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Classical $\mathcal{W}$-algebras

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Classical $\mathcal{W}$-algebras

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Introduction

The first appearence of (quantum) $\mathcal{W}$-algebras as mathematical objects is related to the conformal field theory. The main problem of the conformal field theory is a description of fields having conformal symmetry. Only in dimension $D = 2$, the group of conformal diffeomorphisms is rich enough to give rise to a meaningful theory.

After the fundamental paper by Belavin, Polyakov and Zamolodchikov [4] it was realized by Zamolodchikov [24] that extended symmetries in two dimensional conformal field theory in general do not give rise to (super)algebras with linear defining relations. He constructed the so-called $\mathcal{W}_3$-algebra, which is an extension of the Virasoro algebra obtained adding one primary field of weight 3. Later, this construction was generalized by Fateev and Lukyanov [14] to construct which are known as $\mathcal{W}_n$-algebras. Roughly speaking, these algebras are non-linear extensions of the Virasoro algebra obtained by adding primary fields. An exhaustive reference about extended symmetries can be found in [5].

The key point in the construction of the algebras $\mathcal{W}_n$ by Fateev and Lukyanov was the relation between $\mathcal{W}$-algebras and integrable systems. By a work of Gervais [18], the Virasoro algebra was found hidden in the second Poisson structure of the Korteweg-de Vries (KdV) equation, which is the so-called Virasoro-Magri structure. Magri [21] first revealed the bi-Hamiltonian nature of the KdV equation. Fateev and Lukyanov identified Zamolodchikov's $\mathcal{W}_3$ algebra with the so-called second Poisson structure of the equations of n-th KdV type, for $n = 3$ (the KdV equation corresponds to $n = 2$). These Poisson structures are known as Gelfand-Dickey algebras, after pioneristic works on the subject by them (one can have a look to [10] for a review of these works and a lot of material on the argument) and are obtained as Poisson algebras of local functionals on the algebra of differential operators. After quantizing the $n = 3$ structure, Fateev and Lukyanov found the same commutator formulas of Zamolodchikov $\mathcal{W}_3$ algebra. This observation enabled them to construct all the $\mathcal{W}_n$, since the second Poisson structure of the n-th KdV type equation were known for any n. At the classical level, Gelfand-Dickey algebras are the first examples of classical $\mathcal{W}$-algebras.

After, the second Poisson structure of Gelfand-Dickey type was considered in a more general setup, namely, in a similar fashion, it can be defined a Poisson structure on the space of local functionals on the larger algebra of pseudodifferential operators. The corresponding Poisson structures are related to the Kadomtsev-Petviashvili (KP) equation and the n-th KdV equations can be obtained with a reduction procedure from the KP hierarchy of equations. A quick reference is given by the lecture notes [9].

The fact that $\mathcal{W}$-algebras have in general non-linear defining relations puts them outside of the scope of the Lie algebra theory. However, they are intimately related to Lie algebras via the Drinfeld-Sokolov Hamiltonian reduction [12]. Given a Lie algebra $\mathfrak{g}$ and its principal nilpotent, this reduction allows us to construct a classical $\mathcal{W}$-algebra. Furthermore, this procedure also emphasizes again the fact that these structures are related to certain hierarchies of partial differential equations. Moreover, they proved that Gelfand-Dickey algebras correspond to the Drinfeld-Sokolov Hamiltonian reduction performed for the Lie algebra of n by n traceless matrices $\mathfrak{sl}_n$.

In this work we will be interested in the classical aspect of the theory, rather than the quantum one. So, from now on, we can skip the adjective classical. We have seen that $\mathcal{W}$-algebras appear in at least three interrelated contexts. In particular, for which concerns integrable systems, there is also another Poisson structure for the equations of n-th KdV type (or for the KP equation in general). In this case we say that we have a bi-Hamiltonian structure and as pointed out in [21] this is one of the main tool to prove integrability for such equations.

Recently, Barakat, De Sole and Kac [3] established a deep relation between Poisson vertex algebras and Hamiltonian equations and proved that Poisson vertex algebras provide a very convenient framework to study (both classical and quantum) Hamiltonian systems.

The aim of this thesis is to develop the theory of classical $\mathcal{W}$-algebras in the Poisson vertex algebras language. This leads to a better understanding of the Hamiltonian structure underlying $\mathcal{W}$-algebras and to a generalization of results, both in Gelfand-Dickey and Drinfeld-Sokolov approach to integrability of Hamiltonian eqautions.

In the first chapter, we review the basic notions and foundations of the theory of Poisson vertex algebras aimed to the study of Hamiltonian equations as laid down in [3].

The second chapter is about Drinfeld-Sokolov Hamiltonian reduction. The orginal construction given in [12] involved a semisimple Lie algebra $\mathfrak{g}$ and its principal nilpotent element f. After reviewing the Drinfeld-Sokolov Hamiltonian reduction, we will define it in a purely Poisosn vertex algebra language. This will enable us to perform such reduction for any nilpotent element of $\mathfrak{g}$. After showing that our construction is equivalent to the original one, we will construct, in the case of the classical Lie algebra B_n, C_n and D_n, the corresponding $\mathcal{W}$-algebras as quotients of particular Poisson vertex subalgebras of the $\mathcal{W}$-algebra corresponding to $\mathfrak{gl}_n$.

The third chapter is devoted to the analysis of the Gelfand-Dickey algebras approach to $\mathcal{W}$-algebras, although we radically change point of view. Instead of considering a pseudodifferential operator and then defining the Poisson structure on the algebra of local functionals on it, we attach to any differential algebra a particular pseudodifferential operator, which we call "general" and prove that in this case the Adler map [1] gives rise to a Hamiltonian operator, using the generating series of its matrix entries. The use of the λ-bracket language surprisingly simplifies the proof if compared to the usual one [10]. This allows us to think about the Adler map as a map from pseudodifferential operator to λ-bracket structures. When the λ-bracket corresponding under this map to a pseudodifferential operator defines a Poisson vertex algebra structure then the Adler map gives rise to a Hamiltonian structure. We will give an example of an operator in which this does not happen and one in which it happens. In the first case we can still modify the Ader map and get a Hamiltonian structure. In the Drinfeld-Sokolov Hamiltonian reduction, this structure corresponds to the Hamiltonian reduction of $\mathfrak{sl}_n$ and its principal nilpotent element. In the other case we will be able to recover and to give a very simple proof of a famous theorem of Kupershmidt and Wilspon [20].

The fourth section is devoted to establish the well known fact that Gelfand-Dickey algebras are $\mathcal{W}$-algebras corresponding to some special cases of Drinfeld-Sokolov Hamiltonian reduction. Namely, we will prove that the Poisson vertex algebras we construct using Drinfeld-Sokolov approach, in the case of the Lie algebra $\mathfrak{gl}_n$ (respectively $\mathfrak{sl}_n$) and its principal nilpotent element, is isomorphic to the Poisson vertex algebra we got, using Gelfand-Dickey approach, in the case of a general differential operator of order n (respectively the same differential operator with missing ∂^{n-1} term).

In the last chapter we will be interested in finding integrable systems attached to $\mathcal{W}$-algebras and proving their integrability. First, we will consider the homogeneous case, namely we will find integrable hierarchies for affine Poisson vertex algebras (see Example 1.6 for the definition). Then we will generalize the results of [12], about integrable systems attached to $\mathcal{W}$-algebras via Drinfeld-Sokolov Hamiltonian reduction, to a larger class of nilpotent elements.

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CHAPTER 1

Poisson vertex algebras and Hamiltonian equations

In this chapter we review the connection between Poisson vertex algebras and the theory of Hamiltonian equations as laid down in [3]. It will be shown that Poisson vertex algebras provide a very convenient framework for systems of Hamiltonian equations associated to a Hamiltonian operator. As the main application we explain how to establish integrability of such partial differential equations using the so called Lenard scheme.

1.1. Algebras of differential functions and Poisson vertex algebras

By a *differential algebra* we shall mean a unital commutative associative algebra R over $\mathbb{C}$ with a derivation ∂, that is a $\mathbb{C}$-linear map from R to itself such that, for $a, b \in R$

$$\partial(ab) = \partial(a)b + a\partial(b).$$

In particular $\partial 1 = 0$.

One of the most important examples we are interested in is the *algebra of differential polynomials* in one independent variable x and l dependent variables u_i (l may also be infinite)

$$R_l[x] = \mathbb{C}[x, u_i^{(n)} \mid i \in \{1, \dots, l\} = I, n \in \mathbb{Z}_+],$$

where the derivation ∂ is defined by $\partial(u_i^{(n)}) = u_i^{(n+1)}$ and $\partial x = 1$. One can also consider the *algebra of translation invariant differential polynomials* in l variables u_i

$$R_l = \mathbb{C}[u_i^{(n)} \mid i \in \{1, \dots, l\} = I, n \in \mathbb{Z}_+],$$

where $\partial(u_i^{(n)}) = u_i^{(n+1)}$.

Definition 1.1. An *algebra of differential functions* $\mathcal{V}$ in one independent variable x and a set of dependent variables $\{u_i\}_{i\in I}$ is a differential algebra with a derivation ∂ endowed with linear maps $\frac{\partial}{\partial u_i^{(n)}} : \mathcal{V} \longrightarrow \mathcal{V}$, for all $i \in I$ and $n \in \mathbb{Z}_+$, which are commuting derivations of the product in $\mathcal{V}$ such that, given $f \in \mathcal{V}$, $\frac{\partial f}{\partial u_i^{(n)}} = 0$ for all but finitely many $i \in I$ and $n \in \mathbb{Z}_+$ and the following commutation relations hold

$$\left[\frac{\partial}{\partial u_i^{(n)}}, \partial\right] = \frac{\partial}{\partial u_i^{(n-1)}}, \tag{1.1}$$

where the RHS is considered to be zero if $n = 0$.

We call $\mathcal{C} = \ker(\partial) \subset \mathcal{V}$ the subalgebra of *constant functions* and denote by $\mathcal{F} \subset \mathcal{V}$ the subalgebra of *quasiconstant functions*, defined by

$$\mathcal{F} = \{f \in \mathcal{V} \mid \frac{\partial f}{\partial u_i^{(n)}} = 0 \,\forall i \in I, n \in \mathbb{Z}_+\}.$$

One says that $f \in \mathcal{V}$ has *differential order* n in the variable u_i if $\frac{\partial f}{\partial u_i^{(n)}} \neq 0$ and $\frac{\partial f}{\partial u_i^{(m)}} = 0$ for all $m > n$. It follows by (1.1) that $\mathcal{C} \subset \mathcal{F}$. Indeed, suppose that $f \in \mathcal{C}$ has order $n \in \mathbb{Z}_+$ in some variable u_i, then $0 = \left[\frac{\partial}{\partial u_i^{(n+1)}}, \partial\right] f = \frac{\partial f}{\partial u_i^{(n)}}$ which contracdicts our hypothesis. Furthermore, clearly, $\partial\mathcal{F} \subset \mathcal{F}$.

The differential algebras $R_l[x]$ and R_l are examples of algebras of differential functions. Other examples can be constructed starting from $R_l[x]$ or R_l by taking a localization by some multiplicative subset S, or an algebraic extension obtained by adding solutions of some polynomial equations, or a differential extension obtained by adding solutions of some differential equations.

In all these examples, but more generally in any algebra of differential functions which is an extension of $R_l[x]$, the action of the derivation $\partial : \mathcal{V} \longrightarrow \mathcal{V}$, which extends the usual derivation in $R_l[x]$, is given by

$$\partial = \frac{\partial}{\partial x} + \sum_{i\in I, n\in\mathbb{Z}_+} u_i^{(n+1)} \frac{\partial}{\partial u_i^{(n)}}, \tag{1.2}$$

which implies that $\mathcal{F} \cap \partial\mathcal{V} = \partial\mathcal{F}$. Indeed, if $f \in \mathcal{V}$ has differential order $n \in \mathbb{Z}_+$ in some variable u_i, then ∂f has differential order $n+1$, hence, it does not lie in $\mathcal{F}$.

The commutation relations (1.1) imply the following lemma ([3, Lemma 1.2]).

Lemma 1.2. *Let $D_i(z) = \sum_{n\in\mathbb{Z}_+} z^n \frac{\partial}{\partial u_i^{(n)}}$. Then for every $h(\lambda) = \sum_{m=0}^{N} h_m \lambda^m \in \mathbb{C}[\lambda] \otimes \mathcal{V}$ and $f \in \mathcal{V}$ the following identity holds*

$$D_i(z)(h(\partial) f) = D_i(z)(h(\partial)) f + h(z+\partial)(D_i(z) f),$$

where $D_i(z)(h(\partial))$ is the differential operator obtained by applying $D_i(z)$ to the coefficients of $h(\partial)$.

Proof. Multiplying by z^n and summing over $n \in \mathbb{Z}_+$ both sides of (1.1) we get $D_i(z) \circ \partial = (z+\partial) D_i(z)$. It follows that

$$D_i(z) \circ \partial^n = (z+\partial)^n D_i(z),$$

for every $n \in \mathbb{Z}_+$. Thus, if $h(\lambda) = h_n \lambda^n$, this implies that

$$\begin{aligned} D_i(z) \circ h(\partial) = D_i(z) \circ h_n \partial^n = D_i(z)(h_n) \circ \partial^n + h_n D_i(z) \circ \partial^n = D_i(z)(h_n) \circ \partial^n + h_n (z+\partial)^n D_i(z) = \\ = D_i(z)(h(\partial)) + h(z+\partial) D_i(z). \end{aligned}$$

By linearity the general case follows. $\square$

We denote by $\mathcal{V}^{\oplus l} \subset \mathcal{V}^l$ the subspace of all $F = (F_i)_{i\in I}$ with finitely many non-zero entries (l may also be infinite) and introduce a pairing $\mathcal{V}^l \times \mathcal{V}^{\oplus l} \longrightarrow \mathcal{V}/\partial\mathcal{V}$

$$(P, F) \longrightarrow \textstyle\int PF, \tag{1.3}$$

where $\int$ denotes the canonical map $\mathcal{V} \longrightarrow \mathcal{V}/\partial\mathcal{V}$. The pairing (1.3) is non-degenerate [3, Proposition 1.3], namely $\int PF = 0$ for every $F \in \mathcal{V}^{\oplus l}$ if and only if $P = 0$.

Let us define the operator of *variational derivative* $\frac{\delta}{\delta u} : \mathcal{V} \longrightarrow \mathcal{V}^{\oplus l}$ by $\frac{\delta f}{\delta u} = \left(\frac{\delta f}{\delta u_i}\right)_{i\in I} \in \mathcal{V}^{\oplus l}$, where

$$\frac{\delta f}{\delta u_i} = \sum_{n\in\mathbb{Z}_+} (-\partial)^n \frac{\partial f}{\partial u_i^{(n)}}.$$

By (1.1), it follows immediately that $\frac{\delta}{\delta u_i} \cdot \partial = 0$, for each $i \in I$, then $\partial\mathcal{V} \subset \ker \frac{\delta}{\delta u}$.

We let $\mathrm{Vect}(\mathcal{V})$ be the space of all vector fields of $\mathcal{V}$, which is a Lie subalgebra of $\mathrm{Der}(\mathcal{V})$, the Lie algebra of all derivations of $\mathcal{V}$. An element $X \in \mathrm{Vect}(\mathcal{V})$ is of the form

$$X = h\frac{\partial}{\partial x} + \sum_{i\in I, n\in\mathbb{Z}_+} h_{i,n} \frac{\partial}{\partial u_i^{(n)}}, \quad h, h_{i,n} \in \mathcal{V}. \tag{1.4}$$

By (1.2), ∂ is an element of $\mathrm{Vect}(\mathcal{V})$ and we denote by $\mathrm{Vect}^{\partial}(\mathcal{V})$ the centralizer of ∂ in $\mathrm{Vect}(\mathcal{V})$, namely $\mathrm{Vect}^{\partial}(\mathcal{V}) = \mathrm{Vect}(\mathcal{V}) \cap \mathrm{Der}^{\partial}(\mathcal{V})$. Elements $X \in \mathrm{Vect}^{\partial}(\mathcal{V})$ are called *evolutionary vector fields*. For $X \in \mathrm{Vect}^{\partial}(\mathcal{V})$ we have $X(u_i^{(n)}) = X(\partial^n u_i) = \partial^n X(u_i)$, so that, by (1.4) and $\left[\frac{\partial}{\partial x}, \partial\right] = 0$, X is completely determined by its values $X(u_i) = P_i$, $i \in I$. Thus, we have a vector space isomorphism $\mathcal{V}^l \cong \mathrm{Vect}^{\partial}(\mathcal{V})$ given by

$$\mathcal{V}^l \ni P = (P_i)_{i\in I} \longrightarrow X_P = \sum_{i\in I, n\in\mathbb{Z}_+} (\partial^n P_i) \frac{\partial}{\partial u_i^{(n)}} \in \mathrm{Vect}^{\partial}(\mathcal{V}). \tag{1.5}$$

The l-tuple P is called the *characteristic* of the vector field X_P.

The *Fréchet derivative* D_f of $f \in \mathcal{V}$ is defined as the following differential operator from $\mathcal{V}^l$ to $\mathcal{V}$:

$$D_f(\partial) P = X_P(f) = \sum_{i\in I, n\in\mathbb{Z}_+} \frac{\partial f}{\partial u_i^{(n)}} \partial^n P_i.$$

We note that $D_f(\partial)P$ is just the first-order differential of the function $f(u)$, indeed $f(u + \varepsilon P) = f(u) + \varepsilon D_f(\partial) P + o(\varepsilon^2)$.

More generally, for any collection $F = (f_\alpha)_{\alpha\in\mathcal{A}}$, where $\mathcal{A}$ is an index set, the corresponding *Fréchet derivative* is the linear map $D_F(\partial) : \mathcal{V}^l \longrightarrow \mathcal{V}^{\mathcal{A}}$ given by

$$(D_F(\partial)P)_\alpha = D_{f_\alpha}(\partial)P = X_P(f_\alpha),\ \alpha \in \mathcal{A}.$$

Its adjoint map with respect to the pairing (1.3) is the linear map $D_F^*(\partial) : \mathcal{V}^{\oplus\mathcal{A}} \longrightarrow \mathcal{V}^{\oplus l}$ defined by

$$(D_F^*(\partial)G)_i = \sum_{\alpha\in\mathcal{A},n\in\mathbb{Z}_+} (-\partial)^n \left(\frac{\partial F_\alpha}{\partial u_i^{(n)}} G_\alpha\right),\ i \in I.$$

We have the following formula for the commutator of evolutionary vector fields in terms of the Fréchet derivatives:

$$[X_P, X_Q] = X_{D_Q(\partial)P-D_P(\partial)Q}.$$

Elements of the form $\frac{\delta f}{\delta u} \in \mathcal{V}^{\oplus l}$ are called *exact*. An element $F \in \mathcal{V}^{\oplus l}$ is called *closed* if its Fréchet derivative is a self-adjoint differential operator, that is $D_F(\partial) = D_F^*(\partial)$. It is well-known and not hard to check, applying Lemma 1.2 twice, that any exact element in $\mathcal{V}^{\oplus l}$ is closed, namely $D_{\frac{\delta f}{\delta u}}(\partial) = D^*_{\frac{\delta f}{\delta u}}(\partial)$ for every $\int f \in \mathcal{V}/\partial\mathcal{V}$.

The main ingredient in the definition of Poisson vertex algebras is the notion of λ-bracket that we are going to define.

Definition 1.3. Let $\mathcal{V}$ be a $\mathbb{C}[\partial]$-module. A *λ-bracket* on $\mathcal{V}$ is a $\mathbb{C}$-linear map

$$\begin{aligned}\{\cdot_\lambda\cdot\} : \mathcal{V}\otimes\mathcal{V} &\longrightarrow \mathbb{C}[\lambda]\otimes\mathcal{V}\\ f\otimes g &\longrightarrow \{f_\lambda g\},\end{aligned}$$

which is *sesquilinear*, that is, for $f, g \in \mathcal{V}$

$$\{\partial f_\lambda g\} = -\lambda\{f_\lambda g\},\ \{f_\lambda \partial g\} = (\lambda+\partial)\{f_\lambda g\}. \tag{1.6}$$

If, moreover, $\mathcal{V}$ is a commutative associative unital differential algebra with a derivation ∂, a λ-bracket on $\mathcal{V}$ is defined to obey for any $f, g, h \in \mathcal{V}$, in addition, the *left Leibniz rule*

$$\{f_\lambda gh\} = \{f_\lambda g\}h + \{f_\lambda h\}g, \tag{1.7}$$

and the *right Leibniz rule*

$$\{fg_\lambda h\} = \{f_{\lambda+\partial}h\}_\to g + \{g_{\lambda+\partial}h\}_\to f. \tag{1.8}$$

We note that the sesquilinearity property (1.6) means that ∂ is a derivation for the λ-bracket. We should also explain the meaning of the arrow in (1.8). Usually, one writes

$$\{f_\lambda g\} = \sum_{n\in\mathbb{Z}_+} \frac{\lambda^n}{n!}(f_{(n)}g).$$

The $\mathbb{C}$-bilinear products $f_{(n)}g$ are called n-th products on $\mathcal{V}$ and we note that $f_{(n)}g = 0$ for n sufficiently large. In (1.8) and furhter on, the arrow means where $\lambda+\partial$ should be moved. For example

$$\{f_{\lambda+\partial}h\}_\to g = \sum_{n\in\mathbb{Z}_+} (f_{(n)}g)\frac{(\lambda+\partial)^n}{n!}g.$$

We say that the λ-bracket is *commutative* (respectively *skew-commutative*) if

$$\{g_\lambda f\} = {}_\leftarrow\{f_{-\lambda-\partial}g\}\ (\text{respectively }\ = -{}_\leftarrow\{f_{-\lambda-\partial}g\}). \tag{1.9}$$

We remark that in this case the direction of the arrow tells us that

$${}_\leftarrow\{f_{-\lambda-\partial}g\} = \sum_{n\in\mathbb{Z}_+} \frac{(-\lambda-\partial)^n}{n!}(f_{(n)}g).$$

From this point we assume the convention that in case there is no arrow, it is assumed to be to the left.

For $f, g, h \in \mathcal{V}$, the following identity

$$\{f_\lambda\{g_\mu h\}\} = \{\{f_\lambda g\}_{\lambda+\mu}h\} + \{g_\mu\{f_\lambda h\}\} \tag{1.10}$$

is called *Jacobi identity* for the λ-bracket.

Definition 1.4. A $\mathbb{C}[\partial]$-module $\mathcal{V}$ endowed with a λ-bracket which satisfies skew-commutativity (1.9) and Jacobi identity (1.10) is called *Lie conformal algebra* [19]. If, moreover, $\mathcal{V}$ is a differential algebra, then $\mathcal{V}$ is called *Poisson vertex algebra* [3].

Next theorem explains how to extend an arbitrary "non-linear" λ-bracket on a set of variables $\{u_i\}_{i\in I}$ with value in some algebra $\mathcal{V}$ of differential functions to a Poisson vertex algebra structure on $\mathcal{V}$.

Theorem 1.5 ([3, Theorem 1.15]). *Let $\mathcal{V}$ be an algebra of differential functions, which is an extension of the algebra of differential polynomials $R_l[x] = \mathbb{C}[x, u_i^{(n)} \mid i \in I, n \in \mathbb{Z}_+]$. For each pair $i, j \in I$, choose $\{u_{i\lambda}u_j\} \in \mathbb{C}[\lambda] \otimes \mathcal{V}$.*

(a) *Formula*

$$\{f_\lambda g\} = \sum_{\substack{i,j\in I\\ m,n\in\mathbb{Z}_+}} \frac{\partial g}{\partial u_j^{(n)}} (\lambda+\partial)^n \{u_{i\lambda+\partial}u_j\}_\to (-\lambda-\partial)^m \frac{\partial f}{\partial u_i^{(m)}} \tag{1.11}$$

defines a λ-bracket on $\mathcal{V}$, which extends the given λ-brackets on the generators u_i, $i \in I$.

(b) *The λ-bracket (1.11) on $\mathcal{V}$ satisfies the commutativity (respectively skew-commutativity) condition (1.9), provided that the same holds on generators:*

$$\{u_{i\lambda}u_j\} = {}_\leftarrow\{u_{j\,-\lambda-\partial}u_i\} \text{ (respectively } = -{}_\leftarrow\{u_{j\,-\lambda-\partial}u_i\}), \tag{1.12}$$

for all $i, j \in I$.

(c) *Assuming that the skew-commutativity condition (1.12) holds, the λ-bracket (1.11) satisfies the Jacobi identity (1.10), thus making $\mathcal{V}$ a Poisson vertex algebra, provided that the Jacobi identity holds on any triple of generators*

$$\{u_{i\lambda}\{u_{j\mu}u_k\}\} = \{\{u_{i\lambda}u_j\}_{\lambda+\mu}u_k\} + \{u_{j\mu}\{u_{i\lambda}u_k\}\},$$

for all $i, j, k \in I$.

This theorem allows us to make some examples of Poisson vertex algebras.

Example 1.6. Let $\mathfrak{g}$ be a Lie algebra with a symmetric invariant bilinear form $(\cdot \mid \cdot)$ and let s be an element of $\mathfrak{g}$. Let $\{u_i\}_{i\in I}$ be a basis of $\mathfrak{g}$. The *affine Poisson vertex algebra* associated to the triple $(\mathfrak{g}, (\cdot \mid \cdot), s)$ is the differential algebra $R = \mathbb{C}[u_i^{(n)} \mid i \in I, n \in \mathbb{Z}_+]$ togheter with the following λ-bracket, defined for $a, b \in \mathfrak{g}$ by

$$\{a_\lambda b\} = [a, b] + (s \mid [a, b]) + \lambda(a \mid b) \tag{1.13}$$

and extended to the whole R by formula (1.11).

We point out that in the RHS of (1.13) any of the three summands, or more generally, any linear combination of them, endows R with a Poisson vertex algebra structure.

Example 1.7. When $\mathfrak{g} = \mathbb{C}u$ is the 1-dimensional abelian Lie algebra, if we assume the bilinear form normalized by the condition $(u \mid u) = 1$, then, for any $s \in \mathfrak{g}$, the affine Poisson vertex algebra associated to the triple $(\mathbb{C}u, (\cdot \mid \cdot), s)$ is the differential algebra $R = \mathbb{C}[u, u', u'', \dots]$ togheter with the following λ-bracket:

$$\{u_\lambda u\} = \lambda 1 \tag{1.14}$$

(one usually drops 1). This is known as *Gardner-Fadeev-Zakharov (GFZ)* λ-bracket.

More generally, one can replace λ in the RHS of (1.14) by any odd polynomial $p(\lambda) \in \mathbb{C}[\lambda]$ and still get a Poisson vertex algebra structure on R. Indeed, the bracket in (1.14) is skew-commutative and sastisfies the Jacobi identity for the triple u, u, u, since each triple commutator in the Jacobi identity is zero.

Example 1.8. The *Virasoro-Magri Poisson vertex algebra* on $R = \mathbb{C}[u, u', u'', \dots]$, with *central charge* $c \in \mathbb{C}$, is defined by

$$\{u_\lambda u\} = (\partial + 2\lambda)u + \lambda^3 c. \tag{1.15}$$

It is easily seen that the bracket (1.15) is skew-commutative and it satisfies the Jacobi identity for the triple u, u, u.

The following theorem further generalizes the results from Theorem 1.5, as it allows us to consider not only extensions of $R_l[x]$, but also quotients of such extensions by ideals.

Theorem 1.9 ([3, Theorem 1.21]). *Let $\mathcal{V}$ be an algebra of differential functions, which is an extension of the algebra of differential polynomials $R_l[x] = \mathbb{C}[x, u_i^{(n)} \mid i \in I, n \in \mathbb{Z}_+]$. For each pair $i, j \in I$, let $\{u_{i\lambda} u_j\} \in \mathbb{C}[\lambda] \otimes \mathcal{V}$ and consider the induced λ-bracket on $\mathcal{V}$ defined by formula (1.11). Suppose that $\mathcal{J} \subset \mathcal{V}$ is a subspace such that $\partial \mathcal{J} \subset \mathcal{J}$, $\mathcal{J}\mathcal{V} \subset \mathcal{J}$, $\{\mathcal{V}_\lambda \mathcal{J}\} \subset \mathbb{C}[\lambda] \otimes \mathcal{J}$, $\{\mathcal{J}_\lambda \mathcal{V}\} \subset \mathbb{C}[\lambda] \otimes \mathcal{J}$ and consider the quotient space $\mathcal{V}/\mathcal{J}$ with the induced action of ∂, the induced commutative associative product and the induced λ-bracket.*

(a) *The λ-bracket on the quotient space $\mathcal{V}/\mathcal{J}$ satisfies the commutativity (respectively skew-commutativity) condition (1.9), provided that*
$$\{u_{i\lambda} u_j\} \mp {}_{\leftarrow}\{u_{j\,-\lambda-\partial} u_i\} \in \mathbb{C}[\lambda] \otimes \mathcal{J}, \tag{1.16}$$
for all $i, j \in I$.

(b) *Furthermore, assuming that the skew-commutativity condition (1.16) holds, the λ-bracket on $\mathcal{V}/\mathcal{J}$ satisfies the Jacobi identity (1.10), thus making $\mathcal{V}/\mathcal{J}$ a Poisson vertex algebra, provided that*
$$\{u_{i\lambda}\{u_{j\mu} u_k\}\} - \{u_{j\mu}\{u_{i\lambda} u_k\}\} - \{\{u_{i\lambda} u_j\}_{\lambda+\mu} u_k\} \in \mathbb{C}[\lambda] \otimes \mathcal{J},$$
for all $i, j, k \in I$.

This theorem will be needed to prove that *classical $\mathcal{W}$-algebras* are Poisson vertex algebras.

Example 1.10. Let A be a Lie conformal algebra with a central element K such that $\partial K = 0$. Then $\mathcal{V}^k(A) = S(A)/(K - k1)$, $k \in \mathbb{C}$, carries the usual structure of a unital commutative associative differential algebra endowed with the λ-bracket, extending that from A by the left and right Leibniz rules, making it a Poisson vertex algebra. This generalization of Examples 1.6, 1.7, 1.8 may be viewed as a Poisson vertex algebras analogue of the Lie-Kirillov-Kostant Poisson algebra $S(\mathfrak{g})$ associated to a Lie algebra $\mathfrak{g}$.

1.2. Hamiltonian operators and Hamiltonian equations

Theorem 1.5(a) says that, in order to define a λ-bracket on an algebra of differential functions $\mathcal{V}$ exending $R_l[x] = \mathbb{C}[x, u_i^{(n)} \mid i \in I, n \in \mathbb{Z}_+]$ one only needs to define for any pair $i, j \in I$ the λ-bracket
$$\{u_{i\lambda} u_j\} = H_{ji}(\lambda) \in \mathbb{C}[\lambda] \otimes \mathcal{V}. \tag{1.17}$$
In particular, λ-brackets on $\mathcal{V}$ are in one-to-one correspondence with $l \times l$-matrices $H(\lambda) = (H_{ij}(\lambda))_{i,j\in I}$, with $H_{ij}(\lambda) = \sum_{n=0}^{N} H_{ij;n}\lambda^n \in \mathcal{V}[\lambda]$, or, equivalently, with the corresponding $l \times l$-matrix valued differential operators $H(\partial) = (H_{ij}(\partial))_{i,j\in I} : \mathcal{V}^{\oplus l} \longrightarrow \mathcal{V}^l$. We denote by $\{\cdot_\lambda \cdot\}_H$ the λ-bracket on $\mathcal{V}$ corresponding to the operator $H(\partial)$ via equation (1.17).

We recall that the formal adjoint of $H(\partial)$ is the $l \times l$-matrix valued differential operator $H^*(\partial) = (H^*_{ij}(\partial))_{i,j\in I}$, where $H^*_{ij}(\partial) = \sum_{n=0}^{N} (-\partial)^n \circ H_{ji;n}$.

The next proposition relates Poisson vertex algebra structures with a special class of matrix valued differential operators.

Proposition 1.11 ([3, Proposition 1.16]). *Let $H(\partial) = (H_{ij}(\partial))_{i,j\in I}$ be an $l \times l$-matrix valued differential operator.*

(a) *The λ-bracket $\{\cdot_\lambda \cdot\}_H$ satisfies the (skew-)commutativity condition (1.9) if and only if the differential operator $H(\partial)$ is self(skew-)adjoint, that is $H(\partial) = \pm H^*(\partial)$.*

(b) *If $H(\partial)$ is skew-adjoint, the following conditions are equivalent:*

  (i) *the λ-bracket $\{\cdot_\lambda \cdot\}_H$ defines a Poisson vertex algebra structure on $\mathcal{V}$,*

  (ii) *the following identity holds for every $i, j, k \in I$:*
$$\sum_{h\in I, n\in\mathbb{Z}_+} \left(\frac{\partial H_{kj}(\mu)}{\partial u_h^{(n)}} (\lambda + \partial)^n H_{hi}(\lambda) - \frac{\partial H_{ki}(\lambda)}{\partial u_h^{(n)}} (\mu + \partial)^n H_{hj}(\mu) \right) =$$
$$= \sum_{h\in I, n\in\mathbb{Z}_+} H_{kh}(\lambda + \mu + \partial)(-\lambda - \mu - \partial)^n \frac{\partial H_{ji}(\lambda)}{\partial u_h^{(n)}},$$

  (iii) *the following identity holds for every $F, G \in \mathcal{V}^{\oplus l}$:*
$$H(\partial) D_G(\partial) H(\partial) F + H(\partial) D^*_{H(\partial)F}(\partial) G - H(\partial) D_F(\partial) H(\partial) G +$$
$$+ H(\partial) D^*_F(\partial) H(\partial) G = D_{H(\partial)G} H(\partial) F - D_{H(\partial)F}(\partial) H(\partial) G.$$

Definition 1.12. A matrix valued differential operator $H(\partial) = (H_{ij}(\partial))_{i,j\in I}$, which is skew-adjoint and satisfies one of the three equivalent condition (i)-(iii) of Proposition 1.11(b), is called *Hamiltonian operator.*

It follows from Proposition 1.11 that Poisson vertex algebra structures on $\mathcal{V}$ are in one-to-one correspondence with Hamiltonian operators.

Example 1.13 (cf. Example 1.6)**.** Let $\mathfrak{g}$ be a Lie algebra with a non-degenerate symmetric invariant bilinear form $(\cdot \mid \cdot)$. Let $\{u_i\}_{i\in I}$ be a basis of $\mathfrak{g}$ and $\{u^i\}_{i\in I}$ its dual basis with respect to $(\cdot \mid \cdot)$ and $[u_i, u_j] = \sum_k c_{ij}^k u_k$. Take $s \in \mathfrak{g}$, then $s = \sum_k (s \mid u_k) u^k$. The λ-bracket (1.13) on generators reads as

$$\{u_{i\lambda} u_j\} = \sum_{k\in I} c_{ij}^k \left(u_k + (s \mid u_k)\right) + (u_i \mid u_j)\lambda.$$

The corresponding Hamiltonian operator is $H(\partial) = \left(H_{ij}(\partial)\right)_{i,j\in I}$, where

$$H_{ij}(\partial) = -\sum_{k\in I} c_{ij}^k \left(u_k + (s \mid u_k)\right) + (u_i \mid u_j)\partial.$$

Example 1.14 (cf. Example 1.7 and Example 1.8)**.** The Hamiltonian operator corresponding to the GFZ λ-bracket (1.14) is $H(\partial) = \partial$, while to the Virasoro-Magri λ-bracket (1.15) corresponds the Hamiltonian operator $H(\partial) = u' + 2u\partial + c\partial^3$.

The relation between Poisson vertex algebras and systems of Hamiltonian equations associated to a Hamiltonian operator is based on the following result.

Proposition 1.15 ([3, Proposition 1.24])**.** *Let $\mathcal{V}$ be a $\mathbb{C}[\partial]$-module endowed with a λ-bracket $\{\cdot{}_\lambda\cdot\} : \mathcal{V}\otimes\mathcal{V} \longrightarrow \mathbb{C}[\lambda]\otimes\mathcal{V}$ and consider the bracket on $\mathcal{V}$ obtained by setting $\lambda = 0$, that is, for $f, g \in \mathcal{V}$*

$$\{f, g\} = \left.\{f_\lambda g\}\right|_{\lambda=0}. \tag{1.18}$$

- (a) *The bracket (1.18) induces a well defined bracket on the quotient space $\mathcal{V}/\partial\mathcal{V}$.*
- (b) *If $\mathcal{V}$ is a Lie conformal algebra, then the λ-bracket (1.18) induces a structure of a Lie algebra on $\mathcal{V}/\partial\mathcal{V}$ and a structure of left $\mathcal{V}/\partial\mathcal{V}$-module on $\mathcal{V}$.*
- (c) *If $\mathcal{V}$ is a Poisson vertex algebra, then the corresponding Lie algebra $\mathcal{V}/\partial\mathcal{V}$ acts on $\mathcal{V}$ via (1.18) by derivations of the commutative associative product on $\mathcal{V}$, commuting with the derivation ∂ and this defines a Lie algebra homomorphism from $\mathcal{V}/\partial\mathcal{V}$ to the Lie algebra of derivations of $\mathcal{V}$.*

Proposition 1.15 motivates the next definition.

Definition 1.16 ([3, Definition 1.25])**.**

- (a) Elements of $\mathcal{V}/\partial\mathcal{V}$ are called *local functionals*. Given $f \in \mathcal{V}$, its image in $\mathcal{V}/\partial\mathcal{V}$ is denoted by $\int f$.
- (b) Given a local functional $\int h \in \mathcal{V}/\partial\mathcal{V}$, the corresponding *Hamiltonian equation* is

$$\frac{du}{dt} = \left.\{h_\lambda u\}\right|_{\lambda=0} = \{{\textstyle\int} h, u\} \tag{1.19}$$

  and $\{\int h, \cdot\}$ is the corresponding *Hamiltonian vector field.*
- (c) A local functional $\int f \in \mathcal{V}/\partial\mathcal{V}$ is called an *integral of motion* of equation (1.19) if $\frac{df}{dt} = 0$ mod $\partial\mathcal{V}$, or, equivalently, if

$$\{{\textstyle\int} h, {\textstyle\int} f\} = 0.$$

- (d) The local functionals $\int h_n$, $n \in \mathbb{Z}_+$ are *in involution* if $\{\int h_m, \int h_n\} = 0$ for all $m, n \in \mathbb{Z}_+$. The corresponding *hierarchy of Hamiltonian equations* is

$$\frac{du}{dt_n} = \left.\{h_{n\lambda} u\}\right|_{\lambda=0} = \{{\textstyle\int} h_n, u\},\ n \in \mathbb{Z}_+.$$

  In particular, all $\int h_n$'s are integrals of motion of each equation of the hierarchy.

From now on we restrict to the case in which the Poisson vertex algebra $\mathcal{V}$ is an algebra of differential functions in the variables $\{u_i\}_{i\in I}$. In this case, we have already seen that the λ-bracket $\{\cdot{}_\lambda\cdot\}_H$ is uniquely defined by the corresponding Hamiltonian operator $H(\partial) = \left(H_{ij}(\partial)\right)_{i,j\in I}$ and extended to a λ-bracket on $\mathcal{V}$ by formula (1.11). In this case the Hamiltonian vector field $\{\int h, \cdot\}$ is equal to the evolutionary vector field $X_{H(\partial)\frac{\delta h}{\delta u}}$ and the Hamiltonian equation has the form

$$\frac{du}{dt} = H(\partial)\frac{\delta h}{\delta u},$$

where $\frac{\delta h}{\delta u} = \left(\frac{\delta h}{\delta u_i}\right)_{i\in I} \in \mathcal{V}^{\oplus l}$ is the variational derivative of h. Moreover, the corresponding Lie algebra structure of $\mathcal{V}/\partial\mathcal{V}$ is given by

$$\{\textstyle\int f, \int g\} = \displaystyle\int \frac{\delta g}{\delta u}\left(H(\partial)\frac{\delta f}{\delta u}\right) = \sum_{i,j\in I}\int \frac{\delta g}{\delta u_j} H_{ji}(\partial)\frac{\delta f}{\delta u_i}. \tag{1.20}$$

Remark 1.17. Since $\int h \longrightarrow X_{H(\partial)\frac{\delta h}{\delta u}}$ is a Lie algebra homomorphism, local functionals in involution correspond to commuting evolutionary vector fields. If a sequence $\int h_n \in \mathcal{V}/\partial\mathcal{V}$ is such that $\frac{\delta h_n}{\delta u} \in \mathcal{V}^{\oplus l}$ span an infinite dimensional subspace and $\dim\ker H(\partial) < \infty$, then the vector fields $X_{H(\partial)\frac{\delta h_n}{\delta u}}$ span an infinite dimensional space as well.

Definition 1.18. The Hamiltonian equation (1.19) is called *integrable* if there exists an infinite sequence of local functionals $\int h_n$, including $\int h$, which span an infinite dimensional abelian subspace in the Lie algebra $\mathcal{V}/\partial\mathcal{V}$, with Lie bracket defined by (1.18), and such that the evolutionary vector fields $X_{H(\partial)\frac{\delta h_n}{\delta u}}$ span an infinite dimensional space (they commute by Remark 1.17).

1.3. Compatible Poisson vertex algebra structures and integrability of Hamiltonian equations

Definition 1.19. Several λ-brackets $\{\cdot{}_\lambda\cdot\}_n$, $n = 1,2,\dots,N$, on a differential algebra $\mathcal{V}$ are called *compatible* if any $\mathbb{C}$-linear combination of them, $\{\cdot{}_\lambda\cdot\} = \sum_{n=1}^N \{\cdot{}_\lambda\cdot\}_n$, makes it a Poisson vertex algebra. If $\mathcal{V}$ is an algebra of differential functions and $H_n(\partial)$, $n = 1,2,\dots,N$ are the Hamiltonian operators, defined by (1.17), corresponding to the λ-brackets, we say that they are *compatible* as well. A *bi-Hamiltonian pair* (H,K) is a pair of compatible Hamiltonian operators $H(\partial), K(\partial)$.

Example 1.20. (cf. Examples 1.7, 1.8 and 1.14)Let $R = \mathbb{C}[u,u',u'',\dots]$. The λ-brackets

$$\{u_\lambda u\}_1 = (\partial + 2\lambda)u, \qquad \{u_\lambda u\}_2 = \lambda, \qquad \{u_\lambda u\}_3 = \lambda^3,$$

are compatible. The corresponding compatible Hamiltonian operators are

$$H_1(\partial) = u' + 2u\partial, \qquad H_2(\partial) = \partial, \qquad H_3(\partial) = \partial^3.$$

Example 1.21. (cf. Examples 1.6 and 1.13) Let $\mathfrak{g}$ be a Lie algebra with a nondegenerate symmetric bilinear form $(\cdot\mid\cdot)$, let $\{u_i\}_{i\in I}$ be an orthonormal basis of $\mathfrak{g}$ and let $[u_i,u_j] = \sum_k c_{ij}^k u_k$. Let $R = \mathbb{C}[u_i^{(n)} \mid i \in I, n \in \mathbb{Z}_+]$, then the following λ-brackets on R are compatible:

$$\{u_{i\lambda} u_j\}' = \sum_k c_{ij}^k u_k, \qquad \{u_{i\lambda} u_j\}'' = \delta_{i,j}\lambda, \qquad \{u_{i\lambda} u_j\} = c_{ij}^k, k \in I.$$

The corresponding compatible Hamiltonian operators, H', H'' and H^k, $k \in I$, are given by

$$H'_{ij}(\partial) = -\sum_{k\in I} c_{ij}^k u_k, \qquad H''_{ij}(\partial) = \delta_{ij}\partial, \qquad H_{ij}^k = c_{ij}^k.$$

Definition 1.22. Let $\mathcal{V}$ be an algebra of differential functions and let $H(\partial) = (H_{ij}(\partial))_{i,j\in I}$ and $K(\partial) = (K_{ij}(\partial))_{i,j\in I}$ be any two differential operators on $\mathcal{V}^{\oplus l}$. An (H,K)-*sequence* is a collection $\{F_n\}_{0\le n\le N} \subset \mathcal{V}^{\oplus l}$ such that

$$H(\partial)F_n = K(\partial)F_{n+1}, \qquad 0 \le n \le N-1. \tag{1.21}$$

If $N = \infty$, we say that $\{F_n\}_{n\in\mathbb{Z}_+}$ is an *infinite* (H,K)-*sequence.*

Equation (1.21) for an infinite (H,K)-sequence can be rewritten using the generating series $F(z) = \sum_{n\in\mathbb{Z}_+} F_n z^{-n}$ as follows:

$$\left(H(\partial) - zK(\partial)\right)F(z) = -zK(\partial)F_0. \tag{1.22}$$

Note that in the special case in which $F_n = \frac{\delta h_n}{\delta u}$, for some $\int h_n \in \mathcal{V}/\partial\mathcal{V}$, equation (1.21) can be written in terms of the λ-brackets associated to the operators $H(\partial)$ and $K(\partial)$ (see (1.17)):

$$\{h_{n\lambda} u_i\}_H\big|_{\lambda=0} = \{h_{n+1\lambda} u_i\}_K\big|_{\lambda=0}, \tag{1.23}$$

for any $i \in I$.

Lemma 1.23. *Suppose that the operators $H(\partial)$ and $K(\partial)$, acting on $\mathcal{V}^{\oplus l}$, are skew-adjoint. Then, any (H,K)-sequence $\{F_n\}_{0\le n\le N}$ satisfies the orthogonality relations*

$$\textstyle\int F_m \cdot H(\partial)F_n = \int F_m \cdot K(\partial)F_n = 0, \qquad 0 \le m,n \le N. \tag{1.24}$$

Proof. Without loss of generality, we can assume $m \leq n$ and prove (1.24) by induction on $n-m$. If $m=n$, (1.24) clearly holds, since both $H(\partial)$ and $K(\partial)$ are skewadjoint operators. Let us assume $m-n>0$, by (1.21) and inductive assumption we have

$$\int F_m \cdot K(\partial)F_n = \int F_m \cdot H(\partial)F_{n-1} = 0,$$

and similarly, since $H(\partial)$ is skew-adjoint,

$$\int F_m \cdot H(\partial)F_n = -\int F_n \cdot H(\partial)F_m = -\int F_n \cdot K(\partial)F_{m+1} = 0.$$

□

We have a way to construct an infinite hierarchy of Hamiltonian equations, $\frac{du}{dt_n} = \{\int h_n, u\}_H$, and the associated infinite sequence of integrals of motion $\int h_n, n \in \mathbb{Z}_+$. In order to do this, we have to solve two problems. First, given a bi-Hamiltonian pair (H,K), acting on $\mathcal{V}^{\oplus l}$, we need to find an infinite (H,K)-sequence $\{F_n\}_{n\in\mathbb{Z}_+}$. Second, we need to prove that F_n, $n \in \mathbb{Z}_+$ is an exact element, namely, $F_n = \frac{\delta h_n}{\delta u}$, for some local functional $\int h_n \in \mathcal{V}/\partial\mathcal{V}$ and we want to find an explicit formula for it. By Lemma 1.23, the corresponding local functionals $\int h_n$ are pairwise in involution with respect to both Lie brackets associated to both Hamiltonian operators $H(\partial)$ and $K(\partial)$:

$$\{\int h_m, \int h_n\}_H = \{\int h_m, \int h_n\}_K = 0, \qquad \text{for all } m,n \in \mathbb{Z}_+. \tag{1.25}$$

CHAPTER 2

Classical $\mathcal{W}$-algebras via Drinfeld-Sokolov Hamiltonian reduction

In the seminal paper [12] a hierarchy of integrable Hamiltonian equations is attached to semisimple Lie algebras. This construction, called *classical Drinfeld-Sokolov Hamiltonian reduction*, is obtained starting from any principal nilpotent element in the Lie algebra. The algebraic structure arising from this Hamiltonian reduction is nowadays known as *classical $\mathcal{W}$-algebra*. Later, classical $\mathcal{W}$-algebras were constructed starting from any nilpotent element. In this chapter, we review the classical Drinfeld-Sokolov Hamiltonian reduction extending it to the case of an arbitrary nilpotent element and give an interpretation in the language of Poisson vertex algebras (cf. Chapter 1). Then we realize classical $\mathcal{W}$-algebras corresponding to Lie algebras of type B_n, C_n and D_n as quotients of some Poisson vertex subalgebras of the classical $\mathcal{W}$-algebra attached to $\mathfrak{gl}_n$.

2.1. Review of classical Drinfeld-Sokolov Hamiltonian reduction

Let $\mathfrak{g}$ be a reductive finite dimensional Lie algebra with a nondegenerate symmetric invariant bilinear form $(\cdot\mid\cdot)$ and $f\in\mathfrak{g}$ a nilpotent element. By Jacobson-Morozov theorem [6, Theorem 3.3.1], it is possible to embed f in an $\mathfrak{sl}_2$-triple $\{f,h=2x,e\}\subset\mathfrak{g}$. Then we can write

$$\mathfrak{g}=\bigoplus_{i\in\frac{1}{2}\mathbb{Z}}\mathfrak{g}_i \tag{2.1}$$

for its $\operatorname{ad}x$-decomposition. It follows that $f\in\mathfrak{g}_{-1}$, $h\in\mathfrak{g}_0$ and $e\in\mathfrak{g}_1$.

We consider the following subalgebras of $\mathfrak{g}$:

$$\mathfrak{m}_+=\bigoplus_{i\geq 1}\mathfrak{g}_i\subset\mathfrak{n}_+=\bigoplus_{i\geq\frac{1}{2}}\mathfrak{g}_i\subset\mathfrak{b}_+=\bigoplus_{i\in\mathbb{Z}_+}\mathfrak{g}_i\subset\mathfrak{B}_+=\bigoplus_{i\geq-\frac{1}{2}}\mathfrak{g}_i. \tag{2.2}$$

Let $C^\infty(S^1,\mathfrak{g})$ be the space of functions from $S^1=\{z\in\mathbb{C}\mid |z|=1\}$ with values in $\mathfrak{g}$. This space inherits a Lie algebra structure from $\mathfrak{g}$ and the bilinear form of $\mathfrak{g}$ extends to a bilinear form on $C^\infty(S^1,\mathfrak{g})$, which we still denote $(\cdot\mid\cdot)$. Moreover, set $\partial=\frac{d}{dx}$ the total derivative of functions, then ∂ naturally acts on $C^\infty(S^1,\mathfrak{g})$. This allow us to consider the semidirect product Lie algebra $\mathbb{C}\partial\ltimes C^\infty(S^1,\mathfrak{g})$, where $[\partial,u]=u'$, for any $u\in C^\infty(S^1,\mathfrak{g})$.

Let $\mathscr{L}\in\mathbb{C}\partial\ltimes C^\infty(S^1,\mathfrak{B}_+)$ be a first order differential operator of the form

$$\mathscr{L}=\partial+q+\Lambda(z), \tag{2.3}$$

where $q\in C^\infty(S^1,\mathfrak{B}_+)$ and $\Lambda(z)=f+zs$, with $z\in\mathbb{C}$ and $s\in\ker\operatorname{ad}\mathfrak{n}_+$. Clearly $\partial\Lambda(z)=0$.

We define a *gauge transformation* to be a transformation of the type

$$\widetilde{\mathscr{L}}=e^{\operatorname{ad}S}(\mathscr{L}),$$

where $S\in C^\infty(S^1,\mathfrak{n}_+)$. From the fact that $[\mathfrak{n}_+,\mathfrak{B}_+]\subseteq\mathfrak{b}_+$, $[\mathfrak{n}_+,s]=0$ and $[\mathfrak{n}_+,f]\subseteq\mathfrak{B}_+$ it follows that $\widetilde{\mathscr{L}}=\partial+\widetilde{q}+\Lambda(z)$, where $\widetilde{q}=e^{\operatorname{ad}S}(\mathcal{L})-\partial-\Lambda(z)\in C^\infty(S^1,\mathfrak{B}_+)$. Then $\mathscr{L}$ and $\widetilde{\mathscr{L}}$ have the same form. We call them *gauge equivalent operators* and usually write $\mathscr{L}\overset{S}{\sim}\widetilde{\mathscr{L}}$.

Now, let R be the ring of differential polynomials in q that are invariants for gauge transformations. The expression *p is a differential polynomial in q*, where p and q are functions in some vector spaces V and W, means that for some, thus any, choice of bases in V and W, the coordinates of p are differential polynomials in the coordinates of q. Hence, a differential polynomial p belongs to R if and only if $p(q)=p(\widetilde{q})$, for any $\mathscr{L}\overset{S}{\sim}\widetilde{\mathscr{L}}$. We emphasize that $\widetilde{q}=\widetilde{q}(q)$.

For any $i>0$ the operator $\operatorname{ad}f$ acts from $\mathfrak{g}_i$ to $\mathfrak{g}_{i-1}$ injectively (since the grading (2.1) is given by the $\mathfrak{sl}_2$ triple in which f is embedded; such a grading is called good [13]). When it is not surjective, we can choose a vector subspace $V_i\subset\mathfrak{g}_i$ such that $\mathfrak{g}_i=V_i\oplus[f,\mathfrak{g}_{i+1}]$. We set $V=\oplus_iV_i$. Since $\mathfrak{B}_+=V\oplus[f,\mathfrak{n}_+]$ and $\operatorname{ad}f:\mathfrak{n}_+\longrightarrow\mathfrak{B}_+$ is injective, it follows that $\dim V=\dim\mathfrak{B}_+-\dim\mathfrak{n}_+$. By representation theory of $\mathfrak{sl}_2$, it follows that we can choose, for example, $V=\ker\operatorname{ad}e$.

The next proposition is a slight generalization of [12, Proposition 6.1].

