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Inequalities with angular integrability and applications

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Introduction

We study the improvements due to the angular regularity in the context of Sobolev embeddings and PDEs. It is well known that many fundamental inequalities in mathematical analysis get improvements under some additional symmetry assumptions. Such improvements are related to the geometric nature of the space and in particular to the action of a certain group of symmetry. This is not surprising because a symmetric function on a differentiable manifold can be considered as a function defined on lower dimensional manifold on wich stronger estimates are often available. Then such improved estimates can be extended on the whole manifold by the action of a certain group. In particular we work in a very simple setting by considering radially symmetric functions defined on $\mathbb{R}^n$. Such functions are indeed defined on $\mathbb{R}^+$ and $SO(n)$ acts to go back to $\mathbb{R}^n$.

For instance the Hardy-Littelwood ([**24**], [**13**], [**9**], [**7**]), Caffarelli-Kohn-Nirenberg ([**6**]) and Strichartz ([**3**], [**16**], [**22**]) inequality on $\mathbb{R}^n$ get improvements if one restricts to consider just radially symmetric functions. A natural question is if and how this phenomenon occurs if we replace the symmetry hypotesis with merely higher integrability in the angular variables. We show that improvements can be obtained by working with the norms:

$$
\begin{array}{rcl}
\|f\|_{L^p_{|x|} L^{\tilde{p}}_\theta} & = & \left(\int_0^{+\infty} \|f(\rho \, \cdot \,)\|^p_{L^{\tilde{p}}(\mathbb{S}^{n-1})} \rho^{n-1} d\rho \right)^{\frac{1}{p}}, \\
\|f\|_{L^\infty_{|x|} L^{\tilde{p}}_\theta} & = & \sup_{\rho > 0} \|f(\rho \, \cdot \,)\|_{L^{\tilde{p}}(\mathbb{S}^{n-1})};
\end{array}
$$

we observe that such results interpolate beetween the improved versions for radially symmetric functions and the classical ones.

We develop widely this technology in the first chapter. The main results proved are extensions of Hardy-Littelwood-Sobolev (theorem 1.3, corollary 1.4) and Caffarelli-Kohn-Nirenberg inequality (theorem 1.11). Another interesting aspect is that the inequalities on which we focus are fundamental tools in the study of PDEs. For instance the Caffarelli-Kohn-Nirenberg inequality is really important in the study of regularity of the Navier-Stokes equation's solutions, because it provides a priori estimates through interpolation of quantities related to the energy dissipation. So it is expectable that the technology developed can lead to succesfully results also in this areas. We focus basically on the small data theory and on regularity criteria. We actually extended the criteria in [**33**] and we made a conjecture on a possible extension of the small data result in [**27**].

As well known the well posedness of the Cauchy Problem for the Navier-Stokes equation

$$
\begin{cases}
\partial_t u + (u \cdot \nabla) u + \nabla p & = & \Delta u & \text{in} & \mathbb{R}^+ \times \mathbb{R}^n \\
\nabla \cdot u & = & 0 & \text{in} & \mathbb{R}^+ \times \mathbb{R}^n \\
u & = & u_0 & \text{in} & \{0\} \times \mathbb{R}^n,
\end{cases}
$$

is a big mathematical challenge, and only partial results have been obtained. Leray has proved global existence of weak solutions for L^2 initial data in his pioneering work [**26**]. On the other hand the uniqeness of Leray's solutions is still open, as it is the propagation of regularity of the initial datum u_0. The well posedness theory

is well developed for short times, or if one restricts to small u_0. In this scenario is useful to estabilish at least regularity or uniqeness criteria. We mean to find some a priori assumptions on the solutions under which the regularity (or the uniqeness) is guaranteed. We focus on the regularity in space variables, being the time regularity a different and more difficult problem. This is actually expectable by a physical viewpoint, because of the assumption of incompressibility of the fluid; see [29]. It turns out that the a priori assumptions requested in order to get regularity are basically boundedness assumptions on $u, \nabla u$, or $\nabla \times u$.

As mentioned we refer basically to the criteria in [33], in which the main novelty is to consider boundedness in weighted L^p spaces with weights $|x|^\alpha$. More precisely the author works with solutions $u : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, equipped with the norms:

$$\||x|^\alpha u\|_{L^s_T L^p_x}, \qquad \frac{2}{s} + \frac{n}{p} = 1 - \alpha, \tag{0.1}$$

where, of course, the indexes relations follow by scaling considerations. Under boundedness assumptions the regularity in the segment $(0, T) \times \{0\}$ is achieved, more precisely u is C^∞ in the space variables. So the introduction of weights allows to get local regularity criteria, where we mean only in a neighborhood of the origin. At first we show how for some choices of the indexes the criteria in [33] are indeed global (the regularity is achieved in $(0, T) \times \mathbb{R}^n$). Then we get improvements by working with spaces with different integrability in radial and angular directions; so instead of the norms (0.1) we use:

$$\||x|^\alpha u\|_{L^s_T L^p_{|x|} L^{\widetilde{p}}_\theta} = \left(\int_0^T \left(\int_0^{+\infty} \|u(t, \rho \cdot)\|^p_{L^{\widetilde{p}}(\mathbb{S}^{n-1})} \rho^{\alpha p + n - 1} \, d\rho \right)^{\frac{s}{p}} dt \right)^{\frac{1}{s}},$$

with $\frac{2}{s} + \frac{n}{p} = 1 - \alpha$. In this setting we get global regularity if sufficiently high values of $\widetilde{p}$ are considered. We observe, as expectable, two different behaviour in the ranges $\alpha < 0$ and $\alpha > 0$; we show regularity in the case $\widetilde{p} \geq \widetilde{p}_G$ where

$$\widetilde{p}_G = \frac{(n-1)p}{\alpha p + n - 1};$$

and of course:

$$p < \widetilde{p}_G, \qquad \text{if} \quad \alpha < 0,$$

$$\widetilde{p}_G < p, \qquad \text{if} \quad \alpha > 0.$$

A similar analysis has been performed about the well posedness with small data in mixed angular-radial weighted spaces equipped with the norms:

$$\||x|^\alpha u_0\|_{L^p_{|x|} L^{\widetilde{p}}_\theta} = \left(\int_0^{+\infty} \|u_0(\rho \cdot)\|^p_{L^{\widetilde{p}}(\mathbb{S}^{n-1})} \rho^{\alpha p + n - 1} \, d\rho \right)^{\frac{1}{p}}, \tag{0.2}$$

with again the critical scaling relationship $\alpha = 1 + \frac{n}{p}$. Here we find out the critical value

$$\widetilde{p}_G = \frac{(n-1)p}{p-1},$$

and the well posedness is achieved for small data, with $\widetilde{p} \geq \widetilde{p}_G$. Actually we have closely looked at the following heuristic: the weights $|x|^\alpha$, $\alpha < 0$ localize, in some sense, the norms of the data (or of the solutions) near to the origin. In such a way local results are still available, but a loss of informations far from the origin occurs. These informations can be recovered every times by a suitbale amount of angular integrability.

On the other hand the local (in the sense of localized near to the origin) results have an intrinsic interest, and we also look at this problem. In theorem 3.6 we prove local regularity for bounded solutions in

$$\left\| |x|^{\alpha} u \right\|_{L_T^s L_{|x|}^p L_{\theta}^{\widetilde{p}}}, \qquad \text{with} \quad \widetilde{p} \leq p.$$

This improves in a different direction the results in 2.15. We get local regularity under the assumption of a sufficiently high angular regularity, i.e for $\widetilde{p} \geq \widetilde{p}_L$ where

$$\widetilde{p}_L = \begin{cases} \dfrac{2(n-1)p}{(2\alpha+1)p+2(n-1)} & \text{if} \quad -\frac{1}{2} \leq \alpha < 0 \\[3mm] \dfrac{2(n-1)p}{p+2(n-1)} & \text{if} \quad 0 < \alpha < 1. \end{cases}$$

The main technical tools we use consist in decay estimates for convolutions with the heat and Oseen kernels in the context of weighted $L_{|x|}^p L_{\theta}^{\widetilde{p}}$ spaces. Estimates in weighted spaces have been considered in literature, but the information provided by the angular integrability leads to a really satisfactory admissibility range for the weights. The precise relation between the weights and the angular integrability is basically contained in the relation (1.7) in the corollary 1.4.

The small data theory in the context of weighted $L_{|x|}^p L_{\theta}^{\widetilde{p}}$ spaces with $\widetilde{p} < p$ is more delicate and we just make a conjecture about a possible improvement of theorem 2.24, in which the authors show regularity of the Leray's solutions in the interior of the space-time parabola:

$$\Pi = \left\{ (t,x) \quad \text{s.t.} \quad t > \frac{|x|^2}{\varepsilon - \varepsilon} \right\},$$

for a sufficiently small $\varepsilon_0 > 0$ and data with $\left\| |x|^{-1/2} \cdot \right\|_{L_x^2} = \varepsilon < \varepsilon_0$. The conjecture in section 3 is made in order to cover the gap between this localized result and the classical well posedness results (in particular we refer to theorem 3.7 that's a particular case of the Koch-Tataru theorem 2.23).

Remark 0.1. Of course by translations all the results are still valid if the norms and the weights are centered in a point $\bar{x} \neq 0$. So if we consider

$$\left\| f \right\|_{L_{|x-\bar{x}|}^p L_{\theta}^{\widetilde{p}}} = \left(\int_0^{+\infty} \| f(\bar{x}+\rho\theta) \|_{L^{\widetilde{p}}(\mathbb{S}^{n-1})}^p \rho^{n-1} d\rho \right)^{\frac{1}{p}},$$

$$\left\| |x-\bar{x}|^{\alpha} u \right\|_{L_T^s L_{|x-\bar{x}|}^p L_{\theta}^{\widetilde{p}}} = \left(\int_0^T \left(\int_0^{+\infty} \| u(\bar{x}+\rho\theta) \|_{L^{\widetilde{p}}(\mathbb{S}^{n-1})}^p \rho^{\alpha p+n-1} d\rho \right)^{\frac{s}{p}} dt \right)^{\frac{1}{s}},$$

$$\left\| |x-\bar{x}|^{\alpha} u_0 \right\|_{L_{|x-\bar{x}|}^p L_{\theta}^{\widetilde{p}}} = \left(\int_0^{+\infty} \| u_0(\bar{x}+\rho\theta) \|_{L^{\widetilde{p}}(\mathbb{S}^{n-1})}^p \rho^{\alpha p+n-1} d\rho \right)^{\frac{1}{p}},$$

and so on.

The content of the Chapter 1 is taken from [4].

Classical inequalities with angular integrability

The goal of this section is to extend some classical estimates in the context of L^p spaces to a setting in which the role of the angular and radial integrability is well distinguished. In order of explaining our purpose we start by a well known estimate of Walter Strauss [23] that proves

$$|x|^{\frac{n-1}{2}}|u(x)| \le C\|\nabla u\|_{L^2}, \qquad |x| \ge 1, \tag{1.1}$$

for radial functions $u \in \dot{H}^1(\mathbb{R}^n)$, $n \ge 2$, This is an example of a well known general phenomenon: under suitable assumptions of symmetry, notably radial symmetry, classical estimates and embeddings of spaces admit substantial improvements. In the case of (1.1), a control on the H^1 norm of u gives a pointwise bound and decay of u. Both are false in the general case. Radial and more general symmetric estimates have been extensively investigated, in view of their relevance for applications, especially to differential equations.

This phenomenon is quite natural; indeed, symmetric functions can be regarded as functions defined on lower dimensional manifolds, on which stronger estimates are available, then extended by the action of some group of symmetries. Radial functions are essentially functions on $\mathbb{R}^+$, while the norms on $\mathbb{R}^n$ are recovered by the action of $SO(n)$ that introduces suitable dimensional weights connected to the volume form.

In view of the gap between the symmetric and the non symmetric case, an interesting question arises: is it possible to quantify the defect of symmetry of functions and prove more general estimates which encompass all cases, and in particular reduce to radial estimates when applied to radial functions? Heuristically, one should be able to improve on the general case by introducing some measure of the distance from the maximizers of the inequality, which typically have the greatest symmetry. The aim of this paper is to give a partial positive answer to this question, through the use of the following type of mixed radial-angular norms:

$$\begin{aligned} \|f\|_{L^p_{|x|}L^{\widetilde{p}}_\theta} &= \left(\int_0^{+\infty} \|f(\rho\,\cdot\,)\|^p_{L^{\widetilde{p}}(\mathbb{S}^{n-1})}\rho^{n-1}d\rho\right)^{\frac{1}{p}}, \\ \|f\|_{L^\infty_{|x|}L^{\widetilde{p}}_\theta} &= \sup_{\rho>0}\|f(\rho\,\cdot\,)\|_{L^{\widetilde{p}}(\mathbb{S}^{n-1})}. \end{aligned}$$

When the context is clear we shall write simply $L^pL^{\widetilde{p}}$. For $p = \widetilde{p}$ the norms reduce to the usual L^p norms

$$\|u\|_{L^p_{|x|}L^p_\theta} \equiv \|u\|_{L^p(\mathbb{R}^n)},$$

while for radial functions the value of $\widetilde{p}$ is irrelevant:

$$u \text{ radial} \quad \Longrightarrow \quad \|u\|_{L^pL^{\widetilde{p}}} \simeq \|u\|_{L^p(\mathbb{R}^n)} \quad \forall p, \widetilde{p} \in [1, \infty].$$

Notice also that the norms are increasing in $\widetilde{p}$. The idea of distinguishing radial and angular directions is not new and has proved successful in the context of Strichartz estimates and dispersive equations (see [16], [22], [3]; see also [2]). To give a flavour of the results which can be obtained, Strauss' estimate (1.1) can be extended as

follows:

$$|x|^{\frac{n}{p}-\sigma}|u(x)| \lesssim \||D|^\sigma u\|_{L^p L^{\widetilde{p}}}, \qquad \frac{n-1}{\widetilde{p}} + \frac{1}{p} < \sigma < \frac{n}{p}$$

for arbitrary non radial functions u and all $1 < p < \infty$, $1 \leq \widetilde{p} \leq \infty$.

Remark 1.1. Of course by translations all the results we will prove hold with the norm

$$\begin{array}{rcl}
\|f\|_{L^p_{|x-\bar{x}|} L^{\widetilde{p}}_\theta} & = & \left(\int_0^{+\infty} \|f(\bar{x} + \rho\theta)\|^p_{L^{\widetilde{p}}(\mathbb{S}^{n-1})} \rho^{n-1} d\rho \right)^{\frac{1}{p}}, \\
\|f\|_{L^\infty_{|x-\bar{x}|} L^{\widetilde{p}}_\theta} & = & \sup_{\rho>0} \|f(\bar{x} + \rho\theta)\|_{L^{\widetilde{p}}(\mathbb{S}^{n-1})};
\end{array}$$

1. The Stein-Weiss inequality

A central role in our approach will be played by the fractional integrals

$$(T_\gamma \phi)(x) = \int_{\mathbb{R}^n} \frac{\phi(y)}{|x-y|^\gamma} dy, \qquad 0 < \gamma < n.$$

Weighted L^p estimates for T_γ are a fundamental problem of harmonic analysis, with a wide range of applications. Starting from the classical one dimensional case studied by Hardy and Littlewood, an exhaustive analysis has been made of the admissible classes of weights and ranges of indices (see [20] and the references therein). In the special case of power weights the optimal result is due to Stein and Weiss:

Theorem 1.1 ([21]). *Let $n \geq 1$ and $1 < p \leq q < \infty$. Assume α, β, γ satisfy the set of conditions $(1 = 1/p + 1/p')$*

$$\begin{aligned}
& \beta < \frac{n}{q}, \quad \alpha < \frac{n}{p'}, \quad 0 < \gamma < n \\
& \alpha + \beta + \gamma = n + \frac{n}{q} - \frac{n}{p} \\
& \alpha + \beta \geq 0.
\end{aligned} \tag{1.2}$$

Then the following inequality holds

$$\||x|^{-\beta} T_\gamma \phi\|_{L^q} \leq C(\alpha, \beta, p, q) \cdot \||x|^\alpha \phi\|_{L^p}. \tag{1.3}$$

Conditions in the first line of (1.2) are necessary to ensure integrability, while the necessity of the condition on the second line is due to scaling. On the other hand, the sharpness of $\alpha + \beta \geq 0$ is less obvious and follows from the results of [18].
In the radial case the last condition can be relaxed and $\alpha + \beta$ is allowed to assume negative values. Radial improvements were noticed in [24], [13], and the sharp result was obtained by Rubin [9] and more recently by De Napoli, Dreichman and Durán:

Theorem 1.2 ([9],[5]). *Let $n, p, q, \alpha, \beta, \gamma$ be as in the statement of Theorem (1.1) but with the condition $\alpha + \beta \geq 0$ relaxed to*

$$\alpha + \beta \geq (n-1)\left(\frac{1}{q} - \frac{1}{p}\right). \tag{1.4}$$

Then estimate (1.3) is valid for all radial functions $\phi = \phi(|x|)$.

Using the $L^p_{|x|} L^{\widetilde{p}}_\theta$ norms we are able prove the following general result which extends both theorems:

Theorem 1.3. *Let $n \geq 2$ and $1 < p \leq q < \infty$, $1 \leq \widetilde{p} \leq \widetilde{q} \leq \infty$. Assume α, β, γ satisfy the set of conditions*

$$\beta < \frac{n}{q}, \quad \alpha < \frac{n}{p'}, \quad 0 < \gamma < n$$

$$\alpha + \beta + \gamma = n + \frac{n}{q} - \frac{n}{p} \tag{1.5}$$

$$\alpha + \beta \geq (n-1)\left(\frac{1}{q} - \frac{1}{p} + \frac{1}{\widetilde{p}} - \frac{1}{\widetilde{q}}\right).$$

Then the following estimate holds:

$$\left\| |x|^{-\beta} T_\gamma \phi \right\|_{L^q_{|x|} L^{\widetilde{q}}_\theta} \leq C \left\| |x|^{\alpha} \phi \right\|_{L^p_{|x|} L^{\widetilde{p}}_\theta}. \tag{1.6}$$

The range of admissible p, q indices can be relaxed to $1 \leq p \leq q \leq \infty$ in two cases:
  (i) when the third inequality in (1.5) is strict, or
  (ii) when the Fourier transform $\widehat{\phi}$ has support contained in an annulus $c_1 R \leq |\xi| \leq c_2 R$ ($c_2 \geq c_1 > 0$, $R > 0$); in this case (1.6) holds with a constant independent of R.

Remark 1.2. Notice that:
  (a) with the choices $q = \widetilde{q}$ and $p = \widetilde{p}$ (i.e. in the usual L^p norms) Theorem 1.3 reduces to Theorem 1.1;
  (b) if ϕ is radially symmetric, with the choice $\widetilde{q} = \widetilde{p}$, Theorem 1.3 reduces to Theorem 1.2. Indeed, if ϕ is radially symmetric then $T_\gamma \phi$ is radially symmetric too, so that all choices for $\widetilde{q}, \widetilde{p}$ are equivalent;
  (c) obviously, the same estimate is true for general operators T_F with nonradial kernels $F(x)$ satisfying

$$T_F \phi(x) = \int F(x-y)\phi(y)dy, \qquad |F| \leq C|x|^{-\gamma}.$$

The proof of Theorem 1.3 is based on two successive applications of Young's inequality for convolutions on suitable Lie groups: first we use the strong inequality on the rotation group $SO(n)$; then we use a Young inequality in the radial variable, which in some cases must be replaced by the weak Young-Marcinkewicz inequality on the multiplicative group $(\mathbb{R}^+, \cdot)$ with the Haar measure $d\rho/\rho$. The convenient idea of using convolution in the measure $d\rho/\rho$ was introduced in [5].

Remark 1.3. The operator T_γ is a convolution with the homogenous kernel $|x|^{-\gamma}$. Consider instead the convolution with a nonhomogeneous kernel

$$S_\gamma \phi(x) = \int \frac{\phi(y)}{\langle x-y \rangle^\gamma} dy.$$

By the obvious pointwise bound

$$|S_\gamma \phi(x)| \leq T_\gamma |\phi|(x)$$

it is clear that S_γ satisfies the same estimates as T_γ. However the scaling invariance of the estimate is broken, and indeed something more can be proved, thanks to the smoothness of the kernel (see Lemma 1.7):

Corollary 1.4. *Let $n \geq 2$ and $1 \leq p \leq q \leq \infty$, $1 \leq \widetilde{p} \leq \widetilde{q} \leq \infty$. Assume α, β, γ satisfy the set of conditions*

$$\beta < \frac{n}{q}, \qquad \alpha < \frac{n}{p'}, \qquad \alpha + \beta \geq (n-1)\left(\frac{1}{q} - \frac{1}{p} + \frac{1}{\widetilde{p}} - \frac{1}{\widetilde{q}}\right), \tag{1.7}$$

$$\alpha + \beta + \gamma > n\left(1 + \frac{1}{q} - \frac{1}{p}\right). \tag{1.8}$$

Then the following estimate holds:

$$\||x|^{-\beta} S_\gamma \phi\|_{L^q_{|x|} L^{\tilde{q}}_\theta} \leq C\||x|^\alpha \phi\|_{L^p_{|x|} L^{\tilde{p}}_\theta}. \tag{1.9}$$

The first result we need is an explicit estimate of the angular part of the fractional integral $T_\gamma \phi$. Notice that a similar analysis in the radial case was done in [5] (see Lemma 4.2 there). The following estimates are sharp:

Lemma 1.5. *Let $n \geq 2$, $\nu > 0$, and write $\langle x \rangle = (1 + |x|^2)^{1/2}$. Then the integral*

$$I_\nu(x) = \int_{\mathbb{S}^{n-1}} |x - y|^{-\nu} dS(y) \qquad x \in \mathbb{R}^n$$

satisfies

$$|I_\nu(x)| \simeq \langle x \rangle^{-\nu} \qquad for \quad |x| \geq 2, \tag{1.10}$$

while for $|x| \leq 2$ we have

$$|I_\nu(x)| \simeq \begin{cases} 1 & if \ \nu < n - 1 \\ |\log||x| - 1|| + 1 & if \ \nu = n - 1 \\ ||x| - 1|^{n-1-\nu} & if \ \nu > n - 1. \end{cases} \tag{1.11}$$

PROOF. We consider four different regimes according to the size of $|x|$. We write for brevity I instead of I_ν.

First case: $|x| \geq 2$. For x large and $|y| = 1$ we have $|x - y| \simeq |x|$, hence $|I(x)| \simeq |x|^{-\nu} \simeq \langle x \rangle^{-\nu}$. This proves (1.10).

Second case: $0 \leq |x| \leq \frac{1}{2}$. Clearly we have $|x - y| \simeq 1$ when $|y| = 1$, and this implies $|I(x)| \simeq 1 \simeq \langle x \rangle^{-\nu}$. This is equivalent to (1.11) when $|x| \leq 1/2$.

Third case: $1 \leq |x| \leq 2$. This is the bulk of the computation since it contains the singular part of the integral, as $|x| \to 1$. We write the integral in polar coordinates using the spherical angles $(\theta_1, \theta_2, ..., \theta_{n-1})$ on $\mathbb{S}^{n-1}$, oriented in such a way that θ_1 is the angle between x and y. Using the notation $\sigma = |x - y|$, by the symmetry of $I(x)$ in $(\theta_2, ..., \theta_{n-1})$ we have

$$|I(x)| \simeq \int_0^\pi \sigma^{-\nu} (\sin\theta_1)^{n-2} d\theta_1.$$

In order to rewrite the integral using σ as a new variable, we compute

$$2\sigma d\sigma = d(|x - y|^2) = d(|x| + 1 - 2|x| \cos\theta_1) = 2|x| \sin\theta_1 d\theta_1$$

so we have

$$(\sin\theta_1)^{n-2} d\theta_1 = \frac{\sigma(\sin\theta_1)^{n-3}}{|x|} d\sigma$$

and, noticing that $0 \leq |x| - 1 \leq |x - y| = \sigma \leq |x| + 1$,

$$|I(x)| \simeq \int_{|x|-1}^{|x|+1} \sigma^{1-\nu} \frac{(\sin\theta_1)^{n-3}}{|x|} d\sigma.$$

Now let A be the area of the triangle with vertices $0, x$ andd y: we have $2A = |x| \sin\theta_1$ so that

$$|I(x)| \simeq |x|^{2-n} \int_{|x|-1}^{|x|+1} \sigma^{1-\nu} A^{n-3} d\sigma.$$

Recalling Heron's formula for the area of a triangle as a function of the length of its sides we obtain

$$|I(x)| \simeq |x|^{2-n} \int_{|x|-1}^{|x|+1} \sigma^{1-\nu} \left[(|x| + \sigma + 1)(|x| + \sigma - 1)(|x| + 1 - \sigma)(\sigma + 1 - |x|) \right]^{\frac{n-3}{2}} d\sigma.$$

Notice that this formula is correct for all dimensions $n \geq 2$.

Now we split the integral as $I \simeq I_1 + I_2$ with

$$I_1(x) = |x|^{2-n} \int_{|x|-1}^{|x|} \sigma^{1-\nu} \Big[(|x|+\sigma+1)(|x|+\sigma-1)(|x|+1-\sigma)(\sigma+1-|x|) \Big]^{\frac{n-3}{2}} d\sigma$$

and

$$I_2(x) = |x|^{2-n} \int_{|x|}^{|x|+1} \sigma^{1-\nu} \Big[(|x|+\sigma+1)(|x|+\sigma-1)(|x|+1-\sigma)(\sigma+1-|x|) \Big]^{\frac{n-3}{2}} d\sigma.$$

In the second integral I_2, recalling that $1 \leq |x| \leq 2$, we have

$$|x| \simeq \sigma \simeq |x| + \sigma + 1 \simeq |x| + \sigma - 1 \simeq \sigma + 1 - |x| \simeq 1$$

so that

$$I_2 \simeq \int_{|x|}^{|x|+1} (|x|+1-\sigma)^{\frac{n-3}{2}} d\sigma = \int_0^1 (1-\sigma)^{\frac{n-3}{2}} d\sigma \simeq 1.$$

In the first integral I_1, using that $1 \leq |x| \leq 2$ and $|x|-1 \leq \sigma \leq |x|$, we see that

$$|x| \simeq (|x|+\sigma+1) \simeq (|x|+1-\sigma) \simeq 1;$$

moreover,

$$1 \leq \frac{|x|+\sigma-1}{\sigma} \leq 2 \quad \text{so that} \quad |x|+\sigma-1 \simeq \sigma$$

and we have

$$I_1(x) \simeq \int_{|x|-1}^{|x|} \sigma^{1-\nu+\frac{n-3}{2}} (\sigma+1-|x|)^{\frac{n-2}{2}} d\sigma$$

or, after the change of variable $\sigma \to \sigma(|x|-1)$,

$$I_1(x) \simeq (|x|-1)^{n-1-\nu} \int_1^{1+\frac{1}{|x|-1}} (\sigma-1)^{\frac{n-3}{2}} \sigma^{\frac{n-1}{2}-\nu} d\sigma.$$

Now split the last integral as $A + B$ where

$$A = (|x|-1)^{n-1-\nu} \int_1^2 (\sigma-1)^{\frac{n-3}{2}} \sigma^{\frac{n-1}{2}-\nu} d\sigma$$

and

$$B = (|x|-1)^{n-1-\nu} \int_2^{1+\frac{1}{|x|-1}} (\sigma-1)^{\frac{n-3}{2}} \sigma^{\frac{n-1}{2}-\nu} d\sigma;$$

we have immediately

$$A \simeq (|x|-1)^{n-1-\nu}$$

while, keeping into account that $\sigma \simeq \sigma - 1$ for σ in $(2, 1+\frac{1}{|x|-1})$,

$$B = (|x|-1)^{n-1-\nu} \int_2^{1+\frac{1}{|x|-1}} \sigma^{n-2-\nu} d\sigma$$

which gives

$$B \simeq \begin{cases} 1 & \text{if } \nu < n-1 \\ |\log||x|-1|| + 1 & \text{if } \nu = n-1 \\ ||x|-1|^{n-1-\nu} & \text{if } \nu > n-1 \end{cases} \tag{1.12}$$

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Fourth case: $\frac{1}{2} \leq |x| \leq 1$. Using the change of variable $|x'| = 1/|x|$, we see that $|I(x)| \simeq |I(1/|x'|)|$, thus the fourth case follows immediately from the third one, and this concludes the proof of the Lemma. $\qquad\square$

We shall also need the following estimate which is proved in a similar way:

Lemma 1.6. *Let $n \geq 2$, $\nu > 0$. Then the integral*

$$J_\nu(x,\rho) = \int_{\mathbb{S}^{n-1}} \langle x - \rho\theta \rangle^{-\nu} dS(\theta) \qquad x \in \mathbb{R}^n,\ \rho \geq 0$$

satisfies:

$$|J_\nu(x,\rho)| \simeq \langle x \rangle^{-\nu} \qquad \text{for } \rho \leq 1 \text{ or } |x| \geq 2\rho, \tag{1.13}$$

$$|J_\nu(x,\rho)| \simeq \langle \rho \rangle^{-\nu} \qquad \text{for } |x| \leq 1 \text{ or } \rho \geq 2|x|, \tag{1.14}$$

while in the remaining case, i.e. when $|x| \geq 1$ and $\rho \geq 1$ and $2^{-1}|x| \leq \rho \leq 2|x|$,

$$|J_\nu(x,\rho)| \simeq \begin{cases} \langle \rho \rangle^{-\nu} & \text{if } \nu < n-1 \\ \langle \rho \rangle^{-\nu} \log\left(\frac{2\langle \rho \rangle}{\langle |x| - \rho \rangle} \right) & \text{if } \nu = n-1 \\ \langle \rho \rangle^{1-n} \langle |x| - \rho \rangle^{n-1-\nu} & \text{if } \nu > n-1. \end{cases} \tag{1.15}$$

As a consequence, one has $J_\nu \lesssim \langle \rho + |x| \rangle^{-\nu}$ when $\nu < n-1$ and $J_\nu \lesssim \langle \rho + |x| \rangle^{-\nu} \log(2\langle \rho \rangle + |x|)$ when $\nu = n-1$.

