQ(B_0)$ itself because there is no relation to be imposed. A $kQ(B_0)$-module M is said to be *rigid* if $\mathrm{Ext}^1_{kQ(B_0)}(M,M)=0$.

The following two theorems completely describe the c- and d-vectors in this case:

THEOREM 3.1.12 ([**CK06**, Theorem 4], [**BMRT07**, Theorem 2.3]). *For any acyclic skew-symmetric matrix B_0, the set $\mathcal{D}(B_0)$ coincides with the set of the dimension vectors of all the rigid indecomposable $kQ(B_0)$-modules.*

THEOREM 3.1.13 ([**NC12**, Theorem 1]). *For any acyclic skew-symmetric matrix B, the set $\mathcal{C}_+(B)$ coincides with the set of the dimension vectors of all the rigid indecomposable $kQ(B)$-modules.*

Recall that, when $Q(B_0)$ is acyclic, the following formula holds [**ASS06**]:

$$\frac{1}{2}(\underline{\dim}\, M, \underline{\dim}\, M)_{A(B_0)} = \dim \mathrm{End}_{kQ(B_0)}(M) - \dim \mathrm{Ext}^1_{kQ(B_0)}(M,M). \tag{3.1.5}$$

It follows that α is the dimension vector of a rigid indecomposable $kQ(B_0)$-module if and only if it is a real Schur root. Therefore, we have an alternative form of Theorems 3.1.12 and 3.1.13.

Corollary 3.1.14 ([**NC12**, Theorem 1])**.** *For any acyclic skew-symmetric matrix B_0, both the sets $\mathcal{D}(B_0)$ and $\mathcal{C}_+(B_0)$ coincide with the set of all the real Schur roots of $\Delta(A(B_0))$.*

Theorem 3.1.13 and Corollary 3.1.14 are partially extended to the acyclic skew-symmetrizable matrices. (The sign-coherence of c-vectors is covered by [**Dem10**].)

Theorem 3.1.15 ([**RS11**, Theorem 1.1], [**ST12**, Theorem 1]). *For any acyclic skew-symmetrizable matrix B_0, any positive c-vector is a real positive root of $\Delta(A(B_0))$; moreover, it is the dimension vector of a rigid indecomposable representation of the valued quiver $Q(B_0)$.*

Finally, beyond finite type and the acyclic case, the following result is so far the most general result on c-vectors; in particular, it ensures and strengthens Conjecture 3.1.3 for any skew-symmetric matrix B_0.

Theorem 3.1.16 ([**NC12**, Theorem 4]). *For any skew-symmetric matrix B_0, any positive c-vector of $\mathcal{A}_\bullet(B_0)$ is the dimension vector of some rigid indecomposable module M of the Jacobian algebra $J(Q(B_0), W)$ of the quiver $Q(B_0)$ with generic potential W such that $\mathrm{End}_{J(Q(B_0),W)}(M) = k$. In particular, any positive c-vector of $\mathcal{A}_\bullet(B_0)$ is a Schur root of $\Delta(A(B_0))$.*

On the other hand the behavior of the d-vectors is rather complicated as studied in [**BMR09, BM10, MR**]. What was observed therein is a deficiency phenomenon: in some situations the d-vector of a cluster variable x'_i is smaller than the dimension vector of the rigid indecomposable $\Lambda(B_0)$-module associated with x'_i.

We conclude this short survey by presenting two illuminating examples beyond finite type and the acyclic case.

Example 3.1.17. *Type $A_2^{(1)}$.* Consider the skew-symmetric matrix B_0 corresponding the following non-acyclic quiver:

$$\begin{tikzcd} & \overset{2}{\circ} \arrow[dl] & \\ 1\ \circ \arrow[rr, shift left] \arrow[rr, shift right] & & \circ\ 3 \arrow[ul] \end{tikzcd}$$

It is mutation equivalent to the following acyclic quiver whose Cartan counterpart is the Cartan matrix of affine type $A_2^{(1)}$.

$$\begin{tikzcd} & \overset{2}{\circ} \arrow[dr] & \\ 1\ \circ \arrow[ur] \arrow[rr] & & \circ\ 3 \end{tikzcd}$$

This cluster algebra $\mathcal{A}_\bullet(B_0)$ is studied in detail by [**CI12**]. In particular, the non-initial d-vectors of $\mathcal{A}_\bullet(B_0)$ are given by [**CI12**, Lemma 3.3]:

(3.1.6)
$$(0,1,0),(1,1,1),(a,0,a+1),(a+1,0,a),(a,1,a+1),(a+1,1,a), \quad a \geq 0.$$

Moreover, it is not difficult to show that the positive c-vectors of $\mathcal{A}_\bullet(B_0)$ are also given by the same list. Therefore, in this case $\mathcal{C}_+(B_0) = \mathcal{D}(B_0)$ holds, even though B_0 is not acyclic. Thus, any non-initial d-vector is a Schur root of $\Delta(A(B_0))$ by Theorem 3.1.16. Note that the d-vector $(1,1,1)$ in (3.1.6) is the simplest example which shows the deficiency phenomenon [**BMR09**, Example 7.2], where the "proper" dimension vector is $(1,2,1)$. Nevertheless, the d-vector $(1,1,1)$ is still a Schur root. We also note that among the vectors in (3.1.6), the last two are imaginary roots for $a \geq 1$.

Example 3.1.18. *Markov quiver.* We consider the skew-symmetric matrix B_0 such that the corresponding quiver is the following non-acyclic one.

$$\begin{tikzcd}
 & \overset{2}{\circ} \arrow[dr, shift left] \arrow[dr, shift right] & \\
1\ \circ \arrow[ur, shift left] \arrow[ur, shift right] & & \circ\ 3 \arrow[ll, shift left] \arrow[ll, shift right]
\end{tikzcd}$$

This is known as the *Markov quiver*, and the positive c-vectors of $\mathcal{A}_\bullet(B_0)$ are given by the permutations of the following vectors [**NC11**, Theorem 3.1.2]:

(3.1.7) $$(1,2,2),(a+1,b+1,a+b+1),(a-1,b-1,a+b-1),$$

where $1 \leq a \leq b$, and a and b are coprime. The cluster algebra $\mathcal{A}_\bullet(B_0)$ has a surface realization by a once-punctured torus. Using the same technique as in Section 3.3, it can be shown that the non-initial d-vectors are given by the permutations of the vectors in (3.1.7) of the form

$$(a-1, b-1, a+b-1). \tag{3.1.8}$$

So this gives the first example in which the sets $\mathcal{D}(B_0)$ and $\mathcal{C}_+(B_0)$ do not coincide. Nevertheless, $\mathcal{D}(B_0) \subset \mathcal{C}_+(B_0)$ so any non-initial d-vector is still a Schur root of $\Delta(A(B_0))$ by Theorem 3.1.16.

The above examples may suggest that the property $\mathcal{D}(B_0) \subset \mathcal{C}_+(B_0)$ holds in general but this is not true due to the counterexample of [**MR**].

3.2. The sets $\mathcal{X}(Z)$ and $\mathcal{W}(Z)$

Let Z be any finite type. In this section we provide a description of all the diagrams in $\mathcal{X}(Z)$ and define the list $\mathcal{W}(Z)$ of allowed weighted diagram for each type required by Theorem 1.3.1.

3.2.1. Classical types.

3.2.1.1. *Type A_n.* The following is a direct consequence of Proposition 2.4 in [**BV08**].

Proposition 3.2.1. *A diagram X is in $\mathcal{X}(A_n)$ if and only if the following conditions are satisfied:*

- *X has n vertices, is simply laced and connected;*
- *every cycle in X is a triangle;*
- *each vertex in X has at most four neighbours;*
- *if a vertex has three neighbours then exactly two of them are adjacent;*
- *if a vertex has four neighbours then they can be partitioned into two disjoint sets, containing two elements each, and such that the two neighbouring vertices i and j are adjacent if and only if $\{i, j\}$ is one of those sets.*

An example of Dynkin diagram in $\mathcal{X}(A_n)$ is presented in Figure 3.2.1 to illustrate its "quasi-tree" nature.

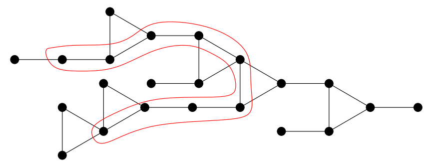


FIGURE 3.2.1. A typical element X in $\mathcal{X}(A_{22})$. The highlighted part is an element of $\mathcal{W}(A_{22})$ embedded in X.

The set $\mathcal{W}(A_n)$ consists of type A Dynkin diagrams (*strings*) with at most n vertices. All the multiplicities are 1. Elements of $\mathcal{W}(A_n)$ are pictorially presented as follows.

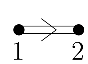
An example of an embedding of such a string in a diagram of $\mathcal{X}(A_n)$ is highlighted in Figure 3.2.1. Note that, as explained in the introduction, an embedding of an element of $\mathcal{W}(A_n)$ in a diagram X is given by a full sub-diagram; therefore at most two vertices of each triangle of X can belong to it. It follows that an embedding of a string is uniquely determined by the positions of its endpoints [**Par11**]. Note that the equality $\mathcal{D}(B) = \mathcal{V}(B)$ is known in this case by [**CCS05, Par11, Tra11**].

The building block of Dynkin diagrams for classical types is given by diagrams of type A_n. While stating the analogous results for other types we will use the convention $\mathcal{X}(A_0) = \emptyset$.

3.2.1.2. *Type B_n.* As usual for Dynkin diagrams we put $a_{ij}a_{ji}$ edges between i and j and the inequality sign on the edges refers to the relation among the lengths of the corresponding simple roots. For example the Cartan matrix

$$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

corresponds to the following Dynkin diagram.



Proposition 3.2.2. *A diagram with n vertices ($n \geq 2$) is in $\mathcal{X}(B_n)$ if and only if it is one of the two in Figure 3.2.2 where $X^{(i)}$ is any diagram in $\mathcal{X}(A_m)$ for a suitable $m \geq 0$.*

We postpone the proof to Section 3.4.2.

The weighted diagrams in $\mathcal{W}(B_n)$ are those in Figure 3.2.3. As we will see they are obtained from (some of) those in $\mathcal{W}(D_{n+1})$ by folding.

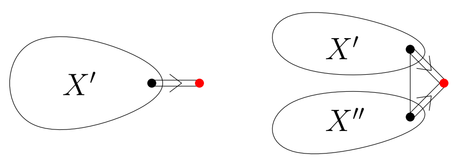


FIGURE 3.2.2. Elements of $\mathcal{X}(B_n)$ for $n \geq 2$; $X^{(i)}$ is any diagram in $\mathcal{X}(A_m)$ for a suitable $m \geq 0$. The highlighted nodes are the images of those permuted by σ in Proposition 3.4.4.

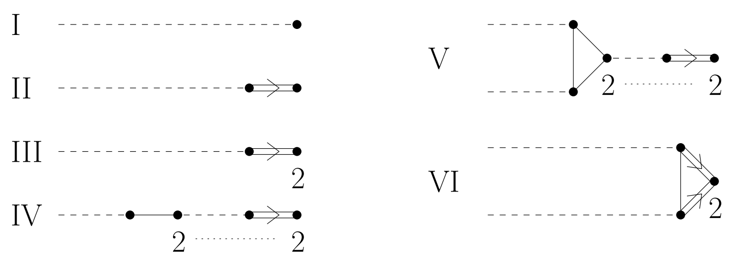


FIGURE 3.2.3. The set $\mathcal{W}(B_n)$. Dotted lines are strings of any length; multiplicity of all the nodes of each such string are the same as their ending points. Solid lines can't be omitted. We will use the above drawing conventions thorough the rest of the paper.

3.2.1.3. *Type C_n.*

Proposition 3.2.3. *A diagram with n vertices ($n \geq 2$) is in $\mathcal{X}(C_n)$ if and only if it is one of the two in Figure 3.2.4 where $X^{(i)}$ is any diagram in $\mathcal{X}(A_m)$ for a suitable $m \geq 0$.*

We postpone the proof to Section 3.4.2.

Remark 3.2.4. The results of Propositions 3.2.2 and 3.2.3 were claimed in [**MS12**] and encoded in the cluster algebra package of Sage. The details will appear in [**Stu**]. The same result also appeared in [**Hen11**].

The weighted diagrams in $\mathcal{W}(C_n)$ are those in Figure 3.2.5; as we will see they are all the weighted diagrams that can be obtained by folding a string embedded on a diagram in $\mathcal{X}(A_{2n-1})$.

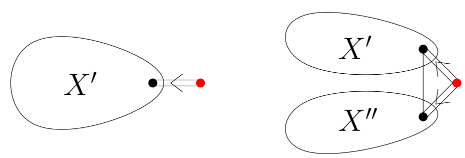


FIGURE 3.2.4. Elements of $\mathcal{X}(C_n)$ for $n \geq 2$; $X^{(i)}$ is any diagram in $\mathcal{X}(A_m)$ for a suitable $m \geq 0$. The highlighted nodes are the images of the fixed point under the action of σ in Proposition 3.4.4.

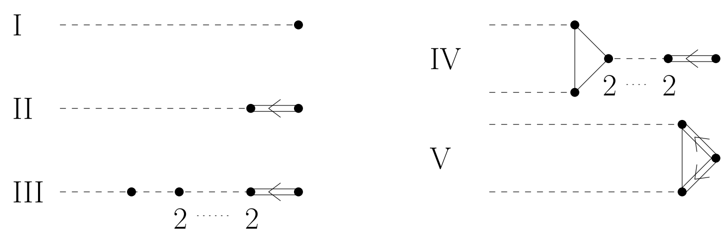


FIGURE 3.2.5. The set $\mathcal{W}(C_n)$. We use the same drawing conventions of Figure 3.2.3.

3.2.1.4. *Type D_n.* From Theorem 3.1 in [**Vat10**] together with Proposition 3.2.1 we get the following description of $\mathcal{X}(D_n)$. Note that the same result can also be obtained easily from the surface realization we use in Section 3.3.

Proposition 3.2.5. *A diagram with n vertices ($n \geq 4$) is in $\mathcal{X}(D_n)$ if and only if it is one of the four in Figure 3.2.6 where $X^{(i)}$ is any diagram in $\mathcal{X}(A_m)$ for a suitable $m \geq 0$.*

The set $\mathcal{W}(D_n)$ consists of all the weighted diagrams in Figure 3.2.7.

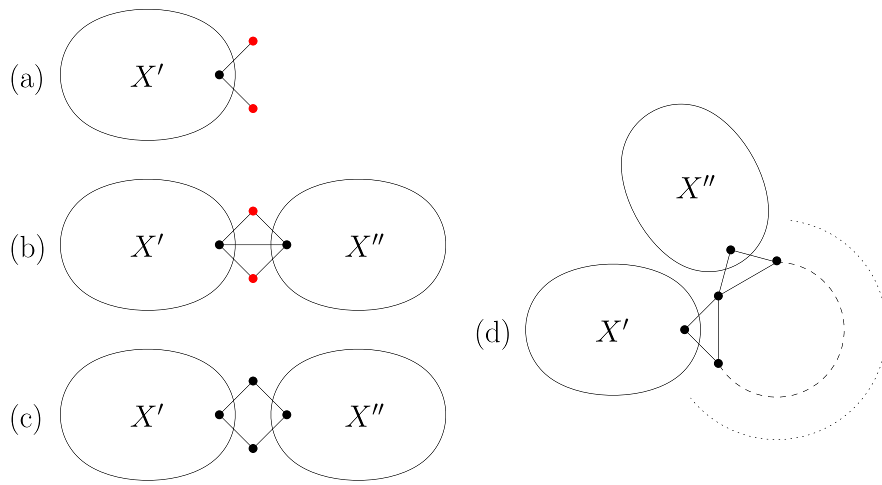


FIGURE 3.2.6. Elements of $\mathcal{X}(D_n)$ for $n \geq 4$; $X^{(i)}$ is any diagram in $\mathcal{X}(A_m)$ for a suitable $m \geq 0$. The highlighted nodes are the one permuted by σ in Remark 3.4.4.

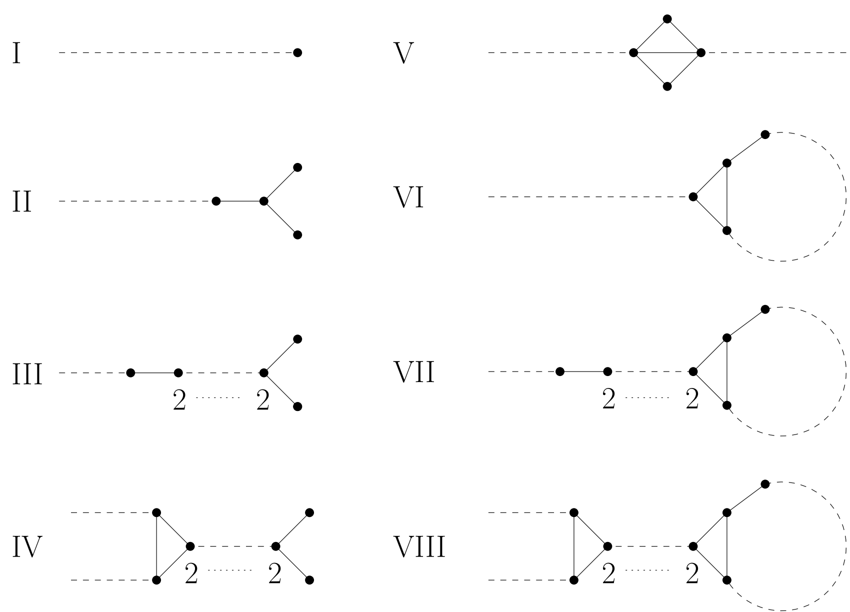


FIGURE 3.2.7. The set $\mathcal{W}(D_n)$. We use the same drawing conventions of Figure 3.2.3.

3.2.2. Exceptional types. For exceptional cases we obtain a description of $\mathcal{X}(Z)$ by direct inspection using [**Kel, MS12**]. Similarly we obtain $\mathcal{W}(Z)$ and check Theorems 1.3.1, 1.3.2, and 1.3.3.

Proposition 3.2.6. *Let Z be any type among E_6, E_7, E_8, F_4, and G_2. The set $\mathcal{X}(Z)$ is the set of diagrams in Figures 3.2.12, 3.2.14-3.2.15, 3.2.21-3.2.24, 3.2.10, and 3.2.8 respectively.*

The sets $\mathcal{W}(Z)$ are given by Figures 3.2.13, 3.2.16-3.2.20, 3.2.25-3.2.64, 3.2.11, and 3.2.9.

3.2.2.1. *Type G_2.*

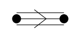


FIGURE 3.2.8. The only diagram in $\mathcal{X}(G_2)$.

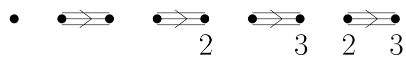


FIGURE 3.2.9. The set $\mathcal{W}(G_2)$.

3.2.2.2. *Type F_4.*

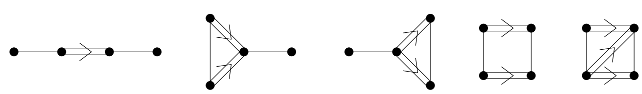


FIGURE 3.2.10. Diagrams in $\mathcal{X}(F_4)$.

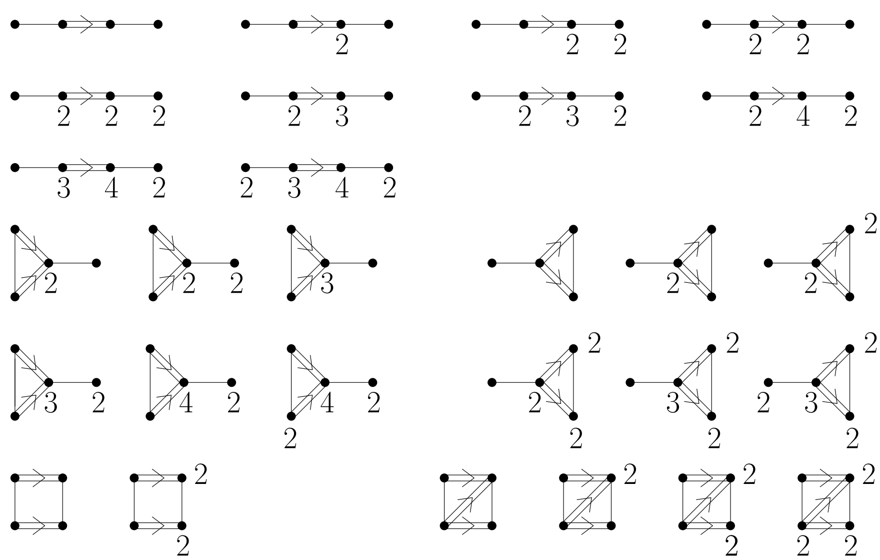


FIGURE 3.2.11. The set $\mathcal{W}(F_4)$ consists of the above weighed diagrams together with all the elements of $\mathcal{W}(B_3)$ and $\mathcal{W}(C_3)$.

3.2.2.3. *Type E_6.*

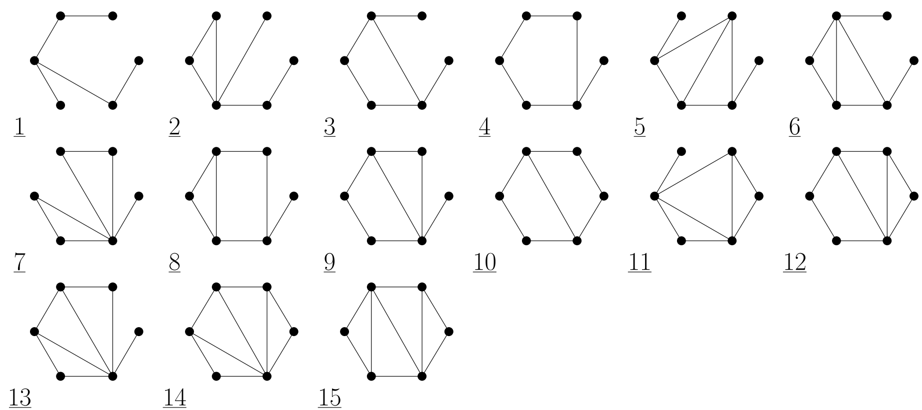


FIGURE 3.2.12. Diagrams in $\mathcal{X}(E_6)$.

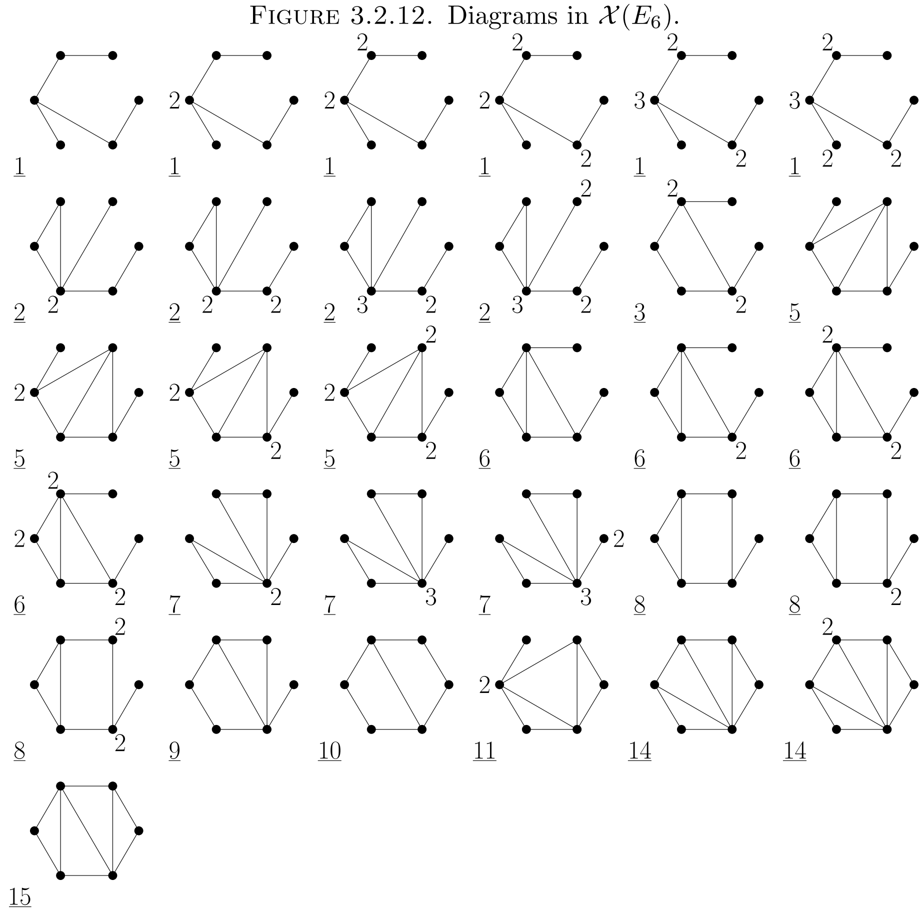


FIGURE 3.2.13. The set $\mathcal{W}(E_6)$ consists of the above weighted diagrams together with all the elements of $\mathcal{W}(A_5)$ and $\mathcal{W}(D_5)$.

3.2.2.4. *Type E_7.*

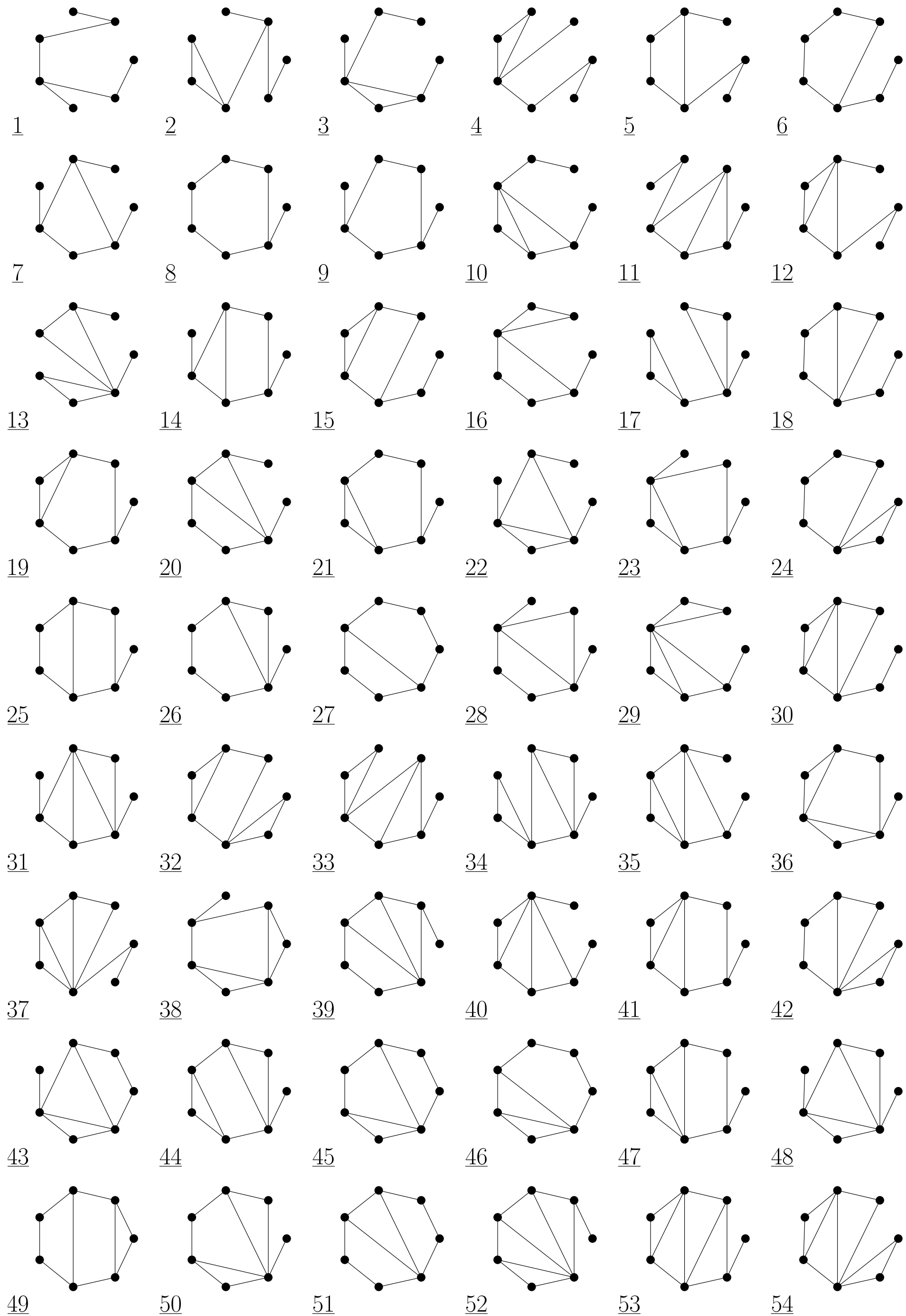


FIGURE 3.2.14. Diagrams in $\mathcal{X}(E_7)$.

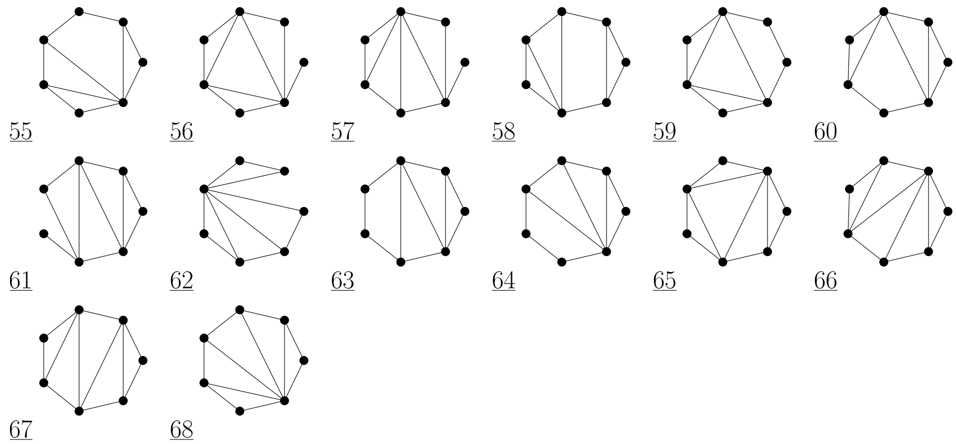


Figure 3.2.15. Diagrams in $\mathcal{X}(E_7)$ (continued).

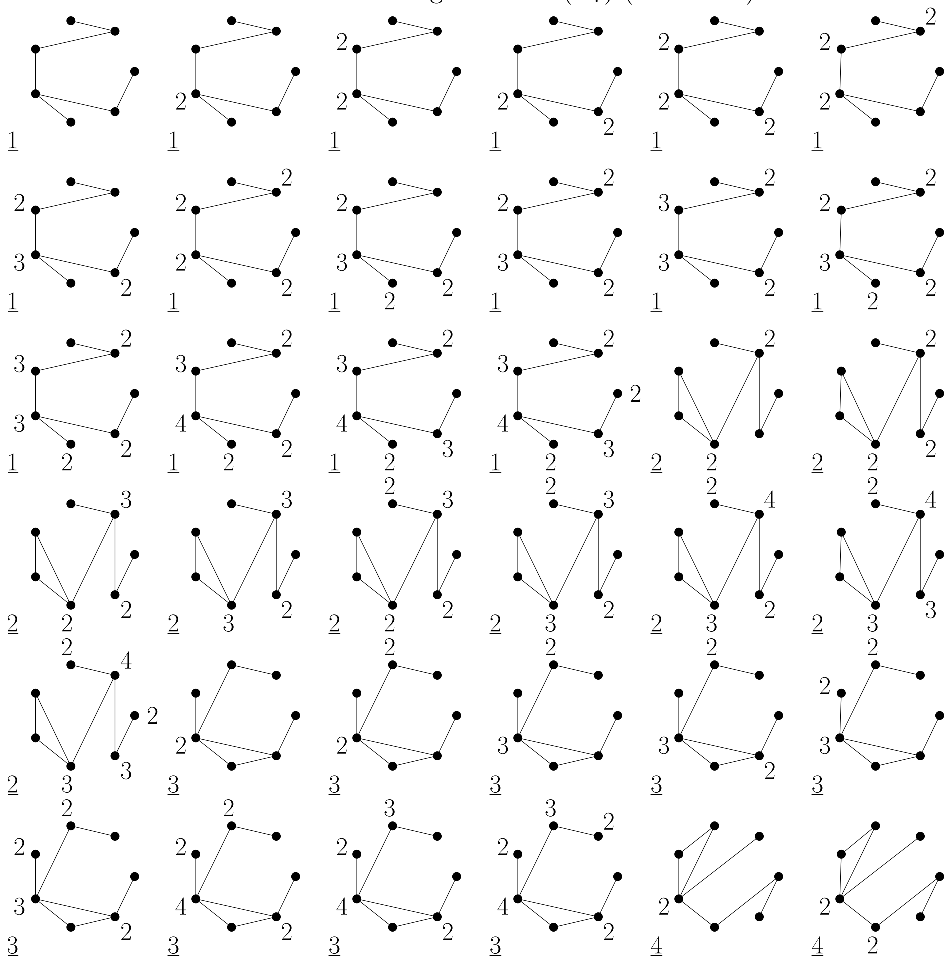


Figure 3.2.16. The set $\mathcal{W}(E_7)$ consists of the above weighted diagrams and all the elements of $\mathcal{W}(A_6)$, $\mathcal{W}(D_6)$, and $\mathcal{W}(E_6)$.

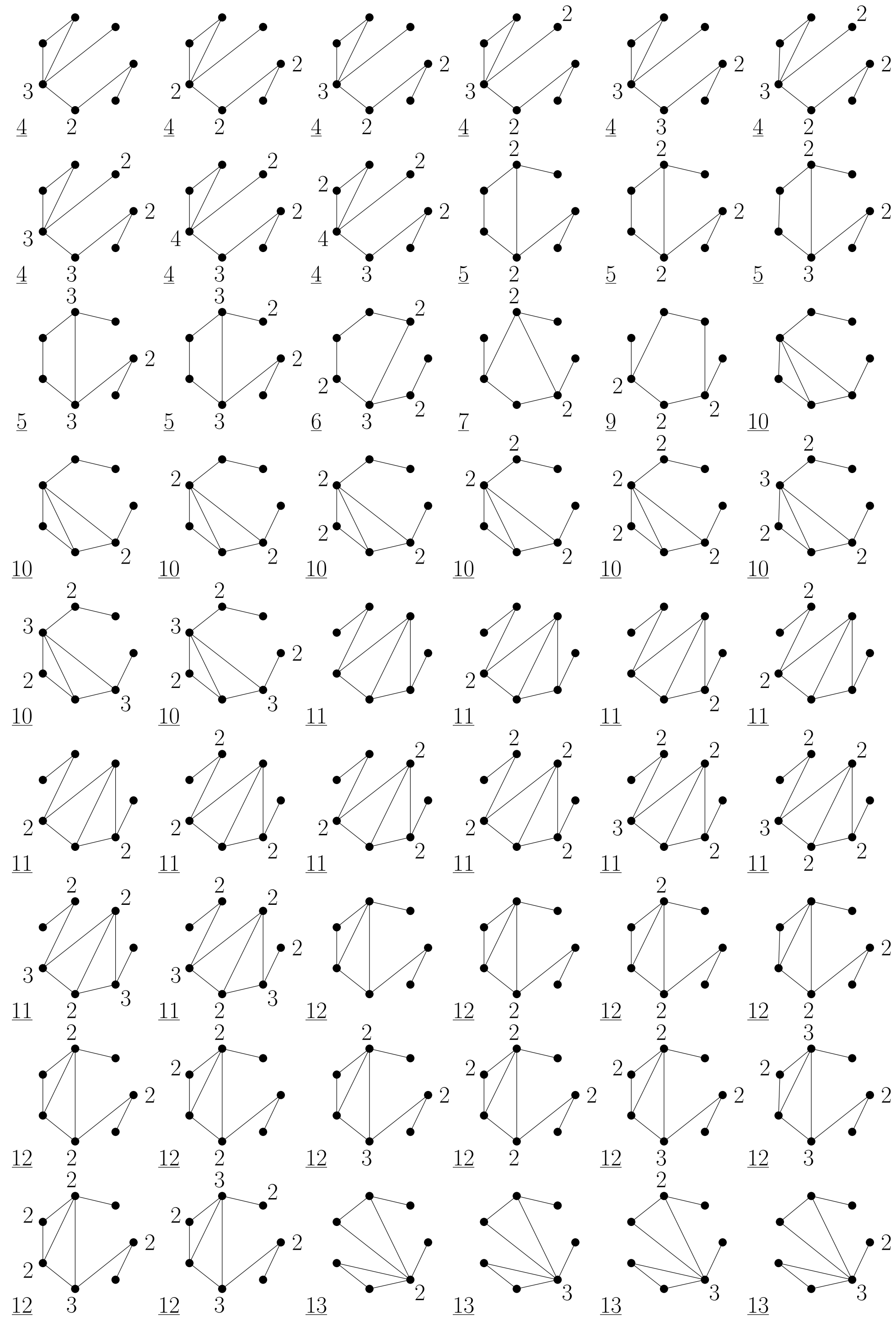


FIGURE 3.2.17. The set $\mathcal{W}(E_7)$ (continued).