Proposition 2.1. *For any operator $\mathscr{L}$ of the form (2.3) there exists a unique $S \in C^\infty(S^1, \mathfrak{n}_+)$ such that $\mathscr{L} \overset{S}{\sim} \mathscr{L}^{can} = \partial + q^{can} + \Lambda(z)$, where $q^{can} \in C^\infty(S^1, V)$. The elements S and q^{can} are differential polynomials in q.*

Proof. Since $[S, s] = 0$, we are left to show that we can find S and q^{can} such that

$$\partial + q^{can} + f = e^{\operatorname{ad} S}(\partial + q + f). \tag{2.4}$$

Let us write $q = \sum_{i \geq -\frac{1}{2}} q_i$, $q^{can} = \sum_{i \geq -\frac{1}{2}} q_i^{can}$ and $S = \sum_{i \geq \frac{1}{2}} S_i$, where q_i, q_i^{can} and S_i are in $C^\infty(S^1, \mathfrak{g}_i)$. Equating coefficients lying in $\mathfrak{g}_i$ in the relation (2.4) we should have $q^{can}_{-\frac{1}{2}} + [f, S_1] = q_{-\frac{1}{2}}$ and, for $i \in \mathbb{Z}_+$,

$$\begin{aligned} q_i^{can} + [f, S_{i+1}] = &- \sum_{k \in \mathbb{Z}_+} \sum_{\substack{h_1 + \cdots + h_{k+1} = i \\ h_1, \ldots, h_{k+1} \geq \frac{1}{2}}} \frac{1}{(k+1)!} [S_{h_1}, [S_{h_2}, [\ldots, [S_{h_k}, S'_{h_{k+1}}]] \ldots]] \\ &+ \sum_{k \in \mathbb{Z}_+} \sum_{\substack{h_0 + \cdots + h_k = i \\ h_0 \geq -\frac{1}{2}, h_1, \ldots, h_k \geq \frac{1}{2}}} \frac{1}{k!} [S_{h_1}, [S_{h_2}, [\ldots, [S_{h_k}, q_{h_0}]] \ldots]] \\ &+ \sum_{k \in \mathbb{Z}_+} \sum_{\substack{h_1 + \cdots + h_{k+1} = i+1 \\ h_1, \ldots, h_{k+1} \geq \frac{1}{2}}} \frac{1}{(k+1)!} [S_{h_1}, [S_{h_2}, [\ldots, [S_{h_{k+1}}, f]] \ldots]]. \end{aligned} \tag{2.5}$$

Since $\operatorname{ad} f$ is injective, $q^{can}_{-\frac{1}{2}}$ and S_1 are uniquely determined. Then we can find uniquely q_i^{can} and S_{i+1} when the previous ones have already been determined. $\square$

Corollary 2.2. *The choice of the space V provides a differential basis for the differential algebra R. Thus, if $u_1, \ldots, u_r$ are the coordinates of q^{can} in V, $R = \mathbb{C}[u_i^{(m)} \mid i \in \{1, \ldots, r\}, m \in \mathbb{Z}_+]$.*

Proof. It suffices to show that if $\mathscr{L} \overset{S}{\sim} \mathscr{L}_1$ and $\mathscr{L}_1 \overset{T}{\sim} \mathscr{L}_2$, then $\mathscr{L} \overset{U}{\sim} \mathscr{L}_2$. This implies that if $\mathscr{L} \overset{S}{\sim} \widetilde{\mathscr{L}}$, then $\mathscr{L}^{can} = \widetilde{\mathscr{L}}^{can}$. Thus, for a gauge invariant polynomial p, $p(q) = p(\widetilde{q}) = p(q^{can})$. We have $\mathscr{L}_2 = e^{\operatorname{ad} T}(\mathscr{L}_1) = e^{\operatorname{ad} T} e^{\operatorname{ad} S}(\mathscr{L})$. By Campbell-Haussdorff formula [23] there exists $U \in C^\infty(S^1, \mathfrak{n}_+)$ such that $e^{\operatorname{ad} T} e^{\operatorname{ad} S} = e^{\operatorname{ad} U}$ proving the assertion. $\square$

For any $u, v \in C^\infty(S^1, \mathfrak{g})$ we define

$$(u, v) = \int_{S^1} (u \mid v). \tag{2.6}$$

This bilinear form is still invariant and integration by parts gives

$$(u, v') = -(u', v). \tag{2.7}$$

We introduce the Hamiltonian structure on the set of equivalence classes of operators $\mathscr{L}$, which we denote $\mathcal{M}(\mathfrak{g})$. Namely,

$$\mathcal{M}(\mathfrak{g}) = \left\{ \mathscr{L} = \partial + q + \Lambda(z) \mid q \in C^\infty(S^1, \mathfrak{B}_+) \right\} / \overset{S}{\sim} .$$

The set of functionals on $\mathcal{M}(\mathfrak{g})$ is

$$\mathcal{F}(\mathfrak{g}) = \left\{ l(q) = \textstyle\int_{S^1} p(q) \mid p \in R \right\}.$$

If $l \in \mathcal{F}(\mathfrak{g})$ and $q \in C^\infty(S^1, \mathfrak{B}_+)$, then by $\operatorname{grad}_q l$ we denote any element of $C^\infty(S^1, \mathfrak{g})$ satisying the relation

$$\left. \frac{d}{d\varepsilon} l(q + \varepsilon h) \right|_{\varepsilon = 0} = (\operatorname{grad}_q l, h) \tag{2.8}$$

for any $h \in C^\infty(S^1, \mathfrak{B}_+)$. Since the orthonormal complement to $\mathfrak{B}_+$ is $\mathfrak{m}_+$, then $\operatorname{grad}_q l$ is defined up to the addition of elements of $C^\infty(S^1, \mathfrak{m}_+)$.

We define the following Poisson bracket on $\mathcal{F}(\mathfrak{g})$: if φ, $\psi \in \mathcal{F}(\mathfrak{g})$ and $q \in C^\infty(S^1, \mathfrak{B}_+)$, then

$$\{\varphi, \psi\}_z^{DS}(q) = (\operatorname{grad}_q \varphi, [\operatorname{grad}_q \psi, \mathscr{L}]). \tag{2.9}$$

We can also write $\{\cdot, \cdot\}_z^{DS} = \{\cdot, \cdot\}_0^{DS} - z\{\cdot, \cdot\}_\infty^{DS}$, where

$$\begin{aligned} \{\varphi, \psi\}_0^{DS}(q) &= (\operatorname{grad}_q \varphi, [\operatorname{grad}_q \psi, \partial + q + f]), \\ \{\varphi, \psi\}_\infty^{DS}(q) &= -(\operatorname{grad}_q \varphi, [\operatorname{grad}_q \psi, s]). \end{aligned}$$

It can be verified that:

a) definition is well posed, that is, it does not depend on the choice of the gradient;
b) gauge invariance of φ and ψ implies gauge invariance of $\{\varphi,\psi\}_z^{DS}$;
c) the brackets $\{\cdot,\cdot\}_0^{DS}$ and $\{\cdot,\cdot\}_\infty^{DS}$ are skewsymmetric, coordinated and verify the Jacoby identity.

Verification of Jacobi identity in c) involves a long computation that can be avoid using Poisson vertex algebras theory, then we will prove it later. Now we prove a) and b).

For a) we have to verify for any $\theta\in C^\infty(S^1,\mathfrak{m}_+)$ the following equalities

$$(\operatorname{grad}_q\varphi,[\theta,\mathscr{L}])=0; \tag{2.10}$$

$$(\theta,[\operatorname{grad}_q\varphi,\mathscr{L}])=0. \tag{2.11}$$

By invariance of the scalar product and (2.7) it follows that (2.10) implies (2.11). Since $[\mathfrak{m}_+,s]=0$, we are left to show that

$$(\operatorname{grad}_q\varphi,[\theta,\partial+q+f])=0.$$

This equality follows from the gauge invariance of φ. Indeed, take $S(\varepsilon)=\varepsilon\theta$ and define

$$\mathscr{L}(\varepsilon)=e^{\operatorname{ad}S(\varepsilon)}(\mathscr{L})=\partial+q(\varepsilon)+\Lambda(z)=\mathscr{L}+\varepsilon[\theta,\mathscr{L}]+o(\varepsilon^2).$$

Then $q(\varepsilon)=q+\varepsilon[\theta,\mathscr{L}]+o(\varepsilon^2)=q+\varepsilon[\theta,\partial+q+f]+o(\varepsilon^2)$. By gauge invariance it follows that

$$0=\left.\frac{d\varphi(q)}{d\varepsilon}\right|_{\varepsilon=0}=\left.\frac{d\varphi(q(\varepsilon))}{d\varepsilon}\right|_{\varepsilon=0}=(\operatorname{grad}_q\varphi,[\theta,\partial+q+f]).$$

To prove b) it suffices to show that if $\mathscr{L}\overset{S}{\sim}\widetilde{\mathscr{L}}$, then

$$e^{\operatorname{ad}S}(\operatorname{grad}_q\varphi)=\operatorname{grad}_{\widetilde{q}}\varphi, \tag{2.12}$$

where $\operatorname{grad}_{\widetilde{q}}\varphi=\operatorname{grad}_q\varphi\big|_{q=\widetilde{q}}$. In fact, it will follow that

$$\begin{aligned}\{\varphi,\psi\}_z^{DS}(\widetilde{q})&=\left(e^{\operatorname{ad}S}(\operatorname{grad}_q\varphi),\left[e^{\operatorname{ad}S}(\operatorname{grad}_q\psi),\widetilde{\mathscr{L}}\right]\right)=\left(e^{\operatorname{ad}S}(\operatorname{grad}_q\varphi),e^{\operatorname{ad}S}\left[\operatorname{grad}_q\psi,\mathscr{L}\right]\right)=\\&=(\operatorname{grad}_q\varphi,[\operatorname{grad}_q\psi,\mathscr{L}])=\{\varphi,\psi\}_z^{DS}(q).\end{aligned}$$

To prove (2.12) we note that, if $\widetilde{\mathscr{L}}=e^{\operatorname{ad}S}(\mathscr{L})$, then $\widetilde{q+\varepsilon h}=\widetilde{q}+\varepsilon\widetilde{h}=\widetilde{q}+\varepsilon e^{\operatorname{ad}S}(h)$, from which we derive, using gauge invariance, the equality

$$\begin{aligned}(\operatorname{grad}_q\varphi,h)&=\left.\frac{d\varphi(q+\varepsilon h)}{d\varepsilon}\right|_{\varepsilon=0}=\left.\frac{d\varphi(\widetilde{q+\varepsilon h})}{d\varepsilon}\right|_{\varepsilon=0}=\left.\frac{d\varphi(\widetilde{q}+\varepsilon e^{\operatorname{ad}S}(h))}{d\varepsilon}\right|_{\varepsilon=0}=\\&=\left(\operatorname{grad}_{\widetilde{q}}\varphi,e^{\operatorname{ad}S}(h)\right)=\left(e^{-\operatorname{ad}S}(\operatorname{grad}_{\widetilde{q}}\varphi),h\right).\end{aligned}$$

2.2. Classical $\mathcal{W}$-algebras in the Poisson vertex algebra theory

Let $\mathfrak{g}$, f and $(\cdot\mid\cdot)$ be as in Section 2.1. Let $\{e,h=2x,f\}\subset\mathfrak{g}$ be a $\mathfrak{sl}_2$-triple associated to f, then we have the $\operatorname{ad}x$-decompositon (2.1). We keep the notation as in (2.2).

Let $\mathcal{V}(\mathfrak{g})=S(\mathbb{C}[\partial]\otimes\mathfrak{g})$ be the symmetric algebra over $\mathbb{C}[\partial]\otimes\mathfrak{g}$. We use the notation $a^{(i)}=\partial^i\otimes a$, $a\in\mathfrak{g}$ to indicate monomials of $\mathbb{C}[\partial]\otimes\mathfrak{g}$.

Given $z\in\mathbb{C}$ and $s\in\mathfrak{g}$ we give $\mathcal{V}(\mathfrak{g})$ the structure of Poisson vertex algebra (cf. Example 1.6) defining for $a,b\in\mathfrak{g}$

$$\{a_\lambda b\}_z=[a,b]+(a\mid b)\lambda+z(s\mid[a,b]) \tag{2.13}$$

and extending the λ-bracket to $\mathcal{V}(\mathfrak{g})$ by Theorem 1.5. We denote this Poisson vertex algebra by $\mathcal{V}_z(\mathfrak{g},s)$ and note that for $z=0$ there is no dependence on s, hence we will denote it $\mathcal{V}_0(\mathfrak{g})$.

We let $\widetilde{\mathfrak{J}}(\mathfrak{g},f)\subset\mathcal{V}(\mathfrak{g})$ be the differential ideal generated by the elements of the form $m-(f\mid m)$ with $m\in\mathfrak{m}_+$, namely

$$\widetilde{\mathfrak{J}}(\mathfrak{g},f)=\langle m-(f\mid m)\mid m\in\mathfrak{m}_+\rangle_{\mathcal{V}(\mathfrak{g})}.$$

This is not a Poisson vertex algebra ideal since we can choose $a\in\mathfrak{g}$ such that $[a,m]\notin\mathfrak{m}_+$ for some $m\in\mathfrak{m}_+$. Then $\{a_\lambda m\}_z\notin\widetilde{\mathfrak{J}}(\mathfrak{g},f)[\lambda]$.

Let us define

$$\widetilde{\mathcal{W}}_z(\mathfrak{g},f,s)=\left\{p\in\mathcal{V}_z(\mathfrak{g},s)\,\middle|\,\{a_\lambda p\}_z\in\widetilde{\mathfrak{J}}(\mathfrak{g},f)[\lambda]\ \forall a\in\mathfrak{n}_+\right\}.$$

As before, for $z=0$ there is no dependence on s and we denote the corresponding space $\widetilde{\mathcal{W}}_0(\mathfrak{g},f)$.

Lemma 2.3. *The following statements hold:*

a) *if $p\in\widetilde{\mathcal{W}}_z(\mathfrak{g},f,s)$ and $q\in\widetilde{\mathfrak{J}}(\mathfrak{g},f)$, then $\{p_\lambda q\}_z$, $\{q_\lambda p\}_z\in\widetilde{\mathfrak{J}}(\mathfrak{g},f)[\lambda]$;*

b) $\widetilde{\mathcal{W}}_z(\mathfrak{g},f,s)\subset\mathcal{V}_z(\mathfrak{g},s)$ *is a Poisson vertex subalgebra;*
c) *if* $s\in\ker\operatorname{ad}\mathfrak{n}_+$, *then* $\widetilde{\mathfrak{J}}(\mathfrak{g},f)\subset\widetilde{\mathcal{W}}_z(\mathfrak{g},f,s)$.

Proof. To prove a) we note that any element of $\widetilde{\mathfrak{J}}(\mathfrak{g},f)$ is a finite sum of elements of the form $r\partial^i(m-(f\mid m))$, with $r\in\mathcal{V}(\mathfrak{g})$ and $i\in\mathbb{Z}_+$. Then we get

$$\begin{aligned}\{p_\lambda q\}_z&=\sum\{p_\lambda r\partial^i(m-(f\mid m))\}_z=\\&=\sum\{p_\lambda r\}_z\partial^i(m-(f\mid m))+\sum r(\lambda+\partial)^i\{p_\lambda m\}_z\in\widetilde{\mathfrak{J}}(\mathfrak{g},f)[\lambda],\end{aligned}$$

since $p\in\widetilde{\mathcal{W}}_z(\mathfrak{g},f,s)$ and $m\in\mathfrak{m}_+\subset\mathfrak{n}_+$. From the fact that $\widetilde{\mathfrak{J}}(\mathfrak{g},f)$ is a differential ideal, using skew-commutativity we get also $\{q_\lambda p\}_z\in\widetilde{\mathfrak{J}}(\mathfrak{g},f)[\lambda]$.

For b), first we prove that $\widetilde{\mathcal{W}}_z(\mathfrak{g},f,s)\subset\mathcal{V}(\mathfrak{g})$ is a differential subalgebra. Indeed, if $p,\,q\in\widetilde{\mathcal{W}}_z(\mathfrak{g},f,s)$ and $a\in\mathfrak{n}_+$, we have

$$\{a_\lambda pq\}_z=p\{a_\lambda q\}_z+q\{a_\lambda p\}_z\in\widetilde{\mathfrak{J}}(\mathfrak{g},f)[\lambda]$$

and

$$\{a_\lambda\partial p\}_z=(\lambda+\partial)\{a_\lambda p\}_z\in\widetilde{\mathfrak{J}}(\mathfrak{g},f)[\lambda],$$

so that pq and ∂p lie in $\widetilde{\mathcal{W}}_z(\mathfrak{g},f,s)$.

Next, we show that $\widetilde{\mathcal{W}}_z(\mathfrak{g},f,s)$ is closed for the λ-bracket. Let $a\in\mathfrak{n}_+$ and $p,\,q\in\widetilde{\mathcal{W}}_z(\mathfrak{g},f,s)$. By the Jacobi identity

$$\{a_\lambda\{p_\mu q\}_z\}_z=\{p_\mu\{a_\lambda q\}_z\}_z+\{\{a_\lambda p\}_{z\,\lambda+\mu}q\}_z.$$

By a) both terms in the right hand side lie in $\widetilde{\mathfrak{J}}(\mathfrak{g},f)[\lambda,\mu]$. Hence $\{p_\lambda q\}_z\in\widetilde{\mathcal{W}}_z(\mathfrak{g},f,s)[\lambda]$.

Finally, let us prove c). By repeated use of left and right Leibniz rule we can reduce to show that elements of the form $m-(f\mid m)$, with $m\in\mathfrak{m}_+$, lie in $\widetilde{\mathcal{W}}_z(\mathfrak{g},f,s)$. Take $a\in\mathfrak{n}_+$, then

$$\{a_\lambda m-(f\mid m)\}_z=\{a_\lambda m\}_z=[a,m]\in\widetilde{\mathfrak{J}}(\mathfrak{g},f)[\lambda].$$

Indeed, $[a,m]\in\bigoplus_{j>1}\mathfrak{g}_j$, then $(f\mid[a,m])=0$. Hence, $m-(f\mid m)\in\widetilde{\mathcal{W}}_z(\mathfrak{g},f,s)$. $\square$

If $s\in\ker\operatorname{ad}\mathfrak{n}_+$, by the above lemma, $\widetilde{\mathfrak{J}}(\mathfrak{g},f)\subset\widetilde{\mathcal{W}}_z(\mathfrak{g},f,s)$ is a Poisson vertex algebra ideal and, by Theorem 1.9, the quotient has an iduced Poisson vertex algebra structure. Thus, we give the following:

Definition 2.4. The *classical* $\mathcal{W}$*-algebra* associated to the triple $(\mathfrak{g},f,s)$ is

$$\mathcal{W}_z(\mathfrak{g},f,s)=\widetilde{\mathcal{W}}_z(\mathfrak{g},f,s)\big/\widetilde{\mathfrak{J}}(\mathfrak{g},f).$$

Remark 2.5. In [17] is proved that $\mathcal{W}_z(\mathfrak{g},f,s)$ depends only on the nilpotent orbit of f and not on the $\mathfrak{sl}_2$-triple we choose.

2.2.1. Equivalence with the classical Drinfeld-Sokolov Hamiltonian reduction. We want to show that the Lie algebra structure on the quotient space $\mathcal{W}_z(\mathfrak{g},f,s)\big/\partial\mathcal{W}_z(\mathfrak{g},f,s)$ with Lie bracket defined by (1.18) is exactly the same structure obtained by the classical Drinfeld-Sokolov Hamiltonian reduction performed in Section 2.1. For our purpose we substitute the loop algebra $C^\infty(S^1,\mathfrak{g})$ with the Lie algebra $\mathfrak{g}\otimes\mathcal{V}(\mathfrak{g})$. The Lie algebra structure on $\mathfrak{g}\otimes\mathcal{V}(\mathfrak{g})$ is defined by $[a\otimes g,b\otimes h]=[a,b]\otimes gh\in\mathfrak{g}\otimes\mathcal{V}(\mathfrak{g})$, for any monomials $a\otimes g,b\otimes h\in\mathfrak{g}\otimes\mathcal{V}(\mathfrak{g})$ and extended by linearity to the whole $\mathfrak{g}\otimes\mathcal{V}(\mathfrak{g})$. As for $C^\infty(S^1,\mathfrak{g})$, we can extend the bilinear form of $\mathfrak{g}$ to a bilinear form on $\mathfrak{g}\otimes\mathcal{V}(\mathfrak{g})$, which we still denote $(\cdot\mid\cdot)$ by $(a\otimes g\mid b\otimes h)=(a\mid b)gh\in\mathcal{V}(\mathfrak{g})$, for any monomials $a\otimes g,b\otimes h\in\mathfrak{g}\otimes\mathcal{V}(\mathfrak{g})$. This bilinear form is still nondegenerate invariant and symmetric. Moreover, let ∂ be the derivation $\mathcal{V}(\mathfrak{g})$, we define an action of the abelian Lie algebra $\mathbb{C}\partial$ on $\mathfrak{g}\otimes\mathcal{V}(\mathfrak{g})$ by

$$\partial.(a\otimes g)=a\otimes\partial g,\tag{2.14}$$

for any monomial $a\otimes g\in\mathfrak{g}\otimes\mathcal{V}(\mathfrak{g})$. Clearly, ∂ acts as a derivation of $\mathfrak{g}\otimes\mathcal{V}(\mathfrak{g})$. Indeed, since it is a derivation of $\mathcal{V}(\mathfrak{g})$, we have

$$\begin{aligned}\partial.[a\otimes g,b\otimes h]&=\partial.([a,b]\otimes gh)=[a,b]\otimes\partial(gh)=[a,b]\otimes(\partial g)h+[a,b]\otimes g(\partial h)=\\&=[a\otimes\partial g,b\otimes h]+[a\otimes g,b\otimes\partial h]=[\partial.(a\otimes g),b\otimes h]+[a\otimes g,\partial.(b\otimes h)],\end{aligned}$$

for any $a\otimes g,b\otimes h\in\mathfrak{g}\otimes\mathcal{V}(\mathfrak{g})$. Thus we can define the semidirect product Lie algebra $\mathbb{C}\partial\ltimes\mathfrak{g}\otimes\mathcal{V}(\mathfrak{g})$, where the commutator of ∂ against elements of $\mathfrak{g}\otimes\mathcal{V}(\mathfrak{g})$ is given by (2.14), namely $[\partial,a\otimes g]=\partial.(a\otimes g)=a\otimes\partial g$, for any monomial $a\otimes g\in\mathfrak{g}\otimes\mathcal{V}(\mathfrak{g})$.

We set $I = \{1, 2, \ldots, h + 2m = \dim \mathfrak{g}\}$ and we fix a basis $\{Q_i \mid i \in I, \}$ of $\mathfrak{g}$ and its dual basis $\{Q^i \mid i \in I\}$ with respect to the form $(\cdot \mid \cdot)$, in the following way

$$\begin{aligned}
&Q_i = Q^i,\ i = 1, \ldots, r \qquad \text{basis of } \mathfrak{g}_0 = \mathfrak{h};\\
&Q_i = Q^{m+i},\ i = r+1, \ldots, r+d \qquad \text{basis of } \mathfrak{g}_{-\frac{1}{2}};\\
&Q_i = Q^{m+i},\ i = r+d+1, \ldots, r+m \qquad \text{basis of } \mathfrak{g}_{\leq -1};\\
&Q_i = Q^{i-m},\ i = r+m+1, \ldots, r+m+d \qquad \text{basis of } \mathfrak{g}_{\frac{1}{2}};\\
&Q_i = Q^{i-m},\ i = r+m+d+1, \ldots, r+2m \qquad \text{basis of } \mathfrak{g}_{\geq 1}.
\end{aligned}$$

Then $\mathcal{V}(\mathfrak{g}) \cong \mathbb{C}[Q_i^{(n)} \mid i \in I, n \in \mathbb{Z}_+]$. We also set $\bar{I} = \{1, \ldots, r+m+d\}$ and

$$\mathfrak{B}_- = \bigoplus_{i \leq \frac{1}{2}} \mathfrak{g}_i.$$

We want to understand how gauge transformations act on the differential algebra $\mathcal{V}(\mathfrak{g})$. We consider the operator

$$\mathcal{L} = \partial + Q + zs \otimes 1 \in \mathbb{C}\partial \ltimes \mathfrak{g} \otimes \mathcal{V}(\mathfrak{g}), \tag{2.15}$$

where $Q = \sum_{i \in I} Q^i \otimes Q_i \in \mathfrak{g} \otimes \mathcal{V}(\mathfrak{g})$, $z \in \mathbb{C}$ and $s \in \ker \operatorname{ad} \mathfrak{n}_+$.

Given $S \in \mathfrak{n}_+ \otimes \mathcal{V}(\mathfrak{g})$, a gauge transformation, as defined in Section 2.1, is a trasformation of the form

$$\widetilde{\mathcal{L}} = e^{\operatorname{ad} S}(\mathcal{L}).$$

Let us set $S(\varepsilon) = \varepsilon a \otimes p \in (\mathfrak{n}_+ \otimes \mathcal{V}(\mathfrak{g}))[[\varepsilon]]$, then gauge transformations read as

$$\mathcal{L}(\varepsilon) = e^{\operatorname{ad} S(\varepsilon)}(\mathcal{L}) = \mathcal{L} + \varepsilon[a \otimes p, \mathcal{L}] + o(\varepsilon^2) \in (\mathbb{C}\partial \ltimes \mathfrak{g} \otimes \mathcal{V}(\mathfrak{g}))[[\varepsilon]]. \tag{2.16}$$

We write $\mathcal{L}(\varepsilon) = \partial + Q(\varepsilon) + zs \otimes 1$, where $Q(\varepsilon) = Q + \varepsilon[a \otimes r, \mathcal{L}] + o(\varepsilon^2) \in (\mathfrak{g} \otimes \mathcal{V}(\mathfrak{g}))[[\varepsilon]]$. Then we state the following result.

Lemma 2.6. *If $p \in \mathcal{V}(\mathfrak{g})$, then*

$$\frac{d}{d\varepsilon} p(Q(\varepsilon))\Big|_{\varepsilon=0} = -\{a_\partial p\}_{z \to} r$$

for all $a \in \mathfrak{n}_+$ and $r \in \mathcal{V}(\mathfrak{g})$.

Proof. By Taylor formula we get

$$p(Q(\varepsilon)) = p(Q) + \varepsilon \sum_{i \in I, n \in \mathbb{Z}_+} \frac{\partial p(Q)}{\partial Q_i^{(n)}} \partial^n([a \otimes r, \mathcal{L}] \mid Q_i \otimes 1) + o(\varepsilon^2).$$

Hence,

$$\frac{d}{d\varepsilon} p(Q(\varepsilon))\Big|_{\varepsilon=0} = \sum_{i \in I, n \in \mathbb{Z}_+} \frac{\partial p(Q)}{\partial Q_i^{(n)}} \partial^n([a \otimes r, \mathcal{L}] \mid Q_i \otimes 1) + o(\varepsilon^2).$$

On the other hand, using (1.11), we have

$$\{a_\partial p\}_{\to} r = \sum_{i \in I, n \in \mathbb{Z}_+} \frac{\partial p(Q)}{\partial Q_i^{(n)}} \partial^n \{a_\partial Q_i\}_{\to} r = \sum_{i \in I, n \in \mathbb{Z}_+} \frac{\partial p(Q)}{\partial Q_i^{(n)}} \partial^n \left(([a, Q_i] + (a \mid Q_i)\partial) r\right).$$

The proof is finished if we prove that

$$-([a, Q_i] + (a \mid Q_i)\partial) r = ([a \otimes r, \mathcal{L}] \mid Q_i \otimes 1),$$

which follows easily from the computation of the right hand side. Indeed, we have

$$\begin{aligned}
([a \otimes r, \mathcal{L}] \mid Q_i \otimes 1) &= (-a \otimes \partial r + \sum_{i \in I} [a, Q_i] Q_i r \mid Q_i \otimes 1) = -(a \mid Q_i)\partial r + \sum_{i \in I} ([a, Q^i] \mid Q_i) Q_i r = \\
&= -(a \mid Q_i)\partial r - \sum_{i \in I} ([a, Q_i] \mid Q^i) Q_i r = -(a \mid Q_i)\partial r - [a, Q_i] r.
\end{aligned}$$

$\square$

Let $\pi_- : \mathfrak{g} \twoheadrightarrow \mathfrak{B}_-$ be the projection map, then we define a differential algebra homomorphism $\pi : \mathcal{V}(\mathfrak{g}) \twoheadrightarrow \mathcal{V}(\mathfrak{B}_-)$, defining $\pi(a^{(n)}) = \partial^n(\pi_-(a)) + \delta_{n0}(a \mid f)$, for any $a \in \mathfrak{g}$ and $n \in \mathbb{Z}_+$ and then we extend to the whole differential algebra $\mathcal{V}(\mathfrak{g})$. It is easy to check that

$$\ker \pi = \langle m - (m \mid f) \mid m \in \mathfrak{m}_+ \rangle_{\mathcal{V}(\mathfrak{g})}.$$

In the notation of Section 2.2, $\ker \pi = \widetilde{\mathcal{J}}(\mathfrak{g}, f)$. If $i \in \bar{I}$, let us denote $\pi(Q_i) = q_i \in \mathfrak{B}_-$, then we identify $\mathcal{V}(\mathfrak{B}_-) = \mathbb{C}[q_i^{(n)} \mid i \in \bar{I}, n \in \mathbb{Z}_+]$ and rewrite $\pi : \mathbb{C}[Q_i^{(n)} \mid i \in I, n \in \mathbb{Z}_+] \twoheadrightarrow \mathbb{C}[q_i^{(n)} \mid i \in \bar{I}, n \in \mathbb{Z}_+]$, where $\pi(Q_i^{(n)}) = \partial^n(\pi_-(Q_i)) + \delta_{n0}(Q_i \mid f)$ and

$$\pi_-(Q_i) = \begin{cases} q_i & \text{if } i \in \bar{I}, \\ 0 & \text{if } i \notin \bar{I}. \end{cases}$$

We define the map $\mathbb{1} \otimes \pi : \mathbb{C}\partial \ltimes \mathfrak{g} \otimes \mathcal{V}(\mathfrak{g}) \longrightarrow \mathbb{C}\partial \ltimes \mathfrak{g} \otimes \mathcal{V}(\mathfrak{g})$, setting $(\mathbb{1} \otimes \pi)(\partial) = \partial$ and $(\mathbb{1} \otimes \pi)(a \otimes g) = a \otimes \pi(g)$, for any $a \otimes g \in \mathfrak{g} \otimes \mathcal{V}(\mathfrak{g})$. We note that $\pi(Q) = q + f \otimes 1$, where $q = \sum_{i \in \bar{I}} Q^i \otimes q_i \in \mathfrak{B}_+ \otimes \mathcal{V}(\mathfrak{B}_-)$. Hence, if $\mathcal{L}$ is an operator of the form (2.15), then

$$\mathscr{L} = (\mathbb{1} \otimes \pi)(\mathcal{L}) = \partial + q + \Lambda(z) \otimes 1$$

is an operator of the type defined in (2.3). Moreover, applying $\mathbb{1} \otimes \pi$ to $\widetilde{\mathcal{L}}$, gauge transformations read

$$\widetilde{\mathscr{L}} = (\mathbb{1} \otimes \pi) \left(e^{\operatorname{ad} S}(\mathcal{L}) \right) = e^{\operatorname{ad}((\mathbb{1} \otimes \pi)(S))}((\mathbb{1} \otimes \pi)(\mathcal{L})) = e^{\operatorname{ad}((\mathbb{1} \otimes \pi)(S))}(\mathscr{L}),$$

where $(\mathbb{1} \otimes \pi)(S) \in \mathfrak{n}_+ \otimes \mathcal{V}(\mathcal{B}_-)$ as in Section 2.1. If we consider $S(\varepsilon) = \varepsilon a \otimes r \in \mathfrak{n}_+ \otimes \mathcal{V}(\mathfrak{g})$, applying $\mathbb{1} \otimes \pi$ to (2.16), we get $q(\varepsilon) = q + \varepsilon[a \otimes \pi(r)] + o(\varepsilon^2)$.

Recall that, by definition of gauge invariance, we can state that $p(q) \in R$ if and only if $\frac{d}{d\varepsilon} p(q(\varepsilon))\big|_{\varepsilon=0} = 0$. Let $p \in \mathcal{W}_z(\mathfrak{g}, f, s)$ and let $\widetilde{p}$ be a lift in $\widetilde{\mathcal{W}}_z(\mathfrak{g}, f, s)$, that is, $\{a_\lambda \widetilde{p}\}_z \in \widetilde{\mathcal{J}}(\mathfrak{g}, f)[\lambda]$, for any $a \in \mathfrak{n}_+$. In particular, $\{a_\partial \widetilde{p}\}_z \in \widetilde{\mathcal{J}}[\mathfrak{g}, f][\partial]$. Thus, $\{a_\partial \widetilde{p}\}_\rightarrow r \in \widetilde{\mathcal{J}}(\mathfrak{g}, f)$ for any $r \in \mathcal{V}(\mathfrak{g})$, since $\widetilde{\mathcal{J}}(\mathfrak{g}, f)$ is a differential ideal. By Lemma (2.6), it follows that

$$\pi \left(\left. \frac{d\widetilde{p}(Q(\varepsilon))}{d\varepsilon} \right|_{\varepsilon=0} \right) = 0.$$

Since π commutes with the derivation with respect to ε, we also have

$$\pi \left(\left. \frac{d\widetilde{p}(Q(\varepsilon))}{d\varepsilon} \right|_{\varepsilon=0} \right) = \left. \frac{d\pi\left(\widetilde{p}(Q(\varepsilon))\right)}{d\varepsilon} \right|_{\varepsilon=0} = \left. \frac{dp(q(\varepsilon))}{d\varepsilon} \right|_{\varepsilon=0}.$$

Hence, $p \in R$. By Lemma 2.6, it follows also that if $p \in R$, then $p \in \mathcal{W}_z(\mathfrak{g}, f, s)$. Thus, as differential algebras $R \cong \mathcal{W}_z(\mathfrak{g}, f, s)$.

By construction, for any $f, g \in \mathcal{W}_z(\mathfrak{g}, f, s)$, the induced λ-bracket, which we denote $\{\cdot_\lambda \cdot\}_{\Lambda(z)}$, is given by

$$\{f_\lambda g\}_{\Lambda(z)} = \pi(\{\widetilde{f}_\lambda \widetilde{g}\}_z),$$

for any $\widetilde{f}, \widetilde{g} \in \widetilde{\mathcal{W}}_z(\mathfrak{g}, f, s)$ liftings of f and g. Since $\pi|_{\mathcal{V}(\mathfrak{B}_-)} = \mathbb{1}$ and $\mathcal{W}_z(\mathfrak{g}, f, s) \subset \mathcal{V}(\mathfrak{B}_-)$, we can choose f and g as their liftings. Hence, by (1.11),

$$\pi(\{f_\lambda g\}_z) = \sum_{\substack{i,j \in \bar{I} \\ n,m \in \mathbb{Z}_+}} \frac{\partial g}{\partial q_j^{(n)}} (\lambda + \partial)^n \{q_{i\,\lambda+\partial} q_j\}_{\Lambda(z)\rightarrow} (-\lambda - \partial)^m \frac{\partial f}{\partial u_i^{(m)}}, \tag{2.17}$$

where, for any $a, b \in \mathfrak{B}_-$,

$$\{a_\lambda b\}_{\Lambda(z)} = \pi(\{a_\lambda b\}_z) = \pi([a, b]) + (a \mid b)\lambda + z(s \mid [a, b]) = \pi_-([a, b]) + (a \mid b)\lambda + (\Lambda(z) \mid [a, b]). \tag{2.18}$$

By Proposition 1.15 the quotient $\mathcal{W}_z(\mathfrak{g}, f, s) / \partial\mathcal{W}_z(\mathfrak{g}, f, s)$ is a Lie algebra with Lie bracket

$$\{{\textstyle\int} p, {\textstyle\int} r\}_{\Lambda(z)} = {\textstyle\int} \{p_\lambda r\}_{\Lambda(z)}\big|_{\lambda=0},$$

for any $\int p, \int r \in \mathcal{W}_z(\mathfrak{g}, f, s) / \partial\mathcal{W}_z(\mathfrak{g}, f, s)$. We want to show that this Lie bracket coincides with the Lie bracket $\{\cdot, \cdot\}_z^{DS}$ defined in Section 2.1 for $R / \partial R$.

We consider the following identifications

$$\begin{aligned} \mathcal{V}(\mathfrak{B}_-)^{\oplus \bar{I}} &\xrightarrow{\sim} \mathfrak{B}_- \otimes \mathcal{V}(\mathfrak{B}_-) \\ F = (F_i)_{i \in \bar{I}} &\longrightarrow \underline{F} = \sum_{i \in \bar{I}} Q_i \otimes F_i \end{aligned} \tag{2.19}$$

and

$$\begin{aligned}\mathcal{V}(\mathfrak{B}_-)^{\bar{I}} &\xrightarrow{\sim} \mathfrak{B}_+\otimes\mathcal{V}(\mathfrak{B}_-)\\ P=(P_i)_{i\in\bar{I}} &\longrightarrow \overline{P}=\sum_{i\in\bar{I}}Q^i\otimes P_i.\end{aligned}\tag{2.20}$$

As we did in (2.6), we define, using the bilinear form on $\mathfrak{g}\otimes\mathcal{V}(\mathfrak{g})$,

$$(a\otimes g, b\otimes h)=\int(a\otimes g\mid b\otimes h)=\int(a\mid b)gh,$$

for any $a\otimes g, b\otimes h\in\mathfrak{g}\otimes\mathcal{V}(\mathfrak{g})$. Using integration by parts and (5.2), we have $(\partial.(a\otimes g), b\otimes h) = -(a\otimes g, \partial.(b\otimes h))$. By identifications (2.19) and (2.20), we note that

$$(\underline{F},\overline{P})=\int(\underline{F}\mid\overline{P})=\int\sum_{i,j\in I}(Q_i\mid Q^j)F_iP_j=\int\sum_{i\in I}F_iP_i,$$

which coincides with (1.3), namely $\mathfrak{B}_-\otimes\mathcal{V}(\mathfrak{B}_-)$ and $\mathfrak{B}_+\otimes\mathcal{V}(\mathfrak{B}_)$ are duals with respect to $(\cdot,\cdot)$. Thus, if $p\in\mathcal{V}(\mathfrak{B}_-)$, then, using (5.4), its variational derivative is identified with

$$\underline{\frac{\delta p}{\delta q}}=\sum_{i\in\bar{I}}^{n}Q_i\otimes\frac{\delta p}{\delta q_i}\in\mathfrak{B}_-\otimes\mathcal{V}(\mathfrak{B}_-).$$

In the sequel, we denote it simply by $\frac{\delta p}{\delta q}$.

We note that the definition of the variational derivative is nothing other than the definition of grad_q in (2.8). Indeed, take $h\in\mathfrak{B}_+\otimes\mathcal{V}(\mathfrak{B}_-)$, say

$$h=\sum_{i\in\bar{I}}Q^i\otimes h_i.$$

For $l=\int p\in {\mathcal{W}_z(\mathfrak{g},f,s)}/{\partial\mathcal{W}_z(\mathfrak{g},f,s)}$, we have

$$\begin{aligned}\frac{d}{d\varepsilon}l(q+\varepsilon h)\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}l(\dots,q_i+\varepsilon h_i,\dots)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int\left(f(q)+\varepsilon\sum_{i\in\bar{I},n\in\mathbb{Z}_+}\frac{\partial f}{\partial q_i^{(n)}}\partial^n h_i+o(\varepsilon^2)\right)\Big|_{\varepsilon=0}=\\ &=\int\sum_{i\in\bar{I},n\in\mathbb{Z}_+}(-\partial)^n\frac{\partial f}{\partial q_i^{(n)}}h_i=\int\sum_{i\in\bar{I}}\frac{\delta f}{\delta q_i}h_i=\left(\frac{\delta f}{\delta q},h\right).\end{aligned}$$

We are now ready to prove that

$$\{\textstyle\int p,\int r\}_z^{DS}=\{\int p,\int r\}_{\Lambda(z)}\tag{2.21}$$

for any $\int p,\ \int r\in {\mathcal{W}_z(\mathfrak{g},f,s)}/{\partial\mathcal{W}_z(\mathfrak{g},f,s)}$. We recall that the left hand side of (2.21), given by (2.9), is

$$\{\textstyle\int p,\int r\}_z^{DS}=\left(\frac{\delta p}{\delta q},\left[\frac{\delta r}{\delta q},\mathscr{L}\right]\right)=\left(\frac{\delta p}{\delta q},\left[\frac{\delta r}{\delta q},\partial+q+\Lambda(z)\right]\right).$$

To compute the left hand side, we split it into three terms. We get

$$\begin{aligned}\left(\frac{\delta p}{\delta q},\left[\frac{\delta r}{\delta q},\partial\right]\right) &= \sum_{i,j\in\bar{I}}\left(Q_j\otimes\frac{\delta p}{\delta q_j},\left[Q_i\otimes\frac{\delta r}{\delta q_i},\partial\right]\right)=-\int\sum_{i,j\in\bar{I}}\frac{\delta p}{\delta q_j}(Q_j\mid Q_i)\partial\frac{\delta r}{\delta q_i}=\\ &=\int\sum_{i,j\in\bar{I}}\frac{\delta r}{\delta q_i}(Q_j\mid Q_i)\partial\frac{\delta p}{\delta q_j};\end{aligned}\tag{2.22}$$

$$\begin{aligned}\left(\frac{\delta p}{\delta q},\left[\frac{\delta r}{\delta q},q\right]\right) &= \sum_{i,j,k\in\bar{I}}\left(Q_j\otimes\frac{\delta p}{\delta q_j},\left[Q_i\otimes\frac{\delta r}{\delta q_i},Q^k\otimes q_k\right]\right)=\int\sum_{i,j,k\in\bar{I}}(Q_j\mid[Q_i,Q^k])q_k\frac{\delta p}{\delta q_j}\frac{\delta r}{\delta q_i}=\\ &=\int\sum_{i,j\in\bar{I}}\frac{\delta r}{\delta q_i}\pi_-([Q_j,Q_i])\frac{\delta p}{\delta q_j},\end{aligned}\tag{2.23}$$

where in the last equality, by invariance of the bilinear form, it follows that $\sum_{k\in\bar{I}}(Q_j \mid [Q_i,Q^k])q_k = \pi_-([Q_j,Q_i])$. Finally,

$$
\begin{aligned}
\left(\frac{\delta p}{\delta q},\left[\frac{\delta r}{\delta q},\Lambda(z)\otimes 1\right]\right) &= \sum_{i,j\in\bar{I}}\left(Q_j\otimes\frac{\delta p}{\delta q_j},\left[Q_i\otimes\frac{\delta r}{\delta q_i},\Lambda(z)\otimes 1\right]\right) = \\
&= \int\sum_{i,j\in\bar{I}}(Q_j \mid [Q_i,\Lambda(z)])\frac{\delta p}{\delta q_j}\frac{\delta r}{\delta q_i} = \int\sum_{i,j\in\bar{I}}\frac{\delta r}{\delta q_i}([Q_j,Q_i] \mid \Lambda(z))\frac{\delta p}{\delta q_j},
\end{aligned}
\tag{2.24}
$$

where again, in the last equality we used invariance of the bilinear form. Summing (2.22), (2.23) and (2.24), we get

$$
\begin{aligned}
\{{\textstyle\int} p,{\textstyle\int} r\}_z^{DS} &= \int\sum_{i,j\in\bar{I}}\frac{\delta r}{\delta q_i}\left(\pi_-([Q_j,Q_i])+(Q_j \mid Q_i)\partial+(\Lambda(z) \mid [Q_j,Q_i])\right)\frac{\delta p}{\delta q_j} = \\
&= \int\sum_{i,j\in\bar{I}}\frac{\delta r}{\delta q_i}\{q_j{}_\partial q_i\}_{\Lambda(z)\to}\frac{\delta p}{\delta q_j} = \int\{p_\lambda r\}_{\Lambda(z)}\big|_{\lambda=0},
\end{aligned}
$$

which follows from (1.11) and (2.18) computed on generators q_i of $\mathcal{V}(\mathfrak{B}_-)=\mathbb{C}[q_i^{(n)} \mid i\in\bar{I},n\in\mathbb{Z}_+]$. In particular, thanks to Proposition 1.15, we have proved that the Lie bracket $\{\cdot,\cdot\}_z^{DS}$ is skew-commutative and satisfies Jacobi identity, thus proving that $\{\cdot,\cdot\}_0^{DS}$ and $\{\cdot,\cdot\}_\infty^{DS}$ are coordinated.

2.3. Explicit construction of $\mathcal{W}_z(\mathfrak{g},f,s)$, where $\mathfrak{g}$ is of type B_n, C_n

Let $\sigma:\mathfrak{gl}_n\longrightarrow\mathfrak{gl}_n$ be a linear map such that

$$
\sigma(AB)=-\sigma(B)\sigma(A) \qquad\text{and}\qquad \sigma^2(A)=A \tag{2.25}
$$

for all A, $B\in\mathfrak{gl}_n$.

Consider the following subspaces of $\mathfrak{gl}_n$:

$$
\mathfrak{o}_\pm=\{A\in\mathfrak{gl}_n \mid \sigma(A)=\pm A\}\,.
$$

Proposition 2.7. *We have*

i) $[\mathfrak{o}_\varepsilon,\mathfrak{o}_{\varepsilon'}]\subset\mathfrak{o}_{\varepsilon\varepsilon'}$, *for all* ε, $\varepsilon'\in\{\pm\}$;
ii) *if* $A\in\mathfrak{o}_+$ *and* $B\in\mathfrak{o}_-$, *then* $\operatorname{tr}(AB)=0$.

Proof. Let A, $B\in\mathfrak{gl}_n$ such that $\sigma(A)=\varepsilon A$ and $\sigma(B)=\varepsilon' B$, with ε, $\varepsilon'\in\{\pm\}$. We have, by (2.25),

$$
\sigma[A,B]=-\sigma(B)\sigma(A)+\sigma(A)\sigma(B)=\varepsilon\varepsilon'[A,B].
$$

This proves i). For ii), note that $\operatorname{tr}(\sigma(AB))=-\operatorname{tr}(I(AB)^\alpha I)=-\operatorname{tr}(AB)$, by invariance of the trace by antitranspositions and cyclic permutations. On the other hand, if $A\in\mathfrak{o}_+$ and $B\in\mathfrak{o}_-$, then $\sigma(AB)=-\sigma(B)\sigma(A)=BA$ so that $\operatorname{tr}(\sigma(AB))=\operatorname{tr}(AB)$. It follows that $\operatorname{tr}(AB)=0$. ∎

Let $\mathcal{V}_z(\mathfrak{gl}_n,s)$ be as in Section 2.2 where in (2.13) as nondegenerate invariant symmetric bilinear form we take the trace of matrices. From the above proposition we get the following easy result.

Corollary 2.8. $\{\mathfrak{o}_\varepsilon{}_\lambda\mathfrak{o}_{\varepsilon'}\}_z\subset\mathfrak{o}_{\varepsilon\varepsilon'}+\delta_{\varepsilon\varepsilon'}\mathbb{C}\lambda+\mathbb{C}z$. *Moreover, if* $s\in\mathfrak{o}_+$, *then* $\{\mathfrak{o}_\varepsilon{}_\lambda\mathfrak{o}_{\varepsilon'}\}_z\subset\mathfrak{o}_{\varepsilon\varepsilon'}+\delta_{\varepsilon\varepsilon'}(\mathbb{C}\lambda+\mathbb{C}z)$.

Proof. It can be checked directly from the definition of the λ-bracket given in (2.13) using Proposition 2.7. ∎

We have a vector space decomposition

$$
\mathfrak{gl}_n=\mathfrak{o}_+\oplus\mathfrak{o}_- \tag{2.26}
$$

and we can consider the projection map

$$
\rho:\mathfrak{gl}_n\twoheadrightarrow\mathfrak{o}_+\cong\mathfrak{gl}_n/\mathfrak{o}_-\,.
$$

The decomposition (2.26) extends to the following decomposition of differential algebras

$$
\mathcal{V}(\mathfrak{gl}_n)=\mathcal{V}(\mathfrak{o}_+)\oplus\langle\mathfrak{o}_-\rangle, \tag{2.27}
$$

where $\langle\mathfrak{o}_-\rangle\subset\mathcal{V}(\mathfrak{gl}_n)$ is the differential ideal generated by $\mathfrak{o}_-\subset\mathfrak{gl}_n$. Hence, we get a homomorphism of differential algebras, which we also denote by ρ

$$
\rho:\mathcal{V}(\mathfrak{gl}_n)\twoheadrightarrow\mathcal{V}(\mathfrak{o}_+)\cong\mathcal{V}(\mathfrak{gl}_n)/\langle\mathfrak{o}_-\rangle.
$$

Remark 2.9. This map is not a Poisson vertex algebras homomorphism from $\mathcal{V}_z(\mathfrak{gl}_n, s)$ to $\mathcal{V}_{z'}(\mathfrak{o}_+, s')$. Indeed, we have

$$\rho(\{\mathbb{1}_{\mathfrak{gl}_n}{}_\lambda \mathbb{1}_{\mathfrak{gl}_n}\}_z) = \rho(n\lambda) = n\lambda,$$

while $\{\rho(\mathbb{1}_{\mathfrak{gl}_n})_\lambda \rho(\mathbb{1}_{\mathfrak{gl}_n})\}_z = 0$, since $\mathbb{1}_{\mathfrak{gl}_n} \in \mathfrak{o}_-$.

We also extend σ to a homomorphism of differential algebras, which again denote by $\sigma : \mathcal{V}(\mathfrak{gl}_n) \longrightarrow \mathcal{V}(\mathfrak{gl}_n)$ and we let

$$\mathcal{V}(\mathfrak{gl}_n)^\sigma = \{p \in \mathcal{V}(\mathfrak{gl}_n) \mid \sigma(p) = p\}\,.$$

Proposition 2.10. *If $s \in \mathfrak{o}_+$, then $\mathcal{V}(\mathfrak{gl}_n)^\sigma \subset \mathcal{V}_z(\mathfrak{gl}_n, s)$ is a Poisson vertex subalgebra which we will denote $\mathcal{V}_z(\mathfrak{gl}_n, s)^\sigma$. In particular $\mathcal{V}_0(\mathfrak{gl}_n)^\sigma \subset \mathcal{V}_0(\mathfrak{gl}_n)$ is a Poisson vertex subalgebra.*

Proof. Since, by definition, σ is a homomorphism of differential algebras, $\mathcal{V}(\mathfrak{gl}_n)^\sigma \subset \mathcal{V}(\mathfrak{gl}_n)$ is a differential subalgebra. Now, given an element $p \in \mathcal{V}(\mathfrak{gl}_n)$ we can decompose it according to (2.27) as $p = p_+ + p_-$, where $p_+ \in \mathcal{V}(\mathfrak{o}_+)$ and $p_- \in \langle \mathfrak{o}_- \rangle$. Clearly $\sigma(p_+) = p_+$ and p_- is a finite sum of elements of the type

$$r s_1^{(m_1)} \cdots s_h^{(m_h)},$$

with $h \geq 1$, $m_i \in \mathbb{Z}_+$, $r \in \mathcal{V}(\mathfrak{o}_+)$ and $s_i \in \mathfrak{o}_-$. If $s \in \mathfrak{o}_-$ and $m \in \mathbb{Z}_+$, then $\sigma(s^{(m)}) = -s^{(m)}$. Thus, if $p \in \mathcal{V}(\mathfrak{gl}_n)^\sigma$, we have $\sigma(p_-) = p_-$. Hence, each summand in p_- has an even number of s_i's.

We should prove that $\{p_\lambda q\}_z \in \mathcal{V}(\mathfrak{gl}_n)^\sigma[\lambda]$ for any p and $q \in \mathcal{V}(\mathfrak{gl}_n)^\sigma$. We have

$$\begin{aligned}\{p_\lambda q\}_z &= \{p_+ + p_-{}_\lambda q_+ + q_-\}_z = \\ &= \{p_+{}_\lambda q_+\}_z + \{p_-{}_\lambda q_+\}_z + \{p_+{}_\lambda q_-\}_z + \{p_-{}_\lambda q_-\}_z.\end{aligned}$$

Since $\mathcal{V}_z(\mathfrak{o}_+, s)$ is a Poisson vertex algebra, it follows that $\{p_+{}_\lambda q_+\}_z \in \mathcal{V}_z(\mathfrak{o}_+, s)[\lambda] \subset \mathcal{V}(\mathfrak{gl}_n)^\sigma[\lambda]$, hence it is σ-invariant. It remains to prove that the other summands are σ-invariant. By linearity of the λ-bracket we can reduce to consider

$$p_- = P s_1^{(m_1)} \cdots s_{2h}^{(m_{2h})} \qquad \text{and} \qquad q_- = Q \bar{s}_1^{(l_1)} \cdots \bar{s}_{2j}^{(l_{2j})}, \tag{2.28}$$

where $P, Q \in \mathcal{V}(\mathfrak{o}_+)$, $s_i, \bar{s}_i \in \mathfrak{s}$ and $m_i, l_i \in \mathbb{Z}_+$. Moreover, by (1.6), (1.8), (1.7) and the σ-invariance of elements in $\mathcal{V}(\mathfrak{o})$, we can reduce to assume $p_+ = r$ and $q_+ = t$, with r, $t \in \mathfrak{o}_+$. We have

$$\begin{aligned}\{p_+{}_\lambda q_-\}_z &= \{r_\lambda Q \bar{s}_1^{(l_1)} \cdots \bar{s}_{2j}^{(l_{2j})}\}_z = \\ &= \bar{s}_1^{(l_1)} \cdots \bar{s}_{2j}^{(l_{2j})} \{r_\lambda Q\}_z + \sum_{i=1}^{2j} Q \bar{s}_1^{(l_1)} \overset{i}{\check{\cdots}} \bar{s}_{2j}^{(l_{2j})} (\lambda + \partial)^{l_i} \{r_\lambda \bar{s}_i\}_z.\end{aligned} \tag{2.29}$$

Clearly $\{r_\lambda Q\}_z \in \mathcal{V}(\mathfrak{o}_+)[\lambda]$. By Corollary 2.8, since by assumption $s \in \mathfrak{o}$, we have $\{r_\lambda \bar{s}_i\}_z \in \mathfrak{o}_-$. Thus, in the second term of (2.29) there are an even number of factors from $\mathfrak{o}_-$. Hence, also $\{p_+{}_\lambda q_-\}_z$ is σ-invariant. Using skew-commutativity of the λ-bracket we get that also elements of the type $\{p_-{}_\lambda q_+\}_z$ are σ-invariant. Let us consider now the last term. Again, by the above observations, we let p_- and q_- as in (2.28). Moreover, by (1.6), (1.8) and (1.7), it is enough to consider the case P, $Q \in \mathfrak{o}_+$. We have

$$\begin{aligned}\{p_-{}_\lambda q_-\}_z &= \{P s_1^{(m_1)} \cdots s_{2h}^{(m_{2h})}{}_\lambda Q \bar{s}_1^{(l_1)} \cdots \bar{s}_{2j}^{(l_{2j})}\}_z = \\ &= \bar{s}_1^{(l_1)} \cdots \bar{s}_{2j}^{(l_{2j})} \{P_{\lambda+\partial} Q\}_z s_1^{(m_1)} \cdots s_{2h}^{(m_{2h})} + \\ &+ \sum_{\alpha=1}^{2h} \bar{s}_1^{(l_1)} \cdots \bar{s}_{2j}^{(l_{2j})} \{s_\alpha{}_{\lambda+\partial} Q\}_z (-\lambda-\partial)^{m_\alpha} P s_1^{(m_1)} \overset{\alpha}{\check{\cdots}} s_{2h}^{(m_{2h})} + \\ &+ \sum_{\beta=1}^{2j} Q \bar{s}_1^{(l_1)} \overset{\beta}{\check{\cdots}} \bar{s}_{2j}^{(l_{2j})} (\lambda+\partial) \{P_{\lambda+\partial} \bar{s}_\beta\}_{z\,\to} s_1^{(m_1)} \cdots s_{2h}^{(m_{2h})} + \\ &+ \sum_{\alpha=1}^{2h} \sum_{\beta=1}^{2j} Q \bar{s}_1^{(l_1)} \overset{\beta}{\check{\cdots}} \bar{s}_{2j}^{(l_{2j})} (\lambda+\partial)^{l_\beta} \{s_\alpha{}_{\lambda+\partial} \bar{s}_\beta\}_{z\,\to} (-\lambda-\partial)^{m_\alpha} P s_1^{(m_1)} \overset{\alpha}{\check{\cdots}} s_{2h}^{(m_{2h})}.\end{aligned}$$

By Corollary 2.8, $\{P_{\lambda+\partial} Q\}_z$ and $\{s_\alpha{}_{\lambda+\partial} \bar{s}_\beta\}_z$ are σ-invariant, while $\{s_\alpha{}_{\lambda+\partial} Q\}_z$ and $\{P_{\lambda+\partial} \bar{s}_\beta\}_z$ lie in $\mathfrak{o}_-$, since $s \in \mathfrak{o}_+$. Hence, the corresponding terms in the sum are σ-invariant because there are an even number of factors from $\mathfrak{o}_-$. ☐

Proposition 2.11. *If $s \in \mathfrak{o}_+$, then the homomorphism of differential algebras $\rho : \mathcal{V}(\mathfrak{gl}_n) \longrightarrow \mathcal{V}(\mathfrak{o}_+)$, restricted to $\mathcal{V}(\mathfrak{gl}_n)^\sigma$ is a homomorphism of Poisson vertex algebras $\rho : \mathcal{V}_z(\mathfrak{gl}_n, s)^\sigma \longrightarrow \mathcal{V}_z(\mathfrak{o}_+, s)$. In particular, $\rho : \mathcal{V}_0(\mathfrak{gl}_n)^\sigma \longrightarrow \mathcal{V}_0(\mathfrak{o}_+)$ is a homomorphism of Poisson vertex algebras.*

Proof. We note that the λ-bracket on $\mathcal{V}_z(\mathfrak{o}_+, s)$ is, by hyphotesis, the restriction of the λ-bracket on $\mathcal{V}_z(\mathfrak{gl}_n, s)$. Moreover, by Proposition 2.10, $\mathcal{V}_z(\mathfrak{gl}_n, s)^\sigma \subset \mathcal{V}_z(\mathfrak{gl}_n, s)$ is a Poisson vertex subalgebra. Take $p, q \in \mathcal{V}_z(\mathfrak{gl}_n, s)^\sigma$. According to decomposition (2.27), we can write $p = p_+ + p_-$ and $q = q_+ + q_-$ and $\rho(p) = p_+$, $\rho(q) = q_+$. We have

$$\begin{aligned}\rho(\{p_\lambda q\}_z) &= \rho(\{p_+ + p_-{}_\lambda q_+ + q_-\}_z) = \\ &= \rho(\{p_+{}_\lambda q_+\}_z) + \rho(\{p_+{}_\lambda q_-\}_z) + \rho(\{p_-{}_\lambda q_+\}_z) + \rho(\{p_-{}_\lambda q_-\}_z) = \\ &= \{p_+{}_\lambda q_+\}_z,\end{aligned}$$

since $\{p_+{}_\lambda q_+\}_z \in \mathcal{V}(\mathfrak{o}_+)[\lambda]$, while the other terms are in $\langle \mathfrak{o}_- \rangle[\lambda]$, because p_- and q_- are finite sums of terms eache of which has at least two factors in $\mathfrak{o}_-$. Hence, by (1.8) and (1.7), they all belong to $\langle \mathfrak{o}_- \rangle[\lambda]$. On the other hand we clearly have $\{\rho(p)_\lambda \rho(q)\}_z = \{p_+{}_\lambda q_+\}_z$. $\square$

In particular the map ρ is surjective and it follows that

$$\mathcal{V}_z(\mathfrak{o}_+, s) \cong \mathcal{V}_z(\mathfrak{gl}_n, s)^\sigma / (\langle \mathfrak{o}_- \rangle \cap \mathcal{V}_z(\mathfrak{gl}_n, s)^\sigma).$$

Let $f \in \mathfrak{gl}_n$ be a nilpotent element and $\{e, h = 2x, f\} \subset \mathfrak{gl}_n$ a $\mathfrak{sl}_2$-triple. We write

$$\mathfrak{gl}_n = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} (\mathfrak{gl}_n)_j$$

for its $\operatorname{ad} x$-decomposition. In the notation of Section 2.2 we have

$$\mathfrak{m}_+ = \bigoplus_{j \geq 1} (\mathfrak{gl}_n)_j, \qquad \mathfrak{n}_+ = \bigoplus_{j \geq \frac{1}{2}} (\mathfrak{gl}_n)_j, \qquad \mathfrak{b}_+ = \bigoplus_{j \in \mathbb{Z}_+} (\mathfrak{gl}_n)_j,$$

with inclusions $\mathfrak{m}_+ \subset \mathfrak{n}_+ \subset \mathfrak{b}_+$.