PROOF. The proof is similar to the proof of Lemma 1.5; we sketch the main steps. Estimates (1.13) and (1.14) are obvious, thus we focus on (1.15). Write $r = |x|$, so that we are in the region $1/2 \leq r/\rho \leq 2$; we shall consider in detail the case

$$1 \leq \frac{r}{\rho} \leq 2,$$

the remaining region being similar. Using the same coordinates as before, the integral is reduced to

$$J_\nu(|x|,\rho) = |x|^{2-n} \int_{|x|-1}^{|x|+1} \langle \rho\sigma \rangle^{-\nu} A^{n-3} \sigma \, d\sigma$$

where A is given by Heron's formula

$$A(|x|,\sigma)^2 = (|x| + \sigma + 1)(|x| + \sigma - 1)(|x| + 1 - \sigma)(\sigma + 1 - |x|).$$

We split the integral on the intervals $|x| \leq \sigma \leq |x| + 1$ and $|x| - 1 \leq \sigma \leq |x|$. The first piece gives

$$I_1 \simeq \langle \rho \rangle^{-\nu} \int_{|x|}^{|x|+1} (|x| + 1 - \sigma)^{\frac{n-3}{2}} d\sigma$$

and by the change of variable $\sigma \to \sigma(|x| + 1)$ we obtain

$$I_1(|x|,\rho) \simeq \langle \rho \rangle^{-\nu}.$$

For the second integral on $|x| - 1 \leq \sigma \leq |x|$, noticing that

$$1 \leq \frac{|x| + \sigma - 1}{\sigma} \leq 2$$

we have

$$I_2 \simeq \int_{|x|-1}^{|x|} \langle \rho\sigma \rangle^{-\nu} \sigma^{\frac{n-1}{2}} (\sigma + 1 - |x|)^{\frac{n-3}{2}} d\sigma$$

$$= (|x| - 1)^{n-1} \int_1^{\frac{|x|}{|x|-1}} \langle (r - \rho)\sigma \rangle^{-\nu} \sigma^{\frac{n-1}{2}} (\sigma - 1)^{\frac{n-3}{2}} d\sigma$$

via the change of variables $\sigma \to \sigma(|x| - 1)$ which gives $\rho\sigma \to (r - \rho)\sigma$. The part of the integral between 1 and 2 produces

$$\simeq (|x| - 1)^{n-1} \langle r - \rho \rangle^{-\nu} = \rho^{1-n} (r - \rho)^{n-1} \langle r - \rho \rangle^{-\nu}$$

while the remaining part between 2 and $|x|/(|x| - 1)$ gives

$$\simeq (|x| - 1)^{n-1} \int_2^{\frac{r}{r-\rho}} \langle (r - \rho)\sigma \rangle^{-\nu} \sigma^{n-2} d\sigma$$

$$= \rho^{1-n} \int_{2(r-\rho)}^r \langle \sigma \rangle^{-\nu} \sigma^{n-2} d\sigma$$

$$\simeq \rho^{1-n} \int_{2(r-\rho)}^r \frac{\sigma^{n-2}}{1 + \sigma^\nu} d\sigma$$

which can be computed explicitly. Summing up we obtain (1.15). □

We are ready for the main part of the proof. By the isomorphism

$$\mathbb{S}^{n-1} \simeq SO(n)/SO(n-1)$$

we can represent integrals on $\mathbb{S}^{n-1}$ in the form

$$\int_{\mathbb{S}^{n-1}} g(y) dS(y) = c_n \int_{SO(n)} g(Ae) dA, \qquad n \geq 2$$

where dA is the left Haar measure on $SO(n)$, and $e \in \mathbb{S}^{n-1}$ is a fixed arbitrary unit vector. Thus, via polar coordinates, a convolution integral can be written as follows (apart from inessential constants depending only on the space dimension n):

$$F * \phi(x) = \int_{\mathbb{R}^n} F(x - y)\phi(y) dy = \int_0^\infty \int_{\mathbb{S}^{n-1}} F(x - \rho\omega)\phi(\rho\omega) dS_\omega \rho^{n-1} d\rho$$

$$\simeq \int_0^\infty \int_{SO(n)} F(x - \rho Be)\phi(\rho Be) dB \rho^{n-1} d\rho$$

Hence the $L^{\widetilde{q}}$ norm of the convolution on the sphere can be written as

$$\| F * \phi(|x|\theta) \|_{L_\theta^{\widetilde{q}}(\mathbb{S}^{n-1})} \simeq \| F * \phi(|x|Ae) \|_{L_A^{\widetilde{q}}(SO(n))}$$

$$\leq \int_0^\infty \left\| \int_{SO(n)} F(|x|Ae - \rho Be)\phi(\rho Be) dB \right\|_{L_A^{\widetilde{q}}(SO(n))} \rho^{n-1} d\rho$$

where e is any fixed unit vector. By the change of variables $B \to AB^{-1}$ in the inner integral (and the invariance of the measure) this is equivalent to

$$= \int_0^\infty \left\| \int_{SO(n)} F(AB^{-1}(|x|Be - \rho e))\phi(\rho AB^{-1}e) dB \right\|_{L_A^{\widetilde{q}}(SO(n))} \rho^{n-1} d\rho$$

If F satisfies

$$|F(x)| \leq Cf(|x|) \tag{1.16}$$

for a radial function f, we can write

$$|F(AB^{-1}(|x|Be - \rho e))| \leq Cf\left(\big||x|Be - \rho e\big|\right)$$

and we notice that the integral

$$\int_{SO(n)} f\left(\big||x|Be - \rho e\big|\right) |\phi(\rho AB^{-1}e)| dB = g * h(A)$$

is a convolution on $SO(n)$ of the functions

$$g(A) = f\left(\big||x|Ae - \rho e\big|\right), \qquad h(A) = |\phi(\rho Ae)|.$$

We can thus apply the Young's inequality on $SO(n)$ (see e.g. Theorem 1.2.12 in [12]) and we obtain, for any

$$\widetilde{q}, \widetilde{r}, \widetilde{p} \in [1, +\infty] \quad \text{with} \quad 1 + \frac{1}{\widetilde{q}} = \frac{1}{\widetilde{r}} + \frac{1}{\widetilde{p}},$$

the estimate

$$\|F * \phi(|x|\theta)\|_{L_\theta^{\tilde{q}}(\mathbb{S}^{n-1})} \lesssim \int_0^\infty \|f(||x|e - \rho\theta|)\|_{L_\theta^{\tilde{r}}(\mathbb{S}^{n-1})} \|\phi(\rho\theta)\|_{L_\theta^{\tilde{p}}(\mathbb{S}^{n-1})} \rho^{n-1} d\rho \quad (1.17)$$

where we switched back to the coordinates of $\mathbb{S}^{n-1}$. Notice that the conditions on the indices imply in particular

$$\widetilde{q} \geq \widetilde{p}.$$

Specializing f to the choice

$$f(|x|) = |x|^{-\gamma}$$

we get

$$\|F * \phi(|x|\theta)\|_{L_\theta^{\tilde{q}}} \lesssim \int_0^\infty \rho^{-\gamma} \||\rho^{-1}|x|e - \theta|^{-\gamma}\|_{L_\theta^{\tilde{r}}} \|\phi(\rho\theta)\|_{L_\theta^{\tilde{p}}} \rho^{n-1} d\rho$$

which can be written in the form

$$= |x|^{n-\alpha-\frac{n}{p}-\gamma} \int_0^\infty \left(\frac{|x|}{\rho}\right)^{\alpha+\frac{n}{p}-n+\gamma} \||\rho^{-1}|x|e - \theta|^{-\gamma}\|_{L_\theta^{\tilde{r}}} \rho^{\alpha+\frac{n}{p}} \|\phi(\rho\theta)\|_{L_\theta^{\tilde{p}}} \frac{d\rho}{\rho}$$

or equivalently, recalling (1.2),

$$= |x|^{\beta-\frac{n}{q}} \int_0^\infty \left(\frac{|x|}{\rho}\right)^{-\beta+\frac{n}{q}} \||\rho^{-1}|x|e - \theta|^{-\gamma}\|_{L_\theta^{\tilde{r}}} \rho^{\alpha+\frac{n}{p}} \|\phi(\rho\theta)\|_{L_\theta^{\tilde{p}}} \frac{d\rho}{\rho}$$

Following [5], we recognize that the last integral is a convolution in the multiplicative group $(\mathbb{R}, \cdot)$ with the Haar measure $d\rho/\rho$, which implies

$$|x|^{-\beta+\frac{n}{q}} \|F * \phi(|x|\theta)\|_{L_\theta^{\tilde{q}}} \lesssim g_1 * h_1(|x|),$$

with

$$g_1(\rho) = \rho^{-\beta+\frac{n}{q}} \||\rho e - \theta|^{-\gamma}\|_{L_\theta^{\tilde{r}}}, \qquad h_1(\rho) = \rho^{\alpha+\frac{n}{p}} \|\phi(\rho\theta)\|_{L_\theta^{\tilde{p}}}.$$

By the weak Young's inequality in the measure $d\rho/\rho$ (Theorem 1.4.24 in [12]) we obtain

$$\||x|^{-\beta} F * \phi\|_{L^q L^{\tilde{q}}} \equiv \left\| |x|^{-\beta+\frac{n}{q}} \|F * \phi(|x|\theta)\|_{L_\theta^{\tilde{q}}} \right\|_{L^q(\rho^{-1} d\rho)}$$

$$\lesssim \|h_1\|_{L^p(\rho^{-1} d\rho)} \|g_1\|_{L^{r,\infty}(\rho^{-1} d\rho)}$$

that is to say

$$\||x|^{-\beta} F * \phi\|_{L^q L^{\tilde{q}}} \lesssim \|\phi\|_{L^p L^{\tilde{p}}} \left\| \rho^{-\beta+\frac{n}{q}} \||\rho e - \theta|^{-\gamma}\|_{L_\theta^{\tilde{r}}} \right\|_{L^{r,\infty}(\rho^{-1} d\rho)}. \quad (1.18)$$

provided

$$q, r, p \in (1, +\infty) \qquad 1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}.$$

In particular this implies

$$q > p. \quad (1.19)$$

In order to achieve the proof, it remains to check that the last norm in (1.18) is finite. Notice that, when $\widetilde{r} < \infty$,

$$\||\rho e - \theta|^{-\gamma}\|_{L_\theta^{\tilde{r}}} = I_{\gamma\tilde{r}}(\rho e)^{\frac{1}{\tilde{r}}}$$

where I_ν was defined and estimated in Lemma 1.5. On the other hand, when $\widetilde{r} = \infty$ one has directly

$$\widetilde{r} = \infty \quad \Longrightarrow \quad \||\rho e - \theta|^{-\gamma}\|_{L_\theta^{\tilde{r}}} \simeq |\rho - 1|^{-\gamma}. \quad (1.20)$$

Using cutoffs, we split the $L^{r,\infty}$ norm in three regions $0 \leq \rho \leq 1/2$, $\rho \geq 2$ and $1/2 \leq \rho \leq 2$.

In the region $0 \leq \rho \leq 1/2$, recalling (1.10)-(1.11) or (1.20), we have

$$I_{\gamma\tilde{r}}(\rho e)^{\frac{1}{\tilde{r}}} \simeq 1 \quad \Longrightarrow \quad \rho^{-\beta+\frac{n}{q}} I_{\gamma\tilde{r}}(\rho e)^{\frac{1}{\tilde{r}}} \in L^1(0, 1/2; d\rho/\rho)$$

since by assumption $\beta < n/q$; thus the contribution of this part to the $L^{r,\infty}(d\rho/\rho)$ norm is finite.

In the region $\rho \geq 2$ we have

$$I_{\gamma\widetilde{r}}(\rho e)^{\frac{1}{\widetilde{r}}} \simeq \rho^{-\gamma} \quad \Longrightarrow \quad \rho^{-\beta + \frac{n}{q}} I_{\gamma\widetilde{r}}(\rho e)^{\frac{1}{\widetilde{r}}} \simeq \rho^{-\beta - \gamma + \frac{n}{q}} \in L^1(2, \infty; d\rho/\rho)$$

since the condition

$$-\beta - \gamma + \frac{n}{q} < 0 \quad \Longleftrightarrow \quad \alpha < \frac{n}{p'}$$

is satisfied by (1.7), and again the contribution to the $L^{r,\infty}$ norm is finite.

For the third region $1/2 \leq \rho \leq 2$, by estimate (1.11), we see that in the case $\gamma\widetilde{r} \leq n - 1$ one has again, for some $\sigma \geq 0$,

$$I_{\gamma\widetilde{r}}(\rho e)^{\frac{1}{\widetilde{r}}} \simeq |\log||\rho| - 1|^{\sigma} \quad \Longrightarrow \quad \rho^{-\beta + \frac{n}{q}} I_{\gamma\widetilde{r}}(\rho e)^{\frac{1}{\widetilde{r}}} \in L^1(1/2, 2; d\rho/\rho)$$

On the other hand, in the case $\gamma\widetilde{r} > n - 1$ (which includes the choice $\widetilde{r} = \infty$), we see that

$$\rho^{-\beta + \frac{n}{q}} I_{\gamma\widetilde{r}}(\rho e)^{\frac{1}{\widetilde{r}}} \simeq |\rho - 1|^{\frac{n-1}{\widetilde{r}} - \gamma} \in L^{r,\infty}(1/2, 2; d\rho/\rho) \quad \Longleftrightarrow \quad \frac{n-1}{\widetilde{r}} - \gamma \geq -\frac{1}{r}.$$

Recalling the relation between q, r, p (resp. $\widetilde{q}, \widetilde{r}, \widetilde{p}$) the last condition is equivalent to

$$-\gamma \geq (n-1)\left(\frac{1}{q} - \frac{1}{p} - \frac{1}{\widetilde{q}} + \frac{1}{\widetilde{p}}\right) - \frac{n}{q} + \frac{n}{p} - n$$

which is precisely the third of conditions (1.7).

The weak Young inequality can be used in (1.18) only in the range $q, r, p \in (1, +\infty)$, which forces

$$1 < p < q < \infty.$$

To cover the cases

$$1 \leq p < q \leq \infty$$

we use instead the strong Young inequality: we can write

$$\||x|^{-\beta} F * \phi\|_{L^q L^{\widetilde{q}}} \lesssim \|\phi\|_{L^p L^{\widetilde{p}}} \left\|\rho^{-\beta + \frac{n}{q}} \||\rho e - \theta|^{-\gamma}\|_{L_\theta^{\widetilde{r}}}\right\|_{L^r(\rho^{-1} d\rho)} \tag{1.21}$$

for the full range $q, r, p \in [1, +\infty]$. The previous arguments are still valid apart from the last step which must be replaced by

$$\rho^{-\beta + \frac{n}{q}} I_{\gamma\widetilde{r}}(\rho e)^{\frac{1}{\widetilde{r}}} \simeq |\rho - 1|^{\frac{n-1}{\widetilde{r}} - \gamma} \in L^r(1/2, 2; d\rho/\rho) \quad \Longleftrightarrow \quad \frac{n-1}{\widetilde{r}} - \gamma > -\frac{1}{r}$$

and this implies that the inequality in the last condition (1.7) must be strict.

The case

$$1 < p = q < \infty$$

has already been covered. Indeed, in this case the scaling condition (1.7) implies

$$\alpha + \beta + \gamma = n \quad \Longrightarrow \quad \alpha + \beta > 0$$

since $\gamma < n$. Thus when $\widetilde{p} = \widetilde{q}$ the last inequality in (1.7) is strict and we can apply the second part of the proof; the cases $\widetilde{p} \leq \widetilde{q}$ follow from the case $\widetilde{p} = \widetilde{q}$.

To complete the proof, it remains to consider the case (ii) where we assume that the support of the Fourier transform $\widehat{\phi}$ is contained in an annular region of size R. By scaling invariance of the inequality, it is sufficient to consider the case $R = 1$. Now let $\psi(x)$ be such that $\widehat{\psi} \in C_c^\infty$ and precisely

$$\widehat{\psi}(\xi) = 1 \quad \text{for } c_1' \leq |\xi| \leq c_2', \qquad \widehat{\psi}(\xi) = 0 \quad \text{for } |\xi| > 2c_1' \text{ and } |\xi| < \frac{1}{2}c_2',$$

for some constants $c_2' > c_2 \geq c_1 > c_1' > 0$. This implies

$$\phi = \mathsf{F}^{-1}(\widehat{\psi}\widehat{\phi}) = \psi * \phi$$

and we can write
$$T_\gamma \phi = |x|^{-\gamma} * \psi * \phi = (T_\gamma \psi) * \phi.$$
Since $T_\gamma \psi = c\mathsf{F}^{-1}(|\xi|^{\gamma-n}\widehat{\psi}(\xi))$ is a Schwartz class function, we arrive at the estimates
$$|T_\gamma \phi(x)| \le C_{\mu,\gamma}\langle x\rangle^{-\mu} * |\phi| \qquad \forall \mu \ge 1. \tag{1.22}$$
Here we can take μ arbitrarily large. Thus the proof of case (ii) is concluded by applying the following Lemma:

Lemma 1.7. *Let $n \ge 2$. Assume $1 \le p \le q \le \infty$, $1 \le \widetilde{p} \le \widetilde{q} \le \infty$ and α, β, μ satisfy*

$$\beta < \frac{n}{q}, \qquad \alpha < \frac{n}{p'}, \qquad \alpha + \beta \ge (n-1)\left(\frac{1}{q} - \frac{1}{p} + \frac{1}{\widetilde{p}} - \frac{1}{\widetilde{q}}\right), \tag{1.23}$$

$$\mu > -\alpha - \beta + n\left(1 + \frac{1}{q} - \frac{1}{p}\right). \tag{1.24}$$

Then the following estimate holds:

$$\||x|^{-\beta}\langle x\rangle^{-\mu} * \phi\|_{L^q_{|x|}L^{\widetilde{q}}_\theta} \lesssim \|\phi\|_{L^p_{|x|}L^{\widetilde{p}}_\theta}. \tag{1.25}$$

PROOF. Notice that, by (1.23), the right hand side in (1.24) is always strictly positive and never larger than $n - 1$, thus it is sufficient to prove the lemma for μ in the range
$$0 < \mu \le n.$$
By (1.17) we have, for all $\widetilde{p}, \widetilde{q}, \widetilde{r} \in [1, +\infty]$ with $1 + 1/\widetilde{q} = 1/\widetilde{r} + 1/\widetilde{p}$,

$$\|\langle\cdot\rangle^{-\mu} * |\phi|(|x|\theta)\|_{L^{\widetilde{q}}_\theta(\mathbb{S}^{n-1})} \lesssim \int_0^\infty J_{\mu\widetilde{r}}(|x|,\rho)^{\frac{1}{\widetilde{r}}} \|\phi(\rho\theta)\|_{L^{\widetilde{p}}_\theta(\mathbb{S}^{n-1})}\rho^{n-1}d\rho. \tag{1.26}$$

Notice that when $\widetilde{r} = \infty$ we have
$$\|\langle|x|e - \rho\theta\rangle^{-\mu}\|_{L^\infty_\theta} \lesssim \langle|x| - \rho\rangle^{-\mu}.$$
We write for brevity
$$Q(|x|) \equiv |x|^{-\beta+\frac{n-1}{q}}\|\langle\cdot\rangle^{-\mu} * |\phi|(|x|\theta)\|_{L^{\widetilde{q}}_\theta}, \qquad P(\rho) = \rho^{\alpha+\frac{n-1}{p}}\|\phi(\rho\theta)\|_{L^{\widetilde{p}}_\theta}$$

$$J(|x|,\rho) = J_{\mu\widetilde{r}}^{\frac{1}{\widetilde{r}}}(x,\rho) \quad (\text{resp. } \langle|x| - \rho\rangle^{-\mu} \text{ if } \widetilde{r} = \infty).$$
Thus (1.26) becomes

$$Q(\sigma) \lesssim \sigma^{-\beta+\frac{n-1}{q}}\int_0^\infty J(\sigma,\rho)\rho^{\frac{n-1}{p'}-\alpha}P(\rho)d\rho \tag{1.27}$$

and the estimate to be proved (1.25) can be written as

$$\|Q\|_{L^q(0,+\infty)} \lesssim \|P\|_{L^p(0,+\infty)} \tag{1.28}$$

Recall that the integrals of the form $J(\sigma,\rho)$ have been estimated in Lemma 1.6. We split Q into the sum of several terms corresponding to different regions of ρ, σ. In the region $\sigma \le 1$ we have $J(\sigma,\rho) \lesssim \langle\rho\rangle^{-\mu}$ so that

$$Q_1(\sigma) \lesssim \sigma^{-\beta+\frac{n-1}{q}}\int_0^\infty \langle\rho\rangle^{-\mu}\rho^{\frac{n-1}{p'}-\alpha}P(\rho)d\rho \tag{1.29}$$

Thus we see that in this region (1.28) follows simply from Hölder's inequality and the fact that $\alpha < n/p'$ and $\beta < n/q$. Similarly, it is easy to handle the part of the integral with $\rho \le 1$ since we have then $J(\sigma,\rho) \lesssim \langle\sigma\rangle^{-\mu}$. Thus in the following we can restrict to $\sigma \gtrsim 1, \rho \gtrsim 1$.
When $1 \lesssim \sigma \le \rho/2$ we have again $J(\sigma,\rho) \lesssim \langle\rho\rangle^{-\mu}$ and (1.27) becomes

$$Q_2(\sigma) \lesssim \sigma^{-\beta+\frac{n-1}{q}}\int_\sigma^\infty \langle\rho\rangle^{-\mu}\rho^{\frac{n-1}{p'}-\alpha}P(\rho)d\rho \tag{1.30}$$

If we assume
$$\mu > \frac{n}{p'} - \alpha \tag{1.31}$$
we can apply Hölder's inequality and we get
$$Q_3(\sigma) \lesssim \sigma^{-\beta + \frac{n-1}{q}} \sigma^{\frac{n}{p'} - \mu - \alpha} \|P\|_{L^p}.$$
Now the right hand side is in $L^q(\sigma \geq 1)$ provided
$$\mu > \frac{n}{p'} - \alpha + \frac{n}{q} - \beta \equiv -\alpha - \beta + n\left(1 + \frac{1}{q} - \frac{1}{p}\right) \tag{1.32}$$
and we see that (1.32) implies (1.31) since $\beta < n/q$ by assumption.

When $1 \lesssim \rho \leq \sigma/2$ we have $J(\sigma, \rho) \lesssim \langle\sigma\rangle^{-\mu}$ and (1.27) becomes
$$Q_4(\sigma) \lesssim \sigma^{-\beta + \frac{n-1}{q}} \sigma^{-\mu} \int_0^\sigma \rho^{\frac{n-1}{p'} - \alpha} P(\rho) d\rho \tag{1.33}$$
and by Hölder's inequality we have as before
$$\lesssim \sigma^{-\beta + \frac{n-1}{q}} \sigma^{\frac{n}{p'} - \mu - \alpha} \|P\|_{L^p}$$
so that (1.32) is again sufficient to obtain (1.28).

Finally, let $\sigma \gtrsim 1$, $\rho \gtrsim 1$ and $2^{-1}\sigma \leq \rho \leq 2\sigma$. In this region we must treat differently the values of $\mu\widetilde{r}$ larger or smaller than $n - 1$, and the case $\widetilde{r} = \infty$ is considered at the end. Assume that $n - 1 < \mu\widetilde{r} \leq n$; then $J(\sigma, \rho) \lesssim \langle\rho\rangle^{1-n}\langle\sigma - \rho\rangle^{\frac{n-1}{\widetilde{r}} - \mu}$, and using the relations
$$\sigma \simeq \rho, \qquad \frac{1}{\widetilde{r}} = 1 + \frac{1}{\widetilde{p}} - \frac{1}{\widetilde{q}}$$
we see that (1.27) reduces to
$$Q_5(\sigma) \lesssim \sigma^{-\alpha - \beta + (n-1)\left(\frac{1}{q} - \frac{1}{p} + \frac{1}{\widetilde{p}} - \frac{1}{\widetilde{q}}\right)} \int_{\sigma/2}^{2\sigma} \langle\sigma - \rho\rangle^{\frac{n-1}{\widetilde{r}} - \mu} P(\rho) d\rho. \tag{1.34}$$

The last integral is (bounded by) a convolution of $P(\rho)$ with the function $\langle\rho\rangle^{\frac{n-1}{\widetilde{r}} - \mu}$. In order to estimate the $L^q(\sigma \geq 1)$ norm of Q_5, we use first Hölder's then Young's inequality:
$$\|Q_5\|_{L^q} \lesssim \|\langle\sigma\rangle^{-\epsilon}\|_{L^{q_0}} \|\langle\rho\rangle^{\frac{n-1}{\widetilde{r}} - \mu}\|_{L^{q_1}} \|P\|_{L^p}$$
where
$$\epsilon = -\alpha - \beta + (n-1)\left(\frac{1}{q} - \frac{1}{p} + \frac{1}{\widetilde{p}} - \frac{1}{\widetilde{q}}\right), \qquad \frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1} + \frac{1}{p} - 1.$$
By assumption we have $\epsilon \geq 0$. When $\epsilon > 0$, in order for the norms to be finite we need
$$\epsilon q_0 > 1, \qquad \frac{n-1}{\widetilde{r}} - \mu < -\frac{1}{q_1}$$
which can be rewritten
$$(n-1)\left(1 + \frac{1}{\widetilde{q}} - \frac{1}{\widetilde{p}}\right) - \mu + 1 + \frac{1}{q} - \frac{1}{p} < \frac{1}{q_0} < \epsilon$$
and we see that we can find a suitable q_0 provided the first side is strictly smaller than the last side; this condition is precisely equivalent to (1.32) again (recall also that $n - 1 < \mu \leq n$). The argument works also in the case $\epsilon = 0$ by choosing $q_0 = \infty$.