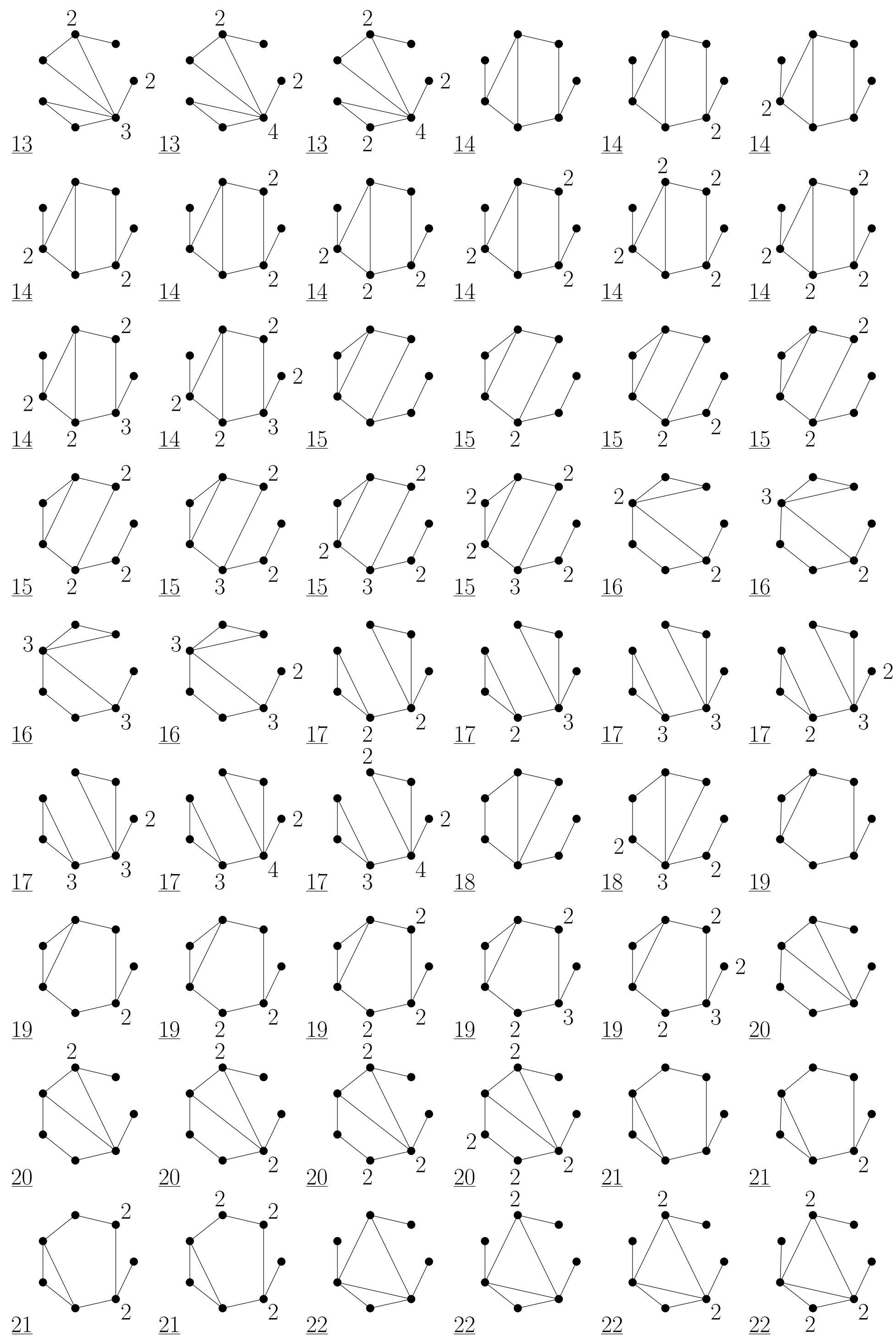


FIGURE 3.2.18. The set $\mathcal{W}(E_7)$ (continued).

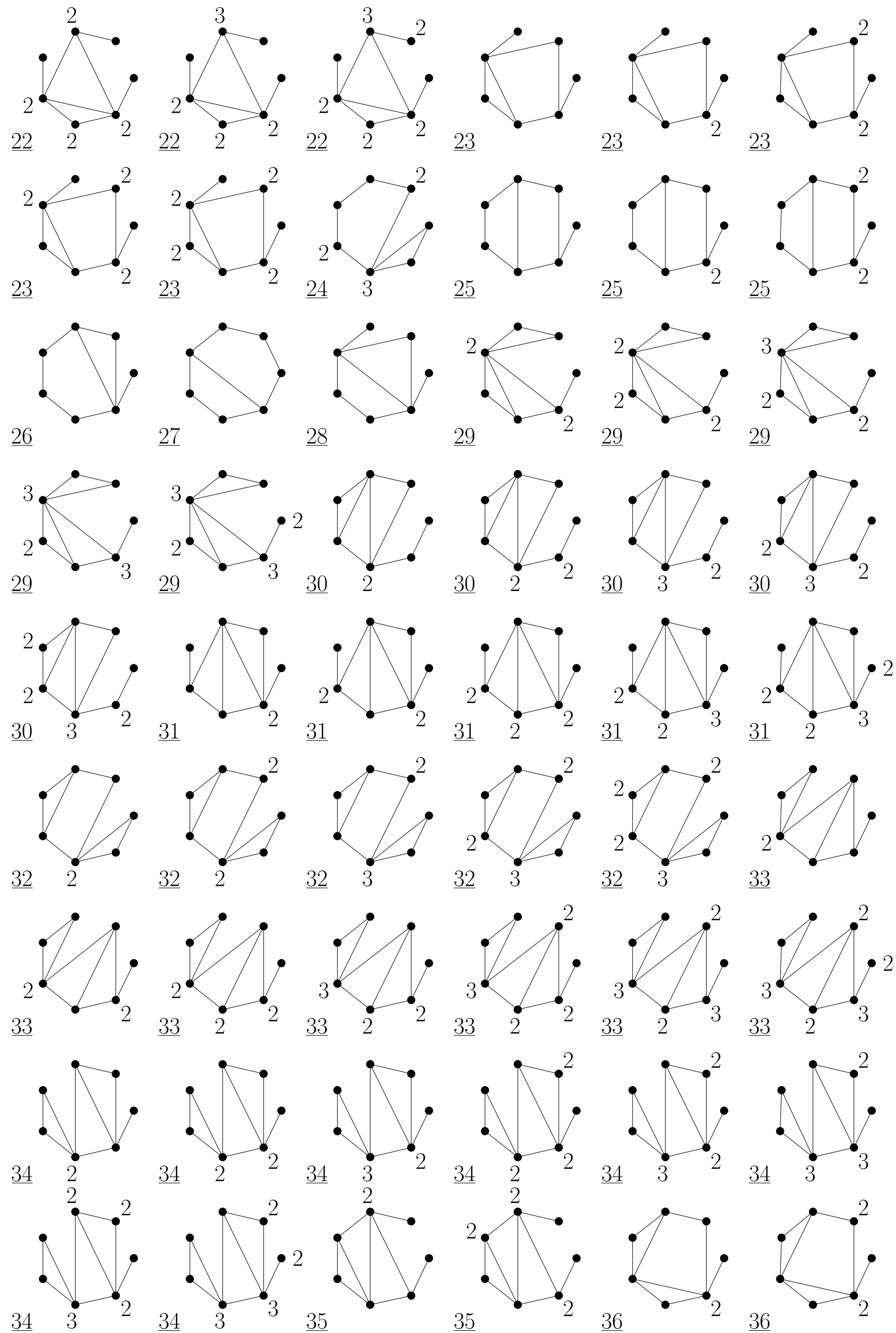


Figure 3.2.19. The set $\mathcal{W}(E_7)$ (continued).

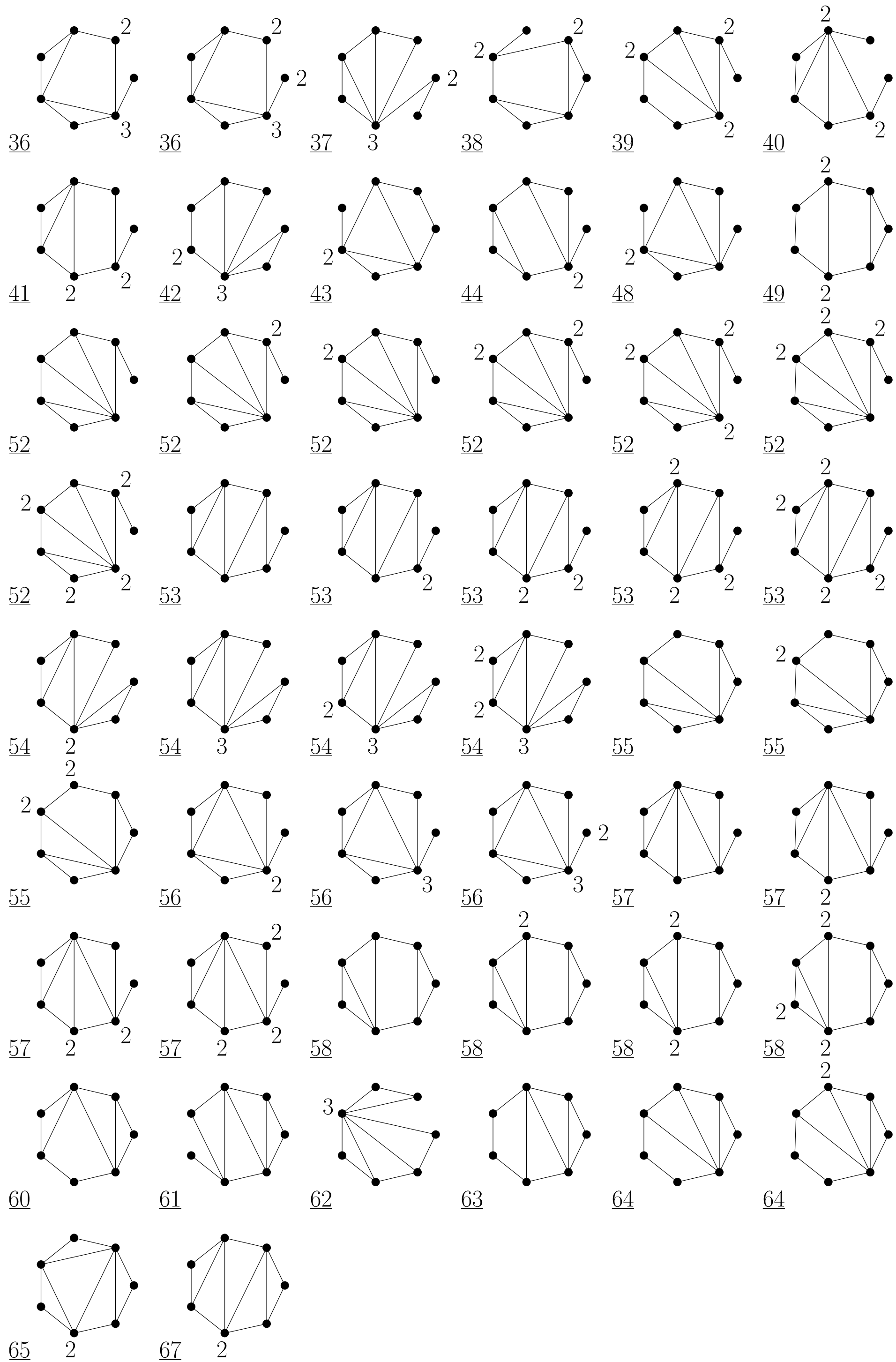


FIGURE 3.2.20. The set $\mathcal{W}(E_7)$ (continued).

3.2.2.5. *Type E_8.*

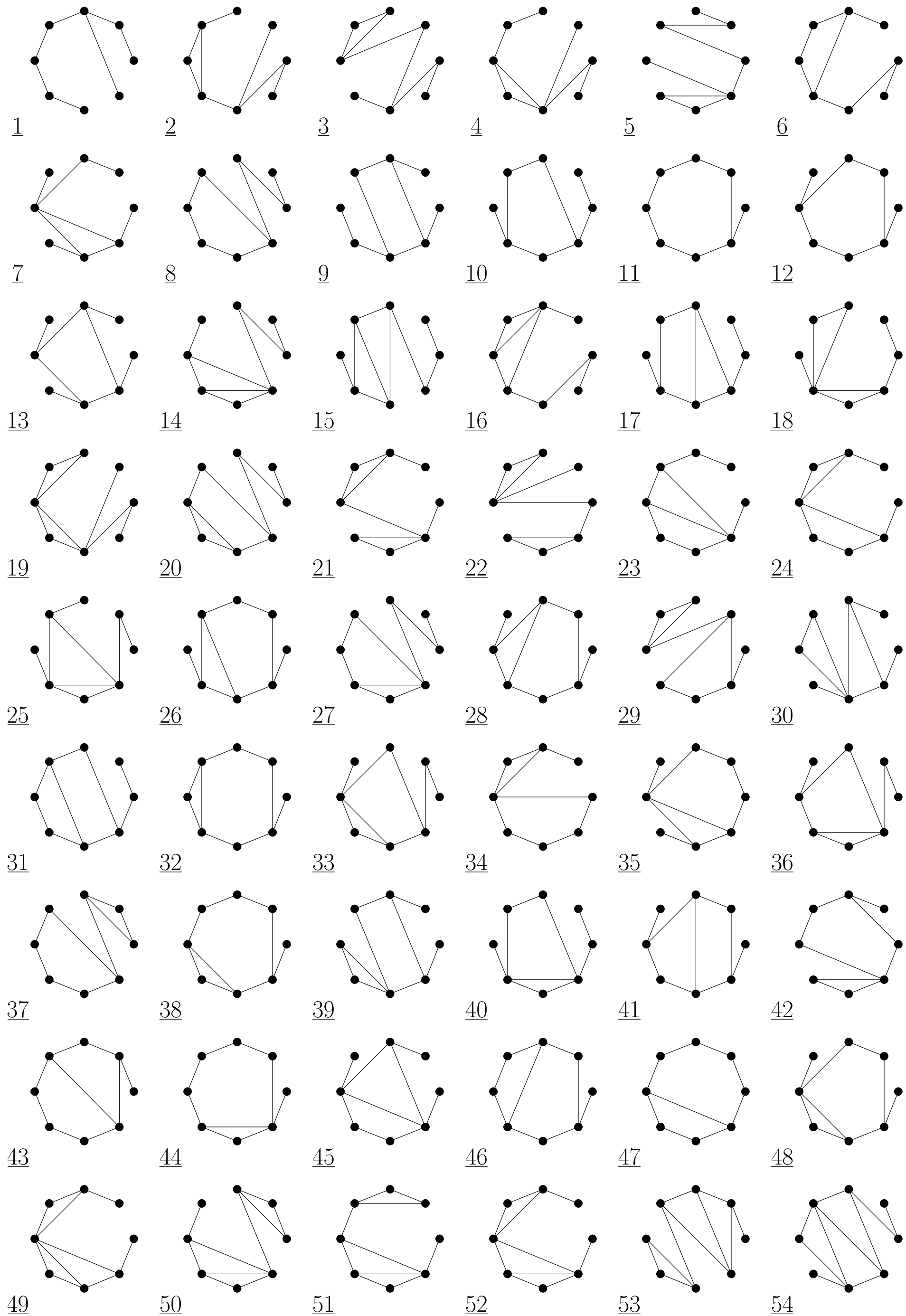


FIGURE 3.2.21. Diagrams in $\mathcal{X}(E_8)$.

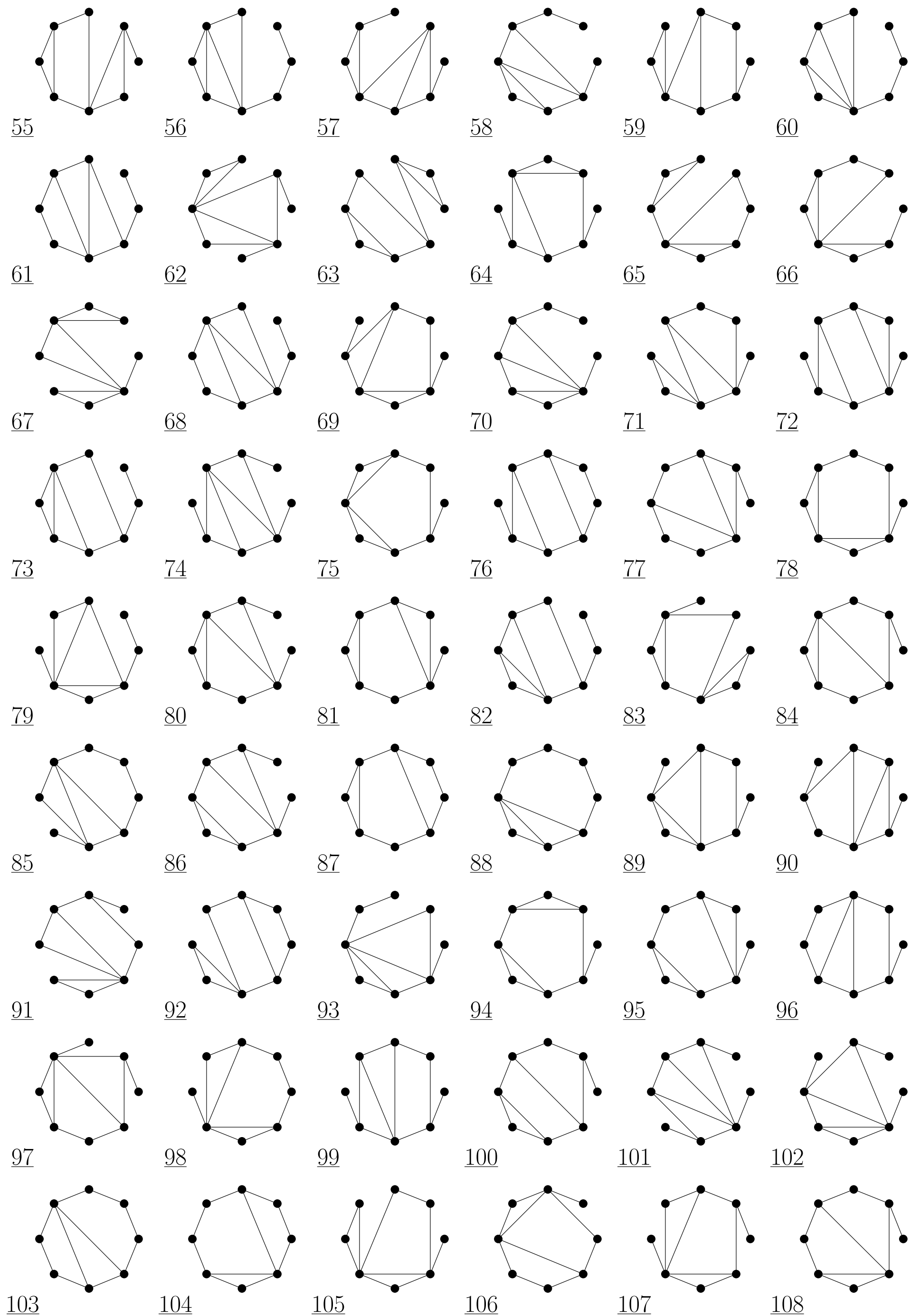


FIGURE 3.2.22. Diagrams in $\mathcal{X}(E_8)$ (continued).

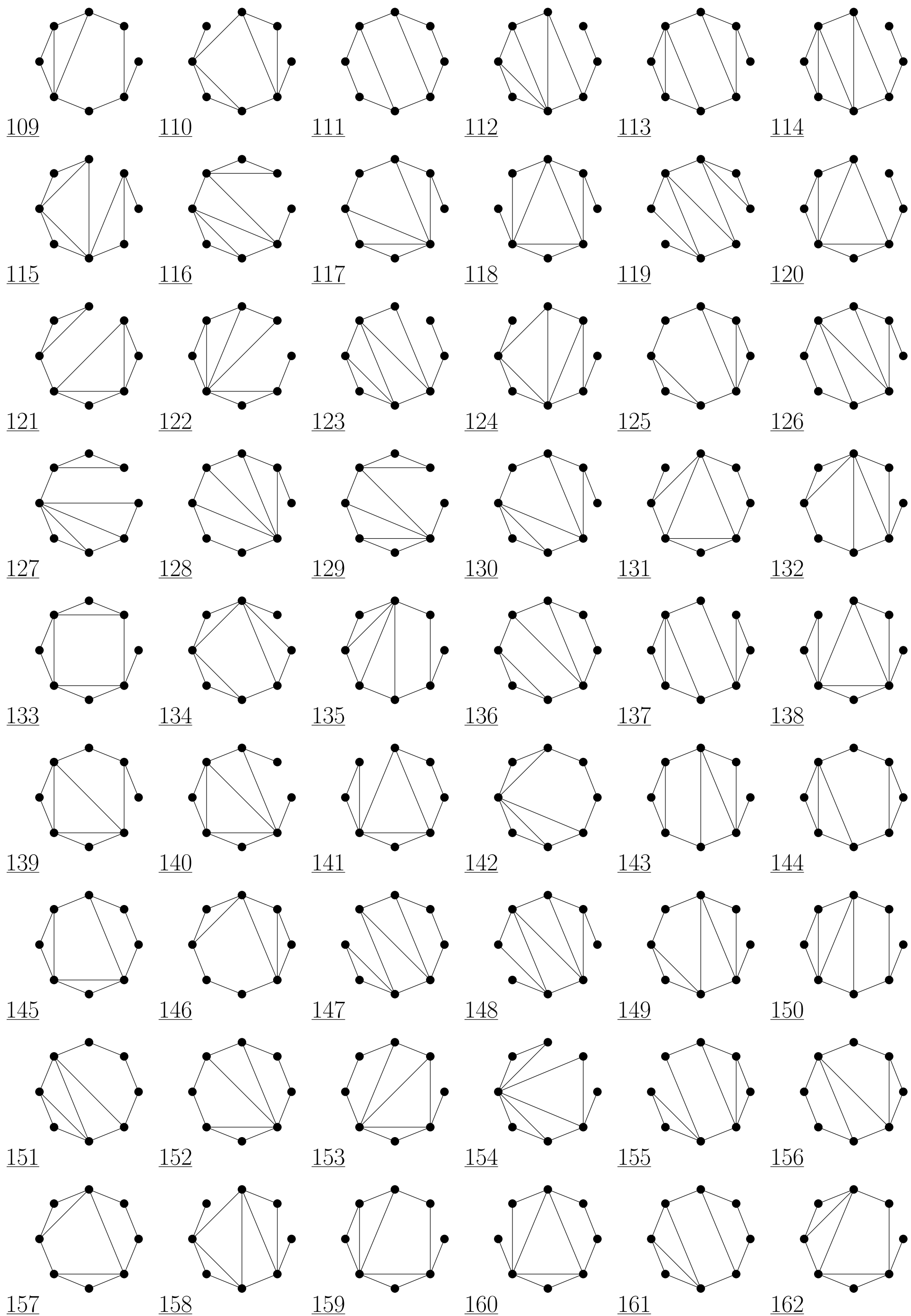


FIGURE 3.2.23. Diagrams in $\mathcal{X}(E_8)$ (continued).

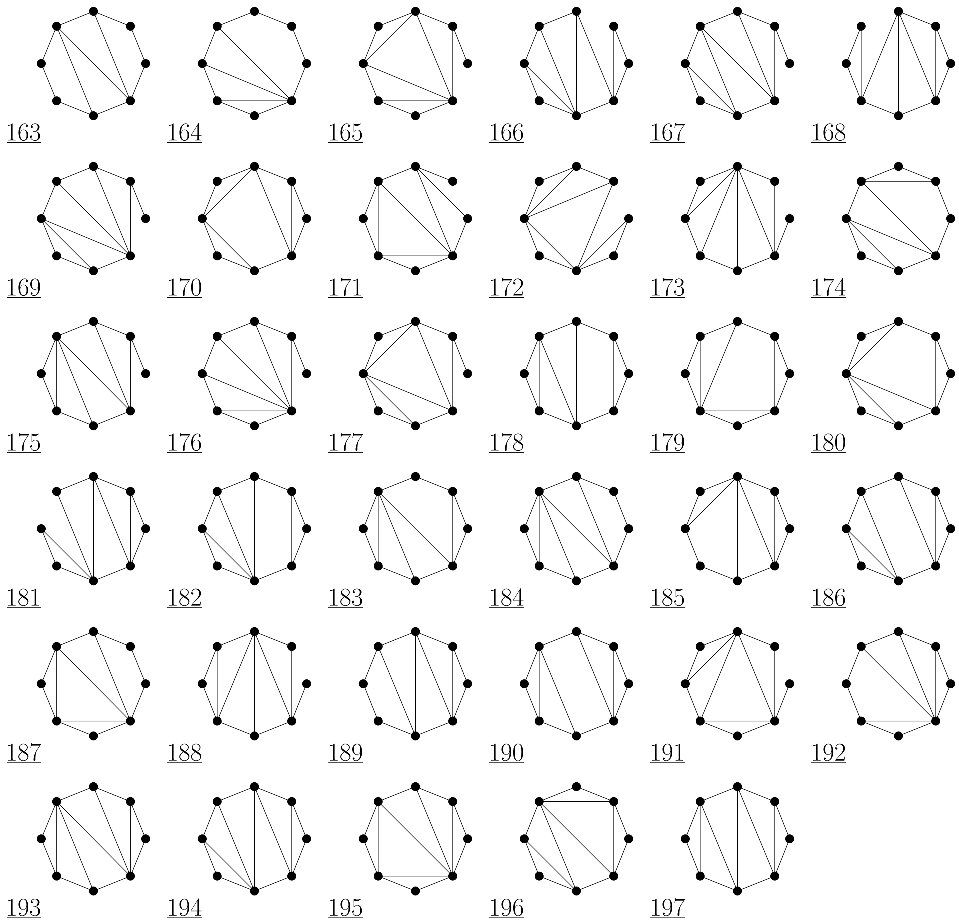


FIGURE 3.2.24. Diagrams in $\mathcal{X}(E_8)$ (continued).

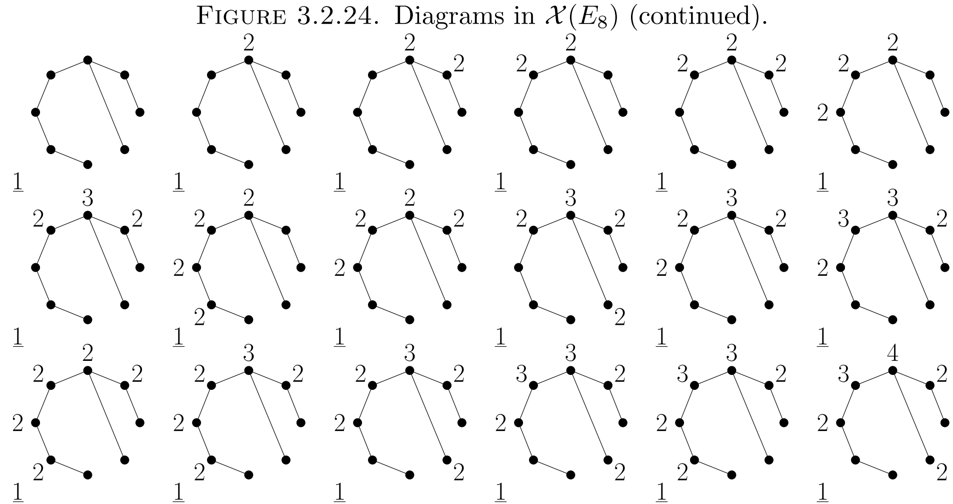


FIGURE 3.2.25. The set $\mathcal{W}(E_8)$ consists of the above weighted diagrams and all the elements of $\mathcal{W}(A_7)$, $\mathcal{W}(D_7)$, and $\mathcal{W}(E_7)$.

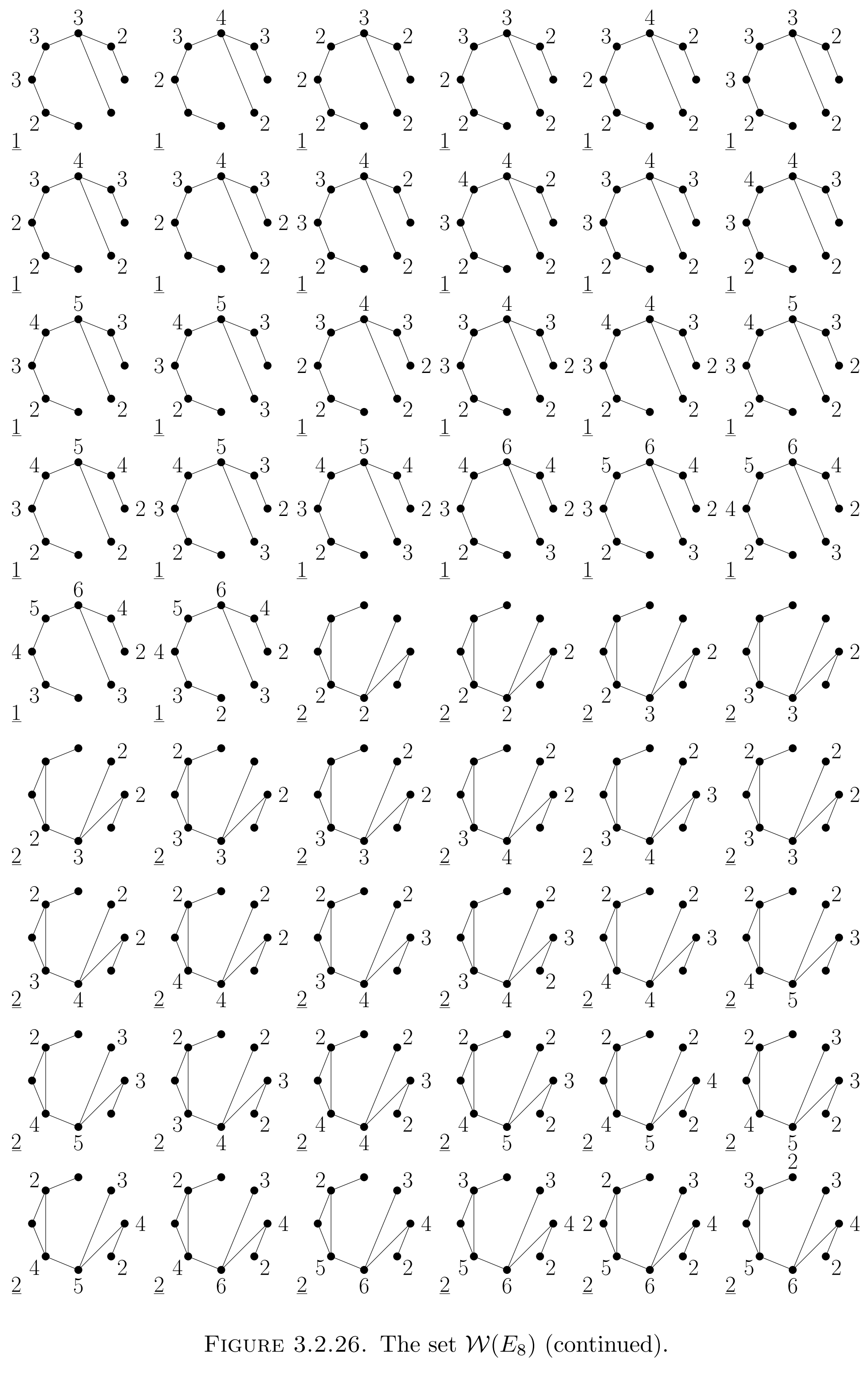


FIGURE 3.2.26. The set $\mathcal{W}(E_8)$ (continued).

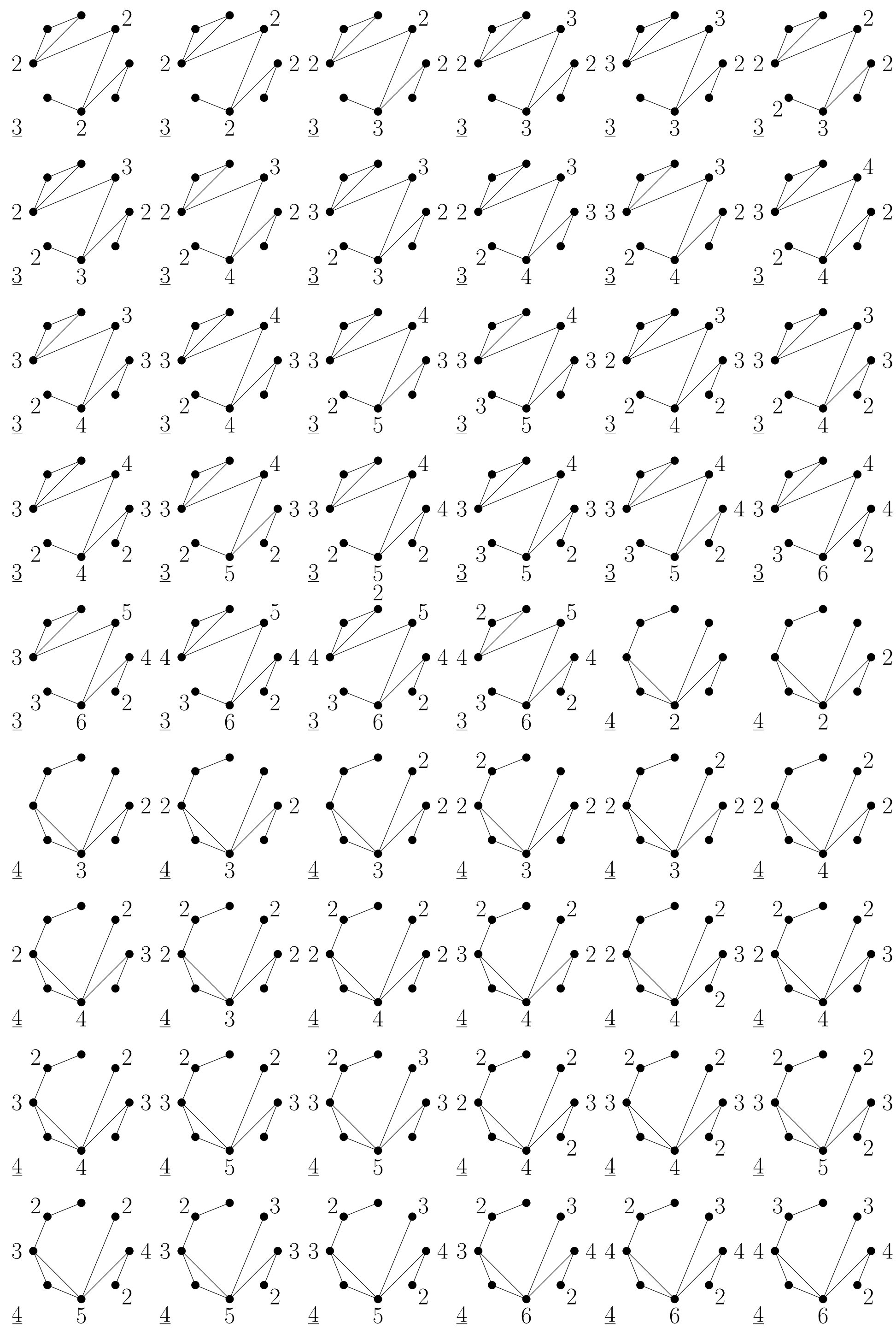


FIGURE 3.2.27. The set $\mathcal{W}(E_8)$ (continued).

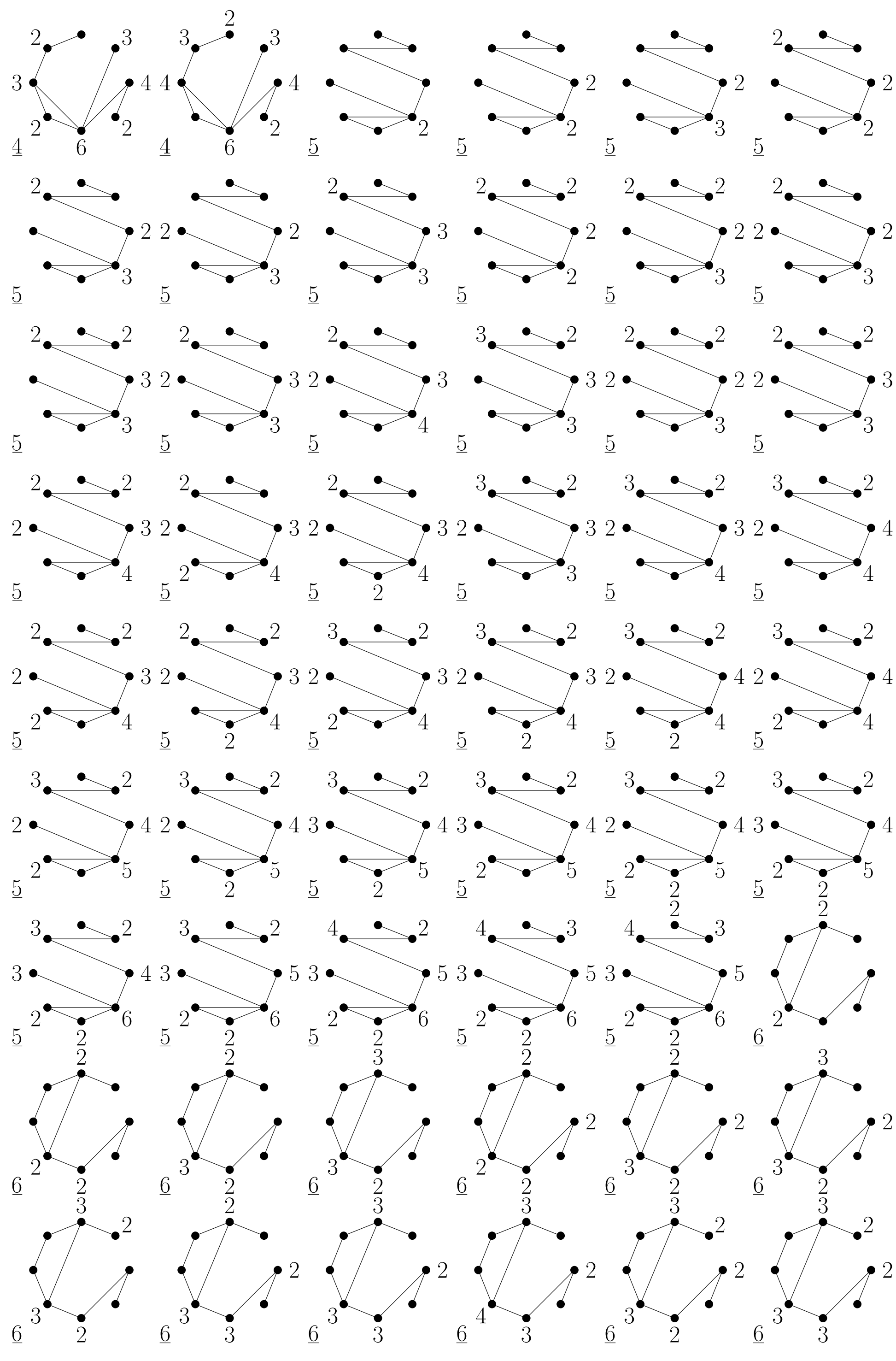


FIGURE 3.2.28. The set $\mathcal{W}(E_8)$ (continued).