Suppose that e, h and f are fixed by σ. Hence, f, considered as an element of $\mathfrak{o}_+$, can be embedded in the same $\mathfrak{sl}_2$-triple. This means that we have the following decomposition for $\mathfrak{o}_+$:

$$\mathfrak{o}_+ = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} (\mathfrak{o}_+)_j,$$

where $(\mathfrak{o}_+)_j = (\mathfrak{gl}_n)_j \cap \mathfrak{o}_+$. We set $\mathfrak{m}_+(\mathfrak{o}_\pm) = \mathfrak{m}_+ \cap \mathfrak{o}_\pm$, $\mathfrak{n}_+(\mathfrak{o}_\pm) = \mathfrak{n}_+ \cap \mathfrak{o}_\pm$ and $\mathfrak{b}_+(\mathfrak{o}_\pm) = \mathfrak{b}_+ \cap \mathfrak{o}_\pm$. Then we have $\mathfrak{m}_+ = \mathfrak{m}_+(\mathfrak{o}_+) \oplus \mathfrak{m}_+(\mathfrak{o}_-)$, $\mathfrak{n}_+ = \mathfrak{n}_+(\mathfrak{o}_+) \oplus \mathfrak{n}_+(\mathfrak{o}_-)$ and $\mathfrak{b}_+ = \mathfrak{b}_+(\mathfrak{o}_+) \oplus \mathfrak{b}_+(\mathfrak{o}_-)$.

Lemma 2.12. *We have $\widetilde{\mathfrak{J}}(\mathfrak{o}_+, f) = \widetilde{\mathfrak{J}}(\mathfrak{gl}_n, f) \cap \mathcal{V}(\mathfrak{o}_+)$. In particular, $\rho(\widetilde{\mathfrak{J}}(\mathfrak{gl}_n, f)) = \widetilde{\mathfrak{J}}(\mathfrak{o}_+, f)$.*

Proof. The inclusion $\widetilde{\mathfrak{J}}(\mathfrak{o}_+, f) \subset \widetilde{\mathfrak{J}}(\mathfrak{gl}_n, f) \cap \mathcal{V}(\mathfrak{o}_+)$ is obvious. Let us prove the other inclusion. Take an element

$$\sum p_i \partial^i (m - (f \mid m)) \in \widetilde{\mathfrak{J}}(\mathfrak{gl}_n, f),$$

where $p_i \in \mathcal{V}(\mathfrak{gl}_n)$ and $m \in \mathfrak{m}_+$. By hyphotesis, this element also belongs to $\mathcal{V}(\mathfrak{o}_+)$, then

$$\begin{aligned}\sum p_i \partial^i (m - (f \mid m)) &= \rho\left(\sum p_i \partial^i (m - (f \mid m))\right) = \\ &= \sum \rho(p_i) \partial^i (\rho(m) - (f \mid m)).\end{aligned} \tag{2.30}$$

We can write $m = m_+ + m_-$, where $\mathfrak{m}_+ \in \mathfrak{o}_+$ and $\mathfrak{m}_- \in \mathfrak{o}_-$. By Proposition 2.7, it follows that $(f \mid m) = (f \mid n_+)$. Then (2.30) becomes

$$\sum p_i \partial^i (m - (f \mid m)) = \sum \rho(p_i) \partial^i (m_+ - (f \mid m_+)).$$

This means that $m \in \mathfrak{m}_+(\mathfrak{o}_+)$ and $p_i \in \mathcal{V}(\mathfrak{o}_+)$. Hence $\sum p_i \partial^i (m - (f \mid m)) \in \widetilde{\mathfrak{J}}(\mathfrak{o}_+, f)$. $\square$

Then we have the following result:

Corollary 2.13. *We get the decomposition*

$$\widetilde{\mathfrak{J}}(\mathfrak{gl}_n, f) = \widetilde{\mathfrak{J}}(\mathfrak{o}_+, f) \oplus \left(\widetilde{\mathfrak{J}}(\mathfrak{gl}_n, f) \cap \langle \mathfrak{o}_- \rangle\right).$$

Proof. Take an element $\sum p_i \partial^i (n - (f \mid n)) \in \widetilde{\mathfrak{J}}(\mathfrak{gl}_n, f)$. We can write $p_i = p_i^+ + p_i^-$, where $p_i^+ \in \mathcal{V}(\mathfrak{o}_+)$, $p_i^- \in \langle \mathfrak{o}_- \rangle$, and $n = n_+ + n_-$, where $n_+ \in \mathfrak{o}_+$, $n_- \in \mathfrak{o}_-$. Then

$$\sum p_i \partial^i (n - (f \mid n)) = \sum p_i^+ \partial^i (n_+ + (f \mid n_+)) + \sum p_i^- \partial^i (n_+ - (f \mid n_+)) + \sum p_i \partial^i n_-.$$

The first summand belongs to $\widetilde{\mathfrak{J}}(\mathfrak{o}_+)$, while the others belong to $\langle \mathfrak{o}_- \rangle$. $\square$

Lemma 2.14. *If $s \in \mathfrak{o}_+$, then $\rho(\widetilde{\mathcal{W}}_z(\mathfrak{gl}_n, f, s)) \subset \widetilde{\mathcal{W}}_z(\mathfrak{o}_+, f, s)$. In particular, $\rho(\widetilde{\mathcal{W}}_0(\mathfrak{gl}_n, f)) \subset \widetilde{\mathcal{W}}_0(\mathfrak{o}_+, f)$.*

PROOF. Take $p \in \widetilde{\mathcal{W}}_z(\mathfrak{gl}_n, f, s)$, then, as an element of $\mathcal{V}(\mathfrak{gl}_n) = \mathcal{V}(\mathfrak{o}_+) \oplus \langle \mathfrak{o}_- \rangle$, it has a decomposition $p = p_+ + p_-$. By definition of $\widetilde{\mathcal{W}}_z(\mathfrak{gl}_n, f, s)$, for $a \in \mathfrak{n}_+$, we have

$$\{a_\lambda p\}_z \in \widetilde{\mathfrak{J}}(\mathfrak{gl}_n, f)[\lambda] = \left(\widetilde{\mathfrak{J}}(\mathfrak{o}_+, f) \oplus \left(\widetilde{\mathfrak{J}}(\mathfrak{gl}_n, f) \cap \langle \mathfrak{o}_- \rangle\right)\right)[\lambda].$$

Suppose now $a \in \mathfrak{n}_+(\mathfrak{o}_+)$, then

$$\{a_\lambda p\}_z = \{a_\lambda p_+\}_z + \{a_\lambda p_-\}_z.$$

Clearly $\{a_\lambda p_+\}_z \in \mathcal{V}_z(\mathfrak{o}_+, s)[\lambda]$. Furthermore, we have $\{a_\lambda p_-\}_z \in \langle \mathfrak{o}_- \rangle[\lambda]$. Indeed, p_- is a finite sum of elements of the form $ps_1^{(m_1)} \cdots s_k^{(m_k)}$ and by Corollary 2.8, $\{\mathfrak{o}_{+\lambda}\mathfrak{o}_-\}_z \subset \mathfrak{o}_-$. Hence, we have $\{a_\lambda p_+\}_z \in \widetilde{\mathfrak{J}}(\mathfrak{o}_+, f)[\lambda]$ and $\{a_\lambda p_-\}_z \in \left(\widetilde{\mathfrak{J}}(\mathfrak{gl}_n, f) \cap \langle \mathfrak{o}_- \rangle\right)[\lambda]$. Then $\{a_\lambda \rho(p)\}_z = \{a_\lambda p_+\}_z \in \widetilde{\mathfrak{J}}(\mathfrak{o}_+, f)$, proving that $\rho(p) \in \widetilde{\mathcal{W}}_z(\mathfrak{o}_+, f, s)$. □

Unless otherwise stated we assume $s \in \mathfrak{o}_+$. Clearly, all the following results hold when we consider the λ-bracket obatained setting $z = 0$.

Lemmas 2.12 and 2.14 show that ρ induces a differential map, that we again denote by ρ,

$$\rho : \mathcal{W}_z(\mathfrak{gl}_n, f, s) \longrightarrow \mathcal{W}_z(\mathfrak{o}_+, f, s).$$

This is not a Poisson vertex algebra homomorphism for the same reason explained in Remark 2.9.

Consider $\widetilde{\mathcal{W}}_z(\mathfrak{gl}_n, f, s)^\sigma = \widetilde{\mathcal{W}}_z(\mathfrak{gl}_n, f, s) \cap \mathcal{V}(\mathfrak{gl}_n)^\sigma$ and $\widetilde{\mathfrak{J}}(\mathfrak{gl}_n, f)^\sigma = \widetilde{\mathfrak{J}}(\mathfrak{gl}_n, f) \cap \mathcal{V}(\mathfrak{gl}_n)^\sigma$. Then ρ induces a homomorphism of differential algebras

$$\rho : \mathcal{W}_z(\mathfrak{gl}_n, f, s)^\sigma \longrightarrow \mathcal{W}_z(\mathfrak{o}_+, f, s),$$

where

$$\mathcal{W}_z(\mathfrak{gl}_n, f, s)^\sigma = \widetilde{\mathcal{W}}_z(\mathfrak{gl}_n, f, s)^\sigma / \widetilde{\mathfrak{J}}(\mathfrak{gl}_n, f)^\sigma.$$

Proposition 2.15. *The map $\rho : \mathcal{W}_z(\mathfrak{gl}_n, f, s)^\sigma \longrightarrow \mathcal{W}_z(\mathfrak{o}_+, f, s)$ is surjective.*

PROOF. Let us set for brevity $\mathfrak{g} = \mathfrak{gl}_n$. We fix a basis of $\mathfrak{g}$ and its dual with respect to the trace form in the following way:

- $Q^i = Q_i$, $i = 1, \dots, h$, basis of $\mathfrak{g}_0$;
- $Q^{\frac{n^2-h}{2}+i} = Q_i$, $i = h+1, \dots, h+d$, basis of $\mathfrak{g}_{-\frac{1}{2}}$;
- $Q^{\frac{n^2-h}{2}+i} = Q_i$, $i = h+d+1, \dots, \frac{n^2+h}{2}$, basis of $\mathfrak{m}_-$;
- $Q^{i-\frac{n^2-h}{2}} = Q_i$, $i = \frac{n^2+h}{2}+1, \dots, \frac{n^2+h}{2}+d$, basis of $\mathfrak{g}_{\frac{1}{2}}$;
- $Q^{i-\frac{n^2-h}{2}} = Q_i$, $i = \frac{n^2+h}{2}+d+1, \dots, n^2$, basis of $\mathfrak{m}_+$,

where lower indeces stand for elements of the basis and upper indeces for elements of the dual basis. Since $\mathfrak{g} = \mathfrak{o}_+ \oplus \mathfrak{o}_-$, we can write $\{1, 2, \dots, n^2\} = I \cup J$, where $\#I = \dim \mathfrak{o}_+$ and $\#J = \dim \mathfrak{o}_-$ and choose the basis in such a way that the set $\{Q_i \mid i \in I\}$ is a basis of $\mathfrak{o}_+$ and $\{Q_i \mid i \in J\}$ is a basis of $\mathfrak{o}_-$. We set

$$\mathfrak{B}_+ = \bigoplus_{j \geq -\frac{1}{2}} \mathfrak{g}_j \qquad \text{and} \qquad \mathfrak{B}_- = \bigoplus_{j \leq \frac{1}{2}} \mathfrak{g}_j.$$

Then $\mathfrak{B}_\pm = \mathfrak{B}_\pm(\mathfrak{o}_+) \oplus \mathfrak{B}_\pm(\mathfrak{o}_-)$, where $\mathfrak{B}_\pm(\mathfrak{o}_\pm) = \mathfrak{B}_\pm \cap \mathfrak{o}_\pm$. We also set $I_- = \{1, 2, \dots, \frac{n^2+h}{2}+d\} \cap I$ and $J_- = \{1, 2, \dots, \frac{n^2+h}{2}+d\} \cap J$.

With this choice of the basis we identify $\mathcal{V}(\mathfrak{B}_-)$ with the differential algebra $\mathbb{C}[q_i^{(m)} \mid i = 1, \dots, \frac{n^2+h}{2}+d,\ m \in \mathbb{Z}_+]$ and $\mathcal{V}(\mathfrak{B}_-(\mathfrak{o}_+)) \subset \mathcal{V}(\mathfrak{B}_-)$ with the differential subalgebra $\mathbb{C}[q_i^{(m)} \mid i \in I_+,\ m \in \mathbb{Z}_+]$ via the identification $Q_i \to q_i$.

We want to consider the following elements

$$q = \sum_{i=1}^{\frac{n^2+h}{2}+d} Q^i \otimes q_i \in \mathfrak{B}_+ \otimes \mathfrak{B}_- \subset \mathfrak{B}_+ \otimes \mathcal{V}(\mathfrak{B}_-),$$

$$q_+ = \sum_{i \in I_+} Q^i \otimes q_i \in \mathfrak{B}_+(\mathfrak{o}_+) \otimes \mathfrak{B}_-(\mathfrak{o}_+) \subset \mathfrak{B}_+(\mathfrak{o}_+) \otimes \mathcal{V}(\mathfrak{B}_-(\mathfrak{o}_+)).$$

By an abuse of notation we let $\rho = \rho \otimes \rho : \mathfrak{B}_+ \otimes \mathcal{V}(\mathfrak{B}_-) \longrightarrow \mathfrak{B}_+(\mathfrak{o}_+) \otimes \mathcal{V}(\mathfrak{B}_-(\mathfrak{o}_+))$. It is clear that $\rho(q) = q_+$. Moreover we can write

$$q = q_+ + q_-, \tag{2.31}$$

where $q_- \in \mathfrak{B}_+(\mathfrak{o}_-) \otimes \mathfrak{B}_-(\mathfrak{o}_-) \subset \mathfrak{B}_+(\mathfrak{o}_-) \otimes \langle \mathfrak{B}_-(\mathfrak{o}_-) \rangle_{\mathcal{V}(\mathfrak{B}_-)}$.

As already said, by properties of good gradings [13], $\operatorname{ad} f : \mathfrak{g}_i \longrightarrow \mathfrak{g}_{i-1}$ is injective for $i \geq \frac{1}{2}$. Moreover, $\operatorname{ad} f : (\mathfrak{o}_\pm)_i \longrightarrow (\mathfrak{o}_\pm)_{i-1}$ is injective for $i \geq \frac{1}{2}$. Then, for each $i \geq -\frac{1}{2}$ we can find subspaces $V_i^\pm \subset (\mathfrak{o}_\pm)_i$ such that

$$(\mathfrak{o}_\pm)_i = V_i^\pm \oplus [f, (\mathfrak{o}_\pm)_{i+1}]. \tag{2.32}$$

If we set $V = V^+ \oplus V^-$, where $V^\pm = \oplus_{i \geq -\frac{1}{2}} V_i^\pm$, then $\mathfrak{B}_+ = V \oplus [f, \mathfrak{n}_+]$.

According to Proposition 2.1, we can find $S \in \mathfrak{n}_+ \otimes \mathcal{V}(\mathfrak{B}_-)$ such that the operator $\mathscr{L} = \partial + q + f \otimes 1 \overset{S}{\sim} \mathscr{L}^{can} = \partial + q^{can} + f \otimes 1$, where $q^{can} \in V \otimes \mathcal{V}(\mathfrak{B}_-)$, and $T \in \mathfrak{n}_+(\mathfrak{o}_+) \otimes \mathcal{V}(\mathfrak{B}_-(\mathfrak{o}_+))$ such that the operator $\mathscr{L}_+ = \partial + q_+ + f \otimes 1 \overset{T}{\sim} \mathscr{L}_+^{can} = \partial + q_+^{can} + f \otimes 1$, where $q_+^{can} \in V^+ \otimes \mathcal{V}(\mathfrak{B}_-(\mathfrak{o}_+))$.

We set $\gamma = \dim V$ and $\gamma_+ = \dim V^+$. If $e_1, \dots, e_{\gamma_+}$ is a basis of V^+, then we can complete it to a basis $e_1, \dots, e_\gamma$ of V. Letting

$$q^{can} = \sum_{i=0}^{\gamma} e_i \otimes v_i, \quad v_i \in \mathcal{V}(\mathfrak{B}_-),$$

$$q_+^{can} = \sum_{i=0}^{\gamma_+} e_i \otimes u_i, \quad u_i \in \mathcal{V}(\mathfrak{B}_-(\mathfrak{o}_+)),$$

by Corollary 2.2, we have the following differential algebras isomorphisms

$$\mathcal{W}_z(\mathfrak{g}, f, s) \cong \mathbb{C}[v_i^{(m)} \mid i = 1, \dots, \gamma,\ m \in \mathbb{Z}_+],$$

$$\mathcal{W}_z(\mathfrak{o}_+, f, s) \cong \mathbb{C}[u_i^{(m)} \mid i = 1, \dots, \gamma_+,\ m \in \mathbb{Z}_+].$$

Hence, to prove surjectiveness of $\rho : \mathcal{W}_z(\mathfrak{g}, f, s)^\sigma \longrightarrow \mathcal{W}_z(\mathfrak{o}_+, f, s)$ it suffices to show that $\sigma(v_i) = v_i$ and $\rho(v_i) = u_i$ for $i = 1, \dots, \gamma_+$, or, equivalently

$$\sigma(v_i) = v_i,\ i = 1, \dots, \gamma_+ \text{ and } \rho(q^{can}) = q_+^{can}. \tag{2.33}$$

By abuse of notation, let us set $\rho = \mathbb{1} \otimes \rho$. To prove the second identity in (2.33) it suffices to show that

$$\rho(\mathscr{L}^{can}) = e^{\operatorname{ad} \rho(S)}(\mathscr{L}_+). \tag{2.34}$$

Indeed, since $\rho(q^{can}) \in V^+ \otimes \mathcal{V}(\mathfrak{B}_-(\mathfrak{o}_+))$ and $\rho(S) \in \mathfrak{n}_+(\mathfrak{o}_+) \otimes \mathcal{V}(\mathfrak{B}_-(\mathfrak{o}_+))$ and since by Proposition 2.1, T and q_+^{can} are uniquely determined, it follows by (2.34) that $T = \rho(S)$ and $\rho(q^{can}) = q_+^{can}$.

We set $\mathfrak{s} = \{a \in \langle \mathfrak{B}_-(\mathfrak{o}_-) \rangle_{\mathcal{V}(\mathfrak{B}_-)} \mid \sigma(a) = -a\}$. First, let us show that $S = S_+ + S_-$, with $S_+ \in \mathfrak{n}_+(\mathfrak{o}_+) \otimes \mathcal{V}(\mathfrak{B}_-)^\sigma$ and $S_- \in \mathfrak{n}_+(\mathfrak{o}_-) \otimes \mathfrak{s}$ and u_i are σ-invariant polynomials for $i = 1, \dots, \gamma_+$. Let us consider the recursion given by (2.5). At the first step we have to solve

$$q_{-\frac{1}{2}}^{can} + [f, S_{\frac{1}{2}}] = q_{-\frac{1}{2}}.$$

By (2.31), we can write $q_{-\frac{1}{2}} = q_{-\frac{1}{2}}^+ + q_{-\frac{1}{2}}^-$, where $q_{-\frac{1}{2}}^+ \in (\mathfrak{o}_+)_{-\frac{1}{2}} \otimes \mathfrak{o}_+$ and $q_{-\frac{1}{2}}^- \in (\mathfrak{o}_-)_{-\frac{1}{2}} \otimes \mathfrak{o}_-$. We also write $S_{\frac{1}{2}} = S_{\frac{1}{2}}^+ + S_{\frac{1}{2}}^-$, where $S_{\frac{1}{2}}^+ \in (\mathfrak{o}_+)_{\frac{1}{2}} \otimes \mathcal{V}(\mathfrak{B}_-)$ and $S_{\frac{1}{2}}^- \in (\mathfrak{o}_-)_{\frac{1}{2}} \otimes \mathcal{V}(\mathfrak{B}_-)$. Similarly, we write $q_{-\frac{1}{2}}^{can} = q_{-\frac{1}{2}}^{can,+} + q_{-\frac{1}{2}}^{can,-}$, where $q_{-\frac{1}{2}}^{can,+} \in V_{-\frac{1}{2}}^+ \otimes \mathcal{V}(\mathfrak{B}_-)$ and $q_{\frac{1}{2}}^{can,-} \in V_{-\frac{1}{2}}^- \otimes \mathcal{V}(\mathfrak{B}_-)$. Then we have to solve the system

$$\begin{cases} q_{-\frac{1}{2}}^{can,+} + [f, S_{\frac{1}{2}}^+] = q_{-\frac{1}{2}}^+ \in (\mathfrak{o}_+)_{-\frac{1}{2}} \otimes \mathfrak{o}_+ \\ q_{-\frac{1}{2}}^{can,-} + [f, S_{\frac{1}{2}}^-] = q_{-\frac{1}{2}}^- \in (\mathfrak{o}_-)_{-\frac{1}{2}} \otimes \mathfrak{o}_-. \end{cases}$$

Using decomposition (2.32) we find that $q_{-\frac{1}{2}}^{can,+} \in V_{-\frac{1}{2}}^+ \otimes \mathfrak{o}_+$ and $S_{\frac{1}{2}}^\pm \in (\mathfrak{o}_\pm)_{\frac{1}{2}} \otimes \mathfrak{o}_\pm$.

Assume by induction that we found $S_1, \dots, S_i$ such that each $S_k = S_k^+ + S_k^-$, where $S_k^+ \in (\mathfrak{o}_+)_k \otimes \mathcal{V}(\mathfrak{B}_-)^\sigma$ and $S_k^- \in (\mathfrak{o}_-)_k \otimes \mathfrak{s}$. Let A be the right hand side of (2.5) and write it has $A = A_+ + A_-$, where $A_\pm \in (\mathfrak{o}_\pm)_i \otimes \mathcal{V}(\mathfrak{B}_-)$. Looking at (2.5), we see that A_+ is obtained when in the commutators there are an even number of elements in $\mathfrak{n}_+(\mathfrak{o}_-)$ or all of them are in $\mathfrak{n}_+(\mathfrak{o}_+)$, then $A_+ \in (\mathfrak{o}_+)_i \otimes \mathcal{V}(\mathfrak{B}_-)^\sigma$. On the other hand A_- is obtained when in the commutators there are an odd number of elements in $\mathfrak{n}_-(\mathfrak{o}_-)$. Thus $A_- \in (\mathfrak{o}_-)_i \otimes \mathfrak{s}$. Reasoning as in the first step of the induction we can find $S_{i+1} = S_{i+1}^+ + S_{i+1}^-$, where $S_{i+1}^+ \in (\mathfrak{o}_+)_{i+1} \otimes \mathcal{V}(\mathfrak{B}_-)^\sigma$ and $S_{i+1}^- \in (\mathfrak{o}_-)_{i+1} \otimes \mathfrak{s}$ and $q_i^{can,+} \in (\mathfrak{o}_+)_i \otimes \mathcal{V}(\mathfrak{B}_-)^\sigma$. It follows that S has the desired decomposition and that v_i is σ-invariant for $i = 1, \dots, \gamma_+$.

Finally, we prove (2.34). We have

$$\rho(\mathscr{L}^{can}) = (\rho \otimes \rho)(e^{\operatorname{ad} S}(\mathscr{L})) = (\rho \otimes 1)(e^{\operatorname{ad} S_+}(\mathcal{L}_+)) = e^{\operatorname{ad} \rho(S)}(\mathcal{L}_+).$$

$\square$

Moreover, $\widetilde{\mathcal{W}}_z(\mathfrak{gl}_n,f,s)^\sigma \subset \widetilde{\mathcal{W}}_z(\mathfrak{gl}_n,f,s)$ is a Poisson vertex subalgebra and $\widetilde{\mathcal{J}}(\mathfrak{gl}_n,f)^\sigma \subset \widetilde{\mathcal{W}}_z(\mathfrak{gl}_n,f)^\sigma$ is a Poisson vertex algebra ideal. Hence, $\mathcal{W}_z(\mathfrak{gl}_n,f,s)^\sigma$ is a Poisson vertex algebra.

Proposition 2.16. *The map $\rho : \mathcal{W}_z(\mathfrak{gl}_n,f,s)^\sigma \twoheadrightarrow \mathcal{W}_z(\mathfrak{o}_+,f,s)$ is a Poisson vertex algebra homomorphism.*

Proof. First, by Proposition 2.11, $\rho : \mathcal{V}_z(\mathfrak{gl}_n,s)^\sigma \longrightarrow \mathcal{V}_z(\mathfrak{o}_+,s)$ is a homomorphism of Poisson vertex algebras. Moreover, by Lemma 2.3 $\widetilde{\mathcal{W}}_z(\mathfrak{o}_+,f,s) \subset \mathcal{V}_z(\mathfrak{o}_+,s)$ is a Poisson vertex subalgebra. By the same lemma and the fact that $\widetilde{\mathcal{W}}_z(\mathfrak{gl}_n,f,s)^\sigma$ is intersection of Poisson vertex algebras, also $\widetilde{\mathcal{W}}_z(\mathfrak{gl}_n,f,s)^\sigma \subset \mathcal{V}_z(\mathfrak{gl}_n,s)^\sigma$ is a Poisson vertex algebra. Then $\rho : \widetilde{\mathcal{W}}_z(\mathfrak{gl}_n,f,s)^\sigma \longrightarrow \widetilde{\mathcal{W}}_z(\mathfrak{o}_+,f)$ is a Poisson vertex algebra homomorphism. Furthermore, $\widetilde{\mathcal{J}}(\mathfrak{gl}_n,f) \subset \widetilde{\mathcal{W}}_z(\mathfrak{gl}_n,f,z)^\sigma$ is a Poisson vertex algebra ideal, since it is intersection of a Poisson vertex algebra ideal and a Poisson vertex subalgebra. By Lemma 2.12 $\rho(\widetilde{\mathcal{J}}(\mathfrak{gl}_n,f)^\sigma) \subset \widetilde{\mathcal{J}}(\mathfrak{o}_+,f)$, then we have an induced map of Poisson vertex algebras $\rho : \mathcal{W}_z(\mathfrak{gl}_n,f,s)^\sigma \to \mathcal{W}_z(\mathfrak{o}_+,f,s)$. $\square$

Corollary 2.17. *We have the following isomorphism of Poisson vertex algebras*

$$\mathcal{W}_z(\mathfrak{o}_+,f,s) \cong \mathcal{W}_z(\mathfrak{gl}_n,f,s)^\sigma/\ker\rho.$$

In the next two paragraphs we assume $\sigma : \mathfrak{gl}_n \longrightarrow \mathfrak{gl}_n$ to be the linear map defined by

$$\sigma(A) = -IA^\alpha I, \tag{2.35}$$

where

$$I = \sum_{k=1}^{n}(-1)^{k+1}E_{kk} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & (-1)^{n+1} \end{pmatrix}$$

and A^α is the transposition with respect to the antidiagonal, namely if $A = (a_{ij})_{i,j=1}^n$, then $A^\alpha = (a_{n+1-j,n+1-i})_{i,j=1}^n$.

One easily checks that

$$(AB)^\alpha = B^\alpha A^\alpha \tag{2.36}$$

for all A, $B \in \mathfrak{gl}_n$, and

$$I^\alpha = I. \tag{2.37}$$

Moreover, it immediately follows from (2.36) and (2.37) that the conditions defined in (2.25) are satisfied by this map.

2.3.1. B_n. In $\mathfrak{gl}_{2n+1}$ we consider the principal nilpotent element

$$f = \sum_{k=1}^{2n} E_{k+1,k} = \begin{pmatrix} 0 & 0 & & & & \\ 1 & 0 & 0 & & & \\ & 1 & 0 & 0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \end{pmatrix}.$$

We can embed it in the following $\mathfrak{sl}_2$-triple:

$$h = 2x = \sum_{k=1}^{2n+1} 2(n+1-k)E_{kk} = \begin{pmatrix} 2n & 0 & & & & \\ 0 & 2n-2 & 0 & & & \\ & 0 & 2n-4 & 0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & 2-2n & 0 \\ & & & & 0 & -2n \end{pmatrix},$$

$$e = \sum_{k=1}^{2n} k(2n+1-k)E_{k,k+1} = \begin{pmatrix} 0 & 2n & & & & \\ 0 & 0 & 2(2n-1) & & & \\ & 0 & 0 & 3(2n-2) & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & 0 & 2n \\ & & & & 0 & 0 \end{pmatrix}.$$

With respect to $\operatorname{ad} x$ we have the following decomposition for $\mathfrak{gl}_{2n+1}$:

$$\mathfrak{gl}_{2n+1} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

where

$$\mathfrak{n}_- = \bigoplus_{1\leq j<i\leq 2n+1} \mathbb{C}E_{ij}, \quad \mathfrak{h} = \bigoplus_{k=1}^{2n+1} \mathbb{C}E_{kk} \quad \text{and} \quad \mathfrak{n}_+ = \bigoplus_{1\leq i<j\leq 2n+1} \mathbb{C}E_{ij}.$$

In particular we have no eigenspaces relative to half integers eigenvalues, then $\mathfrak{m}_+ = \mathfrak{n}_+$ (in the notation of Section 2.2). It is easily checked that

$$\ker \operatorname{ad} \mathfrak{n}_+ = \mathbb{C}E_{1,2n+1}.$$

Since $\ker \operatorname{ad} \mathfrak{n}_+$ is a 1-dimensional space, we can fix $s = E_{1,2n+1}$ and let vary $z \in \mathbb{C}$. Then we denote $\mathcal{V}_z(\mathfrak{gl}_{2n+1}) = \mathcal{V}_z(\mathfrak{gl}_{2n+1}, s)$.

The Lie algebra of type B_n is

$$\mathfrak{o}_{2n+1} = \{A \in \mathfrak{gl}_{2n+1} \mid \sigma(A) = A\}.$$

We note that e, h and f are fixed by σ. Hence, f, considered as an element of $\mathfrak{o}_{2n+1}$, can be embedded in the same $\mathfrak{sl}_2$-triple. This means that we have the following decomposition for $\mathfrak{o}_{2n+1}$:

$$\mathfrak{o}_{2n+1} = \mathfrak{n}_-(\mathfrak{o}_{2n+1}) \oplus \mathfrak{h}(\mathfrak{o}_{2n+1}) \oplus \mathfrak{n}_+(\mathfrak{o}_{2n+1}),$$

where

$$\mathfrak{n}_-(\mathfrak{o}_{2n+1}) = \mathfrak{n}_- \cap \mathfrak{o}_{2n+1}, \quad \mathfrak{h}(\mathfrak{o}_{2n+1}) = \mathfrak{h} \cap \mathfrak{o}_{2n+1} \quad \text{and} \quad \mathfrak{n}_+(\mathfrak{o}_{2n+1}) = \mathfrak{n}_+ \cap \mathfrak{o}_{2n+1}.$$

It can be easily checked that

$$\ker \operatorname{ad} \mathfrak{n}_+(\mathfrak{o}_{2n+1}) = \mathbb{C}(E_{1,2n} + E_{2,2n+1})$$

and, as in the previous case, we can fix $s' = E_{1,2n} + E_{2,2n+1}$ in the definition of the λ-bracket for $\mathcal{V}(\mathfrak{o}_{2n+1})$ and let vary $z' \in \mathbb{C}$. We denote $\mathcal{V}_{z'}(\mathfrak{o}_{2n+1}) = \mathcal{V}_{z'}(\mathfrak{o}_{2n+1}, s')$.

By Proposition 2.11, we get a Poisson vertex algebra homomorphism

$$\rho : \mathcal{V}_0(\mathfrak{gl}_{2n+1})^\sigma \twoheadrightarrow \mathcal{V}_0(\mathfrak{o}_{2n+1}).$$

Moreover, by Propositions 2.16 we have a Poisson vertex algebra homomorphism

$$\rho : \mathcal{W}_0(\mathfrak{gl}_{2n+1}, f)^\sigma \longrightarrow \mathcal{W}_0(\mathfrak{o}_{2n+1}, f)$$

and by Corollary 2.17

$$\mathcal{W}_0(\mathfrak{o}_{2n+1}, f) \cong \mathcal{W}_0(\mathfrak{gl}_{2n+1}, f)^\sigma / \ker \rho.$$

Remark 2.18. We have $\mathcal{W}_0(\mathfrak{o}_{2n+1}, f) \not\subset \mathcal{W}_0(\mathfrak{gl}_{2n+1}, f)$. For example, for $n = 1$, $p = E_{2,1} + E_{3,2} + \left(\frac{E_{1,1}-E_{3,3}}{2}\right)^2 + (E_{1,1} - E_{3,3})' \in \mathcal{W}_0(\mathfrak{o}_3, f)$, but $p \notin \mathcal{W}(\mathfrak{gl}_3, f)$. However, as expected by the above proposition, we can see that $r = E_{2,1} + E_{3,2} - E_{2,2}(E_{1,1} + E_{3,3}) - E_{1,1}E_{3,3} + (E_{1,1} - E_{3,3})' \in \mathcal{W}(\mathfrak{gl}_3, f)$ is such that $\rho(r) = p$.

2.3.2. C_n. This is similar to the previous case. The Lie algebra of type C_n is defined as

$$\mathfrak{sp}_{2n} = \{A \in \mathfrak{gl}_{2n} \mid \sigma(A) = A\}.$$

Let us consider the following principal nilpotent element of $\mathfrak{gl}_{2n}$:

$$f = \sum_{k=1}^{2n-1} E_{k+1,k} = \begin{pmatrix} 0 & 0 & & & & \\ 1 & 0 & 0 & & & \\ & 1 & 0 & 0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \end{pmatrix}.$$

We can embed f in the following $\mathfrak{sl}_2$ triple:

$$h = 2x = \sum_{k=1}^{2n}(2n+1-2k)E_{kk} = \begin{pmatrix} 2n-1 & 0 & & & & & \\ 0 & 2n-3 & 0 & & & & \\ & 0 & 2n-5 & 0 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 0 & 3-2n & 0 \\ & & & & 0 & 1-2n \end{pmatrix},$$

$$e = \sum_{k=1}^{2n-1} k(2n-k)E_{k,k+1} = \begin{pmatrix} 0 & 2n-1 & & & & \\ 0 & 0 & 2(2n-2) & & & \\ & 0 & 0 & 3(2n-3) & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & 0 & 2n-1 \\ & & & & 0 & 0 \end{pmatrix}.$$

With respect to $\operatorname{ad} x$ we have the following decomposition for $\mathfrak{gl}_{2n}$:

$$\mathfrak{gl}_{2n} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

where

$$\mathfrak{n}_- = \bigoplus_{1\le j<i\le 2n} \mathbb{C}E_{ij}, \quad \mathfrak{h} = \bigoplus_{k=1}^{2n} \mathbb{C}E_{kk} \quad \text{and} \quad \mathfrak{n}_+ = \bigoplus_{1\le i<j\le 2n} \mathbb{C}E_{ij}.$$

Also in this case there are no eigenspaces relative to half integers eigenvalues, then $\mathfrak{m}_+ = \mathfrak{n}_+$, and $\ker \operatorname{ad} \mathfrak{n}_+$ is the 1-dimensional space generated by the matrix $E_{1,2n}$. As in the case of B_n, we can fix $s = E_{1,2n}$ and let vary $z \in \mathbb{C}$. We denote $\mathcal{V}_z(\mathfrak{gl}_{2n}) = \mathcal{V}_z(\mathfrak{gl}_{2n}, s)$. Moreover, we set $\widetilde{\mathcal{W}}_z(\mathfrak{gl}_{2n}, f) = \widetilde{\mathcal{W}}_z(\mathfrak{gl}_{2n}, f, s)$ and $\mathcal{W}_z(\mathfrak{gl}_{2n}, f) = \mathcal{W}_z(\mathfrak{gl}_{2n}, f, s)$.

Again, e, h and f are σ-invariant. Then the $\mathfrak{sl}_2$-triple for $f \in \mathfrak{gl}_{2n}$ is also an $\mathfrak{sl}_2$-triple if we consider f as an element of $\mathfrak{sp}_{2n}$. Thus we have the following decomposition for $\mathfrak{sp}_{2n}$:

$$\mathfrak{sp}_{2n} = \mathfrak{n}_-(\mathfrak{sp}_{2n}) \oplus \mathfrak{h}(\mathfrak{sp}_{2n}) \oplus \mathfrak{n}_+(\mathfrak{sp}_{2n}),$$

where

$$\mathfrak{n}_-(\mathfrak{sp}_{2n}) = \mathfrak{n}_+ \cap \mathfrak{sp}_{2n}, \quad \mathfrak{h}(\mathfrak{sp}_{2n}) = \mathfrak{h} \cap \mathfrak{sp}_{2n} \quad \text{and} \quad \mathfrak{n}_+(\mathfrak{sp}_{2n}) = \mathfrak{n}_+ \cap \mathfrak{sp}_{2n}.$$

In this case we have $\ker \operatorname{ad} \mathfrak{n}_+(\mathfrak{sp}_{2n}) = \mathbb{C}E_{1,2n} = \ker \operatorname{ad} \mathfrak{n}_+$. We can choose the same s as for $\mathfrak{gl}_{2n}$ and let vary $z' \in \mathbb{C}$. We denote $\mathcal{V}_{z'}(\mathfrak{sp}_{2n}) = \mathcal{V}_{z'}(\mathfrak{sp}_{2n}, s)$. Moreover, we set $\widetilde{\mathcal{W}}_{z'}(\mathfrak{sp}_{2n}, f) = \widetilde{\mathcal{W}}_{z'}(\mathfrak{sp}_{2n}, f, s)$ and $\mathcal{W}_{z'}(\mathfrak{sp}_{2n}, f) = \mathcal{W}_{z'}(\mathfrak{sp}_{2n}, f, s)$.

By Proposition 2.10, we have that $\mathcal{V}_z(\mathfrak{gl}_{2n})^\sigma \subset \mathcal{V}_z(\mathfrak{gl}_{2n})$ is a Poisson vertex subalgebra and the map

$$\rho : \mathcal{V}_z(\mathfrak{gl}_{2n})^\sigma \twoheadrightarrow \mathcal{V}_z(\mathfrak{sp}_{2n})$$

is a Poisson vertex algebra homomorphism.

Moreover, by Proposition 2.16, we have a Poisson vertex algebra homomorphism

$$\rho : \mathcal{W}_z(\mathfrak{gl}_{2n}, f)^\sigma \longrightarrow \mathcal{W}_z(\mathfrak{sp}_{2n}, f)$$

and by Corollary 2.17 we have

$$\mathcal{W}_z(\mathfrak{sp}_{2n}, f, s) \cong \mathcal{W}_z(\mathfrak{gl}_{2n}, f, s)^\sigma / \ker \rho.$$

2.3.3. D_n**.** Let $\sigma : \mathfrak{gl}_{2n} \longrightarrow \mathfrak{gl}_{2n}$ be the linear map defined by

$$\sigma(A) = -IA^\alpha I, \tag{2.38}$$

where

$$I = \sum_{k=1}^{n}(-1)^{k+1}(E_{k\ k} + E_{2n+1-k,2n+1-k})$$

and A^α is the transposition with respect to the antidiagonal. Since $I^\alpha = I$ and σ satisfies (2.36), then σ verifies the conditions defined in (2.25). As in the previous cases it can be verified that $\sigma(AB) = -\sigma(B)\sigma(A)$ for $A, B \in \mathfrak{gl}_{2n}$ and $\sigma^2 = \mathbb{1}_{\mathfrak{gl}_{2n}}$, since $I = I^\alpha$ and $I^2 = \mathbb{1}_{\mathfrak{gl}_{2n}}$. The Lie algebra of type D_n is defined as

$$\mathfrak{o}_{2n} = \{A \in \mathfrak{gl}_{2n} \mid \sigma(A) = A\}.$$

Let us consider the following nilpotent element of $\mathfrak{gl}_{2n}$:

$$
f = \frac{1}{2}(E_{n+1,n-1} + E_{n+1,n-1}) + \sum_{k=1}^{n-1}(E_{kk} + E_{2n+1-k,2n+1-k}) =
$$

$$
= \begin{pmatrix}
0 & 0 & & & & & & & \\
1 & 0 & 0 & & & & & & \\
 & \ddots & \ddots & \ddots & & & & & \\
 & & 1 & 0 & 0 & & & & \\
 & & \frac{1}{2} & 0 & 0 & 0 & & & \\
 & & & \frac{1}{2} & 1 & 0 & 0 & & \\
 & & & & & \ddots & \ddots & \ddots & \\
 & & & & & & 1 & 0 & 0 \\
 & & & & & & & 1 & 0
\end{pmatrix}.
$$

We can embed f in the following $\mathfrak{sl}_2$-triple:

$$
h = 2x = \sum_{k=1}^{n-1} 2(n-k)(E_{kk} - E_{2n+1-k,2n+1-k}) =
$$

$$
= \begin{pmatrix}
2n-2 & 0 & & & & \\
0 & 2n-4 & 0 & & & \\
 & 0 & 2n-6 & 0 & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & 0 & 4-2n & 0 \\
 & & & & 0 & 2-2n
\end{pmatrix},
$$

$$
e = ???
$$

With respect to $\operatorname{ad} x$ we have the following decomposition for $\mathfrak{gl}_{2n}$:

$$
\mathfrak{gl}_{2n} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,
$$

where

$$
\mathfrak{n}_- = \bigoplus_{\substack{1 \le j < i \le 2n \\ (i,j) \neq (n+1,n)}} \mathbb{C}E_{ij}, \quad \mathfrak{n}_+ = \bigoplus_{\substack{1 \le i < j \le 2n \\ (i,j) \neq (n,n+1)}} \mathbb{C}E_{ij} \quad \text{and}
$$

$$
\mathfrak{h} = \left(\bigoplus_{k=1}^{2n} \mathbb{C}E_{kk} \right) \oplus \mathbb{C}E_{n+1,n} \oplus \mathbb{C}E_{n,n+1}.
$$

It follows that $\ker \operatorname{ad} \mathfrak{n}_+ = \mathbb{C}E_{1,2n}$. Thus we can choose $s = E_{1,2n}$ and let vary $z \in \mathbb{C}$. We set $\mathcal{V}_z(\mathfrak{gl}_{2n}) = \mathcal{V}_z(\mathfrak{gl}_{2n}, s)$.

We note that e, h and f are fixed by σ. Moreover f is principal nilpotent in $\mathfrak{o}_{2n}$. We can choose the same $\mathfrak{sl}_2$-triple and get the following $\operatorname{ad} x$-decomposition for $\mathfrak{o}_{2n}$:

$$
\mathfrak{o}_{2n} = \mathfrak{n}_-(\mathfrak{o}_{2n}) \oplus \mathfrak{h}(\mathfrak{o}_{2n}) \oplus \mathfrak{n}_+(\mathfrak{o}_{2n}),
$$

where

$$
\mathfrak{n}_-(\mathfrak{o}_{2n}) = \mathfrak{n}_- \cap \mathfrak{o}_{2n}, \quad \mathfrak{h}(\mathfrak{o}_{2n}) = \mathfrak{h} \cap \mathfrak{o}_{2n} \quad \text{and} \quad \mathfrak{n}_+(\mathfrak{o}_{2n}) = \mathfrak{n}_+ \cap \mathfrak{o}_{2n}.
$$

We see that

$$
\ker \operatorname{ad} \mathfrak{n}_+(\mathfrak{o}_{2n}) = \mathbb{C}(E_{1,2n-1} + E_{2,2n}),
$$

and we choose $s' = E_{1,2n-1} + E_{2,2n}$ and set $\mathcal{V}_{z'}(\mathfrak{o}_{2n}) = \mathcal{V}_{z'}(\mathfrak{o}_{2n}, s')$.

By Proposition 2.11 we get a Poisson vertex algebra homomorphism

$$
\rho : \mathcal{V}_0(\mathfrak{gl}_{2n})^\sigma \twoheadrightarrow \mathcal{V}(\mathfrak{o}_{2n}).
$$

Moreover, by Proposition 2.16, we have also a Poisson vertex algebra homomorphism

$$
\rho : \mathcal{W}_0(\mathfrak{gl}_{2n}, f)^\sigma \longrightarrow \mathcal{W}(\mathfrak{o}_{2n}, f)
$$

and by Corollary 2.17 it follows that

$$
\mathcal{W}_0(\mathfrak{o}_{2n}, f) \cong \mathcal{W}_0(\mathfrak{gl}_{2n}, f)^\sigma / \ker \rho.
$$

2.3.4. Following the examples we made so far, we can apply Corollary 2.17 to any classical Lie algebra of type B_n, C_n and D_n and any nilpotent element.

Indeed, let us consider the case of the classical Lie algebra B_n. We have seen that it is realized as the Lie algebra of fixed point of the linear map σ defined by (2.35), namely

$$\mathfrak{o}_{2n+1} = \{A \in \mathfrak{gl}_{2n+1} \mid \sigma(A) = A\}.$$

Let $\mathcal{W}_z(\mathfrak{o}_{2n+1}, f, s)$ be the classical $\mathcal{W}$-algebra associated to $f \in \mathfrak{o}_{2n+1}$ and $s \in \ker \operatorname{ad} \mathfrak{n}_+(\mathfrak{o}_{2n+1})$ as we did in Section 2.2. Since $\mathfrak{o}_{2n+1}$ is a Lie subalgebra of $\mathfrak{gl}_{2n+1}$, f is also a nilpotent element of $\mathfrak{gl}_{2n+1}$ and we may consider $\mathcal{W}_0(\mathfrak{gl}_{2n+1}, f)$. Let ρ be the surjective Poisson vertex algebra homomorphism defined in (2.16), then, by Corollary 2.17 we get the following general result.

Theorem 2.19. *We have the following isomorphism of Poisson vertex algebras*

$$\mathcal{W}_0(\mathfrak{o}_{2n+1}, f) \cong \mathcal{W}_0(\mathfrak{gl}_{2n+1}, f)^\sigma / \ker \rho.$$

If, moreover, $\ker \operatorname{ad} \mathfrak{n}_+ \cap \ker \operatorname{ad} \mathfrak{n}_+(\mathfrak{o}_{2n+1}) \neq (0)$*, then*

$$\mathcal{W}_z(\mathfrak{o}_{2n+1}, f, s) \cong \mathcal{W}_z(\mathfrak{gl}_{2n+1}, f, s)^\sigma / \ker \rho,$$

for any $s \in \ker \operatorname{ad} \mathfrak{n}_+ \cap \ker \operatorname{ad} \mathfrak{n}_+(\mathfrak{o}_{2n+1})$.

The same considerations apply to the classical Lie algebra of type C_n and D_n. We have seen that C_n, respectively D_n is realized as the Lie algebra of the fixed point of the linear map σ defined by (2.35), respectively (2.38), namely

$$\mathfrak{sp}_{2n} = \{A \in \mathfrak{gl}_{2n} \mid \sigma(A) = A\} \qquad (\text{respectively } \mathfrak{o}_{2n} = \{A \in \mathfrak{gl}_{2n} \mid \sigma(A) = A\}).$$

Let us denote by $\mathfrak{g}$ a classical Lie algebra of type C_n or D_n and let $f \in \mathfrak{g}$ be a nilpotent element. Let $\mathcal{W}_z(\mathfrak{g}, f, s)$ be the classical $\mathcal{W}$-algebra associated to $f \in \mathfrak{g}$ and $s \in \ker \operatorname{ad} \mathfrak{n}_+(\mathfrak{g})$ as we did in Section 2.2. Since $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}_{2n}$, f is also a nilpotent element of $\mathfrak{gl}_{2n}$ and we may consider $\mathcal{W}_0(\mathfrak{gl}_{2n}, f)$. Let ρ be the surjective Poisson vertex algebra homomorphism defined in (2.16), then, by Corollary 2.17 we get the following general result.

Theorem 2.20. *We have the following isomorphism of Poisson vertex algebras*

$$\mathcal{W}_0(\mathfrak{g}, f) \cong \mathcal{W}_0(\mathfrak{gl}_{2n}, f)^\sigma / \ker \rho.$$

If, moreover, $\ker \operatorname{ad} \mathfrak{n}_+ \cap \ker \operatorname{ad} \mathfrak{n}_+(\mathfrak{g}) \neq (0)$*, then*

$$\mathcal{W}_z(\mathfrak{g} f, s) \cong \mathcal{W}_z(\mathfrak{gl}_{2n}, f, s)^\sigma / \ker \rho,$$

for any $s \in \ker \operatorname{ad} \mathfrak{n}_+ \cap \ker \operatorname{ad} \mathfrak{n}_+(\mathfrak{g})$.

CHAPTER 3

Gelfand-Dickey algebras

3.1. The algebra of formal pseudodifferential operators

We recall here some basic facts about the theory of formal pseudodifferential operators (see [2] or[10] for an extended treatment). Let $\mathcal{A}$ be a differential algebra with a derivation ∂. We consider the algebra $\mathcal{A}((\partial^{-1}))$ of formal pseudodifferentials operators with coefficients in $\mathcal{A}$. The formal "integration" symbol ∂^{-1} obeys the following algebraic rules: $\partial^{-1}\partial = \partial\partial^{-1} = 1$ and, for $a \in \mathcal{A}$,

$$\partial^{-1} \circ a = \sum_{k\in\mathbb{Z}_+} (-1)^k a^{(k)} \partial^{-1+k}. \tag{3.1}$$

Rule (3.1) is motivated by integration by parts formula. Indeed, formally, for any $f \in \mathcal{A}$, we have

$$(\partial^{-1} \circ a)f = \partial^{-1}(af) = \int af = \int a\partial(\partial^{-1}f) = a\partial^{-1}f - \int a'\partial^{-1}f = a\partial^{-1}f - a'\partial^{-2}f + \int a''\partial^{-2}f$$

and so on. This allows us to extend Leibniz rule for any integer power of the derivation symbol ∂, namely

$$\partial^n \circ a = \sum_{k\in\mathbb{Z}_+} \binom{n}{k} a^{(k)} \partial^{n-k} \tag{3.2}$$

for any $n \in \mathbb{Z}$, where as usual, $\binom{n}{k} = (-1)^k\binom{k-n-1}{k}$ if $n < 0$.

It can easily be verified that multiplication given by (3.2) is well defined, since after reshuffling with (3.2) only a finite number of terms appear in front of ∂^i for any $i \in \mathbb{Z}$, avoiding any convergence problem, and is associative, thus making the space of formal pseudodifferential operators an associative algebra with unity.

We can write $P \in \mathcal{A}((\partial^{-1}))$ as

$$P = \sum_{k\leq N} p_k \partial^k,$$

with $p_k \in \mathcal{A}$ and $N \in \mathbb{Z}$. The greatest N such that $p_N \neq 0$ is called order of P and we dentote it by $\mathrm{ord}(P)$. We will use the notation $\mathcal{A}(((\partial^{-1}))_n = \{P \in \mathcal{P} \mid \mathrm{ord}(P) \leq n\} \subset A((\partial^{-1}))$.

Using (3.2) one can also bring all the derivatives to the left, that is

$$P = \sum_{k\leq N} p_k \partial^k = \sum_{k\leq N} \partial^k \circ \tilde{p}_k.$$

The algebra of formal pseudodifferential operators has a natural anti-homomorphism, which we denote by *, called formal adjoint, defined by $f^* = f$, for $f \in \mathcal{A}$, and $\partial^* = -\partial$. Thus

$$P^* = \sum_{k\leq N} (-\partial)^k \circ p_k.$$

We have the direct sum decomposition (as vector spaces) $\mathcal{A}((\partial^{-1})) = \mathcal{A}[\partial] \oplus \mathcal{A}[[\partial^{-1}]]\partial^{-1}$, where $\mathcal{A}[\partial]$ is the subalgebra of differential operators and $\mathcal{A}[[\partial^{-1}]]\partial^{-1}$ is the subalgebra of pseudodifferential symbols (also called integral operators or Volterra operators).

Given $P \in \mathcal{A}((\partial^{-1}))$, we decompose $P = P_+ + P_-$, where P_+ (respectively P_-) is its component in $\mathcal{A}[\partial]$ (respectively $\mathcal{A}[[\partial^{-1}]]\partial^{-1}$). We also define its residue to be

$$\mathrm{Res}_\partial P = p_{-1} \ (= \text{ coefficient of } \partial^{-1}).$$

Using the residue we can define a pairing $\langle \cdot \mid \cdot \rangle : \mathcal{A}((\partial^{-1})) \times \mathcal{A}((\partial^{-1})) \longrightarrow \mathcal{A}/_{\partial\mathcal{A}}$ by

$$\langle X \mid Y \rangle = \int \mathrm{Res}_\partial(X \circ Y), \tag{3.3}$$

for any $X, Y \in \mathcal{A}((\partial^{-1}))$. It is an easy computation to prove that $\langle X \mid Y \rangle = \langle Y \mid X \rangle$, thus $\mathrm{Res}_\partial[X, Y] \in \partial\mathcal{A}$, where, as usual, we denote $[X, Y] = X \circ Y - Y \circ X$. Using this pairing, we can think $\mathcal{A}[[\partial^{-1}]]\partial^{-1}$ as the dual of $\mathcal{A}[\partial]$.