If on the other hand $0 < \mu < n - 1$, we have $J_{\mu\widetilde{r}}^{\frac{1}{\widetilde{r}}} \lesssim \langle\rho\rangle^{-\mu}$ also in this region, so that
$$Q_5(\sigma) \lesssim \sigma^{-\beta + \frac{n-1}{q}} \sigma^{\frac{n-1}{p'} - \alpha - \mu} \int_{\sigma/2}^\sigma P(\rho) d\rho$$

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by $\sigma \simeq \rho$. Hölder's inequality gives

$$Q_5(\sigma) \lesssim \sigma^{-\beta + \frac{n-1}{q}} \sigma^{\frac{n-1}{p'} - \alpha - \mu} \sigma^{\frac{1}{p'}} \|P\|_{L^p}$$

which leads to exactly the same computations as above and in the end to (1.32). The case $\mu = n - 1$ introduces a logarithmic term which does not change the integrability properties used here.

It remains the last region when $\widetilde{r} = \infty$ so that $J(\sigma, \rho) = \langle \sigma - \rho \rangle^{-\mu}$ and $1/\widetilde{p} - 1/\widetilde{q} = 1$. Then

$$Q_5(\sigma) \lesssim \sigma^{-\beta + \frac{n-1}{q}} \int_{\sigma/2}^{2\sigma} \langle \sigma - \rho \rangle^{-\mu} \rho^{\frac{n-1}{p'} - \alpha} P(\rho) d\rho$$

which is identical with (1.34) with $\widetilde{r} = \infty$, thus the same computations apply and the proof is concluded. $\qquad\square$

2. Weighted Sobolev embeddings

In this section we write estimate (1.6) in the form of a Sobolev embedding. In this way we get also critical estimates in Besov Spaces.

Recalling the pointwise bound

$$|u(x)| \leq C T_\lambda(\||D|^{n-\lambda} u|), \qquad 0 < \lambda < n \tag{1.35}$$

where $|D|^\sigma = (-\Delta)^{\frac{s}{2}}$, we see that an immediate consequence of (1.9) is the weighted Sobolev inequality

$$\||x|^{-\beta} u\|_{L^q L^{\widetilde{q}}} \lesssim \||x|^\alpha |D|^\sigma u\|_{L^p L^{\widetilde{p}}} \tag{1.36}$$

provided $1 < p \leq q < \infty$, $1 \leq \widetilde{p} \leq \widetilde{q} \leq \infty$ and

$$\beta < \frac{n}{q}, \quad \alpha < \frac{n}{p'}, \quad 0 < \sigma < n$$
$$\alpha + \beta = \sigma + \frac{n}{q} - \frac{n}{p} \tag{1.37}$$
$$\alpha + \beta \geq (n-1) \left(\frac{1}{q} - \frac{1}{p} + \frac{1}{\widetilde{p}} - \frac{1}{\widetilde{q}} \right).$$

As usual, if the last condition is strict we can take p, q in the full range $1 \leq p \leq q \leq \infty$. For instance, this implies the inequality

$$|x|^{-\beta} |u(x)| \lesssim \||x|^\alpha |D|^\sigma u\|_{L^p L^{\widetilde{p}}} \tag{1.38}$$

provided $1 \leq p \leq \infty$ and

$$\beta < 0, \quad \alpha < \frac{n}{p'}, \quad 0 < \sigma < n$$
$$\alpha + \beta = \sigma - \frac{n}{p}$$
$$\alpha + \beta > (n-1) \left(\frac{1}{\widetilde{p}} - \frac{1}{p} \right).$$

If we choose $\alpha = 0$ we have in particular for $p \in (1, \infty)$, $\widetilde{p} \in [1, \infty]$

$$|x|^{\frac{n}{p} - \sigma} |u(x)| \lesssim \||D|^\sigma u\|_{L^p L^{\widetilde{p}}}, \qquad \frac{n-1}{\widetilde{p}} + \frac{1}{p} < \sigma < \frac{n}{p}. \tag{1.39}$$

This extends to the non radial case the radial inequalities in [23], [17], [2] (see also [8]) and many others; notice that in the radial case we can choose $\widetilde{p} = \infty$ to obtain the largest possible range. When σ is an integer we can replace the fractional operator $|D|^\sigma$ with usual derivatives; see Corollary 1.12 below for a similar argument.

By similar techniques it is possible to derive nonhomogeneous estimates in terms of norms of type $\|\langle D \rangle^\sigma u\|_{L^p}$; we omit the details.

Critical estimates in Besov spaces. Case (ii) in Theorem 1.3 is suitable for applications to spaces defined via Fourier decompositions, in particular Besov spaces. We recall the standard machinery: fix a C_c^∞ radial function $\psi_0(\xi)$ equal to 1 for $|\xi| < 1$ and vanishing for $|\xi| > 2$, define a Littlewood-Paley partition of unity via $\phi_0(\xi) = \psi(\xi) - \psi(\xi/2)$, $\phi_j(\xi) = \phi_0(2^{-j}\xi)$, and decompose u as $u = \sum_{j \in \mathbb{Z}} u_j$ where $u_j = \phi_j(D)u = \mathbb{F}^{-1}\phi_j(\xi)\mathbb{F}u$. Then the homogeneous Besov norm $\dot{B}_{p,1}^s$ is defined as

$$\|u\|_{\dot{B}_{p,1}^s} = \sum_{j \in \mathbb{Z}} 2^{js}\|u_j\|_{L^p}. \tag{1.40}$$

We can apply Theorem 1.3-(ii) to each component u_j in the full range of indices $1 \le p \le q \le \infty$, with a constant independent of j. By the standard trick $\widetilde{u}_j = u_{j-1} + u_j + u_{j+1}$, $u_j = \phi_j(D)\widetilde{u}_j$ we obtain the estimate

$$\||x|^{-\beta}T_\gamma u\|_{L^q_{|x|}L^{\widetilde{q}}_\theta} \le C\sum_{j \in \mathbb{Z}} \||x|^\alpha \widetilde{u}_j\|_{L^{\widetilde{p}}_{|x|}L^{\widetilde{p}}_\theta} \tag{1.41}$$

for the full range $1 \le p \le q \le \infty$, $1 \le \widetilde{p} \le \widetilde{q} \le \infty$, with α, β, γ satisfying (1.37). The right hand side can be interpreted as a weighted norm of Besov type with different radial and angular integrability; this kind of spaces were already considered in [8]. In the special case $\alpha = 0$, $p = \widetilde{p} > 1$ we obtain a standard Besov norm (1.40) and hence the estimate (with the optimal choice $\widetilde{q} = \widetilde{p} = p$) reduces to

$$\||x|^{-\beta}T_\gamma u\|_{L^q_{|x|}L^p_\theta} \le C\|u\|_{\dot{B}_{p,1}^0}. \tag{1.42}$$

This estimate is weaker than (1.9) when the third condition in (1.7) is strict, but in the case of equality it gives a new estimate: recalling (1.35), we have proved the following

Corollary 1.8. *For all* $1 < p \le q \le \infty$ *we have*

$$\||x|^{\frac{n-1}{p} - \frac{n-1}{q}}u\|_{L^q_{|x|}L^p_\theta} \le C\|u\|_{\dot{B}_{p,1}^{\frac{1}{p} - \frac{1}{q}}}. \tag{1.43}$$

If we restrict (1.43) to radial functions and $q = \infty$, we obtain the well known radial pointwise estimate

$$|x|^{\frac{n-1}{p}}|u| \le C\|u\|_{\dot{B}_{p,1}^{1/p}} \qquad 1 < p < \infty \tag{1.44}$$

(see [2], [19]).

3. Caffarelli-Kohn-Nirenberg weighted interpolation inequalities

In this section we use the technology outlined before toghether with interpolation. In this way we can extend to $L^p L^{\widetilde{p}}$ setting also the Caffarelli-Kohn-Nirenberg inequalities. Such inequalities are fundamental tools in mathematical analysis and in PDEs theory. In particular are really useful in the context of Navier-Stokes equation because provide a priori estimate for weak solutions by interpolation of quantities related to the energy dissipation. Let's start by the family of inequalities on $\mathbb{R}^n$, $n \ge 1$

$$\||x|^{-\gamma}u\|_{L^r} \le C\||x|^{-\alpha}\nabla u\|_{L^p}^a \||x|^{-\beta}u\|_{L^q}^{1-a}. \tag{1.45}$$

for the range of parameters

$$n \ge 1, \qquad 1 \le p < \infty, \qquad 1 \le q < \infty, \qquad 0 < r < \infty, \qquad 0 < a \le 1. \tag{1.46}$$

Some conditions are immediately seen to be necessary for the validity of (1.45): to ensure local integrability we need

$$\gamma < \frac{n}{r} \qquad \alpha < \frac{n}{p} \qquad \beta < \frac{n}{q} \tag{1.47}$$

and by scaling invariance we need to assume

$$\gamma - \frac{n}{r} = a\left(\alpha + 1 - \frac{n}{p}\right) + (1-a)\left(\beta - \frac{n}{q}\right). \tag{1.48}$$

In [1] the following remarkable result was proved, which improves and extends a number of earlier estimates including weighted Sobolev and Hardy inequalities:

Theorem 1.9 ([1]). *Consider the inequalities* (1.45) *in the range of parameters given by* (1.47), (1.46), (1.48). *Denote with* $\boldsymbol{\Delta}$ *the quantity*

$$\boldsymbol{\Delta} = \gamma - a\alpha - (1-a)\beta \equiv a + n\left(\frac{1}{r} - \frac{1-a}{q} - \frac{a}{p}\right) \tag{1.49}$$

(the identity in (1.49) *is a reformulation of the scaling relation* (1.48)*). Then the inequalities* (1.45) *are true if and only if both the following conditions are satisfied:*

(i) $\boldsymbol{\Delta} \geq 0$
(ii) $\boldsymbol{\Delta} \leq a$ *when* $\gamma - n/r = \alpha + 1 - n/p$.

Remark 1.4. Notice that in the original formulation of [1] also the case $a = 0$ was considered, but with the introduction of an additional parameter forcing $\beta = \gamma$ when $a = 0$. Thus the case $a = 0$ becomes trivial in the original formulation; however, at least for $r > 1$, a much larger range $0 \leq \gamma - \beta < n$ can be obtained by a direct application of the Hardy-Littlewood-Sobolev inequality, so strictly speaking the additional requirement $\beta = \gamma$ is not necessary. We think the formulation adopted here is cleaner.
On the other hand, the necessity of (i) follows from the uniformity of the estimate w.r.to translations, while the necessity of (ii) is proved by testing the inequality on the spikes $|x|^{\gamma-n/r}\log|x|^{-1}$ truncated near $x = 0$.

In [5] the authors prove the following radial improvement of Theorem 1.9:

Theorem 1.10 ([5]). *Let* $n \geq 2$, *let* $\alpha, \beta, \gamma, r, p, q, a$ *be in the range determined by* (1.47), (1.46), (1.48), *define* $\boldsymbol{\Delta}$ *as in* (1.49), *and assume that*

$$a\left(1 - \frac{n}{p}\right) \leq \boldsymbol{\Delta} \leq a, \qquad \alpha < \frac{n}{p} - 1, \tag{1.50}$$

the first inequality being strict when $p = 1$. *Then estimate* (1.45) *is true for all radial functions* $u \in C_c^\infty(\mathbb{R}^n)$.

We somewhat simplified the statement of Theorem 1.1 in [5], and in particular conditions (1.8)-(1.10) in that paper are equivalent to (1.50) here, as it is readily seen. Notice that the condition $\boldsymbol{\Delta} \leq a$ forces r to be larger than 1.
Using the $L^p L^{\widetilde{p}}$ norms we can extend both Theorems 1.9 and 1.10. For greater generality we prove an estimate with fractional derivatives

$$|D|^\sigma = (-\Delta)^{\frac{\sigma}{2}}, \qquad \sigma > 0.$$

Our result is the following:

Theorem 1.11. *Let* $n \geq 2$, $r, \widetilde{r}, p, \widetilde{p}, q, \widetilde{q} \in [1, +\infty)$, $0 < a \leq 1$, $0 < \sigma < n$ *with*

$$\gamma < \frac{n}{r}, \qquad \beta < \frac{n}{q}, \qquad \frac{n}{p} - n < \alpha < \frac{n}{p} - \sigma \tag{1.51}$$

satisfying the scaling condition

$$\gamma - \frac{n}{r} = a\left(\alpha + \sigma - \frac{n}{r}\right) + (1-a)\left(\beta - \frac{n}{q}\right). \tag{1.52}$$

Define the quantities

$$\boldsymbol{\Delta} = a\sigma + n\left(\frac{1}{r} - \frac{1-a}{q} - \frac{a}{p}\right), \qquad \widetilde{\boldsymbol{\Delta}} = a\sigma + n\left(\frac{1}{\widetilde{r}} - \frac{1-a}{\widetilde{q}} - \frac{a}{\widetilde{p}}\right). \tag{1.53}$$

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and assume further that

$$\boldsymbol{\Delta} + (n-1)\widetilde{\boldsymbol{\Delta}} \geq 0, \tag{1.54}$$

$$1 < p, \qquad a\left(\sigma - \frac{n}{p}\right) < \boldsymbol{\Delta} \leq a\sigma, \qquad a\left(\sigma - \frac{n}{\widetilde{p}}\right) \leq \widetilde{\boldsymbol{\Delta}} \leq a\sigma. \tag{1.55}$$

Then the following interpolation inequality holds:

$$\||x|^{-\gamma}u\|_{L^r_{|x|}L^{\widetilde{r}}_\theta} \leq C\||x|^{-\alpha}|D|^\sigma u\|^a_{L^p_{|x|}L^{\widetilde{p}}_\theta}\||x|^{-\beta}u\|^{1-a}_{L^q_{|x|}L^{\widetilde{q}}_\theta}. \tag{1.56}$$

If one assumes strict inequality in (1.54), then the inequalities in (1.55) can be relaxed to non strict inequalities.

When σ is an integer, the condition on α from below can be dropped, and a slightly stronger estimate can be proved. We introduce the notation

$$\||x|^{-\alpha}D^\sigma u\|_{L^p L^{\widetilde{p}}} = \sum_{|\nu|=\sigma} \||x|^{-\alpha}D^\nu u\|_{L^p L^{\widetilde{p}}}, \qquad \nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n.$$

Then we have:

Corollary 1.12. *Assume $\sigma = 1, \ldots, n-1$ is an integer. Then the following estimate holds*

$$\||x|^{-\gamma}u\|_{L^r_{|x|}L^{\widetilde{r}}_\theta} \leq C\||x|^{-\alpha}D^\sigma u\|^a_{L^p_{|x|}L^{\widetilde{p}}_\theta}\||x|^{-\beta}u\|^{1-a}_{L^q_{|x|}L^{\widetilde{q}}_\theta}. \tag{1.57}$$

provided the parameters satisfy the same conditions as in the previous theorem, with the exception of the condition $\alpha > -n + n/p$ which is not necessary.

Remark 1.5. If $\sigma = 1$, Corollary 1.12 contains both the original result of [1] (for $\boldsymbol{\Delta} \leq a$) and the radial improvement of [5].
Indeed, if we choose $p = \widetilde{p}$, $q = \widetilde{q}$, $r = \widetilde{r}$ in Corollary 1.12 we get of course $\boldsymbol{\Delta} = \widetilde{\boldsymbol{\Delta}}$, and selecting $\sigma = 1$ we reobtain the original inequality (1.45) in the range $0 \leq \boldsymbol{\Delta} \leq a$.
On the other hand, if u is a radial function, estimate (1.57) does not depend on the choice of $\widetilde{p}, \widetilde{q}, \widetilde{r}$ and we can let $\widetilde{\boldsymbol{\Delta}}$ assume an arbitrary value in the range (1.55). Thus if $\boldsymbol{\Delta} > a(\sigma - n/p)$ we can choose $\widetilde{\boldsymbol{\Delta}} = 0$, while if $\boldsymbol{\Delta} = a(\sigma - n/p)$ we can choose $\widetilde{\boldsymbol{\Delta}} = \epsilon > 0$ arbitrarily small, recovering the results of Theorem 1.10.

Again we can get theorem 1.11 as a consequence of 1.3.

PROOF. We begin by taking $0 < a \leq 1$, and indices $r, \widetilde{r}, s, \widetilde{s}, q, \widetilde{q} \in [1, +\infty]$ such that

$$\frac{1}{r} = \frac{a}{s} + \frac{1-a}{q}, \qquad \frac{1}{\widetilde{r}} = \frac{a}{\widetilde{s}} + \frac{1-a}{\widetilde{q}}. \tag{1.58}$$

Then by two applications of Hölder's inequality we obtain the interpolation inequality

$$\begin{aligned}
\||x|^{-\gamma}u\|_{L^r L^{\widetilde{r}}} &= \|(|x|^{-\delta}u)^a(|x|^{-\beta}u)^{1-a}\|_{L^r L^{\widetilde{r}}} \\
&\leq \|(|x|^{-\delta}u)^a\|_{L^{s/a}L^{\widetilde{s}/a}}\|(|x|^{-\beta}u)^{1-a}\|_{L^{q/(1-a)}L^{\widetilde{q}/(1-a)}} \\
&= \||x|^{-\delta}u\|^a_{L^s L^{\widetilde{s}}}\||x|^{-\beta}u\|^{1-a}_{L^q L^{\widetilde{q}}}
\end{aligned} \tag{1.59}$$

provided the exponents γ, δ, β are related by

$$\gamma = a\delta + (1-a)\beta. \tag{1.60}$$

Now the main step of the proof. By Theorem (1.3) we know that

$$\||x|^{-\delta}T_\lambda u\|_{L^s L^{\widetilde{s}}} \lesssim \||x|^{-\alpha}u\|_{L^p L^{\widetilde{p}}}$$

under suitable conditions on the indices. Now using the well known estimate

$$|u(x)| \leq C_{\lambda,n}T_\lambda\left(\left||D|^{n-\lambda}u\right|\right) \tag{1.61}$$

the previous inequality can be equivalently written

$$\||x|^{-\delta}u\|_{L^s L^{\widetilde{s}}} \lesssim \||x|^{-\alpha}|D|^\sigma u\|_{L^p L^{\widetilde{p}}}, \qquad \sigma = n - \lambda$$

which together with (1.59) gives

$$\||x|^{-\gamma}u\|_{L^r L^{\widetilde{r}}} \lesssim \||x|^{-\alpha}|D|^\sigma u\|_{L^p L^{\widetilde{p}}}^a \||x|^{-\beta}u\|_{L^q L^{\widetilde{q}}}^{1-a}. \tag{1.62}$$

The conditions on the indices are those given by (1.58), (1.60), plus those listed in the statement of Theorem (1.3) (notice that we are using $-\alpha$ instead of α). The complete list is the following:

$$r, s, q, \widetilde{r}, \widetilde{s}, \widetilde{q} \in [1, +\infty], \qquad a < 0 \le 1, \qquad 0 < \sigma < n, \tag{1.63}$$

$$\frac{1}{r} = \frac{a}{s} + \frac{1-a}{q}, \qquad \frac{1}{\widetilde{r}} = \frac{a}{\widetilde{s}} + \frac{1-a}{\widetilde{q}}. \tag{1.64}$$

$$1 < s \le p < \infty, \qquad 1 \le \widetilde{s} \le \widetilde{p} \le \infty, \tag{1.65}$$

$$\gamma < \frac{n}{r}, \qquad \beta < \frac{n}{q}, \qquad -\alpha < \frac{n}{p'}, \qquad \delta < \frac{n}{s}, \tag{1.66}$$

$$\gamma = a\delta + (1-a)\beta, \tag{1.67}$$

$$-\alpha + \delta + n - \sigma = n + \frac{n}{s} - \frac{n}{p}, \tag{1.68}$$

$$-\alpha + \delta \ge (n-1)\left(\frac{1}{s} - \frac{1}{p} + \frac{1}{\widetilde{p}} - \frac{1}{\widetilde{s}}\right). \tag{1.69}$$

Recall also that, when the last inequality (1.69) is strict, we can allow the full range

$$1 \le s \le p \le \infty.$$

Our final task is to rewrite this set of conditions in a compact form, eliminating the redundant parameters $\delta, s, \widetilde{s}$. Define the two quantities

$$\boldsymbol{\Delta} = a\sigma + n\left(\frac{1}{r} - \frac{1-a}{q} - \frac{a}{p}\right), \qquad \widetilde{\boldsymbol{\Delta}} = a\sigma + n\left(\frac{1}{\widetilde{r}} - \frac{1-a}{\widetilde{q}} - \frac{a}{\widetilde{p}}\right).$$

Then (1.64) are equivalent to

$$\boldsymbol{\Delta} = a\left(\sigma + \frac{n}{s} - \frac{n}{p}\right), \qquad \widetilde{\boldsymbol{\Delta}} = a\left(\sigma + \frac{n}{\widetilde{s}} - \frac{n}{\widetilde{p}}\right) \tag{1.70}$$

while (1.68) is equivalent to

$$\delta = \alpha + \frac{\boldsymbol{\Delta}}{a} \tag{1.71}$$

and we can use (1.70), (1.71) to replace $\delta, s, \widetilde{s}$ in the remaining relations. Condition (1.67) becomes

$$\boldsymbol{\Delta} = \gamma - a\alpha - (1-a)\beta, \tag{1.72}$$

which is precisely the scaling condition, while (1.69) becomes

$$\boldsymbol{\Delta} + (n-1)\widetilde{\boldsymbol{\Delta}} \ge 0. \tag{1.73}$$

The last inequality in (1.66), $\delta < n/s$, can be written

$$\alpha < \frac{n}{p} - \sigma$$

so that (1.66) is replaced by

$$\gamma < \frac{n}{r}, \qquad \beta < \frac{n}{q}, \qquad \frac{n}{p} - n < \alpha < \frac{n}{p} - \sigma. \tag{1.74}$$

Finally, conditions (1.65) translate to

$$1 < p, \qquad a\left(\sigma - \frac{n}{p}\right) < \boldsymbol{\Delta} \le a\sigma, \qquad a\left(\sigma - \frac{n}{\widetilde{p}}\right) \le \widetilde{\boldsymbol{\Delta}} \le a\sigma. \tag{1.75}$$

When the inequality in (1.73) is strict, the last condition can be relaxed to

$$1 \le p, \qquad a\left(\sigma - \frac{n}{p}\right) \le \mathbf{\Delta} \le a\sigma, \qquad a\left(\sigma - \frac{n}{\widetilde{p}}\right) \le \widetilde{\mathbf{\Delta}} \le a\sigma. \qquad (1.76)$$

We pass now to the proof of Corollary 1.12. Assume now σ is integer, and the inequality

$$\||x|^{-\gamma}u\|_{L^r L^{\widetilde{r}}} \le C\||x|^{-\alpha}|D|^\sigma u\|_{L^p L^{\widetilde{p}}}^a \||x|^{-\beta}u\|_{L^q L^{\widetilde{q}}}^{1-a}$$

is true for a certain choice of the parameters as in the theorem, so that in particular

$$\alpha < \frac{n}{p} - \sigma < \frac{n}{p}.$$

Then we shall prove that also the following inequalities are true

$$\||x|^{k-\gamma}u\|_{L^r L^{\widetilde{r}}} \le C\||x|^{k-\alpha}D^\sigma u\|_{L^p L^{\widetilde{p}}}^a \||x|^{k-\beta}u\|_{L^q L^{\widetilde{q}}}^{1-a} \qquad (1.77)$$

for all integers $k \ge 0$, where we are using the shorthand notation

$$\||x|^{k-\alpha}D^\sigma u\|_{L^p L^{\widetilde{p}}} = \sum_{|\nu|=\sigma} \||x|^{k-\alpha}D^\nu u\|_{L^p L^{\widetilde{p}}}, \qquad (\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n).$$

This in particular implies that the condition on α from below can be dropped when σ is integer.

When $k = 0$, (1.77) is obtained just by replacing $|D|^\sigma$ with D^σ in the original inequality. The proof of this estimate is identical to the previous one; the only modification is to use, instead of (1.61), the stronger pointwise bound

$$|u(x)| \le C_{\lambda,n} T_\lambda \left(\left|D^{n-\lambda}u\right|\right) \qquad (1.78)$$

which is valid for all $\lambda = 1, \ldots, n-1$.

Now if we apply (1.77) (with $k = 0$) to a function of the form $|x|^k u$ for some $k \ge 1$, we obtain

$$\||x|^{k-\gamma}u\|_{L^r L^{\widetilde{r}}} \le C\||x|^{-\alpha}D^\sigma(|x|^k u)\|_{L^p L^{\widetilde{p}}}^a \||x|^{k-\beta}u\|_{L^q L^{\widetilde{q}}}^{1-a}$$

and to conclude the proof we see that it is sufficient to prove the inequality

$$\||x|^{-\alpha}D^\sigma(|x|^k u)\|_{L^p L^{\widetilde{p}}} \lesssim \||x|^{k-\alpha}D^\sigma u\|_{L^p L^{\widetilde{p}}} \qquad (1.79)$$

for all $\alpha < n/p$, $1 \le p, \widetilde{p} < \infty$, and integers $\sigma = 1, \ldots, n-1$, $k \ge 1$. Notice indeed that all the conditions on the parameters (apart from $\alpha > -n+n/p$) are unchanged if we decrease γ, α, β by the same quantity.

