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Typical Ranks of ternary cubic forms

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Typical Ranks of ternary cubic forms over $\mathbb{R}$

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Introduction

In this thesis we consider the problem of the classification of real ternary cubics, that is, plane cubic curves with real coefficients, with respect to an arithmetical invariant, the rank, and we give the decomposition of each real ternary cubic form.
We prove a theorem that characterizes the reducible cubic which factors as a product of imaginary conic and a real line with respect to rank and this is a new result in the theory of real plane cubic curves.
We have shown that there are two cases where the rank over the real number is five, while over the complex there is only one case of plane cubic with rank five (see[29]).
More generally, we consider a vector space V of dimension 3 over $\mathbb{R}$ and the action of the group $SL(V)$, that is a change of variables, on the space of cubic forms $S^3(V^\vee) = \mathbb{R}[x, y, z]_3$.
This problem is a special case of a more general framework of the so called Waring problem and arises also from the tensor decomposition that is an active area of research in mathematics and its applications.
In particular, the research of rank of symmetric tensors and problems related to the decomposition of them, is relevant in many applications as Electrical Engineering (in the sub sector of Antenna Array Processing), in Statistic (cumulant tensors, [30]), in Algebraic Statistics, in Computer Science, in Data Analysis.
For application concernig blind identification see ([14]). For all these applications see also the book of J. M. Landsberg ([28]).
We find that the maximal rank over the reals of ternary cubics is five in two cases, namely when the cubic form factors as a real conic and an external line and when a cubic factors as a conic and a tangent line.
Actually, it is known a classification of plane cubics with respect to rank only over $\mathbb{C}$ (see [29]).
In their paper (see [29]), Landsberg and Teitler obtained normal forms for symmetric tensors (homogeneous polynomials) of border rank up to five and they showed that the maximal rank of plane cubics over $\mathbb{C}$ is five in only one case, namely when the cubic factors as a conic and a tangent line.

This work is based also on the classification of the binary cubic forms over $\mathbb{R}$ with respect to rank given by P. Comon and G. Ottaviani in their paper (see[13]), where they were able also to compute the typical ranks for symmetric tensors in $\mathbb{P}(S^d\mathbb{R}^2)$ for $d \leq 5$ with respect to Veronese variety $\nu_d(\mathbb{P}^1)$.
The real decomposition theory is not so well developed as its complex counterpart; one of the reason is that, over the real numbers, there can be more than one generic rank.
A typical rank of $f \in S^3(V^\vee) = \mathbb{R}[x, y, z]_3$ is, following Comon and Ottaviani ([13]), any rank that occurs on an open subset of $\mathbb{R}[x, y, z]_3$ with the euclidean topology.
The binary complex tensor decomposition is well known from J. J. Sylvester who gave an algorithm to find a decomposition. Over the real numbers only recently, Comon and Ottaviani conjectured a list of all typical ranks and G. Blekherman was able to find all typical ranks for binary real forms and to prove the conjecture (see [6]).
In this thesis we have found the following table in which there are all the $SL(3, \mathbb{R})$-orbits of real ternary forms and we compute their rank and border rank:

Description	**normal form**	**R**	$\underline{\mathbf{R}}$	**Hessian**	rank(9x9)
triple line	x^3	1	1		2
1 real line+2 immaginary lines	$x(x^2+y^2)$	2	2		4
3 real concurrent lines	$x(x^2-y^2)$	3	3		4
double line+line	x^2y	3	2		4
Fermat	$x^3+y^3+z^3$	3	3	$216xyz$(triangle)	6
imm. conic+line	$(x^2+y^2+z^2)x$	4	4	$-8x(-3x^2+y^2+z^2)$(conic+external line)	8
real conic+externalline	$(x^2+y^2-z^2)z$	5	4	$-8z(x^2+y^2+3z^2)$(immaginary conic+line)	8
real conic+secantline	$(x^2+y^2-z^2)y$	4	4	$8y(x^2-3y^2-z^2)$(conic+secant line)	8
real conic+tangentline	$(x^2+y^2-z^2)(y-z)$	5	3	$-32(y-z)^3$(triple line)	6
irr. 2 connected component	$y^2z-x^3+xz^2$	4	4	$24x^2z+8z^3-24xy^2$	8
irr. 1 connected component	$y^2z-x^3-xz^2$	4	4	$-24x^2z+8z^3-24xy^2$	8
cusp	y^2z-x^3	4	3	$24xy^2$(double line +line)	6
nodal cubic	$y^2z-x^3-x^2z$	4	4	$24xy^2-8x^2z+8y^2z$(nodal cubic)	8
cubica punctata	$y^2z-x^3+x^2z$	4	4	$8(3xy^2-x^2z-y^2z)$	8
triangle	xyz	4	4	$2xyz$(triangle)	8

Table 1: Ranks and border ranks of plane cubics on $\mathbb{R}$

The numbers in the last column are the ranks of the 9×9 matrix (see Section (10.1)).

Chapter 1

Tensor Decomposition

1.1 Symmetric Tensor Decomposition

Let V be a complex vector space of dimension n+1 and $V^\vee$ be its dual vector space, that is the vector space of linear forms on V.
We denote by $\mathbb{P}(V)$ the projectivization of V, that is, the set of 1-dimensional subspaces of V.
For example if $V = \mathbb{C}^{n+1}$ is a standard complex vector space then $\mathbb{P}^n$=$\mathbb{P}(\mathbb{C}^{n+1})$ is a standard complex projective space.
A point of $\mathbb{P}^n$ is defined by (n+1) homogeneous coordinates $(x_0, x_1, .., x_n)$, with x_i complex numbers, which are not all equal to 0 and are considered up to a scalar multiple.
Consider a symmetric tensor $[a_{j_0}, ..., a_{j_n}]$ of order d and dimension n+1 (we can think to a tensor as a multi-way array, a generalization of a matrix). We associate to this tensor a homogeneous polynomial $f \in S^d(V)$ of degree d in n+1 variables

$$f(\mathbf{x}) = \sum_{j_0+j_1+..+j_n=d} \binom{d}{j_0 \ldots \ j_n} a_{j_0 j_1 .., j_n} x_0^{j_0} x_1^{j_1} \ldots x_n^{j_n}.$$

It is customary to scale the coefficients $a_{j_0 j_1 .., j_n}$ by the multinomial coefficients

$$\binom{d}{j_0 \ldots \ j_n} = \frac{d!}{j_0! \ j_1! \cdots j_n!}.$$

With this notation, an element $f \in S^3(V)$, with dim V=3, can be represented as

$$f(x, y, z) = a_{300}x^3+3a_{210}x^2y+3a_{201}x^2z+3a_{120}xy^2+6a_{111}xyz+3a_{102}xz^2+a_{030}y^3+3a_{021}y^2z+3a_{012}yz^2+a_{003}z^3$$

Decomposing a symmetric tensor means to find a representation of f as a sum of d-powers of linear forms, that is,

$$f(\mathbf{x}) = \sum_{i=1}^{r} \lambda_i(\alpha_i^0 x_0 + \alpha_i^1 x_1 + \cdots + \alpha_i^n x_n)^d$$

with $\lambda_i \neq 0$ and r the smallest possible.
A naive approach is based on a method of counting constants (a so called direct approach used in the XIX century based on elimination theory) (see for example [20]).
We have to consider a polynomial system of $\binom{n+d}{d}$ equations in $r(n+1)$ unknowns (the coefficients of the linear forms) and to resolve it.
But this method has a disadvantage: it introduces $r!$ redundant solutions because the system is an over-constrained system due to the permutations of the summands.
As we will see a more efficient approach is based on (one of) Sylvester theorem and on the classical concept of apolarity.
For setting the problem in a general context we review some facts about tensors.

1.2 Multilinear algebra and tensor products

The tensor product of two vector spaces V and W over a field K is a vector space denoted by $V \otimes W$ endowed with a bilinear map

$$V \times W \to V \otimes W, \quad v \times w \mapsto v \otimes w$$

which is universal that is for any bilinear map $\beta : V \times W \to U$, where U is a vector space there is a unique linear map from the tensor product $V \otimes W$ to U that takes $v \otimes w$ to $\beta(v, w)$.
The universal property determines the tensor product up to isomorphism.
If v_i and w_j are bases for V and W, the elements $v_i \otimes w_j$ constitute a basis for $V \otimes W$. The construction is functorial, that is linear maps from V to V' and from W to W' determine a linear map from $V \otimes W$ to $V' \otimes W'$.

$$\begin{tikzcd}
V \times W \arrow[rr] \arrow[dr, "j"'] & & V \otimes W \arrow[dl, dashed] \\
& U &
\end{tikzcd}$$

The construction of tensor product is commutative, distributive and associative that is we have the proprieties:

1. $V \otimes W \cong W \otimes V$
2. $(V_1 \oplus V_2) \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W)$
3. $(U \otimes V) \otimes W \cong U \otimes (V \otimes W) \cong U \otimes V \otimes W$

There is a different definition of tensor product.
$V \otimes W$ is the vector space of the linear maps $V^\vee \to W$, that is

$$V \otimes W = Hom(V^\vee, W).$$

The space $V^\vee \otimes W$ can be thought as the vector space of all linear maps from V to W,

$$V^\vee \otimes W = Hom(V, W)$$

but also as the vector space of linear maps from $W^\vee$ to $V^\vee$ or as the dual vector space to $V \otimes W^\vee$ or as the space of all bilinear maps $V \times W^\vee \to \mathbb{K}$.
The general linear group

$$GL(V) \times GL(W)$$

acts naturally over $V \otimes W$. If $(g, h) \in GL(V) \times GL(W)$ then

$$(g, h)(v \otimes w) = (gv) \otimes (hw)$$

for all $(v, w) \in V \otimes W$.
The rank of $f \in V \otimes W$ is the dimension of $Im(f)$.
The linear maps f of rank $\leq r$ are defined by minors of size $r + 1$.
If we set

$$D_r = \{f \in Hom(V^\vee, W) | rk(f) \leq r\}$$

then

Proposition 1.1. *Let V,W be vector spaces over $\mathbb{K}$, with dim$V$=n+1, dim$W$=m+1. The singular locus of D_r is D_{r-1} so $D_r \setminus D_{r-1}$ is a smooth variety and the codimension of D_r is*

$$(n + 1 - r)(m + 1 - r).$$

Proof. Following [34], $D_r \setminus D_{r-1}$ is an orbit under the action of $GL(V) \otimes GL(W)$ and under a choise of a basis of V and W, a representant of this orbit is given by the block matrix

$$\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

where I_r is the identity matrix of size r. □

If we choose basis and represent an element $f \in V^\vee \otimes W$ by a matrix $X = (f_{i,j})$ the first action is multiplication by a column vector, the second action is a multiplication by a row vector, the third by taking $\sum_{i,j} f_{i,j} g_{j,k}$, where $(g_{j,k})$ denotes a dimW×dimV matrix and the fourth is defined by $(v, \beta) \mapsto f_{i,j} v_i \beta^j$.
In general, given k vector spaces $V_1, .., V_k$ and a multilinear function

$$f : V_1 \times V_2 \times ... \times V_k \to K$$

the vector space of such multilinear functions is denoted by $V_1^\vee \otimes V_2^\vee .. \otimes V_k^\vee$ and is called tensor product of $V_1^\vee, .., V_k^\vee$.
Elements $T \in V_1^\vee \otimes \cdots \otimes V_k^\vee$ are called tensors, the integer k is called order of T and the sequence of natural numbers $(dimV_1, .., dimV_k)$ is called the dimension of T.
When $V_1 = V_2 = ... = V_k = V$, we use the standard notation $V^{\otimes k} := V \otimes ... \otimes V$, and by convention $V^{\otimes 0}$ is the ground field.
We give the definition of rank and border rank first in case of bilinear maps and homogeneous polynomials and then in case of tensors.

Definition 1.2. For any bilinear map $T : A \times B \to C$, one can represent it as a sum

$$T(a, b) = \sum_{i=1}^{r} \alpha^i(a) \beta^i(b) c_i$$

$\forall a \in A, \forall b \in B$ where $\alpha^i \in A^\vee$,$\beta^i \in B^\vee$, $c_i \in C$ the minimal number r over all such representation is called the rank of T and denoted $R(t)$ or $rk(T)$.

We note that we can identify the space of symmetric tensors with the space of homogeneous polynomials.
More precisely, if τ is a tensor of dimension n and order d, that is an element of the tensor space

$$\mathbb{C}^n \otimes ... \otimes \mathbb{C}^n$$

then the homogeneous polynomial

$$p(x_{i_1}, .., x_{i_k}) = \sum_{i_1,..,i_k=1}^{n} \tau_{i_1 i_2 .. i_k} x_{i_1} x_{i_2} .. x_{i_k}$$

can be associated 1-1 with τ.
The vector space of symmetric tensors $S^d(V)$, with dim V=n+1, has dimension

$$\dim\ S^d(V) = \binom{n+d}{d}.$$

1.3 Rank and border rank of a tensor

We take $\alpha^1 \in V_1^\vee, .., \alpha^k \in V_k^\vee$ and we define a tensor $\alpha^1 \otimes ... \otimes \alpha^k$ in $V_1^\vee \otimes V_2^\vee ... \otimes V_k^\vee$ by the formula:

$$\alpha^1 \otimes \cdots \otimes \alpha^k(v_1, ..., v_k) = \alpha^1(v_1) \cdots \alpha^k(v_k). \tag{1.1}$$

We say that a tensor has rank one if it may be written as in the formula (1.1).
We note that the property for a tensor to have rank one is independent of any choices of basis.
The definition of the rank of a tensor $T \in V_1 \otimes V_2 \otimes ... \otimes V_k$ is:

Definition 1.3. The rank of a tensor $T \in V_1 \otimes V_2 \otimes ... \otimes V_k$, denoted by $\mathbf{R}(T)$ is the minimum natural number r such that

$$T = \sum_{i=1}^{r} v_{i,1} \otimes \cdots \otimes v_{i,k}$$

with each $v_{i,j}$ are indecomponible.

Remark The rank of $f \in Hom(V^\vee, W) = V \otimes W$ is the same than the rank defined in Definition (1.3).
The rank of a tensor is unchanged if one makes changes of bases in the V_i.
We can say that the rank of a tensor is a measure of its complexity;moreover the rank of a tensor is not continuous, that is, the limit of a sequence of rank r tensors need not have rank r.
Therefore, (see [29]), we give the following definition of *border rank* of a tensor

$$T \in V_1 \otimes V_2 \otimes ... \otimes V_k.$$

Definition 1.4. A tensor T has *border rank* r if it is a limit of tensors of rank r but it is not a limit of tensors of rank s for any $s < r$.
We denote with $\underline{\mathbf{R}}(T)$ the border rank of a tensor T.

Example 1.5. Let a_i,b_j,c_k be basis of vector spaces A,B,C.
The tensor:
$T = a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_2 \otimes c_1$ has rank 1, because T can be written in the form:
$T = a_1 \otimes (b_1 + b_2) \otimes c_1$.
The rank of $S = a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_1$ is $\mathbf{R}(S) = 2$.
It's obvious that for all tensors $T \in V_1 \otimes \cdots \otimes V_k$,

$$R(T) \leq \prod_j (dimV_j).$$

The border rank of the tensor T:

$$a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1 + a_1 \otimes b_1 \otimes c_1$$

is $\underline{\mathbf{R}}(T)$=2, but $\mathbf{R}(T)$=3.
To see that $\mathbf{R}(T)$=3 it is sufficient to write the tensor in the form:

$$T = a_1 \otimes b_1 \otimes (c_1 + c_2) + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1,$$

while to see that $\underline{\mathbf{R}}(T)$=2 is sufficient to write the tensor as the limit of $T(\epsilon)$ for $\epsilon \to 0$, where

$$T(\epsilon) = \frac{1}{\epsilon}\left[(\epsilon - 1)a_1 \otimes b_1 \otimes c_1 + (a_1 + \epsilon a_2) \otimes (b_1 + \epsilon b_2) \otimes (c_1 + \epsilon c_2)\right]$$

1.4 Symmetric and skew-symmetric tensors

The symmetric powers $Sym^d V$, also denoted by $S^d V$, come with a universal symmetric multilinear map

$$V \times V ... \times V = V^{\otimes d} \to Sym^d V, \quad v_1 \times ... \times v_d \mapsto v_1 ... v_d.$$

The symmetric powers can be constructed as the quotient space of $V^{\otimes d}$ by the subspace generated by
$v_1 \otimes ... \otimes v_d - v_{\sigma(1)} \otimes ... \otimes v_{\sigma(d)}$, where $\sigma \in \mathfrak{S}$ is an element of the symmetric group.
If e_i is a basis for V, then

$$\{e_{i_1} \cdot e_{i_2} \cdot ... \cdot e_{i_d} : i_1 \leq i_2 \leq ... \leq i_d\}$$

is a basis of $Sym^d V$. So this space can be regarded as the space of homogeneous polynomials of degree d in the variable e_i.
In particular, $S^2 V$ is a subspace of $V^{\otimes 2}$ defined by:

$$S^2 V := \left\{v_i \otimes v_j + v_j \otimes v_i\right\} = \left\{T \in V^{\otimes 2} | \sigma(T) = T, \forall \sigma\right\}$$

The exterior powers $\Lambda^d V$ of a vector space V, can be defined in the following way:
define a map $\pi_\Lambda : V^{\otimes k} \to V^{\otimes k}$ such that

$$v_1 \otimes \cdots \otimes v_k \mapsto v_1 \wedge \cdots \wedge v_k := \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_t} (sgn(\sigma)) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

where $sgn(\sigma) = \pm 1$ denotes the sign of the permutation σ.
The image of this map is denoted by $\Lambda^k V$ and is called the space of alternating

k-tensors.
In particular, $\Lambda^2 V$ denotes the space of skew-symmetric 2-tensors and is a subspace of $V^{\otimes 2}$ given by:

$$\Lambda^2 V := \left\{ v_i \otimes v_j - v_j \otimes v_i \right\} = \left\{ T \in V^{\otimes 2} | \sigma(T) = -T \right\}$$

Strictly related to the rank of bilinear map is the symmetric border rank of homogeneous polynomials.

Definition 1.6. Let F be a homogeneous polynomial of degree d in n variables, the symmetric rank of F, denoted by $R_S(F)$ is the smallest r such that F is expressible as the sum of r d-powers of linear forms.

Definition 1.7. The symmetric border rank of a homogeneous polynomial F, $\underline{R}_S(F)$ is the smallest r such that there exists a sequence of polynomials F_ε each of rank r, such that F is the limit of these polynomials as $\epsilon \to 0$.

We recall the following definition:

Definition 1.8. A tensor $\tau_{i_1 i_2 .. i_k}$ is called symmetric if

$$\tau_{i_{\sigma(1)} .... i_{\sigma(k)}} = \tau_{i_1 .... i_k}$$

for all permutation $\sigma \in \mathfrak{S}_k$.

For example, the three tensor τ_{ijk} belonging to the tensor space
$\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} = \mathbb{C}^{\otimes 3}$ is symmetric if

$$\tau_{ijk} = \tau_{ikj} = \tau_{jik} = \tau_{jki} = \tau_{kij} = \tau_{kji}$$

for all $i, j, k \in 1, .., n$.
We write $T^k(\mathbb{C}^n)$:=$\mathbb{C}^n \otimes \cdot \cdot \mathbb{C}^n$, with k copies of $\mathbb{C}^n$, for the space of k-dimensional tensors over $\mathbb{C}^n$.
The symmetric group $\mathfrak{S}_k$ acts on $T^k(\mathbb{C}^n)$ as follows.
For any $\sigma \in \mathfrak{S}_k$ and for any tensor in $T^k(\mathbb{C}^n)$,

$$\sigma(x_{i_1} \otimes x_{i_2} \cdot \cdot \otimes x_{i_k}) := x_{i_{\sigma(1)}} \otimes \cdot \cdot \otimes x_{i_{\sigma(k)}}$$

and extend this linearly to all of $T^k(\mathbb{C}^n)$.
With this notation we can, more generally, give the following definition of symmetric tensor.

Definition 1.9. A k-tensor T is symmetric if

$$\sigma(T) = T$$

for all permutations $\sigma \in \mathfrak{S}_k$.

If we let $S{:}T^k(\mathbb{C}^n) \to T^k(\mathbb{C}^n)$ be the linear operator given by

$$S := \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}} \sigma.$$

Then we have the following characterization of a symmetric tensor $T \in T^k(\mathbb{C}^n)$. A tensor T is symmetric if and only if $S(T) = T$, or, if and only if it is an eigenvector of the linear operator S with eigenvalue 1.
In fact, if T is symmetric then we have

$$S(T) = \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}} \sigma(T) = \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}} T = T \tag{1.2}$$

and conversely, if $S(T) = T$, then

$$\sigma(T) = \sigma(S(T)) = S(T) = T$$

for all $\sigma \in \mathfrak{S}$ and so T is a symmetric tensor.

1.5 Polarization

Consider the vector space $S^d(V^\vee)$ that is the space of symmetric d-linear forms on V. Let $u_0, u_1, \cdots u_n$ be a basis of V and let $x_0, x_1, \cdots x_n$ be s dual basis in $V^\vee$. This space is the space of homogeneous polynomials of degree d on V.This identification may be done by the process called polarization.The polarization of a form $F \in S^d(V^\vee)$ is the unique symmetric multilinear function $\tilde{F}(x, y, ..., z)$ on $V^{\otimes d}$ such that

$$F(x) = \tilde{F}(x, ..., x).$$

By fixing the first k variable $a, c, \cdots c$ in $\tilde{F}$, and making equal the other ones, we obtain the k-th mixed polar of F with respect to the points $a, b, \cdots c$

$$P_{a,b,\cdots c}(F)(x) = \tilde{F}(a, b, \cdots c, x, \cdots , x).$$

The first polar of F with respect to the point Y=$y_0u_0 + \cdots y_nu_n \in V$ is

$$P_Y(F) = \frac{1}{n}\sum_{i=0}^{n} y_i \frac{\partial F}{\partial x_i}.$$

For example if Q is a quadratic homogeneous polynomial on V, the bilinear form associated, $\bar{Q}$, is given by

$$\bar{Q}(x,y) = \frac{1}{2}[Q(x+y) - Q(x) - Q(y)]$$

In coordinate-free term the polarization of a quadric $Q \in S^2(V^\vee)$ is the bilinear form $b_Q$associated to it:

$$b_Q \in Sym^2(V)$$

given by the above formula.
If F is a cubic form, $F \in S^3(V^\vee)$, with x, y, z coordinates on V, its first polar $P_Y(F)$ with respect to a point Y is a quadratic form, and its second mixed polar form $P_{Y,Z}(F)$ is a linear form. Its total polarization is given by the formula ([28])

$$\bar{F}(x,y,z) = \frac{1}{6}[F(x+y+z) - F(x+y) - F(x+z) - F(y+z) + F(x) + F(y) + F(z)].$$

For general multilinear form, for example for a k-linear form, the polarization formula is

$$\bar{Q}(x_1, x_2, ..., x_k) = \frac{1}{k!}\sum_{I\subset[k]} (-1)^{k-|I|} Q\left(\sum_{i\in I} x_i\right)$$

where $[k] = \{1, ..., k\}$.
For example the total polarization of the cubic polynomial of two variables

$$F(s,t) = s^2t$$

is, with $s = (s_1, s_2, s_3)$ and $t = (t_1, t_2, t_3)$

$$\bar{P} = \frac{1}{3}(s_1s_2t_3 + s_1s_3t_2 + s_2s_3t_1)$$

because we have:
$\bar{P}((s_1, t_1), (s_2, t_2), (s_3, t_3)) =$
$= \frac{1}{6}(P(s_1 + s_2 + s_3, t_1 + t_2 + t_3) - P(s_1 + s_2, t_1 + t_2) - P(s_1 + s_3, t_1 + t_3)$
$- P(s_2 + s_3, t_2 + t_3) + P(s_1, t_1) + P(s_2, t_2) + P(s_3, t_3)).$

1.6 Typical real rank

For the real binary forms we have the following definition of typical rank ([13]):

Definition 1.10. A rank r is typical if the locus

$$S(d,r) = \left\{ f \in Sym^d(\mathbb{R}^2) | rk(f) = r \right\}$$

has non empty interior with the euclidean topology.

An equivalent definition of typical rank (cfr.[19]) for the tensor product

$$V_1 \otimes V_2 \otimes \cdots \otimes V_k$$

is

Definition 1.11. The linear space $V_1 \otimes V_2 \otimes \cdots \otimes V_k$ has typical rank r if r is the smallest integer m such that the set of elements $v \in V_1 \otimes V_2 \otimes \cdots \otimes V_k$ of rank at most m is a dense set in the Euclidean topology.

Over $\mathbb{C}$, for every degree d, there is a unique value of r such that

$$S(d,r) = \left\{ f \in Sym^d(\mathbb{C}^2) | rk(f) = r \right\}$$

is dense and this is called "generic rank" over $\mathbb{C}$.
It is known that the smallest typical rank over $\mathbb{R}$ is equal to the generic rank over $\mathbb{C}$.
Roughly speaking, a property is "generic " if it is true almost everywhere and is "typical" if it is true on a non-zero volume set.
The difference is that there can be several typical ranks of a tensor but only one generic rank.
It was conjectured by Comon and Ottaviani [13] and proved by G. Blekherman [6] that over $\mathbb{R}$ binary forms of degree d take all integer values between $\lfloor \frac{d}{2} + 1 \rfloor \leq r \leq d$.
In general the rank depends on the ground field, and there is a difference between real and complex rank as the following example

$$2x^3 - 6xy^2 = (x+iy)^3 + (x-iy)^3 = (\sqrt[3]{4}x)^3 - (x+y)^3 - (x-y)^3$$

shows.
In the above example,

$$rk_{\mathbb{C}}(2x^3 - 6xy^2) = 2$$

while

$$rk_{\mathbb{R}}(2x^3 - 6xy^2) = 3.$$

If the field is $\mathbb{R}$ the set

$$S_r = \{f | rk(f) = r\}$$

is a semi algebraic set and its interior is non empty for just finitely many values of r.

Definition 1.12. A real semi algebraic set in $\mathbb{R}^n$ is the zero set of m polynomials $p_1, .., p_m \in \mathbb{R}[x]$.