We denote $\mathbb{C}[\partial,\partial^{-1}]\circ\mathcal{A}\subset\mathcal{A}((\partial^{-1}))$ the space of all pseudodifferential operators of the form

$$\sum_{k\in\mathbb{Z}}\partial^k\circ P_k,$$

where all but finitely many elements P_k are zero. Under the pairing given by (3.3), the dual space to $\mathbb{C}[\partial,\partial^{-1}]\circ\mathcal{A}$ is $\mathcal{A}[[\partial,\partial^{-1}]]$.

Proposition 3.1. *Let P be a monic pseudodifferential operator of order N, then there exists exactly one monic pseudodifferential operator M of order one, such that $M^N=P$. We denote $M=P^{\frac{1}{N}}$.*

Proof. Let $P=\sum_{k\leq N}p_k\partial^k$ and $M=\sum_{k\leq 1}m_k\partial^k$, where $p_N=m_1=1$, be two such pseudodifferential operators. Equating the coefficients of powers of ∂ in the expression $P=M^N$, we get the recursion

$$\begin{cases} p_{N-1}=Nm_0 \\ p_{N-k}=Nm_{1-k}+f_k(m_0,\dots,m_{2-k}), \quad k\geq 2\end{cases}$$

where f_k is a differential polynomial in the variables $m_0,\dots,m_{2-k}$. Thus, we get $m_0=\frac{p_N-1}{N}$ and, for $k\geq 2$, we obtain the expression of m_{1-k} as a differential polynomial in the coefficients of P by induction. Indeed, if we know $m_0,\dots,m_{2-k}$, for $k\geq 2$, then $f_{k+1}(m_0,\dots,m_{1-k})=g_{k+1}(p_{N-1},\dots,p_{N-k})$, obtaining $m_{-k}=\frac{1}{N}\left(p_{N-k}-g_{k+1}(p_{N-1},\dots,p_{N-k})\right)$. Each value of m_{1-k} is uniquely determined. □

3.2. Poisson vertex algebra structures attached to a general pseudodifferential operator

Let $\mathcal{A}$ be a differential algebra with a derivation ∂. We consider the following identifications

$$\begin{aligned}\mathbb{C}[\partial,\partial^{-1}]\circ\mathcal{A} &\xleftrightarrow{\sim}\mathcal{A}^{\oplus\mathbb{Z}}\\ \sum_{k\in\mathbb{Z}}\partial^k\circ F_k &\longleftrightarrow (F_k)_{k\in\mathbb{Z}},\end{aligned}\tag{3.4}$$

where we emphasize that the sum on the left is finite, namely all but finitely many F_k are zero, and

$$\begin{aligned}\mathcal{A}[[\partial,\partial^{-1}]] &\xleftrightarrow{\sim}\mathcal{A}^{\mathbb{Z}}\\ P=\sum_{k\in\mathbb{Z}}P_{-k-1}\partial^k &\longleftrightarrow (P_k)_{k\in\mathbb{Z}},\end{aligned}\tag{3.5}$$

that is $P_k=(\mathrm{Res}_\partial(P\partial^k))$, for $k\in\mathbb{Z}$. Let $P\in\mathcal{V}[[\partial,\partial^{-1}]]$ and $F\in\mathbb{C}[\partial,\partial^{-1}]\circ\mathcal{A}$, an explicit computation of (3.3) gives

$$\langle P\mid F\rangle=\int\mathrm{Res}_\partial\sum_{k,l}P_{-k-1}\partial^{k+l}F_l=\int\sum_k P_kF_k.\tag{3.6}$$

We note that, using identifications (3.4) and (3.5), the pairing (3.6) coincides with (1.3).

Given $L\in\mathcal{A}((\partial^{-1}))$, we define the map $A^{(L)}:\mathbb{C}[\partial,\partial^{-1}]\circ\mathcal{A}\longrightarrow\mathcal{V}[[\partial,\partial^{-1}]]$ by

$$A^{(L)}(F)=L(FL)_+-(LF)_+L,\tag{3.7}$$

for any $F\in\mathbb{C}[\partial,\partial^{-1}]\circ\mathcal{A}$. This map was first introduced by Adler (see [1]).

From now on, unless otherwise stated, we assume that L is a fixed pseudodifferential operator of order $\mathrm{ord}(L)=N\in\mathbb{Z}$. In this case we have $A^{(L)}(F)\in\mathcal{A}((\partial^{-1}))_{N-1}$. Indeed, $A^{(L)}(F)=-L(FL)_-+(LF)_-L$, and the product of an operator of order N with an operator of order less than or equal to -1 has order at most $N-1$. Moreover, if $F\in\partial^{-N-1}\mathbb{C}[\partial^{-1}]\circ\mathcal{A}\subset\mathcal{A}((\partial^{-1}))_{-N-1}$, then $A^{(L)}(F)=0$, since $(FL)_+=(LF)_+=0$. In conclusion, $A^{(L)}$ induces a map

$$A^{(L)}:\left(\mathbb{C}[\partial,\partial^{-1}]\circ\mathcal{A}\right)\big/\left(\partial^{-N-1}\mathbb{C}[\partial^{-1}]\circ\mathcal{A}\right)\longrightarrow\mathcal{A}((\partial^{-1}))_{N-1}.$$

We set $I_{-N}=I=\{k\in\mathbb{Z}\mid k\geq -N\}$. Identifications (3.4) and (3.5) induce natural identifications

$$\left(\mathbb{C}[\partial,\partial^{-1}]\circ\mathcal{A}\right)\big/\left(\partial^{-N-1}\mathbb{C}[\partial^{-1}]\circ\mathcal{A}\right)\xrightarrow{\sim}\mathcal{A}^{\oplus I}\tag{3.8}$$

and

$$\mathcal{A}((\partial^{-1}))_{N-1}\xrightarrow{\sim}\mathcal{A}^I.\tag{3.9}$$

Thus we get a matrix $H^{(L)}(\partial) \in \mathrm{Mat}_{I\times I}(\mathcal{A}[\partial])$ by the following commutative diagram

$$\begin{tikzcd}
\left(\mathbb{C}[\partial,\partial^{-1}]\circ\mathcal{A}\right)/\left(\partial^{-N-1}\mathbb{C}[\partial^{-1}]\circ\mathcal{A}\right) \arrow[r, "A^{(L)}"] \arrow[d, leftrightarrow, "\wr"'] & \mathcal{A}((\partial^{-1}))_{N-1} \arrow[d, leftrightarrow, "\wr"] \\
\mathcal{A}^{\oplus I} \arrow[r, "H^{(L)}(\partial)"] & \mathcal{A}^{I}.
\end{tikzcd}$$

In the sequel, if $a(z) \in \mathcal{A}((z^{-1}))$, we denote $a(z+\alpha) = e^{\alpha\partial_z}(z)$ its power expansion in the domain $|z| > |\alpha|$ (see (A.5)). Namely, for any $n \in \mathbb{Z}$, we have

$$(z+\alpha)^n = \sum_{k\in\mathbb{Z}_+} \binom{n}{k}\alpha^k z^{n-k}.$$

In the case we have a meromorphic function in two variables z, w, say f, we denote $i_z f$ the expansion when $|z|$ is big, namely, the expansion in negative powers of z. For example, if $f = (z-w-\alpha)^n$, for $n \in \mathbb{Z}$, we have

$$i_z(z-w-\alpha)^n = \sum_{k\in\mathbb{Z}_+} \binom{k-n-1}{k}(w+\alpha)^k z^{n-k},$$

Similarly, $i_w f$ will denote the expansion in negative powers of w. We can compute explicity the matrix differential operator $H^{(L)}(\partial)$ in terms of generating series. This is given by the following:

Lemma 3.2. *Let $H^{(L)}(\partial)(z,w) = \sum_{i,j\in I} H^{(L)}_{ij}(\partial) z^{-i-1}w^{-j-1}$ be the generating series for the differential operators $H^{(L)}_{ij}(\partial)$. Then*

$$H^{(L)}(\partial)(z,w) = L(z+\partial)i_w(w-z-\partial)^{-1}L^*(\partial-w) - L(w)i_w(w-z-\partial)^{-1}L(z). \tag{3.10}$$

Proof. First we note that we can assume $H^{(L)}(\partial) \in \mathrm{Mat}_{\mathbb{Z}\times\mathbb{Z}}(\mathcal{A}[\partial])$ simply extending it to an infinite matrix in both directions adding infinite rows and coloumns of zeroes. It corresponds to consider the following commutative diagram

$$\begin{tikzcd}
\mathbb{C}[\partial,\partial^{-1}]\otimes\mathcal{A} \arrow[r, "A^{(L)}"] \arrow[d, leftrightarrow, "\wr"'] & \mathcal{A}[[\partial,\partial^{-1}]] \arrow[d, leftrightarrow, "\wr"] \\
\mathcal{A}^{\oplus\mathbb{Z}} \arrow[r, "H^{(L)}(\partial)"] & \mathcal{A}^{\mathbb{Z}}.
\end{tikzcd}$$

Using (3.4) and (3.5), for $F = (F_k) \in \mathcal{A}^{\oplus\mathbb{Z}}$, we have $(H^{(L)}(\partial)(F))_i = \mathrm{Res}_\partial\left(A^{(L)}\left(\sum_k \partial^k\circ F_k\right)\partial^i\right)$, $i \in \mathbb{Z}$, from which we get

$$H^{(L)}_{ij}(\partial)(f) = \mathrm{Res}_\partial\left(A(L)\left(\partial^j\circ f\right)\partial^i\right),$$

for all $f \in \mathcal{A}$ and $i, j \in \mathbb{Z}$.

To complete the proof, it suffices to perform a straightforward computation. For any $f \in \mathcal{A}$, we have

$$\begin{aligned}
H^{(L)}(\partial)(z,w)f &= \sum_{i,j\in\mathbb{Z}} H^{(L)}_{ij}(\partial)(f)z^{-i-1}w^{-j-1} = \sum_{i,j\in\mathbb{Z}} \mathrm{Res}_\partial\left(A^{(L)}\left(\partial^j\circ f\right)\partial^i\right)z^{-i-1}w^{-j-1} = \\
&= \sum_{i,j\in\mathbb{Z}} \mathrm{Res}_\partial\left(\left(L(\partial)(\partial^j\circ fL(\partial))_+ - (L(\partial)\partial^j\circ f)_+L(\partial)\right)\partial^i\right)z^{-i-1}w^{-j-1}.
\end{aligned}$$

We can write the above expression in terms of the formal δ-function defined in (A.3):

$$H^{(L)}(\partial)(z,w)f = \mathrm{Res}_\partial\left((L(\partial)(\delta(w-\partial)\circ fL(\partial))_+ - (L(\partial)\delta(w-\partial)\circ f)_+L(\partial))\,\delta(z-\partial)\right).$$

By Lemma A.1, we have

$$(\delta(w-\partial)\circ fL(\partial))_+ = \delta(w-\partial)_+\circ(L^*(\partial-w)f) = i_w(w-\partial)^{-1}\circ(L^*(\partial-w)f).$$

Similarly,

$$(L(\partial)\delta(w-\partial)\circ f)_+ = L(w)\delta(w-\partial)_+\circ f = L(w)i_w(w-\partial)^{-1}\circ f.$$

Thus we get

$$H^{(L)}(\partial)(z,w) = \mathrm{Res}_\partial\left(\left(L(\partial)i_w(w-\partial)^{-1}\circ(L^*(\partial-w)f) - L(w)i_w(w-\partial)^{-1}\circ fL(\partial)\right)\delta(z-\partial)\right).$$

Applying again Lemma A.1 to $\delta(z-\partial)$, the above expression is equivalent to (3.10), proving the lemma. $\square$

Let $\mathcal{V}$ be an algebra of differential functions in infinitely many variables u_i, $i \in I$. Consider the monic pseudodifferential operator

$$L^{(N)} = L = \partial^N + u_{-N}\partial^{N-1} + u_{-N+1}\partial^{-N-2} + \ldots = \sum_{i \leq N} u_{-i-1}\partial^i \in \mathcal{V}((\partial^{-1})), \tag{3.11}$$

where $u_{-N-1} = 1$. We shall refer to L as the pseudodifferential operator of general type associated to $\mathcal{V}$.

Theorem 3.3. *The operator $H^{(L)}(\partial)$ is a Hamiltonian operator, namely, the associated λ-bracket defines a Poisson vertex algebra structure on $\mathcal{V}$.*

Proof. We set $L(z) = \sum_{i \geq -N-1} u_i z^{-i-1}$ be the symbol of the operator $L(\partial)$. By (1.17) and Lemma 3.2

$$\begin{aligned} &\{L(z)_\lambda L(w)\} = H^{(L)}(\lambda)(w,z) = \\ &= L(w+\lambda+\partial) i_z (z-w-\lambda-\partial)^{-1} L^*(\lambda - z) - L(z) i_z (z-w-\lambda-\partial)^{-1} L(w) \end{aligned} \tag{3.12}$$

defines a λ-bracket among any pairs of generators of $\mathcal{V}$. Indeed, expanding the left hand side and using the fact that z and w are central elements, we get

$$\{L(z)_\lambda L(w)\} = \sum_{i,j \in I} \{u_{i\lambda} u_j\} z^{-i-1} w^{-j-1},$$

that is the generating series of the λ-bracket on any pair of generators of $\mathcal{V}$.

We claim that this λ-bracket satisfies skew-commutativity and Jacobi identity, thus proving the theorem according to Definition 1.12.

For the skew-commutativity case, we should prove that $\{L(z)_\lambda L(w)\} = -_{\leftarrow}\{L(w)_{-\lambda-\partial} L(z)\}$. We have

$$\begin{aligned} -_{\leftarrow}\{L(w)_{-\lambda-z} L(z)\} = L(w+\lambda+\partial) i_w (z-w-\lambda-\partial)^{-1} L^*(\lambda-\partial) \\ - L(z) i_w (z-w-\lambda-\partial)^{-1} L(w). \end{aligned} \tag{3.13}$$

By (3.12) and (3.13), skew-commutativity condition is equivalent to prove the following identity

$$L(w+\lambda+\partial)\delta(z-w-\lambda-\partial)L^*(\lambda-z) = L(z)\delta(z-w-\lambda-\partial)L(w),$$

which follows easily applying twice part 4) of Lemma (A.1) to the left hand side.

We are left to prove Jacobi identity. For generating series it reads as (see (1.10))

$$\{L(z)_\lambda \{L(w)_\mu L(t)\}\} - \{L(w)_\mu \{L(z)_\lambda L(t)\}\} - \{\{L(z)_\lambda L(w)\}_{\lambda+\mu} L(t)\} = 0. \tag{3.14}$$

Using (3.12), Leibniz rule and sesquilinearity, by a straightforward computation we get:

$$\begin{aligned}
&\{L(z)_\lambda \{L(w)_\mu L(t)\}\} = \\
&= L(t+\lambda+\mu+\partial) i_z (z-t-\lambda-\mu-\partial)^{-1} L^*(\lambda-z) i_w (w-t-\mu-\partial)^{-1} L^*(\mu-\partial) && \text{(3.15a)}\\
&- L(z) i_z (z-t-\lambda-\mu-\partial)^{-1} L(t+\mu+\partial) i_w (w-t-\mu-\partial)^{-1} L^*(\mu-w) && \text{(3.15b)}\\
&+ L(t+\lambda+\mu+\partial) i_w (w-t-\lambda-\mu-\partial)^{-1} L^*(\lambda-z) i_z (z-w+\mu+\partial)^{-1} L^*(\mu-w) && \text{(3.15c)}\\
&- L(t+\lambda+\mu+\partial) i_w (w-t-\lambda-\mu-\partial)^{-1} L^*(\lambda+\mu+\partial-w) i_z (z-w+\mu+\partial) L(z) && \text{(3.15d)}\\
&+ L(z) \left(i_z (z-w-\lambda-\partial)^{-1} L(w)\right) i_w (w-t-\mu-\partial)^{-1} L(t) && \text{(3.15e)}\\
&- \left(i_w (w-t-\mu-\partial)^{-1} L(t)\right) L(w+\lambda+\partial) i_z (z-w-\lambda-\partial)^{-1} L^*(\lambda-z) && \text{(3.15f)}\\
&+ L(w) i_w (w-t-\lambda-\mu-\partial)^{-1} L(z) i_z (z-t-\lambda-\partial)^{-1} L(t) && \text{(3.15g)}\\
&- L(w) i_w (w-t-\lambda-\mu-\partial)^{-1} L(t+\lambda+\partial) i_z (z-t-\lambda-\partial)^{-1} L^*(\lambda-z); && \text{(3.15h)}
\end{aligned}$$

$$
\begin{aligned}
&-\{L(w)_\mu\{L(z)_\lambda L(t)\}\} = \\
&= -L(t+\lambda+\mu+\partial)i_w(w-t-\lambda-\mu-\partial)^{-1}L^*(\mu-w)i_z(z-t-\lambda-\partial)^{-1}L^*(\lambda-z) && (3.16a)\\
&+ L(w)i_w(w-t-\lambda-\mu-\partial)^{-1}L(t+\lambda+\partial)i_z(z-t-\lambda-\partial)^{-1}L^*(\lambda-z) && (3.16b)\\
&- L(t+\lambda+\mu+\partial)i_z(z-t-\lambda-\mu-\partial)^{-1}L^*(\mu-w)i_w(w-z+\lambda+\partial)^{-1}L^*(\lambda-z) && (3.16c)\\
&+ L(t+\lambda+\mu+\partial)i_z(z-t-\lambda-\mu-\partial)^{-1}L^*(\lambda+\mu+\partial-z)i_w(w-z+\lambda+\partial)^{-1}L(w) && (3.16d)\\
&- L(w)\left(i_w(w-z-\mu-\partial)^{-1}L(z)\right)i_z(z-t-\lambda-\partial)^{-1}L(t) && (3.16e)\\
&+ \left(i_z(z-t-\lambda-\partial)^{-1}L(t)\right)L(z+\mu+\partial)i_w(w-z-\mu-\partial)^{-1}L^*(\mu-w) && (3.16f)\\
&- L(z)i_z(z-t-\lambda-\mu-\partial)^{-1}L(w)i_w(w-t-\mu-\partial)^{-1}L(t) && (3.16g)\\
&+ L(z)i_z(z-t-\lambda-\mu-\partial)^{-1}L(t+\mu+\partial)i_w(w-t-\mu-\partial)^{-1}L^*(\mu-w); && (3.16h)
\end{aligned}
$$

$$
\begin{aligned}
&-\{\{L(z)_\lambda L(w)\}_{\lambda+\mu}L(t)\} = \\
&= -L(t+\lambda+\mu+\partial)\left(i_w(w-t-\mu-\partial)^{-1}L^*(\mu-w)\right)i_z(z-w-\lambda-\partial)^{-1}L^*(\lambda-z) && (3.17a)\\
&+ \left(i_w(w-t-\mu-\partial)^{-1}L(t)\right)L(w+\lambda+\partial)i_z(z-w-\lambda-\partial)^{-1}L^*(\lambda-z)- && (3.17b)\\
&- L(t+\lambda+\mu+\partial)\left(i_z(z-t-\lambda-\partial)^{-1}L^*(\lambda-z)\right)i_z(z-w+\mu+\partial)^{-1}L^*(\mu-w)+ && (3.17c)\\
&+ \left(i_z(z-t-\lambda-\partial)^{-1}L(t)\right)L(z+\mu+\partial)i_z(z-w+\mu+\partial)^{-1}L^*(\mu-w) && (3.17d)\\
&+ L(t+\lambda+\mu+\partial)i_z(z-t-\lambda-\mu-\partial)^{-1}L^*(\lambda+\mu+\partial-z)i_z(z-w-\lambda-\partial)^{-1}L(w)- && (3.17e)\\
&- L(z)i_z(z-t-\lambda-\mu-\partial)^{-1}L(t)i_z(z-w-\lambda-\partial)^{-1}L(w) && (3.17f)\\
&+ L(t+\lambda+\mu+\partial)i_w(w-t-\lambda-\mu-\partial)^{-1}L^*(\lambda+\mu+\partial-w)i_z(z-w+\mu+\partial)^{-1}L(z) && (3.17g)\\
&- L(w)i_w(w-t-\lambda-\mu-\partial)^{-1}L(t)i_z(z-w+\mu+\partial)^{-1}L(z), && (3.17h)
\end{aligned}
$$

where derivatives act on each term on the right. If some terms are inside parenthesis, this means that the derivatives appearing act only inside the parenthesis, that is, if $a(\partial)=\sum_n a_n\partial^n \in \mathcal{V}((\partial^{-1}))$, then $(a(\partial)b)c=\sum_n a_n b^{(n)}c$, for any $b,c\in\mathcal{V}$.

We note that (3.15b)+(3.16h)=0, (3.15d)+(3.17g)=0, (3.15f)+(3.17b)=0 and(3.15h)+(3.16b)=0, then this terms disappear in the sum in (3.14). To conclude the proof it remains to prove that

$$
\begin{aligned}
&(3.15a)+(3.15c)+(3.15e)+(3.15g)+(3.16a)+(3.16c)+(3.16d)+(3.16e)+(3.16f)+\\
&+(3.16g)+(3.17a)+(3.17c)+(3.17d)+(3.17e)+(3.17f)+(3.17h)=0.
\end{aligned}
\tag{3.18}
$$

We claim that $(3.15e)+(3.16g)+(3.17f)=0$. Indeed, we set $\alpha=z-w-\lambda-\partial$ and $\beta=w-t-\mu-\partial$, where we assume that derivative in α acts only on $L(w)$ while derivative in β acts only on $L(t)$. Then we can write

$$
(3.15e)+(3.16g)+(3.17f)=L(z)\left(i_z\alpha^{-1}i_w\beta^{-1}-i_z(\alpha+\beta)^{-1}\left(i_w\alpha^{-1}+i_z\beta^{-1}\right)\right)L(w)L(t).
$$

Using the fact that

$$
i_w\alpha^{-1}+i_z\beta^{-1}=(\alpha+\beta)i_z\alpha^{-1}i_w\beta^{-1},
\tag{3.19}
$$

our claim is proved. Similarly, we write $(3.17c)+(3.15c)+(3.16a)$ as

$$
L(t+\lambda+\mu+\partial)\left(i_z\alpha^{-1}i_z\beta^{-1}-i_w(\alpha+\beta)^{-1}\left(i_z\alpha^{-1}+i_z\beta^{-1}\right)\right)L^*(\mu-w)L^*(\lambda-z),
$$

where $\alpha=w-z-\mu-\partial$ (respectively $\beta=z-t-\lambda-\partial$) acts only on $L^*(\mu-w)$ (respectively $L^*(\lambda-z)$). Equality (3.19) shows that $(3.17c)+(3.15c)+(3.16a)=0$.

Next, by (A.3) we rewrite

$$(3.17h) = +L(w)i_w(w-t-\lambda-\mu-\partial)^{-1}L(t)i_w(w-z-\mu-\partial)^{-1}L(z) \tag{3.20}$$

$$-L(w)i_w(w-t-\lambda-\mu-\partial)^{-1}L(t)\delta(w-z-\mu-\partial)L(z); \tag{3.21}$$

$$(3.16c) = L(t+\lambda+\mu+\partial)i_z(z-t-\lambda-\mu-\partial)^{-1}L^*(\mu-w)i_z(z-w-\lambda-\partial)^{-1}L^*(\lambda-z) \tag{3.22}$$

$$-L(t+\lambda+\mu+\partial)i_z(z-t-\lambda-\mu-\partial)^{-1}L^*(\mu-w)\delta(z-w-\lambda-\partial)L^*(\lambda-z); \tag{3.23}$$

$$(3.16f)+(3.17d) = -\left(i_z(z-t-\lambda-\partial)^{-1}L(t)\right)L(z+\mu+\partial)\delta(w-z-\mu-\partial)L^*(\mu-w); \tag{3.24}$$

$$(3.16d)+(3.17e) =$$
$$= -L(t+\lambda+\mu+\partial)i_z(z-t-\lambda-\mu-\partial)^{-1}L^*(\lambda+\mu+\partial-z)\delta(z-w-\lambda-\partial)L(w). \tag{3.25}$$

Then we get

$$(3.16e)+(3.20)+(3.15g) = L(w)\left(-i_w\alpha^{-1}i_z\beta^{-1}+i_w(\alpha+\beta)^{-1}\left(i_w\alpha^{-1}+i_z\beta^{-1}\right)\right)L(z)L(t),$$

where $\alpha = w-z-\mu-\partial$ acts only on $L(z)$ and $\beta = z-t-\lambda-\partial$ acts only on $L(t)$ and

$$(3.17a)+(3.15a)+(3.20) =$$
$$L(t+\lambda+\mu+\partial)\left(-i_w\alpha^{-1}i_z\beta^{-1}+i_z(\alpha+\beta)^{-1}\left(i_w\alpha^{-1}+i_z\beta^{-1}\right)\right)L^*(\lambda-z)L^*(\mu-w),$$

where $\alpha = w-t-\mu-\partial$ acts only on $L^*(\mu-w)$ and $\beta = z-w-\lambda-\partial$ acts only on $L^*(\lambda-z)$. Using (3.19), it follows that both terms are zero. Hence, by (3.18), we are left to prove that

$$(3.21)+(3.24)+(3.23)+(3.25) = 0.$$

We claim that $(3.21)+(3.24)=0$. Indeed, using Lemma A.1 we get

$$(3.24) = \left(i_z(z-t-\lambda-\partial)^{-1}L(t)\right)L(w)\delta(w-z-\mu-\partial)L(z) =$$
$$= L(w)i_w(w-t-\lambda-\mu-\partial)^{-1}L(t)\delta(w-z-\mu-\partial)L(z) = -(3.21).$$

Applying again Lemma A.1, a similar computation shows that $(3.23)+(3.25)=0$ thus concluding the proof. $\square$

We denote this Poisson vertex algebra $\mathcal{V}^{(N,\infty)}$. We give an application of Theorem 3.3 that we are going to use in Chapter 5 to derive integrable hierarchies attached to the Poisson vertex algebra $\mathcal{V}^{(N,\infty)}$. Let us assume $\mathrm{ord}(L) = N \in \mathbb{Z}_+$ and write, for $c \in \mathbb{C}$, $H^{(L-c)}(\partial) = H^{(0)}(\partial) - cH^{(\infty)}(\partial)$, after expanding (3.7).

Corollary 3.4. *For any $c \in \mathbb{C}$, the operators $H^{(L-c)}(\partial)$ are Hamiltonian operators, namely, they define a bi-Hamiltonian structure $(H^{(0)}, H^{(\infty)})$ on $\mathcal{V}$.*

Proof. Since $\mathrm{ord}(L-c) = N \in \mathbb{Z}+$, identifications (3.8) and (3.9) still hold. Thus, by (1.17) and Lemma 3.2

$$\{L(z)_\lambda L(w)\}_c = H^{(L-c)}(\lambda)(w,z) = H^{(0)}(\lambda)(w,z) + cH^{(\infty)}(\lambda)(w,z) = \{L(z)_\lambda L(w)\}_0 + c\{L(z)_\lambda L(w)\}_\infty,$$

where $L(z) = \sum_{i\geq -N-1} u_i z^{-i-1}$ defines a λ-bracket among any pairs of generators of $\mathcal{V}$. Theorem 3.3 shows that $\{\cdot_\lambda\cdot\}_c$ satisfies skew-commutativity and Jacobi identity, thus proving that brackets $\{\cdot_\lambda\cdot\}_0$ and $\{\cdot_\lambda\cdot\}_\infty$ define two compatible Poisson vertex structures on $\mathcal{V}$, namely, the pair $(H^{(0)}, H^{(\infty)})$ is a bi-Hamiltonian pair. $\square$

We denote this bi-Poisson vertex algebra $\mathcal{V}_c^{(N,\infty)}$. In literature, $H^{(\infty)}$ is usually called *first Gelfand-Dickey structure*, while $H^{(0)}$ is called *second Gelfand-Dickey structure*.

3.3. Reduction to the case $u_{-N} = 0$

It would be interesting to understand if, more generally, any pseudodifferential operators $L \in \mathcal{V}((\partial^{-1}))$ defines a Poisson vertex algebra structure on $\mathcal{V}$. Namely, let $\mathcal{V}$ be an algebra of differential functions in some variables v_i, $i \in J$ an index set, and let $L \in \mathcal{V}((\partial^{-1}))_N$ be a monic pseudodifferential operator. We can define a map $\varphi : \mathcal{V}^{(N,\infty)} \longrightarrow \mathcal{V}$ given by comparing coefficients of the general pseudodifferential operator $L^{(N)}$ (see (3.11)) and coefficients of L. If this map is surjective and its kernel is a Poisson vertex algebra ideal, then $\mathcal{V}$ inherits a Poisson vertex algebra structure. Unfortunately this is not the case in general.

Indeed, let us assume $\mathcal{V} \subset \mathcal{V}^{(N,\infty)}$ to be the subalgebra of differential functions in the variable u_i for $i \geq -N+1$ and set

$$L = \partial^N + u_{-N+1}\partial^{N-2} + \ldots \in \mathcal{V}^{(N,\infty)}((\partial^{-1}))_N.$$

Comparing $L^{(N)}$ and L gives the following map $\varphi : \mathcal{V}^{(N,\infty)} \longrightarrow \mathcal{V}$, defined on generators of $\mathcal{V}^{(N,\infty)}$ by

$$\varphi(u_{-N}) = 0 \qquad \text{and} \qquad \varphi(u_i) = u_i, \text{ for } i \geq -N+1.$$

Namely, φ is the projection map on the variable u_{-N}, hence it is surjective and $\ker\varphi = \langle u_{-N}\rangle_{\mathcal{V}^{(N,\infty)}}$ is the differential ideal in $\mathcal{V}^{(N,\infty)}$ generated by the variable u_{-N}. By an explicit computation of (3.12) we have $\{u_{-N\,\lambda}u_{-N}\} = N\lambda$, from which follows that $\ker\varphi$ is not a Poisson vertex algebra ideal in $\mathcal{V}^{(N,\infty)}$. Hence, $\mathcal{V}$ does not inherit any Poisson vertex algebra structure from $\mathcal{V}^{(N,\infty)}$. Anyway, in this case we can slightly modify the definition of the map $A^{(L)}$ in (3.7) and give another Poisson vertex algebra structure on $\mathcal{V}^{(N,\infty)}$ such that $\ker\varphi$ is a Poisson vertex algebra ideal for this new structure.

Let $\mathcal{A}$ be a differential algebra with a derivation ∂ and consider the identifications (3.4) and (3.5). Given $L \in \mathcal{A}((\partial^{-1}))$ and $d \in \mathbb{C}$, we define a map $A^{(L,d)} : \mathbb{C}[\partial,\partial^{-1}] \circ \mathcal{A} \longrightarrow \mathcal{A}((\partial^{-1}))$ by

$$A^{(L,d)}(F) = A(L)(F) - d[L, \partial^{-1}(\mathrm{Res}_\partial[L,F])]. \tag{3.26}$$

We already noticed that $\mathrm{Res}_\partial[L,F] \in \partial\mathcal{A}$, then it makes sense to consider its antiderivative, that we denote by $\partial^{-1}(\mathrm{Res}_\partial[L,X])$. This antiderivative is defined up to the sum of a constant element $a \in \mathcal{A}$, but this choice is irrelevant in our definition since $[L,a] = 0$. Thus, we can assume $a = 0$.

If $\mathrm{ord}(L) = N \in \mathbb{Z}_+$, then $A^{(L,d)}(F) \in \mathcal{V}((\partial^{-1}))_{N-1}$. Indeed, we already know that $A^{(L)}(F) \in \mathcal{V}((\partial^{-1}))_{N-1}$ and, moreover, the commutator of a function with L has order $N-1$. Moreover, if $F \in \partial^{-N-1}\mathbb{C}[\partial^{-1}] \circ A$, then $A^{(L,d)} = 0$, since $\mathrm{ord}[L,F] \leq -2$, thus this term has no residue. This means that, $A^{(L,d)}$ induces a map

$$A^{(L,d)} : \left(\mathbb{C}[\partial,\partial^{-1}] \circ \mathcal{A}\right)\big/\left(\partial^{-N-1}\mathbb{C}[\partial^{-1}] \circ \mathcal{A}\right) \longrightarrow \mathcal{A}((\partial^{-1}))_{N-1}.$$

We recall that we set $I_{-N} = I = \{k \in \mathbb{Z} \mid k \geq -N\}$, then, using identifications (3.8) and (3.9), we define $H^{(L,d)}(\partial) \in \mathrm{Mat}_{I\times I}(\mathcal{A}[\partial])$ by

$$\begin{tikzcd}
\left(\mathbb{C}[\partial,\partial^{-1}] \circ \mathcal{A}\right)\big/\left(\partial^{-N-1}\mathbb{C}[\partial^{-1}] \circ \mathcal{A}\right) \arrow[r, "A^{(L,d)}"] & \mathcal{A}((\partial^{-1}))_{N-1} \\
\mathcal{A}^{\oplus I} \arrow[u, leftrightarrow, "\wr"] \arrow[r, "H^{(L,d)}(\partial)"] & \mathcal{A}^{I}. \arrow[u, leftrightarrow, "\wr"']
\end{tikzcd}$$

Lemma 3.5. *Let $H^{(L,d)}(\partial)(z,w) = \sum_{i,j\in I} H_{ij}^{(L,d)}(\partial) z^{-i-1}w^{-j-1}$ be the generating series for the differential operators $H_{ij}^{(L,d)}(\partial)$. Then*

$$H^{(L,d)}(\partial)(z,w) = H^{(L)}(\partial)(z,w) + d\left(L(z+\partial) - L(z)\right)\partial^{-1}\left(L^*(\partial - w) - L(w)\right). \tag{3.27}$$

Proof. As we did in the proof of Lemma 3.2 we can consider the following commutative diagram

$$\begin{tikzcd}
\mathbb{C}[\partial,\partial^{-1}] \circ \mathcal{A} \arrow[r, "A^{(L,d)}"] & \mathcal{A}[[\partial,\partial^{-1}]] \\
\mathcal{A}^{\oplus\mathbb{Z}} \arrow[u, leftrightarrow, "\wr"] \arrow[r, "H^{(L,d)}(\partial)"] & \mathcal{A}^{\mathbb{Z}}. \arrow[u, leftrightarrow, "\wr"']
\end{tikzcd}$$

Using (3.4) and (3.5), for $F = (F_k) \in \mathcal{A}^{\oplus\mathbb{Z}}$, we have $\left(H^{(L,d)}(\partial)(F)\right)_i = \mathrm{Res}_\partial\left(A^{(L,d)}\left(\sum_k \partial^k \circ f\right)\partial^i\right)$, $i \in \mathbb{Z}$, from which follows that

$$H_{ij}^{(L,d)}(\partial)(f) = H_{ij}^{(L)}(\partial) - d\,\mathrm{Res}_\partial\left(\left[L, \partial^{-1}\left(\mathrm{Res}_\partial\left[L, \partial^j \circ f\right]\right)\right]\partial^i\right)$$

for all $f \in \mathcal{A}$ and $i,j \in \mathbb{Z}$.

To complete the proof we need only to compute the generating series for the coefficient of d, since we already computed $H_{ij}^{(L)}$ in Lemma 3.2. We have, for any $f \in \mathcal{A}$, after multiplying for $z^{-i-1}w^{-j-1}$ and summing over all $i,j \in \mathbb{Z}$, that this term is given by

$$\begin{aligned}
&-\mathrm{Res}_\partial\left(\left[L(\partial), \partial^{-1}(\mathrm{Res}_\partial\left[L(\partial), \delta(w-\partial)\circ f\right])\right]\delta(z-\partial)\right) = \\
&= -\mathrm{Res}_\partial\left(\left[L(\partial), \partial^{-1}\,\mathrm{Res}_\partial\left(L(\partial)\delta(w-\partial)\circ f - \delta(w-\partial)\circ f L(\partial)\right)\right]\delta(z-\partial)\right).
\end{aligned} \tag{3.28}$$

By Lemma A.1, we have $\mathrm{Res}_\partial\left(L(\partial)\delta(w,\partial)\circ f - \delta(w,\partial)\circ fL(\partial)\right) = \left(L(w) - L^*(\partial - w)\right)f$, thus obtaining,

$$(3.28) = \mathrm{Res}_\partial\left(\left(L(\partial)\partial^{-1}\left(L(w) - L^*(\partial-w)\right)f - \partial^{-1}\left(L(w) - L^*(\partial-w)\right)fL(\partial)\right)\delta(z-\partial)\right).$$

Applying again Lemma (A.1) to $\delta(z-\partial)$ the above expression in equivalent to the coefficient of d in (3.27), proving the Lemma. $\square$

Let $\mathcal{V}$ be an algebra of differential functions in infinitely many variables u_i, $i \in I$ and L be the general pseudodifferential operator we associate to it. We write $H^{(L,d)}(\partial) = H^{(L)}(\partial) + dH^{(d)}(\partial)$.

Theorem 3.6. *The operators $H^{(L,d)}(\partial)$ are Hamiltonian operators, namely, the pair $(H^{(L)}(\partial), H^{(d)})$ defines a bi-Hamiltonian structure on $\mathcal{V}$.*

Proof. We set $L(z) = \sum_{i \geq -N-1} u_i z^{-i-1}$ be the symbol of the pseudodifferential operator $L(\partial)$. By (1.17) and Lemma 3.5

$$
\begin{aligned}
\{L(z)_\lambda L(w)\} &= H^{(L,d)}(\lambda)(w,z) = \\
&= H^{(L)}(\lambda)(z,w) + d\left(L(w+\lambda+\partial) - L(w)\right)(\lambda+\partial)^{-1}\left(L^*(\lambda - z) - L(z)\right)
\end{aligned} \tag{3.29}
$$

defines a λ-bracket among any pair of generators of $\mathcal{V}$ as already discussed in the proof of Theorem 3.3. We write the bracket in (3.5) as $\{\cdot_\lambda\cdot\} = \{\cdot_\lambda\cdot\}_L + d\{\cdot_\lambda\cdot\}_d$, where $\{\cdot_\lambda\cdot\}_L$ (rspectively $\{\cdot_\lambda\cdot\}_d$) is the λ-bracket corresponding to $H^{(L)}$ (respectively $H^{(0,d)}$). Since skew-commutativity of the bracket $\{\cdot_\lambda\cdot\}_d$ is evident and we proved that $\{\cdot_\lambda\cdot\}_L$ is skew-commutative in Theorem 3.3, the whole bracket (3.5) is skew-commutative. To conclude the proof, we are left to show that it satisfies Jacobi identity.

First we prove that $\{\cdot_\lambda\cdot\}_d$ satisfies Jacobi identity (3.14). We use the following notation

$$
\{L(z)_\lambda L(w)\}_d = p(\lambda, \partial_z, \partial_w)\left(L(z), L(w)\right), \tag{3.30}
$$

where $p(\lambda, \partial_z, \partial_w) = \left(e^{(\lambda+\partial)\partial_w} - 1\right)(\lambda+\partial)^{-1}\left(e^{(-\lambda-\partial)\partial_z} - 1\right) \in \mathcal{V}[[\lambda, \partial, \partial_z, \partial_w]]$ and we assume the convention that ∂ acts on the first component. Using (3.29), Leibniz rule and sesquilinearity we get

$$
\begin{aligned}
\{L(z)_\lambda\{L(w)_\mu L(t)\}_d\}_d = p(\lambda+\mu, \partial_w, \partial_t)\big(p(\lambda, \partial_z, \partial_w)\big(L(z), L(w)\big), L(t)\big) + &\qquad (3.31a)\\
+ p(\mu, \partial_w, \partial_t)\big(L(w), p(\lambda, \partial_z, \partial_t)\big(L(z), L(t)\big)\big); &\qquad (3.31b)
\end{aligned}
$$

$$
\begin{aligned}
\{L(w)_\mu\{L(z)_\lambda L(t)\}_d\}_d = p(\lambda+\mu, \partial_z, \partial_t)\big(p(\mu, \partial_w, \partial_z)\big(L(w), L(z)\big), L(t)\big) + &\qquad (3.32a)\\
+ p(\lambda, \partial_z, \partial_t)\big(L(z), p(\mu, \partial_w, \partial_t)\big(L(w), L(t)\big)\big); &\qquad (3.32b)
\end{aligned}
$$

$$
\begin{aligned}
\{\{L(z)_\lambda L(w)\}_{d\,\lambda+\mu} L(t)\}_d = p(\lambda+\mu, \partial_z, \partial_t)\big(p(-\mu, \partial_z, \partial_w)\big(L(z), L(w)\big), L(t)\big) + &\qquad (3.33a)\\
+ p(\lambda+\mu, \partial_w, \partial_t)\big(p(\lambda, \partial_z, \partial_w)\big(L(z), L(w)\big), L(t)\big). &\qquad (3.33b)
\end{aligned}
$$

Since $p(-\mu, \partial_z, \partial_w)(L(z), L(w)) = -p(\mu, \partial_w, \partial_z)(L(w), L(z))$, (3.32a) and (3.33a) cancel out in (3.14). Furthermore, expanding (3.30), we get

$$
\begin{aligned}
& p(\lambda, \partial_z, \partial_t)\big(L(z), p(\mu, \partial_w, \partial_t)\big(L(w), L(t)\big)\big) = \\
&= p(\lambda, \partial_z, \partial_t)\big(L(z), \left(L(t+\mu+\partial) - L(w)\right)(\mu+\partial)^{-1}\left(L^*(\mu - w) - L(w)\right)\big) = \\
&= \left(L(t+\lambda+\mu+\partial) + L(t)\right)\left((\lambda+\partial)^{-1}\left(L^*(\lambda - z) - L(z)\right)\right)(\mu+\partial)^{-1}\left(L^*(\mu - w) - L(w)\right) - \\
&- \left((\lambda+\partial)^{-1}\left(L^*(\lambda - z) - L(z)\right)\right)L(t+\mu+\partial)(\mu+\partial)^{-1}\left(L^*(\mu - w) - L(w)\right) - \\
&- \left((\mu+\partial)^{-1}\left(L^*(\mu - w) - L(w)\right)\right)L(t+\lambda+\partial)(\lambda+\partial)^{-1}\left(L^*(\lambda - z) - L(z)\right) = \\
&= p(\mu, \partial_w, \partial_t)\big(L(w), \left(L(t+\lambda+\partial) - L(t)\right)(\lambda+\partial)^{-1}\left(L^*(\lambda - z) - L(z)\right)\big) = \\
&= p(\mu, \partial_w, \partial_t)\big(L(w), p(\lambda, \partial_z, \partial_t)\big(L(z), L(t)\big)\big).
\end{aligned}
$$

Hence, also (3.31b) and (3.32b) cancel out in Jacobi identity. It remains to note that (3.31a) and (3.33b) coincide, thus proving Jacobi identity for $\{\cdot_\lambda\cdot\}_d$.

Expanding (3.14) and using the fact that both $\{\cdot_\lambda\cdot\}_L$ and $\{\cdot_\lambda\cdot\}_d$ satisfy Jacobi identity, to conclude the proof we are left to prove that

$$
\begin{aligned}
&\{L(z)_\lambda\{L(w)_\mu L(t)\}_L\}_d + \{L(z)_\lambda\{L(w)_\mu L(t)\}_d\}_L - \{L(w)_\mu\{L(z)_\lambda L(t)\}_L\}_d - \\
&- \{L(w)_\mu\{L(z)_\lambda L(t)\}_d\}_L - \{\{L(z)_\lambda L(w)\}_{L\,\lambda+\mu} L(t)\}_d - \{\{L(z)_\lambda L(w)\}_{d\,\lambda+\mu} L(t)\}_L = 0.
\end{aligned} \tag{3.34}
$$

Using (3.29), Leibniz rule and sesquilinearity, we get

$$\{L(z)_\lambda \{L(w)_\mu L(t)\}_L\}_d =$$
$$= p(\lambda, \partial_z, \partial_t)\big(L(z), L(t+\lambda+\partial)\big) i_w(w-t-\mu-\partial)^{-1} L^*(\mu-w) + \tag{3.35a}$$
$$+ L(t+\lambda+\mu+\partial) i_w(w-t-\lambda-\mu-\partial)^{-1} e^{(-\lambda-\mu-\partial)\partial_w} p(\lambda, \partial_z, \partial_w)\big(L(z), L(w)\big) - \tag{3.35b}$$
$$- p(\lambda, \partial_z, \partial_w)\big(L(z), L(w)\big) i_w(w-t-\mu-\partial)^{-1} L(t) - \tag{3.35c}$$
$$- L(w) i_w(w-t-\lambda-\mu-\partial)^{-1} p(\lambda, \partial_z, \partial_t)\big(L(z), L(t)\big); \tag{3.35d}$$
$$\{L(z)_\lambda \{L(w)_\mu L(t)\}_d\}_L =$$
$$= p(\mu, \partial_w, \partial_t)\left(L(w), L(t+\lambda+\partial) i_z(z-t-\lambda-\partial)^{-1} L^*(\lambda-z)\right) - \tag{3.35e}$$
$$- L(z) p(\mu, \partial_w, \partial_t)\left(L(w), i_z(z-t-\lambda-\partial)^{-1} L(t)\right) + \tag{3.35f}$$
$$+ p(\lambda+\mu+\partial, \partial_w, \partial_t)_\rightarrow \left(L(w+\lambda+\partial) i_z(z-w-\lambda-\partial)^{-1}, L(t)\right) L^*(\lambda-z) - \tag{3.35g}$$
$$- p(\lambda+\mu+\partial, \partial_w, \partial_t)_\rightarrow \left(i_z(z-w-\lambda-\partial)^{-1} L(w), L(t)\right) L(z), \tag{3.35h}$$

where the arrow means that ∂ is acting on the right. Furthermore, we have

$$\{L(w)_\mu \{L(z)_\lambda L(t)\}_L\}_d =$$
$$= p(\mu, \partial_w, \partial_t)\big(L(w), L(t+\lambda+\partial)\big) i_z(z-t-\lambda-\partial)^{-1} L^*(\lambda-z) + \tag{3.36a}$$
$$+ L(t+\lambda+\mu+\partial) i_z(z-t-\lambda-\mu-\partial)^{-1} e^{(-\lambda-\mu-\partial)\partial_w} p(\mu, \partial_w, \partial_z)\big(L(w), L(z)\big) - \tag{3.36b}$$
$$- p(\mu+\partial, \partial_w, \partial_z)\big(L(w), L(z)\big) i_z(z-t-\lambda-\partial)^{-1} L(t) - \tag{3.36c}$$
$$- L(z) i_z(z-t-\lambda-\mu-\partial)^{-1} p(\mu, \partial_w, \partial_t)\big(L(w), L(t)\big); \tag{3.36d}$$
$$\{L(w)_\mu \{L(z)_\lambda L(t)\}_d\} =$$
$$= p(\lambda, \partial_z, \partial_t)\left(L(z), L(t+\mu+\partial) i_w(w-t-\mu-\partial)^{-1}\right) L^*(\mu-w) - \tag{3.36e}$$
$$- L(w) p(\lambda, \partial_z, \partial_t)\left(L(z), i_w(w-t-\mu-\partial)^{-1} L(t)\right) + \tag{3.36f}$$
$$+ p(\lambda+\mu+\partial, \partial_z, \partial_t)_\rightarrow \left(L(z+\mu+\partial), i_w(w-z-\mu-\partial)^{-1} L(t)\right) L(\mu-w) - \tag{3.36g}$$
$$- p(\lambda+\mu+\partial, \partial_z, \partial_t)_\rightarrow \left(i_w(w-z-\mu-\partial)^{-1} L(z), L(t)\right) L(w); \tag{3.36h}$$

$$\{\{L(z)_\lambda L(w)\}_{\lambda+\mu} L(t)\}_d =$$
$$= p(\lambda+\mu+\partial, \partial_w, \partial_t)_\rightarrow \big(L(w+\lambda+\partial), L(t)\big) i_z(z-w-\lambda-\partial)^{-1} L^*(\lambda-z) + \tag{3.37a}$$
$$+ p(\lambda+\mu+\partial, \partial_z, \partial_t)_\rightarrow \big(L(z+\mu+\partial), L(t)\big) i_z(z-w+\mu+\partial)^{-1} L^*(\mu-w) - \tag{3.37b}$$
$$- p(\lambda+\mu+\partial, \partial_z, \partial_t)_\rightarrow \big(L(z), L(t)\big) i_z(z-w-\lambda-\partial)^{-1} L(w) - \tag{3.37c}$$
$$- p(\lambda+\mu+\partial, \partial_w, \partial_t)_\rightarrow \big(L(w), L(t)\big) i_z,(z-w+\mu+\partial)^{-1} L(z); \tag{3.37d}$$
$$\{\{L(z)_\lambda L(w)\}_{d\,\lambda+\mu} L(t)\} =$$
$$= L(t+\lambda+\mu+\partial) p(-\mu, \partial_z, \partial_w)\left(i_z(z-t-\lambda-\mu-\partial)^{-1} L^*(\lambda+\mu+\partial-z), L(w)\right) - \tag{3.37e}$$
$$- L(t) p(-\mu, \partial_z, \partial_w)\left(L(z) i_z(z-t-\lambda-\mu-\partial)^{-1}, L(w)\right) + \tag{3.37f}$$
$$+ L(t+\lambda+\mu+\partial) p(\lambda, \partial_z, \partial_w)\left(L(z), {}_\leftarrow e^{(-\lambda-\mu-\partial)\partial_w} i_w(w-t) L(w)\right) - \tag{3.37g}$$
$$- L(t) p(\lambda, \partial_z, \partial_w)\left(L(z), i_w(w-t-\lambda-\mu-\partial)^{-1} L(w)\right) L(t); \tag{3.37h}$$

where the left arrow, means that ∂ acts on the left, namely on the coefficients of $L(t)$ and $L(z)$

We claim that

$$(3.35a) + (3.35b) = (3.36e) + (3.37g).$$

Indeed, using the expansion of (3.30) we get

$$\begin{aligned} LHS = {} & L(t+\lambda+\mu+\partial)\left((\lambda+\partial)^{-1}\left(L^*(\lambda-z) - L(z)\right)\right) i_w(w-t-\mu-\partial)^{-1} L^*(\mu-w) - \\ & - \left((\lambda+\partial)^{-1}\left(L^*(\lambda-z) - L(z)\right)\right) L(t+\mu+\partial) i_w(w-t-\mu-\partial)^{-1} L^*(\mu-w) + \\ & + L(t+\lambda+\mu+\partial) i_w(w-t-\lambda-\mu-\partial)^{-1} L^*(\mu-w)(\lambda+\partial)^{-1}\left(L^*(\lambda-z) - L(z)\right) - \\ & - L(t+\lambda+\mu+\partial) i_w(w-t-\lambda-\mu-\partial)^{-1} L^*(\lambda+\mu+\partial-w)(\lambda+\partial)^{-1}\left(L^*(\lambda-z) - L(z)\right). \end{aligned}$$

On the other hand we have

$$
\begin{aligned}
RHS = &+L(t+\lambda+\mu+\partial)i_w(w-t-\lambda-\mu-\partial)^{-1}L^*(\mu-w)(\lambda+\partial)^{-1}\left(L^*(\lambda-z)-L(z)\right)-\\
&-\left((\lambda+\partial)^{-1}\left(L^*(\lambda-z)-L(z)\right)\right)L(t+\mu+\partial)i_w(w-t-\mu-\partial)^{-1}L^*(\mu-w)+\\
&+L(t+\lambda+\mu+\partial)\left((\lambda+\partial)^{-1}\left(L^*(\lambda-z)-L(z)\right)\right)i_w(w-t-\mu-\partial)^{-1}L^*(\mu-w)-\\
&-L(t+\lambda+\mu+\partial)i_w(w-t-\lambda-\mu-\partial)^{-1}L^*(\lambda+\mu+\partial-w)(\lambda+\partial)^{-1}\left(L^*(\lambda-z)-L(z)\right).
\end{aligned}
$$

In the same way it can be proved that (3.35c) + (3.35d) = (3.36f) + (3.37h). Indeed, we have

$$
\begin{aligned}
LHS = &-\left(L(w+\lambda+\partial)(\lambda+\partial)^{-1}\left(L^*(\lambda-z)-L(z)\right)\right)i_w(w-t-\mu-\partial)^{-1}L(t)+\\
&+L(w)\left((\lambda+\partial)^{-1}\left(L^*(\lambda-z)-L(z)\right)\right)i_w(w-t-\mu-\partial)L(t)-\\
&-L(w)i_w(w-t-\lambda-\mu-\partial)^{-1}L(t+\lambda+\partial)(\lambda+\partial)^{-1}\left(L^*(\lambda-z)-L(z)\right)+\\
&+L(w)i_w(w-t-\lambda-\mu-\partial)^{-1}L(t)(\lambda+\partial)^{-1}\left(L^*(\lambda-z)-L(z)\right),
\end{aligned}
$$

While, on the other hand we have

$$
\begin{aligned}
RHS = &-L(w)i_w(w-t-\lambda-\mu-\partial)^{-1}L(t+\lambda+\partial)(\lambda+\partial)^{-1}\left(L^*(\lambda-z)-L(z)\right)+\\
&+L(w)\left((\lambda+\partial)^{-1}\left(L^*(\lambda-z)-L(z)\right)\right)i_w(w-t-\mu-\partial)L(t)-\\
&-\left(L(w+\lambda+\partial)(\lambda+\partial)^{-1}\left(L^*(\lambda-z)-L(z)\right)\right)i_w(w-t-\mu-\partial)^{-1}L(t)+\\
&+L(w)i_w(w-t-\lambda-\mu-\partial)^{-1}L(t)(\lambda+\partial)^{-1}\left(L^*(\lambda-z)-L(z)\right).
\end{aligned}
$$

Interchanging z and w and λ and μ in the last computations and using the fact that $p(-\mu-\partial,\partial_z,\partial_w)=-p(\mu+\partial,\partial_w,\partial_z)$, it follows that (3.35e) = (3.36a) + (3.36b) + (3.37e) and (3.35f) = (3.36c) + (3.36d) + (3.37f)

Next, we compute (3.35g) − (3.36g) − (3.37a) − (3.37b). We have, using the expansion of (3.30),

$$
\begin{aligned}
&(3.35g) =\\
&=\left(L(t+\lambda+\mu+\partial)-L(t)\right)(\lambda+\mu+\partial)^{-1}\left(i_w(z-w+\mu+\partial)^{-1}L^*(\mu-w)\right)L^*(\lambda-z)-\\
&-\left(L(t+\lambda+\mu+\partial)-L(t)\right)(\lambda+\mu+\partial)^{-1}L(w+\lambda+\partial)i_z(z-w-\lambda-\partial)^{-1}L^*(\lambda-z).
\end{aligned}
$$

On the other hand, using the same expansion, we get

$$
\begin{aligned}
&(3.37a) =\\
&=\left(L(t+\lambda+\mu+\partial)-L(t)\right)(\lambda+\mu+\partial)^{-1}L^*(\mu-w)i_z(z-w-\lambda-\partial)^{-1}L^*(\lambda-z)- \qquad (3.38)\\
&-\left(L(t+\lambda+\mu+\partial)-L(t)\right)(\lambda+\mu+\partial)^{-1}L(w+\lambda+\partial)i_z(z-w-\lambda-\partial)^{-1}L^*(\lambda-z) \qquad (3.39)
\end{aligned}
$$

and

$$
\begin{aligned}
&(3.37b) =\\
&=\left(L(t+\lambda+\mu+\partial)-L(t)\right)(\lambda+\mu+\partial)^{-1}L^*(\lambda-z)i_z(z-w+\mu+\partial)^{-1}L^*(\mu-w)- \qquad (3.40)\\
&-\left(L(t+\lambda+\mu+\partial)-L(t)\right)(\lambda+\mu+\partial)^{-1}L(z+\mu+\partial)i_z(z-w+\mu+\partial)L^*(\mu-w). \qquad (3.41)
\end{aligned}
$$

Thus, we get (3.37g) − (3.39) − (3.40) = 0. Moreover, expanding (3.36g), using the definition of (3.30), we get

$$
\begin{aligned}
&=\left(L(t+\lambda+\mu+\partial)-L(t)\right)(\lambda+\mu+\partial)^{-1}\left(i_w(w-z+\lambda+\partial)L^*(\lambda-z)\right)L^*(\mu-w)- \qquad (3.42)\\
&-\left(L(t+\lambda+\mu+\partial)-L(t)\right)(\lambda+\mu+\partial)^{-1}L(z+\mu+\partial)i_w(w-z-\mu-\partial)^{-1}L^*(\mu-w). \qquad (3.43)
\end{aligned}
$$

Hence, we obtain, using the definition of the formal δ-function in −(3.36g) − (3.38) and −(3.43) − (3.41),

$$
\begin{aligned}
&(3.35g)-(3.36g)-(3.37a)-(3.37b) =\\
&=-\left(L(t+\lambda+\mu+\partial)-L(t)\right)(\lambda+\mu+\partial)^{-1}L^*(\mu-w)\delta(z,w+\lambda+\partial)L^*(\lambda-z)+ \qquad (3.44)\\
&+\left(L(t+\lambda+\mu+\partial)-L(t)\right)(\lambda+\mu+\partial)^{-1}L(z+\mu+\partial)\delta(w,z+\mu+\partial)L^*(\mu-w).
\end{aligned}
$$

We can use a similar procedure to compute (3.35h) − (3.36h) − (3.37c) − (3.37d). Expanding, we have

$$\begin{aligned}
(3.35h) = &-(L(t+\lambda+\mu+\partial)-L(t))(\lambda+\mu+\partial)^{-1}L^*(\lambda+\mu+\partial-w)i_z(z-w+\mu+\partial)L(z)+ && (3.45)\\
&+(L(t+\lambda+\mu+\partial)-L(t))(\lambda+\mu+\partial)^{-1}L(z)i_z(z-w-\lambda-\partial)^{-1}L(w); && (3.46)\\
(3.37c) = &\,(L(t+\lambda+\mu+\partial)-L(t))(\lambda+\mu+\partial)^{-1}L(z)i_z(z-w-\lambda-\partial)^{-1}L(w); && (3.47)\\
&-(L(t+\lambda+\mu+\partial)-L(t))(\lambda+\mu+\partial)^{-1}L^*(\lambda+\mu+\partial-z)i_z(z-w-\lambda-\partial)^{-1}L(w) && (3.48)\\
(3.37d) = &\,(L(t+\lambda+\mu+\partial)-L(t))(\lambda+\mu+\partial)^{-1}L(w)i_z(z-w+\mu+\partial)^{-1}L(z), && (3.49)\\
&-(L(t+\lambda+\mu+\partial)-L(t))(\lambda+\mu+\partial)^{-1}L^*(\lambda+\mu+\partial-w)i_z(z-w+\mu+\partial)^{-1}L(z) && (3.50)
\end{aligned}$$

then (3.35h) − (3.47) − (3.50) = 0. Moreover,

$$\begin{aligned}
(3.36h) = &\,(L(t+\lambda+\mu+\partial)-L(t))(\lambda+\mu+\partial)^{-1}L(w)i_w(w-z-\mu-\partial)^{-1}L(z)- && (3.51)\\
&-(L(t+\lambda+\mu+\partial)-L(t))(\lambda+\mu+\partial)^{-1}L^*(\lambda+\mu+\partial-z)i_w(w-z+\lambda+\partial)^{-1}L(w), && (3.52)
\end{aligned}$$

then, using the formal δ-function in −(3.52) − (3.48) and (3.51) − (3.49), we obtain

$$\begin{aligned}
&(3.35h)-(3.36h)-(3.37c)-(3.37d)=\\
&=(L(t+\lambda+\mu+\partial)-L(t))(\lambda+\mu+\partial)^{-1}L^*(\lambda+\mu+\partial-z)\delta(z,w+\lambda+\partial)L(w)+ \qquad (3.53)\\
&-(L(t+\lambda+\mu+\partial)-L(t))(\lambda+\mu+\partial)^{-1}L(w)\delta(w,z+\mu+\partial)L(z).
\end{aligned}$$

Using the properties of the formal delta-function (Lemma A.1) it follows easily that the terms (3.44) and (3.53) cancel out, thus proving that identity (3.34) holds. □

We denote this Poisson vertex algebra $\mathcal{V}_d^{(N,\infty)}$. Let us assume $\mathrm{ord}(L)=N\in\mathbb{Z}_+$ and write, for any $c\in\mathbb{C}$, $H^{(L-c,d)}(\partial)=H^{(0)}(\partial)-cH^{(\infty)}(\partial)+dH^{(d)}(\partial)$, after expanding (3.26).