By induction on k (and writing $\delta = -\alpha$), we are reduced to prove that for all $p, \widetilde{p} \in [1, \infty)$ and $1 \le \sigma \le n-1$

$$\||x|^\delta D^\sigma(|x|u)\|_{L^p L^{\widetilde{p}}} \lesssim \||x|^{1+\delta}D^\sigma u\|_{L^p L^{\widetilde{p}}}, \qquad \delta > \sigma - \frac{n}{p}. \qquad (1.80)$$

Using Leibnitz' rule we reduce further to

$$\||x|^{1+\delta-\ell}u\|_{L^p L^{\widetilde{p}}} \lesssim \||x|^{1+\delta}D^\ell u\|_{L^p L^{\widetilde{p}}}, \qquad \delta > \ell - \frac{n}{p} \qquad (1.81)$$

for $\ell = 1, \ldots, n-1$, and by induction on ℓ this is implied by

$$\||x|^\delta u\|_{L^p L^{\widetilde{p}}} \lesssim \||x|^{1+\delta}\nabla u\|_{L^p L^{\widetilde{p}}}, \qquad \delta > 1 - \frac{n}{p}. \qquad (1.82)$$

In order to prove (1.82), consider first the radial case. When $u = \phi(|x|)$ is a radial (smooth compactly supported) function, we have

$$\||x|^\delta u\|_{L^p L^{\widetilde{p}}}^p \simeq \int_0^\infty \rho^{\delta p+n-1}|\phi(\rho)|^p d\rho.$$

Integrating by parts we get

$$= -\frac{p}{\delta p + n} \int_0^\infty \rho^{\delta p + n} |\phi|^{p-1} |\phi(\rho)|' d\rho$$

$$\lesssim \int_0^\infty (\rho^{\delta p + n - 1} |\phi|^p)^{\frac{p-1}{p}} (\rho^{\delta p + p + n - 1} |\phi'|^p)^{\frac{1}{p}} d\rho$$

$$\simeq \||x|^\delta u\|_{L^p L^{\widetilde{p}}}^{\frac{p-1}{p}} \||x|^{1+\delta} \nabla u\|_{L^p L^{\widetilde{p}}}$$

which implies (1.82) in the radial case. If u is not radial, define

$$\phi(\rho) = \|u(\rho\theta)\|_{L^{\widetilde{p}}_\theta(\mathbb{S}^{n-1})} = \left(\int_{\mathbb{S}^{n-1}} |u(\rho\theta)|^{\widetilde{p}} dS_\theta \right)^{\frac{1}{\widetilde{p}}}$$

so that

$$\||x|^\delta u\|_{L^p L^{\widetilde{p}}} \simeq \left(\int_0^\infty \rho^{\delta p + n - 1} |\phi(\rho)|^p d\rho \right)^{\frac{1}{p}}.$$

The proof in the radial case implies

$$\||x|^\delta u\|_{L^p L^{\widetilde{p}}} \leq \||x|^{\delta+1} \phi'(|x|)\|_{L^p};$$

moreover we have

$$|\phi'(\rho)| \lesssim \phi^{1-\widetilde{p}} \int_{\mathbb{S}^{n-1}} |u(\rho\theta)|^{\widetilde{p}-1} |\theta \cdot \nabla u| \, dS_\theta$$

$$\leq \phi^{1-\widetilde{p}} \left(\int_{\mathbb{S}} |u|^{\widetilde{p}} \right)^{\frac{\widetilde{p}-1}{\widetilde{p}}} \left(\int_{\mathbb{S}} |\nabla u|^{\widetilde{p}} \right)^{\frac{1}{\widetilde{p}}} = \|\nabla u(\rho\theta)\|_{L^{\widetilde{p}}_\theta(\mathbb{S}^{n-1})}$$

and in conclusion we obtain

$$\||x|^\delta u\|_{L^p L^{\widetilde{p}}} \leq \||x|^{\delta+1} \nabla u\|_{L^p L^{\widetilde{p}}}$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Strichartz estimates for the wave equation

As a last example, we mention an application of our result to Strichartz estimates for the wave equation; a more detailed analysis will be conducted elsewhere. The wave flow $e^{it|D|}$ on $\mathbb{R}^n$, $n \geq 2$, satisfies the estimates, which are usually called *Strichartz estimates*:

$$\||D|^{\frac{n}{r} + \frac{1}{p} - \frac{n}{2}} e^{it|D|} f\|_{L^p_t L^r_x} \lesssim \|f\|_{L^2} \tag{1.83}$$

provided the indices p, r satisfy

$$p \in [2, \infty], \qquad 0 < \frac{1}{r} \leq \frac{1}{2} - \frac{2}{(n-1)p}. \tag{1.84}$$

Here the $L^p_t L^r_x$ norms are defined as

$$\|u(t,x)\|_{L^p_t L^r_x} = \left\| \|u(t,\cdot)\|_{L^r_x} \right\|_{L^p_t}.$$

In their most general version, the estimates were proved in [11], [15]. Notice that in (1.83) we included the extension of the estimates which can be obtained via Sobolev embedding on $\mathbb{R}^n$.

If the initial value f is a radial function, the estimates admit an improvement in the sense that conditions (1.84) can be relaxed to

$$p \in [2, \infty], \qquad 0 < \frac{1}{r} < \frac{1}{2} - \frac{1}{(n-1)p}. \tag{1.85}$$

This phenomenon is connected with the finite speed of propagation for the wave equation and is usually deduced using the space-time decay properties of the equation. For a thourough discussion and a comprehensive history of such estimates see e.g. [14] and the references therein.

A different set of estimates are the *smoothing estimates*, also known as Morawetz-type or weak dispersion estimates. These appear in a large number of versions; a particularly sharp one is the following, from [10]:

$$\||x|^{-\zeta}|D|^{\frac{1}{2}-\zeta}e^{it|D|}f\|_{L_t^2 L_x^2} \lesssim \|\Lambda^{\frac{1}{2}-\zeta}f\|_{L^2}, \qquad \frac{1}{2} < \zeta < \frac{n}{2}. \qquad (1.86)$$

Here the operator

$$\Lambda = (1 - \Delta_{\mathbb{S}^{n-1}})^{1/2}$$

is a function of the Laplace-Beltrami operator on the sphere and acts only on angular variables, thus we see that the flow improves the angular regularity. Morawetz-type estimates are conceptually simpler than (1.83), being related to more basic properties of the operators; indeed L^2 estimates of this type can be proved for quite large classes of equations via multiplier methods.

Corresponding estimates are known for the Schrödinger flow $e^{it\Delta}$, and M.C. Vilela [24] noticed that in the radial case they can be used to deduce Strichartz estimates via the radial Sobolev embedding. Following a similar idea for the wave flow, in combination with our precised estimates (1.36), gives an even better result, which strengthens the standard Strichartz estimates (1.83)-(1.84) in terms of the mixed $L_{|x|}^p L_\theta^{\widetilde{p}}$ norms. Indeed, a special case of (1.36) gives, for arbitrary functions $g(x)$,

$$\|g\|_{L_{|x|}^q L_\theta^{\widetilde{q}}} \lesssim \||x|^\alpha |D|^{\alpha + \frac{n}{2} - \frac{n}{q}}g\|_{L^2}, \qquad q,\widetilde{q} \in [2,\infty), \qquad \frac{n}{2} > \alpha \geq (n-1)\left(\frac{1}{q} - \frac{1}{\widetilde{q}}\right) \qquad (1.87)$$

with the exclusion of the case $\alpha = 0$, $q = \widetilde{q} = 2$. Then by (1.87) and (1.86) we obtain the precised Strichartz estimates

$$\||x|^{-\delta}|D|^{\frac{n}{q} + \frac{1}{2} - \frac{n}{2} - \delta}e^{it|D|}f\|_{L_t^2 L_{|x|}^q L_\theta^{\widetilde{q}}} \lesssim \|\Lambda^{-\epsilon}f\|_{L^2} \qquad (1.88)$$

provided

$$q,\widetilde{q} \in [2,+\infty), \qquad \delta < \frac{n}{q}, \qquad 0 < \epsilon < \frac{n-1}{2}, \qquad 0 < \frac{1}{q} < \frac{1}{\widetilde{q}} - \frac{1}{2(n-1)} \qquad (1.89)$$

and

$$\epsilon \leq \delta + (n-1)\left(\frac{1}{\widetilde{q}} - \frac{1}{2(n-1)} - \frac{1}{q}\right). \qquad (1.90)$$

We will not exploit the consequence of this particular section in the thesis. The results contained can be consider just as an application of our point of view to a different class of problems. We hope to came back on Strichartz estimates in $L^p L^{\widetilde{p}}$ spaces in future works.

Introduction to the regularity problem for the Navier-Stokes equation

In this chapter we introduce the Cauchy problem for the Navier-Stokes approximation of the fluid motion in the whole space, that's

$$
\begin{cases}
\partial_t u + (u \cdot \nabla) u + \nabla p - \Delta u & = & 0 & \text{in} & [0, T) \times \mathbb{R}^n \\
\nabla \cdot u & = & 0 & \text{in} & [0, T) \times \mathbb{R}^n \\
u & = & u_0 & \text{in} & \{0\} \times \mathbb{R}^n.
\end{cases}
\tag{2.1}
$$

Here $u = (u_1, u_2, u_3)$ is the velocity field, p is the pressure and the viscosity have been set to one. No external forces is working. The first equation is the Newton law while the second guarantees the incompressibility of the fluid. To require incompressibility also at time $t = 0$ have to be considered just initial data u_0 such that $\nabla \cdot u_0 = 0$. So is useful to define the space

$$
L_\sigma^2(\mathbb{R}^n) = \overline{\left\{ u_0 \in C_0^\infty(\mathbb{R}^n) \quad \text{s.t.} \quad \int_{\mathbb{R}^n} |u_0|^2 \, dx < +\infty, \, \nabla \cdot u_0 = 0 \right\}}.
\tag{2.2}
$$

We will use the same notation for the norm of scalar, vector or tensor quantities, the meaning will be clear by the situation; for instance we will use $\|p\|_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |p| \, dx$, $\|u\|_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \sum_{i=1}^{3} u_i^2 \, dx$, $\|\nabla u\|_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \sum_{i,j=1}^{3} (\partial_i u_j)^2 \, dx$. We will use also the notation $u \in L^2(\mathbb{R}^n)$ instead of $u \in (L^2(\mathbb{R}^n))^3$, and so on. The well posedness of (2.1) is a well-known mathematical challenge and just partial results have been obtained. The main question is: if we consider an initial datum u_0 in the Schwartz class, there exists a unique global solution of the problem (2.1)?

In this chapter we will give an excursus on classical theorems starting by the pioneering work of Leray, Hopf, Serrin and Kato [26, 28, 29]. This classical results have been improved in many different directions and in the thesis we will focus basically on the weighted norm approach, which also has a wide reference literature, see [33], [32], et al. We will also briefly focus on [35, 27]. The first is particulary relevant because seems to be a sharp version for the existence with small data. The second is a celebrated landmark for the regularity theory.

1. Equivalence between the differential and integral formulation

In this secion we give the integral formulation of problem (2.1). This formulation is very useful in order to study both local (in time) well posedness and global well posedness with small data. In such a case, starting by the integral formulation is immediate to look at the equation (2.1) as a perturbed heat equation, and fixed point techniques are available. Of course this is irrelevant when we look for global solutions with large initial data. We follow basically [45], and we omit, almost completely, the proofs to get a compact presentation. Anyway all the proofs are classical and can be easily found in literature. Let come back on the system:

$$
\begin{cases}
\partial_t u + (u \cdot \nabla) u + \nabla p & = \Delta u & 0 \\
\nabla \cdot u & = & 0 \\
u & = & u_0.
\end{cases}
$$

or in components ($i = 1, ..., n$):

$$\begin{cases} \partial_t u_i + \sum_{j=1}^n u_j \partial_j u_i + \partial_i p &= \sum_{j=1}^n \partial_{jj} u_i \\ \sum_{i=1}^n \partial_i u_i &= 0. \end{cases}$$

By taking the divergence of the first equation and using $\nabla \cdot u = 0$ we get:

$$0 = \sum_{i=1}^n \partial_i \sum_{j=1}^n u_j \partial_j u_i + \Delta p \tag{2.3}$$

$$= \sum_{i,j=1}^n \partial_i \partial_j (u_i u_j) + \Delta p, \tag{2.4}$$

so p can, at least formally, be recovered by u througt:

$$p = -\frac{\sum_{i,j=1}^n \partial_i \partial_j (u_i u_j)}{\Delta}. \tag{2.5}$$

By using his relation the system can be written as:

$$\begin{cases} u &= e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes u) ds & \text{in} & [0, T) \times \mathbb{R}^n \\ \nabla \cdot u &= 0 & \text{in} & [0, T) \times \mathbb{R}^n. \end{cases} \tag{2.6}$$

where $(u \otimes u)_{i,j} = u_i u_j$ and $\mathbb{P}$ is formally:

$$\mathbb{P}f = f - \nabla \frac{1}{\Delta} \nabla \cdot f, \tag{2.7}$$

or in components

$$(\mathbb{P}f)_i = f_i - \frac{\partial_i \partial_j}{\Delta} f_j.$$

The operator $\mathbb{P}$, that's a really useful tool in the study of the Navier-Stokes problem, has been introduced by Leray in [26]. It is a projection operator on the subspace of the divergence free vector fields. It is infact easy to show that $\mathbb{P}f = f \Leftrightarrow \nabla \cdot f = 0$. In order to give precise definition of the formal computation above at first we have to make sense to the operator $\mathbb{P}$. This is easy if we restrict to $f \in L^2(\mathbb{R}^n)$, in this case $\mathbb{P}f$ is defind by:

$$\mathbb{P}f = f - (R \otimes R)f$$

or in components:

$$(\mathbb{P}f)_i = f_i - \sum_j R_i R_j f_j,$$

where R_j is the Riesz transform in the direction j defined by the symbol $i\frac{\xi_j}{|\xi|}$. This is a simple way to define $\mathbb{P}$, even if it can be defined on larger Banach spaces as a Calderon-Zygmund operator; details can be found in [45]. Anyway we are interested basically in the operator $\mathbb{P}(\nabla \cdot f)$, that appears in the integral formulation (2.6); it is defined through components by:

$$(\mathbb{P} \nabla \cdot (u \otimes u))_i = \partial_j (u \otimes u)_{i,j} - \frac{1}{\Delta} \partial_i \partial_j \partial_k (u \otimes u)_{j,k}.$$

In such a case the differentiation allows to extend the definition to a larger class of Banach spaces. To give a precise definition (again in [45]) we need the auxiliary spaces:

Definition 2.1. Let define the dual spaces $WL^\infty(\mathbb{R}^n), L^1_{uloc}(\mathbb{R}^n)$:

- the space $WL^\infty(\mathbb{R}^n)$ is the Banach space of the Lebesgue measurable functions ϕ on $\mathbb{R}^n$ such that

$$\sum_{k \in \mathbb{Z}^n} \sup_{x \in \{k + [0,1]^n\}} |\phi(x)| < +\infty, \tag{2.8}$$

equipped with the norm (2.8);

- the space $L^1_{uloc}(\mathbb{R}^n)$ is the space of locally intgrable functions equipped with the norm:

$$L^1_{uloc}(\mathbb{R}^n) = \sup_{[0,1]^n} \| \mathbb{1}_{[0,1]^n} f \|_{L^1(\mathbb{R}^n)}.$$

It holds the following:

Lemma 2.2 ([45]). *The operator $\frac{1}{\Delta} \partial_i \partial_j \partial_k$ is a convolution operator with a kernel $T_{i,j,k}$ such that the following decomposition holds:*

$$T_{i,j,k} = \alpha_{i,j,k} + \partial_i \partial_j \beta_k$$

where $\alpha_{i,j,k} \in WL^\infty(\mathbb{R}^n)$ and $\beta_k \in L^1_{loc}(\mathbb{R}^n)$.

By lemma (2.2) and inclusions

$$L^1_{loc} * L^1_{uloc} \subset L^1_{uloc}, \qquad WL^\infty * L^\infty,$$

it turns out that $\mathbb{P}(\nabla \cdot)$ can be defined on the space $(L^1_{uloc}(\mathbb{R}^n))^{n \times n}$. Now we focus on some properties of convolutions with the Oseen Kernel, so we consider:

$$\frac{1}{\Delta} \partial_i \partial_j e^{t\Delta}.$$

It holds the following:

Lemma 2.3 ([45]). *Let $1 \le i, j \le n$. The operator $\frac{1}{\Delta} \partial_i \partial_j e^{t\Delta}$ is a convolution operator $O_{i,j}(t) * f_j$, with $O_{i,j}(t) \in (C^\infty(\mathbb{R}^n))^{n \times n}$, the homogeneity:*

$$O_{i,j}(t) = \frac{1}{t^{\frac{n}{2}}} O_{i,j}\left(\frac{x}{\sqrt{t}} \right),$$

and the decay:

$$(1 + |x|)^{n+|\eta|} \partial^\eta O_{i,j} \in (L^\infty(\mathbb{R}^n))^{n \times n},$$

for each multi-index η.

This is the main technical tool we need in order to study the properties of $e^{t\Delta} \mathbb{P}(\nabla \cdot)$ that acts on the tensor $u \otimes u$ through

$$(e^{t\Delta} \mathbb{P}(\nabla \cdot (u \otimes u)))_i = e^{t\Delta} \partial_j (u \otimes u)_{i,j} - e^{t\Delta} \frac{1}{\Delta} \partial_i \partial_j \partial_k (u \otimes u)_{j,k}.$$

It holds the following:

Proposition 2.4 ([45]). *Let $1 \le i, j, k \le n$. The operator $e^{t\Delta} \mathbb{P}(\nabla \cdot)$ is a convolution operator $K_{i,j,k}(t) * f_{j,k}$, with $K_{i,j,k}(t) \in (C^\infty(\mathbb{R}^n))^{n \times n}$, the homogeneity:*

$$K_{i,j,k}(t) = \frac{1}{t^{\frac{n+1}{2}}} K_{i,j,k}\left(\frac{x}{\sqrt{t}} \right),$$

and the decay:

$$(1 + |x|)^{n+1+|\eta|} \partial^\eta K_{i,j,k} \in (L^\infty(\mathbb{R}^n))^{n \times n},$$

for each multi-index η.

We finish by stating an useful equivalence result:

Theorem 2.5 ([45]). *Let $u \in \cap_{s<T} \left(L^2_t L^2_{uloc,x}(0,s) \times \mathbb{R}^n \right)$. Then the following are equivalent:*

(1) *u is a weak solution of:*

$$\begin{cases} \partial_t u + \mathbb{P} \nabla \cdot (u \otimes u) & = & \Delta u & in & [0,T) \times \mathbb{R}^n \\ \nabla \cdot u & = & 0 & in & [0,T) \times \mathbb{R}^n \\ u & = & u_0 & in & \{0\} \times \mathbb{R}^n. \end{cases} \qquad (2.9)$$

(2) u solves the integral problem:

$$\begin{cases} u & = & e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)ds & in & [0,T]\times\mathbb{R}^n \\ \nabla\cdot u & = & 0 & in & [0,T]\times\mathbb{R}^n. \end{cases} \quad (2.10)$$

2. The Leray-Hopf solutions

The modern theory of Navier-Stokes equation starts with the Leray's work [26] in which global existence of weak solutions of (2.1) for L^2 initial data is proved. Such weak solutions have also physical meaning because they respond to the energy dissipation. On the other hand the existence theorem is by compactness and neither regularity nor uniqeness have been proved in the general case. In this section we will briefly sketch the ideas of the Leay's theory. We start by definition of weak solution in the context of [26]

Definition 2.6 (Leray's solutions). The pair (u,p) is a weak Leray solution of the Navier-Stokes system (2.1) in $[0,T]\times\mathbb{R}^n$ if the following holds:

(1) Exist some constants E_0, E_1 such that:

$$\int_{\mathbb{R}^n} |u(t,\cdot)|^2 \ dx \le E_0, \quad (2.11)$$

for almost every $t \in (0,T)$, and

$$\int_0^T \int_{\mathbb{R}^n} |\nabla u|^2 \ dxdt \le E_1; \quad (2.12)$$

(2) (u,p) satisfy (2.1) in the sense of distributions in $[0,T]\times\mathbb{R}^n$, that's

$$\int_0^T \int_{\mathbb{R}^n} (\partial_t\phi + (u\cdot\nabla)\phi)u \ dxdt + \int_{\mathbb{R}^n} u_0\phi(x,0) \ dx = \int_0^T \int_{\mathbb{R}^n} (\nabla\phi\cdot\nabla)u \ dxdt, \ (2.13)$$

for each $\phi \in C_c^\infty([0,T]\times\mathbb{R}^n)$ with $\nabla\cdot\phi = 0$ and

$$\int_0^T \int_{\mathbb{R}^n} u\cdot\nabla\phi \ dxdt = 0, \quad \int_0^T \int_{\mathbb{R}^n} p\Delta\phi + \sum_{i,j=1}^n u_iu_j\partial_i\partial_j\phi = 0, \quad (2.14)$$

for each $\phi \in C_c^\infty([0,T]\times\mathbb{R}^n)$.

(3) u satisfy the energy inequality:

$$\int_{\mathbb{R}^n} |u(t,\cdot)|^2 + 2\int_0^t \int_{\mathbb{R}^n} |\nabla u|^2 \ dxdt \le \int_{\mathbb{R}^n} |u_0|^2 \quad (2.15)$$

for each $t \in (0,T)$.

Condition (2.15) is expression of the dissipation of kinetic energy (the first term of the sum) caused by the frictions (the second term). It can be justified by multiplication of (2.1) with $2\phi u$ and integration by parts.

It is well know that Leray weak solutions are weakly continuous (see [41]), i.e.

$$\lim_{t\to s} \int_{\mathbb{R}^n} u(t,x)w(x) \ dx = \int_{\mathbb{R}^n} u(s,x)w(x) \ dx \quad (2.16)$$

for all $w \in L^2(\mathbb{R}^n)$, and so, if $u_0 \in L^2(\mathbb{R}^n)$ then

$$\lim_{t\to 0} \int_{\mathbb{R}^n} u(t,x)w(x) \ dx = \int_{\mathbb{R}^n} u_0(x)w(x) \ dx, \quad (2.17)$$

for all $w \in L^2(\mathbb{R}^n)$. This is how u attend its initial datum. In [26] is proved the existence of a weak solution $u \in \mathbb{R}^+\times\mathbb{R}^n$ of (2.1) for every $u_0 \in L_\sigma^2(\mathbb{R}^n)$. If we set the problem in $[0,T]\times\Omega$, $\Omega \subset \mathbb{R}^n$ open and bounded, and we require zero Dirichlet condition on $[0,T]\times\partial\Omega$, an analogous result is due to Hopf [28]. Let then state the precise Leray's result:

Theorem 2.7 ([26]). *Let $u_0 \in L^2_\sigma(\mathbb{R}^n)$. There exist a weak solution $u \in L^\infty(\mathbb{R}^+; L^2_\sigma(\mathbb{R}^n)) \cap L^2(\mathbb{R}^+; \dot{H}^1(\mathbb{R}^n))$ of the Cauchy problem (2.1) in $\mathbb{R}^+ \times \mathbb{R}^n$. Then u weakly attend its initial datum, i.e.*

$$\lim_{t \to 0} \int_{\mathbb{R}^n} (u(t,x) - u_{0,x}) w(x) \ dx = 0, \qquad \forall w \in L^2(\mathbb{R}^n).$$

Moreover the energy inequality holds:

$$\int_{\mathbb{R}^n} |u(t, \cdot)|^2 + 2 \int_0^T \int_{\mathbb{R}^n} |\nabla u|^2 \ dxdt \leq \int_{\mathbb{R}^n} |u_0|^2, \qquad \forall t \in \mathbb{R}^+. \tag{2.18}$$

Remark 2.1. The choice $u_0 \in L^2_\sigma$ is of course the more natural and phisically relevant for the problem. In such a way all initial data with bounded energy are covered. Anyway, as noticed, this generality leads to a poor wellposedess theory in which neither uniqeness nor persistence of regularity are guaranteed. Instead, as will seen in the next section, well posedness can be achieved if we restrict to small (in a suitable sense) initial data.

PROOF. We just sketched the proof. Details can be found widely in literature. A possible way to get the Leray's theorem is consider the family of regularized systems:

$$\begin{cases} \partial_t u^\varepsilon + (u^\varepsilon * \rho^\varepsilon \cdot \nabla) u^\varepsilon + \nabla p &= \Delta u^\varepsilon &\text{in} &\mathbb{R}^+ \times \mathbb{R}^n \\ \nabla \cdot u^\varepsilon &= 0 &\text{in} &\mathbb{R}^+ \times \mathbb{R}^n \\ u^\varepsilon &= u_0 &\text{in} &\{0\} \times \mathbb{R}^n, \end{cases} \tag{2.19}$$

where ρ^ε is a standard mollifier of size ε, that's:

$$\rho \in C_c^\infty(\mathbb{R}^n), \qquad \rho^\varepsilon = \frac{1}{\varepsilon^n} \rho(\frac{x}{\varepsilon}).$$

Now if $u_0 \in L^2_\sigma(\mathbb{R}^n)$ exists for each ε a unique global (and smooth in space) solution u^ε of problem (2.19). Furthermore the functions u^ε satisfies the energy inequality:

$$\int_{\mathbb{R}^n} |u^\varepsilon(t, \cdot)|^2 + 2 \int_0^t \int_{\mathbb{R}^n} |\nabla u^\varepsilon|^2 \ dxdt \leq \int_{\mathbb{R}^n} |u_0|^2, \quad \forall t > 0, \tag{2.20}$$

uniformly in ε. This follows simply by taking the scalar product of equation (2.19) with u and by integrating by parts[1]:

$$\begin{aligned} 0 &= \int_0^t \int_{\mathbb{R}^n} \partial_t u^\varepsilon \cdot u^\varepsilon + (u_j^\varepsilon * \rho^\varepsilon \cdot \nabla) u^\varepsilon \cdot u^\varepsilon + \nabla p \cdot u^\varepsilon - \Delta u^\varepsilon \cdot u^\varepsilon \ dxdt \\ &= \int_0^t \int_{\mathbb{R}^n} \sum_{i=1}^n (\partial_t u_i^\varepsilon) u_i^\varepsilon + \sum_{i,j=1}^n (u^\varepsilon * \rho^\varepsilon)(\partial_j u_i^\varepsilon) u_i^\varepsilon \\ &\quad + \sum_{i=1}^n (\partial_i p) u_i^\varepsilon - (\sum_{i,j=1}^n (\partial_j \partial_j u_i^\varepsilon) u_i^\varepsilon \ dxdt \\ &= \int_0^t \int_{\mathbb{R}^n} \frac{1}{2} \partial_t (\sum_{i=1}^n u_i^\varepsilon)^2 \ dxdt + \int_0^t \int_{\mathbb{R}^n} \sum_{i,j=1}^n (\partial_j u_i)^2 \ dxdt \\ &= \int_0^t \int_{\mathbb{R}^n} \frac{1}{2} |u^\varepsilon|^2 \ dxdt + \int_0^t \int_{\mathbb{R}^n} |\nabla u|^2 \ dxdt. \end{aligned}$$

Equation (2.20) allows to recover u by compactness from the sequence u^ε. The solutions u^ε are built up as a Picard sequences related to the integral formulation

---

[1] By integration by parts we get $\int_0^t \int_{\mathbb{R}^n} \sum_{i,j=1}^n (u^\varepsilon * \rho^\varepsilon)(\partial_j u_i^\varepsilon) u_i^\varepsilon$, $\int_0^t \int_{\mathbb{R}^n} \sum_{i=1}^n (\partial_i p) u_i^\varepsilon = 0$.

of ([2.19](#)), i.e.

$$\begin{cases} u^\varepsilon &= e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot ((u^\varepsilon * \rho^\varepsilon) \otimes u^\varepsilon)ds & \text{in} & \mathbb{R}^+ \times \mathbb{R}^n \\ \nabla \cdot u^\varepsilon &= 0 & \text{in} & \mathbb{R}^+ \times \mathbb{R}^n. \end{cases}$$

So u^ε is the limit of:

$$\begin{array}{rcl} u_1^\varepsilon &=& e^{t\Delta}u_0 \\ u_2^\varepsilon &=& e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot ((u_1^\varepsilon * \rho^\varepsilon) \otimes u_1^\varepsilon)(s)ds \\ u_n^\varepsilon &=& e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot ((u_{n-1}^\varepsilon * \rho^\varepsilon) \otimes u_{n-1}^\varepsilon)(s)ds. \end{array} \tag{2.21}$$

$\square$

Notice finally that

$$u \in \cap_{s<T} \left(L_t^2 L_{uloc,x}^2(0,s) \times \mathbb{R}^n \right), \quad \forall t \in \mathbb{R}^+,$$

because of ([2.18](#)); so by theorem [2.5](#) we have the integral representation:

$$u = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot (u \otimes u)ds.$$

3. Regularity criteria

As seen in theorem ([2.7](#)) weak global solutions of problem ([2.1](#)) are always available for initial data u_0 with bounded energy ($u_0 \in L_\sigma^2(\mathbb{R}^n)$). It is not known, on the other hand, if such solutions are unique, neither if they preserve the regularity of u_0. In this section we focus on partial regularity criterions. The philosophy of such a criterions is that we can infer about the uniqeness or regularity of weak solutions by the knowledge of some a priori information on the solution itself. In general a partial regularity criterion is an assertion of the following type:

Let u a Leray's solution of ([2.1](#)). Let furthermore assume some additional a priori properties (tipically these are boundedness conditions) about u, then u is the unique Leray solution of ([2.1](#)). Furthermore u is C^∞ in the space variables at each time $t > 0$.

The first result of this kind goes back to Serrin [29]. The author consider a little different setting. He works with weak solutions in open regions $\Omega \subseteq \mathbb{R}^+ \times \mathbb{R}^n$, that is a couple (u,p) u such that:

$$\int \int (\partial_t \phi + (u \cdot \nabla)\phi)u - (\nabla\phi \cdot \nabla)u = 0, \tag{2.22}$$

for each $\phi \in C_c^\infty(\Omega)$ with $\nabla \cdot \phi = 0$ and

$$\int \int u \cdot \nabla\phi = 0, \quad \int \int p\Delta\phi + \sum_{i,j=1}^n u_i u_j \partial_i \partial_j \phi = 0, \tag{2.23}$$

for each $\phi \in C_c^\infty(\Omega)$. Conditions ([2.22](#), [2.23](#)) are formally justified by taking the scalar product of the equations with the vector field ϕ and then by integrating by parts.