The generic rank of F, grk(F), that depends on the degree d and on the number n of variables is not known in general, that is there is no general expression that gives the exact value of grk(F).

Chapter 2

Sylvester's penthaedral theorem

2.1 Cubic form in four variables

Consider $f \in S^d(\mathbb{V}^\vee)$, a form of degree d in n+1 variables, (n-ary d-ic quantic in the old English literature) where V denote a vector space over a field $\mathbb{K}$ of dimension n+1, and $V^\vee$ denote the dual vector space of V, $V^\vee$=Hom(V,$\mathbb{K}$). The problem concerning the representation of f as a sum of d-power of linear forms can be reduced to the research of the canonical form of a conic or a quadric; a basically generalization of this theory is the discovery of the Pentahedral Theorem of a cubic surface made in 1851 by J.J. Sylvester.
He was able to show that a form cutting out a smooth cubic surface in $\mathbb{P}^3$ can be written as a sum of five cubes of linear forms and moreover these linear forms are unique up to reordering and multiplication by a cube root of unity.
The proof of the Pentahedral theorem, according to I. Dolgachev, was given ten years later by A. Clebsch in 1861.
We observe overall that the equation of a cubic surfaces $S \subset \mathbb{P}^3$ have 20 coefficients because 1,3,6,10 are respectively the coefficients of the monomials of degree 0,1,2,3, so that it depends on 19 essential quantities.
Let $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ be five generic planes of $\mathbb{P}^3$ and let

$$\alpha_i = a_{i0}x_0 + a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 = 0, i = 0, 1, ..4$$

be their equations.
The equation

$$F(x_0, x_1, x_2, x_3) = \lambda_0\alpha_0^3 + \lambda_1\alpha_1^3 + \lambda_2\alpha_2^3 + \lambda_3\alpha_3^3 + \lambda_4\alpha_4^3 = 0 \tag{2.1}$$

represents a cubic surface whose 5 planes α give the Sylvester penthaedron: the above equation contain the 19 parameters.
We can, with a change of variables, take the equations of the five planes as

1. $x_0 = 0$
2. $x_1 = 0$
3. $x_2 = 0$
4. $x_3 = 0$
5. $x_0 + x_1 + x_2 + x_3 = 0$

so that the equation of the cubic surface become the canonic equation:

$$F(x_0, x_1, x_2, x_3) = \lambda_0\, x_0^3 + \lambda_1\, x_1^3 + \lambda_2\, x_2^3 + \lambda_3\, x_3^3 + \lambda_4\, (x_0 + x_1 + x_2 + x_3)^3. \tag{2.2}$$

Theorem 2.1. *The equation of a generic smooth cubic surface $S \subset \mathbb{P}^3$ can be written in one and only one way by equating to zero the sum of five cubes, or more precisely, given $f \in S^3\mathbb{C}^4$ a general cubic form in four variables, there is a unique decomposition*

$$f = \sum_{i=0}^{4} \lambda_i l_i^3,$$

where l_i ,i=0,..4, are five linear forms four by four linearly independent such that

$$\sum_{i=0}^{4} l_i = 0$$

and λ_i,i=0,..,4, are five non zero constants determined up a factor.

Proof. We sketch a proof of the Sylvester's theorem (see [8] or also [2]). Let S be a smooth cubic surface of equation

$$f = \sum_{i=0}^{4} \lambda_i l_i^3 = 0,$$

where l_i are like in the hypothesis of the theorem.
The Hessian surface $He(S)$ of S is a quartic surface with ten double points $Q_1, \dots Q_{10}$(these are the $\binom{5}{3} = 10$ intersections of any three among the five planes l_i) and passes simply through the ten lines $\left\{l_i = l_j = 0\right\}$, with $0 \leq i < j \leq 4$.
A geometric consideration shows that for each Q_i there are only three lines $\leq Q_i, Q_k \geq k \neq i$ that belong to the Hessian surface.
So there is a pentahedron having its verticies in the double points of $He(S)$ and its edges lying on $He(s)$ and each faces are the five planes l_i.

This pentahedron is uniquely determined by the cubic surface because, if there exists another representation of S of the form

$$\sum_{i=1}^{5} \xi_i m_i^3,$$

computing the Hessian of S we find that the plane $l_i = 0$ are the same as $m_i = 0$, up a permutation of indices; so the pentahedron is uniquely determinate by the cubic surface.
By the linear independence of l_i, we have that $\lambda_i = \xi_i$ up to factor.
A dimension count conclude the proof of Sylvester theorem in the smooth case. □

In modern notation we can enunciate the Sylvester Theorem 2.1 in the form:

Theorem 2.2. *Let $f \in S^3\mathbb{C}^4$. Then $R(f) = 5$, that is, there exists a unique decomposition of f in linear forms,*

$$f = l_1^3 + l_2^3 + l_3^3 + l_4^3 + l_5^3,$$

where the 10 vertices points of the 5 planes $(l_i = 0)$ coincide with the 10 points such that $rk\{P_x(f) \le 2\}$.

Proof. Assuming that

$$f = \sum_{i=1}^{5} \lambda_i l_i^3$$

and let

$$P_{ijk} = \{l_i = l_j = l_k = 0\}$$

be a point of $\mathbb{P}^3$ on which three of the linear forms l_i vanish simultaneously.
The number of these points are $\binom{5}{3}$.
Consider the polar form

$$P_{P_{ijk}}(f)(x) = f(P_{ijk}, x, x).$$

This is sum of just two squares hence it is a quadric of rank 2.
In fact we have

$$P_{P_{ijk}}(l_5^3) = 3l_5^2 P_{P_{ijk}}(l_5) = 3l_5^2 l_5(P_{ijk}).$$

Consider the subvariety $X_2 \in \mathbb{P}^9$ parametrizing the quadrics of rank 2. A quadric of rank 2 is the union of two plane and $\dim X_2 = 6$.
To find the degree of X_2 we have to intersect with a 3-plane, that is intersection of 6 hyperplanes.

So the degree of X_2 is the number of quadrics of rank two passing through six general points of $\mathbb{P}^3$. Then these quadrics are $\frac{1}{2}\binom{6}{3}$, so $\deg X_2 = 10$, then the intersection of the linear space

$$\mathbb{P}^3 \simeq \{P_x f | x \in \mathbb{P}^3(\mathbb{C}^4)\} \subset \mathbb{P}(S^3\mathbb{C}^4) \simeq \mathbb{P}^9$$

with the variety

$$X_2 = \{f | \; rk(f) \leq 2\}$$

is given by 10 points.
But they are just obtained by $l_i = l_j = l_k = 0$ hence there are no other points. □

Corollary 2.3. *Every cubic surface F admits a (possibly degenerate) s-polyedron with $s \leq 5$.*

Proof. See [18]. □

Chapter 3

Apolarity

3.1 Quadratic forms

When the degree d is 2, that is when $Q \in S^2(V^\vee)$, it is known from linear algebra that a "generic" quadratic non degenerate form Q on a complex vector space of dimension $n+1$ can be decomposed as a sum of $n+1$ squares of linear forms (on the real numbers can be minus sign also)

$$Q = \sum_{i=1}^{n+1} l_i^2$$

that is,

$$Q = l_1^2 + \cdots + l_{n+1}^2$$

and since a homogeneous polynomial of degree 2 correspond to a symmetric matrix, the minimum numbers of summands in the decomposition as a sum of squares is the rank of the symmetric matrix.
We have the theorem:

Theorem 3.1. *If $f \in S^2(V)$ with V vector space of dimension n+1 over k and char(k)$\neq 2$ then rk(f)=n+1.*

Proof. We recall that every quadratic form $f \in S^2(V)$ can be associated to a symmetric square matrix of dimension n+1 and that every symmetric matrix can be diagonalized. The only invariant is the rank of the matrix, that is the number of non zero diagonal entries. Thus, after the diagonalization of the associated matrix, we see that every quadratic form is a sum of $s \leq n+1$ squares of linear forms and the quadratic forms which are the sum of $\leq n+1$ squares of linear forms are the symmetric matrices of rank $\leq n$.
But a symmetric matrix has rank $s \leq n$ if and only if all minors of size $(s+1)\times(s+1)$

vanish.
Then the entries of the matrix must satisfy equations in the entries and this equations define a proper closed subset of $S^2(V)$. □

For example:

$$x^2 + 8xy + 9y^2 = (x + 4y)^2 - 7y^2$$

can be written as

$$x^2 + 8xy + 9y^2 = (x, y)\begin{pmatrix} 1 & 4 \\ 4 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and this matrix has rank 2.
In particular, we have a theorem of classification for quadratic forms over the complex numbers $\mathbb{C}$:

Theorem 3.2. *Let A be a symmetric matrix $n \times n$ on the complex field and let $SL(n)$ be the special linear group that acts on $P(S^2V)$, V is a vector space over $\mathbb{C}$, $dimV = n$ and the action is definite as follow:*
$g.A = {}^t gAg$, where g belongs to the group and A is a matrix $n \times n$.
Then there exist g such that

$$^t gAg = \begin{pmatrix} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ & & & \\ & 0 & & 0 \\ & & & \end{pmatrix} \tag{3.1}$$

and the 1's are the diagonal elements of a matrix $r \times r$ and $r = rank\ A$.

The real counterpart of this theorem is the so called Sylvester's Law of Inertia that has several expression in the mathematical literature and can be stated in the following form:

Theorem 3.3. *If a real quadratic form of rank r is reduced by two real linear transformations (non-singular) to the forms*

$$\sum_{i=1}^{r} c_i x_i'^2, \tag{3.2}$$

$$\sum_{i=1}^{r} k_i x_i''^2 \tag{3.3}$$

respectively, then the number of positive c's in 3.2 is equal to the number of positive k's in 3.3.

We can thus associate with every real quadratic form a couple of integers (p,q), namely, the number of positive and negative coefficients respectively which we get when we reduce the form by any real non-singular linear transformation to the form (1).
These two numbers are arithmetical invariants of the quadratic form with respect to real linear transformation, since
two real quadratic forms which can be transformed into one another by means of such transformation can be reduced to the
same expression of form (1).
The number p is sometimes called the "index of inertia" of the form.
The two numbers p and q and the arithmetical invariant r are not independent because we have the relation

$$p + q = r.$$

One of the two invariants p and q is therefore superfluous, is more convenient to use neither p nor q but their difference

$$s = p - q$$

which is called the "signature" of the quadratic form.

Definition 3.4. The signature of a real quadratic form is the difference between the number of positive and the number of negative coefficients which we obtain when we reduce the form by any real non-singular linear transformation to the form (1).

From this two equations, we see that p and q may be expressed in terms of the rank and signature of the form by the formulas:

$$p = \frac{r+s}{2}, q = \frac{r-s}{2}.$$

Thus we have the theorem over $\mathbb{R}$:

Theorem 3.5. *A real quadratic form of rank r and signature s can be reduced by a real non-singular transformation to the*
normal form

$$x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_r^2$$

where $p = \frac{r+s}{2}$.

Example 3.6. Consider the quadratic form in $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3):

$$Q = x_1x_2 + x_2x_3 + x_3x_1.$$

We put

$$y_1 = \frac{x_1 + x_2}{2}, y_2 = \frac{x_1 - x_2}{2}$$

to get

$$Q = y_1^2 - y_2^2 + x_3(2x_1).$$

We complete the square to obtain:

$$Q = (y_1 + x_3)^2 - y_1^2 - x_3^2$$

so that if we substitute

$$z_1 = y_1 + x_3, z_2 = y_2, y_3 = x_3$$

we obtain

$$p = 1, q = 2.$$

3.2 Apolarity

The concept of apolarity is an old one and belong to classical invariant theory. It was first developed by German mathematicians (principally Aronhold, Clebsch, Reye but also by the British mathematicians A. Cayley, G. Salmon) that studied how homogeneous polynomials of degree p and in q variables could be represented as sums of pth powers of linear forms.
The problem was to find canonical forms for quantics (in the old English literature see for example ([20]). A quantic is a form that in symbolic notation (for n-binary forms) may be written as

$$(a_1x_1 + a_2x_2)^n = a_1^n x_1^n + n a_1^{n-1} a_2 x_1^{n-1} x_2 + \cdots + a_2^n x_2^n$$

and that may be written symbolically as

$$(a_1x_1 + a_2x_2)^n = a_x^n$$

where a_x is an abbreviation for $a_1x_1 + a_2x_2$. For a ternary form the symbolic notation is

$$a_x^n = (a_1x_1 + a_2x_2 + a_3x_3)^n = a_1^n x_1^n + na_1^{n-1}a_2x_1^{n-1}x_2 + \cdots + \frac{n!}{i!j!k!}a_1^i a_2^j a_3^k x_1^i x_2^j x_3^k + \dots$$

where $i + j + k = n$ and a_x is written as an abbreviation for $a_1x_1 + a_2x_2 + a_3x_3$.

3.3 Symbolic calculus

Briefly, the symbolic calculus is a useful notation to represent concomitants(both invariants and covariants). Formally, we write a ternary cubic like

$$F = a_{x^3} = (a_1x_1 + a_2x_2 + a_3x_3)^3.$$

That is to say, after expansion, $\frac{3!}{i_1!i_2!i_3!}a_1^{i_1}a_2^{i_2}a_3^{i_3}$ stands for the appropriate coefficient a_r in the expression

$$F = a_0x_1^3 + a_2x_1^2x_2 + \cdots a_9x_3^3.$$

The symbol $(\alpha\beta\gamma)$ stands for the determinant

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}$$

For instant, in this symbolic notation, the Hessian is written

$$(\alpha\beta\gamma)^2\alpha_x\beta_x\gamma_x.$$

For the expression of other concomitans in symbolic notation see ([32]) or ([24]).
The apolarity can be summarized as follows:
let $\mathbb{K}$ any field of characteristic zero, consider the polynomial ring $R = \mathbb{K}[x, y]$ graded by degree and the dual ring of differential operators
$D = \mathbb{K}\left[\partial_x, \partial_y\right]$ that acts on R by differentiation.
We have the pairing

$$R_d \otimes D_k \to R_{d-k}$$

so the homogeneous differential operators of degree k acts on homogeneous polynomials of degree d to give homogeneous polynomial of degree $d-k$.
If we have a 1-form, that is a homogeneous polynomial $l \in R_1$ given by $l = ax+by$ its orthogonal operator $l^{\perp} \in D_1$, $l^{\perp} = b\partial_x - a\partial_y$ is such that $l^{\perp}(l) = 0$ is apolar to l.
The most important remark that goes back to the XIXth century is that the form of degree d that is sum of r linear powers is annihilated by the operator given by the product of r apolar operators.
We can say that apolarity is an extension of the canonical bilinear pairing between a vector space V and its dual $V^{\vee}$ given by :

$$V \times V^{\vee} \to \mathbb{K}$$

such that $< v, l >= l(v)$, that is the evalutation of l on v, to a canonical pairing

$$S^d V \times S^s V^{\vee} \to S^{s-d} V^{\vee}$$

definite by

$$(\Phi, F) \longmapsto P_{\Phi}(F)$$

for each $s \geqslant d$, so that Φ and F are called apolar if

$$P_{\Phi}(F) = 0$$

where $P_{\Phi}(F) \in S^{d-s}(V^{\vee})$ is called the apolar of F with respect to Φ. When $\dim V$=2, the apolarity for two polynomials $f \in S^d$ and $g \in S^d$ can be seen in this way:
if $f = (a_0x_0 + a_1x_1)^d$ and $g = (b_0x_0 + b_1x_1)^d$ then the contraction $f \cdot g$ is

$$< f, g >= (a_0b_1 - a_1b_0)^d.$$

We can extend by linearity this contraction to any couple of polynomials f and g so we have the formula for apolarity of two binary forms as:

$$\sum_{k=0}^{d} (-1)^i \binom{d}{k} a_k b_{d-k} \tag{3.4}$$

where $\binom{d}{k} a_k$ and $\binom{d}{k} b_k$ are the coefficients of f and g respectively.
Apolarity can also be seen as the natural pairing between differential operators and polynomials.
If we choose a basis of the vector space V the symmetric algebra $Sym(V^{\vee}) \cong \mathbb{K}[x, y, z]$ and $Sym(V) \cong \mathbb{K}[\partial_x, \partial_y, \partial_z]$, where x, y, z are the indeterminate and $\partial_x := \frac{\partial}{\partial x}$,$\partial_y := \frac{\partial}{\partial y}$, $\partial_z := \frac{\partial}{\partial z}$.
In particular given a point $P = (x_P, y_P, z_P) \in \mathbb{P}^2$, the corresponding linear form

$$\Delta_P := x_P\partial_x + y_P\partial_y + z_P\partial_z$$

is called the polarization operator that give the first polar of a point with respect to a form.
More precisely, the polarization operator defines a linear map

$$\Delta_P : S^d(V^\vee) \to S^{d-1}V^\vee$$

given by

$$\Delta_P(F) = x_P\partial_x(F) + y_P\partial_y(F) + z_P\partial_z(F)$$

that sends a form $F \in S^d(V^\vee)$ of degree d to a form $\Delta_P(F)$ of degree d-1.
In the case of degree 2 this is the well known notion of collinearity that gives an isomorphism between the projective plane and its dual.
This notion give rise to the concept of coniugate points and conjugate lines.
We say that two points are conjugate with respect to a conic if each of them belongs to the polar line of the other and two lines are conjugate with respect to a conic if each contains the pole of the other. We have two propositions that can be considered the dual of each other (see [34]).

Proposition 3.7. *Let l_i homogeneous polynomials of degree 1 be distinct for $i = 1, .., r$. Any $f \in S^dV$ defines a map*

$$A(f)_{r,d-r} : Sym^r(V^\vee) \to Sym^{d-r}(V^\vee).$$

There are c_i numbers of $\mathbb{K}$ such that

$$f = \sum_{i=0}^{r} c_i(l_i)^d$$

if and only if

$$(l_1^\perp \circ \cdots \circ l_r^\perp)f = 0.$$

Proof. The implication $\Rightarrow$ is easy because

$$l^\perp(l^d) = 0.$$

The other is a dimensional calculation because both spaces have dimension r. □

This proposition is now known as Sylvester's theorem, because the differential operators that annihilated the form f allow to decompose f as a sum of d-powers. We have a dual formulation of this proposition.

Proposition 3.8. *Let l_i as in the above proposition. Let $e < d - r$. There are numbers c_i such that*

$$f = \sum_{i=0}^{r} c_i (l_i)^d$$

if and only if

$$ImA(f)_{e,d-e} \subseteq < (l_1^{\perp} \circ \cdots \circ l_r)^{d-e} >$$

Proof. Consider the map

$$A(f)_{r,d-r} : Sym^r(V^{\vee}) \to Sym^{d-r}(V)$$

We have from the proposition 3.7 that f is a sum of distinct d-powers if and only if

$$(l_1^{\perp} \circ \cdots \circ l_r^{\perp}) f \in kerA(f)_{r,d-r}.$$

□

If we observe that the transpose of

$$A(f)_{r,d-r}$$

is

$$A(f)_{d-r,r}$$

we have equalities

$$(ImA(f)_{r,d-r})^{\perp} = kerA(f)_{d-r,r}$$

and

$$< (l_1^{\perp} \circ \cdots \circ l_r^{\perp} \circ S^{d-e-r} >=< ((l_1)^{d-e}, \ldots, (l_r)^{d-e} >^{\perp} .$$

For the last one the equality holds because both spaces have the same dimension.

3.4 Apolarity map and Catalecticant matrix

Following [17] a homogeneous form $\Phi \in S^k(V)$ is apolar to a homogeneous form $F \in S^d(V^{\vee})$ if Φ belong to the kernel of following map:

$$ap_F^k : S^k(V) \to S^{d-k}(V^{\vee})$$

given by

$$\Phi \longmapsto P_{\Phi}(F)$$

This map is called the apolarity map. The kernel of the apolarity map is the linear space (denoted by AP_k in [18]) of apolars to F of class k.
We can give the following definition:

Definition 3.9. A homogeneous form $\Phi \in S^k(V)$ is called apolar to a homogeneous form $F \in S^d(V^\vee)$ if $P_\Phi F = 0$.

The expected dimension of this linear space (denoted by AP_k) in [18]) is N(k)-N(d-k), where $N(k) = dimS^k(V^\vee)$ hence F admits a nonzero apolar of class k if $N(k) > N(d-k)$ that is $k > \frac{n}{2}$.
If we have the equality $k = \frac{n}{2}$ the two spaces have the same dimension so this kernel is different from zero only if the determinant of the catalecticant matrix of F is zero.
On the other hand, if $k \leq \frac{n}{2}$, then $AP_k(F) = \{0\}$.

Example 3.10. Consider a vector space V, $dimV = 3$ and $Q \in Sym^2(V^\vee)$ a homogeneous quadratic form on $V^\vee$ given by $Q(u_0, u_1, u_2) = \sum_{i,j=0}^{2} a_{ij}u_iu_j$ with (a_{ij}) the matrix representing the quadratic form Q.
We have:

$$\frac{\partial(Q)}{\partial u_0} = 2a_{00}u_0 + 2a_{01}u_1 + 2a_{02}u_2 \tag{3.5}$$

$$\frac{\partial(Q)}{\partial u_1} = 2a_{01}u_0 + 2a_{11}u_1 + 2a_{12}u_2 \tag{3.6}$$

$$\frac{\partial(Q)}{\partial u_2} = 2a_{02}u_0 + 2a_{12}u_1 + 2a_{22}u_2 \tag{3.7}$$

If we consider $\phi \in S^1(V)$ given by $\phi(t_0, t_1, t_2) = a_0t_0 + a_1t_1 + a_2t_2$.
So $D_\phi(Q) = a_0\frac{\partial}{\partial u_0} + a_1\frac{\partial}{\partial u_1} + a_2\frac{\partial}{\partial u_2}(Q)$
The apolar map

$$ap_Q^1(\phi) = \sum_{i=0}^{2} \frac{\partial(Q)}{\partial u_i}(\phi)u_i.$$

Definition 3.11. (cfr. [27]) The matrix $Cat(F)$ of the above linear map ap_F^k with respect to two basis of monomials of S^kV and $S^{d-k}(V^\vee)$ is called the k-th catalecticant matrix of the homogeneous form F and if $n = 2k$ the determinant of this matrix is the catalecticant (J.J. Sylvester)of F.
The polynomial function given by this determinant, that is, the map

$$S^{2k}(V^\vee) \to \mathbb{C}$$

given by

$$F \mapsto det(Cat((F))$$

is an invariant with respect to the action of the group $SL(V)$ and is called catalecticant invariant.

3.5 Catalecticant invariant at work

The research of a canonical form for the binary cubic based on apolarity and Sylvester's Theorem allow to decompose a generic symmetric polynomial of degree 3 in two variables as a sum of two cubes:
the catalecticant matrix associated to

$$f = 4x^3 + 9x^2y + 18xy^2 + 17y^3$$

is

$$C_f = \begin{bmatrix} 4 & 3 & 6 \\ 3 & 6 & 17 \end{bmatrix}$$

Now KerC_f is spanned by

$$\begin{bmatrix} 3 \\ -10 \\ 3 \end{bmatrix}$$

which decomposes in

$$3\partial_x^2 - 10\partial_x\partial_y + 3\partial_y^2 = (\partial_x - 3\partial_y)(3\partial_x - \partial_y)$$

so we obtain the decomposition

$$c_1(3x + y)^3 + c_2(x + 3y)^3$$

and solving the linear system we have

$$c_1 = \frac{5}{8} \quad c_2 = \frac{1}{8}$$

and the decomposition

$$4x^3 + 9x^2y + 18xy^2 + 17y^3 = \frac{5}{8}(x + 3y)^3 + \frac{1}{8}(3x + y)^3.$$

3.6 Solution of the cubic equation

Dehomogenizing the above form, consider the cubic equation

$$4x^3 + 9x^2 + 18x + 17 = 0.$$

We have seen that this equation is equivalent to

$$5(x + 3)^3 + (3x + 1)^3 = 0$$

or

$$(\sqrt[3]{5}x + 3\sqrt[3]{5})^3 + (3x+1)^3 = 0$$

that is

$$\left(\frac{\sqrt[3]{5}x + 3\sqrt[3]{5}}{3x+1}\right)^3 = -1$$

so that,

$$\left(\frac{\sqrt[3]{5}x + 3\sqrt[3]{5}}{-3x-1}\right)^3 = 1$$

and finally,

$$\sqrt[3]{5}x + 3\sqrt[3]{5} = (-3x-1)\omega^i$$

for i=0,1,2 with $\omega^i = exp\frac{2\pi i}{3}$, which give us the three solutions of the cubic equation.

3.7 Binary quintic as a sum of three powers

We give a numerical example of the theorem as in [35].
Let

$$f(x,y) = 3x^5 - 20x^3y^3 + 10xy^4$$

a binary quintic. We can write it in the form

$$f(x,y) = \binom{5}{0} \cdot 3x^5 + \binom{5}{1} \cdot 0x^4y + \binom{5}{2} \cdot (-2)x^3y^2 + \tag{3.8}$$

$$\binom{5}{3} \cdot x^2y^3 + \binom{5}{4} \cdot 2xy^4 + \binom{5}{5} \cdot 0y^5 \tag{3.9}$$

Consider the matrix

$$\begin{pmatrix} 3 & 0 & -2 & 0 \\ 0 & -2 & 0 & 2 \\ -2 & 0 & 2 & 0 \end{pmatrix} \tag{3.10}$$

The linear system

$$\begin{pmatrix} 7 & 1 & 0 & -1 \\ 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.11}$$

has ∞^1 solutions, that is the vector $(b_0, b_1, b_2, b_3) = t(0, 1, 0, 1)$
and

$$h(x, y) = y(x + iy)(x - iy)$$

so there exist three numbers $\xi_k \in \mathbb{C}$ such that

$$f(x, y) = \xi_1 x^5 + \xi_2 (x + iy)^5 + \xi_3 (x - iy)^5$$

Now,

$$f(x, y) = 3x^5 - 20x^3y^3 + 10xy^4 = \xi_1 x^5 + \xi_2 (x + iy)^5 + \xi_3 (x - iy)^5$$

and from this equation we may check that $\xi_1 = \xi_2 = \xi_3 = 1$. So

$$3x^5 - 20x^3y^3 + 10xy^4 = x^5 + (x + iy)^5 + (x - iy)^5$$

and

$$rk_{\mathbb{C}}(f) = 3.$$

3.8 The theorem of Comas-Seiguer

The decomposition of a tensor in two variables over $\mathbb{C}$ has been well understood by Sylvester and solved by Comas and Seiguer in ([11]).They gave the following theorem

Theorem 3.12 (Comas-Seiguer)**.** *Consider $v_d(\mathbb{P}^1) \subset \mathbb{P}^d$.*
Let $r \leq \lfloor \frac{d+1}{2} \rfloor$. Then

$$\sigma(v_d(\mathbb{P}^1)) = \{[\phi\} : R(\phi) \leq r\} \cup \{[\phi] : R(\phi) \geqslant d - r + 2\}$$

that says that if the border rank of $\phi \in S^d\mathbb{C}^2$ is r, then there are only two possibility:

- the rank is r
- the rank is d-r+2

but in this case the are no normal forms in general.
In terms of symmetric polynomials, the theorem 3.12 may be stated as:

Theorem 3.13. *Let $\phi \in S^2\mathbb{C}^2$ then the maximum possible $\underline{rk}(\phi)$ is $\lfloor \frac{d+1}{2} \rfloor$.*
If $\underline{rk}(\phi) = r$, then either $rk(\phi) = r$ or $rk(\phi) = d - r + 1$.