Corollary 3.7. *The operators $H^{(L-c,d)}$ are Hamiltonian operators, namely, they define a tri-Hamiltonian structure $(H^{(0)},H^{(\infty)},H^{(d)})$ on $\mathcal{V}$.*

Proof. Apply the same argument of the proof of Corollary 3.4. □

We denote this tri-Poisson vertex algebra $\mathcal{V}_{c,d}^{(N,\infty)}$.

Let us come back to the general pseudodifferential operator case and set $\mathcal{J}=\langle u_{-N}\rangle_{\mathcal{V}}$ to be the differential ideal in $\mathcal{V}$ generated by the differential variable u_{-N}.

Proposition 3.8. *$\mathcal{J}$ is a Poisson vertex algebra ideal in $\mathcal{V}_{\frac{1}{N}}^{(N,\infty)}$.*

Proof. We have

$$\begin{aligned}
A^{(L,\frac{1}{N})}(F)&=L(FL)_+-(LF)_+L-\frac{1}{N}[L,\partial^{-1}(\mathrm{Res}_\partial[L,F])]=\\
&=(LF)_-L-L(FL)_--\frac{1}{N}[L,\partial^{-1}(\mathrm{Res}_\partial[L,F])]=\\
&=(LF)_-\partial^N-\partial^N(FL)_--\frac{1}{N}[\partial^N,\partial^{-1}(\mathrm{Res}_\partial[L,F])]+o(\partial^{N-2})=\\
&=[L,F]_-\partial^N-\mathrm{Res}_\partial[L,F]\partial^{N-1}+o(\partial^{N-2}),
\end{aligned}$$

from which we derive that the coefficient of ∂^{N-1} vanishes, hence $A^{(L,\frac{1}{N})}(F)\in\mathcal{A}((\partial^{-1}))_{N-2}$. It follows that $H_{-N,j}^{(L,\frac{1}{N})}=0$ for all $j\geq -N$. By (1.17), it means that $\{u_j{}_\lambda u_{-N}\}=0$ for all $j\geq -N$. By skew-commutativity and Jacobi identity we see that u_{-N} is central in $\mathcal{V}_{\frac{1}{N}}^{(N,\infty)}$. □

We get that $\widehat{\mathcal{V}}^{(N,\infty)}\cong\mathcal{V}_{\frac{1}{N}}^{(N,\infty)}/\mathcal{J}$ has an induced structure of Poisson vertex algebra. For $N\in\mathbb{Z}_+$, the same proof of Proposition 3.8 holds in the case of the bi-Poisson vertex algebra $\mathcal{V}_{c,\frac{1}{N}}^{(N,\infty)}$, just replacing L by $L-c$. Thus, $\widehat{\mathcal{V}}_c^{(N,\infty)}\cong\mathcal{V}_{c,\frac{1}{N}}^{(N,\infty)}/\mathcal{J}$ has an induced structure of bi-Poisson vertex algebra.

3.4. Poisson vertex algebra structure attached to a general differential operator: Gelfand-Dickey algebras

Now we want to derive a Poisson vertex algebra structure from a monic differential operator of order $N \in \mathbb{Z}$. Let $\mathcal{A}$ be a differential algebbra with a derivation ∂ and use identifications (3.4) and (3.5) to consider the map $A^{(L)}$, defined in (3.7), for $L \in \mathcal{A}((\partial^{-1}))$.

Let us assume $L \in \mathcal{A}[\partial]$, with $\mathrm{ord}(L) = N \in \mathbb{Z}$. If $F \in \mathbb{C}[\partial, \partial^{-1}] \otimes \mathcal{A}$, then $A^{(L)}(F) = L(FL)_+ - (LF)_+L \in \mathcal{A}[\partial]$. On the other hand, we know that $A^{(L)}(F) \in \mathcal{A}((\partial^{-1}))_{N-1}$. Thus, $A^{(L)} \in \mathcal{A}[\partial]_{N-1} = \mathcal{A}[\partial] \cap \mathcal{A}((\partial^{-1}))_{N-1}$. Moreover, if $F \in \mathbb{C}[\partial] \circ \mathcal{A}$, then $(FL)_+ = FL$ and $(LF)_+ = LF$, showing that $A^{(L)}(F) = 0$. On the other hand, we showed that $A^{(L)}(F) = 0$, for $F \in \partial^{-N-1}\mathbb{C}[\partial^{-1}] \circ \mathcal{A}$. This means that $A^{(L)}$ induces a map

$$A^{(L)} : (\mathbb{C}[\partial^{-1}] \circ \mathcal{A})/(\partial^{-N-1}\mathbb{C}[\partial^{-1}] \circ A) \longrightarrow \mathcal{A}[\partial]_{N-1}.$$

Using identifications (3.8) and (3.9), we define $H^{(L)}(\partial) \in \mathrm{Mat}_{N\times N}(\mathcal{A}[\partial])$ by

$$\begin{tikzcd}
(\mathbb{C}[\partial^{-1}] \circ \mathcal{A})/(\partial^{-N-1}\mathbb{C}[\partial^{-1}] \circ A) \arrow[r, "A^{(L)}"] \arrow[d, leftrightarrow, "\wr"'] & \mathcal{A}[\partial]_{N-1} \arrow[d, leftrightarrow, "\wr"] \\
\mathcal{A}^{\oplus N} \arrow[r, "H^{(L)}(\partial)"] & \mathcal{A}^N.
\end{tikzcd}$$

Remark 3.9. The same proof of Lemma 3.2 applies in this case, then the expresion of $H^{(L)}(\partial)(z, w)$ is the same we got in Lemma 3.2. The only difference is that in this case we get an element in $\mathcal{A}[\partial][z, w]$, instead of a Laurent series in z^{-1} and w^{-1}.

Let $\mathcal{V}$ be an algebra of differential functions in the variables u_i, $i \in \{-N, \dots, -1\}$. We define

$$L_+^{(N)} = L = \partial^N + u_{-N}\partial^{N-1} + u_{-N+1}\partial^{N-2} + \dots + u_{-1} = \sum_{k=0}^{N} u_{-k-1}\partial^k \in \mathcal{V}[\partial]$$

to be the differential operator of general type associated to $\mathcal{V}$.

Theorem 3.10. *The operator $H^{(L)}(\partial)$ is an Hamiltonian operator, namely, it defines a Poisson vertex algebra structures on $\mathcal{V}$.*

Proof. After setting $L(z) = \sum_{i=-N}^{-1} u_i z^{-i-1}$ for the symbol of the operator $L(\partial)$, using (1.17) and Remark 3.9, we get

$$\{L(z)_\lambda L(w)\} = L(w)(w + \lambda + \partial)i_z(z - w - \lambda - \partial)^{-1}L^*(\lambda - z) - L(z)i_z(z - w - \lambda - \partial)^{-1}(w). \quad (3.54)$$

The formal proof of Theorem 3.3 holds when $L(z)$ is a Laurent series in z^{-1}, so it works also for $L(z) \in \mathcal{V}[z]$. $\square$

We denote the Poisson vertex algebra structure we got $\mathcal{W}_N$. We want to give another description of $\mathcal{W}_N$. Namely, we want to realize this algebra as a quotient of the Poisson vertex algebra $\mathcal{V}^{(N,\infty)}$. We set

$$\mathcal{J}_+ = \langle u_i \mid i \in \mathbb{Z}_+\rangle_{\mathcal{V}^{(N,\infty)}}.$$

As differential algebras, we have $\mathcal{W}_N \cong \mathcal{V}^{(N,\infty)}/\mathcal{J}_+$.

Proposition 3.11. *$\mathcal{J}_+$ is a Poisson vertex algebra ideal in $\mathcal{V}^{(N,\infty)}$.*

Proof. Let us consider the setup of Lemma 3.2 (now we are considering $L = L^{(N)}$ a general pseudodifferential operator) and compute the generating series for the operators $H_{ij}^{(L)}$, for $i \in \mathbb{Z}_+$ and $j \in \mathbb{Z}$. Using the definition of the formal δ-function and Lemma A.1 to perform a computation silmilar to that one in the proof of Lemma 3.2, we get

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}_+, j\in\mathbb{Z}} H_{ij}^{(L)}(\partial)(f)z^{-i-1}w^{-j-1} = \sum_{i\in\mathbb{Z}_+, j\in\mathbb{Z}} \mathrm{Res}_\partial \left(A^{(L)}(\partial^j \circ f)\partial^i\right) z^{-i-1}w^{-j-1} \\
&= \sum_{i\in\mathbb{Z}_+, j\in\mathbb{Z}} \mathrm{Res}_\partial \left(\left(L(\partial)(\partial^j \circ fL(\partial))_+ - (L(\partial)\partial^j \circ f)_+L(\partial)\right)\partial^i\right) z^{-i-1}w^{-j-1} = \\
&= \mathrm{Res}_\partial \left(\left(L(\partial)(\delta(w, \partial) \circ fL(\partial))_+ - (L(\partial)\delta(w, \partial) \circ f)_+L(\partial)\right) i_z(z - \partial)^{-1}\right) = \\
&= \mathrm{Res}_\partial \left(\left(L_-(\partial)i_w(w - \partial)^{-1} \circ (L^*(\partial - w)f) - L(w)i_w(w - \partial)^{-1} \circ f)_+L_-(\partial)\right) i_z(z - \partial)^{-1}\right),
\end{aligned}$$

where in the last equality the appeareance of $L_-(\partial))$ is due to the fact that both $i_z(z-\partial)^{-1}$ and $i_w(w-\partial)^{-1}$ have only non negative powers of ∂, thus $L_+(\partial)$ does not contribute to the residue. We note that $L_-(\partial) \in \mathcal{J}_+((\partial^{-1}))$, then $\sum_{i\in\mathbb{Z}_+, j\in\mathbb{Z}} H_{ij}^{(L)} z^{-i-1}w^{w-j-1} \in \mathcal{J}_+[\partial][[z^{-1}]]((w^{-1}))$. Using (1.17), we get $\{u_j{}_\lambda u_i\} \in \mathcal{J}_+[\lambda]$, for all $i \in \mathbb{Z}_+$ and $j \in \mathbb{Z}$. By skew-commutativity and Leibniz rule it follows that $\{\mathcal{V}^{(N,\infty)}{}_\lambda \mathcal{J}_+\}, \{\mathcal{J}_+{}_\lambda \mathcal{V}^{(N,\infty)})\} \subset \mathcal{J}_+[\lambda]$, thus concluding the proof. $\square$

We obtain that the quotient space $\mathcal{W}_N \cong \mathcal{V}^{(N,\infty)}/\mathcal{J}_+$ has an induced structure of Poisson vertex algebra. The induced λ-bracket on the quotient is given by Lemma 3.2 after imposing $L_-(\partial) = 0$, thus coincides with the λ-bracket defined by the operator $H^{(L)}(\partial)$ when L is a general differential operator.

When $\mathrm{ord}(L) = N \in \mathbb{Z}_+$, Theorem 3.10 and Proposition 3.11 also apply in the case of the operator $L - c$, for $c \in \mathbb{C}$ (see Corollary 3.4). We denote by $\mathcal{W}_{N,c}$ the bi-Poisson vertex algebra structure we get from the operators $H^{(L-c)}$.

Definition 3.12. The bi-Poisson vertex algebra $\mathcal{W}_{N,c}$ is called *Gelfand-Dickey algebra*.

Sometimes this algebra is also reffered as Gelfand-Dickey algebra of $\mathfrak{gl}_N$ type. Indeed, in next chapter, it will be proved that this Poisson vertex algebra is isomorphic to $\mathcal{W}_z(\mathfrak{gl}_N, f, s)$, where $f \in \mathfrak{gl}_N$ is principal nilpotent.

We want to apply similar considerations to $A^{(L,d)}$, when $L \in \mathcal{A}[\partial]$. As usual, we let $N = \mathrm{ord}(L) \in \mathbb{Z}$. It can be easily show that $A^{(L,d)} : (\mathbb{C}[\partial^{-1}] \circ \mathcal{A})/(\partial^{-N-1}\mathbb{C}[\partial^{-1}] \circ \mathcal{A}) \longrightarrow \mathcal{A}[\partial]_{N-1}$. Using identifications (3.4) and (3.5), we get $H^{(L,d)} \in \mathrm{Mat}_{(N)\times(N)}\, \mathcal{V}[\partial]$ by

$$
\begin{array}{ccc}
(\mathbb{C}[\partial^{-1}] \circ \mathcal{A})/(\partial^{-N-1}\mathbb{C}[\partial^{-1}] \circ \mathcal{A}) & \xrightarrow{A^{(L,d)}} & \mathcal{A}[\partial]_{N-1} \\
\wr\updownarrow & & \updownarrow\wr \\
\mathcal{A}^{\oplus N} & \xrightarrow{\quad H^{(L,d)}(\partial) \quad} & \mathcal{A}^N.
\end{array}
$$

Remark 3.13. The same proof of Lemma 3.5 applies in this case, then the expression of $H^{(L,d)}(\partial)(z,w)$ is the same we got in Lemma 3.5. The only difference is that in this case we get an element in $\mathcal{A}[\partial][z,w]$, instead of a Laurent series in z^{-1} and w^{-1}.

Let $\mathcal{V}$ be an algebra of differential functions in the variables u_i, $i \in \{-N, \dots, -1\}$ and let L be the general differential operator we attach to $\mathcal{V}$.

Theorem 3.14. *The operators $H^{(L,d)}(\partial)$ are Hamiltonian operators, namely, they define a bi-Hamiltonian structure $(H^{(0)}, H^{(d)})$ on $\mathcal{V}$.*

Proof. We can apply the same considerations we made in the proof of Theorem 3.10 to our case, showing that the formal proof of Theorem 3.6 still works in this case. $\square$

We denote the bi-Poisson vertex algebra we got by $\mathcal{W}_{N,d}$. By Proposition 3.8 we get that $\mathcal{J} = \langle u_{-N}\rangle_{\mathcal{V}}$ is a Poisson vertex algebra ideal for $\mathcal{W}_{N,\frac{1}{N}}$. We denote the quotient Poisson vertex algebra $\widehat{\mathcal{W}}_N \cong \mathcal{W}_{N,\frac{1}{N}}/\mathcal{J}$. Moreover, when $N = \mathrm{ord}(L) \in \mathbb{Z}_+$, Theorem 3.14 applies to the operator $H^{(L-c,d)}$, for $c \in \mathbb{C}$. We denote the tri-Poisson vertex algebra we get by $\mathcal{W}_{N,c,d}$ in particular we denote by $\widehat{\mathcal{W}}_{N,c} \cong \mathcal{W}_{N,c,\frac{1}{N}}/\mathcal{J}$. The fact that this quotient is a Poisson vertex algebra can be easily deduced from Proposition 3.8.

The Poisson vertex algebra $\widehat{\mathcal{W}}_{N,c}$ is is also an algebra of Gelfand-Dickey type, sometimes called Gelfand-Dickey algebra of $\mathfrak{sl}_N$ type. In the next chapter it will be shown that it is isomorphic to $\mathcal{W}_z(\mathfrak{sl}_N, f, s)$, where $f \in \mathfrak{sl}_N$ is principal nilpotent.

We want to realize $\mathcal{W}_{N,d}$ as a quotient of the Poisson vertex algebra $\mathcal{V}_d^{(N,\infty)}$. As differential algebras, $\mathcal{W}_N \cong \mathcal{V}_d^{(N,\infty)}/\mathcal{J}_+$.

Proposition 3.15. *$\mathcal{J}_+$ is a Poisson vertex algebra ideal in $\mathcal{V}_d^{(N,\infty)}$.*

Proof. By Proposition 3.11 we know that $\mathcal{J}_+$ is a Poisson vertex algebra ideal for $\{\cdot{}_\lambda\cdot\}_L$. We need to prove that it is also a Poisson vertex algebra ideal for $\{\cdot{}_\lambda\cdot\}_d$. We take the generating series for the corresponding Hamiltonian operator, for $i \in \mathbb{Z}_-+$ and $j \in \mathbb{Z}$, and get, using the definiton and properties

(Lemma A.1) of the formal δ-function

$$\begin{aligned}&\mathrm{Res}_\partial\left(\left[L(\partial),\partial^{-1}(\mathrm{Res}_\partial\left[L(\partial),\delta(w,\partial)\circ f\right])\right]i_z(z-\partial)^{-1}\right)=\\&=\mathrm{Res}_\partial\left(\left[L(\partial),\partial^{-1}\left(L(w)-L^*(\partial-w)\right)f\right]i_z(z-\partial)^{-1}\right)=\\&=\mathrm{Res}_\partial\left(\left[L_-(\partial),\partial^{-1}\left(L(w)-L^*(\partial-w)\right)f\right]i_z(z-\partial)^{-1}\right),\end{aligned}$$

where we used the fact that $L_+(\partial)$ does not contribute to the residue, since $i_z(z-\partial)^{-1}$ has only non-negative powers of ∂. Using (1.17) and the fact that $L_-(\partial)\in\mathcal{J}_+[[\partial^{-1}]]$, it follows that $\{u_j{}_\lambda u_i\}_d\in\mathcal{J}_+[\lambda]$, for all $i\in\mathbb{Z}_+$ and $j\in\mathbb{Z}$. Using Leibniz rule it can be shown that $\{\mathcal{V}_d^{(N,\infty)}{}_\lambda\mathcal{J}_+\}_d\subset\mathcal{J}_+[\lambda]$. By skew-commutativity, also $\{\mathcal{J}_+{}_\lambda\mathcal{V}_d^{(N,\infty)}\}_d\subset\mathcal{J}_+[\lambda]$, thus concluding the proof. $\square$

We obtain that the quotient space $\mathcal{W}_{N,d}\cong\mathcal{V}_d^{(N,\infty)}/\langle\mathcal{J},\mathcal{J}_+\rangle$ has an induced structure of bi-Poisson vertex algebra. The induced λ-bracket on the quotient is given by Lemma 3.5 after imposing $L_-(\partial)=0$, thus coincides with the λ-bracket defined by the operator $H^{(L,d)}(\partial)$ when L is a general differential operator. Furthermore, by Proposition 3.8, $\widehat{\mathcal{W}}_{N,c}\cong\mathcal{V}_{c,\frac{1}{N}}^{(N,\infty)}/\langle\mathcal{J},\mathcal{J}_+\rangle$ has an induced structure of bi-Poisson vertex algebra.

3.5. Some examples

The Poisson vertex algebras $\widehat{\mathcal{W}}_N$ have been some of the first explicit realizations of classical $\mathcal{W}$-algebras. We perform some computations of the λ-bracket on the generators.

```
spiegare meglio la relazione con le $\c W$-algebre...qua o nell'introduzione
```

3.5.1. $N=2$: Virasoro-Magri and Gardner-Fadeev-Zakharov Poisson vertex algebras. Let us consider the case $N=2$. Performing an explicit computation of $H^{(L-c,\frac{1}{N})}$. We get

$$\{u_{-1}{}_\lambda u_{-1}\}_c=-(\partial+2\lambda)u_{-1}-\frac{1}{2}\lambda^3+c(2\lambda).$$

If we set $u=-u_{-1}$ and rescale c then u is the differential generator of a Gardner-Fadeev-Zakharov Poisson vertex algebra with respect to the Hamiltonian structure $H^{(L,\frac{1}{N})}$, while it generates a Virasoro-Magri Poisson vertex algebra of central charge $-\frac{1}{2}$ with respect to the Hamiltonian structure $H^{(\infty)}$ (see Examples 1.7 and 1.8).

3.5.2. $\widehat{\mathcal{W}}_N$ is a Poisson vertex algebra of conformal field theory type. We get these values of the λ-bracket on the generators of $\widehat{\mathcal{W}}_N$:

$$\{u_{-N+1}{}_\lambda u_{-N+1}\}_0=-(\partial+2\lambda)u_{-N+1}-\binom{N}{3}\lambda^3, \tag{3.55}$$

$$\{u_{-N+1}{}_\lambda u_i\}_0=-\left(\partial+(N+i+1)\,\lambda\right)+o(\lambda^2), \tag{3.56}$$

for $-N+2\le i\le-1$. If we set $L=-u_{-N+1}$, by (3.55), this element spans a Virasoro-Magri Poisson vertex subalgebra in $\widehat{\mathcal{W}}_N$ of central charge $\binom{N}{3}$. Moreover, according to (3.56), for $3\le k\le N$, u_{-N-1-k} is a quasiprimary field of conformal weight k with respect to L. In [11] it is shown how to construct $w_3,\dots,w_N$ such that w_k is a primary field of conformal weight k with respect to L and the differential algebra generated by $L,w_3,\dots,w_N$ is the same differential algebra generated by u_i, for $-N+1\le i\le-1$. Hence, $\widehat{\mathcal{W}}_N$ is a Poisson vertex algebra of conformal field theory type.