Remark 2.2. Let $\Omega = (0,T) \times \Omega'$ where Ω' is an open subset of $\mathbb{R}^n$. Let then Ψ a scalar function such that:

$$\Delta\Psi = 0 \quad \text{in} \quad \mathbb{R}^n,$$

and $a : (0,T) \to \mathbb{R}$ is integrable in $(0,T)$. Then it's easy to check that:

$$u(t,x) = a(t)\Psi(x)$$

is a weak solutions of ([2.22](#)).

The remark shows that regularity in space and time have to be analyzed in different ways[2]. While it is reasonable that little a priori assumptions on u are sufficient to get space regularity, stronger assumptions should be necessary in order to get time regularity. In this spirit the following holds:

Theorem 2.8 ([29], [31], [30])**.** *Let u a weak solution of the Navier-Stokes equation in the open space-time region Ω. Then define:*

$$\Omega_t = \{t\} \times \mathbb{R}^n \cap \Omega.$$

If $u \in L_t^\infty L_{\Omega_t}^2$, $\nabla u \in L_t^2 L_{\Omega_t}^2$ and furthermore:

$$\int_0^{+\infty} \|u\|_{L_x^p(\Omega_t)}^s (t) \ dt < +\infty, \quad \frac{2}{s} + \frac{n}{p} \leq 1; \qquad (2.24)$$

then $u \in C^\infty$ in the space variables at each times t such that $\Omega_t \neq 0$. Assume in addition that:

$$\partial_t u \in L_t^p L_{\Omega_t}^2, \qquad p \geq 1. \qquad (2.25)$$

Then $\partial_{x_i} u(t,x)$ are absolute continuous in time and exists a differentiable function $p(t,x)$ such that:

$$\partial_t u - \Delta u + (u \cdot \nabla)u = -\nabla p$$

almost everywhere in Ω.

Actually Serrin has proved the theorem just in the case $\frac{2}{s} + \frac{n}{p} < 1$. The endpoint case has been fixed in [30], [31]. The critical relation $\frac{2}{s} + \frac{n}{p} = 1$ follows by requiring $L_t^s L_x^p$ invariance under the scaling:

$$\lambda \to \lambda u(\lambda^2 t, \lambda x).$$

The regularity and the uniqeness of weak solutions are strictly related. A good example is the fact that under condition (2.24) in $(0,T) \times \mathbb{R}^n$ also uniqeness is easily achieved. This is again due to Serrin (see [45]).

Lemma 2.9 (J.Serrin)**.** *Let $u_0 \in L_\sigma^2(\mathbb{R}^n)$ and u a Leray solution of (2.1). If furthermore*

$$\int_0^T \|u\|_{L_x^p(\mathbb{R}^n)}^s (t) \ dt < +\infty, \quad p \in (n, +\infty], \quad \frac{2}{s} + \frac{n}{p} = 1, \qquad (2.26)$$

then u is unique in $[0,T]$.

Another fundamental regularity result is due to Caffarelli, Kohn and Nirenberg [27]. We will again state it without the proof, that's quite difficult, and can be found in the original paper or, in a simplified version, in [25]. The authors give a local regularity criterion for system (2.1). A local criterion is an assertion of the following type:

Let u a Leray solution of (2.1) and (s,y) a fixed point in the space-time $\mathbb{R}^+ \times \mathbb{R}^n$. If u satisfies some a priori boundedness condition near (in a sense to be specified) the point (s,y), then u is C^∞ in the space variables in the point (s,y).

To state the criterion in [27] we need some preliminaries:

---

[2]This is physically expectable by the incompressibility of fluid. A little change of fluid velocity localized in space causes instantly a global change of fluid velocity, so the time derivatives of the velocity are expected to be more singular.

Definition 2.10. The parabolic cylinder $Q(s, y, r)$ with top centered in (s, y) is the set:

$$Q(s, y, r) = B(y, r) \times (s, s - r^2),$$

where $B(y, r)$ is the n-dimensional ball of radius r centred in y. The set Q is important in the study of parial regularity of (2.1) because is invariant under the scaling: $(s, y) \to (\lambda^2 s, \lambda x)$.

We need also consider a different definition of weak solution than the Leray's one.

Definition 2.11 (Suitable solutions, [27])**.** Let Ω a open subset of $\mathbb{R}^+ \times \mathbb{R}^n$ and:

$$\Omega_t = \{t\} \times \mathbb{R}^n \cap \Omega.$$

The pair (u, p) is a suitable weak solution of the Navier-Stokes equation if:

(1) $p \in L^{\frac{n+2}{4}}(\Omega)$;
(2) Exist some constants E_0, E_1 such that:

$$\int_{\Omega_t} |u(t, \cdot)|^2 \, dx \le E_0, \tag{2.27}$$

for almost every t such that $\Omega_t \ne 0$, and

$$\int \int_{\Omega} |\nabla u|^2 \, dx dt \le E_1; \tag{2.28}$$

(3) (u,p) satisfy (2.1) in the sense of distributions in Ω;
(4) A generalized version of the energy inequality holds:

$$2 \int \int |\nabla u|^2 \phi \le \int \int |u|^2 (\phi_t + \Delta \phi) + (|u|^2 + 2p) u \cdot \nabla \phi \tag{2.29}$$

for each function $\phi \in C_c^\infty(\Omega)$.

The inequality (2.29) can be formally justified by taking the scalar product of the equation (2.1) against the vector field ϕu and then by integrating by parts. It is straightforward to check that Leray solutions are also suitable solutions. Furthermore definition the (2.11) is meaningful, infact:

Theorem 2.12 ([27])**.** *Let $u_0 \in L_\sigma^2(\Omega)$, with Ω an open subset of $\mathbb{R}^n$. Then there exist a pair:*

$$(u, p) : (\mathbb{R}^+ \times \Omega, \mathbb{R}^+ \times \Omega) \to (\mathbb{R}^n, \mathbb{R}),$$

such that (u, p) is a suitable solution of the Navier-Stokes equation in $\mathbb{R}^+ \times \Omega$. Furthermore u attains u_0 as initial datum in the following sense:

$$\int_\Omega u(t, x) \phi(x) \, dx \to \int_\Omega u_0(x) \phi(x) \, dx, \qquad as \quad t \to 0,$$

for each $\phi \in L_\Omega^2$.

Now we are ready to state the fundamental local regularity criterion:

Lemma 2.13 ([27])**.** *Let $n \ge 3$. Let then Ω an open subset of $\mathbb{R}^+ \times \mathbb{R}^n$ and the pair (u, p) a suitable solution of the Navier-Stokes equation in Ω. Let then (s, y) some point in Ω. There exist an absolute constant ε such that if:*

$$\limsup_{r \to 0} \frac{1}{r^{n-2}} \int \int_{Q^*(r,s,y)} |\nabla u|^2 \le \varepsilon, \tag{2.30}$$

where $Q^*(r, s, y) = Q(r, s + \frac{1}{8} r^2, y)$. Then u is regular (C^∞ in the space variables) in a neighborhood of (s, y).

In the following we will use a little different formulation of the lemma, that's more convenient in order to work with weighted spaces:

Lemma 2.14 ([27],[33]). *Let $u_0 \in L^2_\sigma(\mathbb{R}^n)$ and u a Leray solution of (2.1). Let then*

$$\int_0^T \int_{\mathbb{R}^n} |x|^{2-n} |\nabla u|^2 \, dtdx < +\infty; \tag{2.31}$$

then u is regular (C^∞ in the space variables) in the segment $(0,T) \times \{0\}$.

PROOF. Let $t \in (0.T)$, then

$$\limsup_{r \to 0} \frac{1}{r^{n-2}} \int\int_{Q(r,t,0)} |\nabla u|^2(t,x) \, dxdt$$

$$= \limsup_{r \to 0} \frac{1}{r^{n-2}} \int_{t-r^2}^t \int_{B(x,r)} |\nabla u|^2(t,x) \, dxdt$$

$$\leq \limsup_{r \to 0} \int_{t-r^2}^t \int_{B(x,r)} |x|^{2-n} |\nabla u|^2(t,x) \, dxdt$$

$$\leq \limsup_{r \to 0} \int_{t-r^2}^t \int_{\mathbb{R}^n} |x|^{2-n} |\nabla u|^2(t,x) \, dxdt = 0,$$

where the continuity property of the integral (2.31) is used. □

As suggested by this version of the Caffarelli-Kohn-Nirenberg lemma, a local boundedness condition, for instance (2.30), can be replaced by imposing boundedness in a suitable weighted L^p space. We follow this point of view in the next of the thesis. A complete set of partial regularity criteria can be obtained, infact, by working with weighted norms, as shown in [33]. Before stating this results we give some notations:

Notation. Let $\alpha \in \mathbb{R}$, $p, s \in [1, +\infty)$. Let then $f : \mathbb{R}^n \to \mathbb{R}$, $F : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$. We will say:

- $f \in L^p_{|x|^{\alpha p} dx}$ if

$$\left(\int_{\mathbb{R}^n} |x|^{\alpha p} |f(x)|^p \, dx \right)^{\frac{1}{p}} < +\infty,$$

and we denote this norm with $\| \cdot \|_{L^p_{|x|^{\alpha p} dx}}$, or with $\| |x|^\alpha \cdot \|_{L^p_x}$;

- $f \in L^\infty_{|x|^\alpha dx}$ if

$$\sup_{x \in \mathbb{R}^n} |x|^\alpha |f(x)| < +\infty,$$

and we denote this norm with $\| \cdot \|_{L^\infty_{|x|^\alpha dx}}$, or with $\| |x|^\alpha \cdot \|_{L^\infty_x}$;

- $F \in L^s_t L^p_{|x|^{\alpha p} dx}$ if

$$\left(\int_{\mathbb{R}^+} \left| \int_{\mathbb{R}^n} |x|^{\alpha p} |F(t,x)|^p \, dx \right|^{\frac{s}{p}} dt \right)^{\frac{1}{s}} < +\infty,$$

and we denote this norm with $\| \cdot \|_{L^s_t L^p_{|x|^{\alpha p} dx}}$, or with $\| |x|^\alpha \cdot \|_{L^s_t L^p_x}$;

- $F \in L^s_T L^p_{|x|^{\alpha p} dx}$ if

$$\left(\int_0^T \left| \int_{\mathbb{R}^n} |x|^{\alpha p} |F(t,x)|^p \, dx \right|^{\frac{s}{p}} dt \right)^{\frac{1}{s}} < +\infty,$$

and we denote this norm with $\| \cdot \|_{L^s_T L^p_{|x|^{\alpha p} dx}}$, or with $\| |x|^\alpha \cdot \|_{L^s_T L^p_x}$.

We give similar definitions for

$$L^\infty_t L^p_{|x|^{\alpha p} dx}, \quad L^\infty_T L^p_{|x|^{\alpha p} dx}, \quad L^s_t L^\infty_{|x|^\alpha dx}, \quad L^s_T L^\infty_{|x|^\alpha dx}.$$

As told partial regularity criterions in weighted Lebesgue spaces hold:

Theorem 2.15 ([**33**]). *Let $n \geq 3$ and u_0 a divergence free vector field such that $u_0 \in H^2(\mathbb{R}^n)$ and:*

$$\||x|^{1-\frac{n}{2}} u_0\|_{L^2_x} < +\infty. \tag{2.32}$$

If u is a weak Leray's solution of (2.1) such that:

$$\||x|^\alpha u\|_{L^s_T L^p_x} < +\infty, \tag{2.33}$$

with

$$\frac{2}{s} + \frac{n}{p} = 1 - \alpha, \quad \frac{2}{1-\alpha} < s < +\infty, \quad \frac{n}{1-\alpha} < p < +\infty, \quad -1 \leq \alpha < +1; \tag{2.34}$$

or

$$\||x|^\alpha u\|_{L^{2/(1-\alpha)}_T L^\infty_x} < +\infty, \quad -1 < \alpha < +1; \tag{2.35}$$

or

$$\sup_{t \in (0,T)} \||x|^\alpha u\|_{L^{n/(1-\alpha)}_x} = \varepsilon, \quad -1 \leq \alpha \leq +1; \tag{2.36}$$

with ε sufficiently small; then actually u is regular (C^∞ in space variables) in the segment $(0,T) \times \{0\}$.

Of course regularity in $(0,T) \times \bar{x}$ is achieved if the weights and the norms are centered in $\bar{x}$ instead of in the origin. This is infact a slightly different formulation of the theorem. In the original one the author gets global regularity by requiring

$$\sup_{\bar{x} \in \mathbb{R}^n} \||x - \bar{x}|^{1-\frac{n}{2}} u_0\|_{L^2_x} < +\infty,$$

and

$$\sup_{\bar{x} \in Rn} \||x - \bar{x}|^\alpha u\|_{L^s_T L^p_x} < +\infty.$$

Such a formulation is more useful for our purposes. This is a local regularity criterion. The weight $|x|^\alpha$ localizes the norms near to the origin and provides regularity just in the points $(0,T) \times \{0\}$. We will show how it is actually the case just for negative values of α. If $\alpha \geq 0$ then global regularity can be achieved. We will show that it is also the case if, by taking $\alpha < 0$, we assume a suitable amount of angular integrability of the solution.

Remark 2.3. Notice that:

(1) The first equation in (2.34) is the critical relation coming out by requiring $L^s_t L^p_{|x|^{\alpha p} dx}$ invariance under the scaling:

$$u^\lambda : u(t,x) \to \lambda u(\lambda^2 t, \lambda x);$$

(2) The estimates (2.35), (2.36) are the endpoint version of (2.33), obtained by setting (s,p) equal to $(2/(1-\alpha), \infty)$ and $(\infty, 3/1-\alpha)$ respectively. These are consistent with the scaling relation (2.34);

(3) Condition (2.36) in the case $\alpha = 1$ becomes a smallness condition on the norm $\||x|^\alpha u\|_{L^\infty_T L^\infty_x}$, which implies that the possible behaviour of the strong solution can be $|x|^{-1+\varepsilon}$ at the neighbourhood of the origin for small ε. This recovers one of the main results proved in [**27**], [**37**] for suitable weak solutions;

(4) Of course can be set $T = +\infty$ to get regularity for all times;

(5) The range of values for α does not depend on the dimension.

Then an analogous of theorem (2.15) holds by assuming a priori boundedness of ∇u in weighted Lebesgue spaces:

Theorem 2.16 ([**33**]). *Let $n \geq 3$ and u_0 a divergence free vector field such that $u_0 \in H^2(\mathbb{R}^n)$ and:*

$$\||x|^{1-\frac{n}{2}} u_0\|_{L_x^2} < +\infty. \tag{2.37}$$

If u is a weak Leray solution of (2.1) such that:

$$\||x|^\alpha \nabla u\|_{L_T^s L_x^p} < +\infty, \tag{2.38}$$

with

$$\frac{2}{s} + \frac{n}{p} = 2 - \alpha, \quad 1 < s < +\infty, \quad \frac{n}{2-\alpha} < p < +\infty, \quad -1 \leq \alpha < +2; \tag{2.39}$$

or

$$\||x|^\alpha \nabla u\|_{L_T^{2/(2-\alpha)} L_x^\infty} < +\infty, \quad 0 < \alpha \leq +2; \tag{2.40}$$

or

$$\sup_{t \in (0,T)} \||x|^\alpha \nabla u\|_{L_x^{n/(2-\alpha)}} = \varepsilon, \quad -1 \leq \alpha < +2; \tag{2.41}$$

with ε sufficiently small; then actually u is regular (C^∞ in space variables) on the segment $(0,T) \times \{0\}$.

We give again similar remarks.

Remark 2.4. Notice that:

(1) The first equation in (2.39) is assumed again to ensure invariance with respect to the scaling:

$$u^\lambda : u(t,x) \to \lambda u(\lambda^2 t, \lambda x),$$

infact:

$$\begin{aligned}
\||x|^\alpha \nabla u^\lambda\|_{L_t^s L_x^p} &= \||x|^\alpha \lambda^2 (\nabla u)(\lambda^2 t, \lambda x)\|_{L_t^s L_x^p} \\
&= \lambda^{2-\alpha-\frac{2}{s}-\frac{n}{p}} \||x|^\alpha u\|_{L_t^s L_x^p};
\end{aligned}$$

(2) Conditions $1 < s$, $0 < \alpha$ in (2.39),(2.40) seems to be artificial with respect to the more natural $\frac{2}{2-\alpha} \leq s$, $-1 \leq \alpha$, but they are necessay in order to work with $L_t^s L_x^p$ spaces with $s,p \geq 1$;
(3) By setting $\alpha = \frac{1}{2}$, $s = p = 2$ a similar result to lemma (2.13) is achieved;
(4) A global regularity result is again achieved by setting $T = +\infty$.
(5) The range of values for α does not depend again on the dimension.

4. Well posedness with small data

A different approach to the existence and uniqeness of the solutions of (2.1) consists in considering small initial data. Such a big restriction provides complete well posedness, i.e. regularity, uniqeness and decay of solutions. Solutions with small initial data have been deeply investigated since by [**42**], and in [**35**] the sharp case seems to be covered. The key point of the small data theory is that the nonlinear term $(u \cdot \nabla)u$ is negligible with respect to the others, so the equation (2.1) can be interpreted as a perturbed heat equation. This point of view suggests to perform a fixed point algorithm around the Heat propagator. Let's start by the Duhamel's representation:

$$\begin{cases} u &= e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes u)(s) ds \quad \text{in} \quad [0,T] \times \mathbb{R}^n \\ \nabla \cdot u &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{in} \quad [0,T] \times \mathbb{R}^n. \end{cases} \tag{2.42}$$

Where $\mathbb{P}$ is the Leray projection as defined in (2.7). So the solution is the sum of the linear propagator $e^{t\Delta}$ and a bilinear term $B(u,u) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes u)(s) ds$.

Then the Picard iteration can be performed:

$$
\begin{array}{rcl}
u_1 &=& e^{t\Delta}u_0 \\
u_2 &=& e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u_1\otimes u_1)(s)ds \\
u_n &=& e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u_{n-1}\otimes u_{n-1})(s)ds.
\end{array}
\tag{2.43}
$$

It is easy to show the following:

Theorem 2.17 ([**45**])**.** *Let X_T a Banach space of functions defined on $[0,T]\times\mathbb{R}^n$ such that the bilinear form:*

$$
B(u,v) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes v)(s)ds
$$

is bounded by $X_T\times X_T$ to X_T. Let then $X_0\subset\mathbb{S}'(\mathbb{R}^n)$ such that

$$
\|e^{t\Delta}f\|_{X_T} \leq C_{X_0,X_T}\|f\|_{X_0} \qquad \forall t\in(0,T].
$$

Under this assumptions u_n is a Cauchy sequence that converges to a solution of the integral problem (2.42).

Usually (see again [**45**]) X_T is called an admissible path space, while X_0 is called an adapted space. Furthermore under suitable smallness assumption on $\|u_0\|_{X_0}$ in theorem 2.17 can be set $T = +\infty$ to get a global existence result. Many adapted spaces have been considered in literature since by [**42**] where $u_0 \in L^n$. This result have been generalized to the homogeneous Sobolev space $\dot{H}^{\frac{n}{p}-1}$ in [**43**, **44**], to Morrey spaces in [**40**], to Besov spaces in [**38**] and of course a lot of alternative references are possible. The biggest space in which Picard iteration is possible seems to be BMO^{-1}, see [**35**]. For instance the following continuous embeddings of adapted spaces holds:

$$
\dot{H}^{\frac{n}{p}-1} \subset L^n \subset \dot{B}_{p,p<\infty,\infty}^{-1+\frac{n}{p}} \subset BMO^{-1}.
$$

Even if, as mentioned, several choices are possible, we will work basically with weighted L^p spaces.

Remark 2.5. Global well posedness with small data forces the adapted space X_0 to be invariant under the scaling $\lambda\to u_0^\lambda = \lambda u_0(\lambda x)$, i.e

$$
\|u_0^\lambda\|_{X_0} = \|u_0\|_{X_0}, \qquad \forall\lambda\in\mathbb{R}^+.
$$

This easily follows by the similarity propety of equation (2.1). For instance, if we restrict to the L^p case the right-scaling adapted space is $L^n(\mathbb{R}^n)$. Let infact $u_0\in L^p(\mathbb{R}^n)$ with $p>n$. Of course scaling

$$
\begin{array}{rcl}
u^\lambda(t,x) &=& \lambda u(\lambda^2 t,\lambda x) \qquad \lambda>0 \\
u_0^\lambda(x) &=& \lambda u_0(\lambda x)
\end{array}
\tag{2.44}
$$

leads to a one-parameter family of solutions of (2.1). If furthermore a global well posedness result with small data is achieved, by setting $\lambda\to+\infty$ it follows well posedness with arbitrarily large initial data. The same is in the case $p<n$, by setting $\lambda\to 0$.

The first result in small data teory goes back to [**42**]:

Theorem 2.18. *Let u_0 a divergence free vector field on $\mathbb{R}^n$. Exists $\varepsilon>0$ such that if*

$$
\|u_0\|_{L^n(\mathbb{R}^n)} < \varepsilon,
$$

then there is a unique solution $u:\mathbb{R}^+\times\mathbb{R}^n\to\mathbb{R}^n$ of the integral problem (2.6). Furthermore u has the decay:

$$
\|u(t,\cdot)\|_{L^q(\mathbb{R}^n)} \leq \frac{c_q}{t^{(1-\frac{n}{q})/2}}\|u_0\|_{L^n(\mathbb{R}^n)}, \qquad t>0;
\tag{2.45}
$$

$$\|\partial u(t, \cdot)\|_{L^q(\mathbb{R}^n)} \leq \frac{c_q}{t^{(2-\frac{n}{q})/2}} \|u_0\|_{L^n(\mathbb{R}^n)}, \qquad t > 0 \tag{2.46}$$

for all $q \in [n, +\infty]$. Solutions u also obey to the bound:

$$\|u\|_{L_t^r L_x^q} < +\infty, \quad \frac{2}{r} + \frac{n}{q} = 1, \quad n \leq q \leq \frac{n^2}{n-2}. \tag{2.47}$$

This is of course the simplest small data result achievable. As we mentioned the optimal case seems to be covered by Koch and Tataru in [35]. To state their result we need some more definitions:

Definition 2.19 $(BMO(\mathbb{R}^n))$**.** A tempered distribution u belongs to the space $BMO(\mathbb{R}^n)$ if

$$\left(\sup_{x, R>0} \frac{2}{|B(x,R)|} \int_0^R \int_{B(x,R)} t|\nabla(e^{t\Delta} u)|^2(t,y) \, dydt \right)^{\frac{1}{2}} < \infty. \tag{2.48}$$

This is the Carleson charaterization of $BMO(\mathbb{R}^n)$. Other equivalent definitions are in [20]. The square root in (2.48) is taken because in such a way th quantity is a seminorm[3].

Remark 2.6. Let w the solution of the heat equation

$$\begin{cases} \partial_t w - \Delta w &= 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^n \\ w &= v \quad \text{in} \quad \{0\} \times \mathbb{R}^n, \end{cases}$$

of course

$$\|v\|_{BMO(\mathbb{R}^n)} = \left(\sup_{x, R>0} \frac{1}{|B(x,R)|} \int_0^{R^2} \int_{B(x,R)} |\nabla w|^2(t,y) \, dydt \right)^{\frac{1}{2}},$$

where we used the symbol $\|\cdot\|_{BMO(\mathbb{R}^n)}$ even if the quantity is a seminorm.

Then the space $BMO^{-1}(\mathbb{R}^n)$ is defined by:

Definition 2.20 $(BMO^{-1}(\mathbb{R}^n))$**.** A tempered disrtibution v belongs to $BMO^{-1}(\mathbb{R}^n)$ if:

$$\|v\|_{BMO^{-1}(\mathbb{R}^n)} = \left(\sup_{x, R>0} \frac{1}{|B(x,R)|} \int_0^{R^2} \int_{B(x,R)} |e^{t\Delta} v|^2(t,y) \, dydt \right)^{\frac{1}{2}} < \infty. \tag{2.49}$$

In such a case $\|\cdot\|_{BMO^{-1}(\mathbb{R}^n)}$ is actually a norm.

It is easy to show that if v is a vector field such that $v_i \in BMO(\mathbb{R}^n)$ for each i, then $\nabla \cdot v \in BMO^{-1}(\mathbb{R}^n)$. The converse is also true, as stated by the following theorem that establishes the precise relationship between the two spaces:

Theorem 2.21 ([45])**.** *Let u a tempred distribution. Then $u \in BMO^{-1}(\mathbb{R}^n)$ if and only if there exist $v_i \in BMO(\mathbb{R}^n)$ such that:*

$$u = \sum_{i=1}^n \partial_i v_i.$$

We also define the parabolic cylinder[4] centred in x and with radius R:

$$Q(x,R) = B(x,R) \times (0, R^2), \tag{2.50}$$

and introduce the adapted space X by:

---

[3]Of course (2.48) of a constant function is zero.

[4]This is a central object in the study of regulariry properties of (2.1) because it obeys to the scaling: $(t,x) \to (\lambda^2 t, \lambda x)$; we will came back later on this topics.

Definition 2.22 ([35]).

$$\|u\|_X = \sup_{t>0} t^{\frac{1}{2}} \|u(t)\|_{L^\infty(\mathbb{R}^n)} + \left(\sup_{x,R>0} \frac{1}{|B(x,R)|} \int_{Q(x,R)} |u|^2(t,y) \ dydt \right)^{\frac{1}{2}}. \quad (2.51)$$

We are ready to state the theorem:

Theorem 2.23 ([35]). *Let u_0 a divergence free vector field on $\mathbb{R}^n$. Exists $\varepsilon > 0$ such that if*

$$\|u_0\|_{BMO^{-1}(\mathbb{R}^n)} < \varepsilon,$$

then there is a unique solution $u : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ of the integral problem (2.6). Furthermore $u \in X$.

This result works with a translation invariant adapted spaces, and, as mentioned, there is a wide literature on the topics. On the other hand even if X_0 is not translation invariant local regularity results are still available, but this cases have not been so deeply investigated. In the following we focus in particular on weighted spaces with power weights of the kind $|x|^\alpha$. So the translation invariance is broken but we still have to require invariance with respect to the scaling centered in the origin

$$\lambda \to \lambda u_0(\lambda x);$$

In this way the most simple spaces available are endowed with the norms

$$\| |x|^\alpha \cdot \|_{L^p(\mathbb{R}^n)}, \qquad \text{with} \quad \alpha = 1 - \frac{n}{p}.$$

In the case $p = 2$ a very interesting result has been obtained in [27]

Theorem 2.24 (Caffarelli-Kohn-Nirenberg). *Let $u_0 \in L^2_\sigma(\mathbb{R}^n)$ and u a suitable weak solution of (2.1). There exists an absolute constant $\varepsilon_0 > 0$ such that if*

$$\| |x|^{1-n/2} u_0 \|_{L^2(\mathbb{R}^n)} = \varepsilon < \varepsilon_0,$$

then u is regular (C^∞ in space variables) in the interior of the parabola

$$\Pi = \left\{ (t,x) \quad t.c. \quad t > \frac{|x|^2}{\varepsilon_0 - \varepsilon} \right\}.$$

Remark 2.7. Of course by translations the analogous result holds if one consider small data in the norm $\| |x - \bar{x}|^\alpha \cdot \|_{L^p(\mathbb{R}^n)}$, for a fixed $\bar{x} \in \mathbb{R}^n$.

We will come back on this result in the next chapter and we will show how it depends on the amount of angular integrability of u_0. By applying the technology developed in the first chapter we suggest how to quantify precisely the gain of regularity that angular integrability provides.