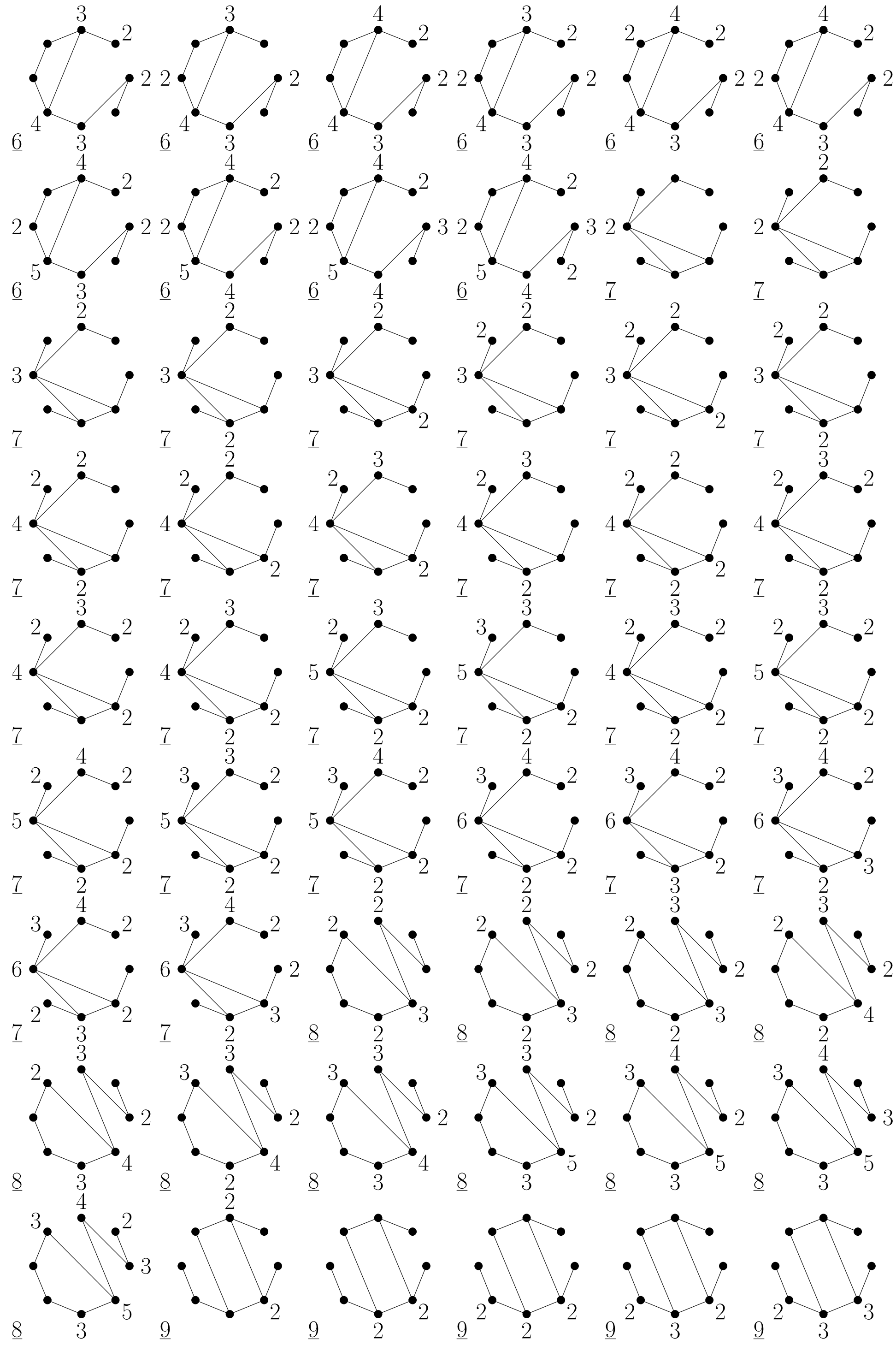


FIGURE 3.2.29. The set $\mathcal{W}(E_8)$ (continued).

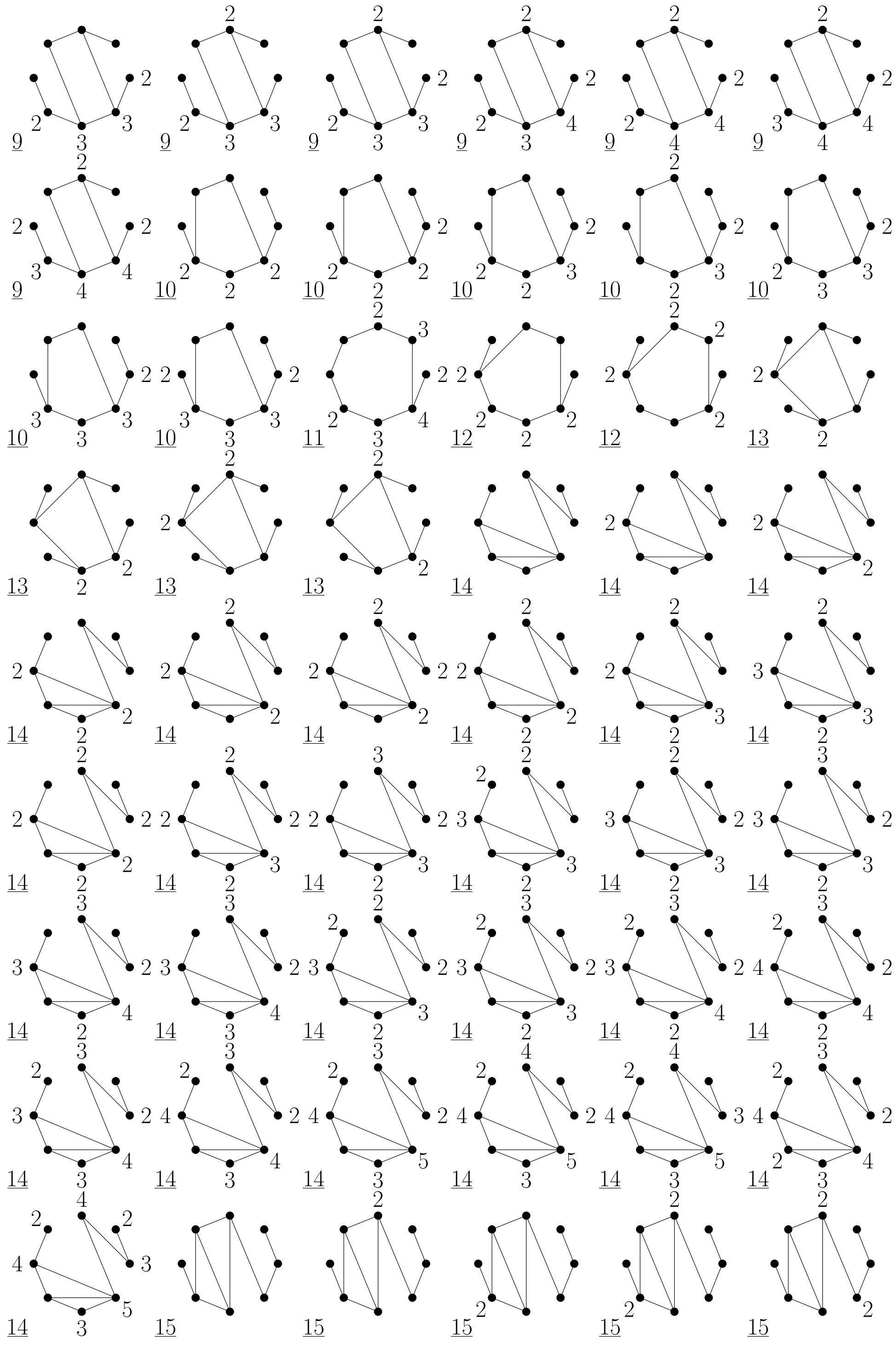


FIGURE 3.2.30. The set $\mathcal{W}(E_8)$ (continued).

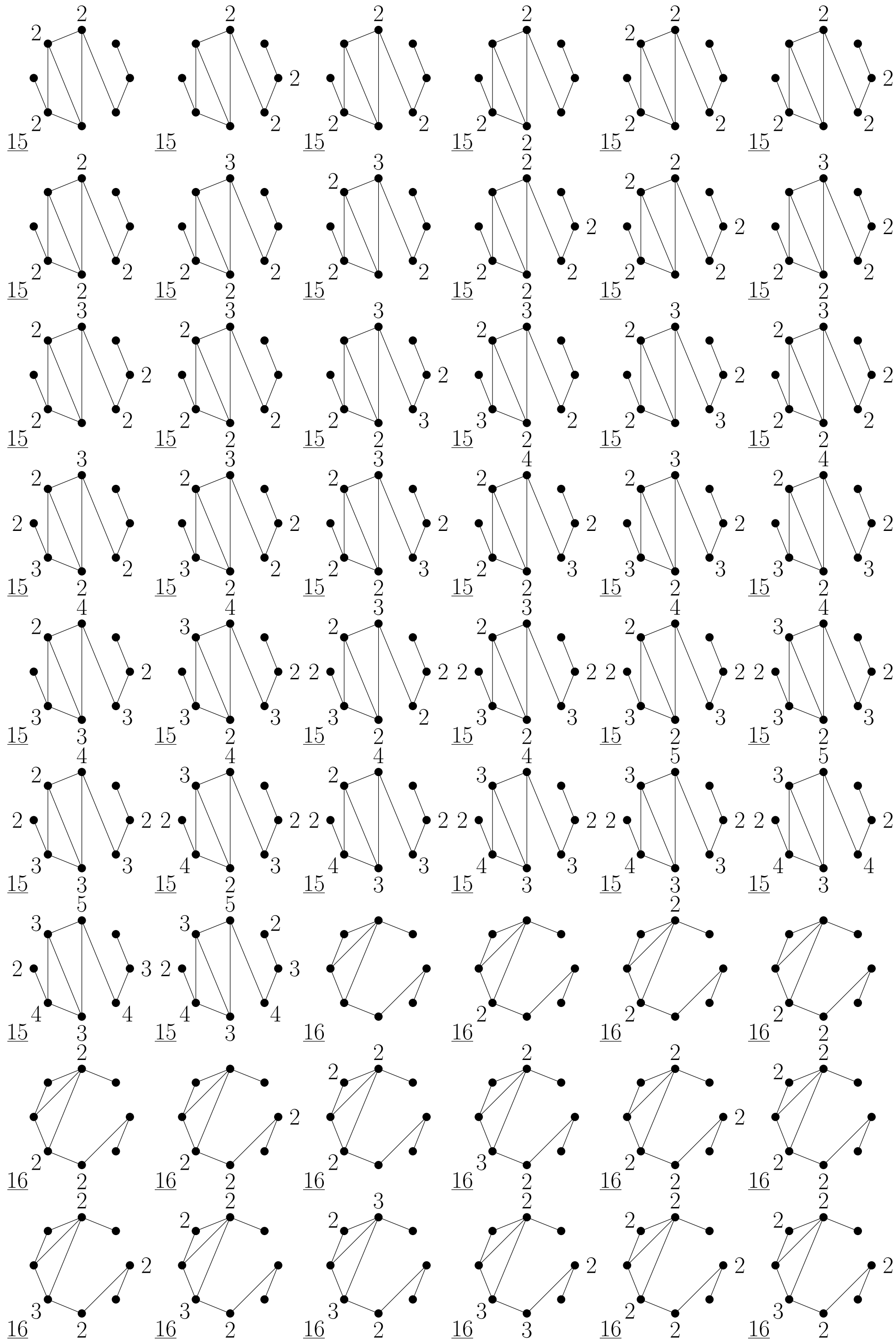


FIGURE 3.2.31. The set $\mathcal{W}(E_8)$ (continued).

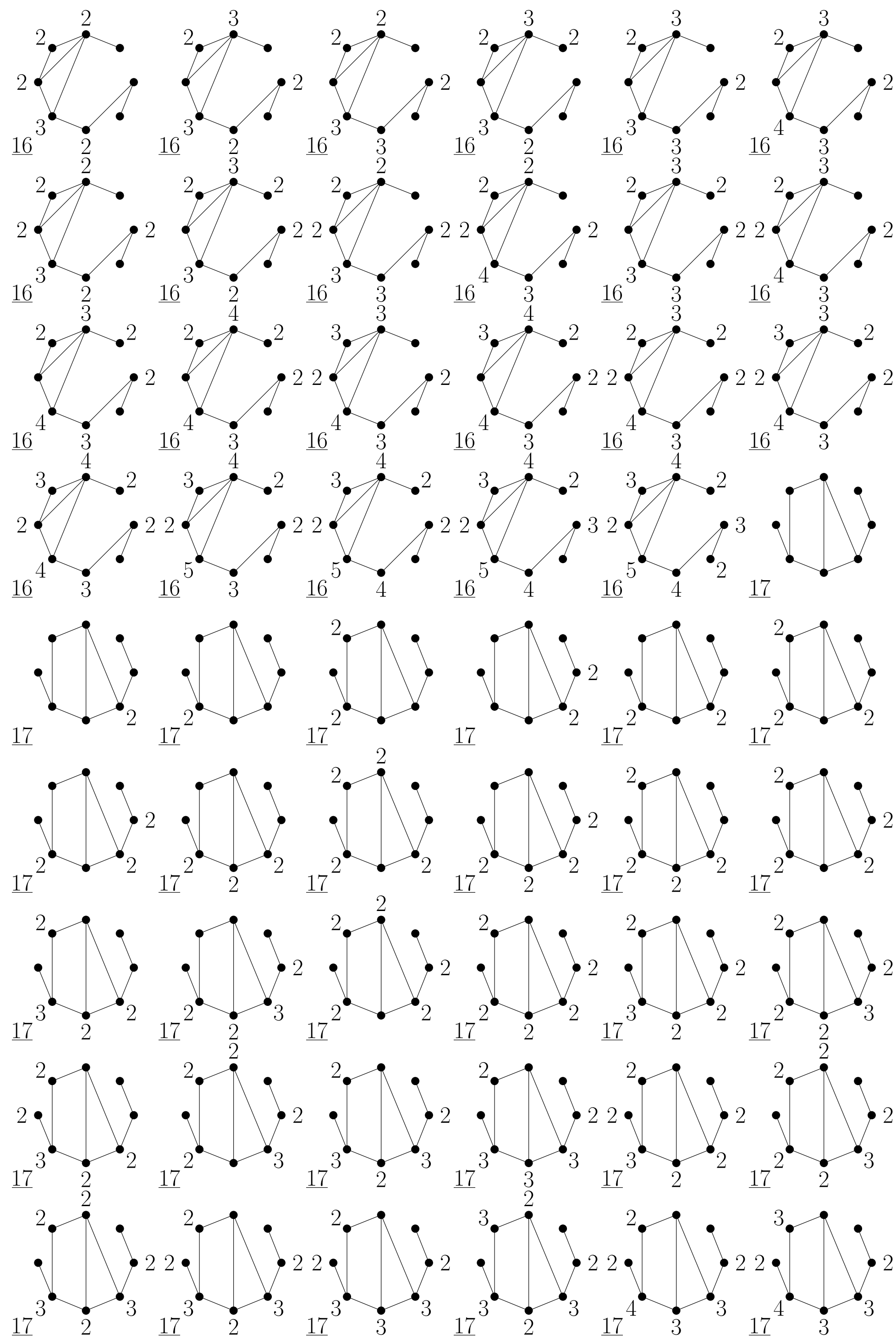


FIGURE 3.2.32. The set $\mathcal{W}(E_8)$ (continued).

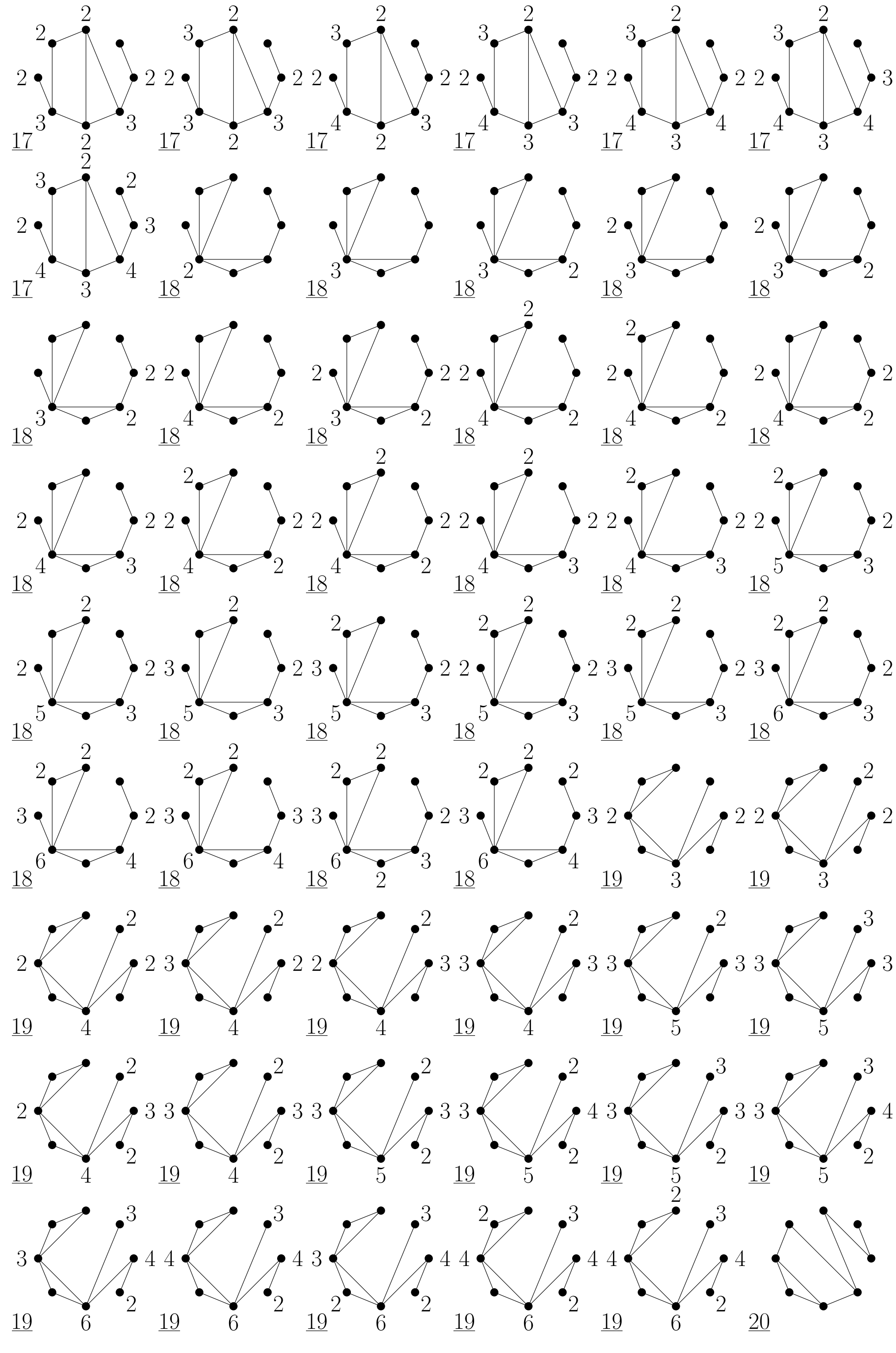


FIGURE 3.2.33. The set $\mathcal{W}(E_8)$ (continued).

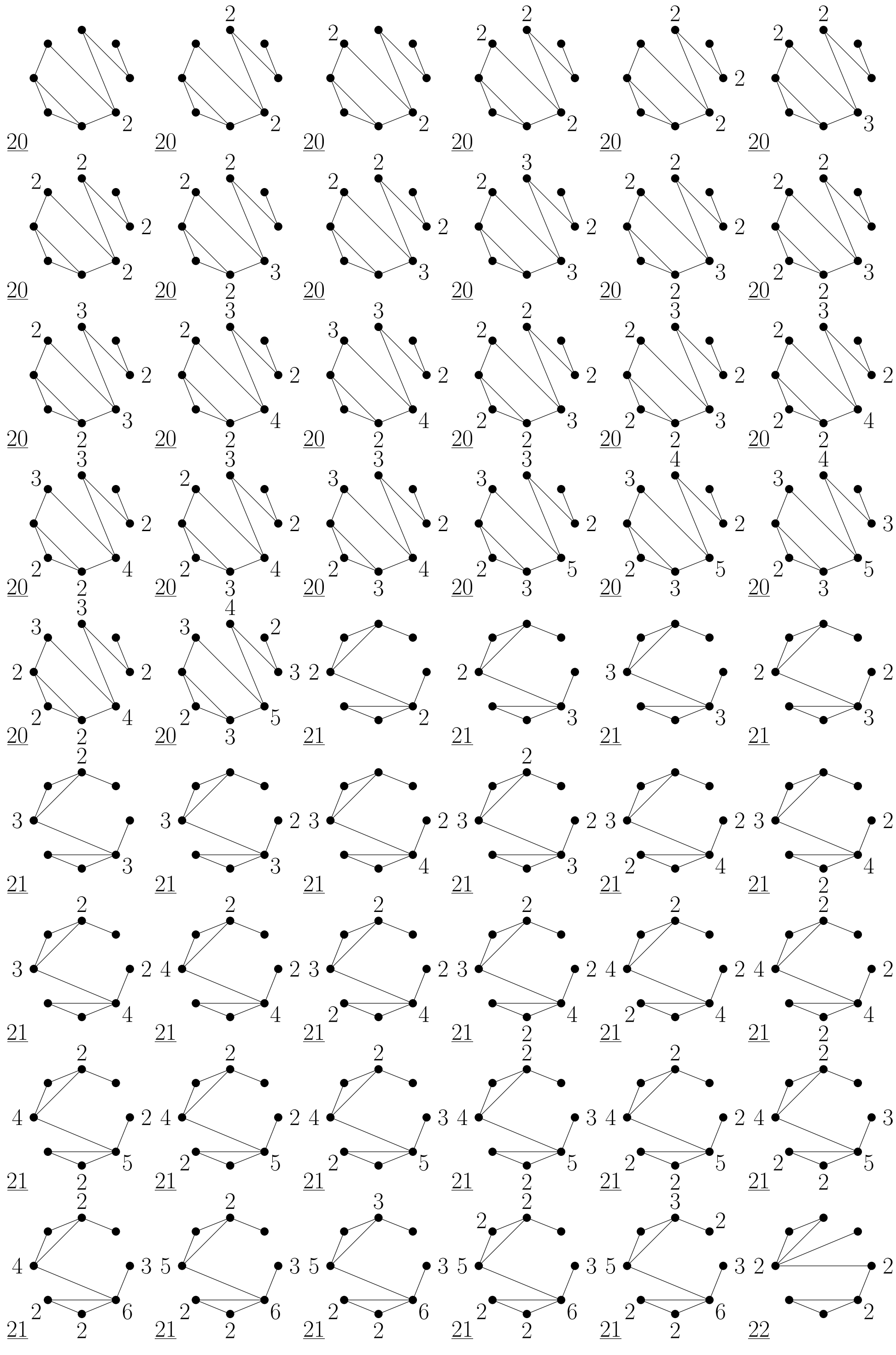


FIGURE 3.2.34. The set $\mathcal{W}(E_8)$ (continued).

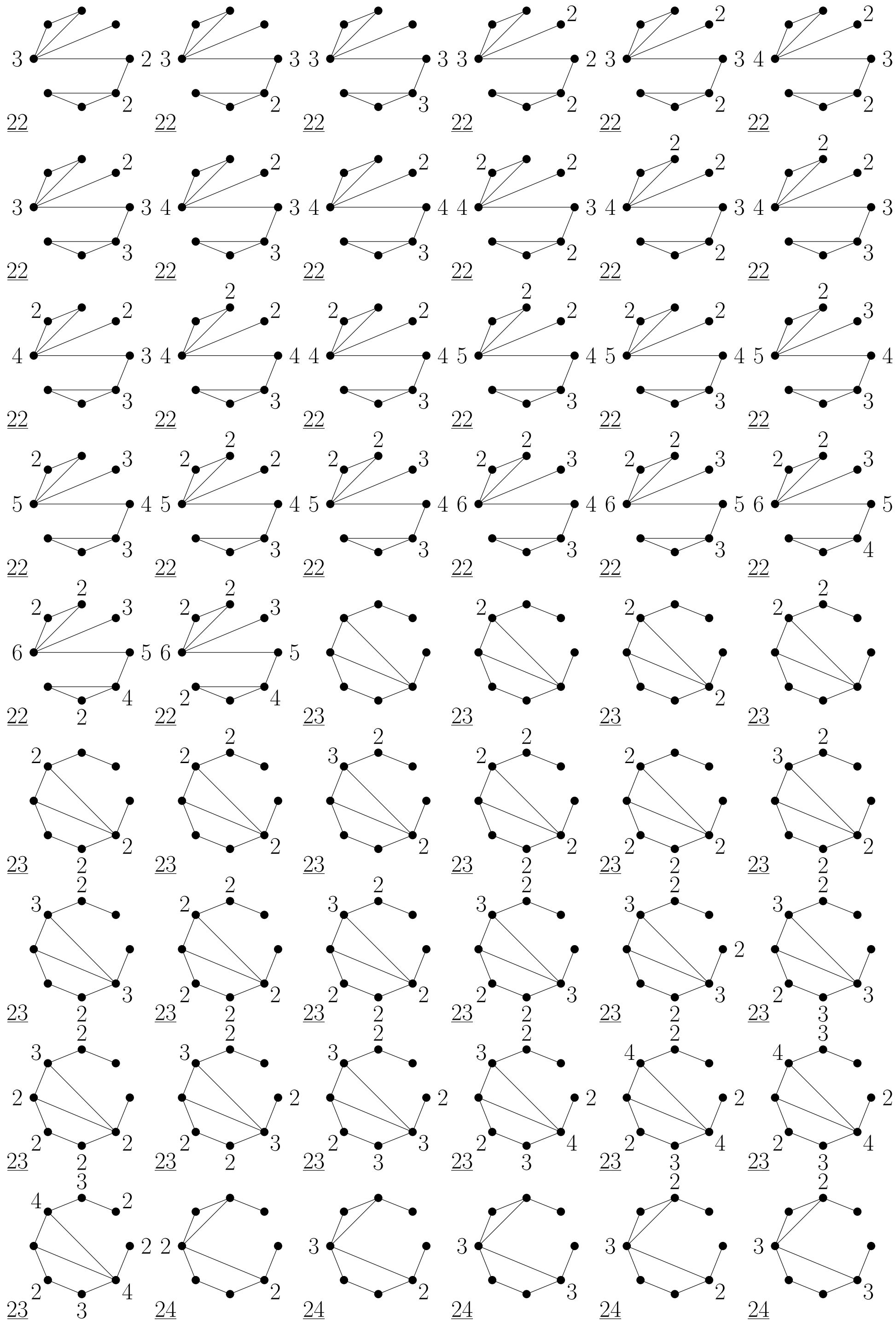


FIGURE 3.2.35. The set $\mathcal{W}(E_8)$ (continued).

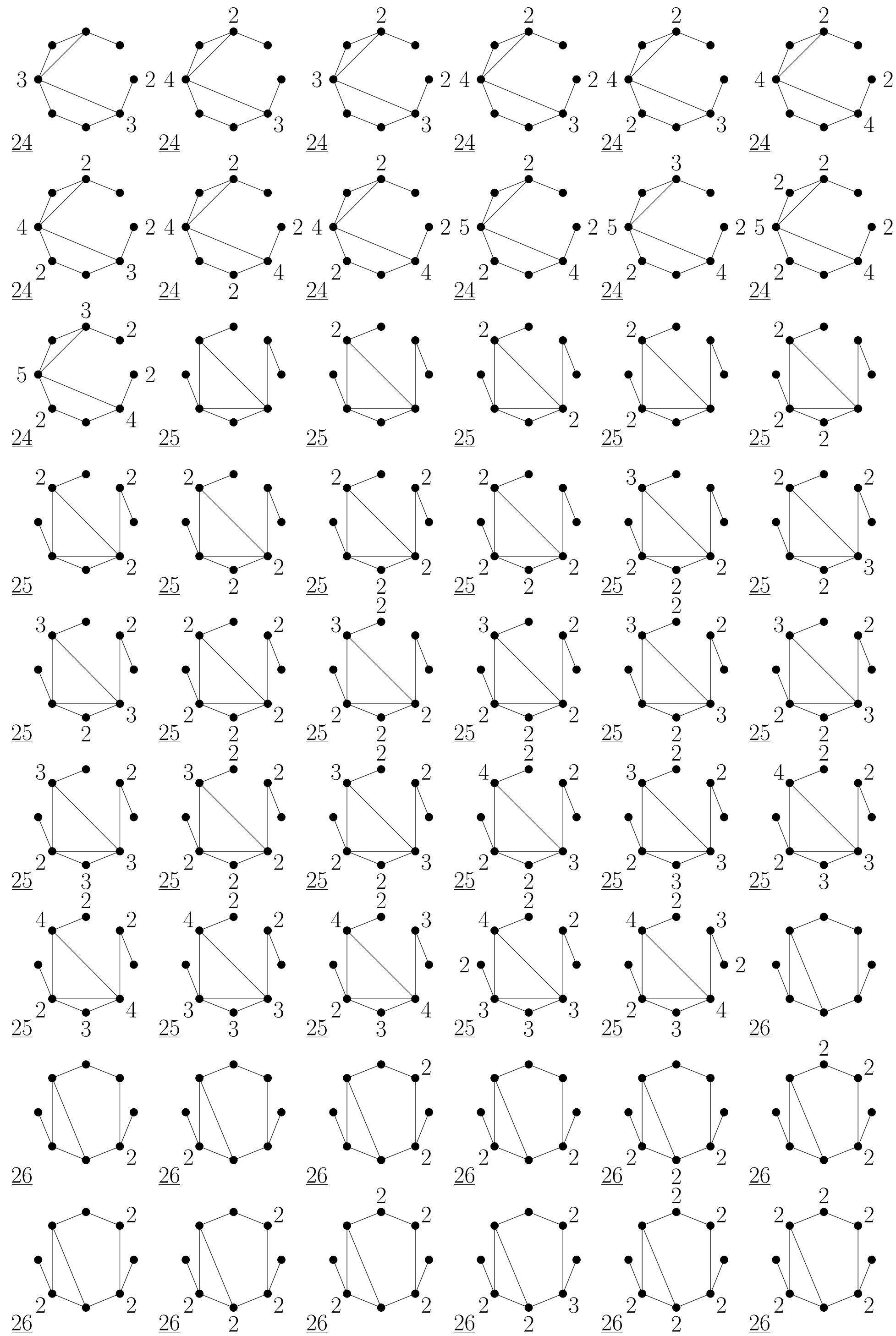


FIGURE 3.2.36. The set $\mathcal{W}(E_8)$ (continued).

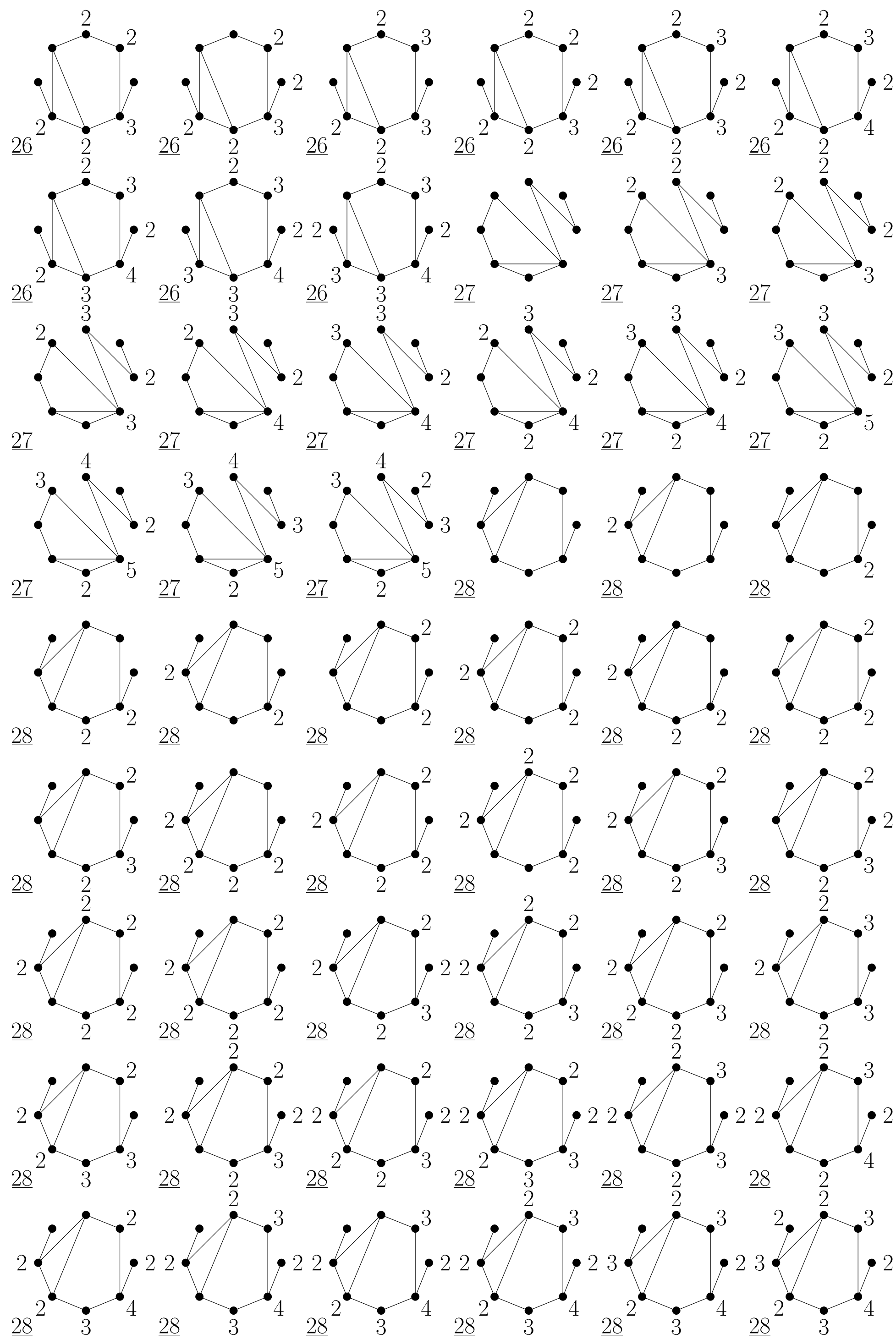


FIGURE 3.2.37. The set $\mathcal{W}(E_8)$ (continued).

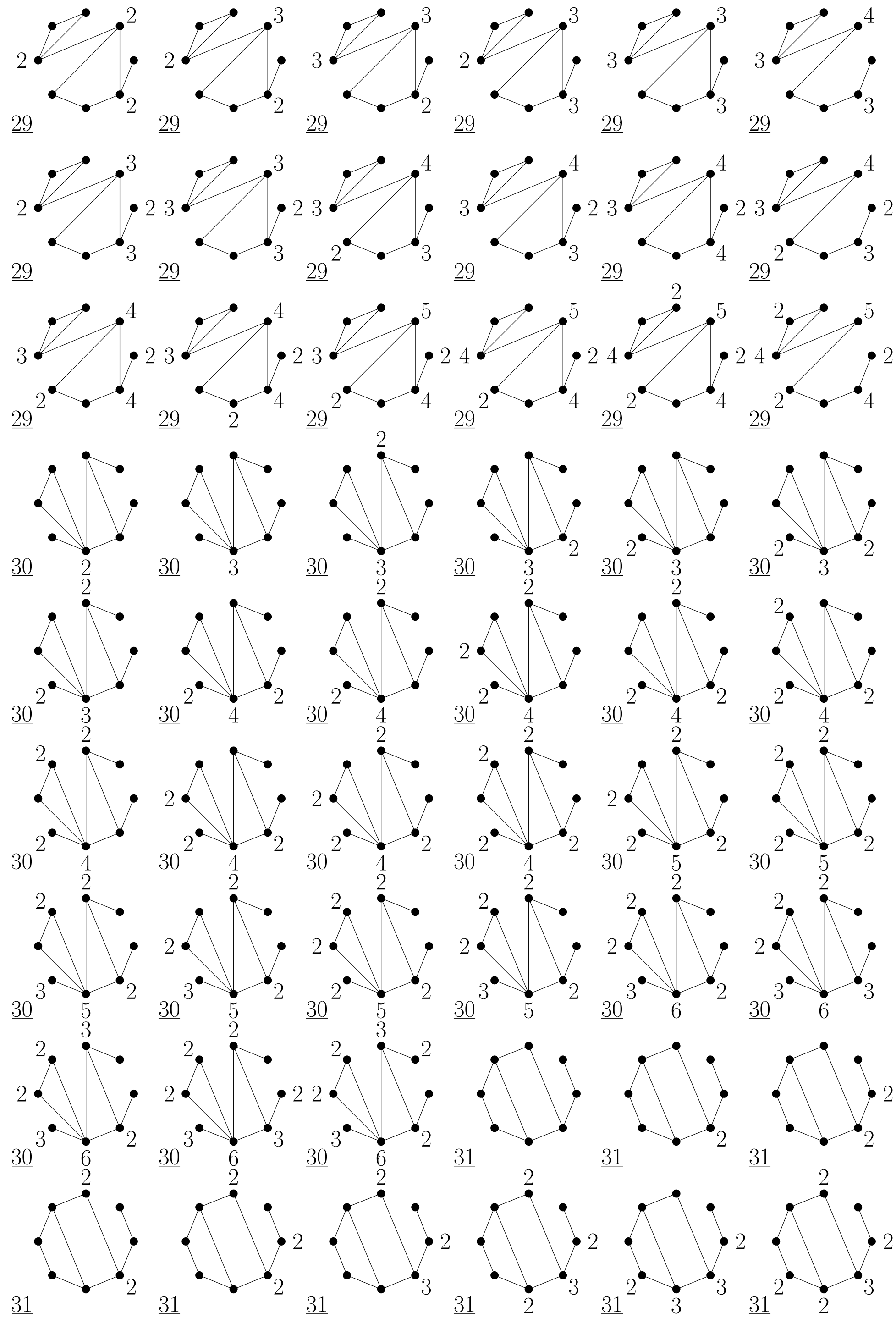


FIGURE 3.2.38. The set $\mathcal{W}(E_8)$ (continued).

