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Generalized Mom-structures and volume estimates for hyperbolic 3-manifolds with geodesic boundary and toric cusps

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Introduction

The issue of understanding the volume of hyperbolic 3-manifolds has been one of the central themes of geometric topology since the pioneering work of Jorgensen and Thurston, who showed that the set of possible volumes is a well-ordered subset of $\mathbb{R}$ (see, *e.g.*, [2]). Many authors, including W. Thurston [19], J. Weeks [20], S. Matveev and A. Fomenko [15], also suggested and supported the idea that volume is a good measure of the complexity of a hyperbolic 3-manifold. This was one of the reasons behind many of the efforts that have been made to identify the hyperbolic 3-manifold of minimum volume within given classes of manifolds.

In 1991 S. Kojima and Y. Miyamoto [14] proved that the compact hyperbolic 3-manifolds with nonempty geodesic boundary of minimum volume are the 8 manifolds that decompose into two truncated regular tetrahedra of dihedral angle $\frac{\pi}{6}$ (first described in [9]). In 2001 C. Cao and G. R. Meyerhoff [5] identified the hyperbolic 3-manifolds with toric cusps and empty boundary of minimum volume, which are the figure-eight knot complement and its sibling (first described in [19] and [4]). More recently (2006-2007) a series of articles by D. Gabai, G. R. Meyerhoff and P. Milley ([10], [11], [16] and [12]) showed that the closed hyperbolic 3-manifold of minimum volume is the Weeks manifold (first described in [4] and [15]). Finding the minimal-volume hyperbolic 3-manifold with both nonempty geodesic boundary and toric cusps still remains an open problem, and one of the aim of this thesis is to start investigating it.

The main inspiration for our work came from the cited articles by D. Gabai, G. R. Meyerhoff and P. Milley, and in particular from the crucial objects used in these articles, *i.e.* the *Mom structures* they introduced in [10]. A Mom-n structure is a triple (M, T, Δ) where M is a 3-manifold with toric boundary and Δ is a handle decomposition of M based on a boundary torus T of M and with exactly n 1- and 2-handles, which in addition are required to have valence (*i.e.*, number of other handles they run over) respectively at least 2 and exactly 3. The connection between these topological structures and the hyperbolic volume is, roughly speaking, the following. Let N be a hyperbolic 3-manifold with toric cusps but empty boundary, and consider a neighborhood U of a cusp bounded by a horospherical cross-section. Inflate U until it touches itself: this gives rise to a 1-handle. Continuing to inflate U, other 1- and 2-handles get created. If N has low volume, the handle structure created in this way is indeed a Mom-n structure (M, T, Δ) which is very simple (*e.g.*, with $n = 2$ or 3) and from which N can be obtained by Dehn-filling all but one of the toric boundary components of M [11]. The chain of implications followed by D. Gabai, G. R. Meyerhoff and P. Milley to prove their result is basically the following: a closed hyperbolic 3-manifold with low volume (in particular, less than or equal to the volume of the Weeks manifold) must be obtainable by Dehn filling on a one-cusped hyperbolic 3-manifold with small enough volume, and all such manifolds can be obtained by Dehn filling all but one of the cusps of M, where M is a hyperbolic Mom-2 or Mom-3 manifold. After enumerating, with computer assistance, all such Mom structures and their low-volume Dehn fillings, the Weeks manifold turned out to be the smallest one [10].

A natural generalization of the notion of Mom structure to the case of manifolds with both nonempty boundary and toric cusps is that of *protoMom structure*, which simply replaces, as the base of the handle structure, the torus T with any nonempty (and not necessarily connected) closed surface Σ without spherical components. The general theory of protoMom structures is due to E. Pervova and was presented in [17].

The first chapter of the thesis is devoted to the study of Mom and protoMom structures. The original results it contains heavily rely on [17], and in fact they would have never come to the mind of the author without the constant discussion with E. Pervova during 2009 and early 2010.