When the symmetric rank is r, ther decomposition is unique except when d is even and $r = \frac{d}{2}$.
In the binary case we can say when a form can be decomposed as a sum of r powers and this decomposition works with a simple algorithm but a property of this decomposition is that in some cases the number of the powers must be grater than in generic case;
for instance the cubic x^2y have rank three insted of two or one because it cannot write in less than a sum of three cubics:

$$x^2y = \frac{1}{6}[(x+y)^3 + (-x+y)^3 - 2y^3].$$

3.9 Landsbeg-Teitler table

The research of a generalization of the decomposition of binary forms to the decomposition of $S^d(V)$ is the so called Waring problem or also "the canonical form problem for homogeneous forms".
This problem was solved by Alexander and Hirschowitz (see paragraph 6.3). Landsberg and Teitler (cfr.[29]) give the explicit list of normal forms for plane cubic curves and their ranks and border ranks and they show how one can use singularities of auxiliary geometric objects (like the Hessian of a cubic ternary form) to determine the rank of a cubic polynomial in three complex variables. The following theorem and the following table is in [29].

Theorem 3.14. *The possible ranks and border ranks of plane cubic curves are described in the following table:*

Description	**normal form**	**rk**	$\underline{\mathbf{rk}}$	**Hessian**
triple line	x^3	1	1	
three concurrent line	$xy(x+y)$	2	2	
double line + line	x^2y	3	2	
conic + secant line	$x(x^2+yz)$	4	4	conic + secant line
conic + tangent line	$y(x^2+yz)$	5	3	triple line
irriducible	$y^2z - x^3 - z^3$	3	3	triangle
irriducible	$y^2z - x^3 - xz^2$	4	4	smooth
cusp	$y^2z - x^3$	4	3	double line +line
irriducible	$y^2z - x^3 - axz^2 - bxz^3$	4	4	irred. cubic, smooth for general a,b
triangle	xyz	4	4	triangle

Table 3.1: Ranks and border ranks of plane cubic curves on $\mathbb{C}$

The proof of 3.14 given in [12] relies on a computation of equations for the secant variety $\sigma_k(\nu_3(\mathbb{P}V))$ for $2 \leq k \leq 3$, with V vector space of dimension 3, which determines all the border ranks in the table.
The case of a conic plus tangent line $y(x^2 + yz)$, where the rank is 5, is the case of maximum rank.

Proof. (see[29])
Upper bounds for the ranks in the table above are given by computing an expression for the sums of cubes.
For example, to show that $rk(xyz) \leq 4$, we observe that

$$xyz = \frac{1}{24}\left((x+y+z)^3 + (x-y-z)^3 - (x-y+z)^3 - (x+y-z)^3\right)$$

Now consider the case of rank 5.
Let $f = x^2y + y^2z = y(x^2 + yz)$. The Hessian of f is given by $y^3 = 0$, so the Hessian is a triple line.
Since it is not a triangle, $rk(y(x^2 + yz)) \geq 4$,but in this case the rank is 5: suppose

$$f = y(x^2 + yz) = l_1^3 + l_2^3 + l_3^3 + l_4^3$$

with $[l_i]$ distinct points in $\mathbb{P}V$,with $[l_i]$ not all collinear.
Therefore there is a unique 2-dimensional linear space of conics through these four points. These quadratic forms are in the kernel of the map

$$\phi_{1,2} : S^1\mathbb{C}^{3\vee} \to S^2\mathbb{C}^3$$

In the plane $\mathbb{P}^2 \cong \mathbb{P}Ker\phi_{1,2}$,

$$H := \mathbb{P}Ker\phi_{1,2} \cap \sigma_2(\nu_2(\mathbb{P}V))$$

is a triple line and the pencil of conics vanishing at each l_i is also a line L. Now or $H = L$ or $H \neq L$. In the first case, we have that L contains the point $\mathbb{P}ker\phi_{1,2} \cap \nu_2(\mathbb{P}V) \simeq \sum_1(\phi)$ where $\sum_1(\phi)$ is the set of singular points of $\sum 0(\phi)$=Zeros(ϕ) and this is impossible because $< f >= V$, so L is disjoint from $\nu_2(\mathbb{P}V)$.
Therefore $H \neq L$. But in this case L contains exactly one reducible conic, that corresponds to a the point of $H \cap L$.This is impossible because a pencil of conics through 4 points contains at least three irreducible conics. Thus $rk(f) = 5$. □

Chapter 4

Classification of binary and ternary complex cubic forms

4.1 Group action

In general we work with a vector space and a group that acts on V:
the action is given by the map

$$G \times V \to V$$

such that $(g, v) \mapsto g \cdot v$ for all $g \in G$ and $v \in V$
with the property:

1. $g(v_1 + v_2) = gv_1 + gv_2$

2. $g(\lambda v) = \lambda(gv)$

3. $g_1(g_2) = (g_1 g_2)v$

4. $1 \cdot v = v$

The orbit of a vector $v \in V$ is the set

$$O_v = \left\{ v' \in V : v' = g \cdot v,\ g \in G \right\}.$$

We will always read the action of the group G on the projective space $\mathbb{P}V$ to have a better geometrical wiew of the situation.Moreover, we work in projective

space over the real or complex numbers although the problems are posed on $\mathbb{R}$ or on $\mathbb{C}$.
In this work we consider expecially the group $SL(3)$ over $\mathbb{R}$ and $\mathbb{C}$ because we want to compute the ranks and border rank and typical ranks for real cubics with respect to the Veronese variety $v_3(\mathbb{P}^2)$.
The group $SL(n)$ is the "special linear group", that is the group of $n \times n$ matrix with determinant equal to 1.
We are interested to the orbits of cubic forms with respect to $SL(2)$ and $SL(3)$. We note that the dimension of $SL(n)$ is $n^2 - 1$ so the dimension of $SL(2)$ and $SL(3)$ is 3 and 8 rispectively.
Let's see some examples:

$$G = SL(2, \mathbb{C})$$

In this case, the group G acts in standard way on $\mathbb{P}^1 = \mathbb{P}(V)$, with V vector space of dimension 2 on the complex numbers. If we represent G as a 2×2 matrix (g_{ij})

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{4.1}$$

the standard action is:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto g \begin{pmatrix} x' \\ y' \end{pmatrix}$$

This is nothing else that the group of projectivity on the line. The action is transitive so we have only one orbit.
Let's see another example:
Let S^2V the 2-power of V, $\dim V_{\mathbb{C}}$=2, that is the set of homogeneous polynomials of degree 2 in x, y.
We can set

$$S^2V = \{Ax^2 + 2Bxy + Cy^2\}$$

so we can interpret (A, B, C) as homogeneous coordinate on $\mathbb{P}(S^2V)$ and

$$Ax^2 + 2Bxy + Cy^2 = (xy) \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

With the linear substitution:

$$\begin{cases} x = g_{11}x' + g_{12}y' \\ y = g_{21}x' + g_{22}y' \end{cases}$$

we have

$$A(g_{11}x' + g_{12}y')^2 + 2B(g_{11}x' + g_{12}y')(g_{21}x' + g_{22}y') + C(g_{21}x' + g_{22}y')^2.$$

If we denote by A', B', C' the new coefficients we may write:

$$g^t \begin{pmatrix} A & B \\ B & C \end{pmatrix} g = \begin{pmatrix} A' & B' \\ B' & C' \end{pmatrix}$$

so we have the equation

$$(AC - B^2)(detg)^2 = (A'C' - B'^2). \tag{4.2}$$

This equation shows that the hypersurface

$$AC - B^2 = 0$$

is defined in projective setting.
If

$$AC - B^2 = 0$$

the polynomial

$$Ax^2 + 2Bxy + Cy^2$$

is a perfect square so it correspond to a double point on the projective line.

Definition 4.1. We say that $AC - B^2$ is invariant of weight 2 for the group action.

The orbits of $SL(2)$ over $\mathbb{P}(S^2V) = S^2(\mathbb{P}^1)$ are $AC - B^2 = 0$ (a conic) and $\mathbb{P}^2 - \{AC - B^2\}$ the complementary set of$\{AC - B^2 = 0\}$.
The action on the coefficients is

$$g(A, B; C) = (A', B', C')$$

given by

$$(A', B', C') = (A, B, C) \begin{pmatrix} g_{11} & g_{11}g_{12} & g_{12}^2 \\ 2g_{11}g_{21} & (g_{11}g_{22} + g_{12}g_{21}) & 2g_{12}g_{22} \\ g_{21}^2 & g_{21}g_{22} & g_{12}^2 \end{pmatrix}$$

This matrix is the second symmetric power of g, S^2g.

Definition 4.2. An invariant of a form F is a polynomial Φ of the coefficients $a_0, a_1, .., a_n$ of the form that changes only by a factor (sometimes called modulus) equal to the certain power of the determinant of the linear transformation on coefficients, that is, in formula

$$\Phi(a_0^{'}, \ldots, a_n^{'}) = (detg)^p \Phi(a_0, \ldots, a_n)$$

is invariant of weight p for the form F.

The classical example is the discriminant of a polynomial of degree 2, that is $ac - b^2$ if the polynomial is $ax^2 + 2bxy + cy^2$.

Definition 4.3. A covariant Ψ of a given form is a polynomial of coefficients $a_0, .., a_n$ and the variables x, y that change only by a factor equal to a power of the determinant δ of a linear transformation if one replace the coefficients $a_0, a_1, .., a_n$ of the given form by the corresponding coefficients $a'_0, a'_1, .., a'_n$ of the linearly transformed form and, at the same time, replaces the variables x, y by the linearly transformed variables x', y'.
The defining equation of the covariant is

$$\Psi(a'_0, ..., a'_n, x', y') = \delta^p \Psi(a_0, ..., a_n, x, y)$$

and p is the weight of the covariant.

4.2 Complex binary cubic forms

Let V be a complex vector space such that dimV=2.
The group $SL(2)$ acts on $\mathbb{P}(S^3V) = \mathbb{P}^3$ with the action on x and y. If V is a vector space of dimension 2 and S^dV is the symmetric d-power of V the action of $SL(V)$ on S^dV is given by

$$v_1 \otimes v_2 \otimes v_3 .... \mapsto (gv_1) \otimes (gv_2) \otimes (gv_3) .... \tag{4.3}$$

where $g \in SL(2)$.
We identify S^dV, the symmetric d-power of V, with the set of homogeneous polynomials of degree d in two variables, that is

$$S^dV = \{A_0x^d + A_1x^{d-1}y + ... + A_dy^d\}.$$

The action of $g \in SL(2)$ is made on the coefficients of polynomials so we can think of the action of the group on d-uple of points in $\mathbb{P}^1$ with the identification

$$\mathbb{P}^d = \mathbb{P}(S^dV) = S^d(\mathbb{P}^1),$$

the d-uple of points in the projective.
The action of this group on $\mathbb{P}^3 = \mathbb{P}(S^3V)$ fix the rational normal cubic in the following sense: let $\mathbb{P}(S^3)$ the set of element of type

$$F = \{Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3\}$$

homogeneous polynomials of degree 3, where (A, B, C, D) are homogeneous projective coordinates in $\mathbb{P}^3$.

The geometric interpretation of the equation $F = 0$ is the following: this equation determines three points over a straight line;they can be all different or some of these may coincide.
If the points so defined are all different, then we can apply a linear transformation that vanish for x=0,y=0,x=y, so we have(cfr.[25])

$$F = kxy(x - y), \quad k = 3\alpha \neq 0 \tag{4.4}$$

With a further linear transformation $x \leftarrow \sqrt[3]{\alpha}x$,$y \leftarrow \sqrt[3]{\alpha}y$
the form becomes the canonical form

$$F = 3xy(x - y).$$

In this case we transform this form in any other binary cubic form that determines three different points on a line and we can take as canonical form the following:

$$F = x^3 + y^3.$$

If two points are the same and the third is different we can choose a form that vanish for x=0 double and for y=0.
The form becomes

$$F = 3\alpha x^2 y$$

and a further transformation given by

$$x \to x, \alpha y \to y$$

gives the canonical form

$$F = 3x^2 y$$

At the end, if a binary form determines three coincident points over a line, that is, if the form is a power, we can reduce it to

$$F = x^3$$

In summary, there are 3 orbits for the action of the group SL(2) on $\mathbb{P}^3 = \mathbb{P}(S^3 V)$: the orbit of dimension 1 that correspond to the polynomial x^3 and is therefore of rank 1;
this orbit correspond to one root of multiplicity 3 and geometrically represent the twisted cubic in $\mathbb{P}^3$.
The orbit of dimension 2 that correspond to a double point plus a point and is the tangent variety of degree 4 to the twisted cubic.
Every point of that variety is the limit of two points of the secant to the cubic so is of the type $3x^2y$.

In this case every point on the tangent to the cubic correspond to a polynomial of the form $Ax^3 + Bx^2y = x^2(Ax + B)$ that is a polynomial with one double root. So all points on the tangent variety to the twisted cubic correspond to polynomials with a double root.
The orbit of dimension 3 that correspond to a polynomial with 3 real distinct roots and in this case the form has rank three.
The degree of the tangent variety, $Tan(C)$, to the twisted cubic, is 4 because the discriminant of the twisted cubic, that is the resultant of F and its derivative F', is a homogeneous polynomial of degree 4 in the variable A,B,C,D and is:

$$\Delta = 4AC^3 + A^2D^2 - 6ABCD - 3B^2C^2 + 4B^3D$$

and this is an invariant (see [38]) of 4th degree and weight 6 that is

$$4A'C'^3+A'^2D'^2-6A'B'C'D'-3B'^2C'^2+4B'^3D' = (detg)^6(4AC^3+A^2D^2-6ABCD-3B^2C^2+4B^3D$$

This invariant can be find in this way:
the Hessian of the cubic form

$$Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3$$

is

$$(Ax + By)(Cx + Dy) - (Bx + Cy)^2$$

that is

$$(AC - B^2)x^2 - (AD - BC)xy + (BD - C^2)y^2.$$

The discriminant of this quadratic form is an invariant of the cubic form;if we compute this discriminant we find

$$\Delta = A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD.$$

The cubic can be represented with the matrix

$$\begin{bmatrix} A & B & C \\ B & C & D \end{bmatrix} \tag{4.5}$$

that has kernel

$$\left(\begin{vmatrix} B & C \\ C & D \end{vmatrix}, - \begin{vmatrix} A & C \\ B & D \end{vmatrix}, \begin{vmatrix} A & B \\ B & C \end{vmatrix} \right)$$

and we have

$$\Delta = \begin{vmatrix} A & C \\ B & D \end{vmatrix}^2 - 4 \begin{vmatrix} B & C \\ C & D \end{vmatrix} \begin{vmatrix} A & B \\ B & C \end{vmatrix}.$$

We can see this fact in another way:
the discriminant of the cubic polynomial is the Sylvester resultant of F and F', that is $\Delta = R(F, F')$ so we have

$$R(F, F') = \frac{1}{A}\begin{pmatrix} A & 3B & 3C & D & 0 \\ 0 & A & 3B & 3C & D \\ 3A & 6B & 3C & 0 & 0 \\ 0 & 3A & 6B & 3C & 0 \\ 0 & 0 & 3A & 6B & 3C \end{pmatrix} \tag{4.6}$$

The determinant of this 5×5 matrix is a polynomial of four degree and we have

$$\Delta = A^2D^2 + 4AC^3 + 4B^3D - 3B^2C^2 - 6ABCD$$

and this is an invariant of weight (order) 6.

4.3 Sylvester's Resultant of two binary forms

The theory of elimination is an old theory developed by geometers of the XVIII and XIX century.
In substance the problem is to find conditions such that two or more polynomials have a common root.
The approach of Sylvester gave rise to the classical concept of resultants. The generalization of this concept to several polynomials give rise to multi polynomial resultant.
Given two polynomials f,g $\in K[x]$ of degree m and n respectively:

$$f(x) = a_0x^m + \cdots + a_m, \quad a_0 \neq 0,$$
$$g(x) = b_0x^n + \cdots + b_n, \quad b_0 \neq 0,$$

the *resultant* of f and g, $Res(f, g)$, is the determinant of the $(m+n)\times(m+n)$ matrix

$$Res(f, g) = \begin{pmatrix} a_0 & & & & b_0 & & & \\ a_1 & a_0 & & & b_1 & b_0 & & \\ a_2 & a_1 & \ddots & & b_2 & b_1 & \ddots & \\ \vdots & a_2 & \ddots & a_0 & \vdots & b_2 & \ddots & b_0 \\ a_m & \vdots & \ddots & a_1 & b_n & \vdots & \ddots & b_1 \\ & a_m & & a_2 & & b_n & & b_2 \\ & & \ddots & \vdots & & & \ddots & \vdots \\ & & & a_m & & & & b_n \end{pmatrix}$$

We mention some basic property of resultants (see [22]).
Product formula.
If $a_m \neq 0$ and $b_m \neq 0$ then

$$R(f,g) = a_m^n b_n^m \prod_{i,j} (x_i - y_j)$$

where x_i and y_j are roots of f and g respectively.
Res(f,g) is an integer polynomial in the coefficients of f and g.
Vanishing of the resultant.
For two concrete binary forms F and G (the homogenization of the polynomials f and g), the vanishing of $R(F,G)$ is equivalent to the fact that F and G gave a common root other than (0,0).
Finding the common root.
Res(f,g)=0 if and only if f and g have a non constant common factor.
If, for given f and g, we have $R(f,g) = 0$ but at least one first partial derivative of R at (f,g) is not zero, then f and g have a unique common root α possibly $\alpha = \infty$ and it can be found from the proportions:

$$(1 : \alpha : \alpha^2 : \cdots : \alpha^m) = \left(\frac{\partial R}{\partial a_0}(f,g) : \frac{\partial R}{\partial a_1}(f,g) : \cdots : \frac{\partial R}{\partial a_m}(f,g) \right),$$
$$(1 : \alpha : \alpha^2 : \cdots : \alpha^n) = \left(\frac{\partial R}{\partial b_0}(f,g) : \frac{\partial R}{\partial b_1}(f,g) : \cdots : \frac{\partial R}{\partial b_n}(f,g) \right).$$

Symmetry.

$$R(f,g) = (-1)^{mn} R(g,f)$$

Quasi-homogeneity.
The polynomial $R(f,g)$ is homogeneous of degree n in the a_i and of degree m in the b_j.It has the following property:

$$R(\lambda^0 a_0, \cdots, \lambda^m a_m, \lambda^0 b_0, \cdots, \lambda^n b_n) = \lambda^{mn} R(a_0, \cdots, a_m, b_0, \cdots, b_n).$$

There are polynomials $A(x)$,$B(x)$ with integer coefficients such that Af+Bg=Res(f,g).
Examples.
For two linear polynomials

$$R(a_0 + a_1 x, b_0 + b_1 x) = a_0 b_1 - a_1 b_0$$

For two quadratic polynomials

$$R(a_0+a_1x+a_2x^2, b_0+b_1x+b_2x^2) = a_0^2b_2^2+a_0a_2b_1^2-a_0a_1b_1b_2+a_1^2b_0b_2-a_1a_2b_0b_1+a_2^2b_0^2-2a_0a_2b_0b_2.$$

If we interpret (x_1, x_2) as homogeneous coordinates, the equations of two binary forms

1. $F(x_1, x_2) = a_0 x_1^n + a_1 x_1^{n-1} x_2 + .. + a_n x_2^n$
2. $G(x_1, x_2) = b_0 x_1^m + b_1 x_1^{m-1} x_2 + .. + b_m x_2^m$

represent sets of n and m points on $\mathbb{P}^1$ respectively.
The points given by the equation $F = 0$ are the points at which the linear factors of F vanish, and the points where $G = 0$ are the points at which the linear factors of G vanish.
Since two binary forms vanish at the same point when, and only when, these forms are proportional, it follows that the loci of the two equations have a point in common when, F and G have a common factor other than a constant.
Hence, by the theory of elimination, we have that a necessary and sufficient condition that the two loci $F = 0$ and $G = 0$ have a point in common is that the resultant $R(F, G)$ of the binary forms F and G vanish.
Let $\Delta(F)$ be the discriminant of the binary cubic$F = a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3$ defined as the Sylvester's resultant R of F_x and F_y, that is $R(\partial F_x, \partial F_y) = \Delta(F)$.
The discriminant vanishes exactly when the two partial derivative F_x and F_y have a noncostant common factor. For the binary cubic above we have

1. $F = a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3$
2. $F' = 3a_0x^2 + 6a_1x^2y + 3a_2y^2$

then the resultant $R(F, F')$ is 4.6:

$$R(F,F') = \frac{1}{a_0}\begin{pmatrix} a_0 & 3a_1 & 3a_2 & a_3 & 0 \\ 0 & a_0 & 3a_1 & 3a_2 & a_3 \\ 3a_0 & 6a_1 & 3a_2 & 0 & 0 \\ 0 & 3a_0 & 6a_1 & 3a_2 & 0 \\ 0 & 0 & 3a_0 & 6a_1 & 3a_2 \end{pmatrix}$$

The determinant of this 5×5 matrix is a polynomial of degree 5 that is:

$$\Delta = 4a_0a_2^3 + a_0^2a_3^2 - 6a_0a_1a_2a_3 - 3a_1^2a_2^2 + 4a_1^3a_3$$

and it is an invariant of index 6. In fact, we have:

$$4a_0^{'}a_2^{'3} + a_0^{'2}a_3^{'2} - 6a_0^{'}a_1^{'}a_2^{'}a_3^{'} - 3a_1^{'2}a_2^{'2} + 4a_1^{'3}a_3^{'} = (detg)^6(4a_0a_2^3 + a_0^2a_3^2 - 6a_0a_1a_2a_3 - 3a_1^2a_2^2 + 4a_1^3a_3)$$

where $g \in SL(2, \mathbb{C})$.
In general the discriminant of a binary form

$$f(x,y) = \sum_{i=0}^{d} a_i \binom{d}{i} x^{d-i} y^i$$

is

$$\Delta = a_0^{2(d-1)} \prod_{i<j} (\alpha_i - \alpha_j)^2$$

where α_i are the roots of the polynomial f dehomogeneized.

Vanishing of the discriminant

For a binary form $F(x,y)$ of degree n, the equation $\Delta(F) = 0$ means that F is divisible by a square of a linear form and for a polynomial $f(x)$ of degree $\leq n$, the vanishing of $\Delta(f)$ means that f satisfies at least one of the following conditions:

1. f has a double root;

2. $deg(f) \leq n-2$.
   The second condition means that the double root is at infinity.

Relation with Resultants

We have

$$\Delta(f) = \frac{1}{a_n} R(f, f').$$

Examples

Discriminant of a quadratic polynomial

$$\Delta(a + bx + cx^2) = 4ac - b^2.$$

Discriminant of a cubic polynomial

$$\Delta(a + bx + cx^2 + dx^3) = 27a^2d^2 + 4ac^3 + 4b^3d - b^2c^2 - 18abcd.$$

In summary we have the following table of $SL(2)$-orbits of binary cubic over $\mathbb{C}$.