CHAPTER 3

Results in weighted setting with angular integrability

We apply the technology developed in Chapter 1 to the study of the well posedness and regularity of (2.1). We closely look at these problems in the setting of weighted L^p spaces with different integrability in radial and angular directions. More precisely we consider well posedness of (2.1) with $u_0 \in L^p_{|x|^{\alpha p}d|x|}L^{\widetilde{p}}_\theta$ that's defined by

$$
L^p_{|x|^{\alpha p}d|x|}L^{\widetilde{p}}_\theta = \left\{ f \in L^1_{loc}(\mathbb{R}^n) \quad \text{s.t.} \\ \left(\int_0^{+\infty} \|f(\rho \, \cdot \,)\|^p_{L^{\widetilde{p}}(\mathbb{S}^{n-1})} \rho^{\alpha p+n-1}d\rho \right)^{\frac{1}{p}} < +\infty \right\};
\tag{3.1}
$$

it is immediate to show that the above quantity is a norm and we use the notation

$$
\| \cdot \|_{L^p_{|x|^{\alpha p}d|x|}L^{\widetilde{p}}_\theta}, \quad \text{or} \quad \||x|^\alpha \cdot \|_{L^p_{|x|}L^{\widetilde{p}}_\theta}.
$$

This space is not translation invariant, as the classical adapted spaces are, but as shown in theorem 2.24 local regularity results are still achieveble in this setting. Then we are interested in regularity criteria in the context of spaces:

$$
L^s_T L^p_{|x|^{\alpha p}d|x|}L^{\widetilde{p}}_\theta = \left\{ u \in L^1_{loc}((0,T) \times \mathbb{R}^n) \quad \text{s.t.} \\ \left(\int_0^T \left(\int_0^{+\infty} \|u(t,\rho \, \cdot \,)\|^p_{L^{\widetilde{p}}(\mathbb{S}^{n-1})} \rho^{\alpha p+n-1}d\rho \right)^{\frac{s}{p}} dt \right)^{\frac{1}{s}} < +\infty \right\},
\tag{3.2}
$$

and again we denote this norms with

$$
\| \cdot \|_{L^s_T L^p_{|x|^{\alpha p}d|x|}L^{\widetilde{p}}_\theta}, \quad \text{or} \quad \||x|^\alpha \cdot \|_{L^p_{|x|}L^{\widetilde{p}}_\theta}.
$$

We use also $\|\cdot\|_{L^s_t L^p_{|x|^{\alpha p}d|x|}L^{\widetilde{p}}_\theta}$ and $\||x|^\alpha \cdot\|_{L^s_t L^p_{|x|}L^{\widetilde{p}}_\theta}$ if $T = +\infty$. Of course if we restrict to radially symmetric functions $u_0 = u_0(|x|)$ (or $u = u(t,|x|)$) the norms reduce to the classical ones. Now it is well known that the problem (2.1) is very much simpler in a symmetric setting, and stronger results are achievable. The idea is try to recover some of this improvements in the case of initial data or solutions with merely higher angular integrability. The functional spaces well suited to this purpose are indeed (3.1), (3.2). We find encouraging results by considering $L^p_{|x|^{\alpha p}d|x|}L^{\widetilde{p}}_\theta$ with $\alpha < 0$ and large value of $\widetilde{p}$. The idea is that while the weight $|x|^\alpha$ localizes the norm near to the origin, the radial $L^{\widetilde{p}}$ integrability, with large $\widetilde{p}$, provides a bound for large $|x|$. This heuristic will be more precisely formulated later, and will be useful to interpret all the following results. Of course by translations can be considered power weights centred in some $\bar{x} \neq 0$. In this case all the norms have to be translated in $\bar{x}$. We give some precise definitions about the spaces we are going to use.

Notation. Let $\alpha \in \mathbb{R}$, $p, s \in [1, +\infty)$. Let then $f : \mathbb{R}^n \to \mathbb{R}$, $F : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$. We will say:

- $f \in L^p_{|x|^{\alpha p} d|x|} L^{\widetilde{p}}_\theta$ if

$$\Big(\int_{\mathbb{R}^+} \|f(\rho \cdot\,)\|^p_{L^{\widetilde{p}}_{(\mathbb{S}^{n-1})}} \rho^{\alpha p + n - 1} \, d\rho \Big)^{\frac{1}{p}} < +\infty,$$

and we denote this norm with $\|\cdot\|_{L^p_{|x|^{\alpha p} d|x|} L^{\widetilde{p}}_\theta}$, or $\||x|^\alpha \cdot\|_{L^p_{|x|} L^{\widetilde{p}}_\theta}$;

- $f \in L^\infty_{|x|^\alpha d|x|} L^{\widetilde{p}}_\theta$ if

$$\sup_{\rho > 0} \rho^\alpha \|f(\rho \cdot\,)\|_{L^{\widetilde{p}}(\mathbb{S}^{n-1})} < +\infty,$$

and we denote this norm with $\|\cdot\|_{L^\infty_{|x|^\alpha d|x|} L^{\widetilde{p}}_\theta}$, or $\||x|^\alpha \cdot\|_{L^\infty_{|x|} L^{\widetilde{p}}_\theta}$;

- $F \in L^s_t L^p_{|x|^{\alpha p} d|x|} L^{\widetilde{p}}_\theta$ if

$$\Big(\int_{\mathbb{R}^+} \Big| \int_{\mathbb{R}^+} \|F(t, \rho \cdot\,)\|^p_{L^{\widetilde{p}}(\mathbb{S}^{n-1})} \rho^{\alpha p + n - 1} \, d\rho \Big|^{\frac{s}{p}} dt \Big)^{\frac{1}{s}} < +\infty,$$

and we denote this norm with $\|\cdot\|_{L^s_t L^p_{|x|^{\alpha p} d|x|} L^{\widetilde{p}}_\theta}$, or $\||x|^\alpha \cdot\|_{L^s_t L^p_{|x|} L^{\widetilde{p}}_\theta}$;

- $F \in L^s_T L^p_{|x|^{\alpha p} d|x|} L^{\widetilde{p}}_\theta$ if

$$\Big(\int_0^T \Big| \int_{\mathbb{R}^+} \|F(t, \rho \cdot\,)\| \rho^{\alpha p + n - 1} \, d\rho \Big|^{\frac{s}{p}} dt \Big)^{\frac{1}{s}} < +\infty,$$

and we denote this norm with $\|\cdot\|_{L^s_T L^p_{|x|^{\alpha p} d|x|} L^{\widetilde{p}}_\theta}$, or $\||x|^\alpha \cdot\|_{L^s_T L^p_{|x|} L^{\widetilde{p}}_\theta}$.

We give similar definitions for

$$L^\infty_t L^p_{|x|^{\alpha p} d|x|} L^{\widetilde{p}}_\theta, \quad L^\infty_T L^p_{|x|^{\alpha p} d|x|} L^{\widetilde{p}}_\theta, \quad L^s_t L^\infty_{|x|^\alpha d|x|} L^{\widetilde{p}}_\theta, \quad L^s_T L^\infty_{|x|^\alpha d|x|} L^{\widetilde{p}}_\theta.$$

1. Decay estimates for convolutions with heat and Oseen kernels

The most important technical tools we need are weighted decay estimates for convolutions with the heat and Oseen kernels. Even if results of this kind already exist in literature we cover a larger set of weights by considering $L^p_{|x|} L^{\widetilde{p}}_\theta$ spaces. In particular we show that infact the higher angular integrability allows to consider a larger set of weights. Corollary (1.4) is the main ingredient in the proofs.

To give a more compact notation it's convenient to define the quantities:

$$\Lambda(\alpha, p, \widetilde{p}) = \alpha + \frac{n-1}{p} - \frac{n-1}{\widetilde{p}} \tag{3.3}$$

$$\Omega(\alpha, p, s) = \alpha + \frac{n}{p} + \frac{2}{s}. \tag{3.4}$$

We will use also the notation Λ_a instead of $\Lambda(\alpha, p, \widetilde{p})$, when the values of $p, \widetilde{p}$ will be clear by the context. Let's start by the punctual decay:

Proposition 3.1. *Let $n \geq 2$, $1 < p \leq q < +\infty$ and $1 \leq \widetilde{p} \leq \widetilde{q} \leq +\infty$. Assume further that α, β satisfy the set of conditions*

$$\beta > -\frac{n}{q}, \qquad \alpha < \frac{n}{p'}, \qquad \Lambda(\alpha, p, \widetilde{p}) \geq \Lambda(\beta, q, \widetilde{q}). \tag{3.5}$$

Then the following estimates hold:

(1)

$$\||x|^\beta \partial^\eta e^{t\Delta} u_0\|_{L^q_{|x|} L^{\widetilde{q}}_\theta} \leq \frac{c_\eta}{t^{(|\eta| + \frac{n}{p} - \frac{n}{q} + \alpha - \beta)/2}} \||x|^\alpha u_0\|_{L^p_{|x|} L^{\widetilde{p}}_\theta}, \qquad t > 0, \tag{3.6}$$

provided that $|\eta| + \frac{n}{p} - \frac{n}{q} + \alpha - \beta \geq 0$,

(2)

$$\||x|^\beta \partial^\eta e^{t\Delta}\mathbb{P}\nabla \cdot F\|_{L_x^q L_\theta^{\tilde{q}}} \leq \frac{d_\eta}{t^{(1+|\eta|+\frac{n}{p}-\frac{n}{q}+\alpha-\beta)/2}}\||x|^\alpha F\|_{L_{|x|}^p L_\theta^{\tilde{p}}}, \qquad t > 0, \quad (3.7)$$

$$\text{provided that } 1 + |\eta| + \tfrac{n}{p} - \tfrac{n}{q} + \alpha - \beta > 0.$$

for each multi index η, so in particular:

(1)

$$\||x|^\beta e^{t\Delta}u_0\|_{L_{|x|}^q L_\theta^{\tilde{q}}} \leq \frac{c_0}{t^{(\frac{n}{p}-\frac{n}{q}+\alpha-\beta)/2}}\||x|^\alpha u_0\|_{L_{|x|}^p L_\theta^{\tilde{p}}}, \qquad t > 0, \qquad (3.8)$$

$$\text{provided that } \tfrac{n}{p} - \tfrac{n}{q} + \alpha - \beta \geq 0,$$

(2)

$$\||x|^\beta e^{t\Delta}\mathbb{P}\nabla \cdot F\|_{L_x^q L_\theta^{\tilde{q}}} \leq \frac{d_0}{t^{(1+\frac{n}{p}-\frac{n}{q}+\alpha-\beta)/2}}\||x|^\alpha F\|_{L_{|x|}^p L_\theta^{\tilde{p}}}, \qquad t > 0, \qquad (3.9)$$

provided that $1 + \frac{n}{p} - \frac{n}{q} + \alpha - \beta > 0$. The range of admissible p, q indices can be relaxed to $1 \leq p \leq q \leq +\infty$ provided that $\Lambda(\alpha, p, \tilde{p}) > \Lambda(\beta, q, \tilde{q})$.

PROOF. The proof follows by proposition (1.4) and scaling considerations. At first notice:

$$e^{t\Delta}\phi = S_{\sqrt{t}}e^{\Delta}S_{1/\sqrt{t}}\phi, \qquad (3.10)$$

where S_λ is defined by:

$$(S_\lambda\phi)(x) = \phi\left(\frac{x}{\lambda}\right). \qquad (3.11)$$

Then also notice:

$$\|\partial^\eta|x|^\beta S_\lambda\phi\|_{L_{|x|}^q L_\theta^{\tilde{q}}} = \lambda^{\frac{n}{q}+\beta-|\eta|}\||x|^\beta\phi\|_{L_{|x|}^q L_\theta^{\tilde{q}}}. \qquad (3.12)$$

For each $u_0 \in L_{|x|^{\alpha p}d|x|}^p L_\theta^{\tilde{p}}$ and $t \in \mathbb{R}^+$ the function $e^{\Delta}S_{1/\sqrt{t}}$ is in the Schwartz class, so we can apply proposition (1.4) with arbitrarily high values of λ and condition (1.8) is trivially satisfied. We get:

$$
\begin{aligned}
\||x|^\beta\partial^\eta e^{t\Delta}u_0\|_{L_{|x|}^q L_\theta^{\tilde{q}}} &= \||x|^\beta\partial^\eta S_{\sqrt{t}}e^{\Delta}S_{1/\sqrt{t}}u_0\|_{L_{|x|}^q L_\theta^{\tilde{q}}} \\
&= t^{(\frac{n}{q}+\beta-|\eta|)/2}\||x|^\beta(\partial^\eta e^{\Delta})S_{1/\sqrt{t}}u_0\|_{L_{|x|}^q L_\theta^{\tilde{q}}} \\
&\leq \frac{c_\eta}{t^{(-\frac{n}{q}-\beta+|\eta|)/2}}\||x|^\alpha S_{1/\sqrt{t}}u_0\|_{L_{|x|}^p L_\theta^{\tilde{p}}} \\
&= \frac{c_\eta}{t^{(|\eta|+\frac{n}{p}-\frac{n}{q}+\alpha-\beta)/2}}\||x|^\alpha u_0\|_{L_{|x|}^p L_\theta^{\tilde{p}}},
\end{aligned}
$$

where we used the condition (1.7), i.e.

$$\Lambda(\alpha, p, \tilde{p}) \geq \Lambda(\beta, q, \tilde{q}).$$

The proof of (3.7) is similar, but we have to work with the operator $e^{t\Delta}\mathbb{P}\nabla\cdot$. As seen in proposition 2.4 it is a convolution with a kernel K such that

$$K_{j,k,m}(t) = K_{j,k,m}\left(\frac{x}{\sqrt{t}}\right), \qquad (3.13)$$

and

$$(1 + |x|)^{1+n+|\mu|}\partial^\mu K_{j,k,m} \in L^\infty(\mathbb{R}^n), \qquad (3.14)$$

for each multi index μ. By the scaling (3.13) follows:

$$K(t) * \phi = \frac{1}{\sqrt{t}}S_{\sqrt{t}}K * S_{1/\sqrt{t}}\phi. \qquad (3.15)$$

So we get

$$
\begin{aligned}
\big\||x|^\beta \partial^\eta e^{t\Delta}\mathbb{P}\nabla\cdot F\big\|_{L^q_{|x|}L^{\widetilde{q}}_\theta}
&= \big\||x|^\beta \partial^\eta K(t)*F\big\|_{L^q_{|x|}L^{\widetilde{q}}_\theta} \\
&= \frac{1}{\sqrt{t}}\big\||x|^\beta \partial^\eta S_{\sqrt{t}}K * S_{1/\sqrt{t}}F\big\|_{L^q_{|x|}L^{\widetilde{q}}_\theta} \\
&= \frac{1}{\sqrt{t}}t^{(\frac{n}{q}+\beta-|\eta|)/2}\big\||x|^\beta (\partial^\eta K) * S_{1/\sqrt{t}}F\big\|_{L^q_{|x|}L^{\widetilde{q}}_\theta} \\
&\leq \frac{d_\eta}{t^{(-\frac{n}{q}-\beta+1+|\eta|)/2}}\big\||x|^\alpha S_{1/\sqrt{t}}F\big\|_{L^p_{|x|}L^{\widetilde{p}}_\theta} \\
&= \frac{d_\eta}{t^{(1+|\eta|+\frac{n}{p}-\frac{n}{q}+\alpha-\beta)/2}}\big\||x|^\alpha F\big\|_{L^p_{|x|}L^{\widetilde{p}}_\theta},
\end{aligned}
$$

provided that $\Lambda_\alpha \geq \Lambda_\beta$. Notice that the optimal choice of γ allowed by (3.14) is $\gamma = 1 + n + |\eta|$ and leads to

$$
\alpha + \beta + 1 + n + |\eta| > n\Big(1 + \frac{1}{q} - \frac{1}{p}\Big) \Rightarrow 1 + |\eta| + \frac{n}{p} - \frac{n}{q} + \alpha - \beta > 0.
$$

$\square$

It's remarkable that the restriction $\Lambda_\alpha \geq \Lambda_\beta$ can be removed if we localize the estimate in a space-time parabola above the origin. The size of the parabola will depend by the values of the difference $\Lambda_\alpha - \Lambda_\beta$ and increases as $\Lambda_\alpha - \Lambda_\beta \to 0^-$. In the limit case $\Lambda_\alpha = \Lambda_\beta$ we infact recover proposition 3.1.

Proposition 3.2. *Let $n \geq 2$, $1 < p \leq q < +\infty$ and $1 \leq \widetilde{p} \leq \widetilde{q} \leq +\infty$. Assume further that α, β satisfy the set of conditions*

$$
\beta > -\frac{n}{q}, \qquad \alpha < \frac{n}{p'}, \qquad \Lambda(\alpha,p,\widetilde{p}) < \Lambda(\beta,q,\widetilde{q}), \tag{3.16}
$$

and define:

$$
\Lambda_{\alpha,\beta} = \Lambda(\alpha,p,\widetilde{p}) - \Lambda(\beta,q,\widetilde{q}).
$$

Let then $\Pi(R)$ the space-time parabola:

$$
\Pi(R) = \Big\{ \frac{|x|}{\sqrt{t}} < R \Big\};
$$

then the following estimates hold:

    (1)

$$
\big\|\mathbb{1}_{\Pi(R)}|x|^\beta \partial^\eta e^{t\Delta}u_0\big\|_{L^q_{|x|}L^{\widetilde{q}}_\theta} \leq \frac{c_\eta R^{-\Lambda_{\alpha,\beta}}}{t^{(|\eta|+\frac{n}{p}-\frac{n}{q}+\alpha-\beta)/2}}\big\||x|^\alpha u_0\big\|_{L^p_{|x|}L^{\widetilde{p}}_\theta}, \qquad t > 0, \tag{3.17}
$$

        provided that $|\eta| + \frac{n}{p} - \frac{n}{q} + \alpha - \beta \geq 0$,

    (2)

$$
\big\|\mathbb{1}_{\Pi(R)}|x|^\beta \partial^\eta e^{t\Delta}\mathbb{P}\nabla\cdot F\big\|_{L^q_x L^{\widetilde{q}}_\theta} \leq \frac{d_\eta R^{-\Lambda_{\alpha,\beta}}}{t^{(1+|\eta|+\frac{n}{p}-\frac{n}{q}+\alpha-\beta)/2}}\big\||x|^\alpha F\big\|_{L^p_{|x|}L^{\widetilde{p}}_\theta}, \qquad t > 0, \tag{3.18}
$$

        provided that $1 + |\eta| + \frac{n}{p} - \frac{n}{q} + \alpha - \beta > 0$.

for each muti index η, so in particular:

    (1)

$$
\big\|\mathbb{1}_{\Pi(R)}|x|^\beta e^{t\Delta}u_0\big\|_{L^q_{|x|}L^{\widetilde{q}}_\theta} \leq \frac{c_0 R^{-\Lambda_{\alpha,\beta}}}{t^{(\frac{n}{p}-\frac{n}{q}+\alpha-\beta)/2}}\big\||x|^\alpha u_0\big\|_{L^p_{|x|}L^{\widetilde{p}}_\theta}, \qquad t > 0, \tag{3.19}
$$

        provided that $\frac{n}{p} - \frac{n}{q} + \alpha - \beta \geq 0$,

(2)

$$\|\mathbb{1}_{\Pi(R)}|x|^{\beta}e^{t\Delta}\mathbb{P}\nabla\cdot F\|_{L_x^q L_\theta^{\tilde{q}}} \leq \frac{d_0 R^{-\Lambda_{\alpha,\beta}}}{t^{(1+\frac{n}{p}-\frac{n}{q}+\alpha-\beta)/2}}\||x|^{\alpha}F\|_{L_{|x|}^p L_\theta^{\tilde{p}}}, \qquad t > 0, \quad (3.20)$$

provided that $1 + \frac{n}{p} - \frac{n}{q} + \alpha - \beta > 0$.

These estimates can also be differently formulated by setting $R^{-\Lambda_{\alpha,\beta}} = K$. For instance (3.17) becomes:

$$\|\mathbb{1}_{\Pi(K^{-1/\Lambda_{\alpha,\beta}})}|x|^{\beta}\partial^{\eta}e^{t\Delta}u_0\|_{L_{|x|}^q L_\theta^{\tilde{q}}} \leq \frac{c_\eta K}{t^{(|\eta|+\frac{n}{p}-\frac{n}{q}+\alpha-\beta)/2}}\||x|^{\alpha}u_0\|_{L_{|x|}^p L_\theta^{\tilde{p}}}, \qquad t > 0,$$
$$(3.21)$$

and similarly for the other estimates.

PROOF. We use simply Λ instead of $\Lambda_{\alpha,\beta}$. Of course:

$$\Lambda < 0 \quad \Rightarrow \quad R^{-\Lambda}\left|\frac{x}{\sqrt{t}}\right|^{\Lambda} \geq 1, \quad \text{if} \quad (t,x) \in \Pi(R).$$

Then we get, as in (3.1):

$$
\begin{aligned}
\|\mathbb{1}_{\Pi(R)}|x|^{\beta}\partial^{\eta}e^{t\Delta}u_0\|_{L_{|x|}^q L_\theta^{\tilde{q}}} &= \|\mathbb{1}_{\Pi(R)}|x|^{\beta}\partial^{\eta}S_{\sqrt{t}}e^{\Delta}S_{1/\sqrt{t}}u_0\|_{L_{|x|}^q L_\theta^{\tilde{q}}} \\
&\leq \frac{R^{-\Lambda}}{t^{\Lambda/2}}\||x|^{\beta+\Lambda}\partial^{\eta}S_{\sqrt{t}}e^{\Delta}S_{1/\sqrt{t}}u_0\|_{L_{|x|}^q L_\theta^{\tilde{q}}} \\
&= \frac{R^{-\Lambda}}{t^{\Lambda/2}}t^{(\frac{n}{q}+\beta+\Lambda-|\eta|)/2}\||x|^{\beta+\Lambda}e^{\Delta}S_{1/\sqrt{t}}u_0\|_{L_{|x|}^q L_\theta^{\tilde{q}}} \\
&\leq \frac{c_\eta}{t^{(-\frac{n}{q}-\beta+|\eta|)/2}}\||x|^{\alpha}S_{1/\sqrt{t}}u_0\|_{L_{|x|}^p L_\theta^{\tilde{p}}} \\
&= \frac{c_\eta}{t^{(|\eta|+\frac{n}{p}-\frac{n}{q}+\alpha-\beta)/2}}\||x|^{\alpha}u_0\|_{L_{|x|}^p L_\theta^{\tilde{p}}},
\end{aligned}
$$

where the indexes relationships are consistent because:

$$\Lambda_\alpha \geq \Lambda(\Lambda_{\alpha,\beta}+\beta, p, \tilde{p}) = \Lambda(\Lambda_\alpha - \Lambda_\beta + \beta, p, \tilde{p}) = \Lambda_\alpha.$$

The other inequalities can be proved in the same way. $\qquad\square$

Remark 3.1. We precisely observed that the inequalities hold with an additional factor $R^{-\Lambda}$ after localization in a space-time parabola. Notice that this factor goes to 1 as $\Lambda \to 0^-$. To get a uniformly in Λ constant, it is instead necessary to restrict the size of the parabola: if we chose the constant equal to K, we need to restrict to:

$$\left|\frac{x}{\sqrt{t}}\right| \leq K^{-\frac{1}{\Lambda}}.$$

Here as $\Lambda \to 0^-$ the parabola fills the whole space-time.

By the punctual decay integral estimates for the heat equation and for the Duhamel's term (2.6) can be easily obtained:

Proposition 3.3. Let $n \geq 2$, $1 < p \leq q < \frac{np}{(|\eta|+\alpha-\beta)p+n-2}$, $r \in (1,+\infty)$ and $1 \leq \tilde{p} \leq \tilde{q} \leq +\infty$. Assume further that α, β satisfy the set of conditions

$$\beta > -\frac{n}{q}, \qquad \alpha < \frac{n}{p'}. \tag{3.22}$$

The following estimates hold:

$$\||x|^{\beta}\partial^{\eta}e^{t\Delta}u_0\|_{L_t^r L_{|x|}^q L_\theta^{\tilde{q}}} \leq c_\eta\||x|^{\alpha}u_0\|_{L_{|x|}^p L_\theta^{\tilde{p}}}, \qquad t > 0, \tag{3.23}$$

for each multi index η, provided that:

$$|\eta| + \Omega(\alpha, p, \infty) = \Omega(\beta, q, r), \quad \Lambda(\alpha, p, \tilde{p}) \geq \Lambda(\beta, q, \tilde{q}). \tag{3.24}$$

And the localized version

$$\|\mathbb{1}_{\Pi(R)}|x|^\beta \partial^\eta e^{t\Delta} u_0\|_{L_t^r L_{|x|}^q L_\theta^{\tilde{q}}} \leq c_\eta R^{-\Lambda_{\alpha,\beta}} \||x|^\alpha u_0\|_{L_{|x|}^p L_\theta^{\tilde{p}}}, \qquad t > 0, \qquad (3.25)$$

for each multi index η, provided that:

$$|\eta| + \Omega(\alpha, p, \infty) = \Omega(\beta, q, r), \quad \Lambda_{\alpha,\beta} = \Lambda(\alpha, p, \tilde{p}) - \Lambda(\beta, q, \tilde{q}) < 0, \qquad (3.26)$$

where we remember the definition of $\Pi(R)$:

$$\Pi(R) = \Big\{ \frac{|x|}{\sqrt{t}} < R \Big\}.$$

PROOF. By the punctual decay:

$$\||x|^\beta \partial^\eta e^{t\Delta} u_0\|_{L_{|x|}^q L_\theta^{\tilde{q}}} \leq \frac{c_\eta}{t^{(|\eta| + \frac{n}{p} - \frac{n}{q} + \alpha - \beta)/2}} \||x|^\alpha u_0\|_{L_{|x|}^p L_\theta^{\tilde{p}}}, \qquad t > 0, \qquad (3.27)$$

follows that $\partial^\eta e^{t\Delta} u_0$ is bounded in the Lorentz space $L^{r,\infty}(\mathbb{R}^+; L_{|x|^{\beta q} d|x|}^q L_\theta^{\tilde{q}})$ if $|\eta| + \Omega(\alpha, p, \infty) = \Omega(\beta, q, r)$. Infact:

$$
\begin{aligned}
\big\|\||x|^\beta \partial^\eta e^{t\Delta} u_0\|_{L_{|x|}^q L_\theta^{\tilde{q}}}\big\|_{L_t^{r,\infty}} &\leq c_\eta \left\| \frac{1}{t^{(|\eta| + \frac{n}{p} - \frac{n}{q} + \alpha - \beta)/2}} \||x|^\alpha u_0\|_{L_{|x|}^p L_\theta^{\tilde{p}}} \right\|_{L_t^{r,\infty}} \\
&\leq c_\eta \left\| \frac{1}{t^{(|\eta| + \frac{n}{p} - \frac{n}{q} + \alpha - \beta)/2}} \right\|_{L^{r,\infty}} \|u_0\|_{L_{|x|}^p L_\theta^{\tilde{p}}} \\
&\leq c_\eta \|u_0\|_{L_{|x|}^p L_\theta^{\tilde{p}}},
\end{aligned}
$$

provided that:

$$(|\eta| + \frac{n}{p} - \frac{n}{q} + \alpha - \beta)/2 = \frac{1}{r} \Rightarrow |\eta| + \Omega(\alpha, p, \infty) = \Omega(\beta, q, r).$$

Let now consider a couple $(\alpha_0, \beta_0, p_0, \tilde{p}_0, q_0, \tilde{q}_0, r_0)$, $(\alpha_1, \beta_1, p_1, \tilde{p}_1, q_1, \tilde{q}_1, r_1)$ such that the assumptions of theorem are satisfied, and in particular (3.22, 3.24) holds. Then we have the couples of linear operators:

$$
\partial^\eta e^{t\Delta} :
\begin{aligned}
L_{|x|^{\alpha_0 p_0} d|x|}^{p_0} L_\theta^{\tilde{p}_0} &\longrightarrow L_t^{r_0, \infty} L_{|x|^{\beta_0 q_0} d|x|}^{q_0} L_\theta^{\tilde{q}_0} \\
\\
L_{|x|^{\alpha_1 p_1} d|x|}^{p_1} L_\theta^{\tilde{p}_1} &\longrightarrow L_t^{r_1, \infty} L_{|x|^{\beta_1 q_1} d|x|}^{q_1} L_\theta^{\tilde{q}_1}.
\end{aligned}
\qquad (3.28)
$$

if we fix $\xi \in (0, 1)$ we can perform real interpolation of operators with parameters (ξ, r), provided that:

$$p_\xi < r_\xi, \qquad (3.29)$$

where

$$\frac{1}{p_\xi} = (1 - \xi) \frac{1}{p_0} + \frac{1}{\xi p_1},$$

in the same way are defined $q_\xi, r_\xi, \tilde{q}_\xi, \tilde{p}_\xi$, while

$$\alpha_\xi = (1 - \xi) \alpha_0 + \xi \alpha_1,$$

and the same for β_ξ. So we get the bounded operators:

$$\partial^\eta e^{t\Delta} u_0 :$$

$$L_{|x|^{\alpha_\xi p_\xi} d|x|}^{p_\xi} L_\theta^{\tilde{p}_\xi} \to \Big(L_t^{r_0, \infty} L_{|x|^{\beta_0 q_0} d|x|}^{q_0} L_\theta^{\tilde{q}_0}, L_t^{r_1, \infty} L_{|x|^{\beta_1 q_1} d|x|}^{q_1} L_\theta^{\tilde{q}_1} \Big)_{\xi, r} = L_t^r L_{|x|^{\beta_\xi q_\xi} d|x|}^{q_\xi} L_\theta^{\tilde{q}_\xi}.$$

It is now straightforward to check that indeces $(\alpha_\xi, \beta_\xi, \text{etc})$ satisfy the relations (3.22, 3.24) and other assumptions. Furthetmore constrains (3.24) and (3.29) are equivalent to $q_\xi < \frac{n p_\xi}{(|\eta| + \alpha_\xi - \beta_\xi) p_\xi + n - 2}$. Of course this method misses the endpoints

$p, q = 1$. The estimates (3.25) are proved in the same way by using the localized punctual decay.