Section 1.1 contains the basic definitions concerning Mom structures, protoMom structures and ideal triangulations that will be used through the whole chapter. In particular, we recall the notion of *weak* protoMom (or Mom) structure, which generalizes protoMom structures allowing also 1-handles of valence 1 or 0, and the definition of an *internal* protoMom (or Mom) structure (M, Σ, Δ) on a 3-manifold N with nonempty boundary, which basically means that N can be recovered from $M \subset N$ by Dehn-filling some of its boundary tori.

In **Section 1.2** we show two algorithmic procedures (called *standard procedure* and *reduced procedure*) that, starting from a 3-manifold N whose boundary consists of a surface Σ without spherical components and from an ideal triangulation τ of N, allow us to construct many protoMom structures internal on N, which we will refer to as τ-*induced* protoMom structures. A τ-induced protoMom structure is generally weak, but it is homeomorphic to one with no 1-handles of valence 1, where the homeomorphism is given by a sequence of *elementary collapses*, each one removing a 1-handle of valence 1 together with the 2-handle incident to it. The connected components of the union of all the handles removed in this way are called *chains*, and we briefly describe the shapes they can have. The procedure used to obtain a τ-induced protoMom structure will be essentially encoded into an appropriate coloring of the 4-valent graph Γ dual to τ: the graph Γ with such a coloring will be denoted as a *protoMom subgraph*.

The standard procedure we use is basically the following one:

1. Remove a small open ball from each tetrahedron of τ.

2. Choose a maximal forest $T(\Gamma)$ in Γ, color its edges by t and remove the corresponding 2-handles. Then collapse all chains until no 1-handle of valence 1 is left, and color by d the edges corresponding to 2-handles removed in this way.

3. Remove a 2-handle α with both sides on the same spherical boundary component (thus creating a boundary torus). If in the subsequent step 4 a 1-handle of valence 0 is removed, color the edge corresponding to α by z, otherwise color it by c.

4. Collapse all chains until no 1-handle of valence 1 is left, color by d the corresponding edges, and then remove all 1-handles of valence 0. If the boundary of the manifold does not consists only of ∂N and some tori, restart from step 3.

5. Color by f all the remaining edges of Γ, which correspond to the non-removed 2-handles.

Such a procedure always ends, and gives as a result a protoMom structure internal on N. The reduced procedure presented in Subsection 1.2.3 seems very less general than the standard one, but we will prove that they are equivalent.

In **Section 1.3** we explicitly construct, given a full weak protoMom structure (M, Σ, Δ) internal on a manifold N, a triangulation τ of N such that M is τ-induced (we recall that a protoMom structure is *full* if every connected component of $\partial M \cap \Sigma \times \{1\}$ is a disc). Roughly speaking, Δ induces a triangulation τ_i' on each boundary torus of M that bounds a solid torus H_i in N, and we fill such toric boundary components with a triangulation τ_i of the whole solid torus H_i such that $\tau_i|_{\partial H_i} = \tau_i'$, and $\tau_i^{(0)} = \tau_i'^{(0)}$ (we also explicitly present a reduced procedure on (N, τ) that gives rise to (M, Σ, Δ)). Moreover, the way we construct the triangulation τ of N is such that it tries to minimize the number of tetrahedra employed. Using this construction we get in **Section 1.4** a good estimate for the complexity $c(N)$ of N:

Theorem 1. *Every full protoMom structure (M, Σ, Δ) arises from an ideal triangulation of some manifold N such that ∂N is homeomorphic to Σ and $c(N) \leq 4n$, where n in the number of 2-handles of Δ.*

We end the chapter by recalling, in **Section 1.5**, the set of combinatorial moves used in [17] to relate to each other any two full weak protoMom structures internal on the same manifold N, adding only a few remarks concerning the presence of 1-handles of valence 0 and the coloring we used for protoMom subgraphs.