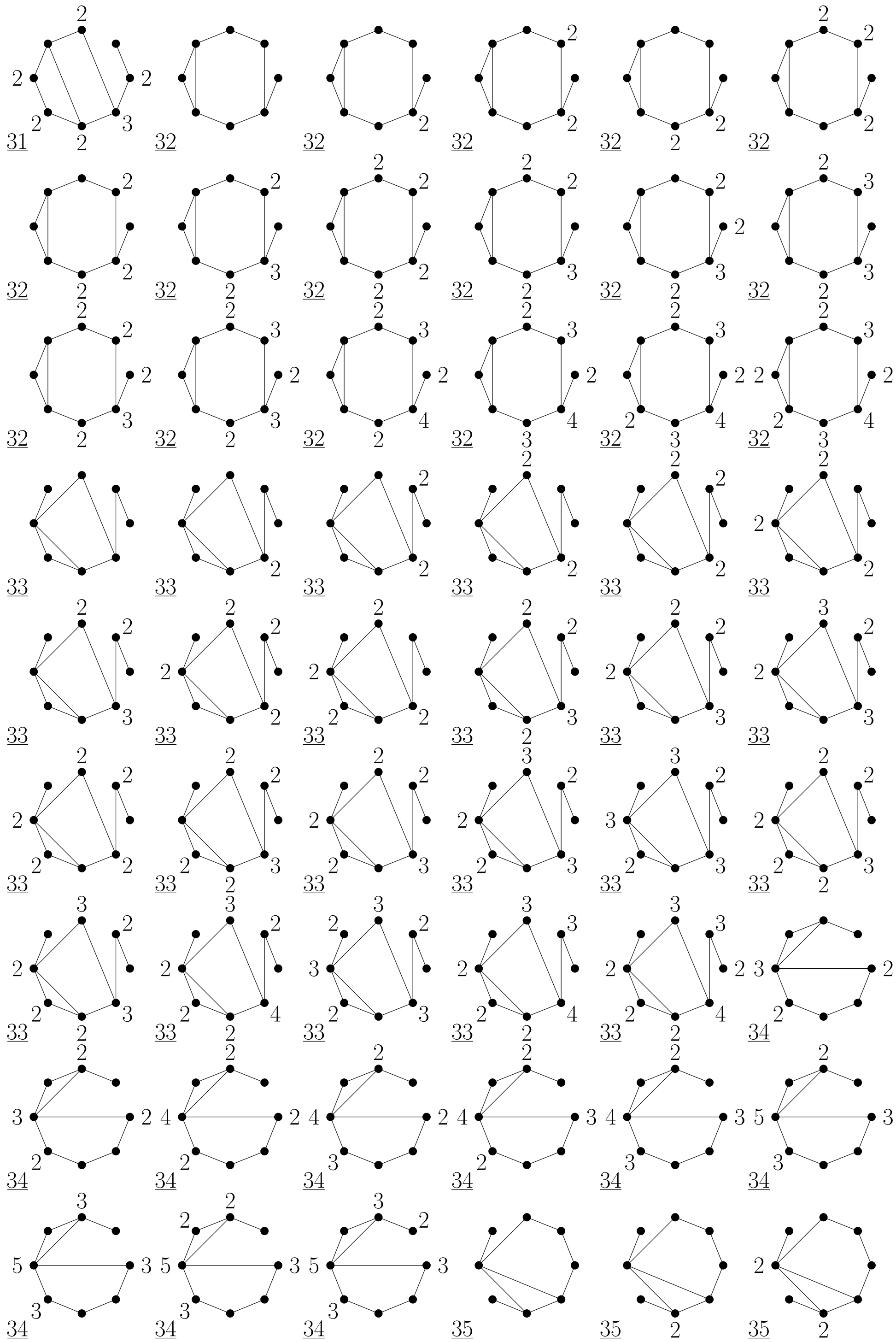


FIGURE 3.2.39. The set $\mathcal{W}(E_8)$ (continued).

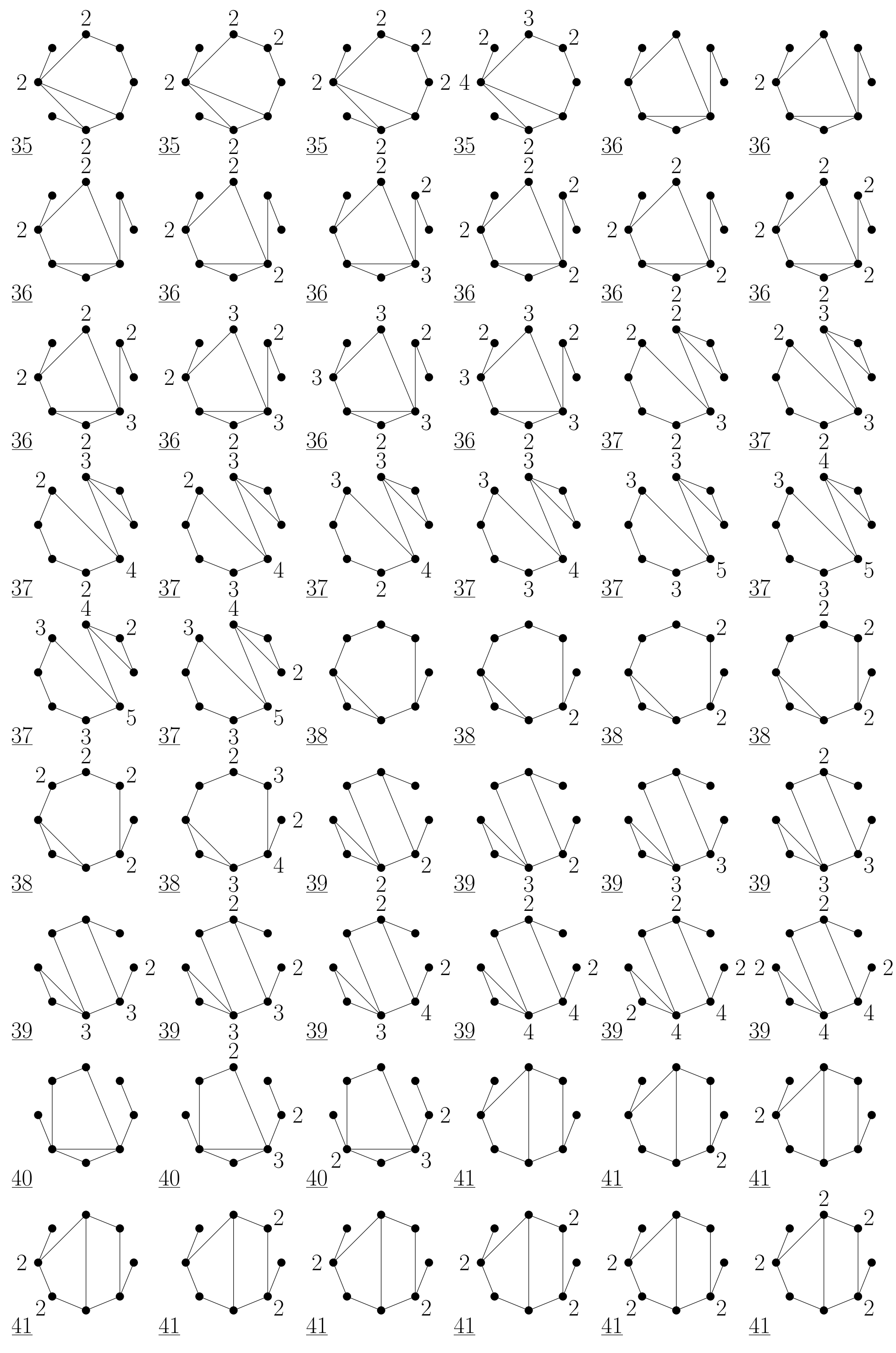


FIGURE 3.2.40. The set $\mathcal{W}(E_8)$ (continued).

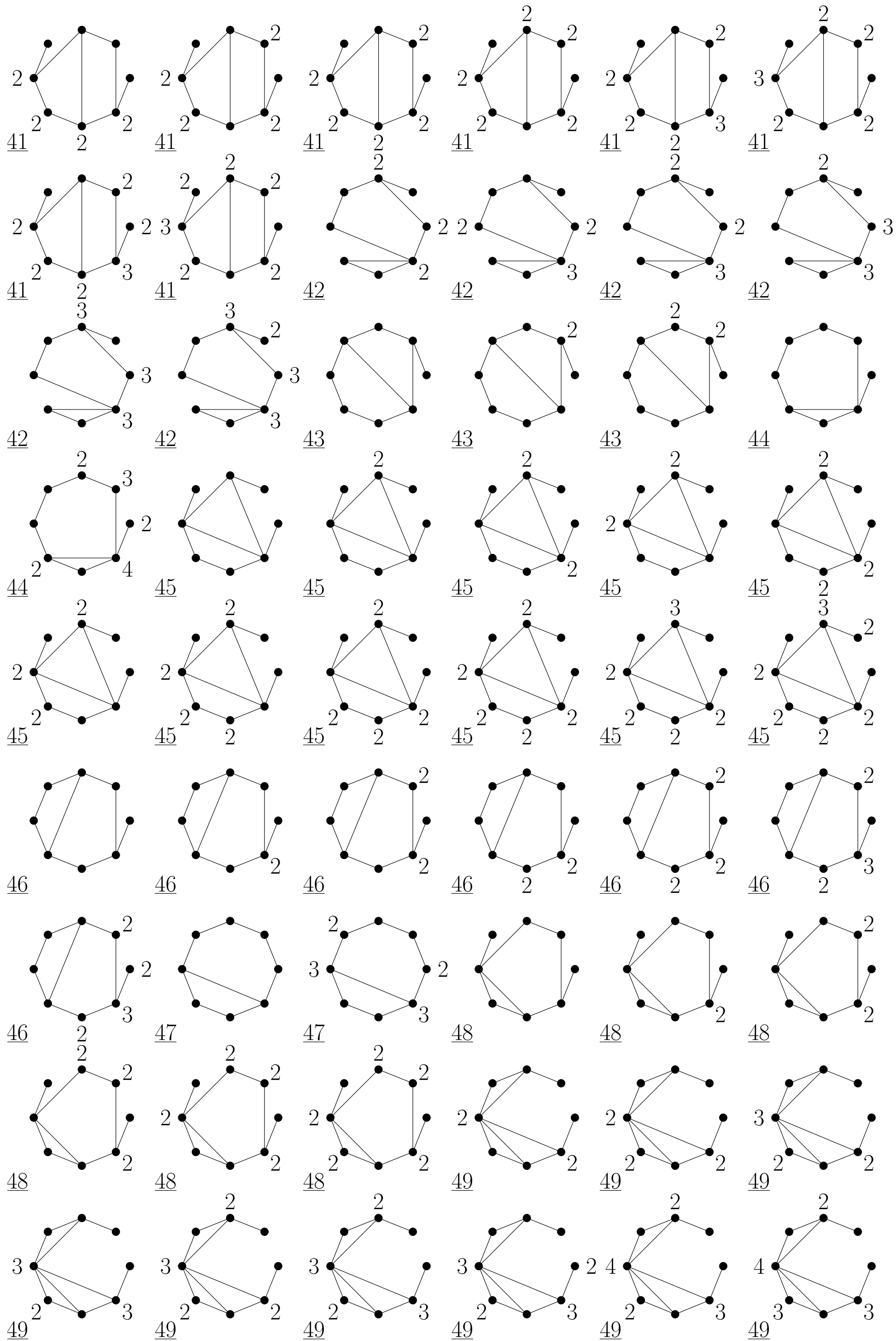


FIGURE 3.2.41. The set $\mathcal{W}(E_8)$ (continued).

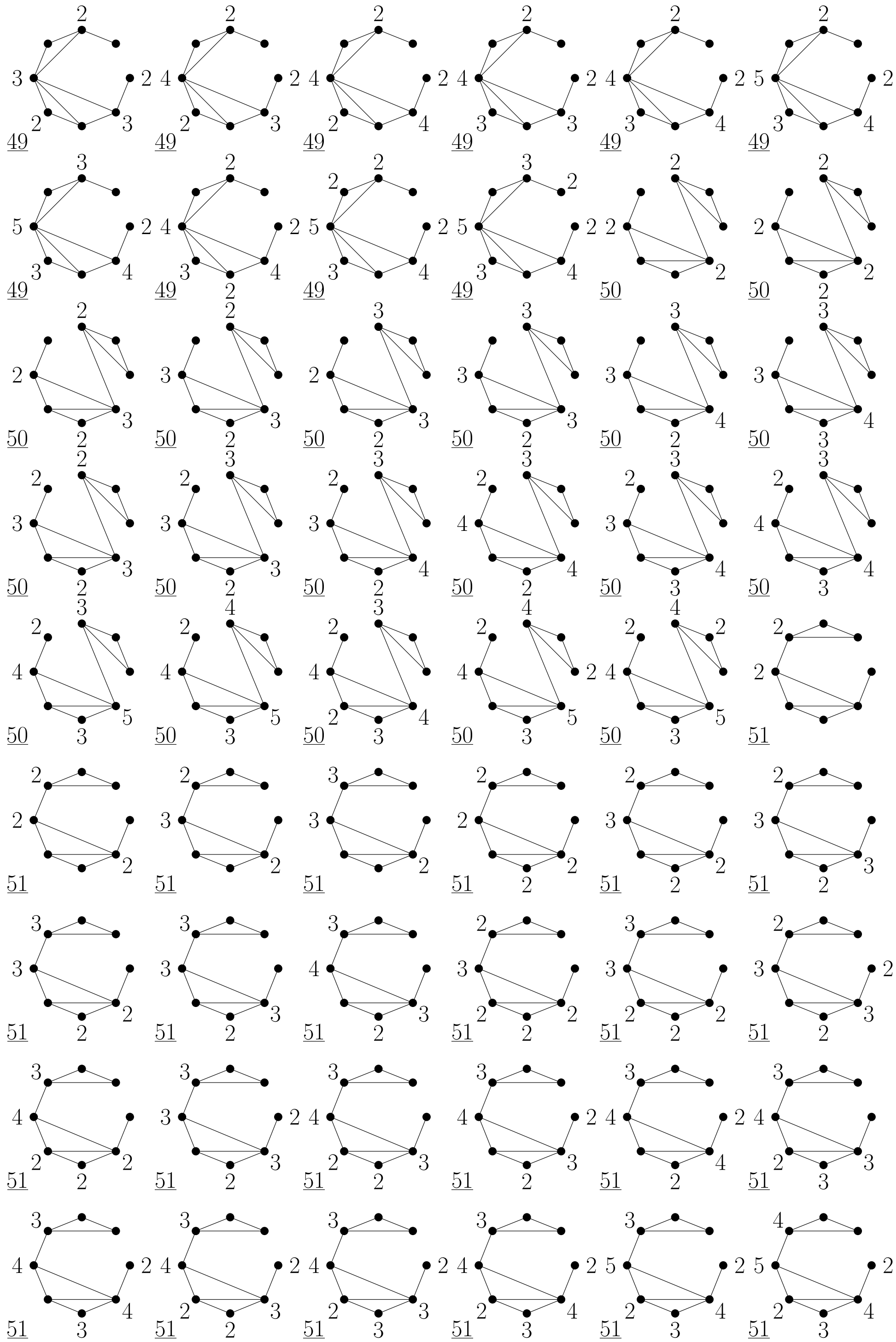


FIGURE 3.2.42. The set $\mathcal{W}(E_8)$ (continued).

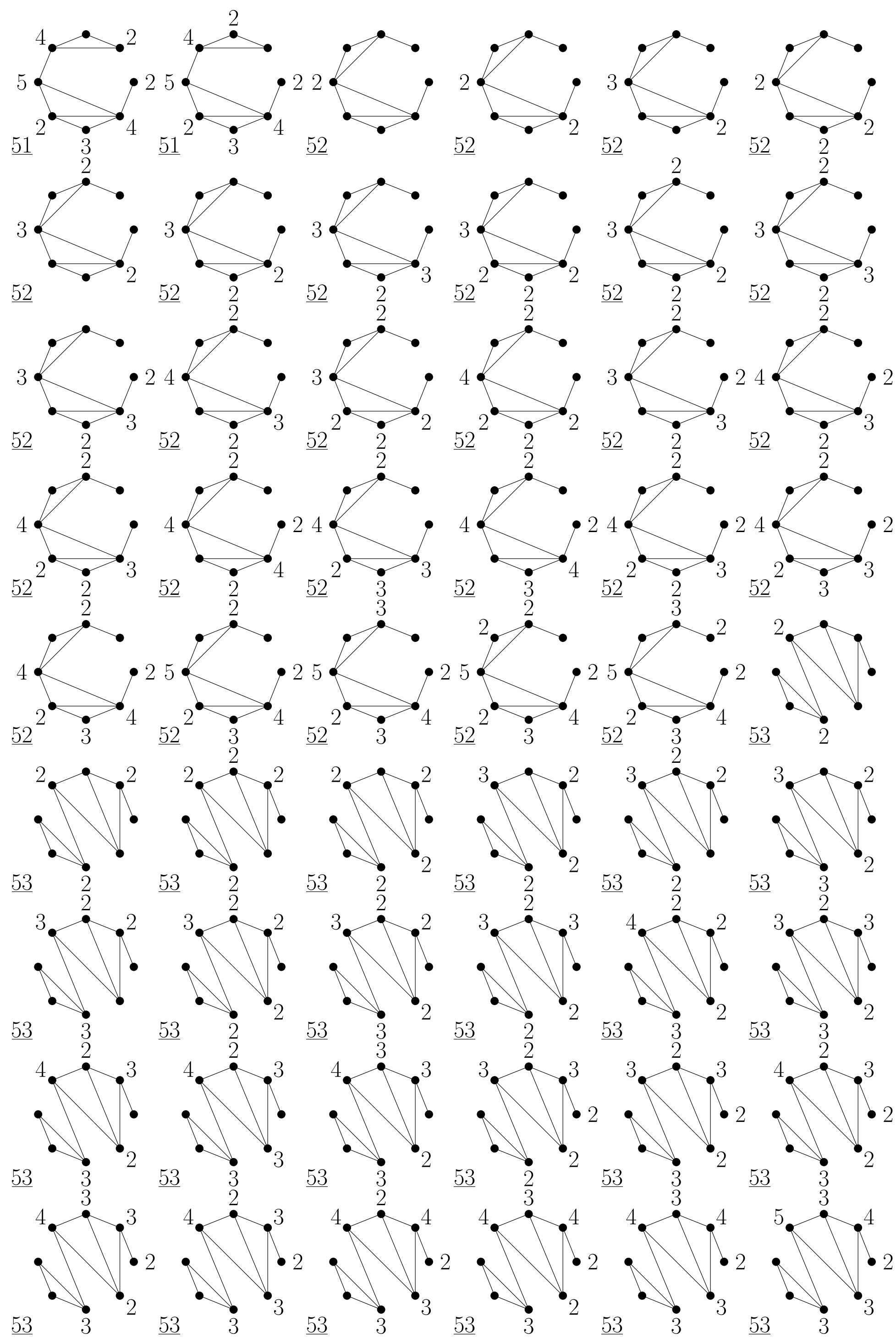


FIGURE 3.2.43. The set $\mathcal{W}(E_8)$ (continued).

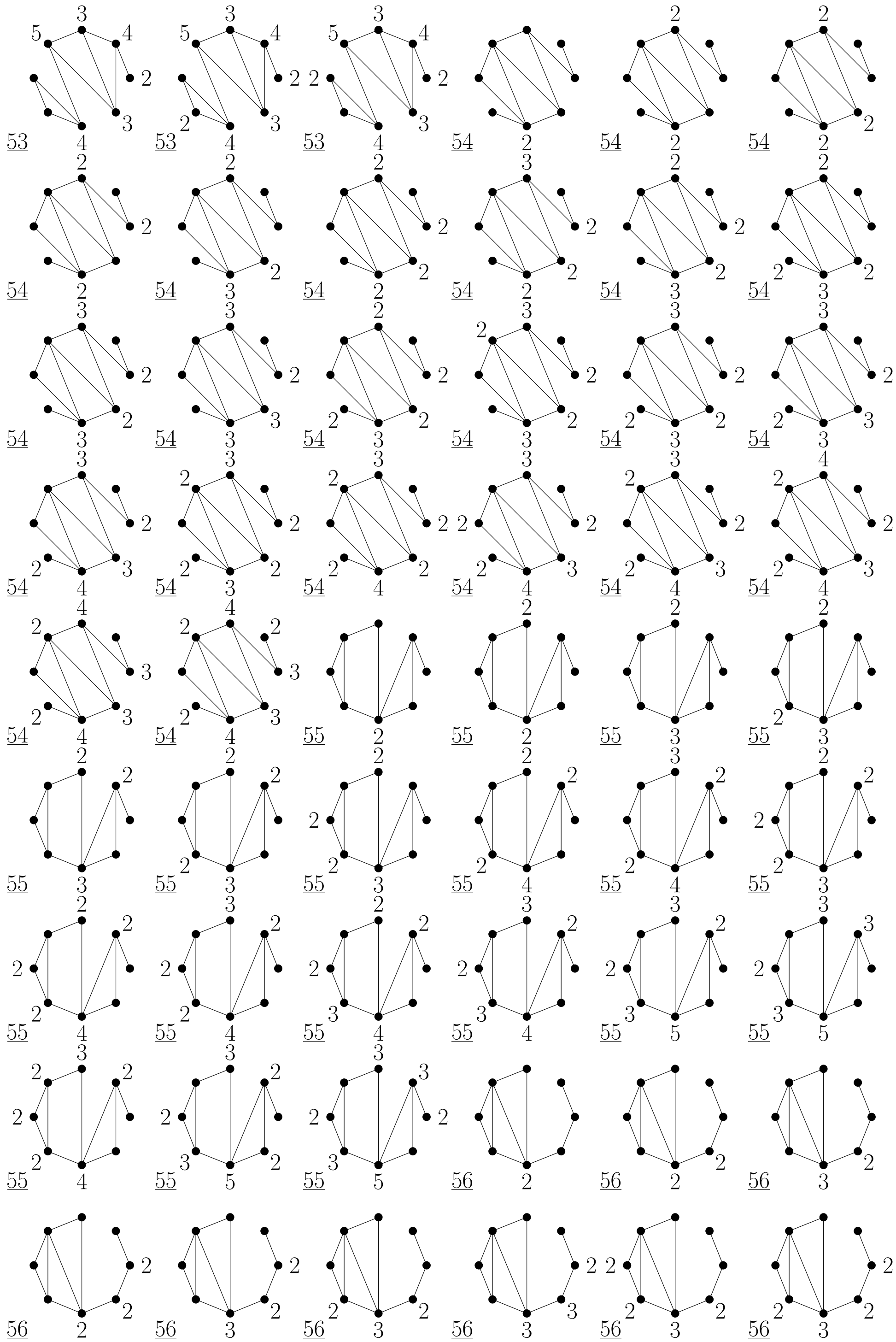


FIGURE 3.2.44. The set $\mathcal{W}(E_8)$ (continued).

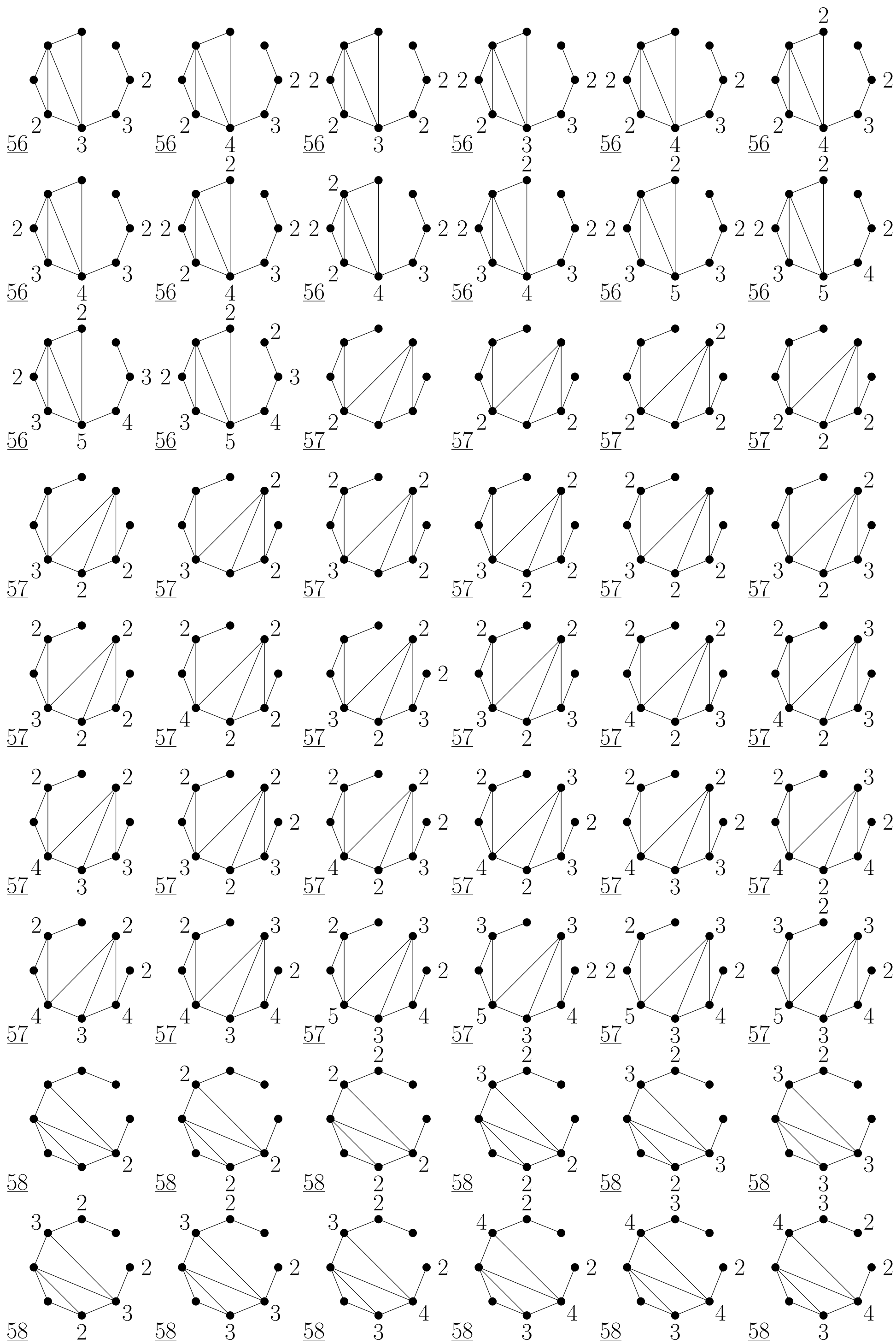


FIGURE 3.2.45. The set $\mathcal{W}(E_8)$ (continued).

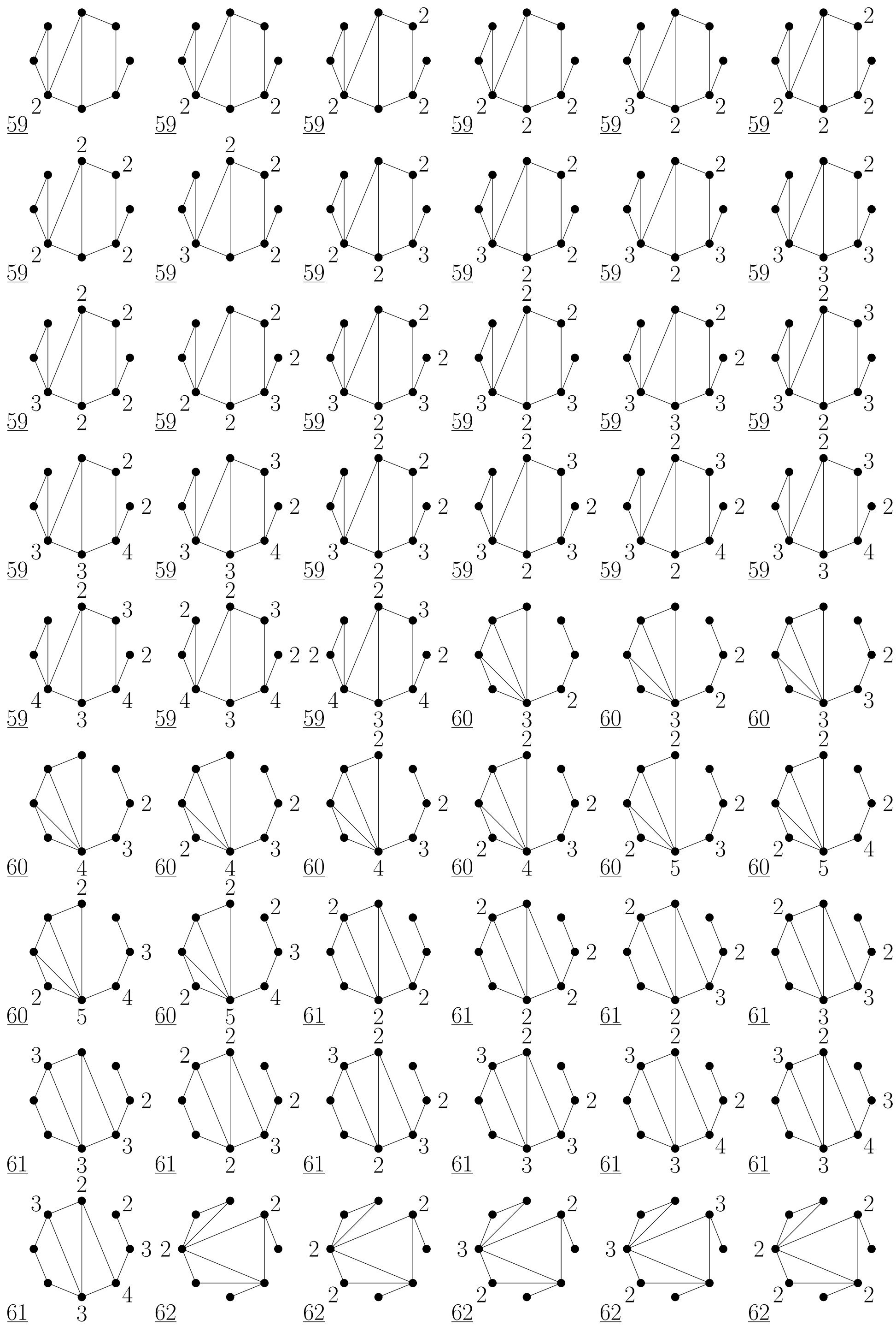


FIGURE 3.2.46. The set $\mathcal{W}(E_8)$ (continued).

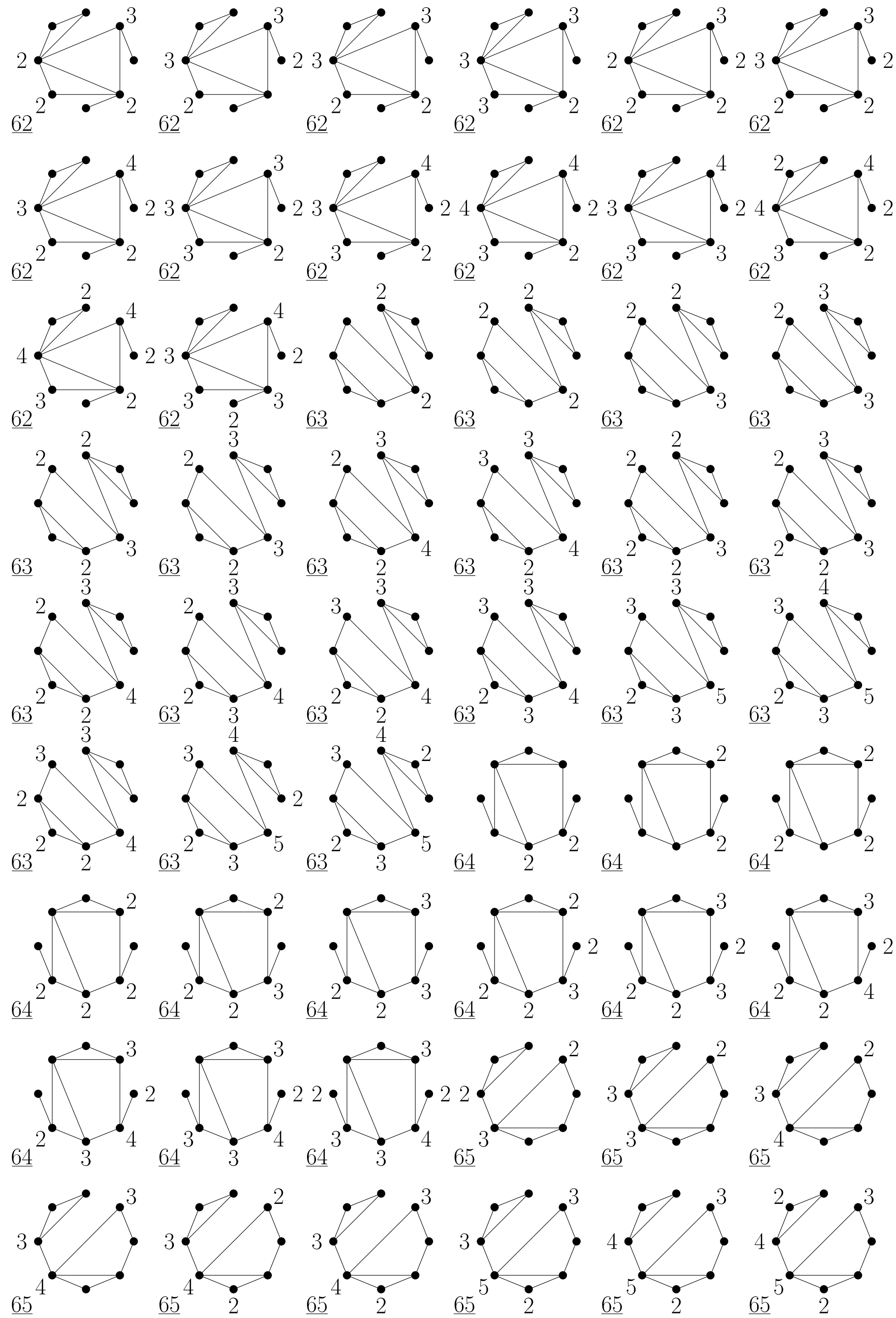


FIGURE 3.2.47. The set $\mathcal{W}(E_8)$ (continued).

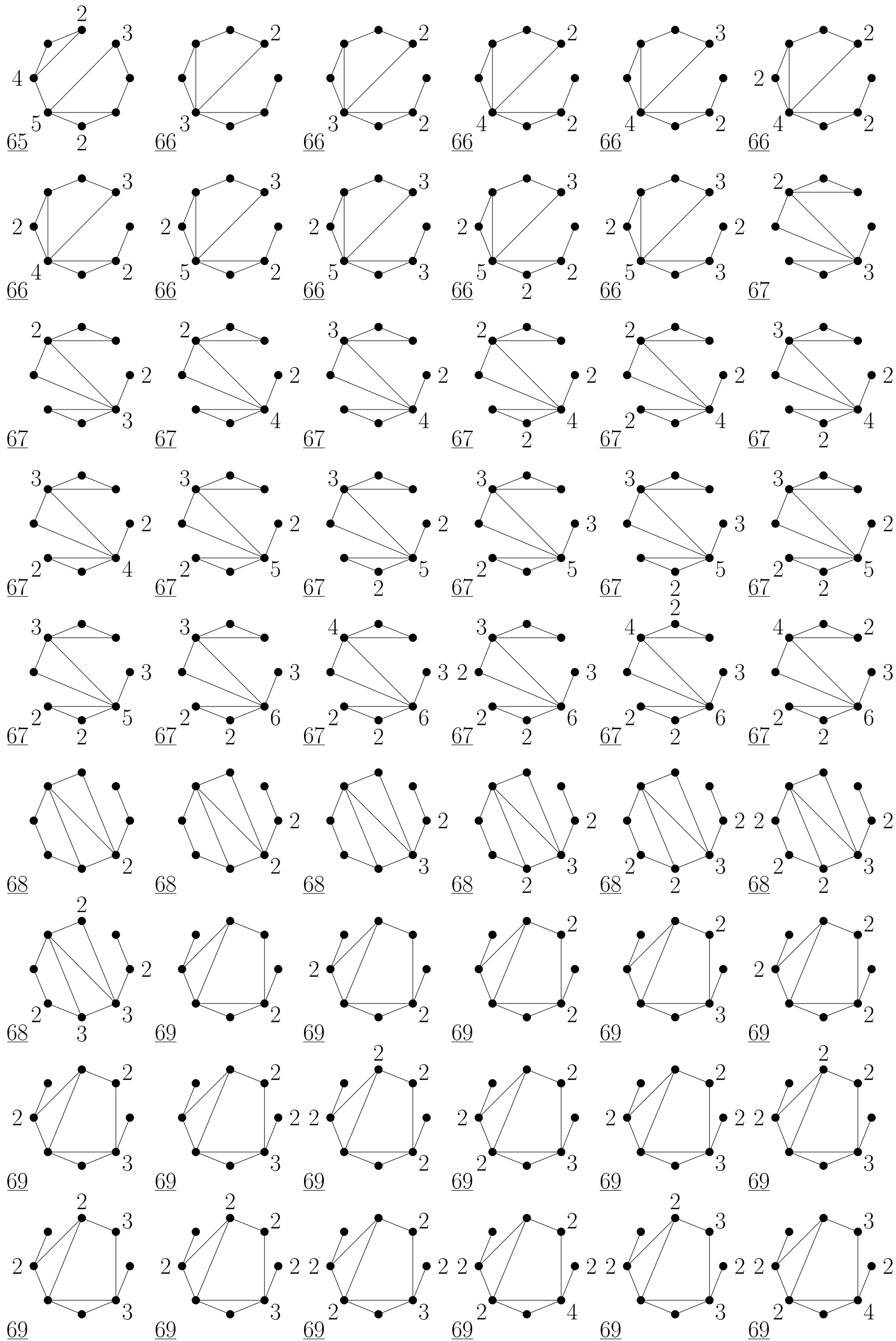


FIGURE 3.2.48. The set $\mathcal{W}(E_8)$ (continued).