$\square$

The estimates of the Duhamel term needs no interpolation notions:

Proposition 3.4. *Let $n \geq 2$, $1 < p \leq 2q < +\infty$, $1 < s \leq 2r < +\infty$ and $1 \leq p \leq 2q \leq +\infty$. Assume further that α, β satisfy the set of conditions*

$$\beta > -\frac{n}{q}, \qquad \alpha < \frac{n}{p'}, \qquad 2\Lambda(\alpha, p, \widetilde{p}) \geq \Lambda(\beta, q, \widetilde{q}) \tag{3.30}$$

Then the following estimates holds:

$$\left\| |x|^\beta \partial^\eta \int_0^t e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (u \otimes u) \, ds \right\|_{L_t^r L_x^q L_\theta^{\widetilde{q}}} \leq d_\eta \||x|^\alpha u\|^2_{L_t^s L_{|x|}^p L_\theta^{\widetilde{p}}}, \tag{3.31}$$

for each multi index η, provided that:

$$2\Omega(\alpha, p, s) = \Omega(\beta, q, r) + 1 - |\eta|. \tag{3.32}$$

In particular holds:

$$\left\| |x|^\beta \partial^\eta \int_0^t e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (u \otimes u) \, ds \right\|_{L_t^r L_x^q L_\theta^{\widetilde{q}}} \leq d_\eta \||x|^\beta u\|^2_{L_t^r L_{|x|}^q L_\theta^{\widetilde{q}}}, \tag{3.33}$$

for each multi index η, provided that:

$$\frac{2}{r} + \frac{n}{q} = 1 - \beta - |\eta|, \qquad \Lambda(\beta, q, \widetilde{q}) \geq 0. \tag{3.34}$$

The range of admissible p, q indices can be relaxed to $1 \leq p \leq q \leq +\infty$ provided that $2\Lambda(\alpha, p, \widetilde{p}) > \Lambda(\beta, q, \widetilde{q})$.

PROOF. By Minkowsky inequality and estimates (3.6):

$$\left\| |x|^\beta \partial^\eta \int_0^t e^{(t-s)\Delta} \mathbb{P}\nabla \cdot F(x, s) \, ds \right\|_{L_t^r L_x^q L_\theta^{\widetilde{q}}}$$

$$\leq \left\| \int_{\mathbb{R}^+} \||x|^\beta \partial^\eta e^{(t-s)\Delta} \mathbb{P}\nabla \cdot F\|_{L_{|x|}^q L_\theta^{\widetilde{q}}} \, ds \right\|_{L_t^r}$$

$$\leq d_\eta \left\| \int_{\mathbb{R}^+} \frac{1}{(t-s)^{(1+|\eta|+\frac{n}{p_0}-\frac{n}{q}+\alpha_0-\beta)/2}} \||x|^{\alpha_0} F\|_{\mathbb{L}_{|x|}^{q_0} L_\theta^{\widetilde{q}_0}} \, ds \right\|_{L_t^r},$$

provided that:

$$\widetilde{p}_0 \leq \widetilde{q}, \quad p_0 \leq q \quad 1 + |\eta| + \frac{n}{p_0} - \frac{n}{q_0} + \alpha_0 - \beta > 0, \quad \Lambda_{\alpha_0} \geq \Lambda_\beta. \tag{3.35}$$

Let then

$$1 + \frac{1}{r} = \frac{1}{s_0} + \frac{1}{k}, \tag{3.36}$$

and use the Young inequality in the following Lorentz Spaces:

$$\| \cdot \|_{L^r} \leq \| \cdot \|_{L^{s_0}} \| \cdot \|_{L^{k,\infty}},$$

that's allowed if $1 < r, s_0, k < +\infty$. It is assured by assumptions $1 < r, s < +\infty$. We get

$$\left\| |x|^\beta \partial^\eta \int_0^t e^{(t-s)\Delta} \mathbb{P}\nabla \cdot F(x, s) \, ds \right\|_{L_t^r L_x^q L_\theta^{\widetilde{q}}}$$

$$\leq d_\eta \||x|^{\alpha_0} F\|_{L_t^{s_0} L_{|x|}^{p_0} L_\theta^{\widetilde{p}_0}} \left\| \int_{\mathbb{R}^+} \frac{dt}{t^{(1+|\eta|+\frac{n}{p_0}-\frac{n}{q_0}+\alpha_0-\beta)/2}} \right\|_{L_t^{k,\infty}}$$

$$\leq d_\eta \||x|^{\alpha_0} F\|_{L_t^{s_0} L_{|x|}^{p_0} L_\theta^{\widetilde{p}_0}},$$

provided that:

$$p_0 \le q \quad (1 + |\eta| + \frac{n}{p_0} - \frac{n}{q_0} + \alpha_0 - \beta)/2 = \frac{1}{k}, \quad \Lambda_{\alpha_0} \ge \Lambda_\beta, \qquad (3.37)$$

because of

$$\left\| \int_{\mathbb{R}^+} \frac{dt}{t^{1/k}} \right\|_{L_t^{k,\infty}} = 1.$$

Conditions (3.37), (3.36) lead to:

$$\Omega(\alpha_0, p_0, s_0) = 1 - |\eta| + \Omega(\beta, q, r). \qquad (3.38)$$

We now specify $F = u \otimes u$:

$$\left\| |x|^\beta \partial^\eta \int_0^t e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (u \otimes u)(x, s) \, ds \right\|_{L_t^r L_x^q L_\theta^{\widetilde{q}}}$$

$$\le \quad c_\eta \| |x|^{\alpha_0} |u|^2 \|_{L_t^{s_0} L_{|x|}^{p_0} L_\theta^{\widetilde{p}_0}}$$

$$\le \quad c_\eta \| |x|^{\alpha_0/2} |u| \|^2_{L_t^{2s_0} L_{|x|}^{2p_0} L_\theta^{2\widetilde{p}_0}} \qquad (3.39)$$

$$\le \quad c_\eta \| |x|^\alpha u \|^2_{L_t^s L_x^p L_\theta^{\widetilde{p}}},$$

where we have set

$$(\alpha_0/2, 2s_0, 2p_0, 2\widetilde{p}_0) = (\alpha, s, p, \widetilde{p}). \qquad (3.40)$$

Notice that $2\Omega(\alpha, s, p) = \Omega(\alpha_0, s_0, p_0)$, so (3.38) and (3.40) lead to (3.32)

$$2\Omega(\alpha, p, s) = \Omega(\beta, q, r) + 1 - |\eta|;$$

while condition (3.37) leads to

$$2\Lambda_\alpha \ge \Lambda_\beta.$$

Finally notice that (3.36) and (3.37) imply the relationships:

$$r \ge s_0 = s/2, \qquad q \ge p_0 = p/2, \qquad \widetilde{q} \ge \widetilde{p}_0 = \widetilde{p}/2.$$

These conditions are furthermore consistent with the choice $(\alpha, s, p, \widetilde{p}) = (\beta, r, q, \widetilde{q})$; in such a way we recover inequality (3.33):

$$\left\| |x|^\beta \partial^\eta \int_0^t e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (u \otimes u) \, ds \right\|_{L_t^r L_x^q L_\theta^{\widetilde{q}}} \le d_\eta \| |x|^\beta u \|^2_{L_t^r L_{|x|}^q L_\theta^{\widetilde{q}}},$$

provided that

$$\Omega(\beta, q, r) = 1 - |\eta|, \qquad \Lambda(\beta, q, \widetilde{q}) \ge 0.$$

$\square$

2. Regularity criteria in weighted spaces with angular integrability

As told the technology developed until now is well suited to study the regularity property of weak solutions of (2.1) in weighted spaces. In particular we focus on theorem (2.15) in which is shown the regularity of a weak solution u in the segment $(0, T) \times \{0\}$ provided that is satisfied the a priori bound:

$$\| |x|^\alpha u \|_{L_T^s L_x^p} \le +\infty, \quad \frac{2}{s} + \frac{n}{p} = 1 - \alpha. \qquad (3.41)$$

Of course such norms are invariant with respect the natural scaling of (2.1) centred in the origin:

$$\lambda \to \lambda u(\lambda^2 t, \lambda x).$$

Now we are interested in what happens if we consider more or less angular integrability in (3.41). We have results in two different directions:
If we consider weights $|x|^\alpha$ with $\alpha < 0$:

- we get regularity in the segment $(0,T) \times \{0\}$ as in (2.15), but by requiring boundedness in $L_t^s L_{|x|^\alpha p d|x|}^p L_\theta^{\widetilde{p}}$ with $\widetilde{p} < p$;
- we actually get global regularity (in $(0,T) \times \mathbb{R}^n$) by requiring boundedness in $L_t^s L_{|x|^\alpha p d|x|}^p L_\theta^{\widetilde{p}}$ with $\widetilde{p} > p$ and $\widetilde{p}$ large enough.

In the case $|x|^\alpha$, $\alpha \geq 0$ we show that:

- assumptions in (2.15) actually lead to global regularity (in $(0,T) \times \mathbb{R}^n$);
- global regularity is achieved also if $\widetilde{p} < p$ with $\widetilde{p}$ large enough; precisely $\widetilde{p}_G \leq \widetilde{p} < p$, with $\widetilde{p}_G$ depending on the other parameters;
- regularity in $(0,T) \times \{0\}$ is achieved in the range $\widetilde{p}_L \leq \widetilde{p} < \widetilde{p}_G$, with $\widetilde{p}_L$ depending on the other parameters.

The notation $\widetilde{p}_L, \widetilde{p}_G$ would remind to $\widetilde{p}_{Local}, \widetilde{p}_{Global}$. Such a scheme is explained by the following heuristic: if $\alpha < 0$ the weight $|x|^\alpha$ localizes in some sense the norm near to the origin, so local regularity is expectable, also for $\widetilde{p} < p$, but some boundedness condition at infinity are necessary in order to get global regularity. Such a condition is actually high $L^{\widetilde{p}}$ integrability in the angular direction. If we consider instead $|x|^\alpha$ with $\alpha \geq 0$, the weight suffices to guarantee also boundedness for large $|x|$, furthermore the integrability in the angular direction can be relaxed to small values of $\widetilde{p}$. In this spirit we prove two extensions of theorem (2.15). In the first under the hypothesis of higher angular integrability we get global regularity. In the second we get regularity in the segment $(0,T) \times \{0\}$, as in (2.15), but with weaker angular integrability. It is convenient to introduce the quantity:

$$\widetilde{p}_L = \begin{cases} \frac{2(n-1)p}{(2\alpha+1)p+2(n-1)} & \text{if} \quad -\frac{1}{2} \leq \alpha < 0 \\ \\ \frac{2(n-1)p}{p+2(n-1)} & \text{if} \quad 0 \leq \alpha < 1, \end{cases} \tag{3.42}$$

$$\widetilde{p}_G = \begin{cases} \max\left(4, \frac{(n-1)p}{\alpha p+n-1}\right) & \text{if} \quad \frac{1-n}{2} \leq \alpha < 0 \\ \\ \frac{(n-1)p}{\alpha p+n-1} & \text{if} \quad 0 \leq \alpha \leq \frac{1}{2}, \end{cases} \tag{3.43}$$

as told $\widetilde{p}_L, \widetilde{p}_G$ would remind to $\widetilde{p}_{Local}, \widetilde{p}_{Global}$. This quantities are infact sufficiently high angular integrability in order to to get respectively local (in $(0,t) \times \{0\}$) and global regularity. Notice that

$$\widetilde{p}_L < \widetilde{p}_G, \quad \text{if} \quad \alpha < 1/2, \qquad \widetilde{p}_L = \widetilde{p}_G \quad \text{if} \quad \alpha = 1/2;$$

and

$$\widetilde{p}_L < p < \widetilde{p}_G, \qquad \text{if} \quad \alpha < 0, \tag{3.44}$$

$$\widetilde{p}_L < \widetilde{p}_G < p, \qquad \text{if} \quad \alpha > 0; \tag{3.45}$$

The range of α in which $\widetilde{p}_L, \widetilde{p}_G$ have been defined is justified in the following proofs.

The following global regularity criterion holds:

Theorem 3.5. *Let $n \geq 3$, $u_0 \in L_\sigma^2(\mathbb{R}^n)$ and u a weak Leray solution of (2.1).*

If $\alpha \in [(1-n)/2, 0)$, $\alpha_0 \in [(2-n)/2, 2/(2+n))$, $\frac{n}{1-\alpha} < p \leq \frac{1-n}{\alpha}$, and

$$\||x|^{\alpha_0} u_0\|_{L_{|x|}^{p_0} L_\theta^{\widetilde{p}_0}} < +\infty$$

with:

$$\alpha_0 = 1 - \frac{n}{p_0}, \quad \widetilde{p}_0 \leq \frac{\widetilde{p}_G}{2}, \tag{3.46}$$

$$\begin{cases} 2 \leq p_0 \leq \widetilde{p}_G/2 & \text{if} \quad \widetilde{p}_G \leq 2n \\ 2 \leq p_0 \leq \widetilde{p}_G/2, \quad p_0 < \frac{2\widetilde{p}_G}{\widetilde{p}_G - 2n} & \text{if} \quad \widetilde{p}_G > 2n; \end{cases} \tag{3.47}$$

and

$$\||x|^\alpha u\|_{L_T^s L_{|x|}^p L_\theta^{\tilde{p}}} < +\infty, \tag{3.48}$$

with:

$$\frac{2}{s} + \frac{n}{p} = 1 - \alpha, \tag{3.49}$$

$$\frac{2}{1-\alpha} < s < +\infty, \tag{3.50}$$

$$\tilde{p} \geq \tilde{p}_G = \max\left(4, \frac{(n-1)p}{\alpha p + n - 1}\right); \tag{3.51}$$

then actually u is regular (C^∞ in space variables) in $(0, T) \times \mathbb{R}^n$.

The same holds if $\alpha \in [\mathbf{0}, \mathbf{1/2}]$, $\alpha_0 \in [(2-n)/2, 2/(2+n))$, $\max\left(4, \frac{n}{1-\alpha}\right) < p < +\infty$, or $p = 4$, and

$$\||x|^{\alpha_0} u_0\|_{L_{|x|}^{p_0} L_\theta^{\tilde{p}_0}} < +\infty$$

with:

$$\alpha_0 = 1 - \frac{n}{p_0}, \quad \tilde{p}_0 \leq \frac{p}{2}, \tag{3.52}$$

$$\begin{cases} 2 \leq p_0 \leq p/2 & if \quad p \leq 2n \\ 2 \leq p_0 \leq p/2, \quad p_0 < \frac{2p}{p-2n} & if \quad p > 2n; \end{cases} \tag{3.53}$$

and

$$\||x|^\alpha u\|_{L_T^s L_{|x|}^p L_\theta^{\tilde{p}}} < +\infty, \tag{3.54}$$

with:

$$\frac{2}{s} + \frac{n}{p} = 1 - \alpha, \tag{3.55}$$

$$\frac{2}{1-\alpha} < s < +\infty, \tag{3.56}$$

$$\tilde{p} \geq \tilde{p}_G = \frac{(n-1)p}{\alpha p + n - 1}. \tag{3.57}$$

Remark 3.2. By relations (3.44, 3.45) it turns out that in the case of localizing weights ($|x|^\alpha, \alpha < 0$) additional angular integrability is requested in order to get global regularity. We will come back on this point also in the next section. On the other hand the weights $|x|^\alpha, \alpha > 0$, by providing an additional boundedness at infinity, allows to get global regularity with less angular integrability.

Of course by translations u is regular provided that the weights and the norms are centered in any $\bar{x}$.

Remark 3.3. We give some remarks about the indexes:

- The first conditions in (3.46, 3.52) and the conditions (3.49, 3.55) follow by requiring invariance with respect to the natural scalings for respectively u_0 and u, i.e.

$$\lambda \to \lambda u_0(\lambda x)$$

$$\lambda \to \lambda u(\lambda^2, \lambda x);$$

- Our method misses the endpoints $s = +\infty$, $p = +\infty$. The last one is recovered just in case $\alpha \geq 0$, if we assume $\tilde{p} > \tilde{p}_G$. This is because of the use of proposition 3.4;
- Of course we can set $T = +\infty$ to get regularity for all times;

- If $n > 3$ we get a gain in the negative values of α with respect to the theorem 2.15. We have infact $\frac{1-n}{2} \leq \alpha$ instead of $-1 \leq \alpha$. This is also more satisfactory because exbits a dependence on the dimension. We have, on the other hand, a loss on the positive values of α, that's $\alpha \leq \frac{1}{2}$ instead of $\alpha < 1$;
- It is interesting to notice that no direct correlation between the radial and angular integrability of the initial datum have to be required.

The assumptions on the initial datum are a bit complicated, and have to be considered as merely technical. For instance can be considered u_0 in the Schwartz class without a real loss in the main contents of the theorem. In this way the formulation becomes simpler. The real information is about the angular integrability of the solution requested in order to get regularity, so it is the hypotesis $p \geq \widetilde{p}_G$. It comes out by requiring $\Lambda(\alpha, p, \widetilde{p}) \geq 0$ in order to apply inequality (3.31) with simply $L_t^r L_x^q$ spaces on the left side.

PROOF. We want to use the regularity criterion (2.8), so we need to show:

$$\|u\|_{L_T^r L_x^q} < +\infty, \quad \text{with} \quad \frac{2}{r} + \frac{n}{q} = 1. \tag{3.58}$$

Let's start by the integral representation

$$u = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes u)(s) ds.$$

In order to get (3.58) we distinguish the cases $\alpha \in [(1-n)/2, 0)$ and $\alpha \in [0, 1/2]$.

Case $\alpha \in [(1-\mathbf{n})/2, 0)$.

$$
\begin{aligned}
\|u\|_{L_T^r L_x^q} &\leq \|e^{t\Delta} u_0\|_{L_T^r L_x^q} + \left\| \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes u)(s) \, ds \right\|_{L_T^r L_x^q} \\
&= I + II.
\end{aligned}
$$

By the scaling assumption and the proposition (3.3) we get:

$$I \leq c_0 \| |x|^{\alpha_0} u_0 \|_{L_{|x|}^{p_0} L_\theta^{\widetilde{p}_0}}, \tag{3.59}$$

provided that

$$p_0 \leq q < \frac{np_0}{p_0 - 2}, \quad \widetilde{p}_0 \leq q, \quad \Lambda_{\alpha_0, p_0, \widetilde{p}_0} \geq 0. \tag{3.60}$$

Actually the condition $\Lambda_{\alpha_0, p_0, \widetilde{p}_0} \geq 0$ is not necessary in order to prove the theorem. We assume it for now to avoid some technicalities in the proof. We will show at the end how it can be removed. To bound II we use proposition (3.4) and scaling, so

$$II \leq d_0 \| |x|^{\alpha} u \|_{L_T^s L_{|x|}^p L_\theta^{\widetilde{p}}}^2, $$

provided that

$$\Lambda(\alpha, p, \widetilde{p}) \geq 0, \tag{3.61}$$

$$p/2, \ \widetilde{p}/2 \leq q, \quad s/2 \leq r. \tag{3.62}$$

Condition (3.61) is ensured by

$$\widetilde{p} \geq \frac{(n-1)p}{\alpha p + n - 1}.$$

Notice also that this condition, the scaling relation and $\alpha < 0$ imply $\frac{n}{1-\alpha} < p \leq \frac{1-n}{\alpha}$. So the widest range for p is atteined as $\alpha \to 0^-$. The last point is to show that a couple (r, q) such that (3.62) is consistent with $\frac{2}{r} + \frac{n}{q} = 1$ actually exists. We

choose $q = \widetilde{p}_G = \max\left(4, \frac{(n-1)p}{\alpha p + n - 1}\right)$. This is allowed by $(1-n)/2 \le \alpha$; infact if $\frac{(n-1)p}{\alpha p + n - 1} \ge 4$ we get

$$\frac{2}{r} = 1 - \frac{n}{q} = 1 - \frac{2n\alpha}{n-1} + \frac{2n}{p} \Rightarrow \frac{2}{r} - \frac{4}{s} = \frac{1 - n - 2\alpha}{n - 1},$$

so

$$(1-n)/2 \le \alpha \Rightarrow s/2 \le r;$$

while if $\frac{(n-1)p}{\alpha p + n - 1} < 4$ we get also $p < 4$ and

$$\frac{2}{r} = 1 - \frac{n}{4}, \quad \frac{4}{s} > 2 - \frac{2n}{p} \Rightarrow \frac{2}{r} - \frac{4}{s} < -1 + \frac{2n}{p} - \frac{n}{4} < 0.$$

Finally the condition (3.60) becomes

$$p_0 \le \frac{\widetilde{p}_G}{2} < \frac{n p_0}{p_0 - 2},$$

that by a straightforward calculation leads to (3.47), $\alpha_0 \in [(2-n)/2, 2/(2+n))$.

Case $\alpha \in [\mathbf{0}, \mathbf{1/2}]$. The only difference is in the choice of (r, q). Here we set $q = p/2$. In such a way the condition (3.62) is ensured by $\alpha \le 1/2$, infact:

$$\frac{2}{r} = 1 - \frac{2n}{p} \Rightarrow \frac{2}{r} - \frac{4}{s} = -1 + 2\alpha,$$

so

$$\alpha \le 1/2 \Rightarrow s/2 \le r.$$

Notice that now we haven't the restriction $p \le \frac{1-n}{\alpha}$. Then the condition (3.60) becomes

$$p_0 \le \frac{q}{2} < \frac{n p_0}{p_0},$$

that by a straightforward calculation leads to (3.53), $\alpha_0 \in [(2-n)/2, 2/(2+n))$ and $\max\left(4, \frac{n}{1-\alpha}\right) < p$ or $p = 4$.

We show how the condition $\Lambda_{\alpha_0, p_0, \widetilde{p}_0} \ge 0$ can be removed. Let call it simply Λ and suppose $\Lambda < 0$. We still can use the localized estimate (3.25) to get the bound

$$\|\mathbb{1}_{\Pi(R)} u\|_{L_T^r L_x^q} \le R^{-\Lambda} c_0 \||x|^{\alpha_0} u_0\|_{L_{|x|}^{p_0} L_\theta^{\widetilde{p}_0}} + d_0 \||x|^\alpha u\|_{L_T^s L_{|x|}^p L_\theta^{\widetilde{p}}}$$

where $\Pi(R)$ is the parabola

$$\Pi(R) = \left\{ (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n \quad \text{s.t.} \quad \left| \frac{x}{\sqrt{t}} \right| \le R \right\}.$$

So regularity is achieved in $\Pi(R)$ and, as $R \to +\infty$, in the whole space-time. $\qquad \square$

Then the following local regularity criterion holds:

Theorem 3.6. *Let $n \ge 3$, $u_0 \in L_\sigma^2(\mathbb{R}^n) \cap H^1(\mathbb{R}^n) \cap L_{|x|^{2-n} dx}^2$. Let u a weak Leray's solution of (2.1).*

If $\alpha \in [\mathbf{-1/2}, \mathbf{0})$, $\alpha_0 \in \left[1 - n, \frac{2-n}{2+n}\right)$, $\max\left(2, \frac{n}{1-\alpha}\right) < p < +\infty$ or $p = 2$, and

$$\||x|^{\alpha_0} u_0\|_{L_{|x|}^{p_0} L_\theta^{\widetilde{p}_0}} < +\infty$$

with:

$$\alpha_0 = 1 - \frac{n}{p_0}, \quad \widetilde{p}_0 \le \frac{p}{2}, \quad \Lambda_{\alpha_0, p_0, \widetilde{p}_0} \ge 0; \qquad (3.63)$$

$$\begin{cases} 1 \le p_0 \le p/2 & \text{if} \quad p \le n \\ 1 \le p_0 \le p/2, \quad p_0 < \frac{p}{p-n} & \text{if} \quad p > n; \end{cases} \qquad (3.64)$$

and

$$\||x|^\alpha u\|_{L^s_T L^p_{|x|} L^{\widetilde{p}}_\theta} < +\infty, \tag{3.65}$$

with:

$$\frac{2}{s} + \frac{n}{p} = 1 - \alpha, \tag{3.66}$$

$$\frac{2}{1-\alpha} < s < +\infty, \tag{3.67}$$

$$\widetilde{p} \geq \widetilde{p}_L = \frac{2(n-1)p}{(2\alpha+1)p + 2(n-1)}; \tag{3.68}$$

then actually u is and regular (C^∞ in space variables) in the segment $(0,T) \times \{0\}$.

The same holds if $\alpha \in [\mathbf{0}, \mathbf{1})$, $\alpha_0 \in \left[1 - (1-\alpha)n, 1 - (1-\alpha)\frac{2n}{2+n}\right)$, $\max\left(2, \frac{n}{1-\alpha}\right) < p < +\infty$ or $p = 2$, and

$$\||x|^{\alpha_0} u_0\|_{L^{p_0}_{|x|} L^{\widetilde{p}_0}_\theta} < +\infty$$

with:

$$\alpha_0 = 1 - \frac{n}{p_0}, \quad \widetilde{p}_0 \leq \frac{p}{2}, \quad \Lambda_{\alpha_0, p_0, \widetilde{p}_0} \geq 0, \tag{3.69}$$

$$\frac{1}{1-\alpha} \leq p_0 \leq \frac{p}{2}, \quad p_0 < \frac{p}{(1-\alpha)p - n}; \tag{3.70}$$

and

$$\||x|^\alpha u\|_{L^s_T L^p_{|x|} L^{\widetilde{p}}_\theta} < +\infty, \tag{3.71}$$

with:

$$\frac{2}{s} + \frac{n}{p} = 1 - \alpha, \tag{3.72}$$

$$\frac{2}{1-\alpha} < s < +\infty, \tag{3.73}$$

$$\widetilde{p} \geq \widetilde{p}_L = \frac{2(n-1)p}{p + 2(n-1)}. \tag{3.74}$$

Of course by translations u is regular in the segment $(0,T) \times \{\bar{x}\}$, provided that all the norms and the weights are centered in $\bar{x}$.