Description	**normal form**	Δ
C_3	x^3	
$Tan(C_3) \setminus C_3$	x^2y	0
$\mathbb{P}^3 \setminus Tan(C_3)$	$x^3 + y^3$	$\neq 0$

Table 4.1: rank over $\mathbb{C}$

4.4 Classification of complex cubic ternary forms

The equation of a complex cubic plane curve has the form

$$\phi(x_1, x_2, x_3) = 0$$

where ϕ is a homogeneous polynomial of degree 3. If we order this equation with respect to x_3, we have

$$u_0 x_3^3 + u_1 x_3^2 + u_2 x_3 + u_3 = 0 \tag{4.7}$$

where $u_i, i = 0, 1, 2, 3$ is a binary form of degree i in the variables x_1, x_2. We let:

- $u_0 = a_{333}$,
- $u_1 = 3\left[a_{133}x_1 + a_{233}x_2\right]$
- $u_2 = 3\left[a_{113}x_1^2 + a_{123}x_1x_2 + a_{233}x_2\right]$,
- $u_3 = a_{111}x_1^3 + 3a_{112}x_1^2x_2 + 3a_{122}x_1x_2^2 + a_{222}x_2^3$

where the a_{ijk} are complex numbers.
We choose the triangle of reference to be $A_1A_2A_3$ where $A_3 = (0, 0, 1) = P$ is a simple point of a curve and for the side $x_1 = 0$ of the triangle the tangent line in P to the cubic curve. The equation (4.8) has to have solution in the vertex $(0, 0, 1)$ so

$$u_0 = a_{333}$$

and the equation of the tangent line in this point is $u_1 = 0$. This equation is to be coincident with the line of equation $x_1 = 0$, we have $a_{233} = 0$. With this choise the equation (4.8) become:

$$3a_{133}x_1x_3^2 + u_2x_3 + u_3 = 0$$

that is

$$3a_{133}x_1x_3^2 + 3\left[a_{113}x_1^2 + a_{123}x_1x_2 + a_{233}x_2\right]x_3 + a_{111}x_1^3 + 3a_{112}x_1^2x_2 + 3a_{122}x_1x_2^2 + a_{222}x_2^3 = 0.$$

The point $P = A_3$ will be a flex point if $a_{223} = 0$, so the equation of the cubic curve will be

$$\phi(x_1, x_2, x_3) = 3a_{133}x_1x_3^2 + 3\left[a_{113}x_1^2 + 2a_{123}x_1x_2\right]x_3 + a_{111}x_1^3 + 3a_{112}x_1^2x_2 + 3a_{122}x_1x_2^2 + a_{222}x_3^3 = 0.$$

If we compute the second derivative ϕ_{ij} in the point P, we have

$$\phi_{11} = 6a_{133}, \phi_{12} = 6a_{123}, \phi_{13} = 6a_{133}, \phi_{22} = 0, \phi_{23} = 0, \phi_{33} = 0$$

so the equation of the polar conic with respect to the point $P = A_3$ is:

$$x_1 \left[a_{113}x_1 + 2a_{123}x_2 + 2a_{133}x_3\right] = 0 \tag{4.8}$$

that is

$$\frac{\partial \phi}{\partial x_3} = 0. \tag{4.9}$$

This polar conic is degenerate in the tangent line $x_1 = 0$ and in the line

$$a_{113}x_1 + 2a_{123}x_2 + 2a_{133}x_3 = 0 \tag{4.10}$$

This line is the ″harmonic polar line“of the flex P.
Now if we choose the line (equation 4.10) as the side $x_3 = 0$ of the reference triangle we particolarize the coefficients and we have

$$a_{113} = a_{123} = 0$$

and finally the equation of the cubic curve has the form

$$\phi = 3a_{133}x_1x_3^2 + \left[a_{111}x_1^3 + 3a_{112}x_1^2x_2 + 3a_{122}x_1x_2^2 + a_{222}x_3^3\right] = 0 \tag{4.11}$$

So we can state the proposition

Proposition 4.4. *The equation of the general plane cubic curve without multiple points (so of genus 1) is, with an appropriate refence system, of the form*

$$x_1x_3^2 + \psi(x_1, x_2) = 0$$

with ψ a binary cubic form.

Landsberg and Titler proposed the following table (Table 4.2) for the classification of cubic plane curves over the complex numbers (cfr. [29]):

Description	normal form	Hessian
triple line	x^3	
three concurrent line	$xy(x+y)$	
double line + line	x^2y	
conic + secant line	$x(x^2+yz)$	conic + secant line
conic + tangent line	$y(x^2+yz)$	triple line
irriducible	$y^2z-x^3-z^3$	triangle
irriducible	$y^2z-x^3-xz^2$	smooth
cusp	y^2z-x^3	double line +line
irriducible	$y^2z-x^3-axz^2-bxz^3$	irred. cubic, smooth for general a,b
triangle	xyz	triangle

Table 4.2: Classification of plane cubic curves on $\mathbb{C}$

4.5 Canonical forms

The problem of the research of canonical forms is, roughly, the following: how much a tensor be simplified under the action of the general linear group $GL(n)$? The point is the lack of uniqueness of canonical form and the following simple example makes evident the problem:
consider the case of a n-ary quadratic form, f, that is $f \in Sym^2V^\vee$ where $\dim V = n$.
As we have seen, a classical result of linear algebra states that such form can be written as a sum of squares of linear forms, namely

$$f = x_1^2 + ... + x_n^2.$$

Moreover, if the dimension n is even, say $n = 2j$, then one can show that a quadratic form can also be generically written as

$$f = x_1x_2 + ... + x_{n-1}x_n$$

after the action of $GL(n)$.
A fundamental concept that play a role in the research of canonical form is the notion of apolarity that was build by several mathematicians of the ninenteen century.

4.6 Example

A generic ternary quartic $F \in K[x_0, x_1, x_2]_4$, with K algebrically closed, can be written in the canonical form $F = Q_1Q_2 + Q_3^2$ and also $F = Q_1^2 + Q_2^2 + Q_3^2$ where

$Q_i \in K[x_0, x_1, x_2]_2$.
The general normal form in the case of ternary cubics is (see [38], [43])

$$F = U^3 + V^3 + W^3 + 6\lambda UVW$$

where U, V, W are linear forms in three variables and λ is a suitable constant. This is the famous Hesse form of the smooth plane cubic.
The canonical forms of the general quaternary cubic ([42], [43])are:

Theorem 4.5. *1. $X_1^3 + X_2^3 + X_3^3 + X_4^3 + X_5^3$ (Sylvester penthaedral Theorem)*

2. $X_1^3 + X_2^3 + X_3^3 + X_4^3 + L_1C_1$

3. $X_1^3 + X_2^3 + X_3^3 + L_1C_1 + L_2C_2$

4. $X_1^3 + X_2^3 + L_1C_1 + L_2C_2 + L_3C_3$

5. $X_1^3 + L_1C_1 + L_2C_2 + L_3C_3 + L_4C_4$

6. $L_1C_1 + L_2C_2 + L_3C_3 + L_4C_4 + L_5C_5$
and

$$L_1L_2L_3 + L_1'L_2'L_3'$$

where C_i and L_i are respectively quaternary quadrics and quaternary linear forms.

A generic binary form of even degree $2j = p$ with $p \geq 4$ can be written in the form $L_1^p + ... + L_j^p + c \cdot L_1^2 L_2^2 \cdots L_j^2$ where L_i are binary linear form and c is a suitable constant.

4.7 Veronese variety

The polar s-polyhedron can be seen geometrically if we introduce the Veronese variety.

Definition 4.6. For any n,d the Veronese map of degree d is the map

$$\nu_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1} \simeq \mathbb{P}^{N(d)}$$

that send

$$[X_0, ..., X_n] \mapsto [X_0^d, X_0^{d-1}X_1, ...X_n^d]$$

where X^I ranges over all monomials of degree d in n+1 variables $X_0, ...X_n$.

This embedding can also be characterized by:

$$\nu_d : (\mathbb{P}^n)^\vee \to \mathbb{P}(S^d)$$

$$[l] \to [l^d]$$

Then we think to the Veronese variety as the variety that parametrizes d-th powers of linear forms. With this position, let ν_n be the Veronese map that sends a hyperplane l to the hypersurface l^n
that is

$$\nu_n(\alpha_0 x_0 + .. + \alpha_n x_n) = \sum \binom{n}{i} \alpha^i x^i$$

where i is a multi index,$i = (i_1, .., i_r)$. So we can interpret F as a point in the projective space $\mathbb{P}S^n(V^\vee)$ and each zero locus $\{l_i^n\}$ as a point in the Veronese variety $\nu_n(\mathbb{P}^r)^\vee$.
Then the hyperplane form a s-polar polyhedron of F if and only if F lies on the secant $(s-1)$-plane of the Veronese containing the points l_i^n.

Example 4.7. The Veronese surface is the image of the map

$$\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5$$

defined by

$$\nu_2 : [X_0, X_1, X_2] \mapsto [X_0^2, X_1^2, X_2^2, X_0X_1, X_0X_2, X_1X_2]$$

We are interested to $\nu_3(\mathbb{P}^2)$, namely to the cubic Veronese in $\mathbb{P}^9$ given by

$$\nu_3 : [X_0, X_1, X_2] :\longmapsto [X_0^3, X_0^2X_1, ..., X_2^3]$$

because we compute the rank with respect to this variety.
The Veronese surface in $\mathbb{P}^5$ is an example of determinantal variety because it can be written as the locus of points $[Z_0, .., Z_5]$ such that the matrix

$$\begin{pmatrix} Z_0 & Z_3 & Z_4 \\ Z_3 & Z_1 & Z_5 \\ Z_4 & Z_5 & Z_2 \end{pmatrix}$$

has rank 1.
Conversely, any 3x3 symmetric matrix of rank one corresponds to a quadratic form which is the square of a linear form.
This example can be generalized: consider $S = K[y_0, y_1, .., y_n]$ and also the space S_d, an affine space over K of dimension $\binom{n+d}{d}$ and a basis given by the monomials of degree d in S.

Remark The elements of rank one in S^dV are the homogeneous polynomials F such that $F = l^d$, with l linear form.
They form an irreducible algebraic variety, which is a cone over the projective variety $\nu_d(\mathbb{P}^n)$.

Chapter 5

Rank and border rank of real binary forms

5.1 Real Binary Cubic forms and typical real rank

The paper of Comon-Ottaviani [13] shows that there are two typical ranks for real cubic polynomials, namely 2 and 3, and the difference depends on the number of real roots, more precisely

Theorem 5.1 (Comon-Ottaviani)**.** *(cfr.[13])*
Let f be a real cubic polynomial without multiple roots.
Then

- $rk(f) = 3$ *if and only if* $\Delta(f) > 0$*, or equivalently, if and only if f has 3 real roots*
- $rk(f) = 2$ *if and only if* $\Delta(f) < 0$*, or equivalently, if and only if f has 1 real root.*

Comon and Ottaviani determined typical rank of a general real binary form of degree 4 and 5; more precisely they showed in the case of real binary quartic that if $f \in S^4(\mathbb{R}^2)$ and f has distinct real roots then

- if f has four real roots then $rk(f) = 4$
- if f has zero or two real roots then $rk(f) = 3$

and if $f \in S^5(\mathbb{R}^2)$ with distinct roots

- if f has five real roots then $rk(f) = 5$
- if f has one or three real roots then $rk(f) = 3$ or $rk(f) = 4$ depends on the sign of an invariant I_{12} of quintic binary forms.

5.2 Table of SL(2)-orbits of real binary cubics

Consider the catalecticant of the cubic form

$$\begin{pmatrix} A & B & C \\ B & C & D \end{pmatrix} \tag{5.1}$$

and the discriminant Δ is

$$\begin{vmatrix} A & C \\ B & D \end{vmatrix}^2 - 4 \begin{vmatrix} B & C \\ C & D \end{vmatrix} \begin{vmatrix} A & B \\ B & C \end{vmatrix}.$$

Over the real the discriminant Δ can be positive or negative corresponding to one real root and two imaginary coniugate roots and to three real distinct roots respectively. The table over $\mathbb{R}$ is the following:

Description	**normal form**	**rk**	**rk catalec**	Δ
C_3	x^3	1	1	
$Tan(C_3) \setminus C_3$	x^2y	3	2	0
$\mathbb{P}^3 \setminus Tan(C_3)$	$x^3 + y^3$	2	2	> 0
$\mathbb{P}^3 \setminus Tan(C_3)$	$x(x^2 - y^2)$	3	2	< 0

Table 5.1: rank over $\mathbb{R}$

Moreover there is a theorem that asserts that the discriminant of every binary form is a closed hypersurface that is a locus of all polynomial with one double root.

We give the proof after the following section that recall some facts about classical resultant of polynomials. For example, $rk(x^3 - 3xy^2) = 3$ on the reals because $x^3 - 3xy^2$ has three real roots. On the contrary, $rk(x^3 + 3xy^2) = 2$ on the reals because $x^3 + 3xy^2$ has one real root. G.Blekherman proved ([6]) a conjecture of Comon and Ottaviani that typical real Waring ranks of binary forms of degree d take all integer values betwen

$$\left\lfloor \frac{d+2}{2} \right\rfloor$$

and d.
That is the set of binary forms of real rank exactly r

$$S^{\mathbb{R}}_{d,r} = \{f \in Sym^d(\mathbb{R}^2|rk(f) = r\}$$

has a non empty interior if and only if

$$\left\lfloor \frac{d+2}{2} \right\rfloor \leq r \leq d$$

In the complement of the hypersurface $\Delta = 0$, the polynomial has distinct roots.
Proof of the theorem (5.1):
The proof is based on apolarity between the graduate polynomial ring

$$R = \mathbb{R}[x, y] = \oplus_{d\geqslant 0} Sym^d(\mathbb{R}^2) = \oplus_{d\geqslant 0} R_d$$

and the dual graduate ring of differential operators

$$D = \mathbb{R}[\partial_x, \partial_y] = \oplus_{k\geqslant 0} D_k.$$

There is an action of D on R that acts with the rules of differentiations.
The operator D_k takes elements of R_d to elements of R_{d-k}.
The space of operators of degree k which annihilate a given homogeneous polynomial f of degree d is the kernel of the linear map

$$A_f : D_k \to R_{d-k}$$

In basis we have that the matrix of A_f is the "catalecticant" (this name was given by Sylvester) of size $(d - k + 1) \times (k + 1)$

$$A_f = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_k \\ a_1 & a_2 & a_3 & \dots & a_{k+1} \\ . & \dots & \dots & \dots & \\ . & \dots & \dots & \dots & \\ a_{d-k} & \dots & \dots & .. & a_d \end{pmatrix} \tag{5.2}$$

So the differential operators of degree 2 that annihilate f are in the kernel of the matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \end{pmatrix} \tag{5.3}$$

The kernel of this matrix has a quadratic equation and its discriminant is $-\Delta(f)$; so the operators have two real roots if $\Delta(f) < 0$ then the rank-2 complex decomposition is actually real.

We observe also that a real cubic of real rank 2 can have only one real root because the equation

$$l_1^3 + l_2^3 = 0$$

splits to the three linear factors

$$l_1 - l_2, l_1 - \varepsilon l_2, l_1 - \epsilon^2 l_2$$

where ϵ is a cubic root of unity.
If, on the other hand, $\Delta(f) > 0$, the quadratic equation has no real roots and the theorem follows from the theorem

Theorem 5.2. *(cfr.[13])*
Let f be a real binary form of degree d with d real distinct roots.Then

$$rk_{\mathbb{R}} f = d.$$

Proof. The proof is by induction on d.
Let $d \geq 3$ and assume that
$rk_{\mathbb{R}} f \leq d - 1$.Then we have

$$f = \sum_{i=1}^{d-1} l_i^d$$

with l_i linear form. We may assume that l_{d-1} does not divide f.
Consider the rational function

$$F(x, y) = \frac{f(x, y)}{l_{d-1}(x, y)}$$

Under the real linear transformation ϕ given by $x' = x, y' = l_{d-1}$, we obtain a rational function G given by

$$G(x', y') = \frac{f(\phi^{-1}(x', y')}{y'^d}$$

so the polynomial $G(x', 1) = \sum_{i=1}^{d-2} m_i(x')^d + 1$,where m_i is a linear form, has d distinct roots since by hypothesis f have d real roots.
The derivative with respect to x'

$$\frac{d}{dx'} G(x', 1) = \sum_{i=1}^{d-2} d \cdot m_i(x')^{d-1} \cdot \frac{d}{dx'} m_i(x')$$

has $d - 1$ distinct real roots. But the above derivative has rank less than equal $d - 2$ since the derivative of the linear forms m_i is constant.This is in contrast with the inductive assumption. So $rk_{\mathbb{R}(f)}$ must exceeded $d - 1$. $\square$

5.3 Sylvester's theorem

One of the first and most important contribute to the solution of Waring's problem for forms was given by J.J. Sylvester in the middle of XIX century when he studied the problem of decompose a symmetric tensor of order d and dimension 2 (a polynomial homogeneous of degree d in two variables or a binary form) as a sum of d-th powers of linear forms.
Moreover, Sylvester was able to prove that a generic binary form of odd degree can be unique decomposed into a sum of powers and he gave also an algorithm to perform this.
The result of Sylvester can be expressed as:
let f $\in Sym^d(\mathbb{C}^2)$ a generic homogeneous polynomial of degree d=2k-1 in two variables, then f may be written as a sum of k terms of d-th powers of linear forms.
But this is true only for a generic polynomial; for example, the degree 3 polynomial x^2y cannot be decomposed as the sum of two cubes.
In fact we have:

$$x^2y = \frac{1}{6}\left((x+y)^3 - (x-y)^3 - 2y^3\right)$$

so we can say that x^2y has rank 3 but has border rank 2 because it is the limit as $\epsilon \to 0$ of polynomials that are sums of two cubes, namely

$$\lim_{\epsilon \to 0} \frac{(x+\epsilon y)^3 - x^3}{\epsilon}$$

More specifically, the theorem of Sylvester is:

Theorem 5.3 (Sylvester 1851). *(see [35]) Let*

$$f(x,y) = \sum_{j=0}^{d} \binom{d}{j} a_j x^{d-j} y^j$$

a binary form of degree d, $f \in S^d(\mathbb{K})$ and let

$$h(x,y) = \sum_{k=0}^{r} b_k x^{r-k} y^k = \prod_{j=1}^{r} (-\beta_j x + \alpha_j y)$$

a product of distinct linear factors.
There exist r (real) numbers ξ_k such that

$$f(x,y)=\sum_{k=1}^{r}\xi_k(\alpha_k x+\beta_k y)^d$$

if and only if

$$\begin{pmatrix} a_0 & a_1 & \dots & a_r \\ a_1 & a_2 & \dots & a_{r+1} \\ . & .. & \dots & . \\ . & .. & .. & . \\ a_{d-r} & a_{d-r+1} & .. & a_d \end{pmatrix} \cdot \begin{pmatrix} b_0 \\ b_1 \\ . \\ . \\ b_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ 0 \end{pmatrix} \tag{5.4}$$

that is, if and only if,

$$\sum_{t=0}^{r} a_{l+t}b_t=0$$

with $l=0,\dots,d-r$.

Proof: the proof is based on apolarity, a method of classical invariant theory, that was developed by J. J. Sylvester, A. Clebsch,E. Lasker, H. W. Richmond, Wakeford and others.
We sketch the proof of the theorem in the specific case of a general binary quintic, this is the first non trivial case of Sylvester'theorem.
Let's see the general binary quintic and let's write it as

$$f(x,y)=a_0x^5+5a_1x^4y+10a_2x^3y^2+10a_3x^2y^3+5a_4xy^4+a_5y^5.$$

We have to find the kernel (b_0,b_1,b_2,b_3) of the matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \end{pmatrix} \tag{5.5}$$

The coefficients b_i, $i=0,1,2,3$ are given by the four maximal minors of the matrix, the first is obtained deleting
the first column and so on.
Now, to to check if $b_0x^3+b_1x^2y+b_2xy^2+b_3y^3$ has three real roots we consider the discriminant δ of the cubic generator of the kernel, that is the invariant of degree 12, I_{12}, called "apple invariant" in [13]. We refer to this work for the explicit expression of the degree 12 invariant I_{12}.
If the field $\mathbb{K}$ is the real field $\mathbb{R}$, the coefficients ξ_k of the theorem of Sylvester can be only $\{1,-1\}$.

Chapter 6

Generalities on symmetric rank over $\mathbb{C}$

6.1 Secant variety and rank

We recall now some basic definition.
Let $X \subset \mathbb{P}^n$ be an irreducible variety and consider a rational map that sends the pair (p, q) to the line $\overline{pq}$, namely

$$\sigma : X \times X - \Delta \dashrightarrow \mathbb{G}(1, n)$$

where $\mathbb{G}(1, n)$ denote the variety of the 1-dimensional linear subspaces of $\mathbb{P}^n$ called the Grassmannian of lines in $\mathbb{P}^n$.
This map is defined in the complement of the diagonal $\Delta \subset X \times X$.
It is called the secant line map and the closure of its image is the *variety of secant lines* to X denoted by $\sigma(X)$.
In general, if $p_1, ..., p_k \in X$ are k points in general position of an irreducible variety X we can define the map

$$\sigma_k : X \times .. \times X \dashrightarrow \mathbb{G}(k-1, n)$$

that sends $p_1, ..., p_k$ in $\langle p_1, .., p_k \rangle$.
For example $\sigma_2(C) = \mathbb{P}^3$, where $C \in \mathbb{P}^3$ is the twisted cubic.
Indeed, let P a generic point not on C, then there exist a unique line l that passes for P and is secant to C.If such line will not exists then we have a contradiction because then the genus of $\overline{C}$, the projection of C from P in $\mathbb{P}^2$, is 3 while the twisted cubic has genus $g = 0$ and if there exist two secant lines to C and containing P the plane containing this two lines would have more than 4 intersection with C, an absurd because $deg(C) = 3$ and C is not contained in a plane.
It is convenient to give a more general definition of secant variety in order to

understand better the concepts of rank and border rank.
Let X a irreducible projective variety, $X \subset \mathbb{P}V$, $V = \mathbb{C}^n$, we give the following

Definition 6.1. The r-th secant variety $\sigma_r(X)$ is

$$\sigma_r(X) = \overline{\left\{ \bigcup_{x_1,..,x_r \in X} \mathbb{P}\langle x_1, .., x_r \rangle \right\}}$$

where $< x_1, .., x_r > \subset V$ denotes the span of the points $x_1, .., x_r$ and over line denotes the Zariski closure.
We get a filtration

$$X = \sigma_1(X) \subset \sigma_2(X) \subset ...$$

We give the following definition of X-rank, $R_X(p)$ and X-border rank, $\underline{R}_X(p)$ of $p \in \mathbb{P}V$:

Definition 6.2.

$$R_X(p) := min\{ r \mid p \in \sigma_r^0(X)\}$$

$$\underline{R}_X(p) := min\{ r \mid \in \sigma_r(X)\}$$

where $\sigma_r^0(X)$ is

$$\sigma_r^0(X) = \left\{ \bigcup_{x_1,..,x_r \in X} \mathbb{P}\langle x_1, .., x_r \rangle \right\}$$

and x_i are all distinct.
If the variety X is the Veronese variety, $\nu_d(\mathbb{P}V)$, then the r-secant variety represents the Zariski closure of the set of homogeneous polynomials that can be written as a sum of r d-th powers.
Then we have a characterization of polynomials in term of secant variety, that is

Definition 6.3. If ϕ is a polynomial of degree d,

$$rk(\phi) = R_{\nu_d(\mathbb{P}V)}([\phi]) \quad \underline{rk}(\phi) = \underline{R}_{\nu_d(\mathbb{P}V)}([\phi])$$

With these notations we can say that a symmetric tensor (homogeneous polynomial) $\phi \in S^dV$ has rank s if and only if

$$[\phi] \in \sigma_s^0(\nu_d(\mathbb{P}V))$$

and

$$[\phi] \notin \sigma_{s-1}^0(\nu_d(\mathbb{P}V)).$$

Definition 6.4. ϕ has border rank s if $[\phi] \in \sigma_s(\nu_d(\mathbb{P}V) \setminus \sigma_{s-1}(\nu_d(\mathbb{P}V))$.

Then the border rank of a symmetric tensor ϕ is the smallest s such that $[\phi] \in \sigma_s(\nu_d(\mathbb{P}V))$ and then $R(\phi) \geqslant \underline{R}(\phi)$. It is useful to give the following interpretation of the rank and border rank to have a better geometric wiew of the questions that follow.
Consider the ν_d, the Veronese map

$$\nu_d : \mathbb{P}(V^*) \to \mathbb{P}(S^d V^*)$$

given by

$$[l] \mapsto [l^d].$$

The r-secant variety

$$\sigma_r(v_d(\mathbb{P}V))$$

of the Veronese variety can be wiewed as the Zariski closure of the set of projectivizations of homogeneous polynomials of degree d in n+1 variables that are expressible as the sum of r d-powers of linear forms.
In this setting, the Waring's problem for polynomials can be stated as:
What is the smallest positive integer r such that

$$\sigma_s(v_d(\mathbb{P}V)) = \mathbb{P}S^d(V^\vee)?$$

So the problem is to find the dimension of $\sigma_r(v_d(\mathbb{P}V))$ for every r.
In the following, we will need the definition:

Definition 6.5. A polar s-polyhedron of $F \in S^d V^\vee$ is a set of s hyperplane $H_1, H_2, .., H_s$ such that there exist non zero numbers $\lambda_1, ..\lambda_s$ satisfying

$$F = \lambda_1 l_1^d + ... + \lambda_s l_s^d$$

where H_i are given by $\{l_i = 0\}$ and the powers l_i^d are linearly independent in $S^d(V^\vee)$.

The name polar s-polyhedron in the space or polar s-side in the plane comes from the theory of conics, that is if

$$C = l^2 + m^2 + n^2$$

is a smooth conic and l, m, n are three coplanar distinct lines then the triangle $\triangle = l + m + n$ formed by the three lines is a polar 3-sides of the conic and the triangle $\triangle$ is self-polar with respect to the conic, that is each side is the polar of its opposite vertex.
A quadratic form F such that rk(F)=k admits a polar k-polyhedron, because of

the diagonalization process.
Let us consider the polar s-sides of plane cubic curve. If F has a polar 1-side, it is a triple line.
If F has a polar 2-sides, it is the union of three concurrent lines.
Assume that F has a polar triangle $\{L, M, N\}$. Then if $L \cap M \cap N \neq 0$, then after a change of variables we can assume $L = x$,$M = y$,$N = -x - y$ hence

$$F = x^3 + y^3 - (x + y)^3$$

is the union of three non concurrent lines, so they are linearly independent forms, forcing F to be projectively equivalent to the Fermat cubic(see [18]).

6.2 Bounds for Rank

We are interested in upper and lower bounds of rank of polynomials.
If X is a variety made of N+1 points, $X \subset \mathbb{P}^N$, the rank of any point with respect to X is bounded by $N + 1$. We give the proof of following theorem [29]:

Theorem 6.6. *Let X be an irreducible projective variety , $dimX = n$, $X \not\subset H$, with H hyperplane. Then*

$$R_X(p) \leq N + 1 - n$$

Proof. The proof is based on induction on the dimension of X. Let p a point such that $p \notin X$, because if $p \in X$ then $R_X(p) = 1$ and $1 \leq N + 1 - n$. If dim$X$=1, let M a general hyperplane through p. By a theorem of Bertini, we know that M intersect X transversally.
We prove that M is spanned by $X \cap M$. If it is not so, let M' be any other hyperplane containing $M \cap X$ and let M and M' be defined by linear forms L and L' respectively.
Then $\frac{L'}{L}$ defines a meromorphic function on X with no poles, since every zero of L is also zero of L'.
So this function is a holomorphic function on X then is constant because X is projective.
Then $M = M'$ and M is spanned by $X \cap M$.
By taking a basis for M of points of $M \cap X$, we have

$$R_X(p) \leq R_{M\cap X}(p) \leq \dim\ M + 1$$

where $\dim\ M + 1 = N + 1 - n$ since $n = \dim\ X = 1$ and the hyperplane M has dimension $N - 1$.