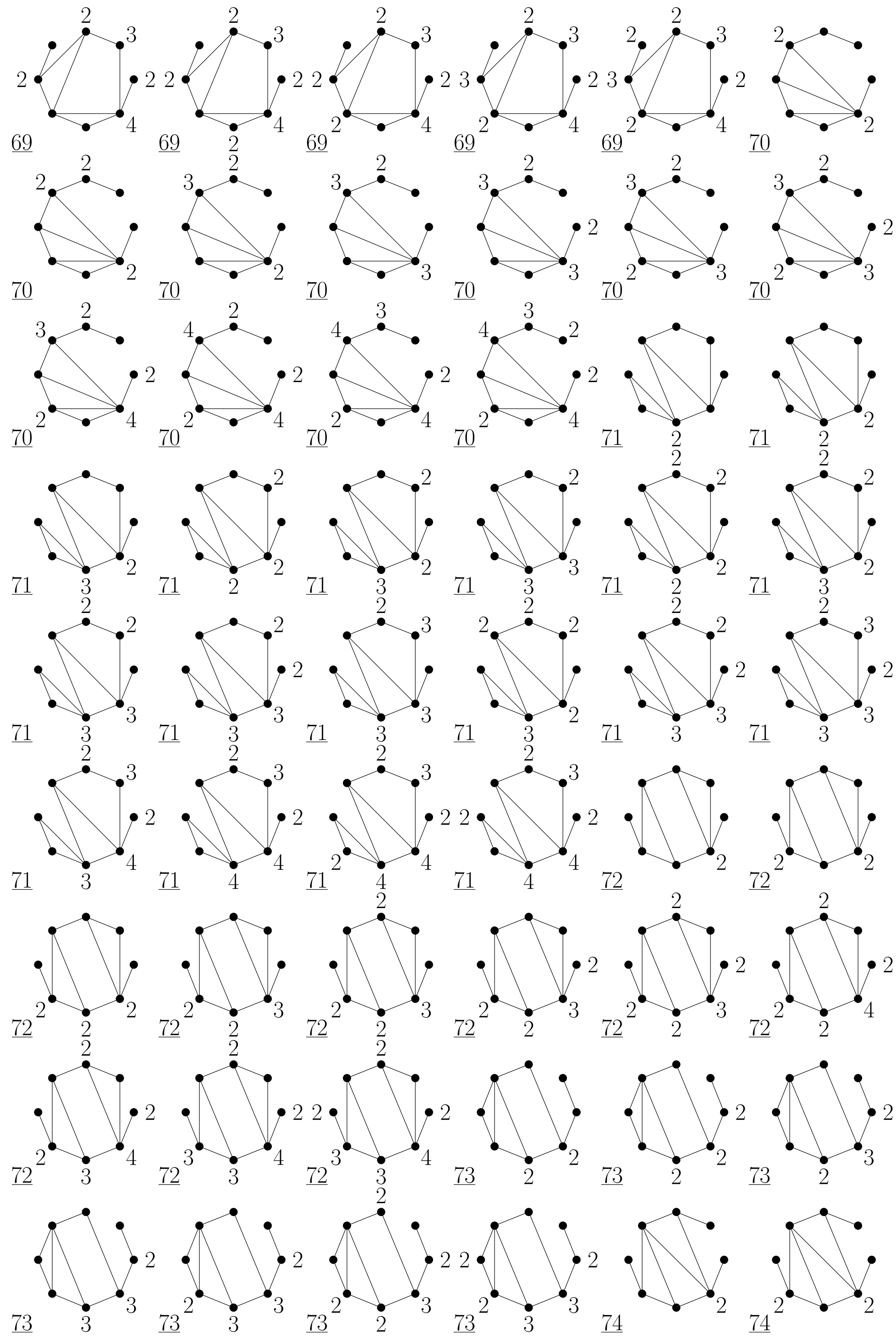


FIGURE 3.2.49. The set $\mathcal{W}(E_8)$ (continued).

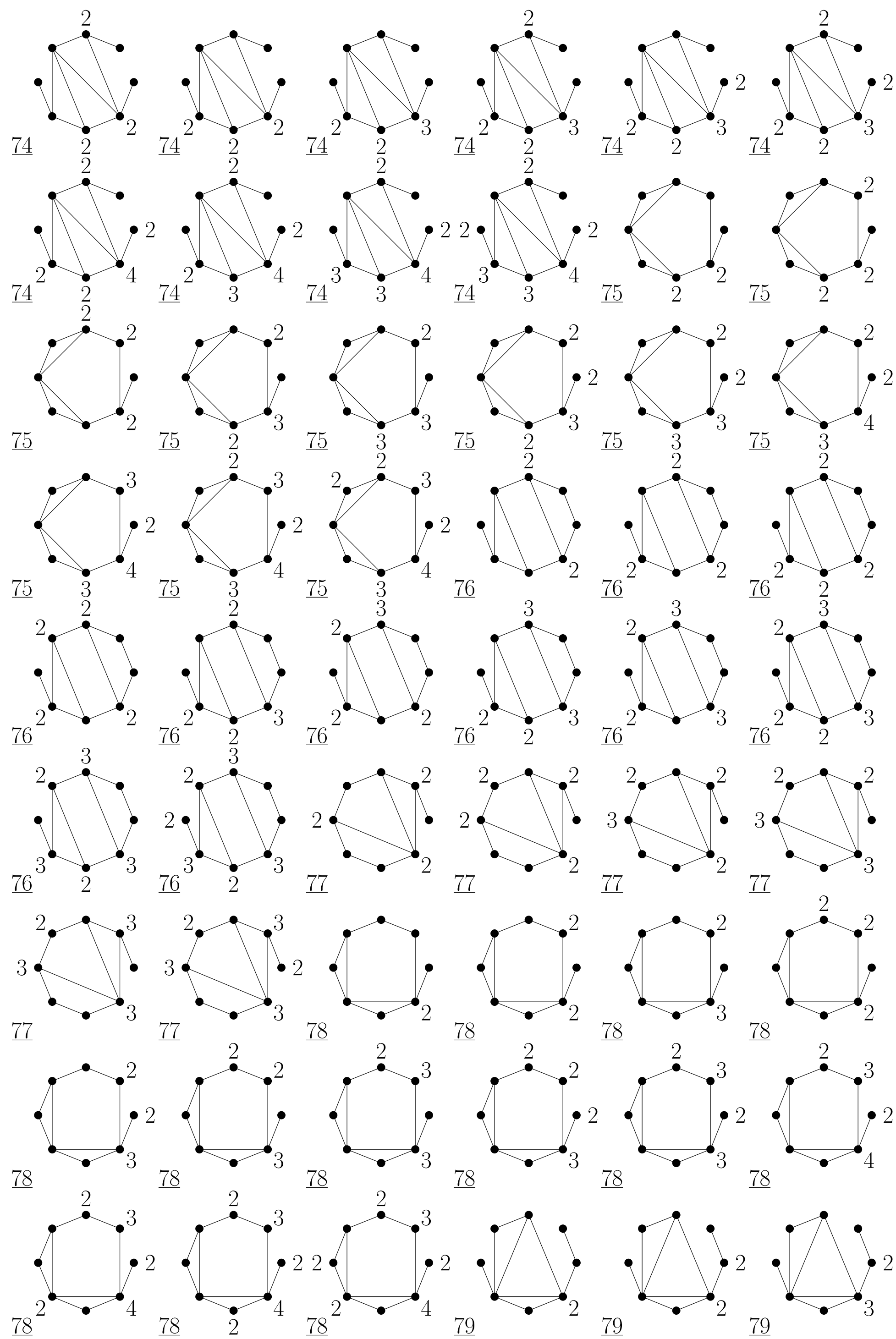


FIGURE 3.2.50. The set $\mathcal{W}(E_8)$ (continued).

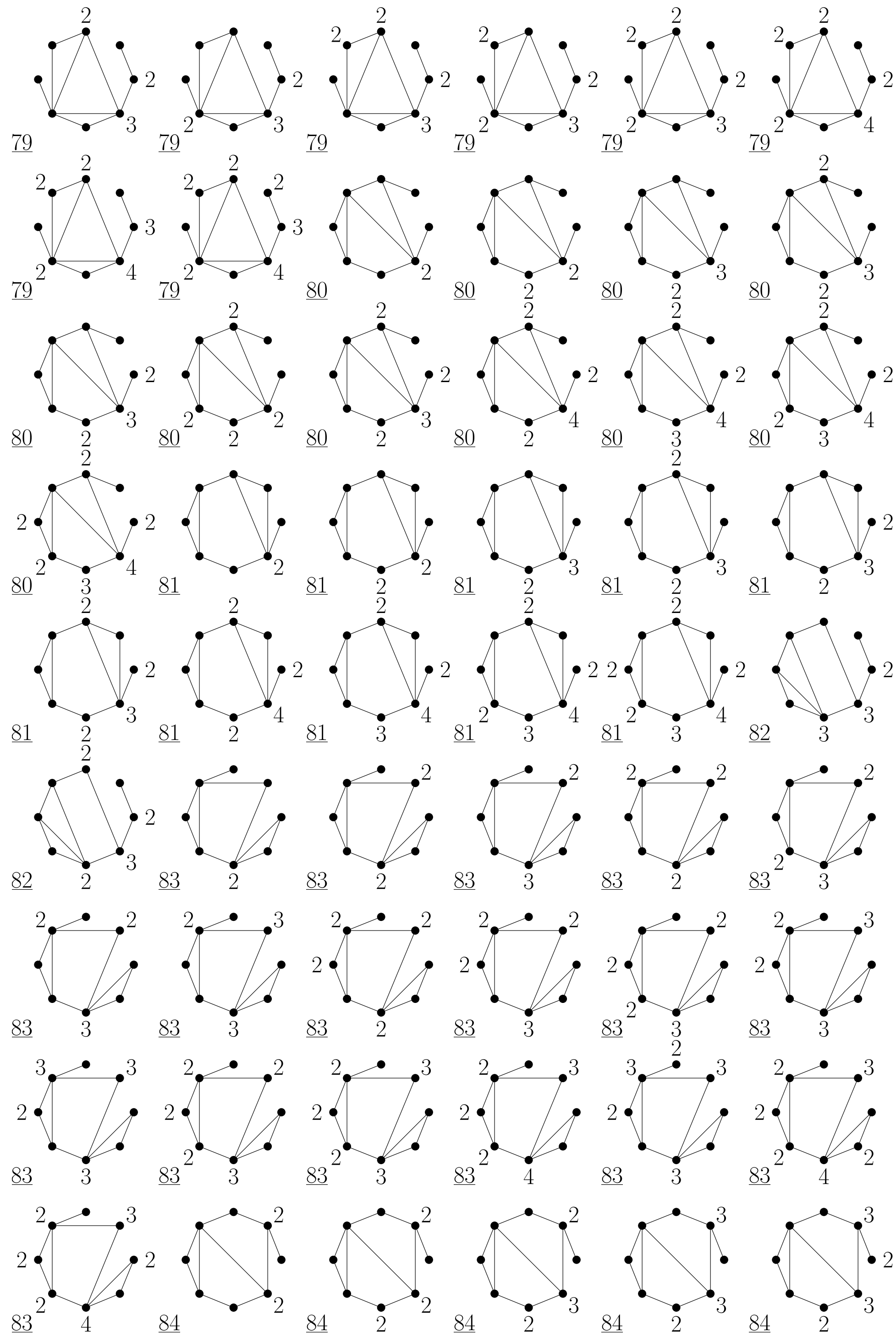


FIGURE 3.2.51. The set $\mathcal{W}(E_8)$ (continued).

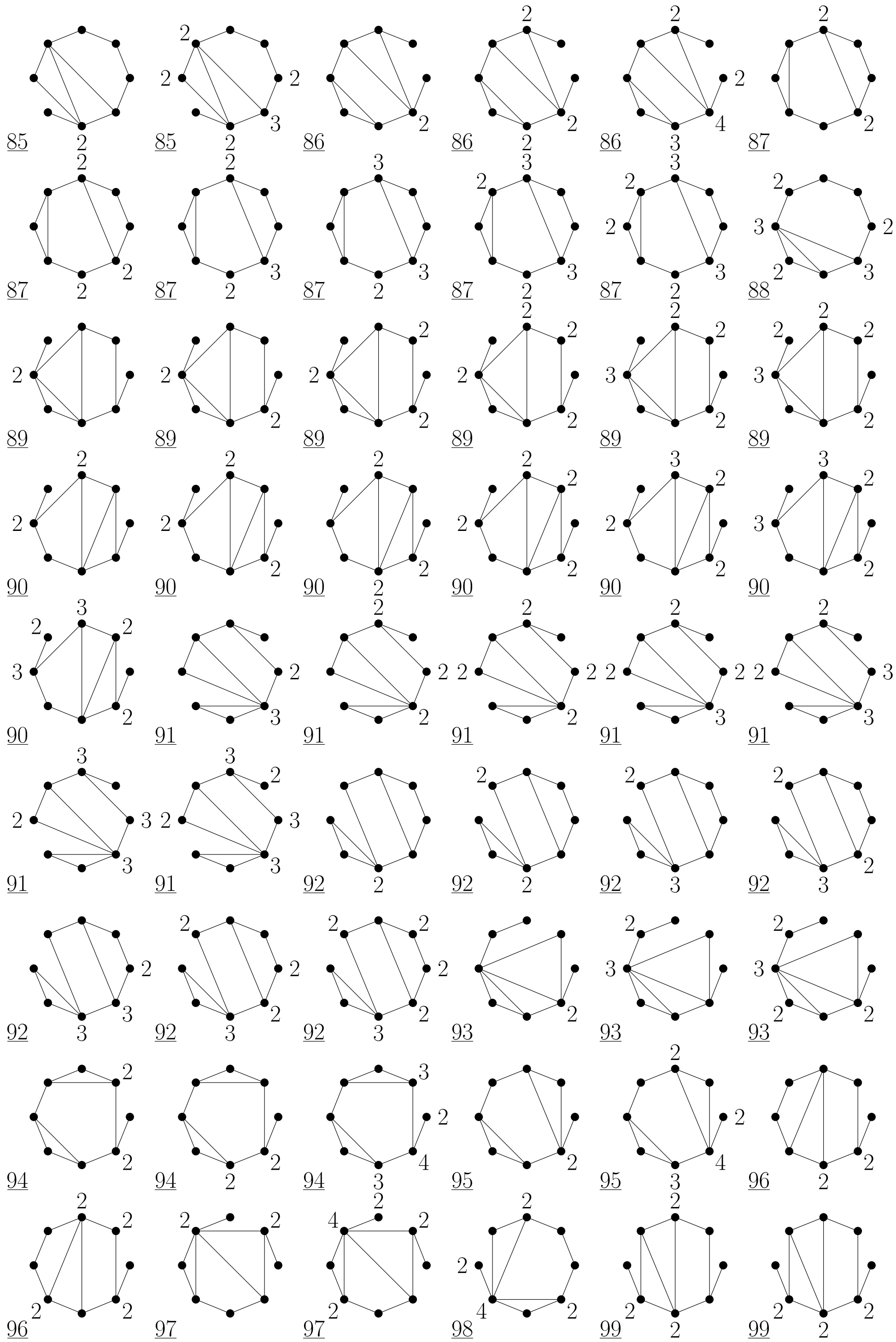


FIGURE 3.2.52. The set $\mathcal{W}(E_8)$ (continued).

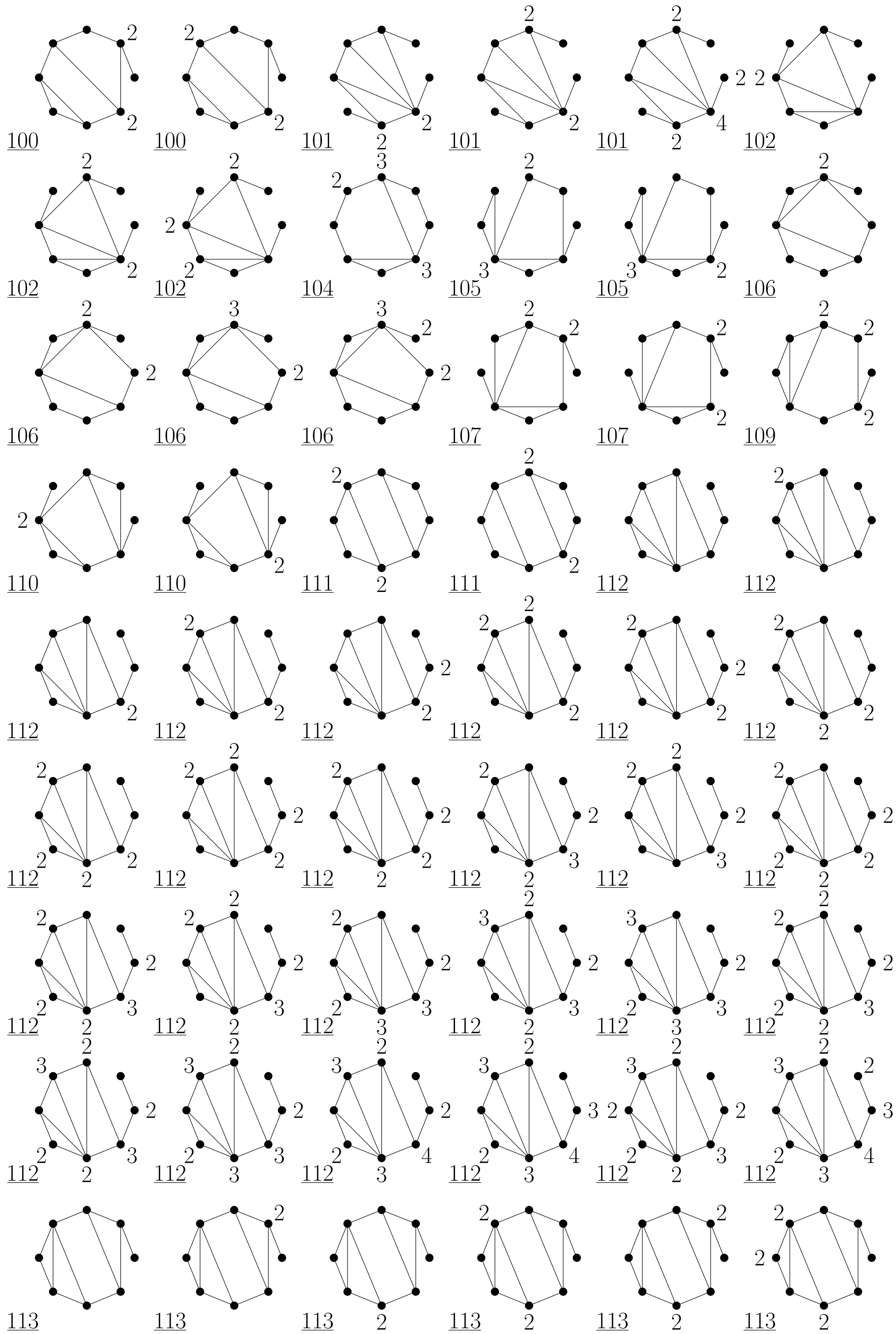


FIGURE 3.2.53. The set $\mathcal{W}(E_8)$ (continued).

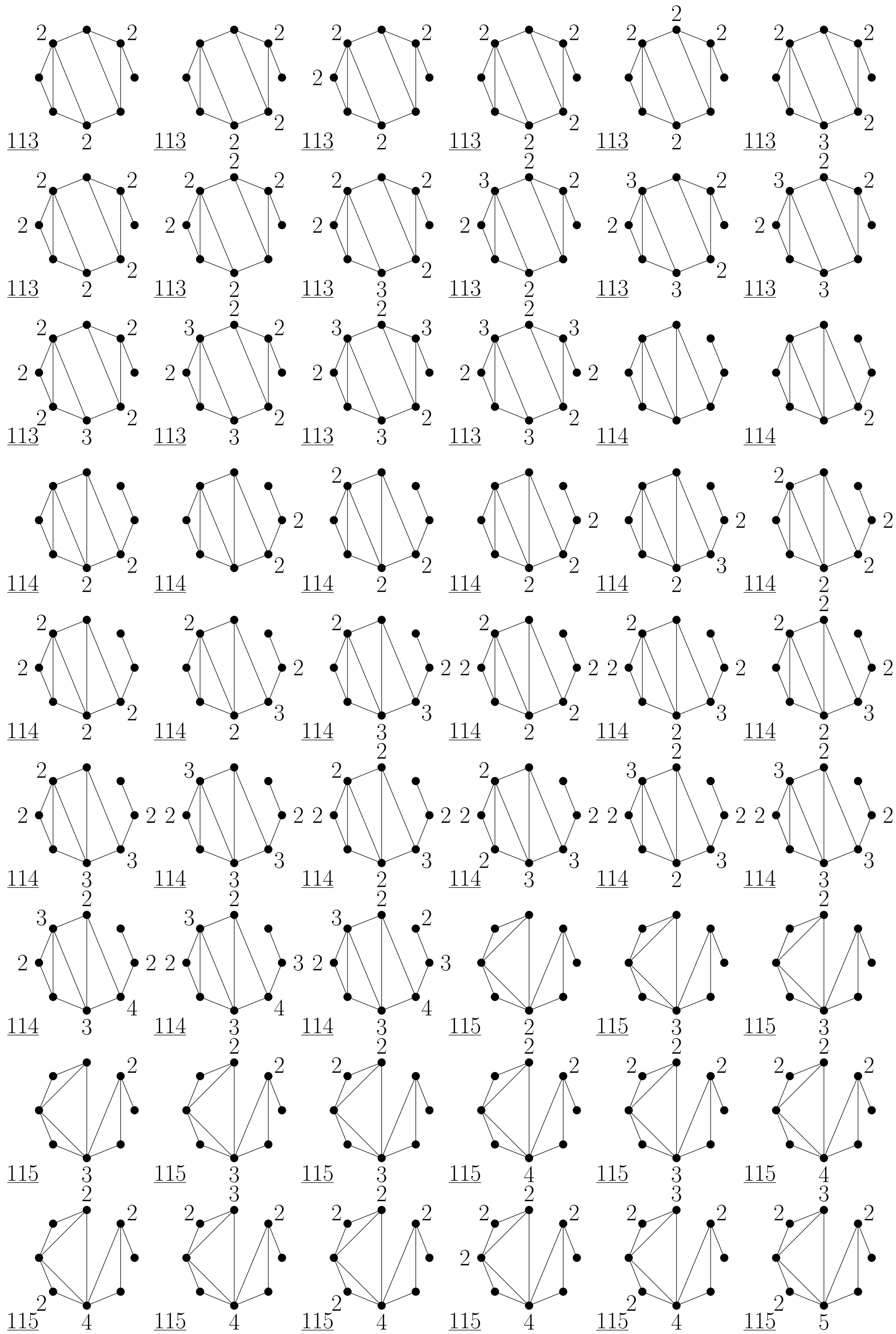


FIGURE 3.2.54. The set $\mathcal{W}(E_8)$ (continued).

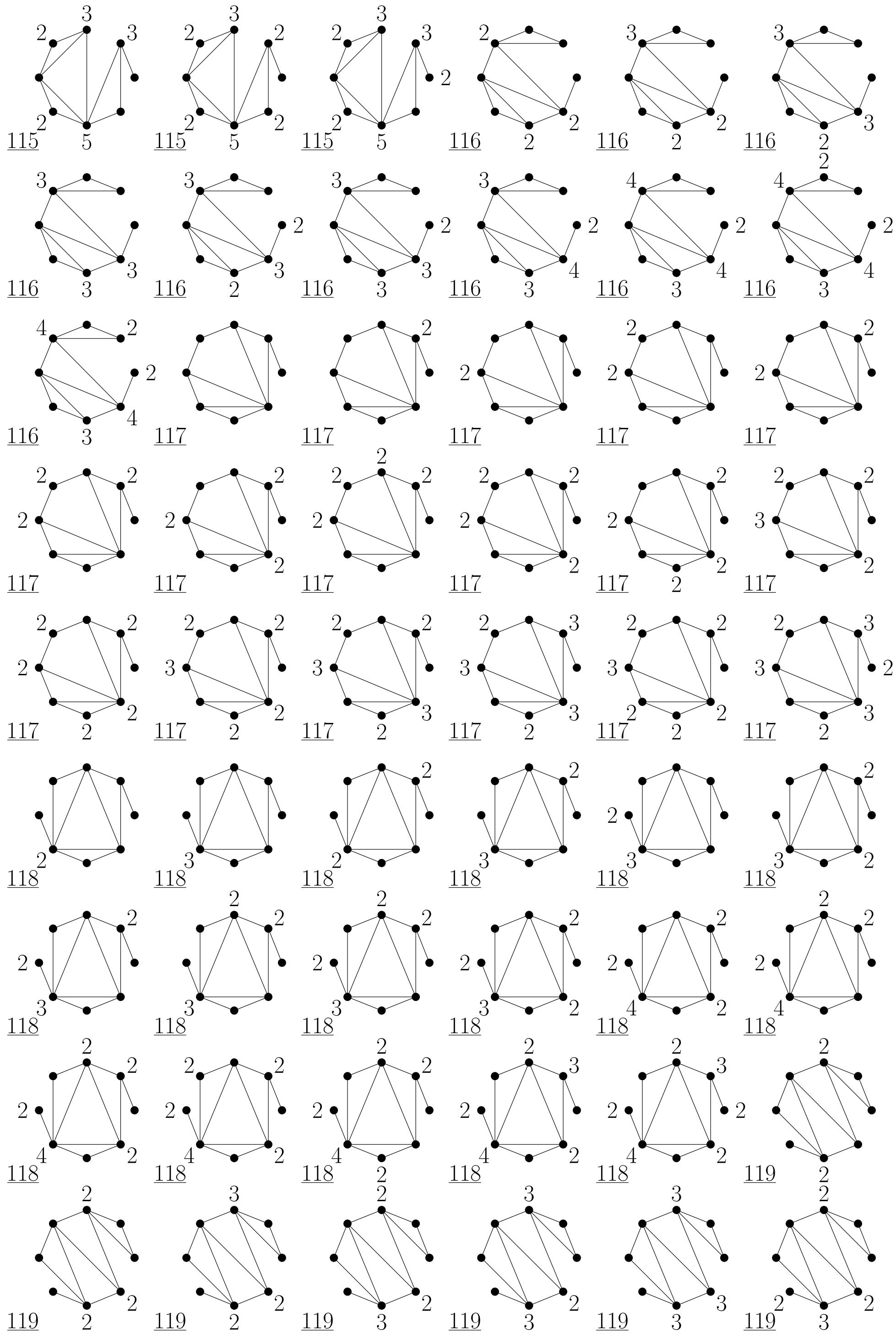


FIGURE 3.2.55. The set $\mathcal{W}(E_8)$ (continued).

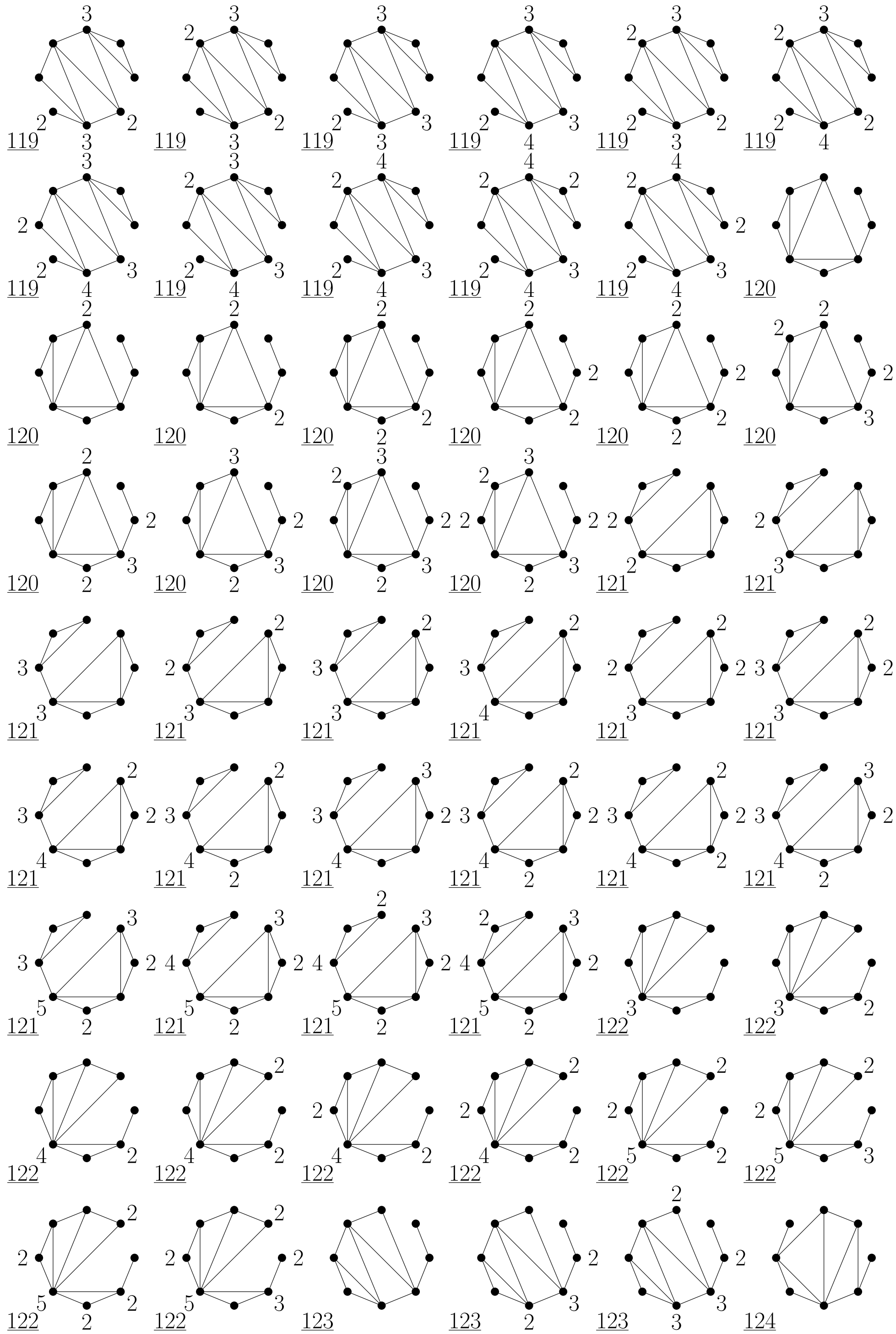


FIGURE 3.2.56. The set $\mathcal{W}(E_8)$ (continued).

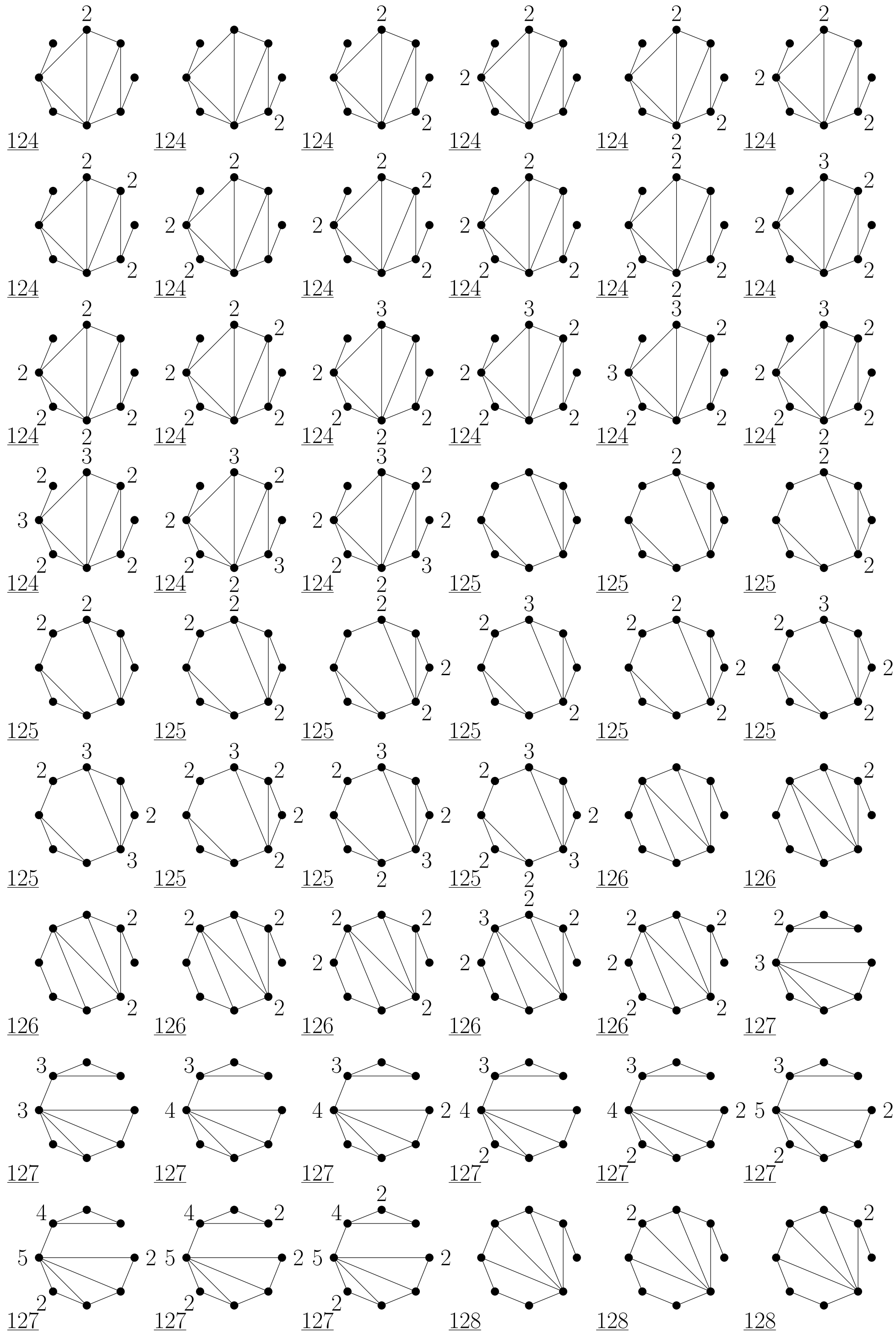


FIGURE 3.2.57. The set $\mathcal{W}(E_8)$ (continued).

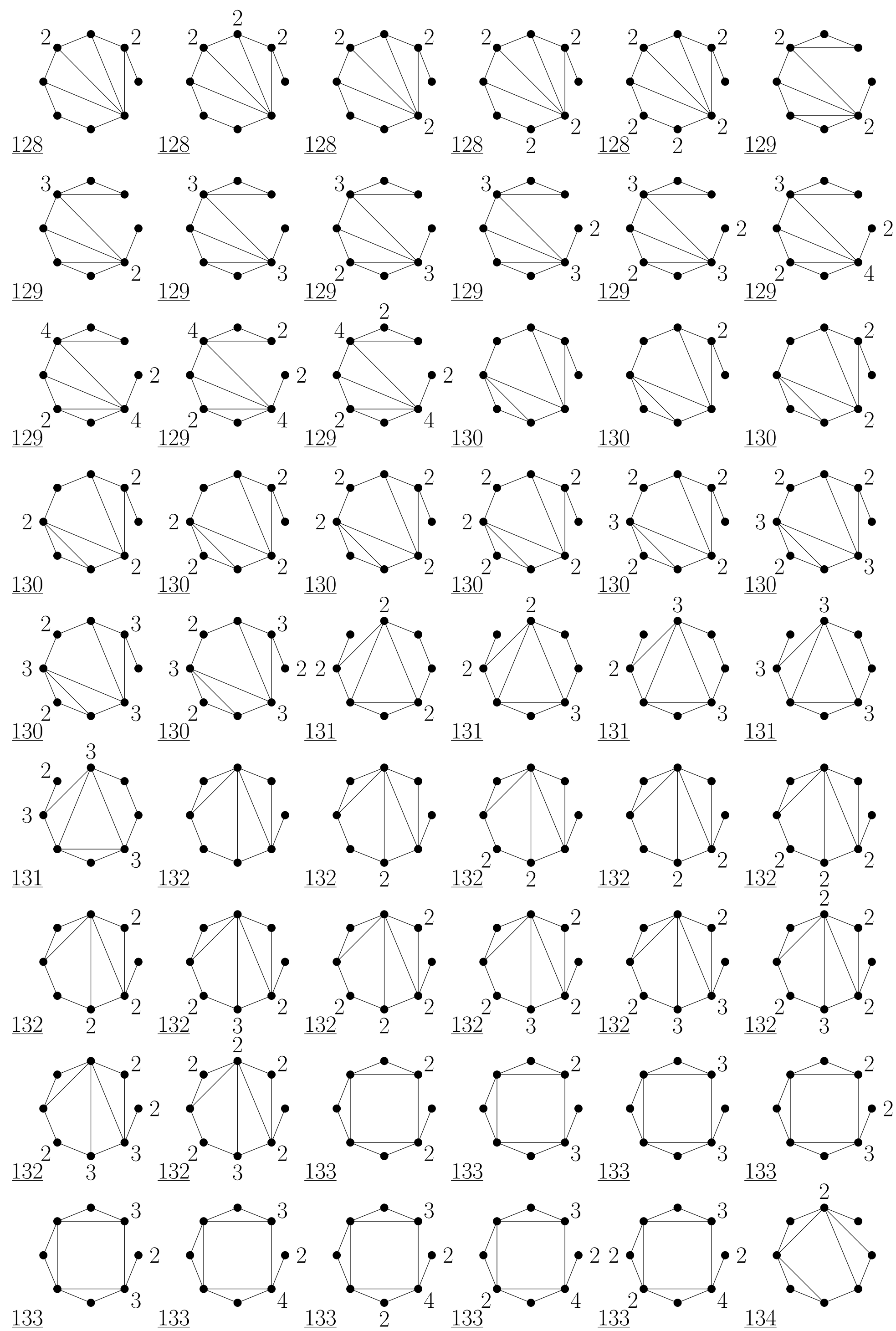


FIGURE 3.2.58. The set $\mathcal{W}(E_8)$ (continued).