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aggiungere la definizione di algebra di vertice di Poisson ti tipo conforme nel primo
capitolo o qua?
```

3.6. The Kupershmidt-Wilson theorem and the Miura map

In this section we want to give another proof of the following theorem due to Kupershmidt and Wilson [20], that we restate according to our formalism.

Theorem 3.16 (Kupershmidt-Wilson Theorem). *Let $\mathcal{V}_N$ be an algebra of differential functions in the variables v_i, $i\in\{1,\dots,N\}$ and make $\mathcal{V}_N$ a Poisson vertex algebra defining $\{v_i{}_\lambda v_j\}=\delta_{ij}\lambda$ (see Example 1.7). Let $\mathcal{V}$ be an algebra of differential functions in the variables u_i, $i\in I=\{-N,\dots,-1\}$, L be the* *general differential operator attached to $\mathcal{V}$ and $\mathcal{W}_N$ be the Poisson vertex algebra structure on $\mathcal{V}$ obtained by $H^{(L)}$. We set*

$$L=\sum_{k=0}^{N}u_{-k-1}\partial^k=(\partial+v_N)(\partial+v_{N-1})\cdots(\partial+v_1). \tag{3.57}$$

Comparing coefficients of powers of ∂ in (3.57) gives a differential algebra inclusion

$$\varphi:\mathcal{W}_N\hookrightarrow\mathcal{V}_N. \tag{3.58}$$

This map is a Poisson vertex algebra homomorphism.

The map φ is called Miura map (or Miura transformation [22]). It allows us to express each differential variable u_i as a differential polynomial in $\mathcal{V}_N$. The original proof given in [20] is very involved and uses circulant matrices. A shorter proof can be found in [8].

We need the following result.

Lemma 3.17. *Let $\mathcal{V}_A$ (respectively $\mathcal{V}_B$) be an algebra of differential functions in the variables a_i, $i\geq -N$ (respectively b_i, $i\geq -M$). Using the generating series $a(z)=\sum_{i\geq -N}a_iz^{-i-1}$ and $b(z)=\sum_{i\geq -M}b_iz^{-i-1}$, we define a λ-bracket, $\{\cdot{}_\lambda\cdot\}_\otimes:\mathcal{V}_A\otimes\mathcal{V}_B\longrightarrow(\mathcal{V}_A\otimes\mathcal{V}_B)[\lambda]$ in the following way:*

- $\{a(z){}_\lambda a(w)\}_\otimes=\{a(z){}_\lambda a(w)\}_L+d\{a(z){}_\lambda a(w)\}_d$;
- $\{b(z){}_\lambda b(w)\}_\otimes=\{b(z){}_\lambda b(w)\}_L+d\{b(z){}_\lambda b(w)\}_d$;
- $\{a(z){}_\lambda b(w)\}_\otimes=d\{a(z){}_\lambda b(w)\}_d$.

This λ-bracket defines a structure of Poisson vertex algebra on $\mathcal{V}_A\otimes\mathcal{V}_B$ which we denote by $\mathcal{V}_d^{(N,\infty)}\otimes\mathcal{V}_d^{(M,\infty)}$ with abuse of notation.

Proof. The λ-bracket we defined is clearly skew-commutative, since both λ-brackets $\{\cdot{}_\lambda\cdot\}_L$ and $\{\cdot{}_\lambda\lambda\}_d$ are. ByTheorem 1.5 we are left to prove that Jacobi identity holds on any triple of generators of $\mathcal{V}_A\otimes\mathcal{V}_B$. We will do it on generatong series (see (3.14)). For the triples $a(z),a(w),a(t)$ and $b(z),b(w),b(t)$ Jacobi identity holds by Theorems 3.3 and 3.6. For the remaining triples, after expanding Jacobi identity, we can use the same computations of Theorem 3.6. After substituting in those computations the Laurent series $L(z),L(w)$ and $L(t)$ with our triple, the same formal proof still works. $\square$

Next, we start proving the following result.

Proposition 3.18. *Let $\mathcal{V}_A$, $\mathcal{V}_B$ and $\mathcal{V}_d^{(N,\infty)}\otimes\mathcal{V}_d^{(M,\infty)}$ as in the previous lemma. Denote by A (respectively B) the pseudodifferential operator of general type associated to $\mathcal{V}_A$ (respectively $\mathcal{V}_B$). Let $\mathcal{V}$ be an algebra of differential functions in the variables u_i, with $i\geq -N-M$, L be the general pseudodifferential operator attached to it and let $\mathcal{V}_d^{(N+M,\infty)}$ be the Poisson vertex algebra structure defined by $H^{(L,d)}$. We set*

$$L=\sum_{k\leq N+M}u_{-k-1}\partial^k=AB. \tag{3.59}$$

Comparing coefficients of powers of ∂ in (3.59) gives an inclusion of differential algebras

$$\varphi:\mathcal{V}_d^{(N+M,\infty)}\hookrightarrow\mathcal{V}_d^{(N,\infty)}\otimes\mathcal{V}_d^{(M,\infty)}.$$

This map is a Poisson vertex algebra homomorphism.

Proof. We start computing $\varphi(u_i)$ for $i\geq -N-M$. We have

$$AB=\sum_{h\leq N,k\leq M}a_{-h-1}\partial^h\circ b_{-k-1}\partial^k=\sum_{\substack{h\leq N,k\leq M\\ \alpha\in\mathbb{Z}_+}}\binom{h}{\alpha}a_{-h-1}b_{-k-1}^{(\alpha)}\partial^{h+k-\alpha}=$$

$$=\sum_{i\geq -N-M-1}\left(\sum_{\substack{h\leq N\\ 0\leq\alpha\leq M+h+i+1}}\binom{h}{\alpha}a_{-h-1}b_{h+i-\alpha}^{(\alpha)}\right)\partial^{-i-1},$$

from which we get, for $i\geq -N-M$,

$$\varphi(u_i)=\sum_{\substack{h\leq N\\ 0\leq\alpha\leq M+h+i+1}}\binom{h}{\alpha}a_{-h-1}b_{h+i-\alpha}^{(\alpha)}. \tag{3.60}$$

Let $L(z)=\sum_{i\geq -N-M-1}u_iz^{-i-1}$ be the symbol of the operator $L(\partial)$ and similarly for $a(z)$ and $b(z)$. Using (3.60) we get the following equality

$$\begin{aligned}\varphi(L(z))&=\sum_{i\geq -N-M-1}\varphi(u_i)z^{-i-1}=\sum_{i\geq -N-M-1}\left(\sum_{\substack{h\leq N\\0\leq\alpha\leq M+h+i+1}}\binom{h}{\alpha}a_{-h-1}b^{(\alpha)}_{h+i-\alpha}\right)z^{-i-1}=\\&=\sum_{\substack{p\geq -N-1\\q\geq -M-1\\\alpha\in\mathbb{Z}_+}}\binom{-p-1}{\alpha}a_pz^{-p-1-\alpha}b^{(\alpha)}_qz^{-q-1}=\sum_{\substack{p\geq -N-1\\q\geq -M-1}}a_p(z+\partial)^{-p-1}b_qz^{-q-1}=\\&=a(z+\partial)b(z).\end{aligned}\tag{3.61}$$

To conclude the proof, it suffices to show that $\varphi\left(\{L(z)_\lambda L(w)\}\right)=\{\varphi(L(z))_\lambda\varphi(L(w))\}_\otimes$. The proof of this fact is straightforward. We have

$$\begin{aligned}&\{\varphi(L(z))_\lambda\varphi(L(w))\}_\otimes=\{a(z+\partial)b(z)_\lambda a(w+\partial)b(w)\}_\otimes=\\&=a(w+\lambda+\partial)b(z)i_z(z-w-\lambda-\partial)^{-1}b(w)a^*(\lambda-z)-\\&-\varphi(L(z))i_z(z-w-\lambda-\partial)^{-1}\varphi(L(w))+\\&+d\left(a(w+\lambda+\partial)b(w)-\varphi(L(w))\right)(\lambda+\partial)^{-1}\left(a^*(\lambda-z)-a(z+\partial)\right)b(z)+\\&+d\left(a(w+\lambda+\partial)b(w)-\varphi(L(w))\right)(\lambda+\partial)^{-1}\left(b^*(\lambda+\partial-z)-b(z)\right)a^*(\lambda-z)+\\&+d\left(\varphi(L(w+\lambda+\partial))-a(w+\lambda+\partial)b(w)\right)(\lambda+\partial)^{-1}\left(a^*(\lambda-z)-a(z+\partial)\right)b(z)+\\&+\varphi(L(w+\lambda+\partial))i_z(z-w-\lambda-\partial)^{-1}\varphi(L^*(\lambda-z))-\\&-a(w+\lambda+\partial)b(z)i_z(z-w-\lambda-\partial)b(w)a^*(\lambda-z)+\\&+d\left(\varphi(L(w+\lambda+\partial))-a(w+\lambda+\partial)b(w)\right)(\lambda+\partial)^{-1}\left(b^*(\lambda+\partial-z)-b(z)\right)a^*(\lambda-z)=\\&=\varphi(L(w+\lambda+\partial))i_z(z-w-\lambda-\partial)^{-1}\varphi(L^*(\lambda-z))-\varphi(L(z))i_z(z-w-\lambda-\partial)^{-1}\varphi(L(w))+\\&+d\varphi\left(L(w+\lambda+\partial)-L(w)\right)(\lambda+\partial)^{-1}\varphi\left(L^*(\lambda-z)-L(z)\right)=\varphi\left(\{L(z)_\lambda L(w)\}\right),\end{aligned}$$

where in the last but one equality we used the identity

$$\varphi(L^*(\lambda-z)-L(z))=\left(a^*(\lambda-z)-a(z+\partial)\right)b(z)+\left(b^*(\lambda+\partial-z)-b(z)\right)a^*(\lambda-z).$$

$\square$

Remark 3.19. We emphasize that, by repeated application, the proof of Proposition 3.18 extends to the case of a multiple factorization $L=L_1L_2\cdots L_k$, where $\operatorname{ord}(L_i)=n_i$ and $\operatorname{ord}L=\sum n_i$.

Corollary 3.20. *The same holds in the case of A and B general differential operators, that is, there is a Poisson vertex algebra inclusion*

$$\mathcal{W}_{N+M,d}\hookrightarrow\mathcal{W}_{N,d}\otimes\mathcal{W}_{M,d}.$$

PROOF. The same formal proof of Theorem 3.18 applies in this case. $\square$

Remark 3.21. We want to give another proof of this corollary. By Proposition 3.15 we have $\mathcal{W}_{N+M,d}\cong\mathcal{V}_d^{(N+M,\infty)}/\mathcal{J}_+$. Furthermore, it follows easily by definition that $\mathcal{W}_{N,d}\otimes\mathcal{W}_{M,d}\cong\mathcal{V}_d^{(N,\infty)}\otimes\mathcal{V}_d^{(M,\infty)}/\mathcal{J}_{++}$, where $\mathcal{J}_{++}=\langle a_i,b_j\mid i,j\in\mathbb{Z}_+\rangle_{\mathcal{V}_d^{(N,\infty)}\otimes\mathcal{V}_d^{(M,\infty)}}$.

We have the following diagram of Poisson vertex algebra homomorphisms

$$\begin{tikzcd}\mathcal{V}_d^{(N+M,\infty)}\arrow[r,hook,"\varphi"]\arrow[d,"\pi_1"'] & \mathcal{V}_d^{(N,\infty)}\otimes\mathcal{V}_d^{(M,\infty)}\arrow[d,"\pi_2"]\\ \mathcal{W}_{N+M,d}\arrow[r,dotted] & \mathcal{W}_{N,d}\otimes\mathcal{W}_{M,d}\end{tikzcd}$$

where $\ker \pi_1 = \mathcal{J}_+$ and $\ker \pi_2 = \mathcal{J}_{++}$. We have a Poisson vertex algebra homomorphism $\mathcal{W}_{N+M,d} \longrightarrow \mathcal{W}_{N,d} \otimes \mathcal{W}_{M,d}$ if $\ker \pi_1 \subset \ker \varphi \circ \pi_2$. This is the case. Indeed, if $i \in \mathbb{Z}_+$,

$$
\begin{aligned}
\varphi(u_i) &= \sum_{\substack{h \leq N \\ 0 \leq \alpha \leq M+h+i+1}} \binom{h}{\alpha} a_{-h-1} b^{(\alpha)}_{h+i-\alpha} = \\
&= \sum_{\substack{h \leq -1 \\ 0 \leq \alpha \leq M+h+i+1}} \binom{h}{\alpha} a_{-h-1} b^{(\alpha)}_{h+i-\alpha} + \sum_{\substack{0 \leq h \leq N \\ 0 \leq \alpha \leq M+h+i+1}} \binom{h}{\alpha} a_{-h-1} b^{(\alpha)}_{h+i-\alpha} = \\
&= \sum_{\substack{h \leq -1 \\ 0 \leq \alpha \leq M+h+i+1}} \binom{h}{\alpha} a_{-h-1} b^{(\alpha)}_{h+i-\alpha} + \sum_{0 \leq h, \alpha \leq N} \binom{h}{\alpha} a_{-h-1} b^{(\alpha)}_{h+i-\alpha},
\end{aligned}
$$

where in the second series we used the fact that $M + h + i + 1 \geq N$, if $0 \leq h \leq N$ and that the binomial coefficient vanishes if $\alpha \geq N$. Thus, in the first sum indeces of variables a_i are non-negative, while in the second sum indeces of variables b_i are non-negative, proving that $\varphi(u_i) \in \mathcal{J}_{++}$. Since u_i are differential generators for $\mathcal{V}_d^{(N+M,\infty)}$ we have also $\ker \pi_1 \supset \ker \varphi \circ \pi_2$. Hence, the induced map is injective.

Corollary 3.22 (Theorem 3.16)**.** *The Kupershmidt-Wilson Theorem holds.*

Proof. Remark 3.19 also applies to Corollary 3.20. Now, considering the structure obtained setting $d = 0$, it suffices to note that, if $L_i = \partial + v_i$, the $H^{(L)}$ Hamiltonian structure is given by $\{v_i{}_\lambda v_i\} = \lambda$, while $\{v_i{}_\lambda v_j\} = 0$, for $i \neq j$, since $d = 0$. $\square$

For the sake of completeness we should mention some facts about the Miura map and the Kupershmidt-Wilson Theorem. In the literature, the aim of this theorem was to prove that the matrix differential operator $H^{(L)}(\partial)$ attached to a general differential operator L is Hamiltonian. Indeed, it is a known fact that the operator $H^{(L_i)}(\partial) = \partial$ is Hamiltonian and this is also true for the operator $H^{(L_1 \cdots L_N)}(\partial) = \partial \mathbb{1}_N$. Kupershmidt-Wilson Theorem shows that $\mathcal{W}_N$ is a closed subspace with respect to the λ-bracket induced by $H^{(L_1 \cdots L_N)}$. Hence it is a Poisson vertex subalgebra, that is Jacobi identity holds on any triple of differential generators of $\mathcal{W}_N$. This gives an easy proof of Theorem 3.10 for $H^{(L)}$. However, we can not apply the same argument in the case of a general pseudodifferential operator, since we do not know such a nice factorization as in the differential case (see (3.57)).

We can give some applications of Proposition 3.18

Corollary 3.23. *Let us assume we are in the same setup of Proposition 3.18. If A and B are such that the coefficient of ∂^{N+M-1} in L vanishes, then we have a Poisson vertex algebra inclusion*

$$
\widehat{\mathcal{V}}^{(N+M,\infty)} \longrightarrow \mathcal{V}_d^{(N,\infty)} \otimes \mathcal{V}_d^{(M,\infty)} / \mathcal{I},
$$

where $\mathcal{I} = \langle a_{-N-1} + b_{-M-1} \rangle_{\mathcal{V}_A \otimes \mathcal{V}_B}$ and $d = \frac{1}{N+M}$.

Proof. By Proposition 3.18 we have the following Poisson vertex algebra homomorphism

$$
\varphi : \mathcal{V}_d^{(N+M,\infty)} \longrightarrow \mathcal{V}_d^{(N,\infty)} \otimes \mathcal{V}_d^{(M,\infty)}.
$$

For $d = \frac{1}{N+M}$, by Proposition 3.8, $\widehat{\mathcal{V}}^{(N+M,\infty)} \cong V_d^{(N+M,\infty)} / \mathcal{J}$, where $\mathcal{J} = \langle u_{-N-M-1} \rangle_{\mathcal{V}^{(N+M,\infty)}}$. Furthermore, it is easy to prove (similar argument of proof of Proposition 3.8) that $\mathcal{I}$ is a Poisson vertex algebras ideal. Since $\varphi(u_{-N-M-1}) = a_{-N-1} + b_{-M-1}$, we got an induced Poisson vertex algebra homomorphism

$$
\widehat{\mathcal{V}}^{(N+M,\infty)} \longrightarrow \mathcal{V}_d^{(N,\infty)} \otimes \mathcal{V}_d^{(M,\infty)} / \mathcal{I}.
$$

$\square$

This proposition holds also in the case of A and B differential operators. Furthermore we can prove the following result, a version of the Kupershmidt-Wilson Theorem for $\widehat{\mathcal{W}}_N$.

Corollary 3.24. *Let $\mathcal{V}_N$ be an algebra of differential functions in the variables v_i, $i \in \{1, \ldots, N\}$ and make $\mathcal{V}_N$ a Poisson vertex algebra defining $\{v_i{}_\lambda v_j\} = \left(\delta_{ij} - \frac{1}{N}\right) \lambda$. We set*

$$
L = \sum_{k=0}^{N} u_{-k-1} \partial^k = (\partial + v_N)(\partial + v_{N-1}) \cdots (\partial + v_1). \tag{3.62}
$$

Let $\mathcal{V}$ to be an algebra of differential functions in these variables, L to be the general differential operator attached to $\mathcal{V}$ and $\mathcal{W}_{N,d}$ to be the Poisson vertex algebra structure on $\mathcal{V}$ obtained by $H^{(L,d)}$. If we assume

$\sum_{i=1}^{N} v_i = 0$, then $\mathcal{V}_N/\mathcal{I}$, where $\mathcal{I} = \langle \sum v_i \rangle_{\mathcal{V}_N}$, is a Poisson vertex algebra and comparing coefficients of (3.62) gives a map

$$\varphi : \widehat{\mathcal{W}}_N \longrightarrow \mathcal{V}_N/\mathcal{I}. \tag{3.63}$$

This map is a Poisson vertex algebra homomorphism.

Proof. By iterating Corollary 3.6 in the case of differential operators, we get a Poisson vertex algebra homomorphism

$$\mathcal{W}_N, d \longrightarrow \mathcal{V}_N, \tag{3.64}$$

noting that, by an explicit computation of $H^{(L_i,d)}$, where $L_i = \partial + v_i$, gives $\{v_{i\lambda} v_j\}_{\otimes} = (\delta_{ij} - d)\,\lambda$. When $d = \frac{1}{N}$, it follows immediatly that $\sum_i v_i$ is central for this λ-bracket, then $\mathcal{I}$ is a Poisson vertex algebra ideal. Moreover, when $d = \frac{1}{N}$, $\widehat{\mathcal{W}}_N = \mathcal{W}_{N,d}/\mathcal{J}$, where $\mathcal{J} = \langle u_{-N} \rangle_{\mathcal{W}_{N,d}}$ is a Poisson vertex algebra ideal. Since the image of u_{-N} under (3.64) is $\sum_i v_i$, we got the desired induced map (3.63). $\square$

CHAPTER 4

Isomorphisms between classical $\mathcal{W}$-algebras and Gelfand-Dickey algebras

We will prove that the classical $\mathcal{W}$-algebra $\mathcal{W}_z(\mathfrak{gl}_n, f, s)$, where f is a principal nilpotent element is isomorphic to the Gelfand-Dickey algebra attached to a general differential operator of order n, which we denoted by $\mathcal{W}_n$, while the classical $\mathcal{W}$-algebra $\mathcal{W}_z(\mathfrak{sl}_n, f, s)$, where, f is a principal nilpotent element too, is isomorphic to $\widehat{\mathcal{W}}_n$.

4.1. From first order matrix differential operators to n-th order pseudodifferential operators

We want to define a map which assigns to gauge equivalent operators of the form (2.3) an operator of the form (3.11). This map is due to [12]. The basic suggestion is that a linear differential equation of order n is equivalent to a system of n first order differential equations.

Let us consider a general setting. Let $\mathcal{B}$ be a noncommutative ring with identity and $F \in \mathrm{Mat}(n, \mathcal{B})$ of the following form

$$F = \begin{pmatrix} \alpha & \beta \\ A & \gamma \end{pmatrix} \tag{4.1}$$

where $A \in \mathrm{Mat}(n-1, \mathcal{B})$ is invertible, $\alpha \in \mathcal{B}^{n-1}$, $\gamma^t \in \mathcal{B}^{n-1}$ and $\beta \in \mathcal{B}$. Let denote by N the set of upper triangular matrices in $\mathrm{Mat}(n, \mathcal{B})$ with ones on the diagonal.

Lemma 4.1. *There exist $S_1, S_2 \in N$ such that*

$$\Phi = S_1 F S_2 = \begin{pmatrix} 0 & \Delta(F) \\ \widetilde{A} & 0 \end{pmatrix}.$$

Moreover $\Delta(F)$ does not depend on S_1 and S_2.

Proof. Since the matrix A is invertible there exist $(x_1, \dots, x_{n-1}) \in \mathcal{B}^{n-1}$ such that $(x_1, \dots, x_{n-1})A = \alpha$ and $(y_1, \dots, y_{n-1}) \in \mathcal{B}^{n-1}$ such that $A(y_1, \dots, y_{n-1})^t = \gamma$. Let E_{ij}, $1 \le i, j \le n$, denote the elementary matrices of order n, then

$$(\mathbb{1}_n - x_1 E_{12} - \dots - x_{n-1} E_{1n}) A (\mathbb{1}_n - y_1 E_{2n} - \dots - y_{n-1} E_{nn}) = \begin{pmatrix} 0 & \beta - \alpha A^{-1} \gamma \\ A & 0 \end{pmatrix}.$$

Now suppose $\Phi_1 = S_1 F S_2$ and $\Phi_2 = \widetilde{S}_1 F \widetilde{S}_2$, then $F = S_1^{-1} \Phi_1 S_2^{-1} = \widetilde{S}_1^{-1} \Phi_2 \widetilde{S}_2^{-1}$. Therefore we are left to show that if $S_1 \Phi_1 = \Phi_2 S_2$, with $S_i \in N$, $i = 1, 2$, then $\Delta(\Phi_1) = \Delta(\Phi_2)$. This is straightforward. We have

$$S_1 \Phi_1 = \begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} 0 & \Delta(\Phi_1) \\ A_1 & 0 \end{pmatrix} = \begin{pmatrix} * & \Delta(\Phi_1) \\ * & 0 \end{pmatrix}$$

and

$$\Phi_2 S_2 = \begin{pmatrix} 0 & \Delta(\Phi_2) \\ A_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} 0 & \Delta(\Phi_2) \\ * & * \end{pmatrix}.$$

$\square$

Lemma 4.2. *Let W be a left module over $\mathcal{B}$, if*

$$F\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} v \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{4.2}$$

where $u_i, v \in W$, then $\Delta(F)u_n = v$.

Proof. Multiplying relation (4.2) by S_1 on the left we get

$$S_1 F\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = S_1 \begin{pmatrix} v \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} v \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

But we have also

$$S_1 F\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = S_1 F S_2 S_2^{-1}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \Phi\begin{pmatrix} u_1 + \dots \\ u_2 + \dots \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} 0 & \Delta(\Phi) \\ A & 0 \end{pmatrix}\begin{pmatrix} u_1 + \dots \\ u_2 + \dots \\ \vdots \\ u_n \end{pmatrix}.$$

Then $\Delta(F)u_n = v$. □

Suppose now there is an antiautomorphism $^* : \mathcal{B} \longrightarrow \mathcal{B}$ such that $(x^*)^* = x$, for any $x \in \mathcal{B}$. For all $A = (a_{ij}) \in \mathrm{Mat}(n, \mathcal{B})$ we define $A^T = (a_{ij}^T)$, where

$$a_{ij}^T = a^*_{n-j+1 n-i+1}.$$

It is easy to verify that $(A_1A_2)^T = A_2^T A_1^T$ for any $A_1, A_2 \in \mathrm{Mat}(n, \mathcal{B})$.

Lemma 4.3. $\Delta(F^T) = \Delta(F)^*$.

Proof. Since $\Phi = S_1 F S_2$, then $\Phi^T = S_2^T F^T S_1^T$. It follows that $\Delta(F^T) = \Phi^T_{1n} = \Phi^*_{1n} = \Delta(F)^*$. □

Now we come back to the situation we are interested in. In our case $\mathcal{B} = \mathcal{V}(\mathfrak{gl}_n)[\partial, z]$, $W = \mathcal{V}(\mathfrak{gl}_n)((z^{-1}))^n$ and * is the formal adjoint, namely, $\partial^* = -\partial$ and $p^* = p$, for any $p \in \mathcal{V}(\mathfrak{gl}_n)[z]$. We want to consider $\Lambda(z) = f + zs = \sum_{k=1}^{n-1} E_{k+1,k} + zE_{1,n}$ in (2.3). Furthermore, in (2.3), we have

$$q = \sum_{1 \leq j \leq i \leq n} E_{ji} \otimes q_{ij}, \tag{4.3}$$

where we assume q_{ij} to be the differential variables corresponding to E_{ij} in $\mathcal{V}(\mathfrak{gl}_n)$.

We introduce the structure of $\mathcal{B}$-module on W by means of the operator $\mathscr{L}$ as follows: if $P = \sum p_i \partial^i \in \mathcal{V}(\mathfrak{gl}_n)[\partial, z]$ and $\eta \in \mathcal{V}(\mathfrak{gl}_n)((z^{-1}))^n$, then

$$P \cdot \eta = \sum p_i \mathscr{L}^i(\eta).$$

The axioms for a module are satisfied, since $[\mathscr{L}, p] = p'$, for $p \in \mathcal{V}(\mathfrak{gl}_n)$. We emphasize that $\partial \cdot \eta = \mathscr{L}(\eta) \neq \eta'$. In particular we can consider $\mathscr{L}$ as a matrix with coefficients in $\mathcal{B}$. If we set $\mathscr{L}_0 = \mathscr{L} - zs$, clearly $\mathscr{L}_0$ does not depend on z and is of the form (3.15f). We denote the elements of the standard basis of W as a module over $\mathcal{V}(\mathfrak{gl}_n)((z^{-1}))$ by $\bar{e}_1, \dots, \bar{e}_n$ and recall that $\psi = \bar{e}_1$.

Proposition 4.4. *Let L be an operator of the form (3.11) such that $L \cdot \psi = z\psi$, then $L = -\Delta(\mathscr{L}_0)^*$.*

Proof. Since $\Lambda(z)^T = \Lambda(z)$, then $\mathscr{L}^T = -\partial + q^T + \Lambda(z)$. Moreover

$$\begin{aligned} \mathscr{L}(\bar{e}_i) &= q_{i1}\bar{e}_1 + \dots + q_{ii}\bar{e}_i + \bar{e}_{i+1}, \quad i \neq n \\ \mathscr{L}(\bar{e}_n) &= q_{n1}\bar{e}_1 + \dots + q_{nn}\bar{e}_n. \end{aligned}$$

Then we get

$$\mathscr{L}^T \begin{pmatrix} \bar{e}_n \\ \bar{e}_{n-1} \\ \vdots \\ \bar{e}_1 \end{pmatrix} = - \begin{pmatrix} \mathscr{L}(\bar{e}_n) \\ \mathscr{L}(\bar{e}_{n-1}) \\ \vdots \\ \mathscr{L}(\bar{e}_1) \end{pmatrix} + q^T \begin{pmatrix} \bar{e}_n \\ \bar{e}_{n-1} \\ \vdots \\ \bar{e}_1 \end{pmatrix} + \begin{pmatrix} z\bar{e}_1 \\ \bar{e}_n \\ \vdots \\ \bar{e}_2 \end{pmatrix} =$$

$$= -\left((\bar{e}_1, \dots, \bar{e}_n)q\right)^T - \begin{pmatrix} z\bar{e}_1 \\ \bar{e}_n \\ \vdots \\ \bar{e}_2 \end{pmatrix} + q^T \begin{pmatrix} \bar{e}_n \\ \bar{e}_{n-1} \\ \vdots \\ \bar{e}_1 \end{pmatrix} + \begin{pmatrix} z\bar{e}_1 \\ \bar{e}_n \\ \vdots \\ \bar{e}_2 \end{pmatrix} = 0,$$

since

$$\left((\bar{e}_1, \dots, \bar{e}_n)q\right)^T = q^T \begin{pmatrix} \bar{e}_n \\ \bar{e}_{n-1} \\ \vdots \\ \bar{e}_1 \end{pmatrix}.$$

This means that

$$\mathscr{L}_0^T \begin{pmatrix} \bar{e}_n \\ \bar{e}_{n-1} \\ \vdots \\ \bar{e}_1 \end{pmatrix} = - \begin{pmatrix} z\bar{e}_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

By Lemma 4.2 it follows that $-\Delta(\mathscr{L}_0^T)\psi = z\psi$. Then $L = -\Delta(\mathscr{L}_0^T) = -\Delta(\mathscr{L}_0)^*$, by Lemma 4.3. □

Thus, we got the desired map $\mathscr{L} \longrightarrow L = -\Delta(\mathscr{L}_0)^*$. By Lemma (4.1), under this map, to gauge equivalent operators $\mathscr{L}$ corresponds the same operator L. Moreover, on the set of gauge equivalence classes of operators $\mathscr{L}$ this map is bijective. Indeed, it is easy to see that we can choose

$$V = \bigoplus_{i=1}^{n} \mathbb{C}E_{i,n}$$

in Proposition 2.1. Then

$$q^{can} = \sum_{i=1}^{n} E_{i,n} \otimes v_i, \quad v_i \in \mathcal{V}(\mathfrak{b}_-).$$

Suppose now, $L = \sum_{k=0}^{n} u_k \partial^k$, with $u_n = 1$, then it easy to see, from relation $L = -\Delta(\mathscr{L}_0^{can})$, that the coefficients of the operator L can be expressed in terms of the coefficients of the matrix q^{can} by the formula $u_i = -v_{i+1}$, for $i = 0, \dots, n-1$.

4.2. $\mathcal{W}_z(\mathfrak{gl}_n, f, s) \cong \mathcal{W}_n$ for f principal nilpotent

If we rename

$$q^{can} = -\sum_{i=1}^{n} E_{i,n} \otimes u_{-i},$$

then we have the assignment

$$\mathscr{L}^{can} \longrightarrow L = \sum_{k=0}^{n} u_{-k-1}\partial^k,$$

with $u_{-n-1} = 1$ and $u_i = u_i(q)$, for $i \in I = \{-n, \dots, -1\}$. By Corollary 2.2, it follows that, as differential algebras,

$$\mathcal{W}_z(\mathfrak{gl}_n, f, s) = \mathbb{C}[u_i^{(m)} \mid i \in I, m \in \mathbb{Z}_+] = \mathcal{W}_n.$$

By gauge invariance, $\{u_i(q)_\lambda u_j(q)\}_z = \{u_i(q)_\lambda u_j(q)\}_z|_{q=q^{can}} = \{u_{i\lambda} u_j\}_z$. If we prove that

$$\{u_{i\lambda} u_j\}_z = \{u_{i\lambda} u_j\}_c,$$

then $\mathcal{W}(\mathfrak{gl}_n, f, s) \cong \mathcal{W}_n$ as Poisson vertex algebras. In order to do that we should set $z = c$, but we keep both in the notation to distinghuish the two structures.

First, explicity computing the coefficients of (3.54), we get, for all $-n \leq i, j \leq -1$,

$$\begin{aligned}\{u_{i}{}_\lambda u_j\}_c = & \sum_{\substack{k,\alpha,\beta\in\mathbb{Z}_+\\ i+k\leq\alpha\leq n+i-k\\ j+k+1\leq\beta\leq n+j+k+1}} (-1)^\beta \binom{i}{\alpha}\binom{j+k}{\beta} u_{j+k-\beta}(\lambda+\partial)^{\alpha+\beta} u_{i-k-\alpha-1} - \\ & - \sum_{\substack{0\leq\alpha\ k\leq n+i\\ j+k+1\leq\alpha\leq n+j+k+1}} \binom{k}{\alpha} u_{i-k-1}(\lambda+\partial)^\alpha u_{j+k-\alpha} + \qquad (4.4a)\\ & + c\sum_{k=0}^{n+i+j+1}\left(\binom{j}{k}(-\lambda)^k - \binom{i}{k}(\lambda+\partial)^k\right) u_{i+j-k}. \qquad (4.4b)\end{aligned}$$

Now we proceed computing $\{u_{i}{}_\lambda u_j\}_z$. By formula (1.11), recalling the definition of q given in (4.3), we hfave,for $-n \leq i, j \leq -1$,

$$\{u_{i}{}_\lambda u_j\}_z = \sum_{\substack{1\leq l\leq k\leq n\\ 1\leq t\leq s\leq n\\ m,p\in\mathbb{Z}_+}} \left.\frac{\partial u_j}{\partial q_{st}^{(p)}}\right|_{q=q^{can}} (\lambda+\partial)^p \left.\{q_{kl}{}_{\lambda+\partial} q_{st}\}_{z_\to}\right|_{q=q^{can}} (-\lambda-\partial)^m \left.\frac{\partial u_i}{\partial q_{kl}^{(m)}}\right|_{q=q^{can}}, \qquad (4.5)$$

where

$$\{q_{kl}{}_\lambda q_{st}\}_z|_{q=q^{can}} = -\delta_{ls}\delta_{kn}u_{-t} + \delta_{tk}\delta_{sn}u_{-l} + \delta_{ls}\delta_{kt}\lambda + z\left(\delta_{ls}\delta_{kn}\delta_{1t} - \delta_{kt}\delta_{sn}\delta_{1l}\right), \qquad (4.6)$$

for all $1 \leq l \leq k \leq n$ and $1 \leq t \leq s \leq n$. Our first goal will be to compute the partial derivatives of the u_k's and specialize them in the case $q = q^{can}$. We can write

$$u_i = \mathrm{Res}(L\partial^i),$$

for $-n \leq i \leq -1$. Then

$$\frac{\partial u_i}{\partial q_{kl}^{(m)}} = \frac{\partial}{\partial q_{kl}^{(m)}}\mathrm{Res}\left(L\partial^i\right) = \mathrm{Res}\left(\frac{\partial}{\partial q_{kl}^{(m)}}\left(L\partial^i\right)\right).$$

By Proposition 4.4, we can write

$$L = -\Delta(\mathcal{L})^* = -\left(\beta - \alpha A^{-1}\gamma\right)^* = \gamma^*\left(A^*\right)^{-1}\alpha^* - \beta^*.$$

Let us denote $\left(\gamma^*\left(A^*\right)^{-1}\alpha^* - \beta^*\right)(\partial) = \gamma^*\left(A^*\right)^{-1}\alpha^* - \beta^*$, to remember it is a differential operator. Thus we have

$$\frac{\partial u_i}{\partial q_{kl}^{(m)}} = \mathrm{Res}\left(\frac{\partial(\gamma^*(A^*)^{-1}\alpha^* - \beta^*)}{\partial q_{kl}^{(m)}}(\partial)\partial^i\right).$$

We want to use Lemma 1.2 to perform the computation of the partial derivatives inside the residue. We get, for any $f \in \mathcal{V}(\mathfrak{gl}_n)$,

$$\begin{aligned}\sum_{m\in\mathbb{Z}_+} z^m \frac{\partial}{\partial q_{kl}^{(m)}}\left(\left(\gamma^*(A^*)^{-1}\alpha^* - \beta^*\right)(\partial)f\right) = & \sum_{m\in\mathbb{Z}_+} z^m \frac{\partial(\gamma^*(A^*)^{-1}\alpha^* - \beta^*)}{\partial q_{kl}^{(m)}}(\partial)f + \\ & + \left(\gamma^*(A^*)^{-1}\alpha^* - \beta^*\right)(z+\partial)\sum_{m\in\mathbb{Z}_+} z^m \frac{\partial f}{\partial q_{kl}^{(m)}}.\end{aligned}$$

If we apply several times Lemma 1.2 to the left hand side we get, for any $f \in \mathcal{V}(\mathfrak{gl}_n)$,

$$\sum_{m\in\mathbb{Z}_+} z^m \frac{\partial}{\partial q_{kl}^{(m)}} \left(\left(\gamma^*(A^*)^{-1}\alpha^* - \beta^*\right)(\partial) f \right) = \sum_{m\in\mathbb{Z}_+} z^m \left[\frac{\partial \gamma^*}{\partial q_{kl}^{(m)}}(\partial) \left(\left((A^*)^{-1}\alpha^*\right)(\partial) f \right) - \frac{\partial \beta^*}{\partial q_{kl}^{(m)}}(\partial) f \right.$$
$$\left. +\gamma^*(z+\partial) \frac{\partial}{\partial q_{kl}^{(m)}} \left(\left((A^*)^{-1}\alpha^*\right)(\partial) f \right) - \beta^*(z+\partial) \frac{\partial f}{\partial q_{kl}^{(m)}} \right] =$$
$$= \sum_{m\in\mathbb{Z}_+} z^m \left[\frac{\partial \gamma^*}{\partial q_{kl}^{(m)}}(\partial) \left(\left((A^*)^{-1}\alpha^*\right)(\partial) f \right) - \frac{\partial \beta^*}{\partial q_{kl}^{(m)}}(\partial) f - \beta^*(z+\partial) \frac{\partial f}{\partial q_{kl}^{(m)}} + \right.$$
$$\left. +\gamma^*(z+\partial) \frac{\partial (A^*)^{-1}}{\partial q_{kl}^{(m)}}(\partial) \left(\alpha^*(\partial) f\right) + \gamma^*(A^*)^{-1}(z+\partial) \frac{\partial}{\partial q_{kl}^{(m)}} \left(\alpha^*(\partial) f\right) \right] =$$
$$= \sum_{m\in\mathbb{Z}_+} z^m \left[\frac{\partial \gamma^*}{\partial q_{kl}^{(m)}}(\partial) \left(\left((A^*)^{-1}\alpha^*\right)(\partial) f \right) + \gamma^*(z+\partial) \frac{\partial (A^*)^{-1}}{\partial q_{kl}^{(m)}}(\partial) \left(\alpha^*(\partial) f\right) + \right.$$
$$\left. +\gamma^*(A^*)^{-1}(z+\partial) \frac{\partial \alpha^*}{\partial q_{kl}^{(m)}}(\partial) f + \left(\gamma^*(A^*)^{-1}\alpha^* - \beta^*\right)(z+\partial) \frac{\partial f}{\partial q_{kl}^{(m)}} - \frac{\partial \beta^*}{\partial q_{kl}^{(m)}}(\partial) f \right].$$

Equating and using the fact that these formulas hold for any $f \in \mathcal{V}(\mathfrak{gl}_n)$, it follows that

$$\sum_{m\in\mathbb{Z}_+} z^m \frac{\partial(\gamma^*(A^*)^{-1}\alpha^* - \beta^*)}{\partial q_{kl}^{(m)}}(\partial) = \sum_{m\in\mathbb{Z}_+} z^m \left[\frac{\partial \gamma^*}{\partial q_{kl}^{(m)}}(\partial) \left((A^*)^{-1}\alpha^*\right)(\partial) + \gamma^*(z+\partial) \frac{\partial (A^*)^{-1}}{\partial q_{kl}^{(m)}}(\partial)\alpha^*(\partial) + \right.$$
$$\left. +\gamma^*(A^*)^{-1}(z+\partial) \frac{\partial \alpha^*}{\partial q_{kl}^{(m)}}(\partial) - \frac{\partial \beta^*}{\partial q_{kl}^{(m)}}(\partial) \right]. \tag{4.7}$$

Let us denote by v_i the vector of order $n-1$ with 1 at the i-th position and 0 elsewhere. Since

$$\alpha = \alpha(\partial) = v_1 \otimes \partial + \sum_{i=1}^{n-1} v_i \otimes q_{i1},$$
$$\beta = \beta(\partial) = q_{n1},$$
$$\gamma = \gamma(\partial) = v_{n-1}^t \otimes \partial + \sum_{i=1}^{n-1} v_i^t \otimes q_{ni+1},$$
$$A = A(\partial) = \sum_{i=1}^{n-1} E_{ii} \otimes 1 + \sum_{i=1}^{n-2} E_{ii+1} \otimes \partial + \sum_{1\le j<i\le n-1} E_{ji} \otimes q_{ij+1}$$

it follows that

$$\alpha^* = \alpha^*(\partial) = -v_1^t \otimes \partial + \sum_{i=1}^{n-1} v_i^t \otimes q_{i1},$$
$$\beta^* = \beta^*(\partial) = q_{n1},$$
$$\gamma^* = \gamma^*(\partial) = -v_{n-1} \otimes \partial + \sum_{i=1}^{n-1} v_i \otimes q_{ni+1},$$
$$A^* = A^*(\partial) = \sum_{i=1}^{n-1} E_{ii} \otimes 1 - \sum_{i=1}^{n-2} E_{i+1i} \otimes \partial + \sum_{1\le j<i\le n-1} E_{ij} \otimes q_{ij+1}.$$

Then applying partial derivatives we get

$$
\begin{aligned}
\frac{\partial \alpha^*}{\partial q_{kl}^{(m)}}(\partial) &= \delta_{m0}\delta_{k\neq n}\delta_{l1} v_k^t \otimes 1 \\
\frac{\partial \beta^*}{\partial q_{kl}^{(m)}}(\partial) &= \delta_{m0}\delta_{kn}\delta_{l1} \\
\frac{\partial \gamma^*}{\partial q_{kl}^{(m)}}(\partial) &= \delta_{m0}\delta_{kn}\delta_{l\neq 1} v_{l-1} \otimes 1 \\
\frac{\partial A^*}{\partial q_{kl}^{(m)}}(\partial) &= \delta_{m0}\delta_{k\neq 1,n}\delta_{l\neq 1,n} E_{kl-1} \otimes 1.
\end{aligned}
\tag{4.8}
$$

We are not interested in formula (4.7) itself, but in specializing it when $q = q^{can}$, namely we have to substitute $q_{ij} = -\delta_{in} u_{-j}$, for all $1 \leq j \leq iw \leq n$. In that case the corresponding operators $\alpha_{can}^*, \beta_{can}^*, \gamma_{can}^*$ and $(A_{can}^*)^{-1}$ are

$$
\begin{aligned}
\alpha_{can}^*(\partial) &= -v_1^t \otimes \partial \\
\beta_{can}^*(\partial) &= -u_{-1} \\
\gamma_{can}^*(\partial) &= -\left(v_{n-1} \otimes \partial + \sum_{i=1}^{n-1} v_i \otimes u_{-i-1} \right) \\
\left(A_{can}^*\right)^{-1}(\partial) &= \sum_{1 \leq j \leq i \leq n-1} E_{ij} \otimes \partial^{i-j}.
\end{aligned}
\tag{4.9}
$$

Thus we want to compute

$$
\sum_{m \in \mathbb{Z}_+} z^m \left. \frac{\partial(\gamma^*(A^*)^{-1}\alpha^* - \beta^*)}{\partial q_{kl}^{(m)}}(\partial) \right|_{q=q^{can}} = \sum_{m \in \mathbb{Z}_+} z^m \left[\frac{\partial \gamma^*}{\partial q_{kl}^{(m)}}(\partial) \left((A_{can}^*)^{-1} \alpha_{can}^* \right)(\partial) + \right.
$$
$$
\left. + \gamma_{can}^*(z+\partial) \frac{\partial (A^*)^{-1}}{\partial q_{kl}^{(m)}}(\partial) \alpha_{can}^*(\partial) + \gamma_{can}^* (A_{can}^*)^{-1}(z+\partial) \frac{\partial \alpha^*}{\partial q_{kl}^{(m)}}(\partial) - \frac{\partial \beta^*}{\partial q_{kl}^{(m)}}(\partial) \right].
$$

Let us compute these terms separately. We have

$$
\frac{\partial \gamma^*}{\partial q_{kl}^{(m)}}(\partial) \left(\left(A_{can}^*\right)^{-1} \alpha_{can}^* \right)(\partial) = -\delta_{m0}\delta_{kn}\delta_{l\neq 1} \sum_{1 \leq j \leq i \leq n-1} v_{l-1} E_{ij} v_1^t \partial^{i-j+1} = -\delta_{m0}\delta_{kn}\delta_{l\neq 1} \partial^{l-1}. \tag{4.10}
$$

For the second term we need to derive a useful formula to compute the partial derivatives of $(A^*)^{-1}(\partial)$. We start considering the trivial relation

$$
\sum_{m \in \mathbb{Z}_+} z^m \frac{\partial}{\partial u_i^{(m)}} \left(A^{-1}(\partial) A(\partial) f \right) = \sum_{m \in \mathbb{Z}_+} z^m \frac{\partial f}{\partial u_i^{(m)}},
$$

for $f \in \mathcal{V}(\mathfrak{gl}_n)$. Applying Lemma 1.2 twice to the left hand side we get, for any $f \in \mathcal{V}(\mathfrak{gl}_n)$,

$$
\sum_{m \in \mathbb{Z}_+} z^m \frac{\partial}{\partial u_i^{(m)}} \left(A^{-1}(\partial) A(\partial) f \right) = \sum_{m \in \mathbb{Z}_+} z^m \frac{\partial A^{-1}}{\partial u_i^{(m)}}(\partial) \left(A(\partial) f \right) + A^{-1}(z+\partial) \sum_{m \in \mathbb{Z}_+} z^m \frac{\partial}{\partial u_i^{(m)}} \left(A(\partial) f \right) =
$$
$$
= \sum_{m \in \mathbb{Z}_+} z^m \frac{\partial A^{-1}}{\partial u_i^{(m)}}(\partial) \left(A(\partial) f \right) + A^{-1}(z+\partial) \sum_{m \in \mathbb{Z}_+} z^m \frac{\partial A}{\partial u_i^{(m)}}(\partial) f + \sum_{m \in \mathbb{Z}_+} z^m \frac{\partial f}{\partial u_i^{(m)}}.
$$

Hence, we have obtained the identity

$$
\sum_{m \in \mathbb{Z}_+} z^m \frac{\partial A^{-1}}{\partial u_i^{(m)}}(\partial) = -A^{-1}(z+\partial) \sum_{m \in \mathbb{Z}_+} \frac{\partial A}{\partial u_i^{(m)}}(\partial) A^{-1}(\partial). \tag{4.11}
$$

We can continue the computation and get

$$
\begin{aligned}
&\gamma^*_{can}(z+\partial)\frac{\partial (A^*)^{-1}}{\partial q^{(m)}_{kl}}(\partial)\alpha^*_{can}(\partial) = -\left(\gamma^*_{can}\left(A^*_{can}\right)^{-1}\right)(z+\partial)\frac{\partial A^*}{\partial q^{(m)}_{kl}}\left(\left(A^*_{can}\right)^{-1}\alpha^*_{can}\right)(\partial) = \\
&= -\delta_{m0}\delta_{k\neq 1,n}\delta_{l\neq 1,n}\sum_{\substack{1\leq i\leq n-1\\ 1\leq h\leq j\leq n\\ 1\leq m\leq p\leq n}} v_i E_{jh}E_{kl-1}E_{pm}v_1^t u_{-i-1}(z+\partial)^{j-h}\partial^{p-m+1}- \\
&- \delta_{m0}\delta_{k\neq 1,n}\delta_{l\neq 1,n}\sum_{\substack{1\leq h\leq j\leq n\\ 1\leq m\leq p\leq n}} v_{n-1}E_{jh}E_{kl-1}E_{pm}v_1^t u_{-i-1}(z+\partial)^{j-h+1}\partial^{p-m+1} = \\
&= -\delta_{m0}\delta_{k\neq 1,n}\delta_{l\neq 1,n}\sum_{i=k}^{n-1} u_{-i-1}(z+\partial)^{i-k}\partial^{l-1} - \delta_{m0}\delta_{k\neq 1,n}\delta_{l\neq 1,n}(z+\partial)^{n-k}\partial^{l-1} = \\
&= -\delta_{m0}\delta_{k\neq 1,n}\delta_{l\neq 1,n}\sum_{i=k}^{n} u_{-i-1}(z+\partial)^{i-k}\partial^{l-1},
\end{aligned}
\tag{4.12}
$$

where we have used formula (4.11) and the fact that $u_{-n-1}=1$. The remaining term to compute is

$$
\begin{aligned}
&\gamma^*_{can}(A^*_{can})^{-1}(z+\partial)\frac{\partial \alpha^*}{\partial q^{(m)}_{kl}}(\partial) = -\delta_{m0}\delta_{k\neq n}\delta_{l1}\sum_{\substack{1\leq i\leq n-1\\ 1\leq h\leq j\leq n-1}} v_i E_{jh} v_k^j u_{-i-1}(z+\partial)^{j-h}- \\
&- \delta_{m0}\delta_{k\neq n}\delta_{l1}\sum_{1\leq h\leq j\leq n-1} v_{n-1}E_{jh}v_k^j(z+\partial)^{j-h+1} = \\
&= -\delta_{m0}\delta_{k\neq n}\delta_{l1}\sum_{i=k}^{n-1} u_{-i-1}(z+\partial)^{i-k} - \delta_{m0}\delta_{k\neq n}\delta_{l1}(z+\partial)^{n-k} = \\
&= -\delta_{m0}\delta_{k\neq n}\delta_{l1}\sum_{i=k}^{n} u_{-i-1}(z+\partial)^{i-k}.
\end{aligned}
\tag{4.13}
$$

By formulas (4.8), (4.10), (4.12) and (4.13) we get

$$
\begin{aligned}
&\sum_{m\in\mathbb{Z}_+} z^m\,\frac{\partial(\gamma^*(A^*)^{-1}\alpha^*-\beta^*)}{\partial q^{(m)}_{kl}}(\partial)\bigg|_{q=q^{can}} = \\
&= -\delta_{kn}\delta_{l\neq 1}\partial^{l-1} - \delta_{k\neq 1,n}\delta_{l\neq 1,n}\sum_{i=k}^{n} u_{-i-1}(z+\partial)^{i-k}\partial^{l-1} - \delta_{k\neq n}\delta_{l1}\sum_{i=k}^{n} u_{-i-1}(z+\partial)^{i-k} - \delta_{kn}\delta_{l1} = \\
&= -\left(\delta_{kn}\delta_{l\neq 1} + \delta_{k\neq 1,n}\delta_{l\neq 1,n} + \delta_{k\neq n}\delta_{l1} + \delta_{kn}\delta_{l1}\right)\sum_{i=k}^{n} u_{-i-1}(z+\partial)^{i-k}\partial^{l-1}.
\end{aligned}
$$

Since $1\leq l\leq k\leq n$, only one term in the parenthesis survives for each pair (k,l), so we can forget about those conditions on the indeces and write

$$
\begin{aligned}
&\sum_{m\in\mathbb{Z}_+} z^m\,\frac{\partial(\gamma^*(A^*)^{-1}\alpha^*-\beta^*)}{\partial q^{(m)}_{kl}}(\partial)\bigg|_{q=q^{can}} = -\sum_{i=k}^{n} u_{-i-1}(z+\partial)^{i-k}\partial^{l-1} = -\sum_{i=0}^{n-k} u_{-i-k-1}(z+\partial)^{i}\partial^{l-1} = \\
&= -\sum_{i=0}^{n-k}\sum_{m=0}^{i}\binom{i}{m}u_{-i-k-1}\partial^{i+l-m-1}z^m = -\sum_{m=0}^{n-k}\left(\sum_{i=m}^{n-k}\binom{i}{m}u_{-i-k-1}\partial^{i+l-m-1}\right)z^m.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\frac{\partial(\gamma^*(A^*)^{-1}\alpha^*-\beta^*)}{\partial q^{(m)}_{kl}}(\partial) &= -\sum_{i=m}^{n-k}\binom{i}{m}u_{-i-k-1}\partial^{i+l-m-1} = \\
&= -\sum_{i=0}^{n-k-m}\binom{m+i}{m}u_{-i-m-k-1}\partial^{i+l-1}.
\end{aligned}
\tag{4.14}
$$

Thanks to formula (4.14) we get

$$
\begin{aligned}
\left.\frac{\partial u_i}{\partial q_{kl}^{(m)}}\right|_{q=q^{can}} &= -\operatorname{Res}\left(\sum_{\alpha=0}^{n-k-m}\binom{m+\alpha}{m}u_{-\alpha-m-k-1}\partial^{\alpha+l+i-1}\right) = \\
&= -\delta_{k+m-n\leq l+i\leq 0}\binom{m-i-l}{m}u_{i+l-m-k-1},
\end{aligned}
$$

where, for $1\leq l\leq k\leq n$, $-n\leq i\leq -1$ and $m\in\mathbb{Z}_+$, we set

$$
\delta_{k+m-n\leq l+i\leq 0} = \begin{cases} 1, & \text{if } k+m-n\leq l+i\leq 0, \\ 0, & \text{otherwise,} \end{cases}
$$

The condition imposed by δ guarantees the contribution of the residue and the fact that $u_k = 0$ for $k<-n$. Now we can substitute in (4.5), using also (4.6) and perform the computation of the λ-bracket in (4.5). We start computing the ∞-λ-bracket. It is

$$
\begin{aligned}
\{u_i{}_\lambda u_j\}_\infty &= \sum_{\substack{1\leq l\leq k\leq n\\ 1\leq t\leq s\leq n\\ m,p\in\mathbb{Z}_+}} \left.\frac{\partial u_j}{\partial q_{st}^{(p)}}\right|_{q=q^{can}} (\lambda+\partial)^p\left(\delta_{ls}\delta_{kn}\delta_{1t}-\delta_{kt}\delta_{sn}\delta_{1l}\right)(-\lambda-\partial)^m \left.\frac{\partial u_i}{\partial q_{kl}^{(m)}}\right|_{q=q^{can}} = \\
&= \sum_{\substack{1\leq l\leq k\leq n\\ 1\leq t\leq s\leq n\\ k+m-n\leq l+i\leq 0\\ s+p-n\leq t+j\leq 0\\ m,p\in\mathbb{Z}_+}} \binom{m-l-i}{m}\binom{p-t-j}{p}u_{t+j-p-s-1}(\lambda+\partial)^p\delta_{ls}\delta_{kn}\delta_{1t}(-\lambda-\partial)^m u_{l+i-m-k-1} - \\
&\quad - \sum_{\substack{1\leq l\leq k\leq n\\ 1\leq t\leq s\leq n\\ k+m-n\leq l+i\leq 0\\ s+p-n\leq t+j\leq 0\\ m,p\in\mathbb{Z}_+}} \binom{m-l-i}{m}\binom{p-t-j}{p}u_{t+j-p-s-1}(\lambda+\partial)^p\delta_{kt}\delta_{sn}\delta_{1l}(-\lambda-\partial)^m u_{l+i-m-k-1} = \\
&= \sum_{p=0}^{n+i+j+1}\binom{p-j-1}{p}u_{i+j-p}\lambda^p - \sum_{m=0}^{n+i+j+1}\binom{m-i-1}{m}(-\lambda-\partial)^m u_{i+j-m} = \\
&= \sum_{l=0}^{n+i+j+1}\left(\binom{j}{l}u_{i+j-l}(-\lambda)^l - \binom{i}{l}(\lambda+\partial)^l u_{i+j-l}\right).
\end{aligned}
$$

This is the same expression of (4.4b). It remains to prove equality of the 0-λ-brackets. We split formula (4.5) relative to the 0-λ-bracket into three terms. For the first we have

$$
\begin{aligned}
&- \sum_{\substack{1\leq l\leq k\leq n\\ 1\leq t\leq s\leq n\\ m,p\in\mathbb{Z}_+}} \left.\frac{\partial u_j}{\partial q_{st}^{(p)}}\right|_{q=q^{can}} (\lambda+\partial)^p\left(\delta_{ls}\delta_{kn}u_{-t}\right)(-\lambda-\partial)^m \left.\frac{\partial u_i}{\partial q_{kl}^{(m)}}\right|_{q=q^{can}} = \\
&= - \sum_{\substack{1\leq l\leq k\leq n\\ 1\leq t\leq s\leq n\\ k+m-n\leq l+i\leq 0\\ s+p-n\leq t+j\leq 0\\ m,p\in\mathbb{Z}_+}} \binom{m-l-i}{m}\binom{p-t-j}{p}u_{t+j-p-s-1}(\lambda+\partial)^p\delta_{ls}\delta_{kn}u_{-t}(-\lambda-\partial)^m u_{l+i-m-k-1} = \\
&= - \sum_{\substack{1\leq t\leq -i\\ p-i-n\leq t+j\\ p\in\mathbb{Z}_+}} \binom{p-t-j}{p}u_{t+i+j-p-1}(\lambda+\partial)^p u_{-t} = - \sum_{\substack{0\leq\alpha\leq k\leq n+i\\ k+j+1\leq\alpha\leq k+j-i}} \binom{k}{\alpha}u_{i-k-1}(\lambda+\partial)^\alpha u_{k+j-\alpha}.
\end{aligned}
$$

While for the second

$$\sum_{\substack{1\le l\le k\le n\\1\le t\le s\le n\\m,p\in\mathbb{Z}_+}}\frac{\partial u_j}{\partial q_{st}^{(p)}}\bigg|_{q=q^{can}}(\lambda+\partial)^p\left(\delta_{tk}\delta_{sn}u_{-l}\right)(-\lambda-\partial)^m\,\frac{\partial u_i}{\partial q_{kl}^{(m)}}\bigg|_{q=q^{can}}=$$

$$=\sum_{\substack{1\le l\le k\le n\\1\le t\le s\le n\\k+m-n\le l+i\le 0\\s+p-n\le t+j\le 0\\m,p\in\mathbb{Z}_+}}\binom{m-l-i}{m}\binom{p-t-j}{p}u_{t+j-p-s-1}(\lambda+\partial)^p\delta_{tk}\delta_{sn}u_{-l}(-\lambda-\partial)^m u_{l+i-m-k-1}=$$

$$=\sum_{\substack{1\le l\le -j\\p-j-n\le l+i\\m\in\mathbb{Z}_+}}\binom{m-l-i}{m}u_l(-\lambda-\partial)^m u_{l+i+j-m-1}=\sum_{\substack{0\le\alpha\le k\le n+j\\k+i+1\le\alpha\le k+i-j}}\binom{k}{\alpha}u_{k+i-\alpha}(-\lambda-\partial)^\alpha u_{j-k-1}.$$

Finally, for the last term we have

$$\sum_{\substack{1\le l\le k\le n\\1\le t\le s\le n\\m,p\in\mathbb{Z}_+}}\frac{\partial u_j}{\partial q_{st}^{(p)}}\bigg|_{q=q^{can}}(\lambda+\partial)^p\left(\delta_{ls}\delta_{kt}\right)(\lambda+\partial)(-\lambda-\partial)^m\,\frac{\partial u_i}{\partial q_{kl}^{(m)}}\bigg|_{q=q^{can}}=$$

$$=\sum_{\substack{1\le l\le k\le n\\1\le t\le s\le n\\k+m-n\le l+i\le 0\\s+p-n\le t+j\le 0\\m,p\in\mathbb{Z}_+}}\binom{m-l-i}{m}\binom{p-t-j}{p}u_{t+j-p-s-1}(\lambda+\partial)^p\delta_{ls}\delta_{kt}(\lambda+\partial)(-\lambda-\partial)^m u_{l+i-m-k-1}=$$

$$=\sum_{\substack{1\le k\le\min(-i,-j)\\0\le m\le n+i\\0\le p\le n+j}}(-1)^m\binom{m-k-i}{m}\binom{p-k-j}{p}u_{j-p-1}(\lambda+\partial)^{m+p+1}u_{i-m-1}.$$

Hence, we have obtained

$$\begin{aligned}\{u_{i\lambda}u_j\}_0=&\sum_{\substack{0\le\alpha\le k\le n+j\\k+i+1\le\alpha\le k+i-j}}\binom{k}{\alpha}u_{k+i-\alpha}(-\lambda-\partial)^\alpha u_{j-k-1}\\&-\sum_{\substack{0\le\alpha\le k\le n+i\\k+j+1\le\alpha\le k+j-i}}\binom{k}{\alpha}u_{i-k-1}(\lambda+\partial)^\alpha u_{k+j-\alpha}+\\&+\sum_{\substack{1\le k\le\min(-i,-j)\\0\le m\le n+i\\0\le p\le n+j}}(-1)^m\binom{m-k-i}{m}\binom{p-k-j}{p}u_{j-p-1}(\lambda+\partial)^{m+p+1}u_{i-m-1}.\end{aligned}\tag{4.15}$$

It is not so evident that this expression is eqivalent to (4.4a). Let us write (4.4a) as $\{u_{i\lambda}u_j\}_0^c=A_1-A_2$ and (4.15) as $\{u_{i\lambda}u_j\}_0^z=B_1-B_2+B_3$ (the c and z index is to distinguish between 0-bracket of Gelfand-Dickey algebras or of classical $\mathcal{W}$-algebras obtained via classical Drinfeld-Sokolov Hamiltonian reduction). We want to prove that

$$A_1-A_2=B_1-B_2+B_3\tag{4.16}$$

We can rewrite

$$
\begin{aligned}
A_2 &= B_2 + \sum_{\substack{0\le\alpha\le k\le n+i\\ k+j-i+1\le\alpha}} \binom{k}{\alpha} u_{i-k-1}(\lambda+\partial)^{\alpha} u_{k+j-\alpha} = \\
&= B_2 + \sum_{\substack{0\le m+p+1\le p+i-j\\ 0\le m\le n+i\\ j\le p\le n+j}} \binom{p+i-j}{m+p+1} u_{j-p-1}(\lambda+\partial)^{m+p+1} u_{i-m-1} = \\
&= B_2 + \sum_{\substack{0\le m+p+1\le p+i-j\\ 0\le m\le n+i\\ 0\le p\le n+j}} \binom{p+i-j}{m+p+1} u_{j-p-1}(\lambda+\partial)^{m+p+1} u_{i-m-1} + \\
&+ \sum_{\substack{0\le m+p+1\le p+i-j\\ 0\le m\le n+i\\ j\le p\le -1}} \binom{p+i-j}{m+p+1} u_{j-p-1}(\lambda+\partial)^{m+p+1} u_{i-m-1} = B_2 + A_2' + A_2''.
\end{aligned}
$$

To prove equality (4.16) is then equivalent to prove the identity

$$
B_3 = A_1 - A_2' - A_2'' - B_1. \tag{4.17}
$$

It is convenient also to rewrite

$$
\begin{aligned}
B_1 &= \sum_{\substack{0\le\alpha\le k\le n+j\\ k+i+1\le\alpha\le k+i-j}} \binom{k}{\alpha} u_{k+i-\alpha}(-\lambda-\partial)^{\alpha} u_{j-k-1} = \\
&= \sum_{\substack{0\le m+p+1\le m+j-i\\ i\le m\le n+i\\ j\le p\le -1}} \binom{m+j-i}{m+p+1} u_{j-p-1}(-\lambda-\partial)^{m+p+1} u_{i-m-1} = \\
&= \sum_{\substack{0\le m+p+1\le m+j-i\\ i\le m\le n+i\\ j\le p\le -1}} \binom{p+i-j}{m+p+1} u_{j-p-1}(\lambda+\partial)^{m+p+1} u_{i-m-1}
\end{aligned}
$$

and

$$
\begin{aligned}
A_1 &= \sum_{\substack{k,\alpha,\beta\in\mathbb{Z}_+\\ i-k\le\alpha\le n+i-k\\ j+k+1\le\beta\le n+j+k+1}} (-1)^{\beta} \binom{i}{\alpha} \binom{j+k}{\beta} u_{j+k-\beta}(\lambda+\partial)^{\alpha+\beta} u_{i-k-\alpha-1} = \\
&= \sum_{\substack{0\le k\le m\le n+i\\ j\le p\le n+j}} (-1)^{k+p+1} \binom{i}{m-k} \binom{j+k}{k+p+1} u_{j-p-1}(\lambda+\partial)^{m+p+1} u_{i-m-1} = \\
&= \sum_{\substack{0\le k\le m\le n+i\\ j\le p\le -1}} \binom{i}{k} \binom{p-j}{m+p+1-k} u_{j-p-1}(\lambda+\partial)^{m+p+1} u_{i-m-1} + \\
&+ \sum_{\substack{0\le k\le m\le n+i\\ 0\le p\le n+j}} \binom{i}{k} \binom{p-j}{m+p+1-k} u_{j-p-1}(\lambda+\partial)^{m+p+1} u_{i-m-1}.
\end{aligned}
$$

Let us denote $c(m,p)=u_{j-p-1}(\lambda+\partial)^{m+p+1}u_{i-m-1}$ ad rewrite equation (4.17)as

$$\sum_{\substack{1\leq k\leq \min(-i,-j)\\0\leq m\leq n+i\\0\leq p\leq n+j}}(-1)^m\binom{m-k-i}{m}\binom{p-k-j}{p}c(m,p)=\sum_{\substack{0\leq k\leq m\leq n+i\\0\leq p\leq n+j}}\binom{i}{k}\binom{p-j}{m+p+1-k}c(m,p)+$$
$$+\sum_{\substack{0\leq k\leq m\leq n+i\\j\leq p\leq -1}}\binom{i}{k}\binom{p-j}{m+p+1-k}c(m,p)-\sum_{\substack{0\leq m+p+1\leq p+i-j\\0\leq m\leq n+i\\0\leq p\leq n+j}}\binom{p+i-j}{m+p+1}c(m,p)$$
$$+\sum_{\substack{0\leq m+p+1\leq p+i-j\\0\leq m\leq n+i\\j\leq p\leq -1}}\binom{p+i-j}{m+p+1}c(m,p)-\sum_{\substack{0\leq m+p+1\leq m+j-i\\i\leq m\leq n+i\\j\leq p\leq -1}}\binom{p+i-j}{m+p+1}c(m,p).$$

Then the proof is finished if we prove the following two identities

$$\begin{aligned}&\sum_{k=1}^{\min(-i,-j)}(-1)^m\binom{m-k-i}{m}\binom{p-k-j}{p}=\\&=\sum_{0\leq k\leq m}\binom{i}{k}\binom{p-j}{m+p+1-k}-\delta_{0\leq m+p+1\leq p+i-j}\binom{p+i-j}{m+p+1}\end{aligned}\tag{4.18}$$

for $0\leq m\leq n+i, 0\leq p\leq n+j$ and