If dim $X \gtrsim 1$, $M \cap X$ spans M again by Bertini Theorem, and it is irreducible (see Griffiths-Harris).
We have that dim $M \cap X = n - 1$ and dim $M = N - 1$.
The theorem follows because by induction $R_{M \cap X}(p) \leq (N-1)+1-(n-1) = N+1-n$ and since $M \cap X \subset X$ we have

$$R_X(p) \leq R_{M \cap X(p)} \leq N + 1 - n.$$

□

Corollary 6.7. *Given $\phi \in S^d\mathbb{C}^{n+1}$*

$$R(\phi) \leq \binom{n+d}{d} - n + 1$$

This inequality is sharp for $n = 1$. So, for $\phi \in S^3\mathbb{C}^2$

$$R(\phi) \leq \binom{4}{3} - 2 + 1 = 3$$

while, for $\phi \in S^3\mathbb{C}^3$

$$R(\phi) \leq \binom{5}{3} - 3 + 1 = 8.$$

With respect to lower bounds for rank, Landsberg and Tietler in the paper [29] found the following inequality for $R(\phi)$:

Theorem 6.8. *Let $\phi \in S^d\mathbb{C}^n$, with span(ϕ)=$\mathbb{C}^n$. Let $1 \leq s \leq d$. Then*

$$R(\phi) \geq rank\phi_{s,d-s} + \dim \sum_s(\phi) + 1$$

Here $\sum_s(\phi)$ are the zero's of ϕ with multiplicity $\geq s+1$ and $\phi_{s,d-s}$ is the linear map given by the polarization

$$\phi_{s,d-s} : S^s\mathbb{C}^{n\vee} \to S^{d-s}\mathbb{C}^n$$

Corollary 6.9. *If $\phi \in S^dW^\vee$ such that $\leq \phi \geq= W$ and $R(\phi) = n = \dim W$, we have that the singular points of Zeros(ϕ) is $\emptyset$. Indeed, we can represent ϕ as a sum of n d-powers as*

$$\phi = \eta_1^d + ... + \eta_n^d$$

so that $\leq \eta_1, .., \eta_n \geq=\leq \phi \geq= W$, so that η_i are a base for W.
Then $\sum_1$, the singular locus of Zeros(ϕ) is the common zero locus of the derivative η_i^{d-1} in $\mathbb{P}W$, which is empty.

Corollary 6.10. *Let $\phi \in S^dW$, with span(ϕ)=W.*
Then

1. *if ϕ is reducible, then $R(\phi) \geq 2n-2$*

2. *if ϕ has a repeated factor, then $R(\phi) \geq 2n-1$.*

Proof. From the hypothesis $\leq \phi \geq = W$ we have that the kernel of the polynomial ker $\phi_{1,d-1} = 0$ so that rank$\phi_{1,d-1} = \dim W = n$(this means geometrically that Zeros(ϕ) is not a cone over a variety in a sub- space of lower dimension).
If ϕ splits in two smaller factors, namely $\phi = \chi \cdot \psi$, the set of singular points of Zeros(ϕ) contains the intersection of $\chi = 0$ and $\psi = 0$ which has codimension 2 in $\mathbb{P}^{n-1}$. Then for the theorem 7.3 we obtain

$$R(\phi) \geq n + (n-3) + 1 = 2n-2$$

If, on the other hand, the polynomial ϕ has a repeated factor, namely $\phi = \psi^2 \cdot (rest)$, then $\sum_1(\phi) \supset \{\ \psi = 0\}$, which has codimension 1 in $\mathbb{P}^{n-1}$. Then

$$R(\phi) \geq n + (n-2) + 1 = 2n-1$$

□

6.3 The Alexander-Hirschowitz theorem

We recall that the Waring rank wrk(F), for $F \in S^dV^\vee$, with dimV=n+1, is the smallest natural number r such that

$$F = l_1^d + ... + l_r^d$$

for linear forms $l_1, .., l_r$.
As we have seen the naive way (a direct approach) to compute the Waring rank is by counting constants. Consider the map

$$(V^\vee)^r \to \mathbb{C}^{\binom{d+n}{n}}$$

given by

$$(l_1, ..., l_r) \mapsto \sum_{i=1}^{r} l_i^d$$

If $\dim(V^\vee)^r \geq dim\mathbb{C}^{\binom{d+n}{n}}$, that is $r(n+1) \geq \binom{d+n}{n}$ one expect that this map is surjective so that wrk(F)$\leq$ r for general F.
The expected generic Waring rank (symmetric tensor rank) is

$$\left\lceil \frac{\binom{d+n}{n}}{n+1} \right\rceil$$

and the expected dimension of

$$\sigma(\nu_d(\mathbb{P}V)$$

is

$$min\left(\binom{n+d}{n} - 1, r(n+1) - 1\right)$$

Alexander and Hirschowitz showed that the varieties $\sigma_r(v_d(PV))$ are all of the expected dimension with few list of exception (see the following paragraph 6.4); thus they solved the problem of the computation of border rank of a generic polynomial of degree d in n+1 variable over $\mathbb{C}$.

Theorem 6.11 (Alexander-Hirshowitz)**.** *Let $d \geqslant 3$. Then $\sigma_r(v_d(PV))$ is defective if and only if*
i) d=3, n=4 and r=7
ii) d=4 and (n,r)=(2,5),(3,9),(4,14)

We can give an equivalent formulation of the Alexander-Hirschowitz theorem for polynomial in the following way:

Corollary 6.12. *The general form $f \in S^d(V)$, dimV=n+1, is a sum of*

$$s = \left\lceil \frac{1}{n+1}\binom{n+d}{n} \right\rceil$$

powers of linear forms, unless
d=2,where s=n+1 instead of $\lceil \frac{n+2}{2} \rceil$,
d=4 and n=2,3,4, where s=6,10,15 instead of s=5,9,14,
d=3 and n=4 where s=8 instead of s=7.

This theorem is difficult to prove and there are simplifications of the proof in a paper of [4].

6.4 Clebsch's theorem

The exception of degree 4 in 3 variables was known for a long time and is the core of a Memoir of Clebsch [10]. In 1860, A. Clebsch wrote a paper "*Über* Curven vierten Ordnung" ("Over curves of fourth order") in which he considered the representation of a form of fourth degree in three variables,

$$f \in S^4\mathbb{C}^3,$$

as a sum of five 4-th powers of linear forms and in order to resolve this problem he thought a quartic f as a "quadric of quadrics"
and gave an example of defective varieties because he showed that $\sigma_5(v_4(\mathbb{P}^2))$ that is contained in $\mathbb{P}(S^4(V^3)) = \mathbb{P}^{14}$ has not the expected dimension.
He represented a quartic f as 6x6 Hankel matrix (Catalecticant) C_f whose entries are the fifteen coefficients of f with respect to basis $(x^2, 2xy, 2xz, y^2, 2yz, z^2)$.
That is, given f
$f = a_{00}x^4 + 4a_{10}x^3y + 4a_{01}x^3z + 6a_{20}x^2y^2 + 12a_{11}x^2yz + 6a_{02}x^2z^2 + 4a_{30}xy^3 + 12a_{21}xy^2z + 12a_{12}xyz^2 + 4a_{03}xz^3 + a_{40}y^4 + 4a_{31}y^3z + 6a_{22}y^2z^2 + 4a_{13}yz^3 + a_{04}z^4 = X^t \cdot C_f \cdot X$
where C_f is given by

$$\begin{pmatrix} a_{00} & a_{10} & a_{01} & a_{20} & a_{11} & a_{02} \\ a_{10} & a_{20} & a_{11} & a_{30} & a_{21} & a_{12} \\ a_{01} & a_{11} & a_{02} & a_{21} & a_{12} & a_{03} \\ a_{20} & a_{30} & a_{21} & a_{40} & a_{31} & a_{22} \\ a_{11} & a_{21} & a_{12} & a_{31} & a_{22} & a_{13} \\ a_{02} & a_{12} & a_{03} & a_{22} & a_{13} & a_{04} \end{pmatrix} \tag{6.1}$$

is it possible to decompose f like $\sum_{i=1}^5 l_i$, where l_i are linear forms?
The expected dimension of such f would be : $3 \times 5 - 1 = 14$, where 3 is the number of variables, 5 is the number of forms and 1 because in the projective all is up to a constant. But this is not so. In fact Clebsch found a condition to represent such forms namely the vanishing of a 6x6 determinant of a Henkel matrix C_f made of the coefficients of the quartic. We observe that if $f = l^4$ the matrix C_f has rank 1, so if $f = \sum_{i=1}^5 l_i^4$ we have

$$rk(C_f) = rk \sum C_{l_i^4} \leq \sum rkC_{l_i^4} = 5,$$

so the five-secant variety of the Veronese $v_4(\mathbb{P}^2)$ is contained in the degree 6 hypersurfaces $\det C_f = 0$.
We can state the theorem of Clebsch of 1860 in the following form:

Theorem 6.13 (Clebsch)**.** *([10]) Let $D_1, D_2, .., D_6$ the differential operators of the second order*
$D_1 = \frac{\partial^2}{\partial x^2}, ...D_6 = \frac{\partial^2}{\partial z^2}$.
Then the quartic f has border rank ≤ 5 and it can be decomposed like a sum of five 4-powers of linear forms if and only if the matrix
with constant coefficients $(D_i D_j f)$ has determinant equal to zero (this determinant is the so called catalecticant invariant) iff does exist a apolar conic with respect to that quartic.

With the words of Clebsch:
"Die Kurven also, deren Gleichungen durch funf Biquadrate darstellbar sind, bilden eine specielle klasse, welche dadurch charactererisirt ist, dass ihre Invariante A verschwindet."
"The quartic curves, that can be represented as a sum of five fouth-powers of linear forms belong to a special set and are characterized by taking to 0 the A invariant". The invariant A is the determinant of the matrix (10).
We can give the following modern geometric interpretation:
the map

$$\mathbb{P}^2 \to V \subset \mathbb{P}^{14}$$

sends the homogeneous coordinates (x, y, z) to the Veronese variety of homogeneous polynomials of degree four that are of the form l_i^4 with l_i linear and that is contained in $\mathbb{P}^{14}$, the hyperspace of all polynomials of degree four.
The secant variety σ_1 is given by the sums of two 4-powers that is

$$\sigma_1(V) = \left\{ \phi = \sum_{i=1}^{2} \phi_i^4 \right\}$$

The secant variety σ_2 is the set of the quartics of the form

$$\sigma_2(V) = \left\{ \phi = \sum_{i=1}^{3} \phi_i^4 \right\}.$$

The variety that is spanned by the 5-uple, is the hypersurfaces of degree 6 given by the catalecticant equal to zero.
In term of polar n-sides we can restate the Clebsch theorem in the following form:

Theorem 6.14 (Clebsch-1861)**.** *If a plane quartic curve of equation $F_4(x, y, z) = 0$ has a polar 5-sides then*

$$det(\partial_i \partial_j F_4)_{1 \leq i \leq j \leq 6} = 0$$

In particular, there aren't general plane quartic curves with polar 5-sides.

The polar 6-sides of quartic curves were intensively studied by G. Scorza at the end of the nintheen century with a memory [39].
Let K be a field of characteristic 0 and let $\mathbb{S}^d K^n$ the space of homogeneous polynomials of degree d in n variables. We give here a definition of rank $R(\phi)$ of a polynomial ϕ of $\mathbb{S}^d K^n$ like

$$R(\phi) = min\{r \in \mathbb{Z} / \phi = \sum_{i=1}^{r} l_i^d,\ l_i \in K^n\}.$$

We define the border rank $\underline{R}(\phi)$ of a polynomial ϕ in such way:
$\underline{R(\phi)}$ is the smallest r such that ϕ is in the Zariski closure of the set of polynomials of rank r in $S^d K^n$.
For example if we have a binary form $f(x, y)$ like

$$f(x, y) = \sum_{i=0}^{d} \binom{d}{i} a_i x^{d-i} y^i$$

of degree d over K the rank of f is the minimum $r \in Z$ such that exist a decomposition

$$f(x, y) = \sum_{j=1}^{r} c_j (l_j)^d$$

where l_j are linear forms in x, y and $c_j \in K$. If $K = \mathbf{R}$ the coefficient c_j can be 1 or -1 because we always put it in $(l_j)^d$.

6.5 Hessian of the cubic f=0

Let f be a nonsingular cubic in the projective plane $\mathbb{P}^2(k)$, where k is an algebraically closed field such that $ch(k) \neq 3$, defined by a homogeneous cubic equation $f(x, y, z) = 0$.
To find degenerate polar conic of f, let us write the equation of the polar conic $P_Y(f)$ with respect to the point Y like:

$$P_Y(f) = \left(\sum x_i \frac{\partial f}{\partial y_i} \right)^2$$

where the symbolic square means the we have taken the polar of the polar. This polar conic is a pair of lines if the discriminat

$$H = \begin{vmatrix} \frac{\partial^2 f}{\partial y_1^2} & \frac{\partial^2 f}{\partial y_1 y_2} & \frac{\partial^2 f}{\partial y_1 y_3} \\ \frac{\partial^2 f}{\partial y_2 y_1} & \frac{\partial^2 f}{\partial y_2^2} & \frac{\partial^2 f}{\partial y_2 y_3} \\ \frac{\partial^2 f}{\partial y_3 y_1} & \frac{\partial^2 f}{\partial y_3 y_2} & \frac{\partial^2 f}{\partial y_3^2} \end{vmatrix}$$

is zero. So there are infinitely many points Y such that the polar conics degenerate in two lines and the locus of such points is given by the above equation that is a cubic curve called the Hessian of f.

Definition 6.15. The Hessian curve $H(F)$ of a plane cubic curve $F = 0$ is the plane cubic curve defined by the equation $H(F) = 0$, where $H(F)$ is the determinant of the matrix of the second partial derivatives of F.
The inflection points of $F = 0$ are the nine points in $F \cap H(F)$.

The Hessian of a cubic polynomial F is a covariant of F:
let's start with $F = Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3$:
the Hessian of F is

$$H(F) = 36 \begin{vmatrix} Ax + By & Bx + Cy \\ Bx + Cy & Cx + Dy \end{vmatrix}$$

that is,

$$x^2(AC - B^2) + 2xy(\frac{AD - BC}{2}) + y^2(BD - C^2)$$

and this is a polynomial in x, y, A, B, C, D.
In general we have

Proposition 6.16. *The hessian of a binary form F is a covariant of weight 2 of the form F (precisely a covariant of weight -2)*

Proof. Let F be a binary form. Then we can write F as:

$$F = xF_x + yFy$$

by Euler's theorem, that is,

$$F = x(xF_{xx} + yF_{xy}) + y(xF_{yx} + yF_{yy})$$

because F_x and F_y are homogeneous polynomial too.
Then

$$F = x^2F_{xx} + 2xyFxy + y^2F_{yy} = XHX^t = X'H'X^{t\prime}$$

where H is the Hessian matrix and $'$ denote the change of coefficients.
Then we obtain

$$(detg)^2(F_{x'x'}F_{y'y'} - F^2_{x'y'}) = F_{xx}F_{yy} - F^2_{xy}.$$

so the Hessian $H(F)$ is a covariant of weight -2. □

If we consider the discriminant of the Hessian H of the form

$$\varphi = a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3$$
$$\begin{vmatrix} a_0a_2 - a_1^2 & a_0a_3 - a_1a_2 \\ a_0a_3 - a_1a_2 & a_1a_3 - a_2^2 \end{vmatrix} = \begin{vmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{vmatrix}$$

this will be an invariant of the form φ of weight 6 because the tensor $h_{ij}h_{kl}$ has weight 4.
Four times the value of this invariant is called the discriminant of the cubic binary form φ; it is denote by the symbol D and is given by:

$$D = 4(a_0a_2 - a_1^2)(a_1a_3 - a_2^2) - (a_0a_3 - a_1a_2)^2 = 3a_1^2a_2^2 + 6a_0a_1a_2a_3 - 4a_0a_2^3 - 4a_1^3a_3 - a_0^2a_3^2$$

Another covariant of the cubic binary form φ is the Jacobian Q of the form itself and of twice its Hessian H.
The expression of the covariant Q is:
$Q = (a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3)x^3 + 3(a_0a_1a_3 + a_1^2a_2 - 2a_0a_2^2)x^2y +$
$+ 3(-a_0a_2a_3 - a_1a_2^2 + 2a_1^2a_3)xy^2 + (-a_0a_3^2 + 3a_1a_2a_3 - 2a_2^3)y^3$

In degree 2 the complete system of forms of a binary quadratic form is given by the form itself and the discriminant of the form.
In degree 3, $S^3\mathbb{K}^2$, the complete system of invariants and covariants is generate by the binary form f, its discriminant D, its Hessian H and Q the jacobian of f and $2H$. We have a relation (a "syzygy")found by A.Cayley:

$$4H^3 + Df^2 + Q^2 = 0$$

This syzygy has a key role in the solution of the cubic equation.

6.6 Cubic equation

Let $f = 0$ a cubic equation, $f = a_0x_1^3 + 3a_1x_1^2x_2 + ...$ We have two cases depending on D.
If $D \neq 0$, from the syzygy of Cayley, we can write:

$$H^3 = \frac{1}{4}(\sqrt{D}f - Q)(\sqrt{D}f + Q).$$

We prove that in this case the two factors haven't a common factor.If this were the case, then both f and Q will have a linear factor and this is no possible by the covariant propriety of f,Q,H,D.
Thus, if, in this case, we set $H = l \cdot m$, with $l \neq m$, then we obtain

- $\frac{1}{2}(\sqrt{D}f - Q) = l^3$
- $\frac{1}{2}(\sqrt{D}f + Q) = m^3$

so that

$$f = \frac{1}{\sqrt{D}}(l + m)(l + \epsilon m)(l + \epsilon^2 m)$$

with $\epsilon = \sqrt[3]{1}, \epsilon \neq 1$.
If $D = 0, H \neq 0$.We have

$$Q^2 = -4H^3$$

and suppose that $f = x_2^2$ ·linear part. This is possible with a suitable linear transformation.
Then the coefficients a_0, a_1 are zero.In this case H contains the factor x_2^2 too, so we can take the square root of H to obtain the double factor of f.
Now it is possible to compute the third factor by division and this is different from the double one because otherwise the hessian H have to be zero.
The third and last case is when $D = 0$ and $H = 0$ identically.Then $Q = 0$ also.
In this case f is a third power of a linear form, $f = l^3$ with l linear form, and we can solve the equation taking the third root of f.
So we get three cases:

- $D \neq 0$ 3 distinct roots
- $D = 0, H \neq 0$ 1 double root
- $D = 0, H = 0$ 1 triple root.

Chapter 7

Classification of real ternary cubics

7.1 Classification of real ternary cubics

If the characteristic of the field $\mathbb{K}$ is $\neq 2$ or 3 (for example $\mathbb{K} = \mathbb{R}$) a non singular cubic admits a Weierstrass canonical form (first canonical form) given by

$$y^2 = x^3 + ax^2 + bx + c \tag{7.1}$$

or,in projective coordinates, after a change of variable that make zero the term in x^2,

$$y^2 z = x^3 + \alpha x + \beta. \tag{7.2}$$

We follow [9] for the classification of real plane cubics:
Let $F(x, y) = 0$ be a real cubic plane curve, that is a cubic curve with real coefficients.
The cubic F has 9 flexes, the points of intersection of $F = 0$ and $He(F) = 0$ the Hessian of F so almost one is necessary real and this happened also if F has a node, and in this case it has 3 real flexes or if F has a cusp, and in this case it has 1 flex.
So the harmonic polar l and the inflexion tangent ,t that is the tangent line through the flex, are real too. This mean that there is a real projectivity that sends the flex in the infinity point of the axis y,
the line l on the axis x, and the line t on the infinity line of the projective plane $\mathbb{P}^2$.
The transformated of the cubic has equation

$$y^2 = ax^3 + bx^2 + cx + d \tag{7.3}$$

because this curve must be symmetric with respect to x axis (invariant for the harmonic homology having the center in the point at infinity of Y_∞ axis and

with axis the improper line $y = 0$).
Now with the projectivity

$$y = \frac{1}{a}Y \quad x = \frac{1}{a}X$$

the equation of the cubic result

$$Y^2 = X^3 + a_1X^2 + a_2X + a_3. \tag{7.4}$$

We can now replace x and y with X and Y to obtain

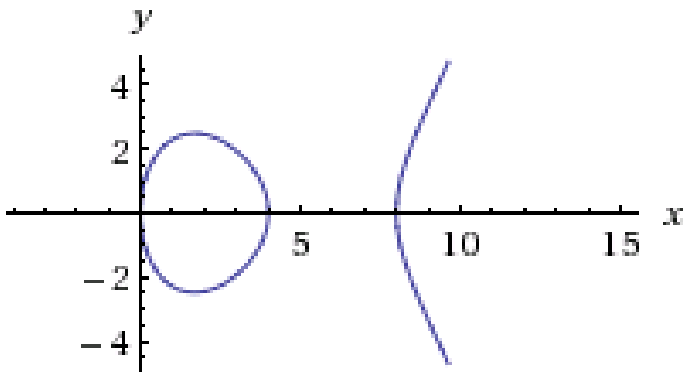
$$y^2 = x^3 + a_1x^2 + a_2x + a_3.$$

If α,β,γ are the three roots of the cubic polynomial of the canonical form we have

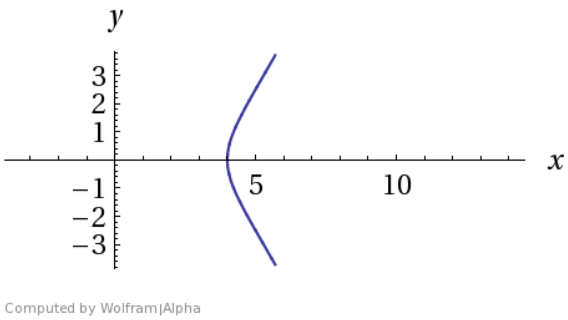
$$y^2 = (x-\alpha)(x-\beta)(x-\gamma). \tag{7.5}$$

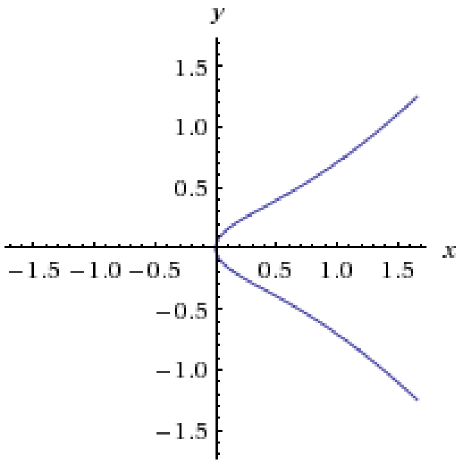
There are five cases which correspond to five different forms of the cubic F:

1. α,β,γ, real and different each other; for example $\alpha < \beta < \gamma$ and the cubic will be
$$y^2 = x(x-4)(x-8)$$
(see picture of "parabola campaniformis cum ovali")



2. α,β,γ real with the first two equal, for example $\alpha = \beta < \gamma$ and the cubic will be
$$y^2 = x^2(x-4)$$
(see picture of "parabola punctata")





3. α,β,γ real with the second two equal, as $\alpha < \beta = \gamma$ and the cubic will be

$$y^2 = x(x-2)^2$$

4. all the roots real and equal with $\alpha = \beta = \gamma$, like the cubic

$$y^2 = x^3$$

5. only one real root, for example the cubic

$$y^2 = x(x-i)(x-i) = x^3 + x,$$

(see picture “parabola campaniforme semplice“) where i is the quadratic root of -1.

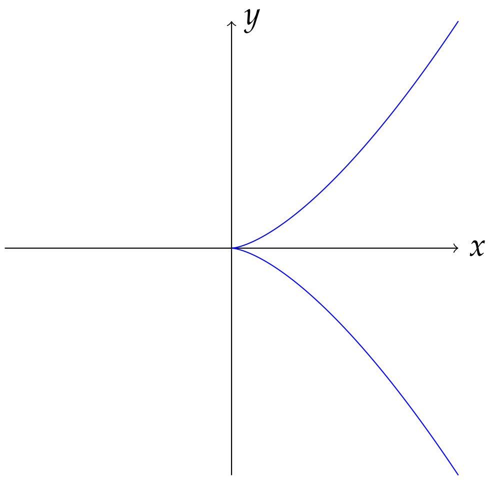


Figure 7.1: Cusp

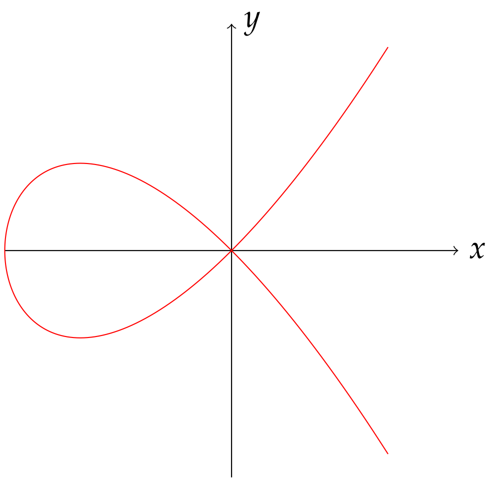


Figure 7.2: node

Now let S and T be the two relative invariant of degree four and sex respectively of a given plane cubic.
From the general expressions of S and T(cfr.[38] or [16]), we have that if the curve is given by the equation

$$ax^3 + 3bx^2 + 3cx + d + 3ey^2 = 0 \tag{7.6}$$

the relative invariant are

$$S = e^2(b^2 - ac); T = 4e^3(2b^3 + a^2d - 3abc)$$

and if R is the discriminant (called -Δ in [41])

$$R = 64S^3 - T^2 \tag{7.7}$$

we will have

$$R = 16e^6[4(b^2 - ac)^3 - (2b^3 + a^2d - 3abc)^2] \tag{7.8}$$

so that

$$R = -16e^6a^2\Delta \tag{7.9}$$

where

$$\Delta = a^2d^2 - 3b^2c^2 + 4b^3d + 4ac^3 - 6abcd \tag{7.10}$$

is the discriminant of the equation

$$ax^3 + 3bx^2 + 3cx + d = 0. \tag{7.11}$$

Now, the last equation 7.11 has 3 real distinct roots or 1 real and two imaginary if Δ is negative or positive respectively; thus if $R > 0$ the cubic is a "parabola campaniformis cum ovali" and if $R < 0$ is a "parabola campaniformis simple" (or "parabola campaniformis pura") If $\Delta = 0$ (and so $R = 0$ too) we can write 7.6 as

$$a(x + \frac{b}{a} + \frac{\sqrt[3]{T}}{ae})(x + \frac{b}{a} - \frac{\sqrt[3]{T}}{2ae})^2 + 3ey^2 = 0 \tag{7.12}$$

or if we set

$$x' = x + \frac{b}{a} + \frac{\sqrt[3]{T}}{ae} \tag{7.13}$$

we obtain

$$a^2x'(x' - \frac{3\sqrt[3]{T}}{2ae})^2 + 3aey^2 = 0 \tag{7.14}$$

The real part of the cubic is in $x' > 0$ or in $x' < 0$ if $ae < 0$ or $ae > 0$ respectively; so the double point

$$x' = \frac{3\sqrt[3]{T}}{2ae}, y = 0 \tag{7.15}$$

is on the same side of the curve or on the opposite side if T is negative or positive. Thus 7.6 represents a nodal cubic if $R = 0$ and $T < 0$ and represent a "parabola punctata "if $R = 0$ and $T > 0$.
If $R = 0$, $T = 0$ (so that $S = 0$ too) the first member of 7.11 is a perfect cube and 7.6 represent a cusp.
The three quantity R,S,T are projective invariant, so if we apply a projectivity to a cubic curve, these three numbers don't change the sign.
Hence, for a generic plane cubic curve,

1. if the discriminant 7.7 is positive, the curve has two different pieces (two connected components).
   One of this component has three real flexes and the other component has no singular points and can be proiected in an "oval", that is a closed smooth curve.