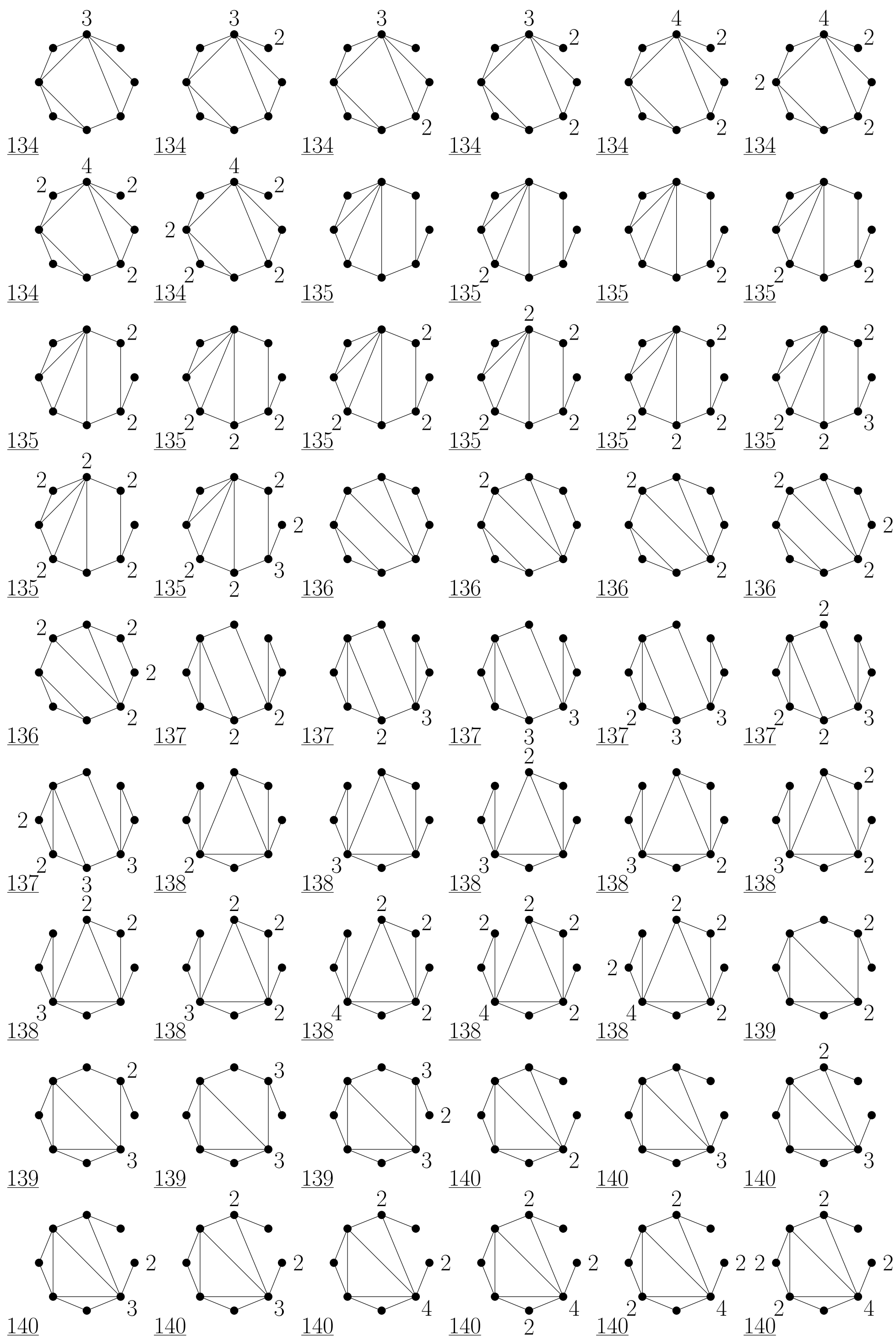


FIGURE 3.2.59. The set $\mathcal{W}(E_8)$ (continued).

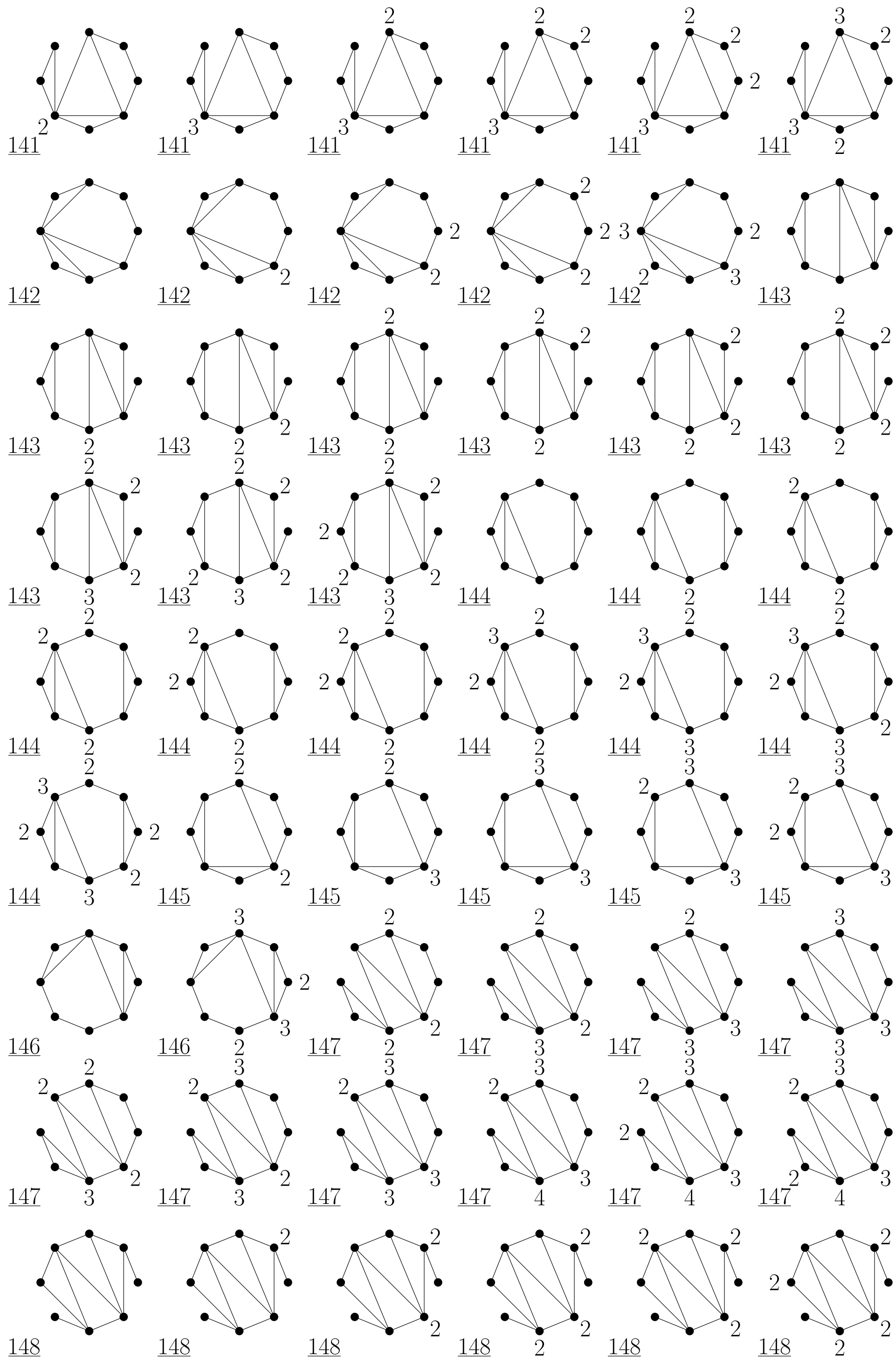


FIGURE 3.2.60. The set $\mathcal{W}(E_8)$ (continued).

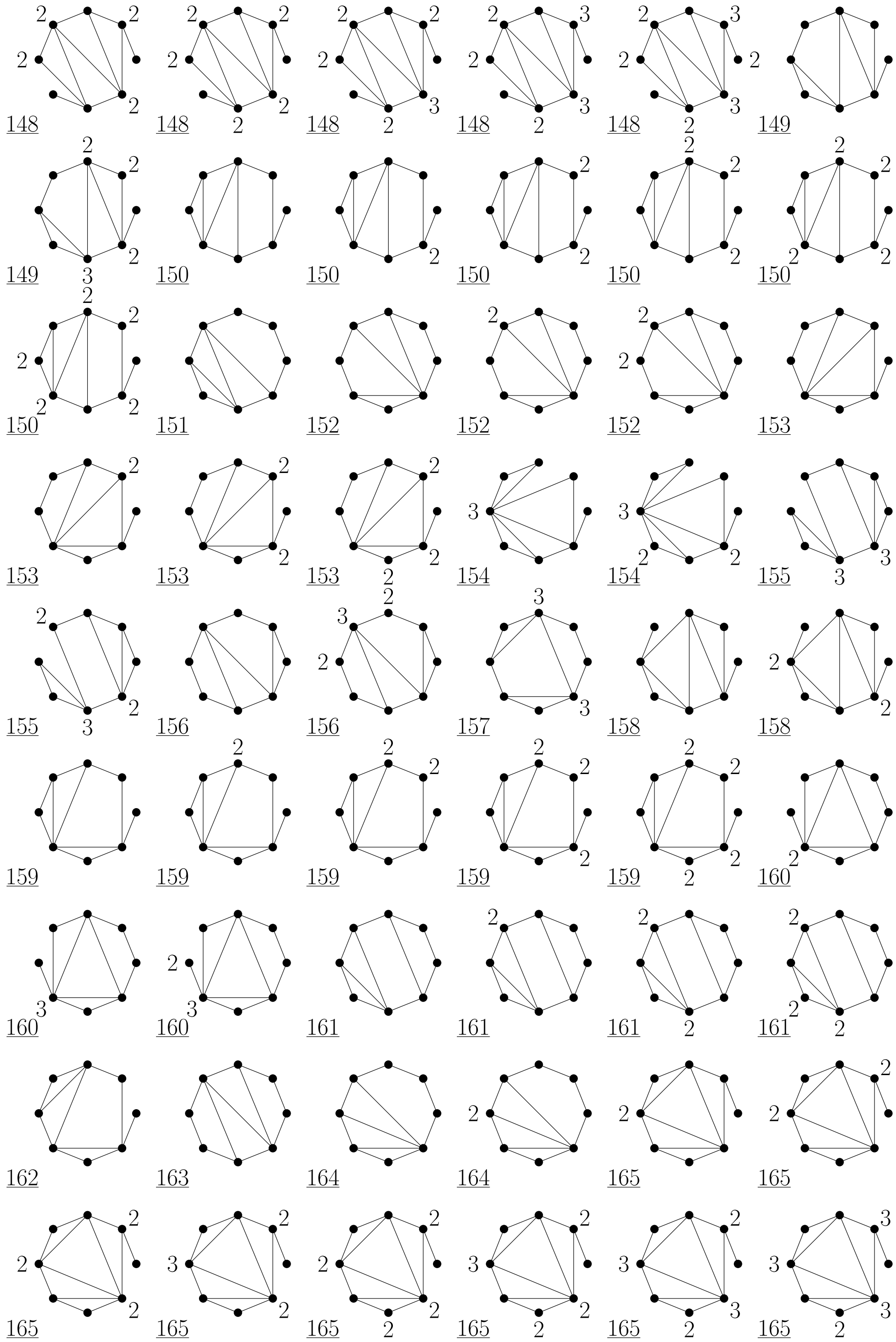


FIGURE 3.2.61. The set $\mathcal{W}(E_8)$ (continued).

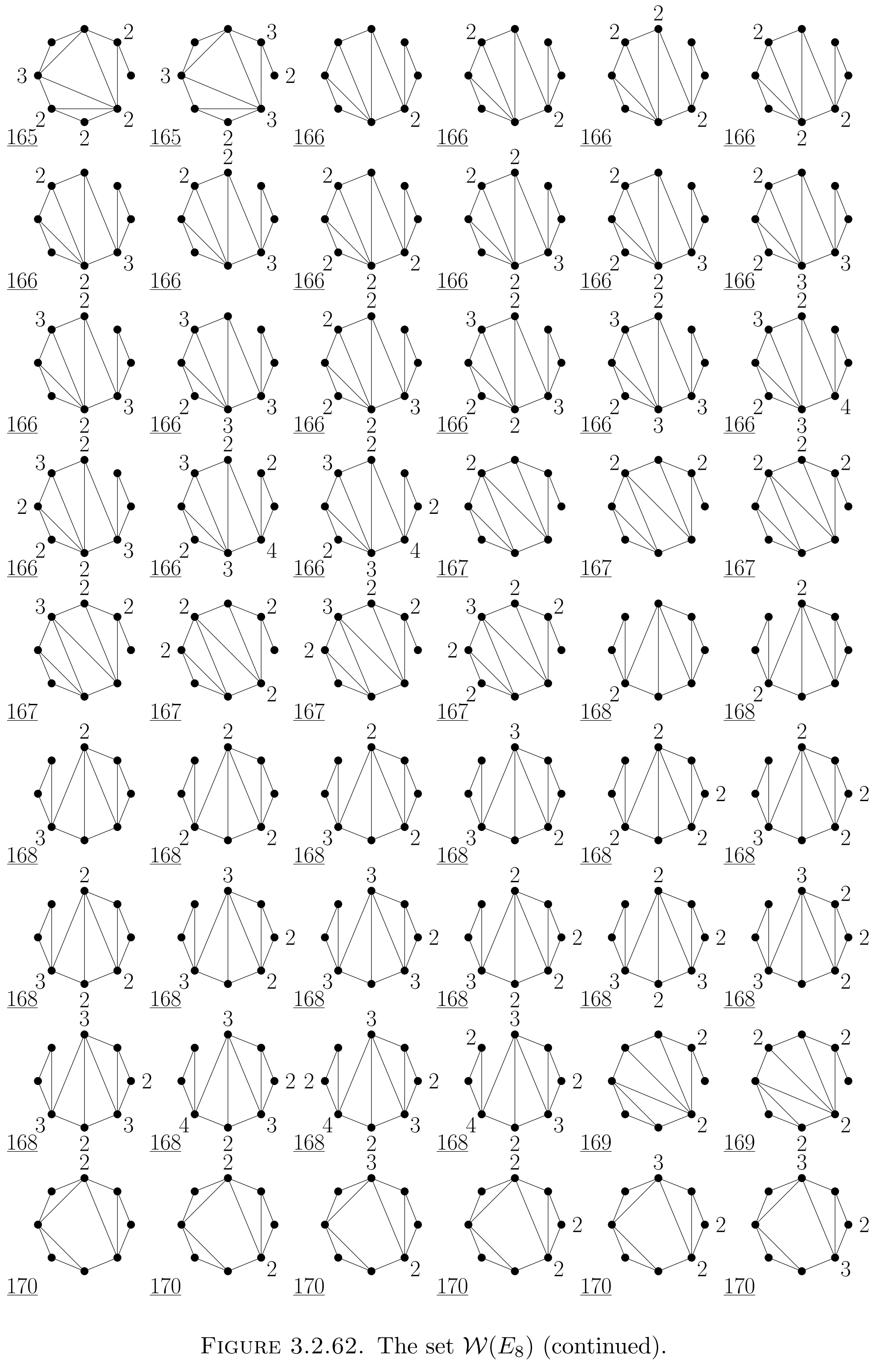


FIGURE 3.2.62. The set $\mathcal{W}(E_8)$ (continued).

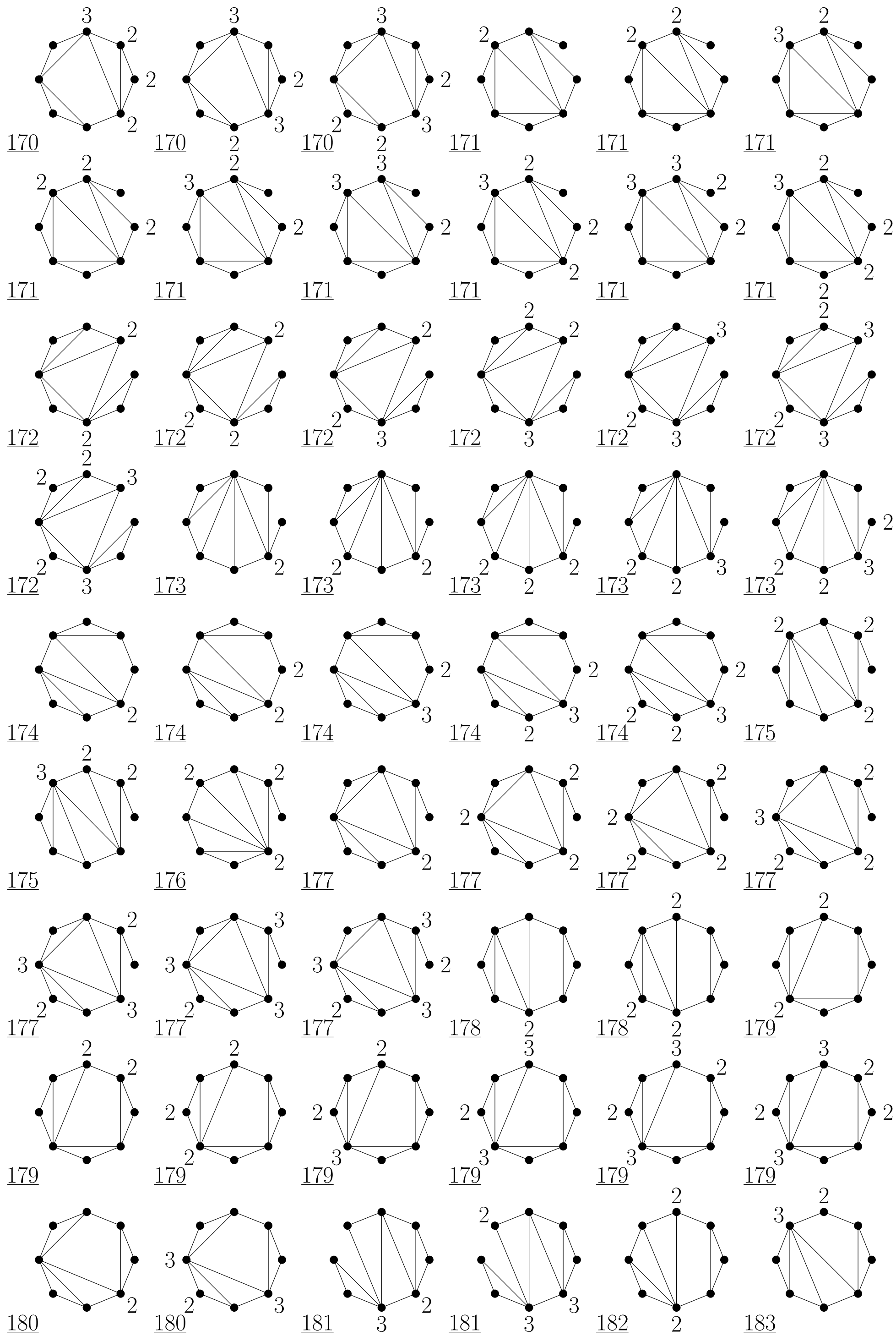


FIGURE 3.2.63. The set $\mathcal{W}(E_8)$ (continued).

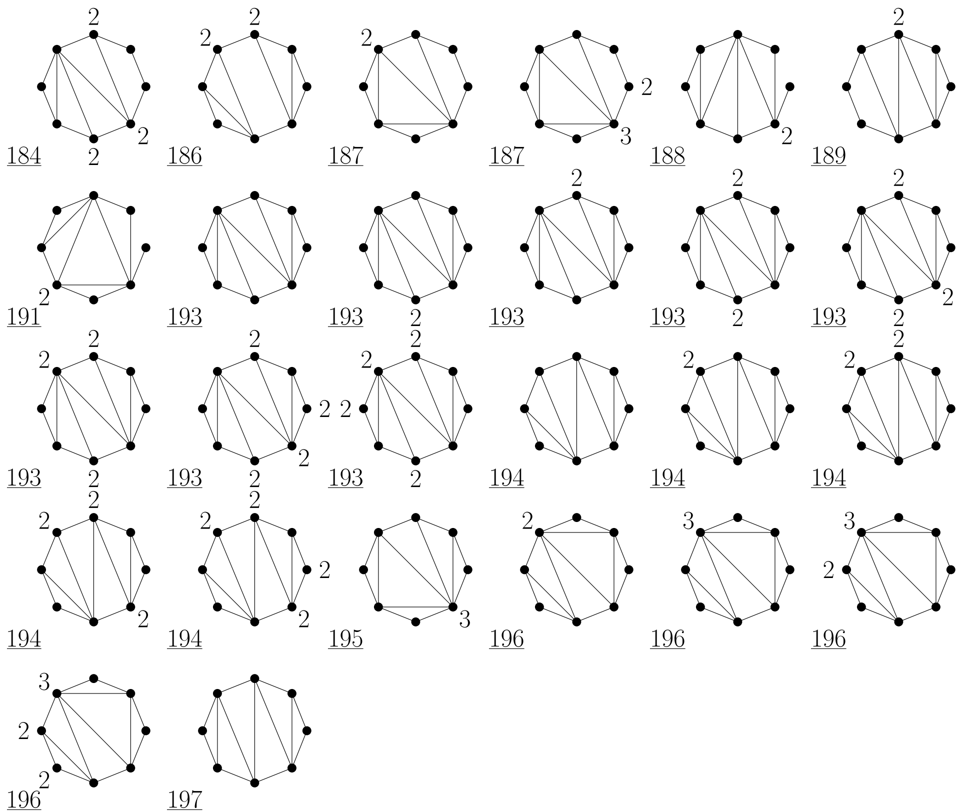


FIGURE 3.2.64. The set $\mathcal{W}(E_8)$ (continued).

3.2.3. Examples. To illustrate how to read the data presented in this subsection let us consider two examples.

Example 3.2.7. Let B be the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}$$

of cluster type D_5. The diagram $X(B)$ and the set of positive c-vectors (and non-initial d-vectors) of $\mathcal{A}_\bullet(B)$ are shown in Figures 3.2.65 and 3.2.66 respectively. Note that any skew-symmetric matrix of cluster type D_5 whose entries are the same as the entries of B in absolute value produces the same $X(B)$ and $\mathcal{V}(B)$.

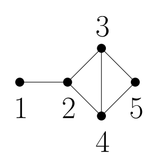


FIGURE 3.2.65. $X(B)$ for Example 3.2.7. Labels refer to the rows of B.

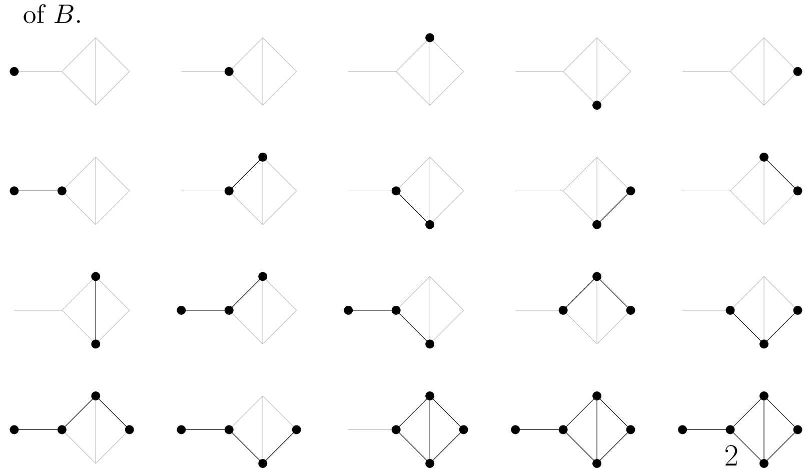


FIGURE 3.2.66. $\mathcal{V}(B)$ for Example 3.2.7.

Example 3.2.8. Let B be the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ -2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 2 \\ -1 & 0 & -1 & 0 \end{pmatrix}$$

of cluster type F_4. The diagram $X(B)$ and the set of positive c-vectors (and non-initial d-vectors) of $\mathcal{A}_\bullet(B)$ are shown in Figures 3.2.67 and 3.2.68 respectively.

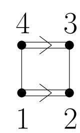


FIGURE 3.2.67. $X(B)$ for Example 3.2.8. Labels refer to the rows of B.

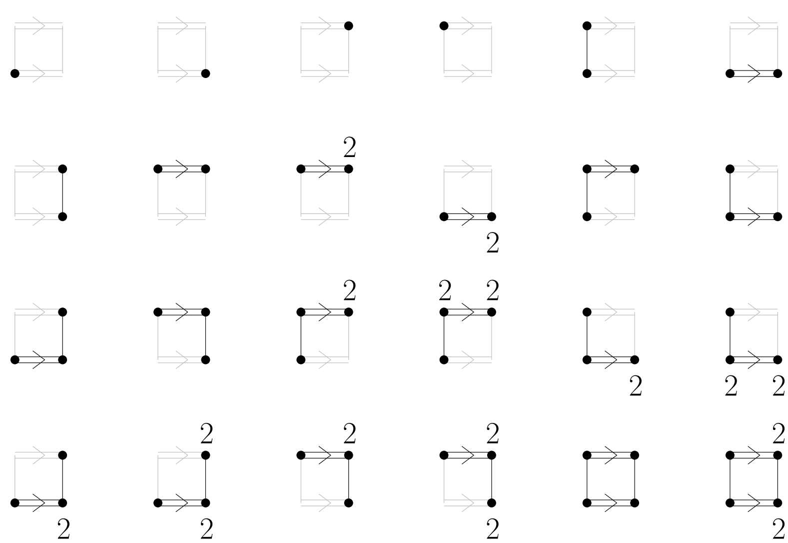


FIGURE 3.2.68. $\mathcal{V}(B)$ for Example 3.2.8.

3.3. Types A_n and D_n: the surface method

In this section we prove Theorem 1.3.1 for types A_n and D_n.

3.3.1. The surface method for types A_n and D_n. To describe c-vectors and d-vectors in types A_n and D_n we do not need to use the construction of [**FST08**] in its full generality so we can slightly simplify the definitions; the reader interested in the general theory can find a comprehensive review in [**MSW11**].

Unless otherwise specified, by *surface* S we mean one of the following:

- (type A_n) a disk with $n+3$ marked points on its boundary ($n \geq 1$);
- (type D_n) a disk with n marked points on the boundary ($n \geq 4$) and one, the *puncture*, in its interior.

We denote the set of marked points by M.

Definition 3.3.1. A *(tagged) arc* is an homotopy class of curves γ in the interior of $S \setminus M$ having no self intersections, connecting two distinct points of M, and not cutting (together with a boundary component of S) an unpunctured bigon. Due to the limitations imposed on the kinds of surfaces we consider there are only two possible types of arcs: *chords*, connecting two marked point on the boundary of S, and *radii*, connecting a point on the boundary with the puncture. Radii comes in two flavours: *plain* and *notched*; to distinguish them in figures we will put a cross on notched arcs.

Remark 3.3.2. Note that this is not the usual definition of tagged arcs, in particular for general surfaces there is a tagging attached to each endpoint of any γ. Another difference from the general case is that we are not allowing loops (arcs with coinciding endpoints).

We need not consider ideal as defined by [**FST08**] arcs so we can drop the adjective "tagged" without generating confusion. To any pair of arcs γ and δ we can associate an integer as follows.

Definition 3.3.3 ([**FST08**, Definition 8.4])**.** The *intersection pairing* of γ and δ is the integer $(\gamma|\delta)$ defined according to these rules:

(1) if γ and δ coincide then $(\gamma|\delta) = -1$;

(2) if γ and δ are homotopic radii with different tagging then $(\gamma|\delta) = 0$;

(3) if γ and δ are non-homotopic radii then $(\gamma|\delta) = 0$ if they are tagged in the same way and $(\gamma|\delta) = 1$ if their tagging is different;

(4) in any other case, set $(\gamma|\delta)$ to be the minimal number of intersections between γ and δ.

Two arcs γ and δ are said to be *compatible* if their intersection pairing is non-positive. A *triangulation* Γ of S is a maximal (by inclusion) set of pairwise compatible arcs.

Remark 3.3.4. Definition 3.3.3 is symmetric; this is not the case for a general surface where loops are allowed (see [**FST08**, Example 8.5]).

In view of [**FST08**, Theorem 7.9] each triangulation of S has n arcs in it and given a triangulation Γ and one of its arcs γ, there is a unique other arc γ' such that

$$\Gamma' = (\Gamma \setminus \{\gamma\}) \cup \{\gamma'\}$$

is again a triangulation of S. The operation of replacing γ with γ' is called a *flip*.

To any triangulation Γ associate a skew-symmetric matrix $B(\Gamma) = (b^{\Gamma}_{\gamma\delta})_{\gamma,\delta\in\Gamma}$ setting

$$b^{\Gamma}_{\gamma\delta} := \begin{cases} 1 & \text{if } \gamma \text{ rotates counterclockwise to } \delta \\ -1 & \text{if } \gamma \text{ rotates clockwise to } \delta \\ 0 & \text{if both or none of the previous conditions hold} \end{cases} \tag{3.3.1}$$

where γ is said to rotate counterclockwise (resp. clockwise) to δ if they are not homotopic, they share an endpoint and, in a neighbourhood of this point, γ can be deformed counterclockwise (resp. clockwise), without crossing any other arc of Γ, to coincide with δ.

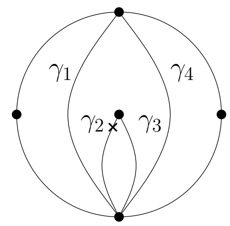


FIGURE 3.3.1. In this triangulation $b_{21} = b_{31} = b_{14} = b_{42} = b_{43} = 1$ while $b_{23} = 0$.

By [**FST08**, Theorem 7.11] the above assignment produces a bijection between triangulations of a type A_n (resp. D_n) surface and unlabeled seeds of the coefficient-free cluster algebra of the same type. In particular cluster variables are in bijection with arcs and if two seeds are obtained from one another exchanging the cluster variables x_γ and $x_{\gamma'}$ then the corresponding triangulations are related by the flip of γ into γ'.

To keep track of principal coefficients we use *laminations* as explained in [**FT12**]. For each marked point p on the boundary of S fix a neighbouring point p' obtained sliding p clockwise on the boundary.

Definition 3.3.5. (see Figure 3.3.2) The *elementary lamination* λ_γ corresponding to an arc γ is the homotopy class of curves, contained in a neighbourhood of γ, defined as follows:

- if γ is a chord connecting p and q then λ_γ connects p' and q';
- if γ is a radius tagged plain (resp. notched) starting from p then λ_γ starts from p' and winds counterclockwise (resp. clockwise) infinitely many times around the puncture.

The *shear coordinates* of an elementary lamination λ with respect to a triangulation Γ are the integers in the n-tuple $(b^\Gamma_{\lambda,\gamma})_{\gamma\in\Gamma}$ defined in terms of intersections between λ and the unique quadrilateral in Γ of which γ is the diagonal. More precisely assume, at first, that Γ contains at most one notched radius; each segment of λ cutting trough the quadrilateral enclosing γ as in Figure 3.3.3 contributes either

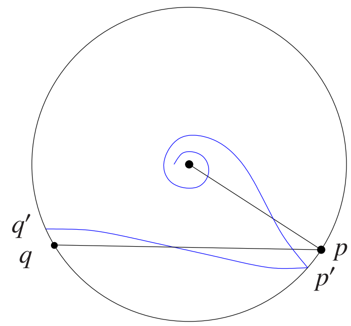


FIGURE 3.3.2. Examples of elementary laminations.

$+1$ or -1 to $b^{\Gamma}_{\lambda,\gamma}$. All other crossings do not contribute. Note that flipping γ interchanges positive and negative crossings. To extend the definition to all possible triangulations it suffices to impose that, if $\Gamma^{\vee}$ is obtained from Γ by changing all the tags at the puncture and $\lambda^{\vee}$ is obtained from λ inverting its winding direction (if any), then for any $\gamma \in \Gamma$

$$b^{\Gamma^{\vee}}_{\lambda^{\vee},\gamma^{\vee}} = b^{\Gamma}_{\lambda,\gamma}. \tag{3.3.2}$$

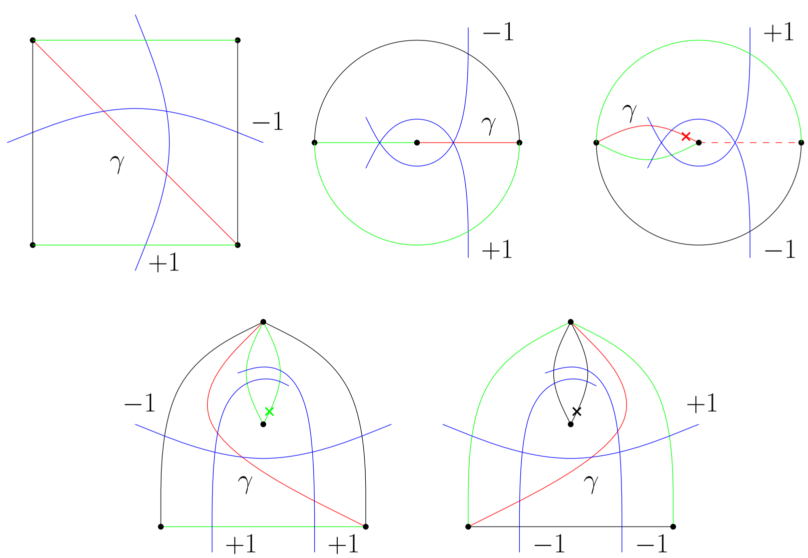


FIGURE 3.3.3. Intersections giving non-zero shear coordinates. The highlighted edges are those crossed by laminations λ giving positive coordinates $b^{\Gamma}_{\lambda,\gamma}$.

Given a triangulation Γ let $\Lambda(\Gamma) = \{\lambda_\gamma\}_{\gamma\in\Gamma}$ be the *multilamination* associated to it, i.e. the collection of the elementary laminations corresponding to the arcs of Γ. Let $\widetilde{B}_\Gamma(\Gamma')$ be the extended B-matrix having top part $B(\Gamma')$ defined by (3.3.1) and bottom part given by the shear coordinates of $\Lambda(\Gamma)$ with respect to Γ'.

Proposition 3.3.6 ([**FT12**, Proposition 16.3])**.** *In the principal-coefficients cluster algebra $\mathcal{A}_\bullet(B(\Gamma))$ the extended exchange matrix corresponding to the triangulation Γ' is given by the above $\widetilde{B}_\Gamma(\Gamma')$.*

We can now describe the sets $\mathcal{C}(B)$ and $\mathcal{D}(B)$. For the rest of this subsection fix a skew-symmetric integer matrix B of type A_n or D_n Let $\Lambda_0 = \{\lambda_i\}_{i\in I}$ be the multilamination corresponding to a triangulation $\Gamma_0 = \{\gamma_i\}_{i\in I}$ of S realizing B. In view of last Proposition the set of c-vectors of the principal-coefficients cluster algebra $\mathcal{A}_\bullet(B)$ is

$$\mathcal{C}(B) = \left\{c_{\gamma,\Gamma} := \left(b^\Gamma_{\lambda_i,\gamma}\right)_{i\in I}\right\}$$

as Γ runs over all possible triangulations of S and γ is an arc in Γ. The parametrization of $\mathcal{C}(B)$ by pairs of arcs and triangulations is not one to one; indeed for any given c-vector there are in general many pairs γ, Γ realizing it. We will see that Γ can always be chosen to be bipartite (see Proposition 3.3.11).

As we already noted, in a cluster algebra coming from a surface, cluster variables are in bijection with tagged arcs. Their denominator vectors can be read directly from the surface: they are given in terms of their intersection pairing with the arcs of the initial triangulation.

Theorem 3.3.7 ([**FST08**, Theorem 8.6]). *Let $\mathcal{A}_\bullet(B)$ be any cluster algebra of type A_n or D_n and let $\Gamma_0 = \{\gamma_i\}_{i\in I}$ be a triangulation corresponding to B.*

If x_γ is the cluster variable corresponding to the tagged arc γ then its d-vector is

$$d_\gamma = ((\gamma_i|\gamma))_{i\in I}\,.$$

The set of non-initial d-vectors of $\mathcal{A}_\bullet(B)$ is therefore

$$\mathcal{D}(B) = \left\{ d_\gamma = ((\gamma_i|\gamma))_{i\in I} \right\}$$

as γ runs over all arcs of S not in Γ_0.

3.3.2. Proof of Theorem 1.3.1. We begin by providing an alternative and immediate proof of (1.3.2) for types A_n and D_n.

Lemma 3.3.8. *All the vectors in $\mathcal{C}(B)$ are sign-coherent.*

Proof. By contradiction let $c_{\gamma,\Gamma}$ be a c-vector that is not sign-coherent i.e. there are two elementary laminations in Λ_0, say λ_i and λ_j such that $b^\Gamma_{\lambda_i,\gamma} > 0$ and $b^\Gamma_{\lambda_j,\gamma} < 0$.

Assume at first that Γ contains at most one notched arc. From Figure 3.3.3 it is clear that λ_i and λ_j intersect and if they both spiral to the puncture then they do not come from homotopic radii. This is in contradiction with the hypothesis that Λ_0 came from a triangulation of S: the intersection pairing of the arcs corresponding to λ_i and λ_j is positive.

The results extends immediately to all the possible triangulation if we observe that changing the windings of all the laminations spiraling to the puncture does not affect the intersection relations among elements of Λ_0. □

Note that, if $c_{\gamma,\Gamma}$ is a c-vector and Γ' is the triangulation obtained from Γ by flipping γ into γ', then

$$c_{\gamma,\Gamma} = -c_{\gamma',\Gamma'}.$$

From now on we concentrate on the set $\mathcal{C}_+(B)$ of positive c-vectors of $\mathcal{A}_\bullet(B)$.

Lemma 3.3.9. *The weighted diagram of any positive c-vector in $\mathcal{A}_\bullet(B)$ is connected.*

Proof. By contradiction assume that the weighted diagram of $c_{\gamma,\Gamma}$ has two disjoint components. Let i be a node in one of them and j a node in the other such that they are at minimal distance in $X(B)$. By hypothesis i and j are not adjacent. Let λ_i and λ_j be the corresponding elementary laminations in Λ_0.

Three cases are possible (in type A_n only the last one occurs).

(1) If λ_i and λ_j have two endpoints in common then they spiral to the puncture in opposite directions. The multilamination Λ_0 contains then a bigon enclosing λ_i and λ_j; at least one side of this bigon (say λ_k) crosses positively the quadrilateral enclosing γ.

(2) If λ_i and λ_j share exactly one endpoint, since i and j are not adjacent, there are two possible configurations. If there is no other lamination sharing that endpoint then they both spiral to the puncture and they are enclosed in a bigon; at least one side of this bigon (again say λ_k) intersects positively the quadrilateral enclosing γ. Otherwise at least one lamination λ_k among those sharing the same endpoint is such that $b^{\Gamma}_{\lambda_k,\gamma} > 0$.

(3) Finally if λ_i and λ_j do not share any endpoint then there is at least one lamination λ_k starting from one of those four points, lying in between λ_i and λ_j and crossing positively the quadrilateral that encloses γ (otherwise such an elementary lamination could be added to Λ_0 in contradiction to the assumption that the multilamination corresponds to a triangulation).

In all of the cases there is a vertex k in between i and j such that the k-th component of $c_{\gamma,\Gamma}$ is non-zero in contradiction with the assumption of minimal distance between i and j. □

Proposition 3.3.10. *In types A_n and D_n we have*

$$\mathcal{C}_+(B) \subset \mathcal{V}(B).$$

PROOF. We deal first with type A_n. It is clear that, having no puncture, any lamination $\lambda \in \Lambda_0$ can intersect any given arc γ at most once so $b^{\Gamma}_{\lambda,\gamma} \in \{0, 1\}$. In view of Proposition 3.2.1 it suffices to show that no c-vector can have a triangle in its weighted diagram. But this follows directly from the fact that, since S has no puncture, at least one of the sides of each triangle in Λ_0 does not intersect any given arc γ.