$$\sum_{0\leq k\leq m}\binom{i}{k}\binom{p-j}{m+p+1-k}=(\delta_{0\leq m+p+1\leq p+i-j}+\delta_{0\leq m+p+1\leq m+j-i})\binom{p+i-j}{m+p+1}\tag{4.19}$$

for $0\leq m\leq n+i, j\leq p\leq -1$.

We note that the left hand side of (4.19) becomes

$$\sum_{k=0}^{m+p+1}\binom{i}{k}\binom{p-j}{m+p+1-k}=\binom{p+i-j}{m+p+1},$$

since the binomial coefficient makes sense for $k\leq m+p+1$ and $p+1\leq 0$, thus $m+p+1\leq m$, and we have used Lemma B.1. Then identity (4.19) is proved because in the right hand side the existence conditions defined by δ are not contemporary satisfied. Indeed, if $p+i-j\in\mathbb{Z}_+$, it follows that $m+p+1\geq m+j-i+1$.

Let us prove now identity (4.18). Let $a=\min(-i,-j)$, we can rewrite (4.19) in this way

$$\sum_{k=1}^{a}\binom{k+i-1}{m}\binom{p-k-j}{p}=\sum_{\alpha=0}^{m}\binom{i}{\alpha}\binom{p-j}{p+m+1-\alpha}-\delta_{m\leq i-j-1}\binom{p+i-j}{m+p+1}.$$

By Lemma B.2, the right hand side becomes

$$\begin{aligned}&\sum_{\alpha=0}^{m}\binom{i}{\alpha}\binom{p-j}{p+m+1-\alpha}-\delta_{m\leq i-j-1}\binom{p+i-j}{m+p+1}=\\
&=\sum_{\alpha=0}^{m}\sum_{k=m+1-\alpha}^{l}\binom{i}{\alpha}\binom{k-1}{m-\alpha}\binom{p-j-k}{p}+\sum_{\alpha=0}^{m}\sum_{h=0}^{m-\alpha}\binom{l}{h}\binom{i}{\alpha}\binom{p-j-l}{m+p+1-\alpha-h}-\\
&-\delta_{m\leq i-j-1}\binom{p+i-j}{m+p+1}=\sum_{k=1}^{l}\left(\sum_{\alpha=0}^{m}\binom{i}{\alpha}\binom{k-1}{m-\alpha}\right)\binom{p-j-k}{p}+\\
&+\sum_{0\leq\alpha\leq t\leq m}\binom{i}{\alpha}\binom{l}{t-\alpha}\binom{p-j-l}{m+p+1-t}-\delta_{m\leq i-j-1}\binom{p+i-j}{m+p+1}=\\
&=\sum_{k=1}^{l}\binom{k+i-1}{m}\binom{p+j-k}{p}+\sum_{t=0}^{m}\binom{i+l}{t}\binom{p-j-l}{m+p+1-t}-\delta_{m\leq i-j-1}\binom{p+i-j}{m+p+1}.\end{aligned}$$

If we choose $l=a$, then we are left to prove that

$$\sum_{t=0}^{m}\binom{i+a}{t}\binom{p-j-a}{m+p+1-t}=\delta_{m\leq i-j-1}\binom{p+i-j}{m+p+1}.$$

If $a=-j$, then the right hand side is zero, while for the left hand side we have

$$\sum_{t=0}^{m}\binom{i-j}{t}\binom{p}{m+p+1-t}=0,$$

since $m+1-t>0$ for all $0\le t\le m$, so the second binomial coefficient in the product is zero. If $a=-i$ the right hand side is

$$\sum_{t=0}^{m}\binom{0}{t}\binom{p+i-j}{m+p+1-t}=\binom{p+i-j}{m+p+1}$$

if and only if $m\le i-j-1$, thus proving the identity and completing the proof.

4.3. $\mathcal{W}_z(\mathfrak{sl}_n,f,s)\cong\widehat{\mathcal{W}}_n$ for f principal nilpotent

Let $\mathfrak{sl}_n=\{a\in\mathfrak{gl}_n\mid \mathrm{tr}(a)=0\}$ be the Lie algebra of traceless matrices and consider $f=\sum_{i=1}^{n-1}E_{i+1i}$ its principal nilpotent element. We fix the following basis of $\mathfrak{sl}_n$:

$$\begin{aligned} q_{ii}&=E_{ii}-\frac{1}{n}\mathbb{1}_n, && 1\le i\le n-1,\\ q_{ij}&=E_{ij}, && 1\le i\neq j\le n. \end{aligned}\tag{4.20}$$

The dual basis with respect to the trace form is

$$\begin{aligned} q^{ii}&=E_{ii}-E_{nn}, && 1\le i\le n-1,\\ q^{ij}&=E_{ji}, && 1\le i\neq j\le n. \end{aligned}\tag{4.21}$$

This gives us the definition of q in (2.3). We fix $s=E_{1n}$, then the λ-bracket (2.13) on the basis elements is

$$\{q_{kl\,\lambda}q_{st}\}_z=\delta_{ls}q_{kt}-\delta_{tk}q_{sl}+\delta_{ls}\delta_{kt}\lambda-\delta_{kl}\delta_{st}\frac{\lambda}{n}+z(\delta_{ls}\delta_{kn}\delta_{1t}-\delta_{kt}\delta_{sn}\delta_{1l}),\tag{4.22}$$

for all $(k,l),(s,t)\in S=\{(i,j)\mid 1\le i,j\le n,(i,j)\neq(n,n)\}$.

As we did in the previous section we can choose $V=\bigoplus_{i=1}^{n-1}\mathbb{C}E_{in}$ in Proposition 2.1 and, if we set

$$q^{can}=-\sum_{i=1}^{n-1}E_{in}\otimes u_{-i},$$

then we have the assignment $\mathcal{L}^{can}\longrightarrow L=\sum_{k=0}^{n}u_{-k-1}\partial^k$, where $u_{-n-1}=1$, $u_{-n}=0$ and $u_i=u_i(q)$, for $i\in I=\{-n+1,\dots,-1\}$. By Corollary 2.2, it follows that, as differential algebras,

$$\mathcal{W}_z(\mathfrak{sl}_n,f,s)=\mathbb{C}[u_i^{(m)}\mid i\in I,m\in\mathbb{Z}_+]=\widehat{\mathcal{W}}_n.$$

By gauge invariance, $\{u_i(q)_\lambda u_j(q)\}_z=\{u_i(q)_\lambda u_j(q)\}_z|_{q=q^{can}}=\{u_{i\lambda}u_j\}_z$. We want to prove that

$$\{u_{i\lambda}u_j\}_z=\{u_{i\lambda}u_j\}_{\widehat{c}}.$$

Expliting computing the coefficients of the generating series for $\{\cdot_\lambda\cdot\}_{\widehat{c}}$ we get

$$\{u_{i\lambda}u_j\}_{\widehat{c}}=\{u_{i\lambda}u_j\}'_c+\frac{1}{n}\sum_{\substack{1\le\alpha\le n+i+1\\1\le\beta\le n+j+1}}(-1)^\beta\binom{i}{\alpha}\binom{j}{\beta}u_{j-\beta}(\lambda+\partial)^{\alpha+\beta+1}u_{i-\alpha}\tag{4.23}$$

where $\{\cdot_\lambda\cdot\}'_c$ has the same expression of (4.4) with the condition $u_{-n}=0$.

By formula (1.11), recalling that

$$q=\sum_{(i,j)\in S_-}q^{ij}\otimes q_{ij},$$

where $S_-=S\cap\{(i,j)\mid 1\le j\le i\le n\}$, we have

$$\{u_{i\lambda}u_j\}_z=\sum_{\substack{(k,l),(s,t)\in S_-\\m.p\in\mathbb{Z}_+}}\left.\frac{\partial u_j}{\partial q_{st}^{(p)}}\right|_{q=q^{can}}(\lambda+\partial)^p\,\{q_{kl\,\lambda+\partial}q_{st}\}_\rightarrow\big|_{q=q^{can}}(-\lambda-\partial)^m\left.\frac{\partial u_i}{\partial q_{kl}^{(m)}}\right|_{q=q^{can}},\tag{4.24}$$

where $\{q_{kl\,\lambda}q_{st}\}_z$ is defined in (4.22).

As done in the previous section we have to compute the partial derivatives of the u_k's and specialize them in the case $q = q^{can}$. In this case we have

$$
\begin{aligned}
\alpha &= \alpha(\partial) = v_1 \otimes \partial + \sum_{i=1}^{n-1} v_i \otimes q_{i1}, \\
\beta &= \beta(\partial) = q_{n1}, \\
\gamma &= \gamma(\partial) = v_{n-1}^t \otimes \partial - \sum_{i=1}^{n-1} v_{n-1}^t \otimes q_{ii} + \sum_{i=1}^{n-2} v_i^t \otimes q_{ni+1}, \\
A &= A(\partial) = \sum_{i=1}^{n-1} E_{ii} \otimes 1 + \sum_{i=1}^{n-2} E_{ii+1} \otimes \partial + \sum_{1 \le j < i \le n-1} E_{ji} \otimes q_{ij+1}
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\alpha^* &= \alpha^*(\partial) = -v_1^t \otimes \partial + \sum_{i=1}^{n-1} v_i^t \otimes q_{i1}, \\
\beta^* &= \beta^*(\partial) = q_{n1}, \\
\gamma^* &= \gamma^*(\partial) = -v_{n-1} \otimes \partial - \sum_{i=1}^{n-1} v_{n-1} \otimes q_{ii} + \sum_{i=1}^{n-2} v_i \otimes q_{ni+1}, \\
A^* &= A^*(\partial) = \sum_{i=1}^{n-1} E_{ii} \otimes 1 - \sum_{i=1}^{n-2} E_{i+1i} \otimes \partial + \sum_{1 \le j < i \le n-1} E_{ij} \otimes q_{ij+1}.
\end{aligned}
$$

Furthermore, since $q_{ij}^{can} = -\delta_{in}\delta_{j\neq n}u_{-j}$, we have

$$
\begin{aligned}
\alpha^*_{can}(\partial) &= -v_1^t \otimes \partial \\
\beta^*_{can}(\partial) &= -u_{-1} \\
\gamma^*_{can}(\partial) &= -\left(v_{n-1} \otimes \partial + \sum_{i=1}^{n-2} v_i \otimes u_{-i-1} \right) \\
\left(A^*_{can}\right)^{-1}(\partial) &= \sum_{1 \le j \le i \le n-1} E_{ij} \otimes \partial^{i-j}.
\end{aligned}
\tag{4.25}
$$

Now, we compute the partial derivatives. They are

$$
\begin{aligned}
\frac{\partial \alpha^*}{\partial q_{kl}^{(m)}}(\partial) &= \delta_{m0}\delta_{k\neq n}\delta_{l1} v_k^t \otimes 1 \\
\frac{\partial \beta^*}{\partial q_{kl}^{(m)}}(\partial) &= \delta_{m0}\delta_{kn}\delta_{l1} \\
\frac{\partial \gamma^*}{\partial q_{kl}^{(m)}}(\partial) &= -\delta_{m0}\delta_{kl} v_{n-1} \otimes 1 + \delta_{m0}\delta_{kn}\delta_{l\neq 1,n} v_{l-1} \otimes 1 \\
\frac{\partial A^*}{\partial q_{kl}^{(m)}}(\partial) &= \delta_{m0}\delta_{k\neq 1,n}\delta_{l\neq 1,n} E_{kl-1} \otimes 1.
\end{aligned}
\tag{4.26}
$$

It is a straightforward computation, using the same strategy of the previous section, to derive

$$
\begin{aligned}
\frac{\partial \gamma^*}{\partial q_{kl}^{(m)}}(\partial) \left(\left(A^*_{can}\right)^{-1} \alpha^*_{can} \right)(\partial) &= \delta_{m0}\delta_{kl}\partial^{n-1} - \delta_{m0}\delta_{kn}\delta_{l\neq 1,n}, \\
\gamma^*_{can}(z+\partial) \frac{\partial (A^*)^{-1}}{\partial q_{kl}^{(m)}}(\partial) \alpha^*_{can}(\partial) &= -\delta_{m0}\delta_{k\neq 1,n}\delta_{l\neq 1,n} \sum_{i=k}^{n} u_{-1-1}(z+\partial)^{i-k}\partial^{l-1}, \\
\gamma^*_{can}(A^*_{can})^{-1}(z+\partial) \frac{\partial \alpha^*}{\partial q_{kl}^{(m)}}(\partial) &= -\delta_{m0}\delta_{l1}\delta_{k\neq n} \sum_{i=k}^{n} u_{-i-1}(z+\partial)^{i-k},
\end{aligned}
$$

where we should remember $u_{-n}=0$ and $u_{-n-1}=1$. Using these identities we get

$$\begin{aligned}\frac{\partial(\gamma^*(A^*)^{-1}\alpha^*-\beta^*)}{\partial q^{(m)}_{kl}}(\partial)\bigg|_{q=q^{can}} &= \delta_{m0}\delta_{kl}\partial^{n-1}-\sum_{i=m}^{n-k}\binom{i}{m}u_{-i-k-1}\partial^{i+l-m-1}= \\ &= \delta_{m0}\delta_{kl}\partial^{n-1}-\sum_{i=0}^{n-k-m}\binom{m+i}{m}u_{-i-m-k-1}\partial^{i+l-1}.\end{aligned}\tag{4.27}$$

By (4.27), for $i\in I$, we have

$$\begin{aligned}\frac{\partial u_i}{\partial q^{(m)}_{kl}}\bigg|_{q=q^{can}} &= \operatorname{Res}\left(\delta_{m0}\delta_{kl}\partial^{n+i-1}-\sum_{\alpha=0}^{n-k-m}\binom{m+\alpha}{m}u_{-\alpha-m-k-1}\partial^{\alpha+l+i-1}\right)= \\ &= -\delta_{k+m-n\le l+i\le 0}\binom{m-i-l}{m}u_{i+l-m-k-1}.\end{aligned}$$

We should put this expression in (4.24). Looking at (4.22) it follows that we get exactly the same expression of (4.4) under the condition $u_{-n}=0$, which we denoted $\{\cdot{}_\lambda\cdot\}'_c$, plus another term corresponding to $\frac{1}{n}\{\cdot{}_\lambda\cdot\}_n$ that we are going to compute. It is

$$\begin{aligned}&-\frac{1}{n}\sum_{\substack{1\le h\le n-1\\ 1\le r\le n-1\\ m,p\in\mathbb{Z}_+\\ h+p-n\le h+i\le 0\\ r+m-n\le r+j\le 0}}\binom{p-h-i}{p}\binom{m-r-j}{m}u_{j-m-1}(\lambda+\partial)^{m+1}(-\lambda-\partial)^p u_{i-p-1}= \\ &=-\frac{1}{n}\sum_{\substack{1\le h\le -i\\ 1\le r\le -j\\ 0\le m\le n+j\\ 0\le p\le n+i}}(-1)^p\binom{p-h-i}{p}\binom{m-r-j}{m}u_{j-m-1}(\lambda+\partial)^{m+p+1}u_{i-p-1}.\end{aligned}\tag{4.28}$$

If we use Lemma B.2 with $j=0$ we obtain

$$\sum_{h=1}^{l}\binom{p-h-i}{p}+\binom{p+i-l}{p+1}=\binom{p-i}{p+1},$$

for all $l\ge 1$. We can choose $l=-i$ and get

$$\sum_{h=1}^{-i}\binom{p-h-i}{p}=\binom{p-i}{p+1}=(-1)^{p+1}\binom{i}{p+1}.$$

In a similar way we get

$$\sum_{h=1}^{-j}\binom{m-r-j}{m}=\binom{m-j}{m+1}=(-1)^{m+1}\binom{j}{m+1}.$$

Hence, after rescaling p and m, we obtain

$$(4.28)=\frac{1}{n}\sum_{\substack{1\le m\le n+j+1\\ 1\le p\le n+i+1}}(-1)^m\binom{i}{p}\binom{j}{m}u_{j-m}(\lambda+\partial)^{m+p-1}u_{i-p},$$

which coincide with the other term appearing in (4.23), thus proving the desired equality.

CHAPTER 5

Integrable hierarchies for classical $\mathcal{W}$-algebras

5.1. The homogeneuos case: integrable hierarchies for affine Poisson vertex algebras

Let us consider $\mathcal{V}_z(\mathfrak{g}, s)$ to be the affine Poisson vertex algebra associated to the triple $(\mathfrak{g}, (\cdot \mid \cdot), s)$ (see Example 1.6 and Section 2.2), where $\mathfrak{g}$ is a reductive finite dimensional Lie algebra, $(\cdot \mid \cdot)$ is a non-degenerate symmetric invariant bilinear form on it and $s \in \mathfrak{g}$. We remind that the λ-bracket on $\mathcal{V}_z(\mathfrak{g}, s)$ is defined by (2.13), namely, for $a, b \in \mathfrak{g}$, it is

$$\{a_\lambda b\}_z = [a, b] + (a \mid b)\lambda + z(s \mid [a, b])$$

and we extend it to a λ-bracket on $\mathcal{V}_z(\mathfrak{g}, s)$ using (1.11).

Let $\{u_i\}_{i\in I} \subset \mathfrak{g}$, where $I = \{1, 2, \dots, n = \dim \mathfrak{g}\}$, be a basis of $\mathfrak{g}$, then, as differential algebras $\mathcal{V}_z(\mathfrak{g}, s) = \mathbb{C}[u_i^{(m)} \mid i \in I, m \in \mathbb{Z}_+]$. We write the λ-bracket on $\mathcal{V}_z(\mathfrak{g}, s)$ as $\{\cdot_\lambda\cdot\}_z = \{\cdot_\lambda\cdot\}_H - z\{\cdot_\lambda\cdot\}_K$, where $H(\partial)$, respectively $K(\partial)$, is the Hamiltonian operator corresponding to $\{u_{i\lambda} u_j\}_H = [u_i, u_j] + (u_i \mid u_j)\lambda$, respectively $\{u_{i\lambda} u_j\}_K = -(s \mid [u_i, u_j])$, by (1.17). Hence,

$$H_{ij}(\partial) = [u_j, u_i] + (u_i \mid u_j)\partial \qquad \text{and} \qquad K_{ij}(\partial) = -(s \mid [u_j, u_i]), \tag{5.1}$$

for all $i, j \in I$.

We endow the space $\mathfrak{g}\otimes\mathcal{V}_z(\mathfrak{g}, s)$ a Lie algebra structure defining $[a\otimes f, b\otimes g] = [a, b]\otimes fg \in \mathfrak{g}\otimes\mathcal{V}_z(\mathfrak{g}, s)$, for any $a, b \in \mathfrak{g}$ and $f, g \in \mathcal{V}_z(\mathfrak{g}, s)$. Moreover, we can extend the bilinear form on $\mathfrak{g}$ to a bilinear form on $\mathfrak{g} \otimes \mathcal{V}_z(\mathfrak{g}, s)$, that we still denote $(\cdot \mid \cdot)$, by $(a \otimes f \mid b \otimes g) = (a \mid b)fg \in \mathcal{V}_z(\mathfrak{g}, s)$, for any $a, b \in \mathfrak{g}$ and $f, g \in \mathcal{V}_z(\mathfrak{g}, s)$. Also the extended bilinear form is non-degenerate symmetric and invariant.

Let ∂ be the derivation of $\mathcal{V}_z(\mathfrak{g}, s)$, we define an action of the abelian Lie algebra $\mathbb{C}\partial$ on $\mathfrak{g} \otimes \mathcal{V}_z(\mathfrak{g}, s)$ by

$$\partial.(a \otimes f) = a \otimes \partial f, \tag{5.2}$$

for any $a \in \mathfrak{g}$ and $f \in \mathcal{V}_z(\mathfrak{g}, s)$. Clearly, ∂ acts as a derivation of $\mathfrak{g} \otimes \mathcal{V}_z(\mathfrak{g}, s)$. Indeed, since it is a derivation of $\mathcal{V}_z(\mathfrak{g}, s)$, we have

$$\begin{aligned}\partial.[a \otimes f, b \otimes g] &= \partial.([a, b] \otimes fg) = [a, b] \otimes \partial(fg) = [a, b] \otimes (\partial f)g + [a, b] \otimes f(\partial g) = \\ &= [a \otimes \partial f, b \otimes g] + [a \otimes f, b \otimes \partial g] = [\partial.(a \otimes f), b \otimes g] + [a \otimes f, \partial.(b \otimes g)],\end{aligned}$$

for any $a, b \in \mathfrak{g}$ and $f, g \in \mathcal{V}_z(\mathfrak{g}, s)$. Thus we can define the semidirect product Lie algebra $\mathbb{C}\partial \ltimes (\mathfrak{g} \otimes \mathcal{V}_z(\mathfrak{g}, s))$, where the commutator of ∂ against elements of $\mathfrak{g} \otimes \mathcal{V}_z(\mathfrak{g}, s)$ is defined by

$$[\partial, a \otimes f] = \partial.(a \otimes f) = a \otimes \partial f,$$

for any $a \in \mathfrak{g}$ and $f \in \mathcal{V}_z(\mathfrak{g}, s)$. We set $\widetilde{\mathfrak{g}} = (\mathbb{C}\partial \ltimes \mathfrak{g} \otimes \mathcal{V}_z(\mathfrak{g}, s))((z^{-1}))$.

Given $U(z) \in z^{-1}(\mathfrak{g} \otimes \mathcal{V}_z(\mathfrak{g}, s))[[z^{-1}]]$, the map $e^{\operatorname{ad} U(z)} : \widetilde{\mathfrak{g}} \longrightarrow \widetilde{\mathfrak{g}}$ is a Lie algebra automorphism. Indeed, it is a well defined map since it does not increase the order of powers of z and, moreover, using the fact that the adjoint action is a derivation, we have, for $A(z), B(z) \in \widetilde{\mathfrak{g}}$,

$$(\operatorname{ad} U(z))^k([A(z), B(z)]) = \sum_{i=0}^{k} \binom{k}{i} \left[(\operatorname{ad} U(z))^i(A(z)), (\operatorname{ad} U(z))^{k-i}(B(z))\right], \tag{5.3}$$

then we get

$$\begin{aligned}e^{\operatorname{ad} U(z)}([A(z), B(z)]) &= \sum_{k\in\mathbb{Z}_+} \frac{(\operatorname{ad} U(z))^k}{k!}([A(z), B(z)]) = \\ &= \sum_{k\in\mathbb{Z}_+} \sum_{i=0}^{k} \frac{1}{i!(k-i)!} \left[(\operatorname{ad} U(z))^i(A(z)), (\operatorname{ad} U(z))^{k-i}(B(z))\right] = \\ &= \sum_{k,i\in\mathbb{Z}_+} \left[\frac{(\operatorname{ad} U(z))^i}{i!}(A(z)), \frac{(\operatorname{ad} U(z))^k}{k!}(B(z))\right] = \left[e^{\operatorname{ad} U(z)}(A(z)), e^{\operatorname{ad} U(z)}(B(z))\right],\end{aligned}$$

for any $A(z), B(z) \in \widetilde{\mathfrak{g}}$, thus proving that $e^{\operatorname{ad} U(z)}$ is a Lie algebra homomorphism. Clearly, its inverse is given by $e^{-\operatorname{ad} U(z)}$. By Campbell-Hausdorff formula [23], it follows that automorphisms of this type form a group.

Let $\{u^i\}_{i\in I} \subset \mathfrak{g}$ be the dual basis with respect to the bilinear form $(\cdot \mid \cdot)$. We consider the following identifications

$$\begin{aligned} \mathcal{V}_z(\mathfrak{g}, s)^{\oplus I} &\xrightarrow{\sim} \mathfrak{g} \otimes \mathcal{V}_z(\mathfrak{g}, s) \\ F = (F_i)_{i\in I} &\longrightarrow \underline{F} = \sum_{i\in I} u_i \otimes F_i \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} \mathcal{V}_z(\mathfrak{g}, s)^{I} &\xrightarrow{\sim} \mathfrak{g} \otimes \mathcal{V}_z(\mathfrak{g}, s) \\ P = (P_i)_{i\in I} &\longrightarrow \overline{P} = \sum_{i\in I} u^i \otimes P_i. \end{aligned} \tag{5.5}$$

We can define a pairing $(\cdot,\cdot) : \mathfrak{g} \otimes \mathcal{V}_z(\mathfrak{g}, s) \times \mathfrak{g} \otimes \mathcal{V}_z(\mathfrak{g}, s) \longrightarrow \mathcal{V}_z(\mathfrak{g}, s)/\partial\mathcal{V}_z(\mathfrak{g}, s)$, using the bilinear form on $\mathfrak{g} \otimes \mathcal{V}_z(\mathfrak{g}, s)$, by

$$(a \otimes f, b \otimes g) = \int (a \otimes f \mid b \otimes g) = \int (a \mid b) f g,$$

for any $a, b \in \mathfrak{g}$ and $f, g \in \mathcal{V}_z(\mathfrak{g}, s)$. Using integration by parts and (5.2), we have $(\partial.(a \otimes f), b \otimes g) = -(a \otimes f, \partial.(b \otimes g))$. By identifications (5.4) and (5.5), we note that

$$(\underline{F}, \overline{P}) = \int (\underline{F} \mid \overline{P}) = \int \sum_{i,j\in I} (u_i \mid u^j) F_i P_j = \int \sum_{i\in I} F_i P_i, \tag{5.6}$$

which coincides with (1.3). Thus, if $f \in \mathcal{V}_z(\mathfrak{g}, s)$, then, using (5.4), its variational derivative is identified with

$$\frac{\delta f}{\underline{\delta u}} = \sum_{i=1}^{n} u_i \otimes \frac{\delta f}{\delta u_i} \in \mathfrak{g} \otimes \mathcal{V}_z(\mathfrak{g}, s).$$

In the sequel, when it is clear from the contest, we will denote this element simply by $\frac{\delta f}{\delta u}$.

We set $u = \sum_{i\in I} u^i \otimes u_i \in \mathfrak{g} \otimes \mathcal{V}_z(\mathfrak{g}, s)$ and define

$$L(z) = \partial + u + zs \otimes 1 \in \widetilde{\mathfrak{g}}.$$

Proposition 5.1. *If $F = (F_i)_{i\in I} \in \mathcal{V}_z(\mathfrak{g}, s)^{\oplus I}$, then*

$$\overline{(H(\partial) - zK(\partial))F} = [L(z), \underline{F}].$$

Proof. By (5.5), we have

$$\overline{(H(\partial) - zK(\partial))F} = \sum_{i\in I} u^i \otimes ((H(\partial) - zK(\partial))F)_i \,.$$

On the other hand, given an element $A \in \mathfrak{g} \otimes \mathcal{V}_z(\mathfrak{g}, s)$ we can write it uniquely as $A = \sum_{i\in I} u^i \otimes (A \mid u_i \otimes 1)$. Indeed, since $\{u^i\}_{i\in I}$ is a basis of $\mathfrak{g}$, we can write $A = \sum_{i\in I} u^i \otimes c_i$, with $c_i \in \mathcal{V}_z(\mathfrak{g}, s)$. Then $(A \mid u_i \otimes 1) = \sum_{j\in I} (u^j \otimes c_j \mid u_i \otimes 1) = \sum_{j\in I} (u^j \mid u_i) c_j = c_i$. It follows that the proposition is proved if we show that

$$((H(\partial) - zK(\partial))F)_i = ([L(z), \underline{F}] \mid u_i \otimes 1)\,. \tag{5.7}$$

Using (5.1), the left hand side of (5.7) is

$$((H(\partial) - zK(\partial))F)_i = \sum_{j\in I} (H_{ij}(\partial) - zK_{ij}(\partial))F_j = \sum_{j\in I} \left((u_i \mid u_j)F_j' + [u_j, u_i]F_j + z(s \mid [u_j, u_i])F_j\right).$$

On the other hand we have

$$[L(z), \underline{F}] = \sum_{j\in I} [\partial + u + zs \otimes 1, u_j \otimes F_j] = \sum_{j\in I} \left(u_j \otimes F_j' + \sum_{k\in I} [u^k, u_j] \otimes u_k F_j + z[s, u_j] \otimes F_j \right).$$

Taking the scalar product of this term with $u_i \otimes 1$ gives the right hand side of (5.7). We get

$$([L(z), \underline{F}] \mid u_i \otimes 1) = \sum_{j\in I} \left((u_j \mid u_i)F_j' + \sum_{k\in I} ([u^k, u_j] \mid u_i) u_k F_j + z([s, u_j] \mid u_i)F_j \right).$$

Using the invariance of the bilinear form we have $\sum_{k\in I}([u^k,u_j]\mid u_i)u_k=\sum_{k\in I}(u^k\mid [u_j,u_i])u_k=[u_j,u_i]$ and $([s,u_j]\mid u_i)=(s\mid [u_j,u_i])$ and by symmetry, $(u_i\mid u_j)=(u_j\mid u_i)$, proving that left hand side and right hand side of (5.7) are equal. $\square$

Let us assume $s\in\mathfrak{g}$ be a semisimple element with nontrivial adjoint action and denote by $\mathfrak{h}=\ker \operatorname{ad} s$. Then,

$$\mathfrak{g}=\mathfrak{h}\oplus \operatorname{im}\operatorname{ad} s. \tag{5.8}$$

Moreover, $\operatorname{im}\operatorname{ad} s=\mathfrak{h}^\perp$. Indeed, by invariance of the bilinear form, $\operatorname{im}\operatorname{ad} s\subset\mathfrak{h}^\perp$. On the other hand, if $b\in\mathfrak{g}$ is such that $(a\mid b)=0$ for all $a\in\mathfrak{h}$ and we write $b=b_{\mathfrak{h}}+b'$, with $b'\in\operatorname{im}\operatorname{ad} s$, then, since $\operatorname{im}\operatorname{ad} s\subset\mathfrak{h}^\perp$, we have $(a\mid b_{\mathfrak{h}})=0$. This forces $b_{\mathfrak{h}}=0$, since the bilinear form is non-degenerate and proves the equality.

According to the discussion in Section 1.3, to find an integrable hierarchy of equations $\frac{du}{dt_k}=\{\int f_k,u\}_H$ and the associated infinite sequence of integrals of motions $\int f_k$, $k\in\mathbb{Z}_+$, we need to find an infinite (H,K)-sequence $\{F_n\}_{n\in\mathbb{Z}_+}\subset\mathcal{V}_z(\mathfrak{g},s)^{\oplus I}$ and prove that each F_n is exact. This means that we need to solve two problems. First, we need to find $F(z)\in\mathcal{V}_z(\mathfrak{g},s)^{\oplus I}[[z^{-1}]]$ such that (1.22) holds. Then, we need to find $\int f(z)\in\left(\mathcal{V}_z(\mathfrak{g},s)/\partial\mathcal{V}_z(\mathfrak{g},s)\right)[[z^{-1}]]$ such that $F(z)=\frac{\delta f(z)}{\delta u}$.

By identifications (5.4), (5.5) and Proposition 5.1, the first problem is equivalent to find $F(z)\in(\mathfrak{g}\otimes\mathcal{V}_z(\mathfrak{g},s))[[z^{-1}]]$ such that $[L(z),F(z)]=0$ and $[s\otimes 1,F_0]=0$, namely $F_0\in\mathfrak{h}\otimes\mathcal{V}_z(\mathfrak{g},s)$. The explicit description of all series commuting with $L(z)$ is given by the following proposition due to Drinfeld and Sokolov ([12, Proposition 4.1]).

Proposition 5.2. *There exists a formal series $U(z)\in z^{-1}(\mathfrak{g}\otimes\mathcal{V}_z(\mathfrak{g},s))[[z^{-1}]]$ such that*

$$L_0(z)=e^{\operatorname{ad} U(z)}(L(z))=\partial+zs\otimes 1+h(z), \tag{5.9}$$

with $h(z)\in(\mathfrak{h}\otimes\mathcal{V}_z(\mathfrak{g},s))[[z^{-1}]]$. The automorphism $e^{\operatorname{ad} U(z)}$ is defined uniquely up to multiplication on the left by automorphisms of the form $e^{\operatorname{ad} S(z)}$, where $S(z)\in z^{-1}(\mathfrak{h}\otimes\mathcal{V}_z(\mathfrak{g},s))[[z^{-1}]]$ and it is possible to choose $U(z)$ uniquely if we require $U(z)\in z^{-1}\left(\mathfrak{h}^\perp\otimes\mathcal{V}_z(\mathfrak{g},s)\right)[[z^{-1}]]$.

Proof. Writing $U(z)=\sum_{i\geq 1}U_iz^{-i}$, with $U_i\in\mathfrak{g}\otimes\mathcal{V}_z(\mathfrak{g},s)$, and equating coefficients of z^{-i} in both sides of (5.9), we find that $h_i+[s\otimes 1,U_{i+1}]$ can be expressed in terms of $U_1,U_2,\dots,U_i$ and $h_0,h_1,\dots,h_{i-1}$. For example, equating the costant term in (5.9) gives the relation $h_0+[s\otimes 1,U_1]=u$, while, equating the coefficients of z^{-1}, gives the relation $h_1+[s\otimes 1,U_2]=-U_1'+[U_1,u]+\frac{1}{2}[U_1,[U_1,s\otimes 1]]$ and so on. Let say $h_i+[s\otimes 1,U_{i+1}]=A\in\mathfrak{g}\otimes\mathcal{V}_z(\mathfrak{g},s)$, where, as already pointed out, we explicitly know A. By (5.8), we can write in a unique way $A=A_{\mathfrak{h}}+A_{\mathfrak{h}^\perp}$, where $A_{\mathfrak{h}}\in\mathfrak{h}\otimes\mathcal{V}_z(\mathfrak{g},s)$ and $A_{\mathfrak{h}^\perp}\in\mathfrak{h}^\perp\otimes\mathcal{V}_z(\mathfrak{g},s)$. Hence, we get $h_i=A_{\mathfrak{h}}$ and $U_{i+1}=(\operatorname{ad}(s\otimes 1))^{-1}(A_{\mathfrak{h}^\perp})$. Since the restriction of $\operatorname{ad} s$ to $\mathfrak{h}^\perp$ is an isomorphism, we can determime uniquely $U_{i+1}\in\mathfrak{h}^\perp\otimes\mathcal{V}_z(\mathfrak{g},s)$. Therefore, we can uniquely determine $h(z)\in(\mathfrak{h}\otimes\mathcal{V}_z(\mathfrak{g},s))[[z^{-1}]]$ and $U(z)\in z^{-1}(\mathfrak{h}^\perp\otimes\mathcal{V}_z(\mathfrak{g},s))[[z^{-1}]]$.

Let $\widetilde{U}(z)\in z^{-1}(\mathfrak{g}\otimes\mathcal{V}_z(\mathfrak{g},s))[[z^{-1}]]$ be such that $e^{\operatorname{ad}\widetilde{U}(z)}(L(z))=\widetilde{L}_0(z)$, where $\widetilde{L}_0(z)$ is of the same type of (5.9), namely, $\widetilde{L}_0(z)=\partial+zs\otimes 1+\widetilde{h}(z)$, with $\widetilde{h}(z)\in(\mathfrak{h}\otimes\mathcal{V}_z(\mathfrak{g},s))[[z^{-1}]]$. Since these automorphisms form a group, there exists $S(z)\in z^{-1}(\mathfrak{g}\otimes\mathcal{V}_z(\mathfrak{g},s))[[z^{-1}]]$ such taht $e^{\operatorname{ad}\widetilde{U}(z)}e^{-\operatorname{ad} U(z)}=e^{\operatorname{ad} S(z)}$. Equating coefficients of powers of z^{-i} in the expression $e^{\operatorname{ad} S(z)}(L_0(z))=\widetilde{L}_0(z)$, we have that $S(z)\in(\mathfrak{h}\otimes\mathcal{V}_z(\mathfrak{g},s))[[z^{-1}]]$, since both coefficients of $L_0(z)$ and $\widetilde{L}_0(z)$ lie in $(\mathfrak{h}\otimes\mathcal{V}_z(\mathfrak{g},s))[[z^{-1}]]$ and $[s\otimes 1,\mathfrak{h}^\perp]\subset\mathfrak{h}^\perp$. $\square$

Let $a\in\mathfrak{Z}(\mathfrak{h})=\{a\in\mathfrak{h}\mid [a,b]=0,\text{ for all } b\in\mathfrak{h}\}$ (note that $\mathfrak{Z}(\mathfrak{h})\neq(0)$, since $s\in\mathfrak{Z}(\mathfrak{h})$), but $a\notin\mathfrak{Z}(\mathfrak{g})$, and set $F(z)=e^{-\operatorname{ad} U(z)}(a\otimes 1)$, where $U(z)$ is the same as in Proposition 5.9. $F(z)$ does not depend on the choice of $U(z)$. Indeed, by Proposition 5.2, if we choose another series $\widetilde{U}(z)$ such that (5.9) holds, then $\widetilde{F}(z)=e^{-\operatorname{ad}\widetilde{U}(z)}(a\otimes 1)=e^{-\operatorname{ad} U(z)}e^{-\operatorname{ad} S(z)}(a\otimes 1)=e^{\operatorname{ad} U(z)}(a\otimes 1)=F(z)$, since $a\in\mathfrak{Z}(\mathfrak{h})$.

Since $e^{-\operatorname{ad} U(z)}$ is an automorphism for $\widetilde{\mathfrak{g}}$ and $a\otimes 1$ commutes with $L_0(z)$, we get, as desired, $[L(z),F(z)]=e^{-\operatorname{ad} U(z)}([L_0(z),a\otimes 1])=0$. Moreover, $F_0=a\otimes 1\in\mathfrak{h}\otimes\mathcal{V}_z(\mathfrak{g},s)$.

Finally, we set $\int f(z)=\int(a\otimes 1\mid h(z))\in\left(\mathcal{V}(\mathfrak{g})/\partial\mathcal{V}_z(\mathfrak{g},s)\right)[[z^{-1}]]$.

Proposition 5.3. *We have*

$$\frac{\delta f(z)}{\delta u}=e^{-\operatorname{ad} U(z)}(a\otimes 1)=F(z). \tag{5.10}$$

Proof. We recall that, by (5.4),

$$\frac{\delta f(z)}{\delta u}=\sum_{i\in I}u_i\otimes\frac{\delta f(z)}{\delta u_i}\in(\mathfrak{g}\otimes\mathcal{V}_z(\mathfrak{g},s))[[z^{-1}]],$$

where $\frac{\delta f(z)}{\delta u_i}=\sum_{m\in\mathbb{Z}_+}(-\partial)^m\frac{\partial f(z)}{\partial u_i^{(m)}}$, for all $i\in I$. We start computing the partial derivatives. First we note that we can extend in a natural way the partial derivatives $\frac{\partial}{\partial u_i^{(m)}}$ of $\mathcal{V}_z(\mathfrak{g},s)$ to linear maps, which by abuse of notation we denote in the same way, $\frac{\partial}{\partial u_i^{(m)}}:\mathfrak{g}\otimes\mathcal{V}_z(\mathfrak{g},s)\longrightarrow\mathfrak{g}\otimes\mathcal{V}_z(\mathfrak{g},s)$ by

$$\frac{\partial}{\partial u_i^{(m)}}(a\otimes f)=a\otimes\frac{\partial f}{\partial u_i^{(m)}},$$

for all $a\in\mathfrak{g}$, $f\in\mathcal{V}_z(\mathfrak{g},s)$ and $i\in I$, $m\in\mathbb{Z}_+$. These linear maps are still derivations. Furthermore, we can extend them to the whole $\widetilde{\mathfrak{g}}$, defining $\frac{\partial}{\partial u_i^{(m)}}(\partial)=0$.

We have, for all $i\in I$ and $m\in\mathbb{Z}_+$,

$$\frac{\partial f(z)}{\partial u_i^{(m)}}=\frac{\partial}{\partial u_i^{(m)}}(a\otimes 1\mid h(z))=\left(a\otimes 1\,\middle|\,\frac{\partial h(z)}{\partial u_i^{(m)}}\right),$$

since partial derivatives act as derivations of the bilinear form (indeed they act only on the right term in the tensor product $\mathfrak{g}\otimes\mathcal{V}_z(\mathfrak{g},s)$). By (5.9), we can write $h(z)=e^{\operatorname{ad}U(z)}(L(z))-\partial-zs\otimes 1$. Hence,

$$\begin{aligned}\frac{\partial h(z)}{\partial u_i^{(m)}}&=\frac{\partial e^{\operatorname{ad}U(z)}(L(z))}{\partial u_i^{(m)}}=\frac{\partial}{\partial u_i^{(m)}}\left(\sum_{k\in\mathbb{Z}_+}\frac{(\operatorname{ad}U(z))^k}{k!}(L(z))\right)=\\&=\frac{\partial L(z)}{\partial u_i^{(m)}}+\frac{\partial}{\partial u_i^{(m)}}\left(\sum_{k\geq 1}\frac{(\operatorname{ad}U(z))^k}{k!}(L(z))\right).\end{aligned}\tag{5.11}$$

We note that

$$\frac{\partial L(z)}{\partial u_i^{(m)}}=\frac{\partial u}{\partial u_i^{(m)}}=\sum_{j\in I}\frac{\partial}{\partial u_i^{(m)}}(u^j\otimes u_j)=\sum_{j\in I}u^j\otimes\frac{\partial u_j}{\partial u_i^{(m)}}=\delta_{m0}u^i\otimes 1.\tag{5.12}$$

Furthermore, let us write $L(z)=\partial+A(z)$, where $A(z)=u+zs\otimes 1$, then we can write

$$\frac{\partial}{\partial u_i^{(m)}}\left(\sum_{k\geq 1}\frac{(\operatorname{ad}U(z))^k}{k!}(L(z))\right)=\frac{\partial}{\partial u_i^{(m)}}\left(\sum_{k\geq 1}\frac{(\operatorname{ad}U(z))^k}{k!}(\partial)\right)+\frac{\partial}{\partial u_i^{(m)}}\left(\sum_{k\geq 1}\frac{(\operatorname{ad}U(z))^k}{k!}(A(z))\right).$$

Since partial derivatives act only on the right of the tensor product $\mathfrak{g}\otimes\mathcal{V}_z(\mathfrak{g},s)$, they are derivations of the Lie bracket on $\mathfrak{g}\otimes\mathcal{V}_z(\mathfrak{g},s)$. Hence,

$$\begin{aligned}\frac{\partial}{\partial u_i^{(m)}}\left(\sum_{k\geq 1}\frac{(\operatorname{ad}U(z))^k}{k!}(A(z))\right)&=\sum_{k\geq 1}\sum_{i=0}^{k-1}\frac{1}{k!}(\operatorname{ad}U(z))^i\frac{\partial\operatorname{ad}U(z)}{\partial u_i^{(m)}}(\operatorname{ad}U(z))^{k-i-1}(L(z))\\&\quad+\sum_{k\geq 1}\frac{(\operatorname{ad}U(z))^k}{k!}\left(\frac{\partial A(z)}{\partial u_i^{(m)}}\right)=\\=\sum_{k\geq 1}\sum_{i=0}^{k-1}\frac{1}{k!}(\operatorname{ad}U(z))^i\frac{\partial\operatorname{ad}U(z)}{\partial u_i^{(m)}}&(\operatorname{ad}U(z))^{k-i-1}(L(z))+\sum_{k\geq 1}\frac{(\operatorname{ad}U(z))^k}{k!}(\delta_{m,0}u^i\otimes 1),\end{aligned}\tag{5.13}$$

where $\frac{\partial A(z)}{\partial u_i^{(m)}}=\delta_{m0}u^i\otimes 1$ by (5.12).

We claim that

$$\frac{\partial}{\partial u_i^{(m)}}(\operatorname{ad}U(z))^k(\partial)=\sum_{i=0}^{k-1}(\operatorname{ad}U(z))^i\frac{\partial\operatorname{ad}U(z)}{\partial u_i^{(m)}}(\operatorname{ad}U(z))^{k-i-1}(\partial)-(\operatorname{ad}U(z))^{k-1}\left(\frac{\partial U(z)}{\partial u_i^{(m-1)}}\right),\tag{5.14}$$

for any $k\geq 1$. We prove formula (5.14) by induction on k. For $k=1$, using (1.1), we have

$$\frac{\partial}{\partial u_i^{(m)}}[U(z),\partial]=-\frac{\partial}{\partial u_i^{(m)}}(\partial U(z))=-\partial\left(\frac{\partial U(z)}{\partial u_i^{(m)}}\right)-\frac{\partial U(z)}{\partial u_i^{(m-1)}}=\operatorname{ad}\left(\frac{\partial U(z)}{\partial u_i^{(m)}}\right)-\frac{\partial U(z)}{\partial u_i^{(m-1)}}.$$

Let us assume that (5.14) holds for $k>1$ and prove it for $k+1$. We have, using the fact that partial derivatives are derivations for the Lie bracket in $\mathfrak{g}\otimes\mathcal{V}_z(\mathfrak{g},s)$ and inductive assumption,

$$\frac{\partial}{\partial u_i^{(m)}}(\operatorname{ad}U(z))^{k+1}(\partial)=\frac{\partial}{\partial u_i^{(m)}}[U(z),\operatorname{ad}U(z)^k(\partial)]=\left[\frac{\partial U(z)}{\partial u_i^{(m)}},(\operatorname{ad}U(z))^k(\partial)\right]$$
$$+\left[U(z),\frac{\partial}{\partial u_i^{(m)}}(\operatorname{ad}U(z))^k(\partial)\right]=\operatorname{ad}\frac{\partial U(z)}{\partial u_i^{(m)}}(\operatorname{ad}U(z))^k(\partial)$$
$$+\operatorname{ad}U(z)\sum_{i=0}^{k-1}(\operatorname{ad}U(z))^i\frac{\partial\operatorname{ad}U(z)}{\partial u_i^{(m)}}(\operatorname{ad}U(z))^{k-i-1}(\partial)-(\operatorname{ad}U(z))^k\left(\frac{\partial U(z)}{\partial u_i^{(m-1)}}\right)=$$
$$\sum_{i=0}^{k}(\operatorname{ad}U(z))^i\frac{\partial\operatorname{ad}U(z)}{\partial u_i^{(m)}}(\operatorname{ad}U(z))^{k-i}(\partial)-(\operatorname{ad}U(z))^k\left(\frac{\partial U(z)}{\partial u_i^{(m-1)}}\right).$$

We can substitute (5.13) and (5.14) in (5.12) and get

$$\frac{\partial h(z)}{\partial u_i^{(m)}}=\sum_{k\geq1}\sum_{i=0}^{k-1}\frac{1}{k!}(\operatorname{ad}U(z))^i\operatorname{ad}\frac{\partial U(z)}{\partial u_i^{(m)}}(\operatorname{ad}U(z))^{k-i-1}(L(z))-\sum_{k\in\mathbb{Z}_+}\frac{(\operatorname{ad}U(z))^k}{(k+1)!}\left(\frac{\partial U(z)}{\partial u_i^{(m-1)}}\right)$$
$$+\delta_{m0}e^{\operatorname{ad}U(z)}(u^i\otimes1).$$

We set

$$A_{i,m}(z)=\sum_{k\in\mathbb{Z}_+}\frac{(\operatorname{ad}U(z))^k}{(k+1)!}\left(\frac{\partial U(z)}{\partial u_i^{(m)}}\right),$$

for any $i\in I$ and $m\in\mathbb{Z}_+$, then we can rewrite

$$\frac{\partial h(z)}{\partial u_i^{(m)}}=[A_{i,m}(z),L_0(z)]+\delta_{m0}e^{\operatorname{ad}U(z)}(u^i\otimes1)-A_{i,m-1}(z).\tag{5.15}$$

Indeed,

$$\sum_{0\leq i\leq k}\frac{1}{(k+1)!}(\operatorname{ad}U(z))^i\left(\operatorname{ad}\frac{\partial U(z)}{\partial u_i^{(m)}}\right)(\operatorname{ad}U(z))^{k-i}(L(z))=$$
$$=\sum_{h,i\in\mathbb{Z}_+}\frac{1}{(h+i+1)!}(\operatorname{ad}U(z))^i\left(\operatorname{ad}\frac{\partial U(z)}{\partial u_i^{(m)}}\right)(\operatorname{ad}U(z))^h(L(z))=$$
$$=\sum_{h,i\in\mathbb{Z}_+}\frac{1}{(h+i+1)!}(\operatorname{ad}U(z))\left[\frac{\partial U(z)}{\partial u_i^{(m)}},(\operatorname{ad}U(z))^h(L(z))\right].$$

Using (5.3), we rewrite the above expression as

$$\sum_{\substack{h\in\mathbb{Z}_+\\0\leq k\leq i}}\binom{i}{k}\frac{1}{(h+i+1)!}\left[(\operatorname{ad}U(z))^k\left(\frac{\partial U(z)}{\partial u_i^{(m)}}\right),(\operatorname{ad}U(z))^{h+i-k}(L(z))\right]=$$
$$=\sum_{k,l\in\mathbb{Z}_+}\sum_{i=k}^{l+k}\binom{i}{k}\frac{1}{\binom{k+l+1}{k+1}}\left[\frac{(\operatorname{ad}U(z))^k}{(k+1)!}\left(\frac{\partial U(z)}{\partial u_i^{(m)}}\right),\frac{(\operatorname{ad}U(z))^l}{l!}\right]=[A_{i,m}(z),L_0(z)],$$

where, in the last equality, we used Lemma B.3 to show that $\sum_{i=k}^{l+k}\binom{i}{k}=\binom{k+l+1}{k+1}$.

By (5.15), for all $i\in I$, we get

$$\frac{\delta f(z)}{\delta u}=\sum_{m\in\mathbb{Z}_+}(-\partial)^m\left(a\otimes1\middle|[A_{i,m}(z),L_0(z)]+\delta_{m0}e^{\operatorname{ad}U(z)}(u^i\otimes1)-A_{i,m-1}(z)\right)$$

By invariance of the bilinear form

$$(a\otimes1\mid[A_{i,m}(z),L_0(z)])=([zs\otimes1+h(z),a\otimes1]\mid A_{i,m})+(a\otimes1\mid[A_{i,m},\partial])=-(a\otimes1\mid\partial A_{i,m}),$$

since $a\in\mathfrak{Z}(\mathfrak{h})$. Finally, it follows that

$$\frac{\delta f(z)}{\delta u_i}=\left(a\otimes1\middle|e^{\operatorname{ad}U(z)}(u^i\otimes1)\right)+\sum_{m\in\mathbb{Z}_+}(-\partial)^m(a\otimes1\mid\partial A_{i,m}-A_{i,m-1})=\left(e^{-\operatorname{ad}U(z)}(a\otimes1)\middle|u^i\otimes1\right),$$

where in the last equality we used the invariance of the bilinear from to bring the exponential of the adjoint action on the left. Hence,

$$\frac{\delta f(z)}{\delta u} = \sum_{i\in I} u_i \otimes \left(e^{-\operatorname{ad} U(z)}(a\otimes 1)\Big| u^i \otimes 1\right) = e^{-\operatorname{ad} U(z)}(a\otimes 1).$$

□

Remark 5.4. Since $F(z)$ does not depend on the choice of $U(z)$, then $\frac{\delta f(z)}{\delta u}$ does not depend on the choice of $U(z)$ too. This means that, if $\widetilde{f}(z) = (a\otimes 1 \mid \widetilde{h}(z))$, where $\widetilde{h}(z)$ is determined by $e^{\operatorname{ad}\widetilde{U}(z)}(L(z)) = \widetilde{L}_0(z)$ (see Proposition 5.2), then $f(z)$ and $\widetilde{f}(z)$ differ by a total derivative. In particular, if $\mathfrak{h}$ is abelian, this is the case, for example, when $s \in \mathfrak{g}$ is regular semisimple, then, by Proposition 5.2, $e^{\operatorname{ad}\widetilde{U}(z)} = e^{\operatorname{ad} S(z)}e^{\operatorname{ad} U(z)}$, where $S(z) \in z^{-1}(\mathfrak{h}\otimes \mathcal{V}_z(\mathfrak{g}, s))[[z^{-1}]]$, and $e^{\operatorname{ad} S(z)}(L_0(z)) = \widetilde{L}(z)$, from which follows that $h(z) - \widetilde{h}(z) = \partial S(z)$ (we always have $h_0 = \widetilde{h}_0$ by the recursion we derived in the proof of Proposition 5.2). We recall that a regular element s in a Lie algebra $\mathfrak{g}$ is an element whose centralizer $\mathfrak{g}^s = \{a \in \mathfrak{g} \mid [a, s] = 0\}$ has minimal dimension among all centralizers of elements of $\mathfrak{g}$. If $\mathfrak{g}$ is reductive and $s \in \mathfrak{g}$ is regular, then $\dim \mathfrak{g}^s = \operatorname{rank}\mathfrak{g}$.

We set $\deg u_i^{(n)} = n + 1$, for all $i \in I$ and $n \in \mathbb{Z}_+$. It is clear from the recurrence we got in the proof of Proposition 5.2 that $\deg U_k = k$, for $k \geq 1$, and $\deg h_k = k + 1$, for $k \in \mathbb{Z}_+$, if we require $U(z) \in z^{-1}(\mathfrak{h}^\perp \otimes \mathcal{V}_z(\mathfrak{g}, s))$. It follows, by Remark 5.4, that f_k, for $k \in \mathbb{Z}_+$ are linearly indipendent modulo total derivatives, that is, $\int f_k$ are linearly independent for all $k \in \mathbb{Z}_+$.

5.1.1. The N-waves equation.

Let us assume $\mathfrak{g} = \mathfrak{gl}_N$ and we consider as symmetric invariant non-degenerate bilinear form the trace form. Let $\{E_{ij}\}_{i,j=1}^N \subset \mathfrak{gl}_N$ be the set of elementary matrices (they form a basis of $\mathfrak{gl}_N$) and write $\{u_{ij}\}_{i,j=1}^N \subset \mathcal{V}_z(\mathfrak{gl}_N, s)$ when we think at them as differential variables of $\mathcal{V}_z(\mathfrak{gl}_N, s)$. Thus

$$u = \sum_{i,j=1}^N E_{ji} \otimes u_{ij}$$

since $\{E_{ji}\}_{i,j=1}^N$ is the dual basis with respect to the trace form.

We take $s \in \mathfrak{gl}_N$ to be a regular semisimple element, namely, s is a diagonal matrix, say $s = \operatorname{diag}(s_1, \ldots, s_N)$, with $s_i \neq s_j$ for all $1 \leq i, j \leq N$. Then $\mathfrak{h} = \ker \operatorname{ad} s = \mathfrak{Diag}_N$ is the Lie subalgebra of diagonal matrices in $\mathfrak{gl}_N$, while $\mathfrak{h}^\perp = \operatorname{im} \operatorname{ad} \mathfrak{h}$ is the subspace of off-diagonal matrices in $\mathfrak{gl}_N$.

Let $U \in \mathfrak{gl}_N$, we denote by L_U, respectively R_U, the operation of multiplication on the left, respectively on the right, by U. Clearly, given $U \in \mathfrak{gl}_N \otimes \mathcal{V}_z(\mathfrak{g}, s)$, we can extend L_U and R_U to the space $\mathfrak{gl}_N \otimes \mathcal{V}_z(\mathfrak{g}, s)$ componentwise and, then, to $\widetilde{\mathfrak{gl}_N}$. An easy computation shows that, for $U(z) \in z^{-1}(\mathfrak{gl}_N \otimes \mathcal{V}_z(\mathfrak{g}, s))[[z^{-1}]]$ and $a \in \widetilde{\mathfrak{gl}}_N$, we have

$$\begin{aligned} e^{\operatorname{ad} U(z)}(a) &= \sum_{k\in\mathbb{Z}_+} \frac{(\operatorname{ad} U(z))^k(a)}{k!} = \sum_{k\in\mathbb{Z}_+} \frac{(L_{U(z)} - R_{U(z)})^k(a)}{k!} = \sum_{\substack{k\in\mathbb{Z}_+\\0\leq l\leq k}} (-1)^{k-l}\frac{L^l_{U(z)}R^{k-l}_{U(z)}(a)}{l!(k-l)!} = \\ &= \sum_{k,l\in\mathbb{Z}_+} (-1)^l \frac{U(z)^k a U(z)^l}{k!l!} = e^{U(z)}ae^{-U(z)}, \end{aligned} \tag{5.16}$$

since $L_{U(z)}$ and $R_{U(z)}$ commute. Thus, we can set $T(z) = e^{\operatorname{ad} U(z)} = \mathbb{1}_N + \sum_{i\geq 1} T_i z^{-1}$ and, by Proposition 5.2, the relation $L_0(z)T(z) = T(z)L(z)$, allows to determine $T(z)$ and $h(z)$ by the recursion

$$\begin{cases} h_0 + [s\otimes 1, T_1] = u, \\ h_n + [s\otimes 1, T_{n+1}] = T_n u - \partial T_n - \sum_{k=0}^{n-1} h_k T_{n-k}, \quad n > 0, \end{cases}$$

from which follows that we can uniquely determine $h(z) \in (\mathfrak{h} \otimes \mathcal{V}_z(\mathfrak{g}, s))[[z^{-1}]]$ and $T(z) \in (\mathfrak{h}^\perp \otimes \mathcal{V}_z(\mathfrak{g}, s))[[z^{-1}]]$. The first terms in the recursion are given by

$$h_0 = u_\mathfrak{h} = \sum_{k=1}^N E_{kk} \otimes u_{kk}, \qquad T_1 = (\mathrm{ad}(s \otimes 1))^{-1}(u_{\mathfrak{h}^\perp}) = \sum_{1 \le i \ne j \le N} E_{ij} \otimes \frac{u_{ji}}{s_{ij}},$$

$$h_1 = (T_1 u)_\mathfrak{h} = \sum_{k=1}^N E_{kk} \otimes \left(\sum_{\substack{i=1 \\ i \ne k}}^N \frac{u_{ik} u_{ki}}{s_{ik}} \right), \qquad T_2 = (\mathrm{ad}(s \otimes 1))^{-1} \left((T_1 u - \partial T_1 - h_0 T_1)_{\mathfrak{h}^\perp} \right) =$$

$$= \sum_{1 \le i \ne j \le N} E_{ij} \otimes \left(-\frac{u'_{ji}}{s_{ij}^2} + \sum_{\substack{k=1 \\ k \ne i}}^N \frac{u_{jk} u_{ki}}{s_{ij} s_{ik}} \right),$$

$$h_2 = (T_2 u)_\mathfrak{h} = \sum_{k=1}^N E_{kk} \otimes \left(-\sum_{\substack{h=1 \\ h \ne k}}^N \frac{u'_{hk} u_{kh}}{s_{hk}^2} + \sum_{\substack{h,l=1 \\ h,l \ne i}}^N \frac{u_{lh} u_{hi} u_{il}}{s_{il} s_{ih}} \right),$$

where $s_{ij} = s_i - s_j$. Since $\mathfrak{h}$ is abelian, we can take $a = \mathrm{diag}(a_1, \dots, a_N) \in \mathfrak{h}$, with $a \ne c\mathbb{1}_N$, for all $c \in \mathbb{C}$. By (5.16), we have $F(z) = T(z)^{-1}(a \otimes 1)T(z) = F_0 + F_1 z^{-1} + F_2 z^{-1} + \dots$, where

$$\begin{aligned} F_0 &= a \otimes 1 \\ F_1 &= [a \otimes 1, T_1] = [a \otimes 1, (\mathrm{ad}(s \otimes 1))^{-1}(u_{\mathfrak{h}^\perp})] = p \\ F_2 &= [a \otimes 1, T_2] - T_1 p \end{aligned}$$

with $p \in \mathfrak{gl}_N \otimes \mathcal{V}_z(\mathfrak{gl}_N, s)$ defined by $p_{ij} = \frac{a_{ij}}{s_{ij}} u_{ji}$ off the diagonal, where $a_{ij} = a_i - a_j$, for all $1 \le i, j \le N$, and its diagonal entries are zero.

The first equations of the hierarchy are given by

$$\frac{du_{\mathfrak{h}^\perp}}{dt_0} = [u, a \otimes 1], \qquad \frac{du_{\mathfrak{h}^\perp}}{dt_1} = p' + [u, p], \qquad \dots$$

We note that in the case in which $a = s$, this last equation reduces to $\frac{du_{\mathfrak{h}^\perp}}{dt_1} = u'_{\mathfrak{h}^\perp}$.

The explicit formulas of the first integrals of motion is given by $\int f(z) = \int (a \otimes 1 \mid h(z)) = \int (f_0 + f_1 z^{-1} + f_2 z^{-1} + \dots)$, where

$$\int f_0 = \int (a \otimes 1 \mid h_0) = \int \sum_{i=1}^N a_i u_{ii},$$

$$\int f_1 = \int (a \otimes 1 \mid h_1) = \int \sum_{\substack{i,j=1 \\ i \ne j}}^N \frac{a_i}{s_{ij}} u_{ij} u_{ji},$$

$$\int f_2 = \int (a \otimes 1 \mid h_2) = \int \sum_{k=1}^N \left(-\sum_{\substack{h=1 \\ h \ne k}}^N \frac{a_k}{s_{hk}^2} u'_{hk} u_{kh} + \sum_{\substack{h,l=1 \\ h,l \ne i}}^N \frac{a_k}{s_{il} s_{ih}} u_{lh} u_{hi} u_{il} \right)$$

5.2. Integrable hierarchies arising from the classical Drinfeld-Sokolov Hamiltonian reduction

Let us briefly recall some basic facts and notations about the construction of classical $\mathcal{W}$-algebras we gave in Section 2.2. Let us assume $\mathfrak{g}$ to be a reductive finite dimensional Lie algebra with a symmetric invariant bilinear form $(\cdot \mid \cdot)$. Let $f \in \mathfrak{g}$ be a nilpotent element, by Jacobson-Morozov theorem we can find a sl_2-triple $\{e, h = 2x, f\} \subset \mathfrak{g}$ and write

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j \tag{5.17}$$

for the $\mathrm{ad}\, x$- eigenspaces decomposition of $\mathfrak{g}$. We set

$$\mathfrak{m}_+ = \mathfrak{g}_{\ge 1} \subset \mathfrak{n}_+ = \mathfrak{g}_{\ge \frac{1}{2}} \subset \mathfrak{b}_+ = \mathfrak{g}_{\ge 0} \subset \mathfrak{B}_+ = \mathfrak{g}_{\ge -\frac{1}{2}} \text{ and } \mathfrak{B}_- = \mathfrak{g}_{\le \frac{1}{2}}.$$

Then $f \in \mathfrak{g}_{-1}$, $h \in \mathfrak{g}_0 = \mathfrak{h}$ and $e \in \mathfrak{g}_1$.

Let us assume $s \in \ker \operatorname{ad} \mathfrak{n}_+$, we endow the space $\mathcal{V}(\mathfrak{g}) = S(\mathbb{C}[\partial] \otimes \mathfrak{g})$ the structure of Poisson vertex algebra defining, for $a, b \in \mathfrak{g}$,

$$\{a_\lambda b\}_z = [a, b] + (a \mid b)\lambda + z(s \mid [a, b]),$$

and extending the λ-bracket to $\mathcal{V}(\mathfrak{g})$ by (1.11). We denoted this Poisson vertex algebra by $\mathcal{V}_z(\mathfrak{g}, s)$.

We set $\widetilde{\mathfrak{J}}(\mathfrak{g}, f) = \langle m - (f \mid m) \mid m \in \mathfrak{m}_+ \rangle_{\mathcal{V}(\mathfrak{g})}$ and $\widetilde{\mathcal{W}}_z(\mathfrak{g}, f, s) = \{p \in \mathcal{V}_z(\mathfrak{g}, s) \mid \{a_\lambda p\}_z \in \widetilde{\mathfrak{J}}(\mathfrak{g}, \lambda)[\lambda] \, \forall a \in \mathfrak{n}_+\}$ and defined the classical $\mathcal{W}$-algebra associated to the triple $(\mathfrak{g}, f, s)$ to be quotient Poisson vertex algebra

$$\mathcal{W}_z(\mathfrak{g}, f, s) = \widetilde{\mathcal{W}}_z(\mathfrak{g}, f, s) \big/ \widetilde{\mathfrak{J}}(\mathfrak{g}, f).$$

In Section 2.2 is proved that the quotient is well defined and has a induced structure of Poisson vertex algebra.

Let us fix a basis of $\mathfrak{g}$ and its dual basis with respect to $(\cdot \mid \cdot)$ in the following way

- $Q^i = Q_i$, $i = 1, \dots, r$, basis of $\mathfrak{h}$;
- $Q^{m+i} = Q_i$, $i = r+1, \dots, r+d$, basis of $\mathfrak{g}_{-\frac{1}{2}}$;
- $Q^{m+i} = Q_i$, $i = r+d+1, \dots, r+m$, basis of $\mathfrak{g}_{\leq -1}$;
- $Q^{i-m} = Q_i$, $i = r+m+1, \dots, r+m+d$, basis of $\mathfrak{g}_{\frac{1}{2}}$;
- $Q^{i-m} = Q_i$, $i = r+m+d+1, \dots, r+2m$, basis of $\mathfrak{g}_{\geq 1}$,

where lower indeces stand for elements of the basis and upper indeces for elements of the dual basis. We denote $\mathcal{V} = \mathcal{V}(\mathfrak{B}_-)$ and set $\overline{I} = \{1, 2, \dots, r+2m\}$, $I = \{1, 2, \dots, r+m+d\}$ and $q_i = \pi_-(Q_i)$, for $i \in I$, when we think at these basis elements as differential generators of $\mathcal{V}$. As differential algebras $\mathcal{V} \cong \mathcal{V}_z(\mathfrak{g}, s) \big/ \widetilde{\mathfrak{J}}(\mathfrak{g}, f)$. Moreover, $\mathcal{V} = \mathbb{C}[q_i^{(n)} \mid i \in I, n \in \mathbb{Z}_+]$. By construction, $\mathcal{W}_z(\mathfrak{g}, f, s) \subset \mathcal{V}$, and the formula for the induced λ-bracket on elements of $\mathcal{W}_z(\mathfrak{g}, f, s)$ is given by (2.17), that we recall. For any $f, g \in \mathcal{W}_z(\mathfrak{g}, f, s)$ we have

$$\{f_\lambda g\}_{\Lambda(z)} = \sum_{\substack{i,j \in \overline{I} \\ n,m \in \mathbb{Z}_+}} \frac{\partial g}{\partial q_j^{(n)}} (\lambda + \partial)^n \{q_{i\,\lambda+\partial} q_j\}_{\Lambda(z)_\to} (-\lambda - \partial)^m \frac{\partial f}{\partial u_i^{(m)}},$$

where, for any $a, b \in \mathfrak{B}_-$,

$$\{a_\lambda b\}_{\Lambda(z)} = \pi_-([a, b]) + (a \mid b)\lambda + (\Lambda(z) \mid [a, b]).$$

We write the λ-bracket on $\mathcal{V}$ as $\{\cdot_\lambda \cdot\}_{\Lambda(z)} = \{\cdot_\lambda \cdot\}_H - \{\cdot_\lambda \cdot\}_K$, where $H(\partial)$, respectively $K(\partial)$, is the matrix valued differential operator corresponding to $\{q_{i\lambda} q_j\}_H = \pi_-([Q_i, Q_j]) + (Q_i \mid Q_j)\lambda + (f \mid [Q_i, Q_j])$, respectively $\{q_{i\lambda} q_j\}_K = -(s \mid [Q_i, Q_j])$, by (1.17). Hence,