2. if the discriminant 7.7 is negative, the cubic curve has only one connected component with three real flexes.

3. if the discriminant 7.7 is 0, so that $S = 0$ too, the curve has only one connected component and has one double point, and this double point is a node, an isolated point or a cusp if the other basic invariant T is negative, positive or 0.

7.2 Real Cubics

If the equation of cubic C has real coefficients the equation of nine degree that define the flexes of C has always a real root (because nine is odd), so it is always possible to riduce C to the form

$$y^2 = k(x^3 - px + q)$$

with a real trasformation.
Now, to reduce the factor k to one, we have to change

$$y \to \sqrt{k} \cdot y$$

and this is a real transformation if $k > 0$; instead if $k < 0$ we can change

$$x \to -x$$

so we have

$$y^2 = k(-x^3 + px + q) = -k(x^3 - px - q)$$

so that we have already the normal form.
From this normal form it's possible to understand the form of C:
indeed, if

$$\varphi(x) = x^3 - px + q$$

has 3 real roots a, b, c with $a < b < c$ and this happened if and only if $\Delta < 0$.
In this case

$$\varphi(x) = (x - a)(x - b)(x - c)$$

and

$$y = \sqrt{\varphi(x)}$$

is real only for

$$a \leqq (x) \leqq (b)$$

and for

$$x \geqslant c.$$

With regard to this we have two real component ("rami") of C: a closed component homeomorphic to S^1 called even component because every line cut S^1 in two or in zero points and an open component called odd component because every line cut this component in one or three points. The first component has no flex while the second has two symmetric flex respect to the x axis and the flex at infinity, so the cubic has 3 real flex.
All of this is based on the theorem:

Theorem 7.1. *(called Mac-Laurin Theorem in [21], see also [3])*
The line that contain two flexes of a cubic curve contain also a third flex.

For the proof of this theorem see [3].
This curve is called "parabola campaniformis cum ovali" by Newton.
If $\varphi(x)$ has only one real root $x = a$ and the other two complex coniugate, so that $\Delta > 0$ we have

$$\varphi(x) = (x - a)\psi(x)$$

where ψ is a polynomial of degree two never 0 for example

$$\psi > 0$$

and we have

$$y = \sqrt{\varphi(x)} = \sqrt{x-a}\,\sqrt{\psi(x)} \tag{7.16}$$

and this is real for $x \geq (a)$ so the cubic is formed by only one component on which there are two real flexes plus the flex at infinity.
This curve is called by Newton "parabola pura campaniformis"
Now we consider the irriducible cubics with a double point (node or cusp).
In this case $\Delta = 0$ so the equation

$$\varphi = 0$$

has one double root $a = b$ and another different real root c so that

$$\varphi(x) = (x-a)^2 \cdot (x-c)$$

and then computing the product

$$(x-a)^2 \cdot (x-c) = x^3 - (2a+c)x^2 + (2ac + a^2)x - a^2c$$

we can put

$$c = -2a$$

so that the quadratic term is zero. So we obtain

$$y = \sqrt{\varphi(x)} = (x-a)\cdot \sqrt{x-c} = (x-a)\cdot \sqrt{x+2a}$$

so the cubic has only one component that exist if $x \geq -2a$.
There is a double point for $x = a$ and this is a node with two real tangent if

$$a > 0$$

or an isolated point if

$$a < 0.$$

These two case were called by Newton "parabola nodata" and "parabola punctata".

Theorem 7.2. *The cubic with a node has only one flex (at infinity).*

Proof. (cfr.[21]) Let C be a cubic with a double point O. If it has two real flexes A, B, it has three flexes A, B, C, all on a line a.
Then exist two harmonic homology that send C to C and change AB and AC, so the product of this two transformation change ciclically A, B, C. □

On the othe end the "parabola punctata" has three flexes, two real (symmetric) and one at infinity.
So we can say that the classification of ternary cubic forms is the following: let

$$T := \left\{ f(X, Y, Z) \in S^3\mathbb{C}[X, Y, Z] \right\}$$

T is the complex vector space of dimension 10 of ternary cubic forms.
On this vector spaces acts the group $SL(3)$ with linear change of variables. Throught the classification of the orbits in T we associate to each form f his zero locus in the projective space $\mathbb{P}^2$:

$$V(f) := \left\{ (x, y, z) \in \mathbb{P}^2 | f(x, y, z) = 0 \right\}$$

7.3 Classification of reducible cubic plane curves

We describe the classification of these plane curves with respect to projective equivalence.

1. f is the product of 3 linear factors $l_1 = l_2 = l_3$: that is $f = X^3$ the picture is a triple line.

2. f is the product of 3 linear factors l_1, l_2, l_3 with $l_2 = l_3$: that is $f = X^2Y$ and a picture is a double line plus a line.

3. f is the product of 3 concorrent lines: that is $f = XY(X + Y)$ and a picture is three lines that go through a point.

4. f is the product of 3 distint linear factors l_1, l_2, l_3 :that is $f = XYZ$ and a pictures is a triangle.

5. f has an irriducible factor of degree 2, so $V(f)$ is composed of a conic Q and a line l.

6. if $\sharp(Q \cap l) = 1$ then $f = (X^2 - YZ)X$.

7. if $\sharp(Q \cap g) = 2$ then $f = (X^2 - YZ)X$
   in the first case the line l is tangent to Q and in the second one l is secant to Q.

8. f is irriducible, so $V(f)$ is a irriducible cubic $\mathrm{C} \subset \mathbb{P}^2$.

9. f is irreducible but singular C has one cusp: f=$Y^2Z - X^3$.

10. f is irreducible but singular C has one double point f=$Y^2Z - X^3 - X^2Z$.

11. C is smooth $f = Y^2Z - X^3 - aX^2Z - bXZ^2 - cZ^3$ with a,b,c $\in \mathbb{C}$.

Chapter 8

Historical note

8.1 Waring's problem

The problem of computing the rank of a homogeneous form is an old one and is strictly connected to the Waring problem in number theory.
E. Waring, in 1770, in his book "*Meditationes Algebraicae*"[44], stated a problem that we can reformulate as:

Waring's problem

Given positive integers d,h may we write any positive integer as a sum of h non-negative d-powers?
The least number of d-th powers, g:=g(d), needed to represent every sufficiently large positive integer is the analog of the rank for forms that we define later.
Waring stated without proof the following (see for example [26]):
a) every positive integer is the sum of (at most) 4 positive squares;
b) every positive integer is a sum of (at most) 9 positive cubes;
c) every positive integer is a sum of (at most) 19 biquadrates; and so on...
Perhaps Waring believed that for every natural number $j \geq 2$, there is a number $n(j)$ such that every positive integer n can be written in the following way:

$$n = a_1^j + ... + a_{n(j)}^j$$

where $a_i \geq 0$.
If such $n(j)$ exists, we call $g(j)$ such minimum.
With the above notations, Waring's statements are the following claims:

1. g(3)=9
2. g(4)=19

3. g(j) exists.

(In this settings the famous Lagrange's theorem says that $g(2) = 4$. Note that this theorem is sharp because there are integers like 7 which cannot be written as a sum of three squares).
We have for example,

$$31 = 5^2 + 2^2 + 1^2 + 1^2$$

$$87 = 7^2 + 5^2 + 3^2 + 2^2$$

The Waring's problem was solved affirmatively by David Hilbert in 1909.

Theorem 8.1 ((Hilbert-1909)**.** *$g(j)$ exists for every $j \geq 2$.*

Now we know that $g(3) = 9$ and $g(4) = 19$ but the determination of $g(j)$ for every j is not yet understood.
The problem above is called "Little Waring problem" by([23]).
G. Hardy and other number theorists observed that although g(3)=9 only 23 and 239 requires 9 cubes in their sum decomposition that is

1. $23 = 2^3 + 2^3 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 2 \times 2^3 + 7 \times 1^3$
2. $239 = 5^3 + 3 \times 3^3 + 4 \times 2^3 + 1 = 2 \times 4^3 + 4 \times 3^3 + 3 \times 1^3$

and only 15 other numbers (8042 for example) requires 8 cubes.
Hardy and Wright in their book "Introduction to number theory"([26]) gave the following definition:

Definition 8.2. let $G(j)$ be the last number such that all sufficiently large integers are the sum of $\leq G(j)$ j-th powers of integers.

From this follow that $G(j) \leq g(j)$, and $G(3) \leq 7$. It is not known if $G(3) \leq 7$,but it is know that $G(4) \leq 16$.
It can happen that $G(j) = g(j)$, for example when $j = 2$ it was stated by Gauss that every natural number $n \equiv 7(\mod 8)$ is a sum of 4 (and not 3) squares, that is $G(2) = g(2) = 4$.

8.2 Newton's classification of cubics

Newton, in his "Enumeratio linearum tertii ordinis" first published in 1704 as an appendix to his "Optics" was able to apply the methods of analytical geometry to an exhaustive classification of cubic curves in the same spirit to that conics were divided into three species.
The first appendix to the optics was the essay on cubics while the second was "De Quadratura Curvarum".
The Treaty is divided into seven sections in which he explains what is the order of a curve, what is the genus and gives some example of curves of degree one, two and three.
The third section is the most important for us, because here Newton shows that the equation of a cubic can be always be written in one of four canonical forms, namely:

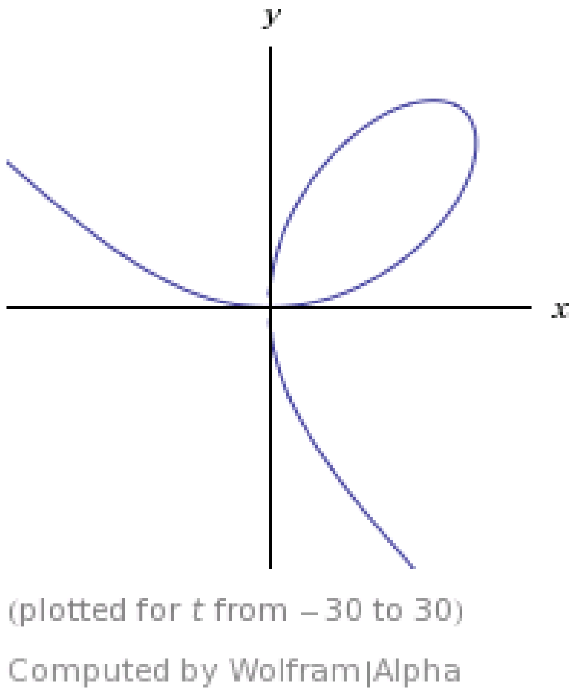
1. i) $xy^2 + ey = ax^3 + bx^2 + cx + d$
2. ii) $xy = ax^3 + bx^2 + cx + d$
3. iii) $y^2 = ax^3 + bx^2 + cx + d$
4. iv) $y = ax^3 + bx^2 + cx + d$

In this section Newton describes his nomenclature for each curve and divides the four canonical forms into seven classes that are subdivided into fourteen genera which contain seventy-two species although the degenerate forms of a conic plus a straight line and three straight lines are not considered. Newton missed six species according to his classification scheme which allows affine coordinate changes. There are 78 species of cubics (cfr. for example[36]). The classification of Newton is respect to the asymptotic behavior of cubics;this approach is natural because the behavior at infinity shows well the shape of the curve. For example the "folium of Descartes"

$$x^3 + y^3 - 3xy = 0$$

has as asymptote the line

$$x + y + 1 = 0$$



Decartes.png

and this simplifies the representation of the curve.
Newton had the idea of regarding an asymptote of a curve as a line to which the curve is tangent at infinity.
This idea works very well for algebraic curves, since such curves extend in a simple fashion to a space that contain points at infinity.
So the search of asymptotes leads in a natural way from $\mathbb{R}^2$ to the projective plane $\mathbb{P}^2$.
Regarding the cubics as lying in the projective plane and enlarging the coordinate change from the affine to the projective one, the Newton's classification can be simplified to 5 different species.
The first canonical form was divided into four great classes, so we have the "Redundant Hyperbolas", the "Defective Hyperbolas or Elliptic Hyperbolas", the "Parabolic Hyperbolas" and the "Hyperbolisms of conics " which correspond respectively to $a > 0$ to $a < 0$, to $a = 0$ and to $a = 0$ and $b = 0$.
The second canonical form was called "The Trident", the third "Diverging Cubic Parabolas or Neilian Parabolas"(in memory of of W. Neil, the first mathematician who succeed to rectifying a curve) and the fourth "The Cubic Parabolas" (sometimes called Wallisian Parabola in memory of John Wallis, professor of mathematics at Oxford).
While in the section four Newton enumerated all the possible species of cubics,

in the section five he made a remarkable discovery, that every cubic curve may be regarded as the shadow of one of the five diverging parabolas cast by a luminous point on a plane properly situated.
No proof of this very remarkable proposition is given by Newton and the earliest mathematician who gave a proof of it was Nicole (communication to French Academy on 1/12/1731).
On 12/12/1731 Clairaut gave another proof that is contained in his paper "Sur les courbes que l'on forme en coupant une surface courbe qualconque par un plan de position" (Memoires de l'Academie des Sciences 1731).
Let us see how Newton arrives to the distinction of four canonical forms; he considers the diametral conic with respect to a point "at infinity" of the cubic. That is, if we take as a ccordinate reference a line to this point of equation $x = 0$, then the equation of a cubic is of the following form:

$$y^2(ax + b) + y(cx^2 + dx + e) - (fx^3 + gx^2 + hx + i) = 0 \tag{8.1}$$

The locus of middle points of chords parallel to Y axis is the conic

$$2axy + 2by + cx^2 + dx + e = 0 \tag{8.2}$$

that is a hyperbolic curve with an asymptot parallel to the vertical axis that is also asymptot for the cubic.
The diametral equilateral hyperbola has equation $xy = k$, so the correspondent coordinate transformation take

$$b = c = d = 0. \tag{8.3}$$

So we have, if $a \neq 0$, the equation of the cubic:

$$I) \quad xy^2 + ey = fx^3 + gx^2 + hx + i. \tag{8.4}$$

When $a = 0$ the diametral conic become a parabola, and two points "at infinity" are the same but the third are different until $c = 0$, so if we take this point as infinity point we have again the type I) of the cubic.
On the other hand, if $a = 0$ and $c = 0$ (so in this case the cubic has the flex tangent to the line at infinity), the diametral conic factorize like the line

$$2cy + dx + e = 0 \tag{8.5}$$

plus the infinity line.
We can now take this line like the x axis and if $b = 1$ and $d = e = 0$ we have

$$II) \quad y^2 = fx^3 + gx^2 + hx + i \tag{8.6}$$

or if $b = e = 0$ and $d = 1$ we have

$$III) \quad xy = fx^3 + gx^2 + hx + i \tag{8.7}$$

and if $b = d = 0$ and $e = 1$ we have

$$IV) \quad y = fx^3 + gx^2 + hx + i. \tag{8.8}$$

8.3 The Hesse equation

The Hesse canonical form of a plane cubic curve, called the second canonical form in [40], after the first canonical form of Weierstass, is

$$f = x^3 + y^3 + z^3 + 6mxyz = 0.$$

The condition of nonsingularity is

$$1 + 8m^3 \neq 0 \tag{8.9}$$

The three partial derivatives of $\frac{1}{3}f$, are

$$x^2 + 2myz, \quad y^2 + 2mxz, \quad z^2 + 2mxy \tag{8.10}$$

so the Hessian $He(f)$ of the curve has the following equation:

$$He(f) = \begin{vmatrix} \mathrm{x} & \mathrm{mz} & \mathrm{my} \\ \mathrm{mz} & \mathrm{y} & \mathrm{mx} \\ \mathrm{my} & \mathrm{mx} & \mathrm{z} \end{vmatrix} = (1 + 2m^3)xyz - m^2(x^3 + y^3 + z^3).$$

In particular, the member of the Hesse pencil corrisponding to the parameter $m \neq 0$ is equal to

$$x^3 + y^3 + z^3 - \frac{1 + 2m^2}{m^2}xyz = 0$$

or if $m = 0$ the Hessian is equal to

$$xyz = 0.$$

We have the

Proposition 8.3. *Any smooth plane cubic C_3 is projectively equivalent to one of the Hesse pencil (Hesse canonical form):*

$$x^3 + y^3 + z^3 + \lambda xyz = 0$$

Proof. We sketch the proof [see Weber Lehrbuch der Algebra II].Let C_3 be a nonsingular cubic. Take three inflexion tangents for C_3 and let it be $x = 0$,$y = 0$ and $ax + by + cz = 0$. Then the equation of C_3 is given by:

$$xy(ax + by + cz) + dz^3 = 0$$

Let $c = 0$ so that $ab \neq 0$ otherwise the curve would be singular.
Recall now that a binary cubic with no repeated roots can be reduced, with a linear change of variables, to the sum of two cubes, so that $xy(ax+by) = x^3 + y^3$;then the equation of c_3 is of the form

$$x^3 + y^3 + dz^3 = 0.$$

After scaling z we get a Hesse equation.
If we assume $c \neq 0$, $c = 3$,and $\omega^3 = 1$, a primitive third root of unity, we define new variables u and v by

$$ax + z = \omega u + \omega^2 v$$

$$by + z = \omega^2 u + \omega v.$$

Then

$$\begin{aligned} abxy(ax + by + z) + dz^3 &= (\omega u + \omega^2 v - z)(\omega^2 u + \omega v - z)(-u - v + z) + dz^3 \\ &= -u^3 - v^3 + (d + 1)z^3 - 3uvz = 0. \end{aligned}$$

For the nonsingularity $d \neq -1$, so after scaling z we get the Hesse equation for C_3:

$$x^3 + y^3 + z^3 + \lambda xyz = 0.$$

□

8.4 Polar Polygons

A general ternary cubic form does not admit polar triangles because for any three general points in $\mathbb{P}^2$ there exists a reducible cubic that is singular at these points, that is the cubic made of three lines.
The cubic form has a polar triangle only when the invariant $S = 0$, that is only when the cubic is equianharmonic.

Proposition 8.4. *A plane cubic C has a polar triangle only in two cases:*

- *C is a Fermat cubic*

- *C is the union of three distinct concurrent lines.*

Proof. Assume $C = \left\{ l_1^3 + l_2^3 + l_3^3 = 0 \right\}$. Let l_1^3 not proportional to l_2^3. Then with a coordinate change we can we can write C as $C = \{x^3 + y^3 + l^3\}$.If the line l doesn't depend on z, the cubic C is the union of three concurrent lines. If, on the other hand, the line l depends on z we can assume that $l = z$ and we get the Fermat cubic. □

For a Fermat cubic its polar triangle is unique and the sises of this triangle are the first polar of the cubic which are double lines.
By counting constants, we have that a general cubic admit a polar quadrangle. A polar quadrangle l_1, l_2, l_3, l_4 is called nondegenerate if the dual of l_i i=1,2,3,4, are 4 points no three of which are collinear. A polar quadrangle is nondegenerate if and only if the linear system of conics in the projective space through the points $[l_i]$ is an irreducible pencil of conics.

Chapter 9

Geometry of plane cubics

9.1 Plane Cubics

In this section we recall some facts on the geometry of plane cubics.
Ternary cubic forms, in Cayley's symbolic notation $(a)(x,y,z)$,

$$\begin{aligned}(a)(x,y,z) = {} & a_{300}x^3 + 3a_{210}x^2y + 3a_{201}x^2z + 3a_{120}xy^2 \\ & + 6a_{111}xyz + 3a_{102}xz^2 + a_{030}y^3 + 3a_{021}y^2z + 3a_{012}yz^2 + a_{003}z^3\end{aligned}$$

or equivalently, plane cubic curves, depends on nine essential parameter, so they live in the vector space

$$V_{2,3} = \langle x^3, y^3, z^3, x^2y, y^2z, z^2x, xy^2, yz^2, zx^2, xyz\rangle.$$

As an affine space $\mathbb{V}_{2,3} \cong \mathbb{A}^{10}$ and it has coordinate ring

$$k[\mathbb{V}_{2,3}] = k[a_{300}, a_{030}, a_{003}, a_{210}, a_{021}, a_{102}, a_{120}, a_{012}, a_{201}, a_{111}].$$

A classical result of Aronhold (cfr.[1]) is that the ring of invariant $k[\mathbb{V}_{2,3}]^{SL(3)}$ is generated by two algebraically independent invariants S and T of degree four and six respectively.
They are called the invariant of plane cubics.
The invariant S is the well known invariant of ternary cubics of degree 4, and is the classical Aronhold invariant (cfr.[41] ,[25] or [16]) and is defined as (in symbolic notation):

$$S = (\alpha\beta\gamma)(\alpha\beta\delta)(\alpha\gamma\delta)(\beta\gamma\delta).$$

The sestic invariant T of plane cubics is defined as:

$$T = (\alpha\beta\gamma)(\alpha\beta\delta)(\beta\gamma\epsilon)(\alpha\gamma\zeta)(\delta\epsilon\zeta)^2.$$

The discriminant Δ of a cubic form in three variables is a linear combination of S^3 and T^2 and

$$\Delta = 0$$

is the closure of the locus of nodal cubic curves and the complete intersection

$$S = 0 \quad T = 0$$

is the locus of cuspidal ones.
To compute explicitly S T of a cubic C, we need of the (second) canonical form of the cubic, the Hesse form:

$$x^3 + y^3 + z^3 + 6\lambda xyz = 0.$$

Then

$$S = \lambda - \lambda^4, \quad T = 1 - 20\lambda^3 - 8\lambda^6, \quad \Delta = (1 + 8\lambda^3)^3. \tag{9.1}$$

The $SL(3)$-invariance of S and T means that they have geometric meaning(see [37] or [31]).
In particular, $S = 0$ means that the Hessian $H(f)$ of a cubic plane curve f factories as three lines the invariant S vanishes on a smooth cubic if and only if this curve is projectively isomorphic to the Fermat cubic (called anharmonic cubic).
The geometric meanings of the invariant T is that the Hessian of the Hessian of the plane cubic curve f is the same cubic up to a scalar, so

$$H(H(f) = f$$

if and only if $T = 0$. In general a plane curve C_n of order n is represented by an equation $f(x, y, z) = 0$, in which f is a polynomial of degree n, homogeneous in x,y,z. Since the number of terms in the polynomial f is $\frac{1}{2}(n + 1)(n + 2) = \binom{n+2}{2}$, the curve depends on the ratios of that number of coefficients and therefore it depends effectively on the values of $\frac{1}{2}n(n + 3)$ parameters.
Since to make C_n pass through a given point we impose a linear condition on the coefficients of the curve, it follows that there exists a C_n through $\frac{1}{2}n(n + 3)$ given points.
We say that a curve of order n has $\frac{1}{2}n(n + 3)$ degrees of freedom. If a cubic is irreducible it has at most one double point, because if a cubic would have two double points the line that go through them has four intersection with the cubic for the Bezout's theorem so it belong to the cubic.
This facts can be deduced from the following theorem that is a simple consequence of Bezout's theorem (9.2):

Theorem 9.1. *[3] If C_n is a irreducible plane curve of degree n then it has at most:*

$$\frac{(n-1)(n-2)}{2}$$

double points.

Proof. Suppose that a plane irriducible curve C_n has

$$\frac{(n-1)(n-2)}{2}+1$$

double points and let us take n-3 simple points P_i on it. We get

$$\frac{(n-1)(n-2)}{2}+1+(n-3)=\frac{1}{2}(n-2)(n+1)=N(n-2).$$

Then there is at least a C_{n-2} going through the double points of C_n and the P_i. The common points of C_n and C_{n-2} given by the double points and by P_i are at least

$$2\left(\frac{1}{2}(n-1)(n-2)+1\right)+(n-3)=n(n-2)+1.$$

Then, for the Bezout's theorem, the two curves must have a common component and this is not possible because C_n is irreducible. □

Theorem 9.2. *[3] Two plane algebraic curves of degree m,n without common components, have mn common points each of which is counted with respectively multeplicity.*

If the double point is a cusp, it is an ordinary cusp.
We recall that a double point P is a cusp if the two tangent lines in P are the same line l and the multiplicity intersection of the line with the curve in P is three.
We have the following theorem that describe the behavior of flexes:

Theorem 9.3. *Let P a non-singular point of a plane cubic curve C. Then P is a flex if and only if the polar conic of P with respect to C splits in two lines: one is the tangent line of C in P, the other does not contains P. The flexes are exactly the simple points of C that belong to the Hessian curve.*

An important theorem of the theory of plane cubic curves is:

Theorem 9.4. *Each cubic passing through eight of the nine points common to two coplanar cubics contains the ninth too.*

(cfr. [21]or [3]) An immediate consequence of this theorem is:

Corollary 9.5. *The line that contains two flexes of a cubic intersect the cubic in a third flex.*

Proof. Let L,M be two flexes of the cubic C and let r be the line through them and through a third point N so that $N = C \cap r$. Consider the pencil of cubics given by the curve C and the line r counted three times.
Then the nine base points of the pencil are L, M, N counted three times, so the cubic given by the three tangent of C in L, M, N contains eight of the nine base points of the pencil. So for the previous theorem it must contain the ninth too:so N is the third flex.