Remark 3.4. We give again some remarks about the indexes:

- The first conditions in (3.63, 3.69) and the conditions (3.66, 3.72) follow by requiring invariance with respect to the natural scalings for respectively u_0 and u, i.e.

$$\lambda \to \lambda u_0(\lambda x)$$

$$\lambda \to \lambda u(\lambda^2, \lambda x);$$

- Of course we can set $T = +\infty$ to get regularity for all times;
- We get a loss in the negative values of α with respect to the theorem 2.15. We have infact $-\frac{1}{2} \leq \alpha$ instead of $-1 \leq \alpha$.

We make again complicated assumptions on the initial datum, but the main point is the angular integrability of the solution requested in order to get regularity in $(0,T) \times \{0\}$, i.e. the hypotesis $p \geq \widetilde{p}_L$. It comes out by requiring $\Lambda(\alpha, p, \widetilde{p}) \geq \beta$ with $\beta \in [-1, 1)$. This is because in such a way (3.31) allows to apply directly the theorem 2.15.

PROOF. Here we want to use directly the theorem (2.15), so we need to show:

$$\||x|^{\beta}u\|_{L_T^r L_x^q} < +\infty, \quad \text{with} \quad \frac{2}{r} + \frac{n}{q} = 1 - \beta. \tag{3.75}$$

Let's start by the integral representation

$$u = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot (u \otimes u)(s)ds.$$

In order to get (3.75) we distinguish the case $\alpha \in [-1/2, 0)$ and $\alpha \in [0, 1)$.

Case $\alpha \in [-1/2, 0)$.

$$
\begin{aligned}
\||x|^{\beta}u\|_{L_T^r L_x^q} &\leq \||x|^{\beta}e^{t\Delta}u_0\|_{L_T^r L_x^q} \\
&+ \left\||x|^{\beta}\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot (u \otimes u)(s) \, ds\right\|_{L_T^r L_x^q} \\
&= I + II.
\end{aligned}
$$

By the scaling assumption and corollary (3.3) we get:

$$I \leq c_0 \||x|^{\alpha_0}u_0\|_{L_{|x|}^{p_0} L_\theta^{\widetilde{p}_0}} \tag{3.76}$$

provided that

$$p_0 \leq q < \frac{np_0}{(\alpha_0 - \beta)p + n - 2}, \quad \widetilde{p}_0 \leq q, \quad \Lambda_{\alpha_0, p_0, \widetilde{p}_0} \geq 0. \tag{3.77}$$

To bound II we use proposition (3.4) and scaling, so

$$II \leq d_0 \||x|^{\alpha}u\|_{L_T^s L_{|x|}^p L_\theta^{\widetilde{p}}}^2,$$

provided that

$$2\Lambda(\alpha, p, \widetilde{p}) \geq \beta, \tag{3.78}$$
$$p/2, \ \widetilde{p}/2 \leq q, \quad s/2 \leq r. \tag{3.79}$$

Condition (3.78) is ensured by

$$\widetilde{p} \geq \frac{2(n-1)}{(2\alpha - \beta)p + 2(n-1)}. \tag{3.80}$$

The last point is to show tha a couple (β, r, q) such that (3.79) is consistent with the scaling relation actually exists.

Because we are using theorem (2.15), we need to restrict to $-1 \leq \beta$ and, in order to get the lower possible value for $\widetilde{p}$, we actually choose $\beta = -1$. In such a way the condition (3.80) becomes the (3.68). With this choice of β we have

$$\widetilde{p} \leq p \quad \text{if} \quad -1/2 \leq \alpha,$$

that's infact the range of α we have restricted on. Then we choose $q = p/2$, so by the scaling relations we get

$$\frac{2}{r} - \frac{4}{s} = 2\alpha - 2 \leq 0,$$

that's consistent with $s/2 \leq r$. Of course because of the choice $q = p/2$ and the scaling we have to require

$$\max\left(2, \frac{n}{1-\alpha}\right) < p, \quad \text{or} \quad p = 2.$$

Then the condition (3.77) becomes

$$p_0 \leq q < \frac{np_0}{2p_0 - 2},$$

that by a straightforward calculation leads to (3.64), $\alpha_0 \in \left[1 - n, \frac{2-n}{2+n}\right)$.

Case $\alpha \in [0,1)$. The only difference is again in the choice of (β, r, q). Because of $\alpha \geq 0$ we can reach smaller values for $\widetilde{p}$, and we do it by requiring $2\alpha - \beta = 1$ in (3.80), in such a way

$$\widetilde{p} \geq \frac{2(n-1)p}{p + 2(n-1)}.$$

More precisely we choose

$$(\beta, r, q) = (2\alpha - 1, s/2, p/2).$$

A simple calculation shows that this choice is consistent with the scaling relation. Now by the condition (3.77) and scaling we have

$$p_0 \leq q < \frac{np_0}{(2 - 2\alpha)p_0 - 2},$$

that by a straightforward calculation leads to (3.70), $\alpha_0 \in \left[1 - (1-\alpha)n, 1 - (1-\alpha)\frac{2n}{2+n}\right)$.
$\square$

3. Well posedness with small data in weighted L^p spaces

In chapter 2 we have introduced the fundamental results and ideas of the small data theory for the solutions of the Navier Stokes equation. In this section we came back on this topic in order to get results in weighted Lebesgue spaces with angular integrability. In particular we will focus on theorem 2.24. As observed the small data theory is very well understood in the case of translation invariant adapted spaces, but not so much is known without this hypothesis. We will consider basically small data u_0 in the weighted space $L^p_{|x|^{\alpha p}d|x|}L^{\widetilde{p}}_\theta$ that's endowed with the norm

$$\left(\int_0^{+\infty} \|f(\rho \, \cdot \,)\|^p_{L^{\widetilde{p}}(\mathbb{S}^{n-1})} \rho^{\alpha p + n - 1} d\rho\right)^{\frac{1}{p}}.$$

Of course by translations all the results we are going to prove still hold if the norms and the weights are centred in some point $\bar{x}$ different from the origin . In order to develop a small data theory we have then to restrict to critical spaces, so we need invariance under the scaling $\lambda \to \lambda u_0(\lambda x)$, that, as observed before, is the right one for the initial datum of system (2.1). This leads to the scaling relation

$$\alpha = 1 - \frac{n}{p}$$

Notice that in this family there are the critical spaces L^3 and $L^2_{|x|^{2-n}dx}$ considered in theorems 2.24 and 2.24. Regular strong solutions are available for small data in L^3, while smallness in $L^2_{|x|^{2-n}dx}$ gives only regularity localized in the interior of a space-time parabola centered on the origin. We conjecture that such behaviour is typical of the power weights $|x|^\alpha$ with $\alpha < 0$. This is not surprising because in such case, even if the norms are defined on the whole space $\mathbb{R}^n$, the weights give a kind of localization near to the origin and a loss of informations at infinity occurs. Then we show that such informations can be recovered by a certain amount of angular integrability, and in this case, global regularity in space and time is availble. This is the case in the limit $\widetilde{p} \to \left(\frac{(n-1)p}{p-1}\right)^-$, as shown by the following

Theorem 3.7. *Let $n \geq 3$, $p \in [2, 2+n]$, $\widetilde{p} \in [1, +\infty]$ and $\alpha, \widetilde{p}$ such that:*

$$\alpha = 1 - \frac{n}{p}, \qquad \widetilde{p} \geq \frac{(n-1)p}{p-1}. \tag{3.81}$$

Let then $u_0 \in L^2_\sigma(\mathbb{R}^n)$. There exists $\varepsilon > 0$, depending on all the parameter, such that if

$$\||x|^\alpha u_0\|_{L^p_{|x|} L^{\widetilde{p}}_\theta} < \varepsilon$$

then exists a unique global solution u to the integral problem (2.42). Furthermore for all $p \leq q < \frac{np}{(\alpha-\beta)p+n-2}$, $\widetilde{p} \leq \widetilde{q}$, $r \in (1, +\infty)$ it holds

$$\||x|^\beta u\|_{L^q_{|x|}L^{\widetilde{q}}_\theta} < 2c_0\varepsilon. \tag{3.82}$$

provided that

$$\Lambda(\alpha, p, \widetilde{p}) > \Lambda(\beta, q, \widetilde{q}) \qquad \frac{2}{r} + \frac{n}{p} = 1 - \beta, \tag{3.83}$$

and for all $p \leq q < \frac{np}{(1+\alpha-\beta)p+n-2}$, $\widetilde{p} \leq \widetilde{q}$, $r \in (1, +\infty)$ it holds

$$\||x|^\beta \nabla u\|_{L^q_{|x|}L^{\widetilde{q}}_\theta} < 2c_1\varepsilon. \tag{3.84}$$

provided that

$$\Lambda(\alpha, p, \widetilde{p}) > \Lambda(\beta, q, \widetilde{q}) \qquad \frac{2}{r} + \frac{n}{p} = 2 - \beta; \tag{3.85}$$

So u is a regular (C^∞ in the space variables) classical solution of (2.1).

PROOF. Let's start by the integral representation:

$$u = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot (u \otimes u)(s)ds$$

and consider the Picard sequence

$$
\begin{array}{rcl}
u_1 & = & e^{t\Delta}u_0 \\
u_2 & = & e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot (u_1 \otimes u_1)(s)ds \\
u_n & = & e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot (u_{n-1} \otimes u_{n-1})(s)ds.
\end{array}
$$

We will show that u_n is a Cauchy sequence in all the Banach spaces $L^r_t L^q_{|x|^\beta q d|x|} L^{\widetilde{q}}_\theta$ such that (3.83) holds. Let's start noticing that it is a straightforward calculation to show that $p \leq q < \frac{np}{(\alpha-\beta)p+n-2}$, $\widetilde{p} \leq \widetilde{q}$, $r \in (1, +\infty)$, (3.83) and (3.81) are satisfied by a non null set of indeces. We use the weighted estimates (3.23), (3.33) with $\eta = 0$ to bound by induction:

$$
\begin{array}{rcl}
\||x|^\beta u_1\|_{L^r_t L^q_{|x|}L^{\widetilde{q}}_\theta} & \leq & c_0\||x|^\alpha u_0\|_{L^p_{|x|}L^{\widetilde{p}}_\theta} = c_0\varepsilon \\
\||x|^\beta u_2\|_{L^r_t L^q_{|x|}L^{\widetilde{q}}_\theta} & \leq & c_0\varepsilon + d_0\||x|^\beta u_1\|^2_{L^r_t L^q_{|x|}L^{\widetilde{q}}_\theta} \\
& \leq & 2c_0\varepsilon
\end{array}
$$

if we take a small ε such that[1] $4d_0 c_0 \varepsilon \leq 1$. Then by induction

$$
\begin{array}{rcl}
\||x|^\beta u_n\|_{L^r_t L^q_{|x|}L^{\widetilde{q}}_\theta} & \leq & c_0\varepsilon + d_0\||x|^\beta u_{n-1}\|^2_{L^r_t L^q_{|x|}L^{\widetilde{q}}_\theta} \\
& \leq & 2c_0\varepsilon
\end{array} \tag{3.86}
$$

again because $4d_0 c_0 \varepsilon \leq 1$. So the sequence is well defined in $L^r_t L^q_{|x|^\beta q d|x|} L^{\widetilde{q}}_\theta$. Now we have to show that u_n is a Cauchy sequence. Agin using (3.23), (3.33) we bound

---

[1] At this step it could suffices to take $d_0 c_0 \varepsilon \leq 1$. The stronger condition $4d_0 c_0 \varepsilon \leq 1$ is used starting by the tirhd step.

the differences:

$$\||x|^\beta (u_n - u_{n-1})\|_{L_t^r L_{|x|}^q L_\theta^{\tilde{q}}}$$

$$= \left\| |x|^\beta \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u_{n-1} \otimes u_{n-1} - u_{n-2} \otimes u_{n-2})(s) ds \right\|_{L_t^r L_{|x|}^q L_\theta^{\tilde{q}}}$$

$$\leq \left\| |x|^\beta \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u_{n-1} \otimes (u_{n-1} - u_{n-2}))(s) ds \right\|_{L_t^r L_{|x|}^q L_\theta^{\tilde{q}}}$$

$$+ \left\| |x|^\beta \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u_{n-2} \otimes (u_{n-1} \otimes u_{n-2}))(s) ds \right\|_{L_t^r L_{|x|}^q L_\theta^{\tilde{q}}}$$

$$\leq d_0 (\||x|^\beta u_{n-1}\|_{L_t^r L_{|x|}^q L_\theta^{\tilde{q}}} \||x|^\beta (u_{n-1} - u_{n-2})\|_{L_t^r L_{|x|}^q L_\theta^{\tilde{q}}}$$

$$+ \||x|^\beta u_{n-2}\|_{L_t^r L_{|x|}^q L_\theta^{\tilde{q}}} \||x|^\beta (u_{n-1} - u_{n-2})\|_{L_t^r L_{|x|}^q L_\theta^{\tilde{q}}})$$

$$\leq d_0 c_0 \varepsilon \||x|^\beta (u_{n-1} - u_{n-2})\|_{L_t^r L_{|x|}^q L_\theta^{\tilde{q}}}$$

where we used the uniform bound (3.86). In $n-2$ steps we get:

$$\||x|^\beta (u_n - u_{n-1})\|_{L_t^r L_{|x|}^q L_\theta^{\tilde{q}}}$$

$$\leq (2 d_0 c_0 \varepsilon)^{n-1} \||x|^\beta (u_2 - u_1)\|_{L_t^r L_{|x|}^q L_\theta^{\tilde{q}}}$$

$$= (2 c_0 \varepsilon)^{n-1} \left\| |x|^\beta \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u_1 \otimes u_1)(s) ds \right\|_{L_t^r L_{|x|}^q L_\theta^{\tilde{q}}}$$

$$\leq (2 d_0 c_0 \varepsilon)^n.$$

So the differences $u_n - u_{n-1}$ are bounded by a gemoetric series, and easily follows that u_n is a Cauchy sequences.

The regularity of u is now a direct consequence of theorem (3.5). In particular by setting[2] $\beta = 0, q = \tilde{q}$ is possible to refer to the original Serrin's result (2.8). Notice that

$$\Lambda(\alpha, p, \tilde{p}) \geq 0 \Rightarrow \tilde{p} \geq \frac{(n-1)p}{p-1},$$

$$p \leq q < \frac{np}{p-2} \Rightarrow p \in [2, 2+n].$$

In order to prove (3.84) we use (3.23), (3.33) with $\eta = 1$, so

$$\||x|^\beta \nabla u_1\|_{L_t^r L_{|x|}^q L_\theta^{\tilde{q}}} \leq c_1 \||x|^\alpha u_0\|_{L_{|x|}^p L_\theta^{\tilde{p}}} = c_1 \varepsilon$$

$$\||x|^\beta \nabla u_2\|_{L_t^r L_{|x|}^q L_\theta^{\tilde{q}}} \leq c_1 \varepsilon + d_1 \||x|^\beta \nabla u_1\|_{L_t^r L_{|x|}^q L_\theta^{\tilde{q}}}^2$$

$$\leq 2 c_1 \varepsilon$$

if we take a small ε such that $4 d_1 c_1 \varepsilon \leq 1$. Then by induction

$$\||x|^\beta \nabla u_n\|_{L_t^r L_{|x|}^q L_\theta^{\tilde{q}}} \leq c_1 \varepsilon + d_1 \||x|^\beta u_{n-1}\|_{L_t^r L_{|x|}^q L_\theta^{\tilde{q}}}^2 \qquad (3.87)$$

$$\leq 2 c_1 \varepsilon.$$

$\square$

Actually this theorem is a particular case of the Koch-Tataru theorem and can be proved directly by the methods in [35] and the estimates in proposition 3.3. Anyway we prefer a more direct proof, in particular of the bounds (3.82, 3.84). Then also

---

[2]It is again straightforward to show that conditions (3.83) are satisfited also under the restriction $\beta = 0, q = \tilde{q}$.

the regularity of u is a really difficult problems for the Koch-Tataru solutionts (see [**36**]), so we prefer to prove it directly in our case.

The theorem show that if we work in $L^p_{|x|^{\alpha p}d|x|}L^{\widetilde{p}}_\theta$ a sufficiently high angular integrability in order to recover the loss of informations for large $|x|$ is

$$\widetilde{p}_G = \frac{(n-1)p}{p-1}.$$

Now it is really interesting to understand what happens in the range

$$p < \widetilde{p} < \widetilde{p}_G.$$

We have no definitive results in this direction but a clear way to pursue. We will show how this problem is connected to the behaviour of the Leray solutions close to the null solutions. In order to simplify as more as possible the notation let consider just the spaces

$$L^2_{|x|^{-1}d|x|}L^{\widetilde{p}}_\theta, \quad 2 < \widetilde{p} < \widetilde{p}_G = 4.$$

So we set $n = 3, p = 2$. This is also the most interesting case because the quantity involved are related to real physical quantities. Let now recall the theorem 2.24

Theorem 3.8 (Caffarelli-Kohn-Nirenberg). *Let $u_0 \in L^2_\sigma(\mathbb{R}^n)$ and u a suitbale weak solution of (2.1). There exists an absolute constant $\varepsilon_0 > 0$ such that if*

$$\||x|^{-1/2}u_0\|_{L^2(\mathbb{R}^n)} = \varepsilon < \varepsilon_0,$$

then u is regular (C^∞ in space variables) in the interior of the parabola

$$\Pi_2 = \left\{(t, x) \quad t.c. \quad t > \frac{|x|^2}{\varepsilon_0 - \varepsilon}\right\}. \tag{3.88}$$

We used the notation Π_2 to remind the fact that th authors work with L^2 integrability in the angular variables. What we expect is a growth in the size of $\Pi_{\widetilde{p}}$ for bigger values of $\widetilde{p}$, and the filling of the whole space-time in the limit $\widetilde{p} \to 4^-$. To be more precise we expect to show regularity in the set

$$\Pi_{\widetilde{p}} = \left\{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n \quad \text{s.t} \quad t > \frac{c(\widetilde{p})|x|^2}{\varepsilon_0 - \varepsilon}\right\}, \tag{3.89}$$

for small data in $L^2_{|x|^{-1}d|x|}L^{\widetilde{p}}_\theta$, and furthermore

$$c(\widetilde{p}) \to 0, \quad \text{as} \quad \widetilde{p} \to 4^-. \tag{3.90}$$

In such a way the gap between theorem 3.8 and 3.7 would be completely covered. It turns out that this behaviour is strictly connected with a possible improvements of theorem 2.24 in the limit $\varepsilon \to 0$. This is a non trivial point and has of course an intrinsic interest. If we take the limit $\varepsilon \to 0$ in (3.88) we get the maximal regularity set

$$\Pi = \left\{t > \frac{|x|^2}{\varepsilon_0}\right\}.$$

On the other hand seems reasonable to conjecture improvements to the size of Π. A possibility is

$$\Pi_\varepsilon = \left\{t > \frac{c(\varepsilon)|x|^2}{\varepsilon_0 - \varepsilon}\right\}, \tag{3.91}$$

with

$$c(\varepsilon) \to 0, \quad \text{as} \quad \varepsilon \to 0. \tag{3.92}$$

Let now show how (3.92) implies (3.89, 3.90). The idea is to perform a decomposition of the initial datum inspired by a similar argument in [**39**]. We split u_0 in

$$u_0 = v_0 + w_0$$

with
$$\nabla \cdot v_0 = \nabla \cdot w_0 = 0,$$
and
$$v_0 \in L^2_{|x|^{-1}dx}, \quad w_0 \in L^2_{|x|^{-1}L^4_\theta}.$$
Moreover we require that
$$\||x|^{-1/2}w_0\|_{L^2_{|x|}L^4_\theta} \to 0 \quad \text{as} \quad \widetilde{p} \to 2^+$$
and
$$\||x|^{-1/2}v_0\|_{L^2_x} \to 0 \quad \text{as} \quad \widetilde{p} \to 4^-$$
In order to achieve such a decomposition we slightly modify the argument in [39]. Let $s > 0$ and define
$$u_0 = u_{0,<s} + u_{0,>s},$$
where $u_{0,<s}$ is equal to u_0 if $|u_0| < s$ and is zero otherwise . Then v_0, w_0 are defined by
$$v_0 = \lim_{t \to 0} e^{t\Delta}\mathbb{P}u_{0,>s},$$
$$w_0 = \lim_{t \to 0} e^{t\Delta}\mathbb{P}u_{0,<s}.$$
It follows easily by the representation of $e^{t\Delta}\mathbb{P}$ as a convolution operator that the limits are attained respectively in $L^2_{|x|^{-1}}dx$ and $L^2_{|x|^{-1}}L^4_\theta$ and the properties $\nabla \cdot v_0 = \nabla \cdot u_0$. Furthermore by a simple interpolation argument[3]
$$\begin{aligned}
\||x|^{-1/2}w_0\|_{L^2_{|x|}L^4_\theta} &\leq Cs^{1-\frac{\widetilde{p}}{4}}\||x|^{-1/2}u_0\|_{L^2_{|x|}L^{\widetilde{p}}_\theta}^{\widetilde{p}/4} \\
\\
\||x|^{-1/2}v_0\|_{L^2_x} &\leq Cs^{1-\frac{\widetilde{p}}{2}}\||x|^{-1/2}u_0\|_{L^2_{|x|}L^{\widetilde{p}}_\theta}^{\widetilde{p}/2},
\end{aligned} \tag{3.93}$$
for each $s > 0$. Then we choose $s = \frac{\theta}{1-\theta}$, where θ is defined by
$$\frac{1}{\widetilde{p}} = \frac{1-\theta}{2} + \frac{\theta}{4}.$$
Notice also that
$$1 - \frac{\widetilde{p}}{4} = \frac{1-\theta}{2-\theta}, \qquad 1 - \frac{\widetilde{p}}{2} = -\frac{\theta}{1-\theta}.$$
To simplify the notation we set
$$A_\theta = C\left(\frac{\theta}{1-\theta}\right)^{\frac{1-\theta}{2-\theta}}, \qquad B_\theta = C\left(\frac{\theta}{1-\theta}\right)^{-\frac{\theta}{2-\theta}},$$
in such a way (3.93) becomes
$$\begin{aligned}
\||x|^{-1/2}w_0\|_{L^2_{|x|}L^4_\theta} &\leq A_\theta\||x|^{-1/2}u_0\|_{L^2_{|x|}L^{\widetilde{p}}_\theta}^{\widetilde{p}/4} \\
\\
\||x|^{-1/2}v_0\|_{L^2_x} &\leq B_\theta\||x|^{-1/2}u_0\|_{L^2_{|x|}L^{\widetilde{p}}_\theta}^{\widetilde{p}/2}.
\end{aligned} \tag{3.94}$$
A straightforward calculation shows that A_θ is an increasing function of θ and
$$\lim_{\theta \to 0} A_0 = 0, \quad \lim_{\theta \to 1} A_\theta = 1; \tag{3.95}$$
while B_θ is a decreasing function of θ such that
$$\lim_{\theta \to 0} B_0 = 1, \quad \lim_{\theta \to 1} B_\theta = 0. \tag{3.96}$$

---

[3]see [39].

Then we consider the Cauchy problems

$$\begin{cases} \partial_t w + (w \cdot \nabla)w + \nabla p_w &= \Delta w \\ \nabla \cdot w &= 0 \\ w &= w_0, \end{cases} \tag{3.97}$$

and

$$\begin{cases} \partial_t v + (v \cdot \nabla)v + (v \cdot \nabla)w + (w \cdot \nabla)v + \nabla p_v &= \Delta v \\ \nabla \cdot v &= 0 \\ v &= v_0. \end{cases} \tag{3.98}$$

Of course $u = v + w$ and the pressure p_v, p_w can be still recovered by v, w through

$$p_w = C \left(\sum_{i,j=1}^{n} R_i R_j(w_i w_j) \right),$$

and

$$p_v = C \left(\sum_{i,j=1}^{n} R_i R_j(v_i v_j) + 2 \sum_{i,j=1}^{n} R_i R_j(v_i w_j) \right).$$

At first we notice that the global regularity of w is ensured by theorem 3.7 provided that

$$A_\theta \|u_0\|_{L^2_{|x|} L^{\widetilde{p}}_\theta}^{\widetilde{p}/4} < \varepsilon_4.$$

We used the notation ε_4 to emphasize the fact that the smallness condition is about the $L^2_{|x|^{-1}d|x|} L^4_\theta$ norm of w_0. Then by (3.82, 3.84) we also get the bound

$$\begin{aligned} \||x|^\beta w\|_{L^r_t L^q_x} &\leq c_0 \varepsilon_4, & \frac{2}{r} + \frac{3}{q} = 1 - \beta \\ \||x|^\beta \nabla w\|_{L^r_t L^q_x} &\leq c_0 \varepsilon_4, & \frac{2}{r} + \frac{3}{q} = 2 - \beta. \end{aligned} \tag{3.99}$$

These bounds are necessary in order to handle the terms $(v \cdot \nabla)w, (w \cdot \nabla)v$ in (3.98). Then if we are able to prove[4] (3.91, 3.92) we get the following regularity set for v:

$$\Pi = \left\{ (t, x) \quad \text{t.c.} \quad t > \frac{c(\varepsilon)|x|^2}{\varepsilon_0 - \varepsilon} \right\},$$

with $\varepsilon = \||x|^{-1/2}v_0\|_{L^2_x} \leq B_\theta \||x|^{-1/2}u_0\|_{L^2_{|x|} L^{\widetilde{p}}_\theta}^{\widetilde{p}/2}$ sufficiently small. Now by using (3.96) we get $c(\varepsilon) \to 0$ as $\widetilde{p} \to 4^-$ (or $\theta \to 1$).

---

[4]Actually we conjecture (3.91, 3.92) for the perturbed system (3.98). Anyway the additional terms are easily handled by using (3.99).

Outlooks and remarks

The consequences of angular integrability in Sobolev embeddings and PDEs have been considered by many authors. We have focused basically on the applications to the Navier-Stokes equation, but, as mentioned in section 2 (Chapter 1), this point of view is natural and useful in the context of Srichartz estimates for the wave and Schrödinger equations on $\mathbb{R}^n$. A comprehensive reference about its application to the wave equation is [14].
The consequences of higer angular integrability have been explored also in te context of the Dirac equation, see [3], [16].
We have in mind to conclue by suggesting some additional consequences of propositions 3.3, 3.4. As we have seen until now the key point in global regularity results is the request

$$\Lambda(\alpha, p, \widetilde{p}) \geq 0 \tag{3.100}$$

for a solution u bounded in

$$\||x|^{\alpha} u\|_{L_t^s L_{|x|}^p L_\theta^{\widetilde{p}}}, \qquad \frac{2}{s} + \frac{n}{p} = 1 - \alpha. \tag{3.101}$$

Then the regularity and uniqeness problems are strictly connected. We have given as example the theorem 2.9 in which the uniqueness of weak solutions bounded in

$$\|u\|_{L_t^s L_x^p} < +\infty$$

is proved. In the same spirit by applying Sobolev embeddings in $L_{|x|^\alpha d|x|^\alpha}^p L_\theta^{\widetilde{p}}$ spaces under condition 3.100, we get uniqueness of weak solutions bounded in the norms 3.101.
Anyway we have basically focused on theorems 3.5, 3.6 to show the difference between local and global regularity results. These theorems can be further extended in some directions. In particular a larger set of indeces can be covered about the L^p integrability. The restriction on p from below, i.e.

$$\frac{n}{1-\alpha} < p, \qquad 4 \leq p \quad \text{or} \quad 2 \leq p,$$

can be relaxed by using respectively the local regularity criteria in [27], [32]. We omit the details.

Acknowledgments

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