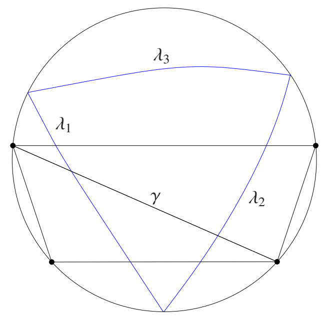


FIGURE 3.3.4. Any triangle in a lamination of a surface of type A_n intersect at most twice any arc γ.

For type D_n the proof proceeds by case analysis. We need first some considerations. In view of condition (3.3.2) we can assume that the quadrilateral enclosing γ is one of those in Figure 3.3.3.

Note that, given a multilamination Λ_0 coming from a triangulation, a once punctured disk can be decomposed into pieces: it will contain exactly one piece in which all the elementary laminations spiral to the puncture (one of the five in Figure 3.3.5); all the other pieces, if any, will contain only elementary laminations corresponding to chords. Any such piece can only be glued to the one containing the puncture as shown in Figure 3.3.6.

Any elementary lamination of Λ_0 not corresponding to a glued edge will be contained, up to a small neighbourhood of one endpoint, in exactly one piece in this decomposition. This implies that any given piece must contain at least a subsection of γ and of two opposite sides of the quadrilateral enclosing γ in order for any of the laminations it contains to give rise to a positive coordinate. In particular a quadrilateral of a triangulation can intersect non trivially at most three pieces in this decomposition.

We need therefore to consider all the possible way a quadrilateral from Figure 3.3.3 can be fitted into a surface with at most three pieces. This is a straightforward

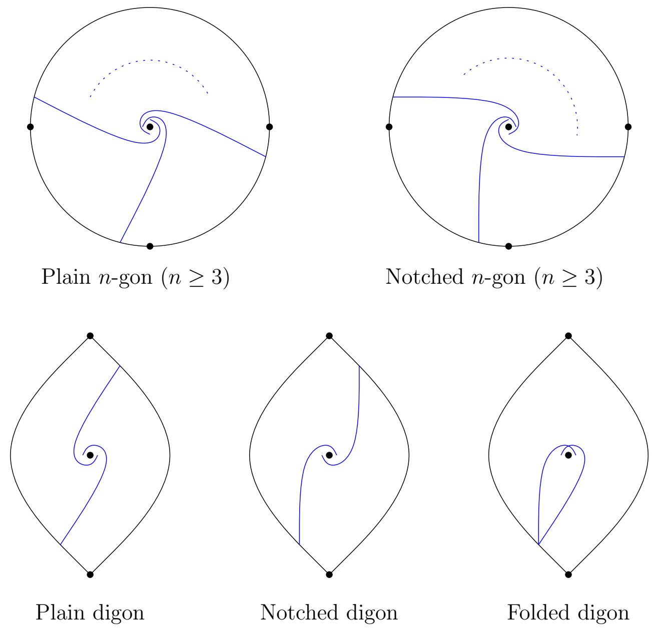


FIGURE 3.3.5. Multilaminations with all elementary laminations spiralling to the puncture.

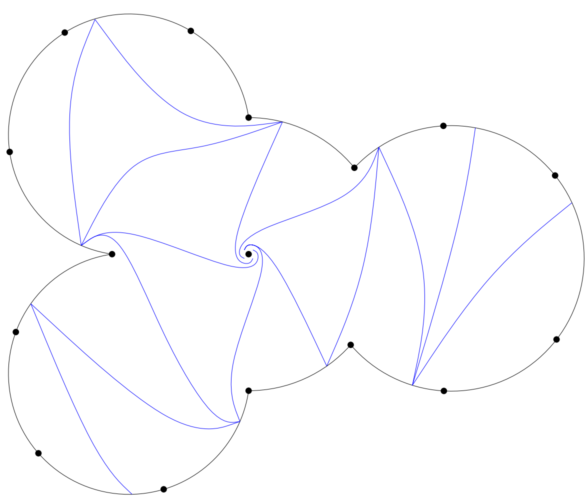


FIGURE 3.3.6. Example of a decomposition of a surface of type D_n according to a multilamination.

but tedious check; a complete analysis of the various cases (87 nontrivial cases in total) is contained in Appendix 3.6. $\square$

To connect $\mathcal{C}_+(B)$ with $\mathcal{D}(B)$ let us improve on the parametrization of c-vectors of $\mathcal{A}_\bullet(B)$. A triangulation Γ of S is said to be *bipartite* if every node of the corresponding quiver is either a sink or a source. Note that, since in finite type any cordless cycle must be oriented ([**BGZ06**] Theorem 1.2), bipartite triangulations correspond to bipartite orientations of the Dynkin diagram of the given type.

Not every quadrilateral can appear in a bipartite triangulation; indeed it is clear from the assignment (3.3.1) that the only allowed one are those in Figure 3.3.7. Moreover any such quadrilateral determines uniquely the bipartite triangulation in which it appears.

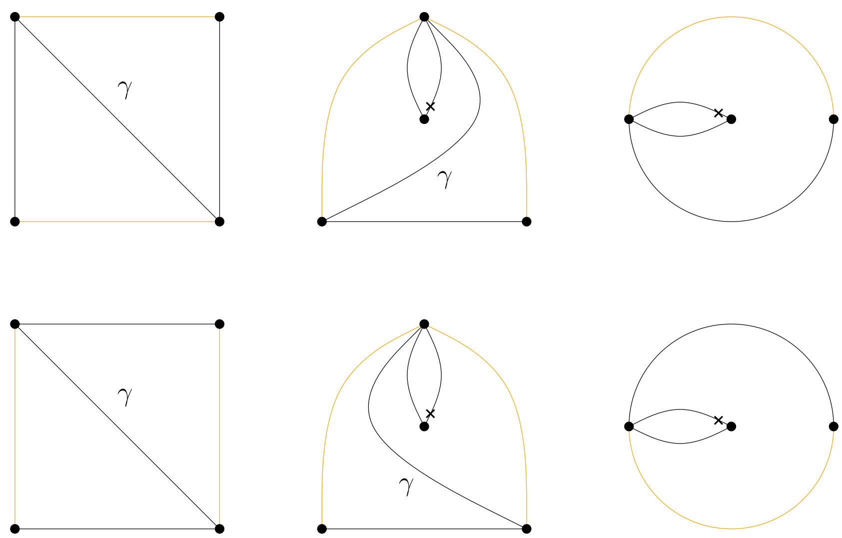


FIGURE 3.3.7. The only quadrilaterals that can appear in a bipartite triangulation of a surface S. The edges on the boundary of S are highlighted. When the quadrilateral is a digon any of the radii can be the diagonal.

Let $\mathcal{C}^b_+(B)$ be the subset of $\mathcal{C}_+(B)$ consisting of c-vectors $c_{\gamma,\Gamma}$ such that Γ is bipartite.

Proposition 3.3.11. *In types A_n and D_n*

$$\mathcal{C}^b_+(B) = \mathcal{C}_+(B).$$

Proof. Let $c_{\gamma,\Gamma}$ be any element of $\mathcal{C}_+(B)$. Let Λ_0 be the multilamination associated to B. In view of the considerations above it suffices to show that we can replace the quadrilateral enclosing γ with a bipartite quadrilateral (possibly enclosing a different diagonal γ') such that the shear coordinates remain unchanged.

We concentrate first on type A_n. The idea is simple: pick a leaf in the support of $c_{\gamma,\Gamma}$ and let λ be the corresponding elementary lamination in Λ_0. Since λ is the "last" lamination intersecting the quadrilateral enclosing γ positively it must belong to a triangle in Λ_0 such that the other two lamination composing it do not give rise to positive shear coordinates. Let p' be the only vertex of the triangle that is not adjacent to λ. We can replace the original quadrilateral with one having the two marked points closest to p' as vertices: all the shear coordinates will be unchanged (cf. the first step in Figure 3.3.8).

This is sufficient in type A_n but not in general in type D_n: we need to deal with folded quadrilaterals as well. The replacement to be performed depends both on Λ_0 and γ but it is straightforward from the pictures. The general procedure is shown in Figure 3.3.8. The precise case analysis is again in Appendix 3.6; there we provide, for each possible quadrilateral and for each multilamination an explicit replacement. □

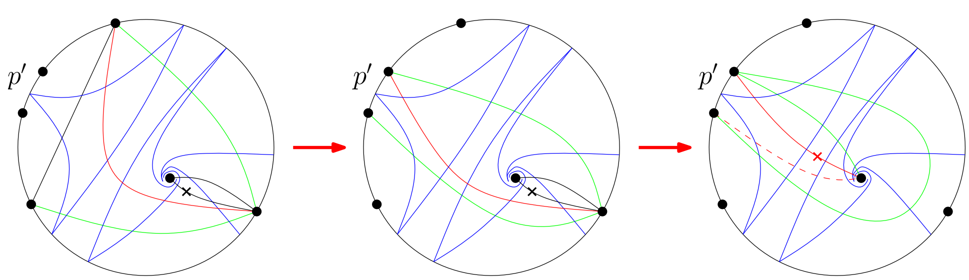


Figure 3.3.8. The reduction of a quadrilateral to a bipartite quadrilateral is in two steps: first move the edges "parallel" to multilamination corresponding to leafs in $c_{\gamma,\Gamma}$ to the boundary; then, if needed, replace the quadrilateral with one among those in Figure 3.3.7.

In analogy with the definition above let $\mathcal{D}^b(B)$ be the subset of all the non-initial d-vectors corresponding to cluster variables appearing in bipartite seeds of $\mathcal{A}_\bullet(B)$. Since any arc on S appears in a bipartite triangulation, in types A_n and D_n we have

$$\mathcal{D}^b(B) = \mathcal{D}(B). \tag{3.3.3}$$

Remark 3.3.12. The above equality, together with Proposition 3.3.11, prove Theorem 1.3.3 for cluster algebras of types A_n and D_n.

Proposition 3.3.13. *In types A_n and D_n*

$$\mathcal{C}_+(B) = \mathcal{D}(B).$$

Proof. In view of the above reductions it suffices to show that

$$\mathcal{C}_+^b(B) = \mathcal{D}^b(B).$$

As before let $\Gamma_0 = \{\gamma_i\}_{i\in I}$ be the triangulation corresponding to B and $\Lambda_0 = \{\lambda_i\}_{i\in I}$ the associated multilamination. In view of Theorem 3.3.7 and Definition 3.3.3 all the vectors in $\mathcal{D}^b(B)$ have non-negative components.

Let γ be any arc not in Γ_0 and consider the d-vector d_γ; we need to distinguish three cases depending on the endpoints of γ (call them p and q).

- If both p and q are on the boundary of S and they are not adjacent then there are two other marked points r and s such that p' is contained on the boundary segment pr and q' is contained in the boundary segment qs. Let γ' be any of the two diagonals of the quadrilateral $prqs$ and complete it to a bipartite triangulation Γ'. We have $d_\gamma = \pm c_{\gamma',\Gamma'}$ and we can choose γ' such that $d_\gamma = c_{\gamma',\Gamma'}$. Note that if S is of type A_n this is the only possible case.
- It both p and q are on the boundary of S and they are adjacent then we can assume (up to relabeling) that q' lies on the boundary segment qp.

Let r be such that p' lies on the boundary segment pr. Let γ' be any of the diagonals of the folded quadrilateral having vertices q, p, r, and the puncture; Let Γ' be a bipartite triangulation containing it. We have again $d_\gamma = \pm c_{\gamma',\Gamma'}$ and we can select the diagonal γ' that yields a positive c-vector.

- If one of the endpoints of γ (say q to fix ideas) is the puncture then let r be the marked point such that p' lies between p and r. Let Γ' be a bipartite triangulation containing the digon with vertices p and r, enclosing the puncture, and such that its radii both start from r. If γ' is the radius with tagging opposite to the tagging of γ then $d_\gamma = c_{\gamma',\Gamma'}$.

Conversely let $c_{\gamma',\Gamma'}$ in $\mathcal{C}^b_+(B)$. The quadrilateral of Γ' enclosing γ' will be exactly one of those constructed above (they are all bipartite). Choosing γ to be the corresponding arc we get $d_\gamma = c_{\gamma',\Gamma'}$. □

We thank Andrei Zelevinsky for providing the idea of using the "bipartite belt" in the above proof.

The Proposition concludes the proof of Theorem 1.3.1 for types A_n and D_n.

Proposition 3.3.14. *In types A_n and D_n we have*

$$\mathcal{V}(B) \subset \mathcal{D}(B).$$

Proof. Let $\Gamma_0 = \{\gamma_i\}_{i\in I}$ be a triangulation realizing B and let $v = (v_i)_{i\in I}$ be any element in $\mathcal{V}(B)$.

In type A_n it is clear how to construct an arc γ crossing exactly one time all the arcs γ_j such that $v_j \neq 0$: suppose i is a leaf in the weighted diagram; the arc γ_i corresponding to it belongs to two triangles. One of them is such that the nodes corresponding to the other two arcs forming it do not belong to the support of weighted diagram. The arc γ we are looking for starts from the vertex of this triangle opposed to γ_i. It crosses then in sequence all the arcs γ_j such that $v_j \neq 0$ and terminates in the vertex opposed to the arc corresponding to the other leaf.

In type D_n the procedure is slightly more involved and depends on the initial triangulation Γ_0 but follows the same basic idea. Suppose at first that Γ_0 does not contain a digon with two homotopic radii. The same procedure described for type A_n works verbatim for diagrams I, IV, and VIII in Figure 3.2.7: the only thing to note is that instead of a leaf we might have to take one of the nodes in the left triangle (for IV we can not use the two rightmost leafs). For diagrams II and VI we need a small fix: γ starts from the vertex opposed to the arc corresponding to the leftmost leaf and ends at the puncture; its tagging is the opposite of the tagging of the radii in Γ_0. For diagrams III and VII we repeat the same argument using the leftmost leaf and the leftmost node with multiplicity 2. Diagrams like V can not be embedded in a $X(B)$ if Γ_0 has not a digon with two homotopic radii in it.

If Γ_0 contains a digon with two homotopic radii then the only diagrams that can arise are I, II, III, IV, and V. For V the procedure is the same as the one for type A_n, we just need to cross both the radii of the digon. For diagrams III and IV the procedure is identical to the above. For diagrams like II γ starts from the vertex opposed to the arc corresponding to the leftmost leaf and ends in the vertex of the digon not adjacent to the radii. For diagrams like I we need to distinguish two cases: if one of the leafs correspond to a radius then the corresponding endpoint of γ is the puncture and its tagging is the opposite of the one of that radius. Otherwise we proceed as in type A_n. $\square$

3.4. Types B_n and C_n: the folding method

Building on the results of last section we will now prove Theorem 1.3.1 for types B_n and C_n. In order to do so we will realize any principal coefficients cluster algebra of type B_n (respectively C_n) as a subquotient of an appropriate cluster algebra of type D_{n+1} (respectively A_{2n-1}) with principal coefficients.

3.4.1. Folding of cluster algebras with trivial coefficients. The construction, for the coefficient-free case, was explained in [**Dup08**]. Since we need to generalize it to work with principal coefficients later on let us begin by recalling in some details its main features.

Let $B = (b_{ij})_{i,j\in I}$ be a skew-symmetrizable integer matrix and σ a permutation of I.

Definition 3.4.1. A permutation σ is an *automorphism* of B if, for any i and j in I,

$$b_{\sigma(i)\sigma(j)} = b_{ij}. \tag{3.4.1}$$

An automorphism of B is said to be *admissible* if, for any i_1 and i_2 in the same σ-orbit $\bar{\imath}$ and for any j in I,

$$b_{i_1,j} b_{i_2 j} \geq 0 \tag{3.4.2}$$

$$b_{i_1,i_2} = 0. \tag{3.4.3}$$

An easy computation shows that, if σ is an admissible automorphism of B and k_1 and k_2 are two points in the same σ-orbit $\overline{k}$, the mutations μ_{k_1} and μ_{k_2} commute; that is

$$\mu_{k_1} \circ \mu_{k_2}(B) = \mu_{k_2} \circ \mu_{k_1}(B).$$

Indeed $\mu_{k_2}(\mu_{k_1}(b_{ij}))$ is either $-b_{ij}$, if at least one among i and j is in $\{k_1, k_2\}$, or

$$b_{ij} + b_{ik_1}[b_{k_1 j}]_+ + [-b_{ik_1}]_+ b_{k_1 j} + b_{ik_2}[b_{k_2 j}]_+ + [-b_{ik_2}]_+ b_{k_2 j}$$

otherwise. Those expressions are clearly independent on the order in which μ_{k_1} and μ_{k_2} are applied. It makes therefore sense to define *orbit-mutations* as the compositions

$$\mu_{\overline{k}}^{\sigma} := \prod_{t \in \overline{k}} \mu_t.$$

Repeating the same reasoning we get

$$\mu_{\overline{k}}^{\sigma}(b_{ij}) = \begin{cases} -b_{ij} & \text{if } i \text{ or } j \in \overline{k} \\ b_{ij} + \sum_{t\in\overline{k}} \left(b_{it}[b_{tj}]_+ + [-b_{it}]_+ b_{tj}\right) & \text{otherwise.} \end{cases} \tag{3.4.4}$$

Note that, given a σ-orbit $\overline{k}$, the permutation σ is always an automorphism of $\mu^{\sigma}_{\overline{k}}(B)$ but it need not be admissible; in particular condition (3.4.2) may be violated.

Definition 3.4.2. An admissible automorphism σ of B is said to be *stable* if, for any finite sequence of σ-orbits $\overline{k_1}, \dots \overline{k_\ell}$, it is an admissible automorphism of

$$\mu^{\sigma}_{\overline{k_\ell}} \circ \cdots \circ \mu^{\sigma}_{\overline{k_1}}(B).$$

Proposition 3.4.3 ([**Dup08**, Proposition 2.22])**.** *If the Cartan counterpart of B is a simply-laced finite type then any admissible automorphism of B is stable.*

Remark 3.4.4. We will need the following incarnations of Proposition 3.4.3:

(1) B has Cartan counterpart of type A_{2n-1} and, using the standard labeling of the nodes of the associated Dynkin diagram,

$$\sigma := \prod_{i=1}^{n} (i, 2n-i)$$

is an admissible automorphism of B.

(2) B has Cartan counterpart of type D_{n+1} and, again in the standard labeling,

$$\sigma = (n, n+1)$$

is an admissible automorphism of B.

Given a skew-symmetrizable integer matrix B and a (stable) admissible automorphism σ we can define a *folded* matrix $\pi(B) := \overline{B} = (b_{\overline{\imath}\overline{\jmath}})$, as $\overline{\imath}$ and $\overline{\jmath}$ vary over all the σ-orbits, by setting

$$b_{\overline{\imath}\overline{\jmath}} := \sum_{s \in \overline{\imath}} b_{sj}. \tag{3.4.5}$$

In view of condition (3.4.1) the value of $b_{\overline{\imath}\overline{\jmath}}$ does not depend on the choice of a representative of $\overline{\jmath}$. The folded matrix $\pi(B)$ is itself skew-symmetrizable (see [**Dup08**, Lemma 2.5]).

The key point here is this: if σ is a stable admissible automorphism of B then for any σ-orbit $\overline{k}$

$$\pi\left(\mu^{\sigma}_{\overline{k}}(B)\right) = \mu_{\overline{k}}(\pi(B))$$

thank to condition (3.4.2) (see [**Dup08**, Theorem 2.24]).

We will use the following obvious converse stating the existence of "unfolding" for the matrices we are interested into.

Proposition 3.4.5. *Let $\overline{B}'$ be any matrix in the same mutation class of a matrix $\overline{B}$ obtained by folding from a skew-symmetrizable matrix B with a stable admissible automorphism σ. There exist a matrix B' and a sequence of σ-orbits $\overline{k_1}, \ldots, \overline{k_\ell}$ such that*

(1) $\mu^{\sigma}_{\overline{k_\ell}} \circ \cdots \circ \mu^{\sigma}_{\overline{k_1}}(B) = B'$
(2) $\overline{B}' = \overline{B'}$.

The folding map can be extended to a morphism of algebras as follows. Fix an initial B-matrix B and a stable admissible automorphism σ. Let

$$\left(B, \{x_i\}_{i\in I}\right)$$

be the initial cluster of the coefficient-free cluster algebra $\mathcal{A}_0(B)$. Write $\mathcal{A}^{\sigma}_0(B)$ for the subalgebra of $\mathcal{A}_0(B)$ generated by all the clusters reachable from the initial one by a sequence of orbit mutations.

Let $\mathcal{A}_0(\overline{B})$ be the coefficient-free cluster algebra with initial B-matrix $\pi(B) = \overline{B}$ and initial cluster variables $\{x_{\overline{\imath}}\}_{\overline{\imath}\in I/\sigma}$. The assignment

$$\pi(x_i) := x_{\overline{\imath}}$$

extends to a surjective map

$$\pi : \mathcal{A}^{\sigma}_0(B) \longrightarrow \mathcal{A}_0(\overline{B}).$$

The algebra $\mathcal{A}_0(\overline{B})$ is the quotient of $\mathcal{A}_0^\sigma(B)$ by the ideal generated by the relations

$$x_i = x_{\sigma(i)}.$$

Moreover, and this is the key point in the construction, the map π preserves the cluster structure: seeds of $\mathcal{A}_0^\sigma(B)$ are mapped to seeds of $\mathcal{A}_0(\overline{B})$.

Combining the above observation with Remark 3.4.4 we get the following statement.

Proposition 3.4.6. *Any matrix of cluster type B_n (respectively C_n) is the image $\pi(B)$ of a matrix B of cluster type D_{n+1} (respectively A_{2n-1}) with automorphism σ from Remark 3.4.4. The coefficient-free cluster algebra $\mathcal{A}_0(\overline{B})$ is the quotient of a subalgebra of $\mathcal{A}_0(B)$ by an ideal preserving the cluster structure. In particular any exchange matrix of $\mathcal{A}_0(\overline{B})$ is the folding of some exchange matrix of $\mathcal{A}_0(B)$.*

3.4.2. Proof of Propositions 3.2.2 and 3.2.3. The results just summarized are enough to describe the sets $\mathcal{X}(B_n)$ and $\mathcal{X}(C_n)$.

Proof of Proposition 3.2.2. In view of Proposition 3.4.6 any element of $\mathcal{X}(B_n)$ can be obtained by folding an element of $\mathcal{X}(D_{n+1})$. On the other hand not every diagram from Figure 3.2.6 can be folded: we know that any cordless cycle in such a diagram correspond to a cyclically oriented cordless cycle in the quiver $Q(B)$ associated to it (see [**FZ03a, BGZ06**]). By definition of admissible automorphism all the vertices in the only non-trivial orbit of σ must be not adjacent and must be connected to all the other adjacent vertices in the same way. This forces us to conclude that diagrams (c) and (d) can not be folded.

The diagrams of Figure 3.2.2 are thus the folding of diagrams (a) and (b) from Figure 3.2.6. □

Proof of Proposition 3.2.3. In view of Proposition 3.4.6 diagrams in $\mathcal{X}(C_n)$ are obtained by folding elements of $\mathcal{X}(A_{2n-1})$. The only requirement a diagram must satisfy to be folded is to be symmetric with respect to the only fixed point of σ from Remark 3.4.4. □

3.4.3. Folding of c-vectors. In order to consider c-vectors we need to extend the above construction to cluster algebras with principal coefficients. We take inspiration from the following example.

Example 3.4.7. Let $\mathcal{A}_\bullet(B)$ be the cluster algebra of type D_4 with principal coefficients at the initial cluster given by

$$B = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

B is invariant under permutation $\sigma = (34)$ and has $b_{34} = 0$. Moreover the mutations in directions 3 and 4 commute; that is

$$\mu_3 \circ \mu_4(B) = \mu_4 \circ \mu_3(B).$$

Let $\mathcal{A}^\sigma_\bullet(B)$ be the subalgebra of all the clusters reachable from the initial one by any sequence of the mutations μ_1, μ_2, and $\mu_3 \circ \mu_4$. All the B-matrices in it have $b_{34} = 0$.

The permutation σ acts on the set of clusters of $\mathcal{A}_\bullet(B)$ by relabeling:

$$\sigma(x_i) := x_{\sigma(i)} \qquad \text{and} \qquad \sigma(y_i) := y_{\sigma(i)}$$

Let $\mathcal{I}$ be the ring ideal of $\mathcal{A}^\sigma_\bullet(B)$ generated by the relations

$$x_3 = x_4 \qquad \text{and} \qquad y_3 = y_4$$

The quotient $\mathcal{A}^\sigma_\bullet(B)/\mathcal{I}$ is a cluster algebra of type B_3 with principal coefficients at the initial cluster given by $\pi(B)$. Under the projection map clusters of $\mathcal{A}^\sigma_\bullet(B)$ are mapped to clusters of $\mathcal{A}^\sigma_\bullet(B)/\mathcal{I}$. Moreover exchange relations in the quotient come from exchange relations of $\mathcal{A}_\bullet(B)$.

For any skew-symmetrizable integer matrix B endowed with a stable admissible automorphism σ let $\mathcal{A}_\bullet(B)$ and $\mathcal{A}_\bullet(\overline{B})$ be the cluster algebras with principal coefficients respectively at B and $\overline{B} = \pi(B)$. Let $\mathcal{A}_\bullet^\sigma(B)$ be the subalgebra of $\mathcal{A}_\bullet(B)$ generated by all clusters reachable from the initial one using orbit-mutations.

In view of the above example it is natural to define folding for a c-vector $c = (c_i)_{i\in I}$ of $\mathcal{A}_\bullet^\sigma(B)$ component wise as follows:

$$c_{\overline{\imath}} := \sum_{s\in\overline{\imath}} c_s. \tag{3.4.6}$$

However this definition turns out to be not so obvious because the tropicalization map (i.e. the map associating to each coefficient y its c-vector) and folding are not compatible in general. Let us clarify the condition required to justify (3.4.6).

Note that if C and C' are two coefficient matrices of $\mathcal{A}_\bullet^\sigma(B)$ connected by a single orbit-mutation $\mu_{\overline{k}}$ then it follows directly from having assumed σ to be a stable admissible automorphism of B that:

$$c'_{ij} = \begin{cases} -c_{ij} & \text{if } j \in \overline{k} \\ c_{ij} + \sum_{t\in\overline{k}} \left(c_{it}[b_{tj}]_+ + [-c_{it}]_+ b_{tj}\right) & \text{otherwise.} \end{cases} \tag{3.4.7}$$

From (3.4.7) we get an important observation: all the C-matrices in $\mathcal{A}_\bullet^\sigma(B)$ are such that

$$c_{\sigma(i)\sigma(j)} = c_{ij}. \tag{3.4.8}$$

Indeed the property holds for the initial C-matrix and we can use the admissibility of σ to propagate it.

We introduce the folded C-matrix $\overline{C}$ for a C-matrix $C = (c_{ij})_{ij\in I}$ of $\mathcal{A}_\bullet^\sigma(B)$ as

$$c_{\overline{\imath}\overline{\jmath}} := \sum_{s\in\overline{\imath}} c_{sj}. \tag{3.4.9}$$

Note that (3.4.9) is independent of the choice of a representative of $\overline{\jmath}$ due to the symmetry (3.4.8).

Proposition 3.4.8. *Let B be any skew-symmetrizable integer matrix and let σ be a stable admissible automorphism of B. The matrix $\overline{C}$ satisfies the recursion relation*

$$c'_{\bar\imath\bar\jmath} = \begin{cases} -c_{\bar\imath\bar\jmath} & \text{if } \bar\jmath = \overline{k} \\ c_{\bar\imath\bar\jmath} + c_{\bar\imath\overline{k}}[b_{\overline{k}\bar\jmath}]_+ + [-c_{\bar\imath\overline{k}}]_+ b_{\overline{k}\bar\jmath} & \text{otherwise} \end{cases} \tag{3.4.10}$$

if and only if the following condition holds: for any i and j the sign of c_{sj} is independent of the choice of representative $s \in \bar\imath$.

Proof. It suffices to establish the proposition for a single mutation; if $\bar\jmath = \overline{k}$ our claim is trivial so we can assume $j \notin \overline{k}$. On the one hand we have

$$c'_{\bar\imath\bar\jmath} = c_{\bar\imath\bar\jmath} + c_{\bar\imath\overline{k}}[b_{\overline{k}\bar\jmath}]_+ + [-c_{\bar\imath\overline{k}}]_+ b_{\overline{k}\bar\jmath} = \sum_{s\in\bar\imath} c_{sj} + \sum_{s\in\bar\imath} c_{sk}\left[\sum_{t\in\overline{k}} b_{tj}\right]_+ + \left[-\sum_{s\in\bar\imath} c_{sk}\right]_+ \sum_{t\in\overline{k}} b_{tj}.$$

On the other hand

$$c'_{\bar\imath\bar\jmath} = \sum_{s\in\bar\imath} c'_{sj} = \sum_{s\in\bar\imath}\left(c_{sj} + \sum_{t\in\overline{k}}\left(c_{st}[b_{tj}]_+ + [-c_{st}]_+ b_{tj}\right)\right).$$

Therefore the recursion 3.4.10 is satisfied if and only if

$$\sum_{s\in\bar\imath} c_{sk}\left[\sum_{t\in\overline{k}} b_{tj}\right]_+ = \sum_{s\in\bar\imath}\sum_{t\in\overline{k}} c_{st}[b_{tj}]_+$$

and

$$\left[-\sum_{s\in\bar\imath} c_{sk}\right]_+ \sum_{t\in\overline{k}} b_{tj} = \sum_{s\in\bar\imath}\sum_{t\in\overline{k}} [-c_{st}]_+ b_{tj}.$$

The first condition is guaranteed by the admissibility of σ; indeed we get

$$\sum_{s\in\bar\imath}\sum_{t\in\overline{k}} c_{sk}\left[b_{tj}\right]_+ = \sum_{s\in\bar\imath}\sum_{t\in\overline{k}} c_{st}[b_{tj}]_+$$

which is true by a simple change of summation index.

Similarly the second condition is equivalent to

$$\sum_{s\in\bar\imath}\sum_{t\in\overline{k}} \left[-c_{sk}\right]_+ b_{tj} = \sum_{s\in\bar\imath}\sum_{t\in\overline{k}} [-c_{st}]_+ b_{tj}.$$

if and only if the sign of c_{sk} is independent on the choice of representative $s \in \bar{\imath}$. $\square$

In our situation the condition of Proposition 3.4.8 is satisfied by the sign-coherence property of c-vectors established in Lemma 3.3.8 or, more generally for skew-symmetric B-matrices, explained in Section 3.1. Thus the folding of c-vectors (3.4.6) is now justified.

Corollary 3.4.9. *Let $\overline{B}$ be any skew-symmetrizable matrix of cluster type B_n (respectively C_n). There exists a matrix B of cluster type D_{n+1} (respectively A_{2n-1}) such that the cluster algebra with principal coefficients $\mathcal{A}_\bullet\left(\overline{B}\right)$ is a subquotient of the cluster algebra with principal coefficients $\mathcal{A}_\bullet(B)$. In particular any c-vector of $\mathcal{A}_\bullet\left(\overline{B}\right)$ is the folding of some c-vector of $\mathcal{A}_\bullet(B)$.*

3.4.4. Folding of d-vectors. Our next goal is to produce a folding rule for d-vectors. From the above example it is natural to fold the vector $d = (d_i)_{i\in I}$ component wise in this way:

$$d_{\bar{\imath}} := \sum_{s\in\bar{\imath}} d_s. \tag{3.4.11}$$

Once again the above definition is not obvious because, in general, folding is not compatible with the tropicalization map (i.e. the map associating to each cluster variable its d-vector).

Recall the definition of D-matrix given in Section 3.1.

Lemma 3.4.10. *If σ is a stable admissible automorphism of B then the entries in any D-matrix of $\mathcal{A}^\sigma(B)$ satisfy*

$$d_{\sigma(i)\sigma(j)} = d_{ij}. \tag{3.4.12}$$

Proof. The property holds for the D-matrix of the initial cluster. Suppose that D and D' correspond to clusters obtained from one another by a single orbit mutation $\mu_{\overline{k}}$ and that the property holds for D. The only non trivial case we need to consider is when j is in $\overline{k}$. By (3.1.1) we have

$$d'_{ij} = -d_{ij} + \max\left(\sum_{t\in I} d_{it}[b_{tk}]_+, \sum_{t\in I} d_{it}[-b_{tk}]_+\right).$$

Using both induction hypotheses and the fact that σ is stable admissible we get

$$d'_{ij} = -d_{\sigma(i)\sigma(j)} + \max\left(\sum_{t\in I} -d_{\sigma(i)\sigma(t)}\left[b_{\sigma(t)\sigma(k)}\right]_+, \sum_{t\in I} -d_{\sigma(i)\sigma(t)}\left[-b_{\sigma(t)\sigma(k)}\right]_+\right)$$

and we can conclude changing the summation index. $\square$

We define the folding $\overline{D}$ of the D-matrix $D = (d_{ij})_{ij\in I}$ as we did for C-matrices:

$$d_{\overline{\imath}\overline{\jmath}} := \sum_{s\in\overline{\imath}} d_{sj}.$$

Thank to the above lemma this definition is independent of the representative j.

Proposition 3.4.11. *The matrix $\overline{D}$ satisfy the recursion*

$$d'_{\overline{\imath}\overline{\jmath}} = \begin{cases} -d_{\overline{\imath}\overline{k}} + \max\left(\displaystyle\sum_{\overline{\ell}\in I/\sigma} d_{\overline{\imath}\overline{\ell}}[b_{\overline{\ell}\overline{k}}]_+, \sum_{\overline{\ell}\in I/\sigma} d_{\overline{\imath}\overline{\ell}}[-b_{\overline{\ell}\overline{k}}]_+,\right) & \overline{\jmath} = \overline{k} \\ d_{\overline{\imath}\overline{\jmath}} & \overline{\jmath} \neq \overline{k}. \end{cases} \tag{3.4.13}$$

if and only if for any σ-orbit $\overline{\imath}$ the sign of

$$\sum_{t\in I} d_{st} b_{tk}$$

is independent of the representative $s \in \overline{\imath}$.

Proof. We proceed again by induction. It suffices to show that the property holds for a single mutation. Fix a σ-orbit $\overline{\imath}$. The only non-trivial case is when $\overline{\jmath} \neq \overline{k}$. On the one hand we have

$$d'_{\overline{\imath}\overline{\jmath}} = \sum_{s\in\overline{\imath}} \left(-d_{sj} + \max\left(\sum_{t\in I} d_{st}[b_{tk}]_+, \sum_{t\in I} d_{st}[-b_{tk}]_+\right)\right).$$

On the other hand, for the recursion to be satisfied, we must have

$$d'_{\bar{\imath}\bar{\jmath}} = -\sum_{s\in\bar{\imath}} d_{sj} + \max\left(\sum_{t\in I}\sum_{s\in\bar{\imath}} d_{st}[b_{tk}]_+, \sum_{t\in I}\sum_{s\in\bar{\imath}} d_{st}[-b_{tk}]_+\right).$$

We need therefore to have

$$\sum_{s\in\bar{\imath}}\left(\max\left(\sum_{t\in I} d_{st}[b_{tk}]_+, \sum_{t\in I} d_{st}[-b_{tk}]_+\right)\right) = \tag{3.4.14}$$

$$\max\left(\sum_{s\in\bar{\imath}}\sum_{t\in I} d_{st}[b_{tk}]_+, \sum_{s\in\bar{\imath}}\sum_{t\in I} d_{st}[-b_{tk}]_+\right) \tag{3.4.15}$$

which holds if and only if the sign of

$$\sum_{t\in I} d_{st}[b_{tk}]_+ - \sum_{t\in I} d_{st}[-b_{tk}]_+ = \sum_{t\in I} d_{st}b_{tk}$$

is independent of the choice of the representative $s \in \bar{\imath}$. □

Remark 3.4.12. Anna Felikson and Pavel Tumarkin found a case of cluster affine type D where the condition of previous proposition does not hold [**FT**]; we thank them for showing us their example.

For our purposes it is enough to show that the condition of Proposition 3.4.11 holds in the cases of Remark 3.4.4. Using Lemma 3.4.10 and the fact that σ is stable admissible, it is equivalent to ask the sign of

$$\sum_{t\in I} d_{it}b_{tr}$$

to be independent of the representative $r \in \overline{k}$. We get therefore that the condition is satisfied whenever k is fixed by σ.

We prefer to work with this third equivalent formulation: the sign of

$$\sum_{t\in I} d_{i\sigma^m(t)}b_{tk}$$

is independent of $m \in \mathbb{Z}$.

Lemma 3.4.13. *The condition of Proposition 3.4.11 holds for B of cluster type D_{n+1} endowed with the automorphism σ of Remark 3.4.4.*

Proof. There is only one non-trivial σ-orbit; in view of previous observations we can assume it is the orbit of k. This forces $t \notin \overline{k}$ to be fixed by σ. Moreover, since σ is a stable admissible automorphism $b_{tk} = 0$ if $t \in \overline{k}$. Therefore

$$\sum_{t\in I} d_{i\sigma^m(t)} b_{tk} = \sum_{t\in I\setminus\overline{k}} d_{it} b_{tk}$$

which is manifestly independent of m. $\square$

Lemma 3.4.14. *The condition of Proposition 3.4.11 holds for B of cluster type A_{2n-1} endowed with the automorphism σ of Remark 3.4.4.*

Proof. Note at first that rows of a D-matrix associated to a B-matrix B' in $\mathcal{A}_0^\sigma(B)$ are again d-vectors: they are the d-vectors of $\mathcal{A}_0^\sigma(B')$ in the D-matrix associated to B. This follows directly from the surface realization (see Theorem 3.3.7). In particular, in this case, they are sign-coherent and their support is either a string (if they are positive) or a single vertex (if they are negative).

As before we can assume that k is not fixed by σ. If the support of the row i does not contain neighbours of both k and $\sigma(k)$ then the statement is clear. We can therefore assume that there is at least one neighbour of each of them in the support of the i-th row of D.

Let t_1 and t_2 be the two neighbours of k and $\sigma(k)$ respectively lying on the shortest path from k to $\sigma(k)$. By the symmetry required for folding $t_2 = \sigma(t_1)$. Moreover if a row of D contains at least one neighbour of both k and $\sigma(k)$ then it contains both t_1 and t_2. We claim that, in this situation,

$$\sum_{t\in I} d_{i\sigma^m(t)} b_{tk}$$

is either 0 or has the same of b_{t_1k}. Indeed each row of D has at most 2 neighbours of k in its support and the entries of B are either 0 or ± 1.