□

From the Pl*ü*cker's formulas (cfr. for example[3])

$$\rho = 3n(n-2) - 6d - 8k$$

and

$$\nu = n(n-1) - 2d - 3k$$

where ρ,ν,n,d,k, are respectively the number of flexes, the class of the curve (i.e. the number of tangent lines from a generic point of the plane), the degree, the number of nodes, the number of cusp, we have three possibility for cubic plane curves:

1. plane cubics without double point so with 9 flexes and class 6
2. plane cubics with a double point with distinct tangent so with 3 flexes and class 3
3. plane cubics with a cusp so with 1 flex and class 3.

We want to look at the action of $PGL(3)$ on the space of cubic polynomials on $\mathbb{P}^2$ up to scalars.
There are a number of orbits of $PGL(3)$ corresponding to reducible cubics:
triple lines (symmetric tensors of the form $v^{\otimes 3}$)that form a single orbit
double lines e line (that correspond to symmetric tensors of the form $v^{\otimes 2} \cdot w$ they also form a single orbit
there are two orbits corresponding to union of three lines because there are 3 concurrent lines and 3 that form a triangle
(from a tensor point of wiew the first one is of the form $u \cdot v \cdot w$ with u, v, w pairwise independent and spanning a plane in the vector space V and the other

of the form $u \cdot v \cdot w$ with u,v,w linearly independent.
There are two orbits that correspond to cubics that are union of a conic and a line, one where the line is secant the conic and one where the line is tangent to the conic.
After, there are the irreducible cubics, the singular one consisting in the cusp and the node. These form two more orbits.
At the end we have the smooth cubics.
They have the following equation (first canonic form or Weierstrass form):

$$Y^2 Z = X \cdot (X - Z) \cdot (X - \lambda Z)$$

and two such curves are projectively equivalent if and only if the J invariant coincide. The J invariant is given by

$$J := \frac{4(\lambda^2 - \lambda + 1)^3}{(\lambda + 1)^2(\lambda - 2)^2(2\lambda - 1)^2} \tag{9.2}$$

where λ is the double ratio of the four tangent lines through a point on it.
If $J = 0$ the plane cubic curve is called equihanarmonic and if $J = \infty$ it is called harmonic.

9.2 J-invariant

Let us see how to compute the absolute invariant 9.2 of a cubic when the equation of the cubic is in normal form.
We recall that by Salmon's theorem ([21]), the cross-ratio of the four tangents, which may be drawn to a plane cubic from a point on it, is constant.
A certain function of this cross-ratio which is rational in the coefficients of the cubic (smooth) is the only absolute projective invariant of the curve and is called J-invariant.
If α is one of the values of the cross-ratio, the J-invariant is

$$J = \frac{4(\alpha^2 - \alpha + 1)^3}{(\alpha + 1)^2(1 - 2\alpha)^2(2 - \alpha)^2}. \tag{9.3}$$

This absolute invariant is called also the modulus of the cubic and it does not depend on the order in which the four tangents are considered.
Let F(x,y)=0 the equation of a cubic C without double points. With a projective transformation we can take a flex of C to be the point at infinity of the y axis and the relative harmonic polar is the line $\{y = 0\}$.

Then the intersection of C with lines perpendicular to x axis will be two symmetric points so y^2 will be a rational function of x namely

$$y^2 = \frac{\varphi(x)}{\psi(x)}$$

so the equation of the cubic will be

$$y^2\psi(x) = \varphi(x)$$

where ψ is a polynomial of the first degree and ψ is a polynomial of third degree. We can put $\psi(x)=x - \alpha$ so for $x = \alpha$ we have $y = \infty$ and the line $x = \alpha$ is the tangent line in the flex.
Then we have

$$y^2(x - \alpha) = x^3 + bx^2 + cx + d.$$

But because $\varphi(x) = 0$ is the tangent variety of degree 4 of the 4 tangents at the cubic C from the infinity point and because the tangent variety is invariant we have

$$J = \frac{S^3}{T^2} = \frac{4i^3}{j^2}. \tag{9.4}$$

where S and T are the Aronhold's invariants of the cubic and i and j are the integer invariants of degree 2 and 3 respectively of the polynomialφ when φ is considered of degree 4 with the leading coefficient equal to zero.
Then we have that

$$S = \rho^2 i$$

and

$$T = \rho^3 \frac{j}{2}.$$

With a new change of coordinates

$$x \to x - \frac{b}{3},$$

the equation of the cubic becomes

$$y^2 = k(x^3 - px + q)$$

and with the change of variable

$$y \to \sqrt{k}y$$

the equation will be in normal form

$$y^2 = x^3 - px + q. \tag{9.5}$$

We get from the notion of double ratio that $i = 0$ means that the group of 4 points on a projective line is equianharmonic and $j = 0$ means that this group is harmonic. In this cade $i = 3p$ and $j = 27q$.
If we choose $\rho = 1$ we have

$$S = 3p \quad T = \frac{27}{2}q$$

then the discriminant of C is

$$D = S^3 - T^2 = 27\left(p^3 - \frac{27}{4}q^2\right)$$

and also

$$J = \frac{S^3}{T^2} = \frac{27p^3 \cdot 4}{27^2 q^2} = \frac{4}{27}\frac{p^3}{q^2}.$$

We have the theorem :

Theorem 9.6 ([1]). *The cubic form in three variables $f(x, y, z)$ such that $S = 0$ (equi-hanarmonic) or $p = 0$ in the normal form is representable as a sum of three cubes, that is for a convenient choise of the unit point:*

$$f(x, y, z) = x^3 + y^3 + z^3.$$

Proof. (We follow [21]). From the normal form 9.5

$$f = x^3 - y^2 z + qz^3$$

then it is sufficient to write the binary form $-y^2 z + z^3$ like a sum of two cubes:

$$z^3 - zy^2 = \frac{1}{2}(x - \frac{y}{\sqrt{3}})^3 + \frac{1}{2}(x + \frac{y}{\sqrt{3}})^3.$$

□

The equation

$$x^3 + y^3 + z^3 = 0$$

say that

$$H(x^3 + y^3 + z^3) = 216xyz$$

that is the Hessian of a anharmonic plane cubic is a triangle. On the other hand

Theorem 9.7. *The plane cubic (harmonic) for which $T = 0$ has the property that it is the hessian of its hessian.*

Proof. In this case,the plane cubic curve has equation

$$y^2 = x^3 - px$$

so the hessian of the ternary cubic form $f = x^3 - y^2z - pxz^2$ is

$$H(x, y, z) = 24px^2z - 24xy^2 + 8p^2z^3$$

and by computing the hessian of H we obtain

$$H(H(f)) = f$$

up to a numerical factor. □

9.3 Aronhold's invariants

Aronhold's invariants of plane cubics, S and T, are introduced by Aronhold in 1849 in a foundamental work over the homogeneus functions of third order in three variables ([1]) and can be written using the symbolic notation as:

$$S = (abc)(abd)(acd)(bcd)$$

and

$$T = (abc)(abd)(ace)(bcf)(def)^2$$

where the plane cubic is given by the symbolic notation by

$$a_x^3 = b_x^3 = c_x^3 = d_x^3 = e_x^3 = f_x^3.$$

The Aronhold's invariant S is a degree four invariant of plane cubic $S \in S^4(S^3\mathbb{C}^3)$ and is the polynomial of minimum degree vanishing on the locus of cubics projectively isomorphic to the Fermat cubic.
Its equals the following polynomial with 25 terms (cfr.[18] or [41] or [37])

$$S(F) = abcj-(bcde+cafg+abhi)-j(agi+bhe+cdf)+(afi^2+ahg^2+bdh^2+bie^2+cgd^2+cef^2)-j^4$$
$$+2j^2(fh + id + eg) - 3j(dgh + efi) - (f^2h^2 + i^2d^2 + e^2g^2) + (ideg + egfh + fhid)$$

where the equation of F is in the form:

$$F(x_0, x_1, x_2) = ax_0^3+bx_1^3+cx_2^3+3dx_0^2x_1+3ex_0^2x_2+3fx_0x_1^2+3gx_1^2x_2+3hx_0x_2^2+3ix_1x_2^2+6jx_0x_1x_2$$

We have the

Proposition 9.8. *Let F be a plane cubic curve. Then the following condition are equivalent[18]:*

1. *S(F)=0;*
2. *F is a Fermat cubic, or a cuspidal one, or a cone or the union of a conic and a tangent line;*
3. *exist a point a such that $P_a(F) = l^2$ that is one of the polar conic of F is a double line;*
4. *F admits a(possibly degenerate) polar s-polygon with $s \leq 3$.*

We sketch the proof that is given in [18].

Proof. If $P_a(F) = l^2$ then there are two cases:$l(a) \neq 0$ and $l(a) = 0$. If $l(a) \neq 0$, that is $a \notin P_a(F)$ and assuming $a = (1,0,0)$ and $l = x_0$ then

$$P_a(F) = \frac{1}{3}\frac{\partial F}{\partial x_0} = x_0^2$$

Then:

$$\int P_a(F)dx_0 = \int x_0^2 dx_0$$

that is,

$$F - x_0^3 = G(x_1, x_2)$$

with $G(x_1, x_2)$ a binary cubic in x_1, x_2.
Then we may assume that only the following cases occur:

$$G(x_1,x_2) = x_1^3 + x_2^3, \quad G(x_1,x_2) = x_1^2 x_2, \quad G(x_1,x_2) = x_1^3, \quad G(x_1,x_2) = 0.$$

It follows

$$F = x_1^3 + x_2^3 + x_3^3, \quad F = x_1^2 x_2 + x_0^3, \quad F = x_0^3 + x_1^3, \quad F = x_0^3$$

If $l(a) = 0$, we can always assume $a = (1,0,0)$, $l = x_1$.Therefore

$$P_a(F) = \frac{1}{3}\frac{\partial F}{\partial x_0} = x_1^2$$

We integrate this equation to obtain

$$\int P_a(F)dx_0 = \frac{1}{3}\int \frac{\partial F}{\partial x_0} = \int x_1^2 dx_0$$

so

$$F(x_0, x_1, x_2) = 3x_0x_1^2 + G(x_1, x_2)$$

with $G(x_1, x_2)$ binary cubic form or 0.The point $(1, 0, 0)$ is a double point with the tangent cone equal to x_1^2.Then F is either a cusp or the union of a smooth conic and its tangent line or a cone with a double irreducible component. □

Definition 9.9. A non singular cubic curve of equation $F = 0$ such that $S(F) = 0$ is called equianharmonic cubic.

The reason is that an irreducible, non singular cubic curve can be reduced to the Weierstrass form:

$$y^2z = x^3 - px - q$$

and we have, up to factor,

$$S = 3p, \qquad T = \frac{27}{2}q.$$

By evaluating the invariant S on this curve we obtain $S(F) = 0$ if and only if $p = 0$.
But the nonsingularity of the cubic means that it is an elliptic curve so the projection

$$\pi : C \to \mathbb{P}^1$$

from a point P$\in C$ is a double cover of $\mathbb{P}^1$ branched at four points $\rho_1, \rho_2, \rho_3, \infty$.
This follows, for example, by the well know Hurwitz formula:

$$\sharp \quad branchpoints = 2n + 2g - 2$$

where n is the degree of the map (n=2), g is the genere of a curve (g=1).
So in this case the number of branch points is 4.
Now the cross ratio of these 4 points is independent (up to order of the 4 points) of the line and also is independent from the projection point P on the cubic curve.
The cross ratio is in the equianharmonic case

$$R = \frac{(\rho_1 - \rho_2)(\rho_3 - \infty)}{\rho_1 - \infty)(\rho_3 - \rho_2)} = \frac{(\rho_1 - \rho_2)}{(\rho_3 - \rho_2)} = \rho,$$

where ρ is a cubic root of -1,$\rho \neq -1$.
This 4-uple is classically called equianharmonic.

9.4 Pfaffian of a skew transformation

Let E be a vector space of dimension $n = 2m$ and define a linear map

$$W : E \otimes E^{\vee} \to \Lambda^2 E$$

by

$$W(\phi) = \sum_{v} e_v \wedge \phi(e_v)$$

for all $\phi \in End(E)$, where e_v is a basis of E. Suppose now that ϕ is a skew transformation of E. Since $W(\phi) \in \Lambda^2(E)$, it follows that

$$W(\phi)^m \in \Lambda^n(E) \simeq \mathbb{K}.$$

Since $\dim(\Lambda^n(E))$=1, there exist a scalar $Pf(\phi)$, called the *Pfaffian* of the skew transformation ϕ, such that

$$\frac{1}{2^m} W(\phi)^m = Pf(\phi)e,$$

where e is the unique unit vector in $\Lambda^n E$ which represent the orientation.
By definition, Pf depends only on a choice of volume form and is invariant under the action of $SL(E)$.
A well-know property of the Pfaffian is the following:

$$Pf(\phi)^2 = \det(\phi). \tag{9.6}$$

As direct consequence of the above formula we obtain that the determinant of a skew matrix of even order is the square of a polynomial in its entries.In fact, if α_{ij} is the given matrix define the skew transformation ϕ by

$$\phi(e_i) = \sum_{j} \alpha_{ij} e_j$$

where e_i is an orthonormal basis of E.Then we have that

$$W(\phi) = \sum_{i,j} \alpha_{ij} e_i \wedge e_j.$$

Now

$$W(\phi)^m = \frac{1}{m!} \sum_{\sigma} \alpha_{\sigma(1)\sigma(2)} \cdots \alpha_{\sigma(n-1)\sigma(n)} e_{\sigma(1)} \cdots e_{\sigma(n)}$$

$$= \frac{1}{m!} e \sum_{\sigma} \epsilon_{\sigma} \alpha_{\sigma(1)\sigma(2)} \cdots \alpha_{\sigma(n-1)\sigma(n)}$$

$$\frac{2^m}{m!} e \sum_{\tau(2j-1)<\tau(2j)} \epsilon_{\tau} \alpha_{\tau(1)\tau(2)} \cdots \alpha_{\tau(n-1)\tau(n)}$$

Now the formula (53) yields

$$\det(\alpha_{ij}) = \left[\frac{1}{m!} \sum_{\tau} \epsilon_{\tau} \alpha_{\tau(1)\tau(2)} \cdots \alpha_{\tau(n-1)\tau(n)} \right]^2$$

For example if $A = (\alpha_{ij})$ is a 4x4 skew matrix,

$$\det\ A = [\alpha_{12}\alpha_{34} - \alpha_{13}\alpha_{24} + \alpha_{14}\alpha_{23}]^2 .$$

9.5 Aronhold invariant revisited

The Aronhold invariant is the degree four equation of the secant variety $\sigma_3(S^3\mathbb{C}^3)$ and can be interpreted as a pfaffian (see [33]).
Let dimV=3 and let e_0, e_1, e_2 be a basis of V and $e_1 \wedge e_2 \wedge e_3$ a base of $\Lambda^3 V = K^3$.
Then consider the space of endomorphisms $End(V)$.
We have

$$End(K^3, K^3) = K^{3^\vee} \otimes K^3 = K \oplus ad(V)$$

with $ad(V)$ the subspace of $End(V)$ corresponding to traceless endomorphisms.
Fix $\omega \in S^3(V)$.
Then map the tensor product

$$K^{3^\vee} \otimes K^3 \to \Lambda^2 K^3 \otimes K^3, \qquad M \in End(K^3) \mapsto (v \mapsto (M(w) \wedge w \wedge v) w$$

Put $\phi \in S^3 V$ with
$\phi = \phi_{000}x_0^3 + \phi_{111}x_1^3 + \phi_{222}x_2^3 + 3\phi_{001}x_0^2x_1 + \phi_{011}x_0x_1^2 + 3\phi_{002}x_0^2x_2 + 3\phi_{022}x_0x_2^2 + 3\phi_{112}x_1^2x_2 + 3\phi_{122}x_1x_2^2 + 6\phi_{012}x_0x_1x_2$.
Then the matrix 9x9 of this map is given by:

$$\begin{bmatrix}
0 & 0 & 0 & \phi_{002} & \phi_{012} & \phi_{022} & -\phi_{001} & -\phi_{011} & -\phi_{012} \\
0 & 0 & 0 & \phi_{012} & \phi_{112} & \phi_{122} & -\phi_{011} & -\phi_{111} & -\phi_{112} \\
0 & 0 & 0 & \phi_{022} & \phi_{112} & \phi_{222} & -\phi_{012} & -\phi_{112} & -\phi_{122} \\
-\phi_{002} & -\phi_{012} & -\phi_{022} & 0 & 0 & 0 & \phi_{000} & \phi_{001} & \phi_{002} \\
-\phi_{012} & -\phi_{112} & -\phi_{122} & 0 & 0 & 0 & \phi_{001} & \phi_{011} & \phi_{012} \\
-\phi_{022} & -\phi_{112} & -\phi_{222} & 0 & 0 & 0 & \phi_{001} & \phi_{011} & \phi_{022} \\
\phi_{001} & \phi_{011} & \phi_{012} & -\phi_{000} & -\phi_{001} & -\phi_{002} & 0 & 0 & 0 \\
\phi_{011} & \phi_{111} & \phi_{112} & -\phi_{001} & -\phi_{011} & -\phi_{012} & 0 & 0 & 0 \\
\phi_{012} & \phi_{112} & \phi_{122} & -\phi_{001} & -\phi_{011} & -\phi_{022} & 0 & 0 & 0
\end{bmatrix}$$

where this matrix 9x9 has the structure:

$$\begin{bmatrix}
0 & \phi_2 & -\phi_1 \\
-\phi_2 & 0 & \phi_0 \\
\phi_1 & -\phi_0 & 0
\end{bmatrix} \tag{9.7}$$

and $\phi_i, i = 0, 1, 2$ denotes the Hessian of the derivative $\frac{\partial \phi}{\partial x_i}$.
All the principal Pfaffians of size 8 coincide, up to scalar factor, with the Aronhold invariant(cfr. [33]).
This result is used to prove the theorem 10.8 that is the theorem about the rank five of the reducible cubic given by an immaginary conic plus a line.

Chapter 10

Rank and border rank of real ternary forms

10.1 Rank of ternary cubics

Let f(x,y,z) a ternary cubic form in $S^3\mathbb{R}^3$ where $S^3\mathbb{R}^3$ is the subspace of symmetric tensors of order 3 and dimension 3.
We compute the typical ranks of $\phi \in \mathbb{P}(S^3\mathbb{R}^3)$ with respect to cubic Veronese surface $\nu_3(\mathbb{P}^2)$.
In this case the generic typical rank is four (see for example [12]), but first we recall some facts from the complex case.
For any $\phi \in S^3\mathbb{C}^3$ there is the contraction morphism that can be thought as

$$A_\phi : Ad(\mathbb{C}^3) \to Ad(\mathbb{C}^3)$$

which is defined for $\phi = v^3$ as

$$A_{v^3}(M)(w) = (M(v) \wedge \ v \wedge \ w)v$$

where $M \in End(\mathbb{C}^3)$ and then extended by linearity to a general ϕ.
Here $Ad(\mathbb{C}^3)$ is the subspace of $End(\mathbb{C}^3)$ with zero trace.

Theorem 10.1. *[33] For every $\phi \in S^3(V)$, let*

$$A_\phi : Ad(\mathbb{C}^3) \to Ad(\mathbb{C}^3)$$

be the SL(V)-invariant contraction operator. Then A_ϕ is skew-symmetric and the pfaffian $Pf(A_\phi)$ is the equation of $\sigma_3(\nu_3(\mathbb{P}^2))$, i.e. it is the Aronhold invariant.

For the proof of the theorem is useful a lemma ([33]):

Lemma 1. *[33] Let $\phi = v^3$ with $v \in V$, with dimV=3. Then $rk(A_\phi) = 2$.*
In particular

$$Im(A_\phi) = \{M \in AdV | ImM \subset < v >\}$$

$$Ker(A_\phi) = \{M \in AdV | v \text{ is an eigenvetor of} M\}.$$

Proof. The statement follows by the equality:

$$A_\phi(M)(v) = 6(M(w) \wedge w \wedge v)w.$$

□

For the proof of the theorem 10.1, consider ϕ belonging to the 3-secant variety of the Veronese variety, that is the Zariski closure of the variety of polynomials which are sum of the powers of three linear forms, so that

$$\phi = \phi_1 + \phi_2 + \phi_3$$

Then, from the Lemma 1 it follows that

$$rkA_\phi \leq rkA_{\sum_{i=1}^3 \phi_i} = rk \sum_{i=1}^{3} A_{\phi_i} \leq \sum_{i=1}^{3} rkA_{\phi_i} = 2 \times 3 = 6.$$

So the pfaffian of A_ϕ vanish on the 3-secant variety. So the Aronhold invariant is the degree 4 equation of $\sigma_3(X)$, that is the $SL(3)$ orbit of the Fermat cubic $x^3 + y^3 + z^3$.
The vanishing of Aronhold invariant gives necessary and sufficient condition to write a ternary cubic form as sum of three cubes.

Corollary 10.2. *If $\underline{R}(\phi) \leq r$, then $R(A_\phi) \leq 2r$.*

Proof. Let $\phi_n \in S^3(\mathbb{R}^3)$ be a sequence such that $R(\phi_n = r)$ and such that

$$\lim_{n \to +\infty} \phi_n = \phi.$$

We have

$$\lim_{n \to +\infty} A(\phi_n) = A(\phi)$$

and with the same argument as above, we conclude that

$$rk(A(\phi)) \leq 2r.$$

□

Remark L. Cremona in the nineteen century (cfr.[21]) gave a proof of the theorem that exist two orbits that may characterized by the sign of the discriminant Δ,

$$\Delta = S^3 - T^2.$$

In fact if we write the cubic in the normal form

$$y^2 = x^3 - px + q$$

where

$$S = 3p, T = \frac{27}{2}q, \Delta = \frac{4}{27}(4p^3 - 27q^2).$$

Then we have that for $\Delta > 0$ the cubic $x^3 - px + q$ has only one real root, while for $\Delta < 0$ it has three real roots (called "casus irriducibilis" for the cubic).
Then the cubic with positive discriminant have "one connected component" and the cubic with negative discriminant have "two connected components".

10.2 Complex De Paolis algorithm

Definition 10.3. (cfr.[15])
Let $C_n := a_x^n = 0$ and $\Gamma_m := u_\alpha^m = 0$ be the (symbolic) equations of two curves of order n and class m respectively with $m < n$.
The curve Γ_m is apolar with respect to the curve C_n when its polar P_{n-m} is indeterminate, that is when, in symbolic notation, $a_\alpha^m a_x^{n-m} = 0$.

For example, if $m = 1$, we have $n - 1$ polar lines of a point with respect to the C_n.
If Γ_m is apolar with respect to C_n, the coefficients of u_α^m must satisfy $N(n-m)+1$ linear homogeneous equations with

$$N(n) = \binom{n+2}{2} - 1 = \frac{n(n+3)}{2},$$

and this is possible if and only if

$$N(m) \geq N(n-m) + 1$$

that is,

$$2m(n+3) \geq (n+1)(n+2).$$

If $2m(n+3) \geq (n+1)(n+2)$, there is a linear system $\infty^{N(m)-N(m-n)-1}$ of Γ_m apolar with respect to C_n.

Let C_3 a general cubic curve definite by a ternary form $F \in \mathbb{S}^3\mathbb{K}^3$, that is

$F = a_{300}x^3+3a_{210}x^2y+3a_{201}x^2z+3a_{120}xy^2+6a_{111}xyz+3a_{102}xz^2+a_{030}y^3+3a_{021}y^2z+3a_{012}yz^2+a_{003}z^3$

Let us write F in the form:

$$F = \sum_{i,j,k}(i,j,k)_3 h_{ijk}x_ix_jx_k$$

where h_{ijk} is symmetric in the i, j, k, and $(i, j, k)_3$ takes the value 1 if all indices are equal, 3 if two indices are equal and 6 if all indices are distinct.Then
the first polar $P_a(F)$ is the polar conic

$$P_a(F) = \sum h_{ijk}a_ix_jx_k$$

the second polar is the linear form

$$P_{a,b}(F) = \sum h_{ijk}a_ib_jx_k$$

and the third polar is the trilinear symmetric form (total polarization)

$$P_{a,b,c}(F) = \sum h_{ijk}a_ib_jc_k.$$

We have the following theorem that characterize the cubics that are cone.

Theorem 10.4. *Let F a cubic plane curve such that its Hessian is the whole plane. Then F is the union of three lines (non necessary all distinct), that is, F is a cone.F depends essentially on only two variables.*

This is also equivalent to the theorem:

Theorem 10.5. *F is the union of three concurrent lines if and only if exist* $(c_0, c_1, c_2) \neq (0, 0, 0)$ *such that*

$$c_0\frac{\partial F}{\partial x} + c_1\frac{\partial F}{\partial y} + c_3\frac{\partial F}{\partial z} = 0$$

that is,

$$\left(c_0\frac{\partial}{\partial x} + c_1\frac{\partial}{\partial y} + c_2\frac{\partial}{\partial z}\right)(F) = 0$$

if and only if the 3×6 *catalecticant matrix*

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{pmatrix}$$

has rank ≤ 2.*[7]*

Let Q be a polar quadric of $X = V(f)$ with respect to a point $a \in \mathbb{P}^n$.
The symmetric matrix relative to the relative quadratic form defining Q is the Hessian matrix of second derivative of F evaluated at the point a:

$$He(f) = \Big(\frac{\partial^2 f}{\partial x_i \partial x_j}\Big)_{i,j=0,\cdots n}, \tag{10.1}$$

The hypersurface

$$He(X) = V(\det He(F)), \tag{10.2}$$

is the locus of the points $a \in \mathbb{P}^n$ such that the polar quadric $P_{a^{d-2}}(X)$ is singular. It is the Hessian hypersurface of X.