$$H_{ij}(\partial) = \pi_-([Q_j, Q_i]) + (Q_i \mid Q_j)\partial + (f \mid [Q_j, Q_i]) \qquad \text{and} \qquad K_{ij}(\partial) = -(s \mid [Q_j, Q_i]), \tag{5.18}$$

for all $i, j \in I$.

As we did in Section 5.1 we endow the space $\mathfrak{g} \otimes \mathcal{V}$ a Lie algebra structure and extend the bilinear form of $\mathfrak{g}$ to a non-degenerate symmetric invariant bilinear form on $\mathfrak{g} \otimes \mathcal{V}$ which we still denote $(\cdot \mid \cdot)$. Furthermore, given ∂ the derivation of $\mathcal{V}$ we define the semidirect product Lie algebra $\mathbb{C}\partial \ltimes (\mathfrak{g} \otimes \mathcal{V})$, where the commutator of ∂ against elements of $\mathfrak{g} \otimes \mathcal{V}$ is defined by $[\partial, a \otimes f] = \partial.(a \otimes f) = a \otimes \partial f$, for any $a \in \mathfrak{g}$ and $f \in \mathcal{V}$. We set $\widetilde{\mathfrak{g}} = (\mathbb{C}\partial \ltimes \mathfrak{g} \otimes \mathcal{V})((z^{-1}))$. As already pointed out in Section 5.1, given $U(z) \in z^{-1}(\mathfrak{g} \otimes \mathcal{V}_z(\mathfrak{g}, s))[[z^{-1}]]$, the map $e^{\operatorname{ad} U(z)} : \widetilde{\mathfrak{g}} \longrightarrow \widetilde{\mathfrak{g}}$ is a Lie algebra automorphism and automorphisms of this type form a group.

We consider the following identifications

$$\begin{aligned} \mathcal{V}^{\oplus \overline{I}} &\xrightarrow{\sim} \mathfrak{g} \otimes \mathcal{V} \\ F = (F_i)_{i \in \overline{I}} &\longrightarrow \underline{F} = \sum_{i \in \overline{I}} Q_i \otimes F_i \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} \mathcal{V}^{\overline{I}} &\xrightarrow{\sim} \mathfrak{g} \otimes \mathcal{V} \\ P = (P_i)_{i \in \overline{I}} &\longrightarrow \overline{P} = \sum_{i \in \overline{I}} Q^i \otimes P_i. \end{aligned} \tag{5.20}$$

In Section 5.1 we defined a pairing $(\cdot, \cdot) : \mathfrak{g} \otimes \mathcal{V} \times \mathfrak{g} \otimes \mathcal{V} \longrightarrow \mathcal{V} \big/ \partial \mathcal{V}$, using the bilinear form on $\mathfrak{g} \otimes \mathcal{V}$. Using (5.19) we can identify $\mathcal{V}^{\oplus I}$ with $\mathfrak{B}_- \otimes \mathcal{V}$ and, using (5.20), we can identify $\mathcal{V}^I$ with $\mathfrak{B}_+ \otimes \mathcal{V}$. These spaces are duals with respect to the pairing $(\cdot,\cdot)$. Thus, if $f\in\mathcal{V}$, we can identify its variational derivative with the following element of $\mathfrak{B}_-\otimes\mathcal{V}$,

$$\frac{\delta f}{\delta q}=\sum_{i\in I}Q_i\otimes\frac{\delta f}{\delta q_i}.$$

We set $q=\sum_{i\in I}Q^i\otimes q_i\in\mathfrak{B}_+\otimes\mathcal{V}$ and define

$$L(z)=\partial+q+\Lambda(z)\otimes 1\in\widetilde{\mathfrak{g}}.$$

Proposition 5.5. *If $F=(F_i)_{i\in I}\in\mathcal{V}^{\oplus I}$, then*

$$\overline{(H(\partial)-zK(\partial))F}=(\pi_+\otimes\mathbb{1})([L(z),\underline{F}]),$$

where $\pi_+:\mathfrak{g}\twoheadrightarrow\mathfrak{B}_+$ is the projection map from $\mathfrak{g}$ to $\mathfrak{B}_+$, and $\pi_+\otimes\mathbb{1}:\mathfrak{g}\otimes\mathcal{V}\longrightarrow\mathfrak{B}_+\otimes\mathcal{V}$.

Proof. Since $I\subset\overline{I}$, and $H(\partial)-zK(\partial)\in\mathrm{Mat}_{I\times I}(\mathcal{V}[\partial])$, by (5.20), we have

$$\overline{(H(\partial)-zK(\partial))F}=\sum_{i\in I}Q^i\otimes((H(\partial)-zK(\partial))F)_i\,.$$

On the other hand, given an element $A\in\mathfrak{g}\otimes\mathcal{V}$ we can write it uniquely as $A=\sum_{i\in\overline{I}}Q^i\otimes(A\mid Q_i\otimes 1)$, from which follows that $(\pi\otimes\mathbb{1})(A)=\sum_{i\in I}Q^i\otimes(A\mid Q_i\otimes 1)$. Then, we are left to show that

$$((H(\partial)-zK(\partial))F)_i=([L(z),\underline{F}]\mid Q_i\otimes 1)\,, \tag{5.21}$$

for all $i\in I$. Using (5.18), the left hand side of (5.21) is

$$\begin{aligned}((H(\partial)-zK(\partial))F)_i&=\sum_{j\in I}(H_{ij}(\partial)-zK_{ij}(\partial))F_j=\\&=\sum_{j\in I}\left((Q_i\mid Q_j)F_j'+\pi_-([Q_j,Q_i])F_j+(\Lambda(z)\mid[Q_j,Q_i])F_j\right).\end{aligned}$$

On the other hand we have

$$[L(z),\underline{F}]=\sum_{j\in I}[\partial+q+\Lambda(z)\otimes 1,Q_j\otimes F_j]=\sum_{j\in I}\left(Q_j\otimes F_j'+\sum_{k\in I}[Q^k,Q_j]\otimes q_kF_j+[\Lambda(z),Q_j]\otimes F_j\right).$$

Taking the scalar product of this term with $Q_i\otimes 1$ gives the right hand side of (5.21). We get

$$([L(z),\underline{F}]\mid Q_i\otimes 1)=\sum_{j\in I}\left((Q_j\mid Q_i)F_j'+\sum_{k\in I}([Q^k,Q_j]\mid Q_i)q_kF_j+([\Lambda(z),Q_j]\mid Q_i)F_j\right).$$

Using the invariance of the bilinear form we have $\sum_{k\in I}([Q^k,Q_j]\mid Q_i)q_k=\sum_{k\in I}(Q^k\mid[Q_j,Q_i])q_k=\pi_-([Q_j,Q_i])$ and $([\Lambda(z),Q_j]\mid Q_i)=(\Lambda(z)\mid[Q_j,Q_i])$ and by symmetry, $(Q_i\mid Q_j)=(Q_j\mid Q_i)$, proving that left hand side and right hand side of (5.21) are equal. ☐

Let us set $\widehat{\mathfrak{g}}=\mathfrak{g}((z^{-1}))$. Let $d_1=z\frac{d}{dz}$ be a degree operator, then d_1 defines a gradation of $\widehat{\mathfrak{g}}$ with respect to powers of z and we denote $\widehat{\mathfrak{g}}^k=\mathfrak{g}z^k$, for $k\in\mathbb{Z}$, the homogeneous component of degree k. We also denote $\widehat{\mathfrak{g}}^+=\mathfrak{g}[z]$. We define a nondegenerate symmetric invariant bilinear form on $\widehat{\mathfrak{g}}$ in the following way. First, for any $a(z)=\sum_i a_iz^i,b(z)=\sum_i b_iz^i\in\widehat{\mathfrak{g}}$, we set

$$(a(z)\mid b(z))=\sum_{i,j}(a_i\mid b_j)z^{i+j}\in\mathbb{C}((z^{-1})). \tag{5.22}$$

The nondegenerate symmetric invariant bilinear form on $\widehat{\mathfrak{g}}$ is obtained taking the costant term of (5.22), namely, we set

$$\langle a(z)\mid b(z)\rangle=\mathrm{Res}_z(a(z)\mid b(z))z^{-1}, \tag{5.23}$$

for all $a(z),b(z)\in\widehat{\mathfrak{g}}$. It is clear from definition that this bilinear form is coordinated with the gradation defined by d_1.

We assume $\Lambda(z)\in\widehat{\mathfrak{g}}$ to be a semisimple element. Then

$$\widehat{\mathfrak{g}}=\mathfrak{H}\oplus\mathrm{im}\,\mathrm{ad}\,\Lambda(z). \tag{5.24}$$

The same argument used in Section 5.1 shows that $\mathfrak{H}^\perp=\mathrm{im}\,\mathrm{ad}\,\Lambda(z)$ with respect to the bilinear form defined in (5.23).

We fix also another gradation on $\widehat{\mathfrak{g}}$ by the condition that $\Lambda(z)$ is a homogeneous element of degree -1. Namely, let us denote $m=\deg(s)$, the degree of s with respect to the $\mathrm{ad}\,x$-decomposition of $\mathfrak{g}$ and consider the gradation of $\widehat{\mathfrak{g}}$ defined by the degree operator $d_2=(-m-1)z\frac{d}{dz}+\mathrm{ad}\,x$. We shall write $\widehat{\mathfrak{g}}_j$ for the component of degree j with respect to this gradation. If $n \in \mathbb{Z}_+$, let us write $n = (m+1)h + a$, where $0 \leq a \leq m + \frac{1}{2}, h \in \mathbb{Z}_+$, then $\widehat{\mathfrak{g}}_n = \mathfrak{g}_a z^{-h} \oplus \mathfrak{g}_{-m-1+a} z^{-h-1}$, where lower indeces, which denotes the $\operatorname{ad} x$-decomposition of $\mathfrak{g}$ (5.17), are in $\frac{1}{2}\mathbb{Z}/m\mathbb{Z}$. Using the same notation, we have $\widehat{\mathfrak{g}}_{-n} = \mathfrak{g}_{-a} z^h \oplus \mathfrak{g}_{m+1-a} z^{h+1}$. Hence, since the bilinear form $(\cdot \mid \cdot)$ on $\mathfrak{g}$ is coordinated with the gradation (5.17) and the bilinear form (5.23) is coordinated with the gradation in powers of z, it follows that (5.23) is also coordinated with the gradation of $\widehat{\mathfrak{g}}$ given by the degree operator d_2. We denote $\widehat{\mathfrak{g}}_- = \oplus_{j \leq \frac{1}{2}} \widehat{\mathfrak{g}}_j = \mathfrak{B}_- \oplus z\mathfrak{g}[z] \subset \widehat{\mathfrak{g}}^+$.

We want to find a sequence f_n, $n \in \mathbb{Z}_+$, such that (1.23) holds, that is, we want that

$$\{f_n{}_\lambda a\}_H|_{\lambda=0} = \{f_{n+1}{}_\lambda a\}_K|_{\lambda=0}\,,$$

for any $a \in \mathcal{W}_z(\mathfrak{g}, f, s)$. By definition of induced λ-braclet, this is equivalent to

$$\pi\left(\{\widetilde{f}_n{}_\lambda \widetilde{a}\}_{\widetilde{H}}\Big|_{\lambda=0}\right) = \pi\left(\{\widetilde{f}_{n+1}{}_\lambda \widetilde{a}\}_{\widetilde{K}}\Big|_{\lambda=0}\right),$$

for any $\widetilde{f}_n, \widetilde{f}_{n+1}, \widetilde{a} \in \widetilde{\mathcal{W}}_z(\mathfrak{g}, f, s)$ lifts of f_n, f_{n+1} and a, and by $\widetilde{H}$ and $\widetilde{K}$ we denote the affine Poisson vertex algebra λ-brackets (which we called H and K in the previous section). In particular, by (1.20), this is equivalent to show that

$$\pi\left(\int \frac{\delta \widetilde{a}}{\delta q} \widetilde{H}(\partial) \frac{\delta \widetilde{f}_n}{\delta q}\right) = \int \frac{\delta a}{\delta q} H(\partial) \frac{\delta f_n}{\delta q} = \int \frac{\delta a}{\delta q} K(\partial) \frac{\delta f_{n+1}}{\delta q} = \pi\left(\int \frac{\delta \widetilde{a}}{\delta q} \widetilde{K}(\partial) \frac{\delta \widetilde{f}_{n+1}}{\delta q}\right), \tag{5.25}$$

since, as already said in Section 2.2.1, we may choose f_n, f_{n+1} and a as their liftings. Furthermore, (5.25), can be stated as

$$\int X_{H(\partial)\frac{\delta f_n}{\delta q}}(a) = \int X_{K(\partial)\frac{\delta f_{n+1}}{\delta q}}(a), \tag{5.26}$$

for any $a \in \mathcal{W}_z(\mathfrak{g}, f, s)$. Thus, our goal will be to find a sequence $f_n \in \mathcal{W}_z(\mathfrak{g}, f, s)$, $n \in \mathbb{Z}_+$, such that (5.26) holds.

Similarly to what we did in the previous section, we start finding an explicit description of $Z_{L(z)} = \{F(z) \in \widehat{\mathfrak{g}} \otimes \mathcal{V} \mid [L(z), F(z)] = 0\}$. The answer is essentially given by the following result of Drinfeld and Sokolov [12, Proposition 6.2]wich we generalize to our situation. We fix the following notation: given a subspace $V \subset \widehat{\mathfrak{g}}$, for all $i \in \mathbb{Z}$, we denote $V^i = V \cap \widehat{\mathfrak{g}}^i$, respectively $V_i = V \cap \widehat{\mathfrak{g}}_i$, the homogeneous component of V of degree i with respect to the grading defined by d_1, respectively d_2.

Proposition 5.6. *There exists a formal series $U(z) \in \widehat{\mathfrak{g}}_{>0} \otimes \mathcal{V}$ such that*

$$L_0(z) = e^{\operatorname{ad} U(z)}(L(z)) = \partial + \Lambda(z) \otimes 1 + h(z), \tag{5.27}$$

with $h(z) \in \mathfrak{H}_{\geq -\frac{1}{2}} \otimes \mathcal{V}$. The automorphism $e^{\operatorname{ad} U(z)}$ is defined uniquely up to multiplication on the left by automorphisms of the form $e^{\operatorname{ad} S(z)}$, where $S(z) \in \mathfrak{H}_{>0} \otimes \mathcal{V}$ and it is possible to choose $U(z)$ uniquely if we require $U(z) \in \mathfrak{H}^{\perp}_{<0} \otimes \mathcal{V}$.

Proof. Writing $U(z) = \sum_{i \geq \frac{1}{2}} U_i$, with $U_i \in \widehat{\mathfrak{g}}_i \otimes \mathcal{V}$, $h(z) = \sum_{i \geq -\frac{1}{2}}$, where $h_i \in \mathfrak{H}_i \otimes \mathcal{V}$ and equating terms which lie in the component of degree i in both sides of (5.27), we find that $h_i + [\Lambda(z) \otimes 1, U_{i+1}]$ can be expressed in terms of $U_{\frac{1}{2}}, U_1, \dots, U_{i+\frac{1}{2}}$ and $h_{-\frac{1}{2}}, h_0, \dots, h_{i-\frac{1}{2}}$. Let us say $h_i + [\Lambda(z) \otimes 1, U_{i+1}] = A \in \widehat{\mathfrak{g}}_i \otimes \mathcal{V}$, where, as already pointed out, we explicitly know A. By (5.24), we can write in a unique way $A = A_{\mathfrak{H}} + A_{\mathfrak{H}^{\perp}}$, where $A_{\mathfrak{H}} \in \mathfrak{H} \otimes \mathcal{V}$ and $A_{\mathfrak{H}^{\perp}} \in \mathfrak{H}^{\perp} \otimes \mathcal{V}$. Hence, we get $h_i = A_{\mathfrak{H}}$ and $U_{i+1} = (\operatorname{ad}(\Lambda(z) \otimes 1))^{-1}(A_{\mathfrak{H}^{\perp}})$. Since the restriction of $\operatorname{ad} \Lambda(z)$ to $\mathfrak{H}^{\perp}$ is an isomorphism, we can determime uniquely $U_{i+1} \in \mathfrak{H}^{\perp} \otimes \mathcal{V}$. Therefore, we can uniquely determine $h(z) \in \mathfrak{H}_{\geq -\frac{1}{2}} \otimes \mathcal{V}$ and $U(z) \in \mathfrak{H}^{\perp}_{<0} \otimes \mathcal{V}$.

Let $\widetilde{U}(z) \in \widehat{\mathfrak{g}}_{<0} \otimes \mathcal{V}$ be such that $e^{\operatorname{ad} \widetilde{U}(z)}(L(z)) = \widetilde{L}_0(z)$, where $\widetilde{L}_0(z)$ is of the same type of (5.27), namely, $\widetilde{L}_0(z) = \partial + \Lambda(z) \otimes 1 + \widetilde{h}(z)$, with $\widetilde{h}(z) \in \mathfrak{H}_{\geq -\frac{1}{2}} \otimes \mathcal{V}$. Since these automorphisms form a group, there exists $S(z) \in \widehat{\mathfrak{g}}_{>0} \otimes \mathcal{V}$ such taht $e^{\operatorname{ad} \widetilde{U}(z)} e^{-\operatorname{ad} U(z)} = e^{\operatorname{ad} S(z)}$. Equating terms with degree i in the expression $e^{\operatorname{ad} S(z)}(L_0(z)) = \widetilde{L}_0(z)$, we have that $S(z) \in (\mathfrak{H}_{>0} \otimes \mathcal{V}$, since both coefficients of $L_0(z)$ and $\widetilde{L}_0(z)$ lie in $(\mathfrak{H}_{\geq -\frac{1}{2}} \otimes \mathcal{V}$ and $[\Lambda(z) \otimes 1, \mathfrak{H}^{\perp}] \subset \mathfrak{H}^{\perp}$. $\square$

Let $U(z) \in \widehat{\mathfrak{g}}_{>0} \otimes \mathcal{V}$ be defined by the previous proposition. Then we have the following corollary.

Corollary 5.7. *$Z_L = e^{-\operatorname{ad} U(z)}(\mathfrak{Z}(\mathfrak{H}) \otimes 1)$ and does not depend on the choice of $U(z)$.*

Proof. By Proposition 5.6 it suffices to show $Z_{L_0(z)} = \mathfrak{Z}(\mathfrak{H}) \otimes 1 \subset \mathcal{V}((z^{-1}))$. Clearly, by definition of $L_0(z)$, given by (5.27), it follows that $\mathfrak{Z}(\mathfrak{H}) \otimes 1 \subset Z_{L_0(z)}$. On the other hand, let us assume $M(z) \in \widehat{\mathfrak{g}} \otimes \mathcal{V}$ and write $M(z) = \sum_{i \geq m} M_i$, with $M_i \in \widehat{\mathfrak{g}}_i$ and $m \in \mathbb{Z}$. Equating to zero the component of degree $m-1$ in $[M(z), L_0(z)]$, we get $[M_m, \Lambda(z)] = 0$. Thus $M_m \in \mathfrak{Z}(\mathfrak{H}) \otimes \mathcal{V}$. While, equating to zero the component of degree $m-\frac{1}{2}$, we get $[M_{m+\frac{1}{2}},\Lambda(z)]=0$, then $M_{m+\frac{1}{2}}\in\mathfrak{Z}(\mathfrak{H})\otimes\mathcal{V}$, fromwhich follows that equating to zero the component of degree m we get the equality $M'_m=[M_{m+1},\Lambda(z)]$. The left hand side of this equality belongs to $\mathfrak{Z}(\mathfrak{H})\otimes\mathcal{V}$, while the right hand side belongs to $\mathfrak{H}^\perp\otimes\mathcal{V}$, which forces $M'_m=0$. Then, we proceed applying analogous considerations to $M(z)-M_m$. The independence from the arbitrariness of the choice of $U(z)$ is clear since, by Proposition5.6, $e^{\operatorname{ad}U(z)}$ is defined up to multiplication on the left by automorphism of the form $e^{\operatorname{ad}S(z)}$, where $S(z)\in\mathfrak{H}\otimes\mathcal{V}$. □

We define the following map $\varphi:\mathfrak{Z}(\mathfrak{H})\otimes 1\longrightarrow Z_{L(z)}$, by $\varphi(a(z)\otimes 1)=e^{-\operatorname{ad}U(z)}(a(z)\otimes 1)$. By Corollary 5.7, this map is well defined. Indeed, it does not depend on the choice of $U(z)$. We want to use this map to define some evolutionary vector fields on $\mathcal{V}$. We need the following properties.

Lemma 5.8. *Let $F(z)\in\widehat{\mathfrak{g}}\otimes\mathcal{V}$ and write $F(z)=F(z)^++F(z)^-$, respectively $F(z)=F(z)_++F(z)_-$, where $F(z)^+\in\widehat{\mathfrak{g}}^+\otimes\mathcal{V}$, respectively $F(z)_-\in\widehat{\mathfrak{g}}_-\otimes\mathcal{V}$, and $F(z)^-=F(z)-F(z)^+$, respectively $F(z)_+=F(z)-F(z)_-$. Then, $F(z)^+-F(z)_-\in\mathfrak{n}_+\otimes\mathcal{V}$. Moreover, If $F(z)\in Z_{L(z)}$, then*

i) $[L(z),F(z)^+],[L(z),F(z)_-]\in\mathfrak{B}_+\otimes\mathcal{V}$;
ii) $[L(z),F(z)]=-[s\otimes 1,\operatorname{Res}_z F(z)]$.

Proof. The first assertion follows from the fact that $\widehat{\mathfrak{g}}^+=\mathfrak{g}[z]$, while $\widehat{\mathfrak{g}}_-=\mathfrak{B}_-\oplus z\mathfrak{g}[z]$. Let us assume $F(z)\in Z_{L(z)}$, then $[L(z),F(z)_-]=-[L(z),F(z)_+]$. The left hand side of this equality belongs to $\mathfrak{g}[z]\otimes\mathcal{V}$, while the right hand side belongs to $\widehat{\mathfrak{g}}_{\geq-\frac{1}{2}}\otimes\mathcal{V}=(\mathfrak{B}_+\oplus z^{-1}\mathfrak{g}[[z^{-1}]])\otimes\mathcal{V}$. Hence, $[L(z),F(z)_-]\in\mathfrak{B}_+\otimes\mathcal{V}=\mathfrak{g}[z]\otimes\mathcal{V}\cap(\mathfrak{B}_+\oplus z^{-1}\mathfrak{g}[[z^{-1}]])\otimes\mathcal{V}$. Since $F(z)^+-F(z)_-\in\mathfrak{n}_+\otimes\mathcal{V}$, we have

$$[L(z),F(z)^+]-[L(z),F(z)_-]=[L(z),F(z)^+-F(z)_-]=[\partial+q+\otimes 1,F(z)^+-F(z)_-]\in\mathfrak{B}_+\otimes\mathcal{V}.$$

Then $[L(z),F(z)^+]\in\mathfrak{B}_+\otimes\mathcal{V}$, which proves i). To prove ii) we simply note that the left hand side of the equality $[L(z),F(z)^+]=-[L(z),F(z)^-]$ is a polynomial in z, while the left hand side is a power series in z^{-1} whose constant term is given by $[s\otimes 1,\operatorname{Res}_z F(z)]$. □

Let $a(z)\in\mathfrak{Z}(\mathfrak{H})$, but $a(z)\notin\mathfrak{Z}(\widehat{\mathfrak{g}})$, by Lemma 5.8, $[L(z),\varphi(a(z))^+],[L(z),\varphi(a(z))_-]\in\mathfrak{B}_+\otimes\mathcal{V}$ and, by identifications (5.20) and (1.5), they define vector fields on $\mathcal{V}$, wich we denote $X_{[\varphi(u(z))^+,L(z)]}$ and $X_{[\varphi(u(z))_-,L(z)]}$.

Proposition 5.9. *We have $X_{[L(z),\varphi(a(z)\otimes 1)^+]}(\mathcal{W}_z(\mathfrak{g},f,s))\subset\mathcal{W}_z(\mathfrak{g},f,s)$ and $X_{[L(z),\varphi(a(z)\otimes 1)_-]}(\mathcal{W}_z(\mathfrak{g},f,s))\subset\mathcal{W}_z(\mathfrak{g},f,s)$. Moreover,*

$$X_{[L(z),\varphi(a(z)\otimes 1)^+]}\big|_{(\mathcal{W}_z(\mathfrak{g},f,s))}=X_{[L(z),\varphi(a(z)\otimes 1)_-]}\big|_{(\mathcal{W}_z(\mathfrak{g},f,s))}.\tag{5.28}$$

Proof. The proof of this proposition heavily use the definition of $\mathcal{W}_z(\mathfrak{g},f,s)$ in terms of gauge invariant polynomials we gave in Section 2.1.

Let $p(q)\in\mathcal{W}_z(\mathfrak{g},f,s)$. First we prove that

$$\left(X_{[L(z),\varphi(a(z)\otimes 1)^+]}(p)\right)(q)=\left(X_{[L(z),\varphi(a(z)\otimes 1)^+]}(p)\right)(q^{can}).\tag{5.29}$$

This means that $X_{[L(z),\varphi(a(z)\otimes 1)^+]}(\mathcal{W}_z(\mathfrak{g},f,s))\subset\mathcal{W}_z(\mathfrak{g},f,s)$. In order to do this we need the following result. Let $P(q)\in\mathfrak{B}_+\otimes\mathcal{V}$, if $P(\widetilde{q})=e^{\operatorname{ad}S}(P(q))$ for any $\widetilde{q}\overset{S}{\sim}q$, then $X_{P(q)}(\mathcal{W}_z(\mathfrak{g},f,s))\subset\mathcal{W}_z(\mathfrak{g},f,s)$. Indeed, we have

$$\begin{aligned}p(q)+t(X_P(p))(q)+o(t^2)&=p(q+tP)=p(\widetilde{q+tP})=p(\widetilde{q}+te^{\operatorname{ad}S}P(q))=p(\widetilde{q}+tP(\widetilde{q}))=\\&=p(\widetilde{q})+t(X_P(p))(\widetilde{q})+o(t^2),\end{aligned}$$

from which follows, since $p(q)=p(\widetilde{q})$, that $(X_P(p))(q)=(X_P(p))(\widetilde{q})$. Hence, we are left to show that, if $q\overset{S}{\sim}\widetilde{q}$, then

$$[\widetilde{L}(z),\widetilde{\varphi}(a(z)\otimes 1)^+]=e^{\operatorname{ad}S}[L(z),\varphi(a(z)\otimes 1)^+].$$

Furthermore, since $e^{\operatorname{ad}S}$ is a Lie algebra automorphism, we reduce to show that $\widetilde{\varphi}(a(z)\otimes 1)^+=e^{\operatorname{ad}S}(\varphi(a(z)\otimes 1)^+)$. Using the fact that $\widetilde{L}(z)=e^{\operatorname{ad}S}(L(z)$, we get

$$\widetilde{L}_0(z)=e^{\operatorname{ad}\widetilde{U}(z)}(\widetilde{L}(z))=e^{\operatorname{ad}\widetilde{U}(z)}e^{\operatorname{ad}S}(L(z)).$$

By Proposition 5.6, $e^{\operatorname{ad}\widetilde{U}(z)}e^{\operatorname{ad}S}=e^{\operatorname{ad}T(z)}e^{\operatorname{ad}U(z)}$, with $T(z)\in\mathfrak{H}_{>0}\otimes\mathcal{V}$. Hence,

$$\widetilde{\varphi}(a(z)\otimes 1)^+=\left(e^{-\operatorname{ad}\widetilde{U}(z)}(a(z)\otimes 1)\right)^+=\left(e^{\operatorname{ad}S}e^{-\operatorname{ad}U(z)}(a(z)\otimes 1)\right)^+=e^{\operatorname{ad}S}\varphi(a(z)\otimes 1)^+,$$

where we can bring $e^{\operatorname{ad}S}$ outside the parenthesis since it does not contain any powers of z. This proves (5.29).

Next, we want to prove that, if $p \in \mathcal{W}_z(\mathfrak{g}, f, s)$, then

$$X_{[L(z),\varphi(a(z)\otimes 1)^+]}(p) = X_{[L(z),\varphi(a(z)\otimes 1)^+]}(p). \tag{5.30}$$

This will conclude the proof. Let us set

$$g(q) = X_{[L(z),\varphi(a(z)\otimes 1)^+]}(p) - X_{[L(z),\varphi(a(z)\otimes 1)^+]}(p) = X_{[L(z),\varphi(a(z)\otimes 1)^+ - \varphi(a(z)\otimes 1)_-]}(p)$$

and $S = t(\varphi(a(z)\otimes 1)^+ - \varphi(a(z)\otimes 1)_-)$. By Lemma 5.8, $S \in \mathfrak{n}_+ \otimes \mathcal{V}$. Moreover, $q \overset{S}{\sim} q(t)$, where $q(t) = q + t[L(z), \varphi(a(z)\otimes 1)^+ - \varphi(a(z)\otimes 1)_-] + o(t^2)$, and, by Taylor expansion, $p(q(t)) = p(q) + tX_{[L(z),\varphi(a(z)\otimes 1)^+ - \varphi(a(z)\otimes 1)_-]} + o(t^2)$. It follows that

$$g = \left.\frac{dp(q(t))}{dt}\right|_{t=0} = \left.\frac{dp(q)}{dt}\right|_{t=0} = 0,$$

where $p(q(t)) = p(q)$, since p is a gauge invariant polynomial. $\square$

We set $f = \langle a(z) \mid h(z)\rangle \in \mathcal{V}$, where $h(z)$ is defined by Proposition 5.6. Since the bilinear form (5.23) is coordinated with the gradation defined by d_2, then $f = 0$, if $u(z) \in \mathfrak{Z}(\mathfrak{H})_{\geq 1}$. Thus without loss of generality we may assume $a(z) \in \mathfrak{Z}(\mathfrak{H})_{\leq \frac{1}{2}} \otimes 1 = (\mathfrak{Z}(\mathfrak{H}) \cap (\mathfrak{B}_- \oplus z\mathfrak{g}[z])) \otimes 1$.

Proposition 5.10.

$$\frac{\delta f}{\delta q} = (\pi_- \otimes \mathbb{1})\left(\varphi(a(z)\otimes 1)^0\right),$$

where the upper index 0 *denotes the constant term in the expansion in powers of* z.

Proof. We recall that, by (5.19),

$$\frac{\delta f}{\delta q} = \sum_{i\in I} Q_i \otimes \frac{\delta f}{\delta q_i} \in \mathfrak{B}_- \otimes \mathcal{V},$$

where $\frac{\delta f(z)}{\delta q_i} = \sum_{m\in\mathbb{Z}_+} (-\partial)^m \frac{\partial f}{\partial q_i^{(m)}}$, for all $i \in I$. We start computing the partial derivatives. As we did in the proof of Proposition 5.3, we have, for all $i \in I$ and $m \in \mathbb{Z}_+$,

$$\frac{\partial f}{\partial q_i^{(m)}} = \frac{\partial}{\partial q_i^{(m)}} (a(z)\otimes 1 \mid h(z))^0 = \left(a(z)\otimes 1 \,\middle|\, \frac{\partial h(z)}{\partial u_i^{(m)}}\right)^0,$$

since partial derivatives act as derivations of the bilinear form. By (5.27), we can write $h(z) = e^{\operatorname{ad} U(z)}(L(z)) - \partial - \Lambda(z)\otimes 1$. Hence,

$$\begin{aligned}\frac{\partial h(z)}{\partial q_i^{(m)}} &= \frac{\partial e^{\operatorname{ad} U(z)}(L(z))}{\partial q_i^{(m)}} = \frac{\partial}{\partial q_i^{(m)}}\left(\sum_{k\in\mathbb{Z}_+} \frac{(\operatorname{ad} U(z))^k}{k!}(L(z))\right) = \\ &= \frac{\partial L(z)}{\partial q_i^{(m)}} + \frac{\partial}{\partial q_i^{(m)}}\left(\sum_{k\geq 1} \frac{(\operatorname{ad} U(z))^k}{k!}(L(z))\right),\end{aligned} \tag{5.31}$$

where

$$\frac{\partial L(z)}{\partial q_i^{(m)}} = \frac{\partial q}{\partial q_i^{(m)}} = \sum_{j\in I} \frac{\partial}{\partial q_i^{(m)}}(Q^j \otimes q_j) = \sum_{j\in I} Q^j \otimes \frac{\partial q_j}{\partial q_i^{(m)}} = \delta_{m0} Q^i \otimes 1 \tag{5.32}$$

and the remaining term in (5.31) is given by

$$[A_{i,m}(z), L_0(z)] - A_{i,m-1}(z), \tag{5.33}$$

where $A_{i,m}(z) = \sum_{k\in\mathbb{Z}_+} \frac{(\operatorname{ad} U(z))^k}{(k+1)!}\left(\frac{\partial U(z)}{\partial q_i^{(m)}}\right)$ (the same computation we made in the proof of Proposition 5.3). Putting terms (5.32) and (5.33) in (5.31), we get

$$\frac{\delta f}{\delta q} = \sum_{m\in\mathbb{Z}_+} (-\partial)^m \left(a\otimes 1 \,\middle|\, [A_{i,m}(z), L_0(z)] + \delta_{m0} e^{\operatorname{ad} U(z)}(Q^i \otimes 1) - A_{i,m-1}(z)\right)$$

By invariance of the bilinear form

$$(a(z)\otimes 1 \mid [A_{i,m}(z), L_0(z)])^0 = ([\Lambda(z)\otimes 1 + h(z), a(z)\otimes 1] \mid A_{i,m}) + (a(z)\otimes 1 \mid [A_{i,m}, \partial]) = -(a\otimes 1 \mid \partial A_{i,m}),$$

since $a(z) \in \mathfrak{Z}(\mathfrak{H})$. Finally, it follows that

$$\frac{\delta f}{\delta q_i} = \left(a(z) \otimes 1 \middle| e^{\operatorname{ad} U(z)}(Q^i \otimes 1)\right) + \sum_{m \in \mathbb{Z}_+} (-\partial)^m (a(z) \otimes 1 \mid \partial A_{i,m} - A_{i,m-1}) =$$
$$= \left(e^{-\operatorname{ad} U(z)}(a(z) \otimes 1) \middle| Q^i \otimes 1\right),$$

for all $i \in I$, where in the last equality we used the invariance of the bilinear from to bring the exponential of the adjoint action on the left. Hence,

$$\frac{\delta f}{\delta q} = \sum_{i \in I} Q_i \otimes \left(e^{-\operatorname{ad} U(z)}(a(z) \otimes 1) \middle| Q^i \otimes 1\right)^0 = (\pi_- \otimes \mathbb{1}) \left(\left(e^{-\operatorname{ad} U(z)}(a(z) \otimes 1)\right)^0\right) =$$
$$= (\pi_- \otimes \mathbb{1}) \left(\varphi(a(z) \otimes 1)^0\right).$$

$\square$

Remark 5.11. Since the map φ does not depend on the choice of $U(z)$, then $\frac{\delta f}{\delta q}$ does not depend on the choice of $U(z)$ too. This means that, if $\widetilde{f} = (a(z) \otimes 1 \mid \widetilde{h}(z))$, where $\widetilde{h}(z)$ is determined by $e^{\operatorname{ad} \widetilde{U}(z)}(L(z)) = \widetilde{L}_0(z)$ (see Proposition 5.6), then f and $\widetilde{f}$ differ by a total derivative. In particular, since $\mathfrak{n}_+ \subset \widehat{\mathfrak{g}}_{>0}$, if we consider $\widetilde{L}(z) = e^{\operatorname{ad} S}(L(z)) = L^{can}(z)$ (see Proposition 2.1), then f is a gauge invariant polynomial up to total derivatives. This means that, if we define $\overline{f}(q) = f(q^{can})$, then, by definition, $\overline{f}$ is a gauge invariant polynomial, namely $\overline{f} \in \mathcal{W}_z(\mathfrak{g}, f, s)$, and it differs from f by a total derivative. Hence, without loss of generality, we may assume $f \in \mathcal{W}_z(\mathfrak{g}, f, s)$.

We set $\deg q_i^{(n)} = n+1$, for all $i \in I$ and $n \in \mathbb{Z}_+$. It is clear from the recurrence we got in the proof of Proposition 5.6 that, if we denote $U_k \in \widehat{\mathfrak{g}}_k$ the homogeneous part of $U(z)$ of degree k, then $\deg U_k = 2k$, for $k \geq 1$, and $\deg h_k = 2(k+1)$, for $k \geq -\frac{1}{2}$, where $h_k \in \mathfrak{Z}(\mathfrak{H})_k$ is the component of $h(z)$ of degree k, if we require $U(z) \in (\mathfrak{H}_{>0})^\perp \otimes \mathcal{V}$. It follows, by Remark 5.11, that h_k, for $k \geq -\frac{1}{2}$ are linearly indipendent modulo total derivatives, that is, $\int h_k$ are linearly independent for all $k \geq -\frac{1}{2}$.

Let us consider the following infinite sequence in $\mathcal{V}$. We set $f_0 = f$ and $f_n = \langle a(z) z^n \otimes 1 \mid h(z)\rangle$, for any $n \geq 1$. By Proposition 5.10, it follows that

$$\frac{\delta f_n}{\delta q} = (\pi_- \otimes \mathbb{1})(\varphi(a(z) z^n \otimes 1)^0) = \varphi(a(z) z^n \otimes 1)^0_-,$$

since $\widehat{\mathfrak{g}}^0 \cap \widehat{\mathfrak{g}}_- = \mathfrak{g} \cap (\mathfrak{B}_- \oplus z\mathfrak{g}[z]) = \mathfrak{B}_-$. Using (2.11), it follows that $(\pi_+ \otimes 1)[L(z), \varphi(a(z) z^n \otimes 1)_-] = [L(z), \varphi(a(z) z^n \otimes 1)_-] \in \mathfrak{B}_+ \otimes \mathcal{V}$. Since this bracket does not depend on z and both $L(z)$ and $\varphi(a(z) z^n \otimes 1)_-$ contain only nonnegative powers of z, we may put $z = 0$ inside the bracket and get $[\partial + q + f \otimes 1, \varphi(a(z) z^n \otimes 1)^0_-] = (\pi_+ \otimes 1) \left[\partial + q + f \otimes 1, \frac{\delta f_n}{\delta q}\right]$. By Proposition 5.9, we get

$$X_{[L(z), \varphi(a(z) z^n \otimes 1)_-]} = X_{H(\partial) \frac{\delta f_n}{\delta q}}.$$

Moreover, by Proposition 5.10, it follows that

$$\frac{\delta f_{n+1}}{\delta q} = (\pi_- \otimes 1) \left(\operatorname{Res}_z \varphi(a(z) z^n \otimes 1)\right).$$

Since $s \in \ker \operatorname{ad} \mathfrak{n}_+$ and using Proposition 5.8 and (2.11), we get

$$(\pi_+ \otimes 1)[L(z), \varphi(a(z) z^n \otimes 1)^+] = -[s, \operatorname{Res}_z \varphi(a(z) z^n \otimes 1)] = -\left[s, \frac{\delta f_{n+1}}{\delta q}\right].$$

It follows, by Proposition 5.10, that

$$X_{[L(z), \varphi(a(z) z^n \otimes 1)^+]} = X_{K(\partial) \frac{\delta f_{n+1}}{\delta q}}.$$

Hence, by Proposition 5.9, for any $n \in \mathbb{Z}_+$, we have

$$X_{H(\partial) \frac{\delta f_n}{\delta q}}\Big|_{\mathcal{W}_z(\mathfrak{g}, f, s)} = X_{K(\partial) \frac{\delta f_{n+1}}{\delta q}}\Big|_{\mathcal{W}_z(\mathfrak{g}, f, s)},$$

as required in (5.26).

APPENDIX A

Remarks on formal calculus

Given a $\mathbb{C}$-vector space $\mathcal{U}$, one usually has:

- $\mathcal{U}[z]$ the space of polynomials with coefficients in $\mathcal{U}$;
- $\mathcal{U}[z, z^{-1}]$ the space of Laurent polynomials with coefficients in $\mathcal{U}$;
- $\mathcal{U}[[z]]$ the space of power series with coefficients in $\mathcal{U}$;
- $\mathcal{U}((z))$ the space of Laurent series with coefficients in $\mathcal{U}$;
- $\mathcal{U}[[z, z^{-1}]]$ the space of infinite series in both directions with coefficients in $\mathcal{U}$.

If $\mathcal{U}$ is an algebra, then $\mathcal{U}[[z, z^{-1}]]$ is a module over $\mathbb{C}[z, z^{-1}]$, while the remaining are all algebras.

A *$\mathcal{U}$-valued formal distribution* in a variable z is a linear function on $\mathbb{C}[z, z^{-1}]$ with values in $\mathcal{U}$. The space $\mathbb{C}[z, z^{-1}]$ is usually referred as the space of thest functions. It can be proved that the space of $\mathcal{U}$-valued formal distribution is canonically identified with $\mathcal{U}[[z, z^{-1}]]$, that is it consists of series of the form

$$\sum_{n\in\mathbb{Z}} u_n z^n,$$

where $u_n \in \mathcal{U}$.

The same is true for many variables. For example, for two variables z_1 and z_2, a $\mathcal{U}$-valued formal distribution is a series of the type

$$a(z_1, z_2) = \sum_{n_1,n_2\in\mathbb{Z}} a_{n_1,n_2} z_1^{n_1} z_2^{n_2}.$$

Thus we call *formal distribution* in the indeterminates $z_1, z_2, \dots, z_m$ with values in $\mathcal{U}$ a formal expression

$$a(z_1, z_2, \dots, z_m) = \sum_{n_1,\dots,n_m\in\mathbb{Z}} a_{n_1,\dots,n_m} z_1^{n_1} z_2^{n_2} \dots z_m^{n_m}$$

where $a_{n_1,\dots,n_m} \in \mathcal{U}$. As already said, $\mathcal{U}[[z_1, z_1^{-1}, \dots, z_m, z_m^{-1}]]$ is not an algebra, thus products of two formal distributions is not always defined. However, if $a(z_1, \dots, z_m) \in \mathcal{U}[[z_1, z_1^{-1}, \dots, z_m, z_m^{-1}]]$ and $b(w_1, \dots, w_r) \in \mathcal{U}[[w_1, w_1^{-1}, \dots, w_r, w_r^{-1}]]$, then the product $a(z_1, \dots, z_m)b(w_1, \dots w_r)$ is a well defined formal distribution in the variables $z_1, \dots, z_m, w_1, \dots, w_r$.

Given a $\mathcal{U}$-valued formal distribution $a(z) = \sum_{n\in\mathbb{Z}} a_n z^n$, its *residue* is the linear function defined by

$$\operatorname{Res}_z a(z) = a_{-1}.$$

The basic property of the residue is that $\operatorname{Res}_z \partial_z a(z) = 0$, that gives the usual integration by parts formula

$$\operatorname{Res}_z \partial_z a(z)b(z) = -\operatorname{Res}_z a(z)\partial_z b(z).$$

Consider the algebras

$$\mathcal{A}_{z,w} = \mathbb{C}[z, z^{-1}, w, w^{-1}][[\frac{w}{z}]] \qquad \text{and} \qquad \mathcal{A}_{w,z} = \mathbb{C}[z, z^{-1}, w, w^{-1}][[\frac{z}{w}]].$$

Clearly $\mathcal{A}_{z,w}, \mathcal{A}_{w,z} \subset \mathbb{C}[[z, z^{-1}, w, w^{-1}]]$. Moreover, let us denote by R the algebra of meromorphic functions with pole only in $z = 0$, $w = 0$ and $|z| = |w|$. Then we have the following homomorphisms of algebras:

$$\begin{aligned} i_{z,w} : R &\longrightarrow \mathcal{A}_{z,w} \\ f &\longrightarrow \text{expansion in the domain } |z| > |w|, \end{aligned}$$

$$\begin{aligned} i_{w,z} : R &\longrightarrow \mathcal{A}_{w,z} \\ f &\longrightarrow \text{expansion in the domain } |w| > |z|. \end{aligned}$$

The expansions of the function $(z-w)^{-1}$ are of great interest:

$$i_{z,w}(z-w)^{-1} = \text{expansion of } \frac{1}{z}\cdot\frac{1}{1-\frac{w}{z}} = z^{-1}\sum_{n\in\mathbb{Z}_+}\left(\frac{w}{z}\right)^n \in \mathfrak{U}_{z,w}; \tag{A.1}$$

$$i_{w,z}(z-w)^{-1} = -\text{expansion of } \frac{1}{w}\cdot\frac{1}{1-\frac{z}{w}} = -w^{-1}\sum_{n\in\mathbb{Z}_+}\left(\frac{z}{w}\right)^n \in \mathfrak{U}_{w,z}. \tag{A.2}$$

We define the *formal δ-function* as the element of $\mathfrak{U}[[z,z^{-1},w,w^{-1}]]$ given by

$$\delta(z-w) = i_{z,w}(z-w)^{-1} - i_{w,z}(z-w)^{-1}. \tag{A.3}$$

Using (A.1) and (A.2) we can rewrite

$$\delta(z-w) = \sum_{n\in\mathbb{Z}}\frac{w^n}{z^{n+1}} = \sum_{n\in\mathbb{Z}}\frac{z^n}{w^{n+1}}.$$

For any $k\in\mathbb{Z}_+$, we also have

$$i_{z,w}(z-w)^k = \sum_{n\in\mathbb{Z}_+}\binom{n-k-1}{n}w^n z^{k-n} \qquad \text{and} \qquad i_{w,z}(z-w)^k = (-1)^k\sum_{n\in Z_+}\binom{n-k-1}{n}z^n w^{k-n},$$

from which follows an important formula for the derivatives of the δ-function:

$$\frac{1}{k!}\partial_w^k\delta(z-w) = \sum_{n\in\mathbb{Z}}\binom{n}{k}w^{n-k}z^{-n-1} = i_{z,w}\frac{1}{(z-w)^{k+1}} - i_{w,z}\frac{1}{(z-w)^{k+1}}. \tag{A.4}$$

Indeed, we can write, for $k\in\mathbb{Z}_+$,

$$i_{z,w}(z-w)^{-k-1} = \sum_{n\in\mathbb{Z}_+}\binom{n+k}{k}w^k z^{-k-1-n} = \sum_{n\in\mathbb{Z}_+}\binom{n}{j}w^{n-k}z^{-n-1}$$

and

$$i_{w,z}(z-w)^{-k-1} = (-1)^{k+1}\sum_{n\in\mathbb{Z}_+}\binom{n+k}{k}z^n w^{-k-1-n} = -\sum_{n\leq -1}\binom{n}{j}z^{-n-1}w^{n-j}.$$

Let $a(z)$ be a formal distribution, then we have

$$e^{w\partial_z}a(z) = \sum_{n\in\mathbb{Z},k\in\mathbb{Z}_+}a_n w^k\frac{\partial_z^k}{k!}z^n = \sum_{n\in\mathbb{Z},k\in\mathbb{Z}_+}\binom{n}{k}a_n w^k z^{n-k} = i_{z,w}a(z+w) \tag{A.5}$$

where we define $i_{z,w}a(z+w) = \sum_{n\in\mathbb{Z}}a_n i_{z,w}(z+w)^n$. In the same way we have that $e^{z\partial_w}a(w) = i_{w,z}a(z+w)$.

Proposition A.1. *The formal δ-function has the following properties:*

1) $\delta(z-w) = \delta(w-z)$;
2) $\partial_z\delta(z-w) = -\partial_w\delta(z-w)$;
3)
$$(z-w)^m\frac{1}{n!}\partial_w^n\delta(z-w) = \begin{cases} \frac{1}{(n-m)!}\partial_w^{n-m}\delta(z-w) & \textit{if } m\leq n,\\ 0 & \textit{if } m>n;\end{cases}$$
4) *if* $a(z)\in\mathfrak{U}[[z,z^{-1}]]$, *then*
$$a(z)\delta(z-w) = a(w)\delta(z-w) \quad \textit{and} \quad \operatorname{Res}_z a(z)\delta(z-w) = a(w);$$
5) $e^{\lambda(z-w)}\partial_w^n\delta(z-w) = (\lambda+\partial_w)^n\delta(z,w)$.

Proof. 1), 2) and 3) follow easily from (A.4). By 3) for $m=n=0$ we know that $z\delta(z,w) = w\delta(z,w)$, then $z^n\delta(z,w) = w^n\delta(z,w)$. By linearity it follows that $a(z)\delta(z,w) = a(w)\delta(z,w)$. If we now take the residue in this last relation, we get

$$\operatorname{Res}_z a(z)\delta(z,w) = a(w)\operatorname{Res}_z\delta(z,w) = a(w),$$

proving 4).

Finally, we prove 5). Let us note that $e^{\lambda(z-w)}\partial_w e^{-\lambda(z-w)} = \lambda+\partial_w$. Indeed, take any function $f(w)$, then

$$\begin{aligned} e^{\lambda(z-w)}\partial_w e^{-\lambda(z-w)}f(w) &= e^{\lambda(z-w)}(\lambda e^{-\lambda(z-w)}f(w) + e^{-\lambda(z-w)}\partial_w f(w))\\ &= (\lambda+\partial_w)f(w). \end{aligned}$$

Then

$$e^{\lambda(z-w)}\partial_w^n e^{-\lambda(z-w)} = (\lambda + \partial_w)^n.$$

Apply now to $\delta(z,w)$, then we get

$$e^{\lambda(z-w)}\partial_w^n e^{-\lambda(z-w)}\delta(z,w) = e^{\lambda(z-w)}\partial_w^n \delta(z-w) = (\lambda + \partial_w)^n \delta(z-w),$$

since by 4), we know that $e^{-\lambda(z-w)}\delta(z-w) = e^{-\lambda(w-w)}\delta(z-w) = \delta(z-w)$. □

If we take $a(z) = \delta(z-x) \in \mathcal{U}[[z, z^{-1}, x, x^{-1}]]$, then, by part 4), we get the identity

$$\delta(z-x)\delta(z-w) = \delta(w-x)\delta(z-w), \tag{A.6}$$

that holds in $\mathcal{U}[[z, z^{-1}, w, w^{-1}, x, x^{-1}]]$.

In particular, from the proof of Proposition A.1, it follows that

$$\mathrm{Res}_z\, \varphi(z)\delta(z-w) = \varphi(w)$$

for any test function $\varphi(z) \in \mathbb{C}[z, z^{-1}]$. This analogy with the Dirac δ-function in distribution theory clarifies why $\delta(z-w)$ is called formal δ-function.

APPENDIX B

Identities involving binomial coefficients

In this appendix we recall and prove the well known identities involving binomial coefficients that we have used throughout this thesis.

We recall that binomial coefficient is extended to negative integers n by $\binom{n}{k} = (-1)^k\binom{k-n-1}{k}$, for all $k \in \mathbb{Z}_+$. Thus, if $n \in \mathbb{Z}$, by Taylor formula, we have

$$(1+x)^n = \sum_{k\in\mathbb{Z}_+} \binom{n}{k} x^k,$$

from which follows that

$$\binom{n}{k} = \operatorname{Res}_x \frac{(1+x)^n}{x^{k+1}}, \tag{B.1}$$

for any $n \in \mathbb{Z}$ and $k \in \mathbb{Z}_+$.

Lemma B.1 (Vandermonde's identity). *For all $n, m \in \mathbb{Z}$ and $k \in \mathbb{Z}_+$,*

$$\sum_{i=0}^{k} \binom{m}{i}\binom{n-m}{k-i} = \binom{n}{k}.$$

Proof. By (B.1) we get

$$\binom{n}{k} = \operatorname{Res}_x \frac{(1+x)^n}{x^{k+1}} = \operatorname{Res}_x \frac{(1+x)^m(1+x)^{n-m}}{x^{k+1}} =$$
$$= \operatorname{Res}_x \left(\sum_{i,j\in\mathbb{Z}_+} \binom{m}{i}\binom{n-m}{j} x^{i+j-k-1} \right) = \sum_{i=0}^{k} \binom{n}{i}\binom{n-m}{k-i},$$

where in the last equality, in order to get the residue of the expression, we have to pick $j = k-i \in \mathbb{Z}_+$. $\square$

Lemma B.2. *For all $0 \le j \le k \le l$,*

$$\sum_{i=j+1}^{l} \binom{n-i}{k-j-1}\binom{i-1}{j} + \sum_{i=0}^{j} \binom{l}{i}\binom{n-l}{k-i} = \binom{n}{k}.$$

Proof. The well known recurrence

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \tag{B.2}$$

defines uniquely the binomial coefficients. Namely, assume $n, k \in \mathbb{Z}_+$, we may define the binomial coefficient $c_k^n \in \mathbb{Z}_+$ as the solution of the recurrence

$$c_k^n = c_k^{n-1} + c_{k-1}^{n-1} \tag{B.3}$$

with initial condition $c_0^0 = 1$ (we assume $c_{-1}^n = 0$, for all $n \in \mathbb{Z}_+$). Indeed, this is a linear recurson in n, then the space of solution is one-dimensional and, since binomial coefficients satisfy (B.3), which is the same as (B.2) and verify the initial condition, they are the unique solution. After, we can extend the definition to negative integers values of n as already recalled.

Let us set

$$c_n^k = \sum_{i=j+1}^{l} \binom{n-i}{k-j-1}\binom{i-1}{j} + \sum_{i=0}^{j} \binom{l}{i}\binom{n-l}{k-i}.$$

Clearly, $c_0^0 = 1$, since in this case the first sum does not give any contribution and the only term which contributes in the second sum is the one corresponding to the value $i = 0$. Hence, the lemma is proved if we show that c_n^k satisfy recurrence (B.3). We have

$$c_k^{n-1} = \sum_{i=j+1}^{l} \binom{n-i-1}{k-j-1}\binom{i-1}{j} + \sum_{i=0}^{j} \binom{l}{i}\binom{n-l-1}{k-i}$$

and

$$c_{k-1}^{n-1} = \sum_{i=j+1}^{l} \binom{n-i-1}{k-j-2}\binom{i-1}{j} + \sum_{i=0}^{j} \binom{l}{i}\binom{n-l-1}{k-i-1}.$$

Finally we get

$$c_k^{n-1} + c_{k-1}^{n-1} = \sum_{i=j+1}^{l} \left(\binom{n-i-1}{k-j-1} + \binom{n-i-1}{k-j-2}\right)\binom{i-1}{j} + \sum_{i=0}^{j} \binom{l}{i}\left(\binom{n-l-1}{k-i} + \binom{n-l-1}{k-i-1}\right).$$

Using (B.2) to sum terms inside parenthesis, we have that c_n^k satisfy recurrence (B.3). $\square$

Lemma B.3. *For all* $0 \leq k \leq n$,

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}.$$

PROOF. By (B.1) and the linearity of the residue, we get

$$\sum_{i=k}^{n} \binom{i}{k} = \sum_{i=0}^{n-k} \binom{i+k}{k} = \sum_{i=0}^{n-k} \operatorname{Res}_x \frac{(1+x)^{i+k}}{x^{k+1}} = \operatorname{Res}_x \frac{(1+x)^k}{x^{k+1}} \left(\sum_{i=0}^{n-k} (1+x)^i\right) =$$
$$= \operatorname{Res}_x \frac{(1+x)^k}{x^{k+1}} \cdot \frac{(1+x)^{n+1-k} - 1}{x} = \operatorname{Res}_x \frac{(1+x)^{n+1}}{x^{k+2}} = \binom{n+1}{k+1},$$

where we used the fact that $\operatorname{Res}_x \frac{(1+x)^k}{x^{k+2}} = 0$, since $\frac{(1+x)^k}{x^{k+2}}$ has order -2. $\square$

APPENDIX C

Virasoro field in classical $\mathcal{W}$-algebras

As stated in the Introduction, in the conformal field theory setup, $\mathcal{W}$-algebras appeared as non-linear extension of the Virasoro algebra by primary fields. We want to make this definition rigourous in the context of Poisson vertex algebras.

In the quantum case, there is a well-known notion of vertex algebra of conformal field theory type. Thus, a Poisson vertex algebra of conformal field theory type is the quasiclassical limit of such vertex algebras (for the quasiclassical limit definition see [7]). We will give more explicit definitions.

Definition C.1. Let $\mathcal{V}$ be a Poisson vertex algebra which contains the Virasoro Poisson vertex algebra (see Example 1.8) as a Poisson vertex subalgebra, namely, there exists $L \in \mathcal{V}$ such that

$$\{L_\lambda L\} = (\partial + 2\lambda)L + c\lambda^3,$$

with $c \in \mathbb{C}$. Then, we give the following definitions:

i) $a \in \mathcal{V}$ is called *quasi-primary field* if $\{L_\lambda a\} = (\partial + \Delta_a \lambda)a + o(\lambda^2)$;

ii) $a \in \mathcal{V}$ is called *primary field* if $\{L_\lambda a\} = (\partial + \Delta_a \lambda)a$.

Δ_a is called *conformal weight* of a.

Once we know the notion of primary field, we can give the definition of Poisson vertex algebras of conformal field theory type.

Definition C.2. Let $\mathcal{V}$ be an algebra of differential functions in the variables $L, W_1, \dots, W_{l-1}$, endowed with a λ-bracket satisfying the axioms of Poisson vertex algebra. We say that $\mathcal{V}$ is a *Poisson vertex algebra of conformal field theory type* if

i) $\{L_\lambda L\} = (\partial + 2\lambda)L + c\lambda^3$, with $c \in \mathbb{C}$;

ii) $\{L_\lambda W_i\} = (\partial + \Delta_j \lambda)W_j$, $j = 1, \dots, l-1$.

This definition says that a Poisson vertex algebra of conformal field theory type is a Poisson vertex algebra generated, as differential algebra, by a Virasoro field and some primary fields.

Classical $\mathcal{W}$-algebras are example of Poisson vertex algebras of conformal field theory type. We want to write explicitly the Virasoro field in the case of the Drinfeld-Sokolov Hamiltonian reduction. In this case the primary fields are obtained choosing $V = \ker \operatorname{ad} e$ in Proposition 2.1 (a proof is given in [15] and [16]).

Let $\mathfrak{g}$ be a reductive Lie algebra with a nondegenerate invariant bilinear form $(\cdot \mid \cdot)$ and let us consider the Poisson vertex algebra $\mathcal{V}(\mathfrak{g}) = \mathcal{V}_0(\mathfrak{g})$ with λ- bracket given by (2.13), setting $z = 0$, which we denote $\{\cdot_\lambda \cdot\}$. Let $\{u_i\}_{i \in I}$, $I = \{1, \dots, n = \dim \mathfrak{g}\}$, be a basis of $\mathfrak{g}$ and $\{u^i\}_{i \in I}$ be the dual basis with respect to $(\cdot \mid \cdot)$.

Lemma C.3. $L = \frac{1}{2}\sum_i u^i u_i \in \mathcal{V}(\mathfrak{g})$ *spans a centerless Virasoro Poisson subalgebra in* $\mathcal{V}(\mathfrak{g})$.

Proof. We have to prove that

$$\{L_\lambda L\} = (\partial + 2\lambda)L.$$

Let us assume $a \in \mathfrak{g}$, then, by Leibniz rule (1.7), we have

$$\begin{aligned}\{a_\lambda L\} &= \frac{1}{2}\sum_{i \in I}\{a_\lambda u^i u_i\} = \frac{1}{2}\sum_{i \in I}\left(\{a_\lambda u^i\}u_i + \{a_\lambda u_i\}u^i\right) = \\ &= \frac{1}{2}\sum_{i \in I}\left([a, u^i]u_i + (a \mid u^i)u_i\lambda + [a, u_i]u^i + (a \mid u_i)u^i\lambda\right).\end{aligned}$$

∎

Since any element $b \in \mathfrak{g}$ can be written as $b = \sum_i (b \mid u_i)u^i = \sum_i (b \mid u^i)u_i$, we have, using invariance and symmetry of the bilinear form

$$\sum_{i \in I}[a, u^i]u_i = \sum_{i,j \in I}([a, u^i] \mid u_j)u^j u_i = -\sum_{i,j \in I}([a, u_j] \mid u^i)u_i u^j = -\sum_{i \in I}[a, u_i]u^i.$$

Hence, we get

$$\{a_\lambda L\} = a\lambda,$$

and, by skew-commutativity (1.9), $\{L_\lambda a\} = (\partial+\lambda)a$. Finally, we have

$$\{L_\lambda L\} = \frac{1}{2}\sum_{i\in I}\{L_\lambda u^i u_i\} = \frac{1}{2}\sum_{i\in I}\big(\{L_\lambda u^i\}u_i + \{L_\lambda u_i\}u^i\big) = \frac{1}{2}\sum_{i\in I}\big(u_i(\partial+\lambda)u^i + u^i(\partial+\lambda)u_i\big) =$$
$$= (\partial+2\lambda)L.$$

In the proof we have also proved that $a\in\mathfrak{g}$ is a primary field of conformal weight 1.

Let $f\in\mathfrak{g}$ be a nilpotent element and $\{f, h=2x, e\}\subset\mathfrak{g}$ be an $\mathfrak{sl}_2$ triple. We define $L^{(x)} = L + x'$.

Lemma C.4. *$L^{(x)}$ spans a Virasoro Poisson subalgebra of central charge $c = -(x\mid x)$.*

Proof. We have to show that

$$\{L^{(x)}{}_\lambda L^{(x)}\} = (\partial+2\lambda)L^{(x)} - (x\mid x)\lambda^3.$$

Using the results from previous lemma and sesquilinearity (1.6), we get

$$\begin{aligned}\{L^{(x)}{}_\lambda L^{(x)}\} &= \{L_\lambda L\} + \{L_\lambda x'\} + \{x'{}_\lambda L\} + \{x'{}_\lambda x'\} = \\ &= \{L_\lambda L\} + (\lambda+\partial)\{L_\lambda x\} - \lambda\{x_\lambda L\} - \lambda(\lambda+\partial)\{x_\lambda x\} = \\ &= (\partial+2\lambda)L + (\lambda+\partial)^2 x - x\lambda^2 - (x\mid x)\lambda^3 = (\partial+2\lambda)L^{(x)} - (x\mid x)\lambda^3.\end{aligned}$$

□

Corollary C.5. $L^{(x)} \in \widetilde{\mathcal{W}}(\mathfrak{g}, f)$.

Proof. We have to show that $\{L^{(x)}{}_\lambda a\} \in \widetilde{\mathcal{J}}(\mathfrak{g}, f)[\lambda]$, for all $a\in\mathfrak{n}_+$. If $a\in\mathfrak{g}$, then we have

$$\{L^{(x)}{}_\lambda x\} = \{L_\lambda a\} - \lambda\{x_\lambda a\} = (\lambda+\partial)a - [x,a] - (x\mid a)\lambda^2 = a' + (a-[x,a])\lambda - (x\mid a)\lambda^2.$$

Hence, if $a\in\mathfrak{n}_+$, then

$$\{L^{(x)}{}_\lambda a\} = a' - (a-[x,a])\lambda.$$

In particular, if $a\in\mathfrak{g}_1$, then $\{L^{(x)}{}_\lambda a\} = a' \in \widetilde{\mathcal{J}}(\mathfrak{g}, f)[\lambda]$, otherwhise $\{L^{(x)}{}_\lambda a\} \in \widetilde{\mathcal{J}}(\mathfrak{g}, f)[\lambda]$, since $(a-[x,a]\mid f) = 0$. □

It follows that the element $\pi(L^{(x)})$ is a Virasoro field in $\mathcal{W}(\mathfrak{g}, f)$.

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