We can therefore conclude or proof: since σ is a stable admissible automorphism of B we have:

$$b_{t_1 k} = b_{\sigma(t_1)\sigma(k)}.$$

$\square$

3.4.5. Proof of Theorem 1.3.1 for types B_n and C_n. To fix the notation observe that any B-matrix of cluster type B_n or C_n uniquely determines a σ-invariant matrix of cluster type respectively D_{n+1} or A_{2n-1} of which it is the folding. We will therefore denote by $\overline{B}$ a matrix of cluster type B_n or C_n and by B its unfolding.

Let $\pi\left(\mathcal{V}(B)\right)$ be the image of the set $\mathcal{V}(B)$ under the folding map

$$\begin{array}{cccc} \pi: & \mathcal{V}(B) & \longrightarrow & \mathbb{Z}^{\overline{I}} \\ & (v_i)_{i\in I} & \longmapsto & \left(\sum_{s\in\overline{\imath}} v_s\right)_{\overline{\imath}\in\overline{I}} \end{array}$$

and recall the definition of the sets $\mathcal{W}(B_n)$ and $\mathcal{W}(C_n)$ from Section 3.2.

Proposition 3.4.15. *For any matrix $\overline{B}$ of cluster type B_n or C_n we have*

$$\mathcal{V}(\overline{B}) = \pi\left(\mathcal{V}(B)\right).$$

Proof. The claim is clear once we observe that the diagrams in $\mathcal{W}(B_n)$ and $\mathcal{W}(C_n)$ are obtained precisely by folding diagrams from $\mathcal{W}(D_{n+1})$ and $\mathcal{W}(A_{2n-1})$ embedded in $X(B)$. $\square$

We have now the tools we need to deduce Theorem 1.3.1 for types B_n and C_n from the same result for types A_n and D_n.

Proposition 3.4.16. *For any matrix $\overline{B}$ of cluster type B_n or C_n we have*

$$\mathcal{C}_+(\overline{B}) \subset \mathcal{V}(\overline{B})$$

Proof. Combining Corollary 3.4.9, Proposition 3.3.10 and Proposition 3.4.15 we have

$$\mathcal{C}_+(\overline{B}) \subset \pi\left(\mathcal{C}_+(B)\right) = \pi\left(\mathcal{V}(B)\right) = \mathcal{V}(\overline{B}).$$

$\square$

Proposition 3.4.17. *For any matrix $\overline{B}$ of cluster type B_n or C_n we have*

$$\mathcal{D}(\overline{B}) \subset \mathcal{V}(\overline{B})$$

Proof. Combining Lemmata 3.4.13 and 3.4.14 with Proposition 3.3.13, Proposition 3.3.10 and Proposition 3.4.15 we have

$$\mathcal{D}(\overline{B}) \subset \pi\left(\mathcal{D}(B)\right) = \pi\left(\mathcal{V}(B)\right) = \mathcal{V}(\overline{B}).$$

$\square$

To conclude we need one last lemma.

Lemma 3.4.18. *For any matrix $\overline{B}$ of cluster type B_n or C_n we have*

$$\mathcal{C}^b_+(\overline{B}) = \pi\left(\mathcal{C}^b_+(B)\right)$$

and

$$\mathcal{D}^b(\overline{B}) = \pi\left(\mathcal{D}^b(B)\right).$$

Proof. The claim follows directly from the following observation: a matrix $\overline{B}$ is bipartite if and only if its unfolding B is bipartite. We get equalities (as opposed to inclusions) because any two bipartite matrices of cluster type D_{n+1} or A_{2n-1} are connected by orbit mutations. $\square$

Proposition 3.4.19. *For any matrix $\overline{B}$ of cluster type B_n or C_n we have*

$$\mathcal{V}(\overline{B}) \subset \mathcal{C}_+(\overline{B})$$

and

$$\mathcal{V}(\overline{B}) \subset \mathcal{D}(\overline{B}).$$

Proof. We show only the second condition; the first one is obtained in the same way. Using Proposition 3.4.15, Proposition 3.3.14, equation (3.3.3), and

Lemma 3.4.18 we get

$$\mathcal{V}(\overline{B}) = \pi\left(\mathcal{V}(B)\right) \subset \pi\left(\mathcal{D}(B)\right) = \pi\left(\mathcal{D}^b(B)\right) = \mathcal{D}^b(\overline{B}) \subset \mathcal{D}(\overline{B}).$$

□

For completeness we record also the following equalities (of which Theorem 1.3.3 is a direct consequence).

Corollary 3.4.20. *For any matrix $\overline{B}$ of cluster type B_n or C_n we have*

$$\mathcal{C}_+^b(\overline{B}) = \mathcal{C}_+(\overline{B}),$$

$$\mathcal{D}^b(\overline{B}) = \mathcal{D}(\overline{B}),$$

$$\mathcal{C}_+^b(\overline{B}) = \mathcal{D}^b(\overline{B}).$$

3.5. Proof of Theorem 1.3.2

Here we derive Theorem 1.3.2. The claim (4) is a direct consequence of our description of c- and d-vectors in Theorem 1.3.1. For simply-laced types claims (1) and (3) follow from Corollaries 3.1.8 and 3.1.11. However, for types A_n and D_n, we provide a direct proof using Theorem 1.3.1 without referring to the representation-theoretic results of Section 3.1.

As we did before we deal with types A_n and D_n first; we will use again a folding argument to deduce the results for types B_n and C_n.

3.5.1. Types A_n and D_n. Let B be any skew-symmetric integer matrix of cluster type either A_n or D_n. Having built an explicit list of all the positive c-vectors and non-initial d-vectors for the cluster algebra $\mathcal{A}_\bullet(B)$ with principal coefficients we can give a combinatorial proof of Theorem 1.3.2.

Proposition 3.5.1. *All c-vectors and d-vectors of $\mathcal{A}_\bullet(B)$ are roots in the root system associated to the Cartan counterpart of B. Each of them is real if and only if its support in $X(B)$ is a tree.*

PROOF. It suffices to establish the claim for positive c-vectors. We are dealing with a local property: since the support of any c-vector c of $\mathcal{A}_\bullet(B)$ is a connected sub-diagram of $X(B)$ it suffices to show that c is a root in the root system associated to its support.

The claim is clear for type A_n and for cases I, II and III of type D_n: they are all roots in the corresponding finite type root system.

Applying in sequence the simple reflections corresponding to the outermost node with multiplicity 2 we can reduce case VII to case VI. We can then "trim the branches" reflecting each time with respect to a leaf of the diagram. After these reductions we are left with the four cases in Figure 3.5.1. They all correspond to

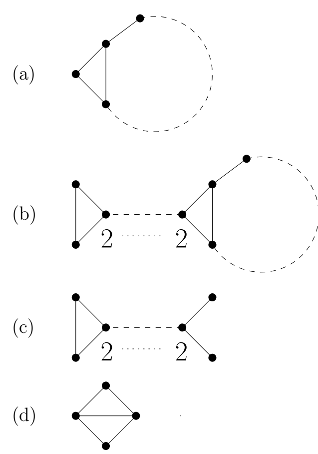


FIGURE 3.5.1. Reduced c-vectors.

imaginary roots (see [**Kac90**, Lemma 5.3]). Indeed let c be any of these reduced c-vectors and let A be the generalized Cartan matrix associated to its support, then all the components of the vector Ac are non-positive. $\square$

Let $\langle \cdot, \cdot \rangle$ be the Euler form of the quiver $Q = Q(B)$ associated to B; it is defined on roots as follows:

$$\langle \sum_{i \in I} c_i \alpha_i, \sum_{i \in I} d_i \alpha_i \rangle := \sum_{i \in I} c_i d_i - \sum_{b_{ij} > 0} b_{ij} c_j d_i.$$

To show that elements of $\mathcal{C}_+(B) = \mathcal{D}(B)$ are Schur roots we will use the following result of A. Schofield ([**Sch92**, Theorem 6.2]).

THEOREM 3.5.2. *Let α be a positive root that is not a Schur root then α satisfies one of the following conditions:*

(1) *$\langle \alpha, \alpha \rangle = 0$ and there are a positive (imaginary) root β and a positive integer k such that $\alpha = k\beta$.*
(2) *α is the sum of two positive roots, one of them (call it β) is real and satisfies*

$$\langle \alpha, \beta \rangle > 0 \quad \text{and} \quad \langle \beta, \alpha \rangle > 0.$$

Let $A = A(B)$ be the Cartan counterpart of B. As noted in [**Sch92**], if α is an imaginary root that is not Schur, there are few possibilities for the positive real root β satisfying (2). Namely, if w is the element of the Weyl group such that all the components of the vector $Aw(\alpha)$ are non positive then $w(\beta)$ has to be a negative real root.

Proposition 3.5.3. *All the vectors in $\mathcal{C}_+(B) = \mathcal{D}(B)$ are Schur roots of $\Delta(A(B))$.*

PROOF. We are dealing still with a local property so we can assume that the c-vector we consider has full support.

It is well known that if $X(B)$ is a finite type Dynkin diagram then any root is a Schur root (every indecomposable $kQ(B)$-module is rigid if $Q(B)$ is an orientation of a Dynkin diagram); therefore we need only to concentrate on cases IV, V, VI, VII, and VIII of type D_n. Let c be any of these c-vectors, they are all imaginary roots. None of them is an integer multiple of a root so case (1) of Theorem 3.5.2 is excluded and we need to show only that we are not in case (2).

As noted in Proposition 3.5.1 the elements w of the Weyl group we need to apply to c are those "trimming the branches"; since the roots β we are looking for change sign when acted on by w their support must be contained in only one of those appendices; we can therefore assume that there is only one appendix in the weighted diagram of c. Label the nodes on such an appendix with $\{1\dots,n-1\}$ starting from the leaf; let n be the node the appendix is connected to and let m be the innermost node with multiplicity 1 in the appendix. It is clear that the element w we are looking for is then $s_{n-1}\dots s_1$ in cases IV, V, VI, and VIII and $s_{n-1}\dots s_1 s_n\dots s_{m+1}$ in case VII. The possible roots β are then

$$\alpha_1+\cdots+\alpha_k$$

for $k\in\{1,\dots,n-1\}$ in cases IV, V, VI, and VIII and

$$\begin{array}{cc}
\alpha_1+\cdots+\alpha_{k+1} & m\le k\le n-1\\
\alpha_1+\cdots+\alpha_k & k<m\\
\alpha_{m+1}+\cdots+\alpha_k & m+1\le k\le n
\end{array}$$

in case VII. By direct inspection we get that in all cases, regardless of the orientations, one of the two integers $\langle c,\beta\rangle$ and $\langle\beta,c\rangle$ is non-positive. □

Proposition 3.5.4. *The cardinality $|\mathcal{C}_+(B)|=|\mathcal{D}(B)|$ depends only on the cluster type of B; it is equal to $n(n+1)/2$ if B is of cluster type A_n and $n(n-1)$ if B is of cluster type D_n.*

Proof. Fix an element $X(B)$ of $\mathcal{X}(B)$. We need to count in how many different ways any diagrams from $\mathcal{W}(B)$ can be embedded in $X(B)$.

This count, for type A_n, was done by Parson ([**Par11**, Lemma 5.8]) by noting that any embedding of a string is determined by the positions of its endpoints.

Let us consider type D_n; there are four cases to be considered depending on which of the four diagrams in Figure 3.2.6 describes $X(B)$. We present case (d): it involves all the techniques and it is the most complex one. The other cases can be dealt in a similar fashion.

The only weighted diagrams that can be embedded in a Dynkin diagram shaped as (d) are I, VI, VII, and VIII from Figure 3.2.7. An embedding of any of those is uniquely determined by a pair of vertices in $X(B)$; for I (with at least two nodes), and VIII they are the two leafs; for VII they are the only leaf and the leftmost node with weight 2. For VI and strings of length 1 the two vertices of $X(B)$ coincide.

We are going to reverse this observation to count embeddings. Suppose that the central cycle contains k vertices.

To each pair of vertices i and j not in the central cycle we can associate precisely two embeddings: if they belong to different components (say X' and X'') we have two strings passing on either side of the central cycle. If i and j belong to the same type-A_m component (say X') and are distinct then we have a string connecting i to j completely contained in X' and a weighted diagram of type VII or VIII depending on the relative position of i and j. Finally if $i = j$ then we have a single point and a weighted diagram of type VI. They sum up to $(n-k)(n-k+1)$ embeddings.

If one of the two vertices, say i, is in the central cycle and j is in the component X' then there are two possibilities: if i is one of the two vertices adjacent to X' then there is only one embedding associated to the pair i and j: the shortest string connecting them. Otherwise there are two strings that we can embed into $X(B)$ depending on the side of the central cycle we cross. Therefore there are $2(k-2)(n-k) + 2(n-k)$ embeddings with one vertex in a type-A_m component and a vertex in the central cycle.

Finally if both i and j are in the central cycle we need to distinguish three cases: they can coincide (yielding embedding of single nodes), they can be adjacent (and produce embedding of strings of length 2). Otherwise they produce precisely two embedding of strings. In total there are $k^2 - k$ embeddings induced by pair of vertices in the central cycle.

Summing up all the contributions we get

$$(n-k)(n-k+1) + 2(k-2)(n-k) + 2(n-k) + k^2 - k = n^2 - n$$

As desired. $\square$

3.5.2. Types B_n and C_n. To extend the above results to types B_n and C_n we will use the following general fact on the folding of root systems.

Proposition 3.5.5. *Let B be a skew-symmetrizable integer matrix together with an admissible automorphism σ and denote by $A = A(B)$ its Cartan counterpart. Let $\overline{B}$ be the image of B under the folding map π and $\overline{A} = A(\overline{B})$ the Cartan counterpart of $\overline{B}$. Let $\{\alpha_i\}_{i\in I}$ be the simple roots for $\Delta(A)$ and $\{\alpha_{\bar{\imath}}\}_{\bar{\imath}\in I/\sigma}$ be the simple roots for $\Delta(\overline{A})$.*

Define the linear map π from the root lattice of $\Delta(A)$ to the root lattice of $\Delta(\overline{A})$ by

$$\pi(\alpha_i) := \alpha_{\bar{\imath}}.$$

Then for any $\alpha \in \Delta(A)$ we have $\pi(\alpha) \in \Delta(\overline{A})$.

Proof. This argument is a refinement of [**Tan02**, Proposition A.7]; there the result is stated only for finite type root systems.

Observe first that the map π commutes with "orbit reflections"

$$s^\sigma_{\bar{\imath}} := \prod_{t\in\bar{\imath}} s_t$$

that is for any root α

$$s_{\bar{\imath}}(\pi(\alpha)) = \pi\left(s^\sigma_{\bar{\imath}}(\alpha)\right). \tag{3.5.1}$$

Orbit reflections are well defined because, by admissibility of σ, we have

$$a_{i_1 i_2} = 0$$

for any pair $i_1 \neq i_2$ in the same σ-orbit $\bar{\imath}$. It is sufficient to verify (3.5.1) on simple roots; we have

$$s_{\bar{\imath}}\left(\pi(\alpha_j)\right) = s_{\bar{\imath}}\left(\alpha_{\bar{\jmath}}\right) = \alpha_{\bar{\jmath}} - a_{\bar{\imath}\bar{\jmath}}\alpha_{\bar{\imath}} = \pi\left(\alpha_j - \sum_{t\in\bar{\imath}} a_{tj}\alpha_t\right) = \pi\left(s^\sigma_{\bar{\imath}}(\alpha_j)\right).$$

Back to our problem, without loss of generality we can assume α to be a positive root; we will proceed by induction on

$$\mathrm{ht}(\alpha) = \mathrm{ht}\left(\sum_{i\in I} c_i\alpha_i\right) := \sum_{i\in I} c_i$$

If $\mathrm{ht}(\alpha) = 1$ then $\alpha = \alpha_i$ for some $i \in I$; thus $\pi(\alpha) = \alpha_{\bar{\imath}}$. Suppose now that $\mathrm{ht}(\alpha) > 1$. If all the components of the vector $\overline{A}\pi(\alpha)$ are negative then $\pi(\alpha)$ is an imaginary root (see [**Kac90**, Lemma 5.3]). Otherwise let $\bar{\imath}$ be such that

$$\left(\overline{A}\pi(\alpha)\right)_{\bar{\imath}} > 0. \tag{3.5.2}$$

Set $\alpha' := s^{\sigma}_{\bar{\imath}}(\alpha)$. Since $\bar{\imath}$ is disconnected, in view of (3.5.2) α' is a positive root and

$$\mathrm{ht}\left(\alpha'\right) < \mathrm{ht}(\alpha).$$

By induction hypothesis then $\pi\left(\alpha'\right)$ is a positive root in the root system of $\Delta(\overline{A})$ therefore so is

$$\pi(\alpha) = s_{\bar{\imath}}\left(s_{\bar{\imath}}(\pi(\alpha))\right) = s_{\bar{\imath}}\left(\pi\left(s^{\sigma}_{\bar{\imath}}(\alpha)\right)\right) = s_{\bar{\imath}}\left(\pi(\alpha')\right).$$

□

Note that the folding of roots agrees with the folding of both c- and $d-$vectors.

Proposition 3.5.6. *Let B be a skew-symmetrizable integer matrix of cluster type B_n or C_n. All the c-vectors and d-vectors of $\mathcal{A}_\bullet(B)$ are roots in the root system $\Delta(A(B))$.*

Proof. It is enough to consider positive c-vectors. By Corollary 3.4.9 any element of $\mathcal{C}_+(B)$ is the image of some c-vector of a cluster algebra of type D_{n+1} or A_{2n-1}. By Proposition 3.5.1 the latter are roots in the root system associated to the unfolding of B. Our claim follows then directly from Proposition 3.5.5. □

Proposition 3.5.7. *Any c-vector (d-vector) of $\mathcal{A}_\bullet(B)$ of type B_n or C_n is a real root if and only if its support is a tree.*

Proof. If the support of the vector we are considering is a tree there is nothing to show. In all other cases we can "trim the branches" and check directly as we did in Proposition 3.5.1. □

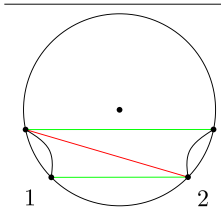
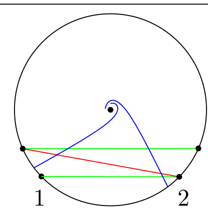
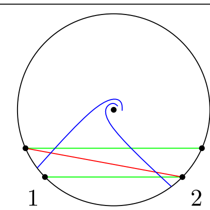
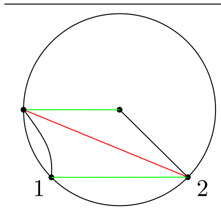
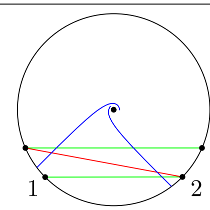
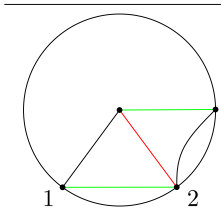
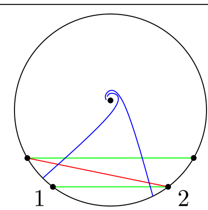
Proposition 3.5.8. *For any B-matrix of cluster type either B_n or C_n the cardinality $|\mathcal{C}_+(B)| = |\mathcal{D}(B)|$ is equal to n^2.*

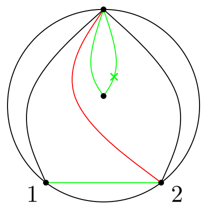
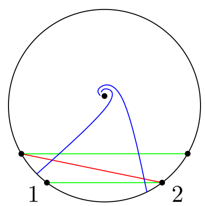
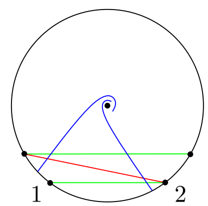
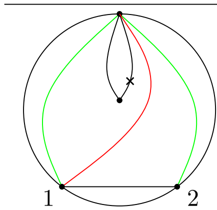
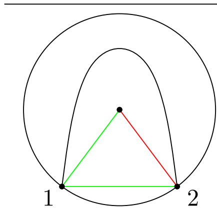
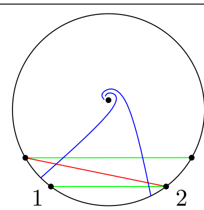
Proof. This claim does not follow directly from folding. Nevertheless it is straightforward to apply the same argument of Proposition 3.5.4 to perform the counting. □

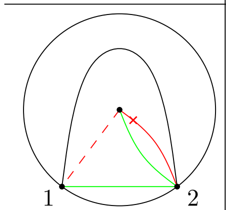
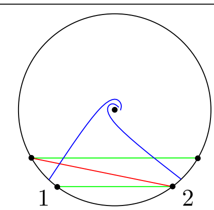
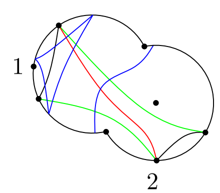
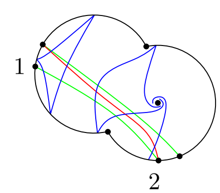
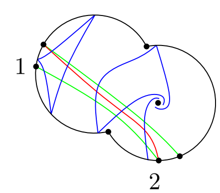
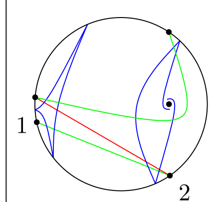
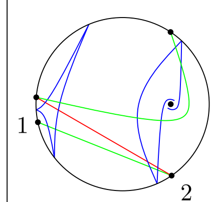
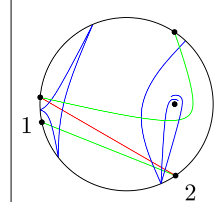
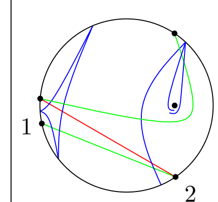
3.6. Appendix: Type D_n analysis

The analysis proceeds as follows: as explained above any multilamination corresponding to an initial triangulation decomposes the surface S into pieces (see Figure 3.3.6 for an example). Any quadrilateral can intersect positively at most laminations contained in three different pieces. We need therefore to consider, for any quadrilateral in Figure 3.3.3, all the possible ways of inscribing it in a surface with at most three pieces. Any such case is divided further into sub-cases depending on how the multilamination looks around the puncture (see Figure 3.3.5) and which marked points are the vertices of the quadrilateral we consider.

For any possible configuration we provide a bipartite quadrilateral giving rise to the same c-vector.

Configuration on the surface	Plain n-gon ($n \geq 3$)	Notched n-gon ($n \geq 3$)	Plain digon	Notched digon	Folded digon
	 Type I	 Type I			
		 Type I			
	 Type I				

Configuration on the surface	Plain n-gon ($n \geq 3$)	Notched n-gon ($n \geq 3$)	Plain digon	Notched digon	Folded digon
	 Type I	 Type I			
					
	 Type I				

Configuration on the surface	Plain n-gon ($n \geq 3$)	Notched n-gon ($n \geq 3$)	Plain digon	Notched digon	Folded digon
		 Type I			
	 Type I	 Type I	 Type I	 Type I	 Type II  Type I

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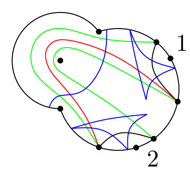
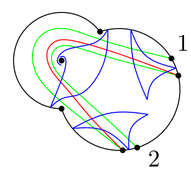
<td></td>
<td>
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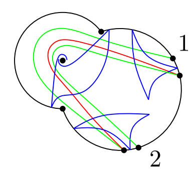
<td>
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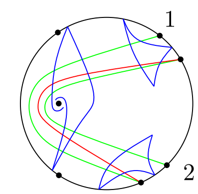
<td>
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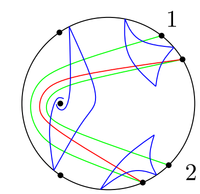
<td>
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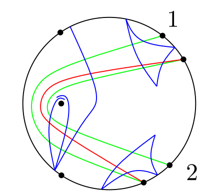
<td>
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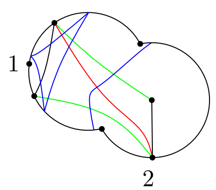
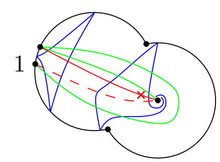
<td></td>
<td>
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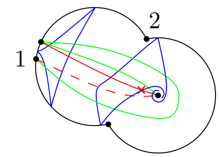
<td>
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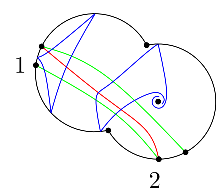
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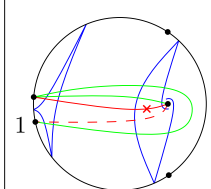
<td>
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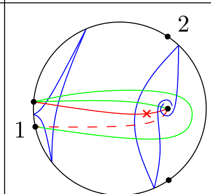
<td>
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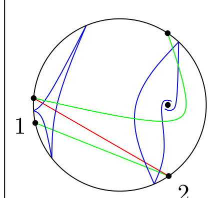
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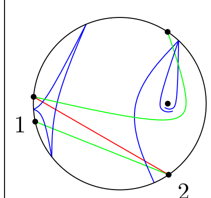
<td>
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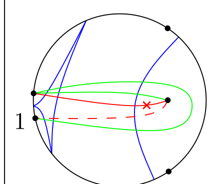
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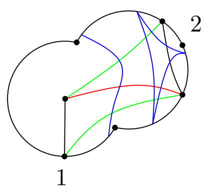
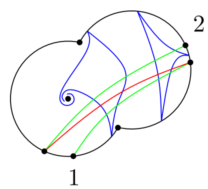
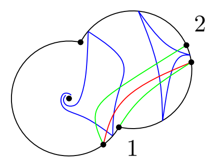
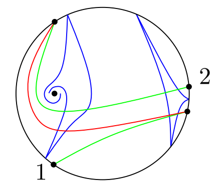
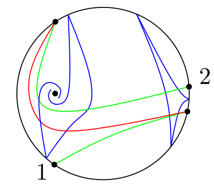
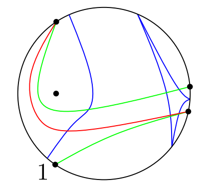
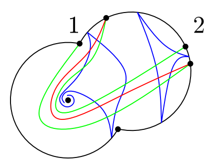
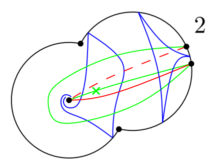
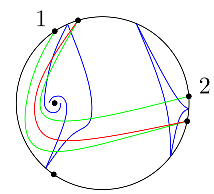
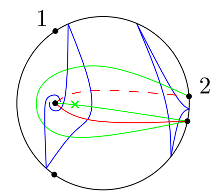
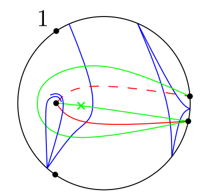
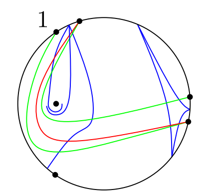
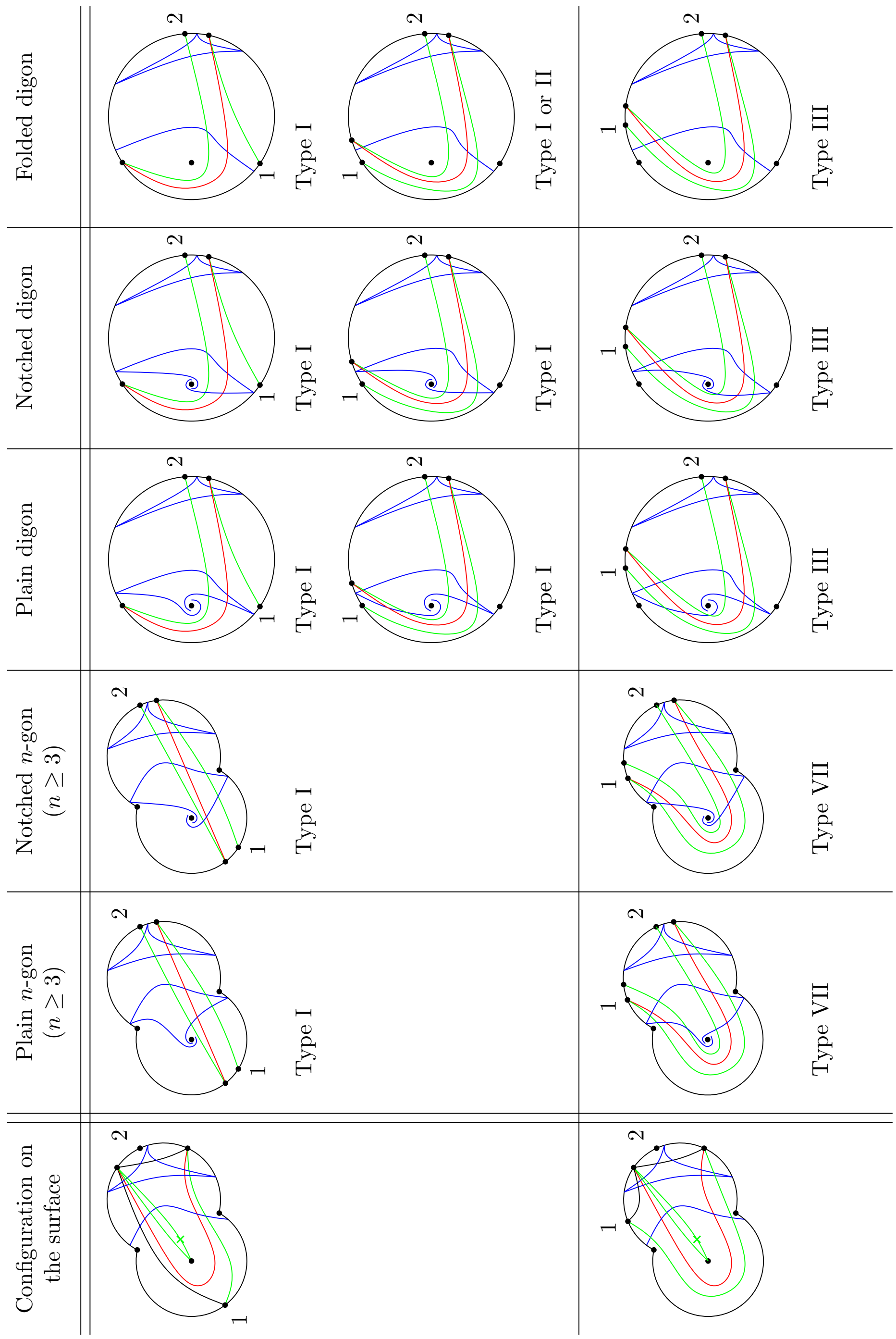
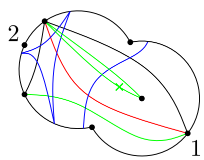
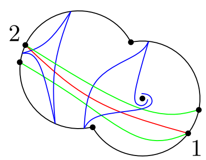
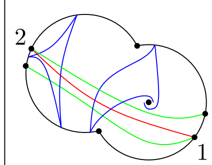
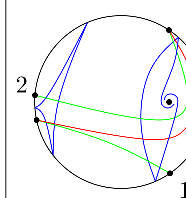
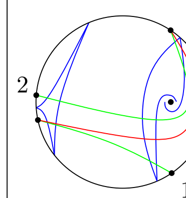
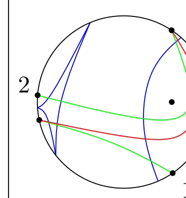
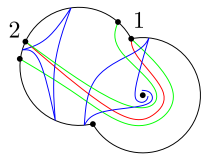
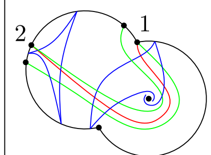
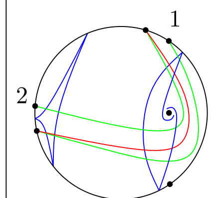
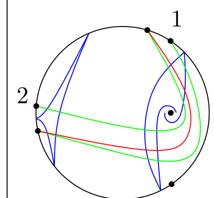
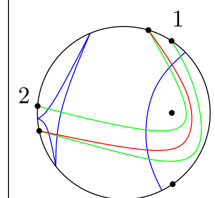
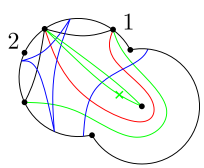
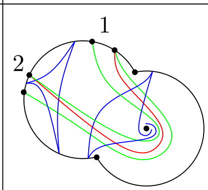
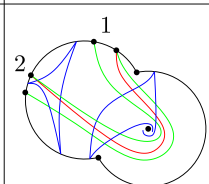
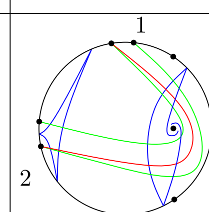
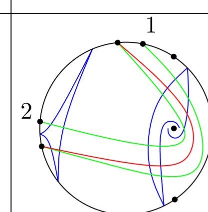
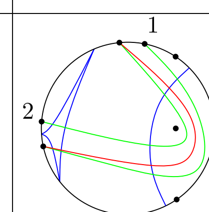
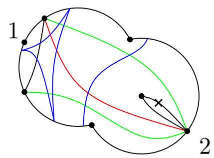
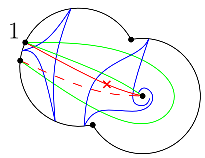
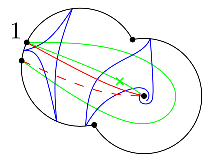
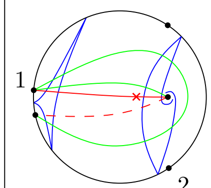
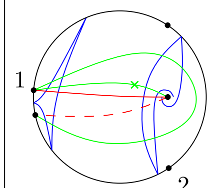
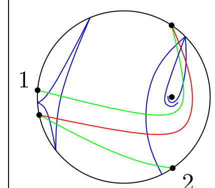
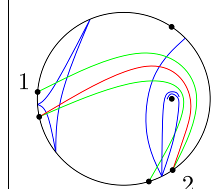
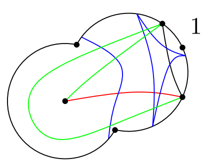
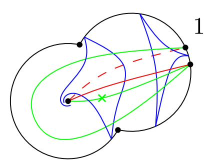
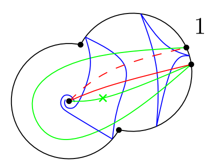
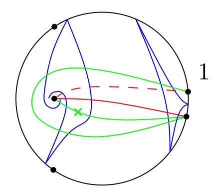
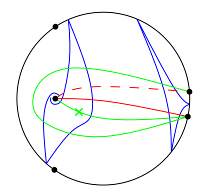
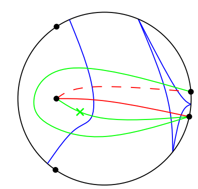
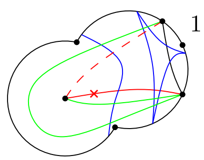
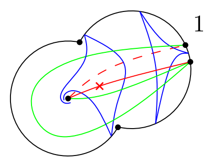
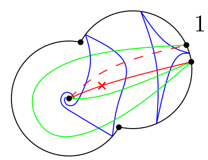
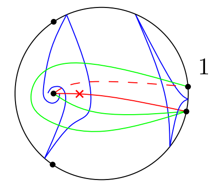
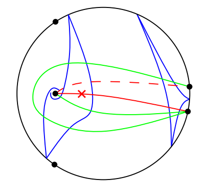
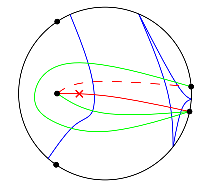
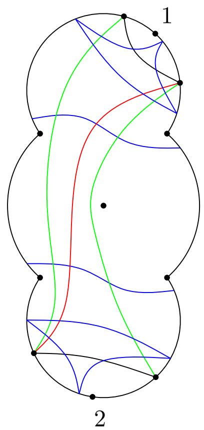
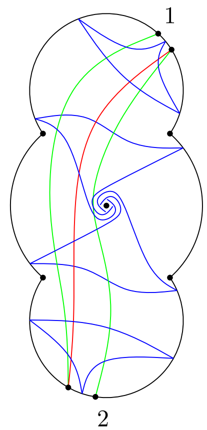
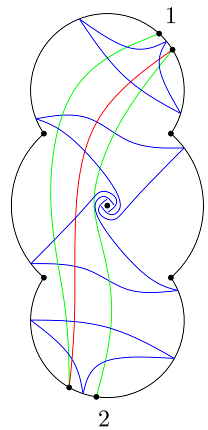
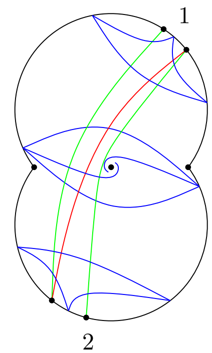
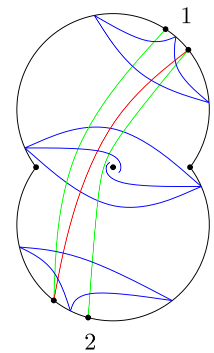
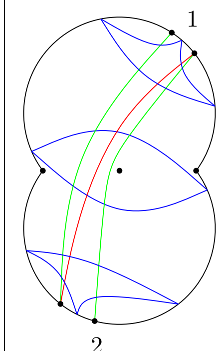
Configuration on the surface	Plain n-gon ($n \geq 3$)	Notched n-gon ($n \geq 3$)	Plain digon	Notched digon	Folded digon
	 Type I	 Type I	 Type I	 Type I	 Type I
	 Type I	 Type I	 Type I	 Type I	 Type I
					 Type I


Configuration on the surface
Plain n-gon ($n \geq 3$)
Notched n-gon ($n \geq 3$)
Plain digon
Notched digon
Folded digon
Type I
Type VII
Type I
Type VII
Type I
Type I
Type III
Type I
Type I
Type III
Type I
Type I or II
Type III

Configuration on the surface	Plain n-gon ($n \geq 3$)	Notched n-gon ($n \geq 3$)	Plain digon	Notched digon	Folded digon
	 Type I	 Type I	 Type I	 Type I	 Type I or II
	 Type VII	 Type VII	 Type III	 Type III	 Type III
	 Type VII	 Type VII	 Type III	 Type III	 Type III

Configuration on the surface	Plain n-gon ($n \geq 3$)	Notched n-gon ($n \geq 3$)	Plain digon	Notched digon	Folded digon
	 Type I	 Type I	 Type I	 Type I	 Type I  Type I
	 Type VI	 Type I	 Type II	 Type I	 Type I

Configuration on the surface	Plain n-gon ($n \geq 3$)	Notched n-gon ($n \geq 3$)	Plain digon	Notched digon	Folded digon
	 Type I	 Type VI	 Type I	 Type II	 Type I
	 Type I	 Type I	 Type I	 Type I	 Type I or V

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