Proposition 10.6. *(cfr. [17]) The hypersurface $He(X)$ is the locus of singular points of the first polar of X.*

Proof. Let $x \in He(X)$ and $y \in Sing(P_{a^{d-2}}(X))$. For the symmmetry of the differentition we have:

$$D_y(D_{x^{d-2}}(f) = D_{x^{d-2}}(D_y(f)) = 0. \tag{10.3}$$

But $degD_y(f) = d - 1$, so

$$\mathbb{T}_x(P_y(X)) = \mathbb{P}^n,$$

where $\mathbb{T}_x(X)$ is the embedded tangent space of a projective subvariety $X \subset \mathbb{P}^n$ at the point x so the point x is a singular point of $P_y(X)$.
On the other hand, if $x \in Sing(P_y(X))$ for $y \in P^n$, then $D_{x^{d-2}}(D_y(f)) = 0$, hence $D_y(D_{x^{d-2}}(f)) = 0$, so the point y is a singular point of the polar quadric with respect to x.Hence $x \in He(X)$. □

Let's return to De Paolis algorithm.
To construct a harmonic quadrilateral with respect to C, a general cubic defined by $F \in S^3C^3$, it sufficient to take one of its lines, l_0, and let P_1, P_2, P_3 the three points in which the Hessian curve H intersect the line l_0.
Let Q_1, Q_2, Q_3 the singular corrispondent points of the conic $P_{P_i}C$.
Then the three lines will be $l_i :=< P_i, P_j >, i \neq j$.
Sketch of the proof: assume that $F = \sum_{i=0}^{3} l_i^3$ and $P_1 = l_0 \cap l_1$, so the polar conic $P_{P_1}C$ has equation $l_2^2 + l_3^2 = 0$ and it is singular on the point $Q_1 = l_2 \cap l_3$, in particular this point belongs to $He(C)$.

10.3 Real De Paolis algorithm

De Paolis's algorithm is an usefull method to find a decomposition of a plane cubic curve as a sum of at most 4 cubes when the first cube is given.
Let C_3 a real cubic curve defined by a cubic polynomial $F \in S^3\mathbb{R}^3$.
We know that a general line l_0 in the real projective plane $\mathbb{P}^2$ meets the Hessian curve in three distinct real point.
In fact we have the classic theorem:

Theorem 10.7. *A nonsingular real cubic has exactly three real inflection points. These points are collinear.*

There are so defined three real lines l_i of the complex algorithm that are real, such that

$$F = \sum_{i=0}^{3} l_i^3$$

Proof: The singular point of a real singular conic is always real.
The algoritm is (see [4]):
INPUT F a irriducible real plane cubic
l_0 a line such that $l_0 \cap He(F)$ consist of three distinct points.
The line l_0 joining three real flexes for example.
COMPUTE $l_0 \cap He(F) = \{P_1, P_2, P_3\}$.
COMPUTE Q_i the singular points of the polar conic $P_{P_i}(F)$ for i=1,2.
COMPUTE $l_1 =< P_1, Q_2 >, l_2 =< P_2, Q_1 >, l_3 =< P_3, Q_1 >$.
SOLVE the linear system $F = \sum_{i=0}^{3} c_i l_i^3$.
OUTPUT lines l_1, l_2, l_3 and numbers $c_i \in \mathbb{R}$, i=0,1,2,3, such that $F = \sum_{i=0}^{3} c_i l_i^3$.
From this algorithm we deduce that for real plane cubics there is only one typical rank which is 4.

10.4 Example

We apply De Paolis algorithm to the Hesse pencil :

$$F_\lambda = x^3 + y^3 + z^3 + 6\lambda xyz.$$

with the condition of non singularity $1 + 8\lambda^2 \neq 0$.
The Hessian of this pencil is again of this form, that is a curve of the pencil: we

have the equation

$$H(F_\lambda) = x^3 + y^3 + z^3 - \frac{1+2\lambda^2}{\lambda^2}xyz = 0$$

and we can choose three real collinear flexes as $P_1 = (0, 1, -1)$,$P_2 = (1, 0, -1)$,$P_3 = (1, -1, 0)$.
This flexes belong to the line

$$l_0 = x + y + z = 0.$$

Compute the equation of the polar conics $P_{P_i}(F_\lambda)$ for i=1,2,3:

$$\begin{aligned} P_{P_1}(F_\lambda) &= 3y^2 + 6\lambda xz - 3z^2 - 6\lambda xy = 0 \\ P_{P_2}(F_\lambda) &= 3x^2 + 6\lambda yz - 3z^2 - 6\lambda xy = 0 \\ P_{P_3}(F_\lambda) &= 3x^2 + 6\lambda yz - 3y^2 - 6\lambda xz = 0 \end{aligned}$$

so we get three singular points

$$\begin{aligned} Q_1 &= (1, 1, \lambda) \\ Q_2 &= (\lambda, 1, \lambda) \\ Q_3 &= (\lambda, \lambda, 1) \end{aligned}$$

and the decomposition

$$F_\lambda = c_0(x+y+z)^3+c_1((1+\lambda)x-\lambda y-\lambda z)^3+c_2(\lambda x-(1+\lambda)y+\lambda z)^3+c_3(-\lambda x-\lambda y+(1+\lambda)z)^3.$$

Solving the linear system

$$F_\lambda = \sum_{i=0}^{3} c_i l_i^3$$

we get the value of the coefficients c_i.
We assume

$$c_1 = c_2 = c_3.$$

Equating the coefficients of the terms containing x^3 we get:

$$1 = c_0 + c_1(-\lambda^3 + 3\lambda^2 + 3\lambda + 1).$$

So we obtain a linear system in the unknown c_0, c_1

$$\begin{cases} c_0 + 3(\lambda+1)\lambda^2 c_1 = \lambda \\ c_0 + (\lambda^3 + 3\lambda^2 + 3\lambda + 1)c_1 = 1 \end{cases}$$

then the solution of this system give:

$$c_0 = \frac{\lambda(1-\lambda^3)}{\lambda-1)(-4\lambda^2-4\lambda-1}$$

and

$$c_1 = \frac{1}{(2\lambda+1)^2}.$$

For example, we apply De Paolis algorithm to decompose the plane cubic curve:

$$x^3+y^3+z^3+12xyz=0.$$

In this case we have $\lambda = 2$ and we get the decomposition

$$x^3+y^3+z^3+12xyz = \frac{14}{25}(x+y+z)^3+\frac{1}{25}(3x-2y-2z)^3+(-2x+3y-2z)^3+(-2x-2y+3z)^3.$$

In the following we look for rank and border rank for all ternary real cubics.

10.5 Imaginary conic plus line

In this case the cubic is $F = (x^2+y^2+z^2)x$ and the Hessian is $H(F) = (9x^2 - y^2 - 3z^2)(8x)$. The singular points of the polar conics are $Q_1 = (0,1,1)$ and $Q_2 = (0,1,-1)$ with respect to $P_1 = (0,1,-1)$, $P_2 = (0,-1,1)$,$P_3 = (0,1,1)$.
The polar conic are:

$$P_{P_1}(F) = 2xy - 2xz = 0$$
$$P_{P_2}(F) = -2xy + 2xz = 0$$
$$P_{P_3}(F) = 2xy + 2xz = 0$$

We can write

$$(x^2+y^2+z^2)x = xz^2 + x(x^2+y^2) = \frac{1}{6}\left[(z+x)^3-(z-x)^3\right] - \frac{1}{3}x^3 + \xi(x,y)$$

where $\xi(x,y) = x(x^2+y^2)$ and

$$rk(\xi(x,y)) = 2$$

because this binary cubic has one real root ([13]).
So

$$rk(F) = 4$$

and a decomposition of f is

$$F = \frac{1}{6}\left[(z+x)^3-(z-x)^3\right] + \frac{1}{2}\left[\frac{1}{3\sqrt{2}}(\sqrt{2}x-y)^3 - \frac{1}{3\sqrt{2}}(-\sqrt{2}x-y)^3\right].$$

10.6 Real conic plus secant line

In this case the cubic is $F = (x^2+y^2-z^2)y$ and the Hessian is $H(F) = 8y(x^2-3y^2-z^2)$. The singular points $P_1 = (1,0,0)$,$P_2 = (0,0,1)$,$P_3 = (1,0,1)$ corrispond to the points $Q_1 = (0,0,1)$,$Q_2 = (1,0,0)$, $Q_3 = (1,0,1)$ because the polar conics are

$$P_{P_1}(F) = 2xy = 0$$

$$P_{P_2}(F) = -2xz = 0$$

$$P_{P_3}(F) = 2y(x-z) = 0$$

that is

$$F_x - F_z = 2y(x-z)$$

With a substitution

$$x + z = x' \quad x - z = z' \quad y = y'$$

we have

$$(x'z' + y'^2)y'$$

and we have the decomposition

$$y(y^2 + xz) = \frac{1}{96}((4y + x + z)^3 + (4y - x - z)^3 - 2(2y + x - z)^3 - 2(2y - x + z)^3)$$

so

$$rk(x^2 + y^2 - z^2)y = 4.$$

10.7 Real conic plus external line

In this case the cubic is

$$F = (x^2 + y^2 - z^2)z$$

and the Hessian of F is

$$H(F) = -8z(x^2 + y^2 + 3z^2),$$

that is the Hessian cubic curve $H(F) = 0$ is a imaginary conic plus a line.
We can write

$$F = x^2z + z(y^2 - z^2) = x^2z + \varphi(y,z).$$

Now

$$x^2z = \frac{1}{6}\left[(x+z)^3 - (x-z)^3\right] - \frac{1}{3}z^3$$

so

$$x^2z + z(y^2 - z^2) = \frac{1}{6}\left[(x+z)^3 - (x-z)^3\right] - \frac{4}{3}z^3 + zy^2$$

and $\varphi(y,z) = -\frac{4}{3}z^3 + zy^2$.
But $\varphi(y,z)$ is a binary cubic form with three real roots because

$$zy^2 - \frac{4}{3}z^3 = z(y - \frac{2}{\sqrt{3}}z)(y + \frac{2}{\sqrt{3}}z)$$

and then

$$rk[\varphi(y,z)] = 3.$$

On the other hand

$$y^2z = \frac{1}{6}[(y+z)^3 - (y-z)^3 - 2z^3]$$

so

$$zy^2 - \frac{4}{3}z^3 = \frac{1}{6}[(y+z)^3 - (y-z)^3] - \frac{1}{3}z^3 - \frac{4}{3}z^3$$

that is

$$zy^2 - \frac{4}{3}z^3 = \frac{1}{6}(y+z)^3 - \frac{1}{6}(y-z)^3 - \frac{5}{3}z^3$$

and finally

$$F = \frac{1}{6}\left[(x+z)^3 - (x-z)^3\right] + \frac{1}{6}(y+z)^3 - \frac{1}{6}(y-z)^3 - \frac{5}{3}z^3$$

so

$$rk(F) = 5.$$

We have to prove that the rank of the above cubic can not be 5.

Theorem 10.8. *The rank of the reducible cubic given by a real conic plus an external line is five,so*

$$rk(x^2 + y^2 - z^2)z = 5.$$

Proof. Let

$$F = (x^2 + y^2 - z^2)z = l_1^3 + l_2^3 + l_3^3 + l_4^3$$

be a conic $C = \{x^2 + y^2 - z^2 = 0\}$ plus a line $L = \{z = 0\}$ that is suppose

$$rk(x^2 + y^2 - z^2)z = 4.$$

Let $Q = l_1 \cap l_2$. Then, up to scalar multiples,

$$P_Q(F) = l_3^2 + l_4^2$$

which is necessarly singular.
Here $P_Q(F)$ is the polar conic of F with respect to Q. Then the point $Q \in H(F)$ and $Q \in L$, for the particular form of the Hessian, which is

$$H(F) = -8z(x^2 + y^2 + 3z^2).$$

Moreover, for the same argument, all the intersections $l_i \cap l_j \in L$ and there are two possibilities: 1) the line L is one of the l_i with $i = 1, 2, 3, 4$ then

$$F - \lambda L^3$$

is a Fermat cubic curve for some λ
or
2) the four lines are concurrent in $\tilde{Q} \in L$ such that $P_{\tilde{Q}}(F) \equiv 0$.
The second case is not possible because we would have that $H(F) = 0$.
The first case also is not possible because

$$S(F - \lambda L^3) = 0$$

is an equation of degree four in λ without real roots. Indeed, computing the Pfaffian of size 8 of the matrix 9x9 given by 9.7, which coincide, up to scale, with the classical Aronhold invariant, (see section 9.3) we get that the generator of the ideal of Groebner basis depends only by d. □

10.8 Nodal cubic

Every plane cubic curve with a real node is projectively equivalent to the cubic

$$F = x^3 + y^3 - xyz = 0.$$

This is a famous cubic curve called "Folium of Decartes".
It has a double point in $(0, 0, 1)$ and there has a node with tangents $x = 0$ and $y = 0$ In this case

$$F_x = 3x^2 - 3yz$$

$$F_y = 3y^2 - 3xz$$

$$F_z = -3xy$$

Let

$$\alpha F_x + \beta F_y + \gamma F_z = 0$$

be the equation of the polar conic with respect to the point (α, β, γ). This is a reducible conic if

$$A = \begin{pmatrix} 3\alpha & \frac{-\gamma}{2} & \frac{-\beta}{2} \\ -\frac{\gamma}{2} & \beta & -\frac{\alpha}{2} \\ -\frac{\beta}{2} & -\frac{\alpha}{2} & 0 \end{pmatrix}$$

We deduce that det A=0 if $(\alpha, \beta, \gamma) = (1, -1, 0)$. So the pencil is

$$3x^2 - 3yz - 3y^2 + 3xz$$

that factorize as

$$3\left[(x + y + z)(x - y)\right].$$

So we can write

$$F_x - F_y = 3\left[(x + y + z)(x - y)\right] = 3\left[\frac{1}{4}(2x + z)^2 - (2y + z)^2\right]$$

because of the identity

$$ab = \frac{(a + b)^2}{4} - \frac{(a - b)^2}{4}$$

So we have, integrating with respect to x and y,

$$F = \frac{1}{8}\left[(2x + z)^3 + (2y - z)^3\right] + (function\ of\ two\ variables).$$

To find this function let us write the equality

$$x^3 + y^3 - 3xyz = \frac{1}{8}\left[(2x + z)^3 + (2y + z)^3\right] - z\left[\frac{3}{2}x^2 + \frac{3}{4}xz + \frac{1}{4}z^2 + \frac{3}{2}y^2 + \frac{3}{4}yz + 3xy\right].$$

The last addend can be written:

$$-\frac{z}{4}\left[6x^2 + 3xz + z^2 + 6y^2 + 3yz + 12xy\right]$$

or

$$\frac{3}{2}x^2z + \frac{3}{4}xz^2 + \frac{1}{4}z^3 + \frac{3}{2}y^2z + \frac{3}{4}yz^2 + 3xyz.$$

Let g be this cubic form; it depends on two essential variable, namely z and $h = x + y$.

If we make the catalecticant of this cubic form, a matrix 3x6, applying a result of E.Carlini(see [7]), we get

$$C_g = \begin{pmatrix} 0 & 0 & 3 & 0 & 3 & \frac{3}{4} \\ 0 & 0 & 3 & 0 & 3 & \frac{3}{4} \\ \frac{3}{2} & 3 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{4} \end{pmatrix}$$

and this matrix has rank 2, so the cubic g depends on two essential variables.
We have

$$g(z, x+y) = g(z, h) = z\left[\frac{z^2}{4} + \frac{3}{4}zh + \frac{3}{2}h^2\right]$$

and the discriminant of the polynomial of degree 2 into the square parentesis

$$z^2 + 3zh + 6h^2$$

is

$$\Delta = 9 - 6 \cdot 4 < 0$$

so that this quadratic polynomial has rank 2 and the cubic nodal form has rank 4.
At the end,

$$g(z, h) = 2\left[\frac{1-\sqrt{5}}{4}z + h\right]^3 - 2\left[\frac{1+\sqrt{5}}{4}z - h\right]^3$$

and

$$x^3+y^3-xyz = \frac{1}{8}\left[(2x+z)^3 + (2y+z)^3\right]+2\left[\frac{1-\sqrt{5}}{4}z + (x+y)\right]^3-2\left[\frac{1+\sqrt{5}}{4}z - (x+y)\right]^3.$$

SO

$$rk(F) = 4.$$

10.9 Cubic with a double complex point

Let

$$f = y^2z - x^3 + x^2z$$

be the so called "parabola punctata" with a double point in the origin with two complex tangent lines $x^2 + y^2 = 0 = (x+iy)(x-iy) = 0$.
We have

$$f_x = -3x^2 + 2xz$$

$$f_y = 2yz$$

$$f_z = x^2 + y^2.$$

If we consider

$$\alpha f_x + \beta f_y + \gamma f_z$$

and we take the determinant A of this pencil of conics

$$A = \begin{pmatrix} -3\alpha\gamma & 0 & \alpha \\ 0 & \gamma & \beta \\ \alpha & \beta & 0 \end{pmatrix}$$

we obtain

$$\det\ A = -\alpha^2\gamma - \beta^2\gamma + 3\alpha\beta^2$$

so that a singular point of the Hessian is $(0, 0, 1)$.
Now the pencil of conics reduce to

$$f_x = x(2z - 3x) = \frac{1}{4}((-2x + 2z)^2 - (4x - 2z)^2)$$

so

$$f = \frac{1}{4}\left(-\frac{1}{2}\frac{(-2x + 2z)^3}{3}\right) - \frac{1}{4}\left(\frac{(4x - 2z)^3}{3}\right) + \phi(y, z)$$

with

$$\phi(y, z) = y^2z + \frac{1}{6}z^3 = z(y^2 + \frac{1}{6}z^2)$$

so that ϕ is a binary cubic with only one real root and for a result of [C.O.] has rank 2. We have

$$\phi(y, z) = \frac{1}{12}\left(\sqrt{2}y + z\right)^3 - \frac{1}{12}\left(\sqrt{2}y - z\right)^3$$

so

$$f = \frac{1}{4}(-\frac{1}{2}\frac{(-2x + 2z)^2}{3} - \frac{1}{4}\frac{(4x - 2z)^3}{3}) + \frac{1}{12}\left(\sqrt{2}y + z\right)^3 - \frac{1}{12}\left(\sqrt{2}y - z\right)^3$$

and finally

$$rk(y^2z - x^3 + x^2z) = 4.$$

10.10 Conic plus tangent line

Let's see the case of the conic plus tangent line where the rank is 5.

$$(x^2 + yz)(y - z) = x^2(y - z) + yz(y - z)$$

We have

$$3y^2z - 3yz^2 = (z - y)^3 + y^3 - z^3$$

that is $y^2z - yz^2$ is a sum of three cubes. On the other hand

$$3x^2(y-z) = \frac{1}{2}((x+y-z)^3 - (x-y+z)^3 - 2(y-z)^3)$$

so

$$(x^2+yz)(y-z) = x^2(y-z)+yz(y-z) = \frac{1}{2}((x+y-z)^3-(x-y+z)^3-2(y-z)^3)+\frac{1}{3}((z-y)^3+y^3-z^3)$$

but

$$\frac{1}{3}(z-y)^3 - (y-z)^3 = -\frac{4}{3}(y-z)^3$$

so the cubic is a sum of 5 cubes and we have the decomposition

$$(x^2+yz)(y-z) = \frac{1}{2}(x+y-z)^3 - \frac{1}{2}(x-y+z)^3 - \frac{4}{3}(y-z)^3 + y^3 - z^3.$$

10.11 Hesse pencil

Proposition 10.9. *Let F be a non-singular plane cubic whose equation is not isomorphic to the equation of a Fermat cubic curve $x^3+y^3+z^3=0$. Then its Hessian is non-singular.*

Proof. From the Hesse canonical form of F we have

$$F = x^3 + y^3 + z^4 + 6\lambda xyz.$$

The cubic is non-singular if and only if $8\lambda^3 + 1 \neq 0$. Computing the Hessian of F we find

$$H(F) = (x^3+y^3+z^3) - \frac{1+2\lambda^3}{\lambda^2}xyz = 0$$

which is always non-singular unless $\lambda^4 = \lambda$ because in this case the S invariant is equal to 0. □

Now assume F is a cubic plane curve irridducible but singular.
If the curve is a nodal cubic curve computing the Hessian we find a nodal cubic curve and if the curve is a cusp the Hessian is the union of a line l_1 and a double line l_2^2.
The line l_1 joins the cuspidal point with the unique non-singular inflexion point, while the line l_2 is the tangent in the cusp. The polar conic of the cuspidal cubic with respect to the cusp is l_2^2 and the polar conic with respect to the point of intersection of the inflectional tangent with l_2 is l_1^2 ([5]).
Let C a curve of degree 3, we have that all the conics that are apolar with respect to C and that are tangent to a generic line are tangent to other 3 so they form an

harmonic quadrilateral.
On the other and, there aren't harmonic trilater (triangle) with respect to a general cubic.
In fact, if there were one such triangle each vertex with a generic point of the opposit edge should be a couple apolar with respect to C, that is a couple of points that are corresponding on the Hessian of C and this fact forces $He(C)$ to factor into three lines.
But this happens only when the invariant S (the Aronhold invariant) vanish and this happen when the cubic C is equianharmonic.
So we can say that it does exist one harmonic trilater (and only one) with respect to a cubic curve when the curve is equianharmonic;
in this case its hessian curve is formed by the three lines of the trilateral.
If so, each of its vertex with a point of the opposit side would be an apolar couple with respect to the cubic then a couple of points on the Hessian curve of the given cubic so this Hessian should be composed of the three sides of the trilater so the condition is that the invariant S=0 and this happened when the cubic is equianharmonic.

10.12 Classification of real plane cubics with respect to rk and brk

The classification of the real cubics with respect to linear projective transformations is

1. the triple line x^3 where the rank is 1
2. three concurrent lines $x(x^2 + y^2)$,1 real + 2 coniugates,where $rk = \underline{rk} = 2$
3. three real concurrent lines $x(x^2 - y^2)$, where $rk = \underline{rk} = 3$
4. double line + line x^2y where $rk = 3$ but $\underline{rk} = 2$
5. the Fermat cubic $x^3 + y^3 + z^3$ where $rk = \underline{rk} = 3$
6. immaginary conic + line $(x^2 + y^2 + z^2)x$ where $rk = \underline{rk} = 4$

7. real conic + external line $(x^2 + y^2 - z^2)z$ where $rk = 5$ and $\underline{rk} = 4$

8. real conic + secant line $(x^2 + y^2 - z^2)y$ where $rk = \underline{rk} = 4$

9. real conic + tangent line $(x^2 + y^2 - z^2)(y - z)$ where $rk = 5$ and $\underline{rk} = 3$

10. irriducible with 3 real roots $y^2z - x^3 - xz^2$ where $rk = \underline{rk} = 4$

11. irriducible with 1 real root $y^2z - x^3 + xz^2$ where $rk = \underline{rk} = 4$

12. cusp $y^2z - x^3$ where $rk = 4$ and $\underline{rk} = 3$

13. nodal cubic $y^2z - x^3 - x^2$ where $rk = \underline{rk} = 4$

14. cubica punctata $y^2z - x^3 + x^2z$ where $rk = \underline{rk} = 4$

15. triangle xyz where $rk = \underline{rk} = 4$

Decomposition of real conic plus secant line:

$$y(x^2 + y^2 - z^2) = \frac{1}{96}\left[(4x + 2y)^3 + (4y - 2x)^3 - 2(2y - 2z)^3 - 2(2y - 2z)^3\right].$$

Decomposition of cusp:

$$y^2z - x^3 = \frac{1}{6}\left[(y + z)^3 + (y - z)^3 - 2z^3\right] - x^3$$

Decomposition of triangle:

$$xyz = -\frac{1}{24}\left[(x - y + z)^3 - (x + y + z)^3 + (x + y - z)^3 + (-x + y + z)^3\right].$$

We observe that if we start with a imaginary conic plus line

$$f = x(x^2 + y^2 + z^2)$$

the Hessian is,up to factor,

$$H(f) = x(-3x^2 + y^2 + z^2)$$

that is a real conic plus external line and if we take the Hessian of the Hessian

$$H(H(f) = H(x(-3x^2 + y^2 + z^2))$$

we obtain, up to factor,

$$x(9x^2 + y^2 + z^2)$$

that is an imaginary conic +line.
This process is an involution and at the i-th step we have, up to a numerical factor,

$$H(H...H(f)) = ((-3)^i x^2 + y^2 + z^2)x.$$

This explain why the real conic plus external line is really a new case of rank 5 in the classification of real ternary cubics. In the following table there are the values of the invariants S and T and the discriminant Δ for each normal form.

Description	**normal form**	**S**	**T**	Δ	rk(9x9)
triple line	x^3	0	0	0	2
three concurrent line	$x(x^2 + y^2)$	0	0	0	4
three concurrent line	$x(x^2 - y^2)$	0	0	0	4
double line + line	x^2y	0	0	0	4
Fermat	$x^3 + y^3 + z^3$	0	≥ 0		6
imaginary conic+ line	$(x^2 + y^2 + z^2)x$	≥ 0	≥ 0		8
real conic + external line	$(x^2 + y^2 - z^2)z$	≥ 0	≥ 0		8
real conic +secant line	$(x^2 + y^2 - z^2)y$	≥ 0		0	8
real conic +tangent line	$(x^2 + y^2 - z^2)(y - z)$	≥ 0		0	6
irriducible 2 connected comp.	$y^2z - x^3 + xz^2$	≥ 0	≥ 0		8
irriducible 1 connected comp.	$y^2z - x^3 - xz^2$	≤ 0	≤ 0		8
cusp	$y^2z - x^3$	0	0	0	6
nodal cubic	$y^2z - x^3 - x^2z$	≥ 0	≤ 0	0	8
cubica puntata	$y^2z - x^3 + x^2z$	≥ 0	≥ 0		8
triangle	xyz	≥ 0	≤ 0	0	8

Table 10.1: Aronhold Invariants S,T and Δ of cubic plane curves on $\mathbb{R}$

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