In the second chapter of the thesis we address directly the computation of the volume of a hyperbolic 3-manifold M with both nonempty geodesic boundary and a number $k \geqslant 1$ of toric cusps. We want to find a (lower) estimate for the volume of M by analyzing what portion of the volume of M can be covered with a boundary collar and a cusp neighborhood with disjoint embedded interiors. Many of the ideas contained in this chapter arose while explicitly studying the simplest examples of hyperbolic 3-manifolds with both cusps and geodesic boundary, which are the 32 manifolds of complexity $c \leqslant 4$ censed in [18] and recalled in the Appendix. For this reason most sections of the chapter include the numerical computations on these 32 manifolds as explicit examples.

For our computations we will often use an arbitrary ideal triangulation τ of M given by geometric (hyper)ideal tetrahedra, *i.e.* ideal tetrahedra which can have all, some or none of their vertices truncated. In order to make our volume estimate explicit with respect to the dihedral angles of τ we usually realize one or more of its (hyper)ideal tetrahedra in the half-space model, using the notation introduced in **Section 2.1** and summarized in Figure 2.1.

Section 2.2 is devoted to the study of a *d-collar* B_d of ∂M, which roughly speaking is the union of all the geodesic arcs of length d orthogonal to ∂M, provided they have disjoint interiors; in other words, B_d is the set of points of M with distance at most d from ∂M, provided d is less than or equal to a certain $d_{\max}$ corresponding to a collar self-tangency. The volume $V_\partial(d)$ of a d-collar of the boundary is given by [6]

$$V_\partial(d) = \mathcal{A}_\partial \frac{2d + \sinh(2d)}{4}$$

where the area $\mathcal{A}_\partial$ of the geodesic boundary of M can be easily computed from the genera g_i of the b connected components of ∂M or from the numbers e and n of edges and tetrahedra of any ideal triangulation τ of the whole M:

$$\mathcal{A}_\partial = -2\pi \cdot \sum_{i=1}^{b} \chi(\partial M_i) = 4\pi \sum_{i=1}^{b} (g_i - 1) = 4\pi(n - e)$$

The maximal width $d_{\max}$ of a boundary collar is half the length of the shortest geodesic arc in M with both endpoints on ∂M and orthogonal to it: in some specific situations, which include all the examples with $c \leq 4$, this turns out to coincide with half the length of the shortest truncated edge of τ, which can be computed using only the dihedral angles of τ [8] (we carry out explicit numerical computations for the 32 examples of low complexity).

In **Section 2.3** we start dealing with a cusp neighborhood bounded by horospherical cross-sections. A cusp neighborhood that is inflated up to touching itself, the boundary of M or a d-collar of ∂M is said to be *maximal*; in particular it is said *d-maximal* in the latter case. If M has a number $k > 1$ of toric cusps, we consider a single cusp neighborhood U given by a horospherical neighborhood of each cusp with equal volume and disjoint interiors. We start by analyzing cusp self-tangencies, and we show a sufficient condition, involving only the dihedral angles of τ, for a cusp neighborhood to be 0-maximal (*i.e.*, tangent to ∂M). If τ has some additional symmetries, as all the examples with $c \leq 4$ have, we can further simplify this condition and check it more easily in the 32 examples of low complexity: it turns out that for $c \leq 4$ no cusp self-tangencies ever happens (*i.e.*, the cusp neighborhood can always be inflated up to touching ∂M).

In **Section 2.4** we compute the volume of a cusp neighborhood, with particular care in handling the differences between a d-maximal and a self-tangent one. In particular, the volume $\mathrm{Vol}_c(d)$ of a d-maximal cusp neighborhood is given by

$$\mathrm{Vol}_c(d) = \frac{\mathrm{Area}(\mathbb{E}^2/\Lambda)}{2\bar{r}^2}e^{-2d}$$

where, roughly speaking, the cusp C tangent to B_d is centered at ∞ in the half space model $\mathbb{E}^2 \times (0, +\infty)$, Λ is the lattice acting horizontally on $\mathbb{H}^3$ to give the cusp C, and $\bar{r}$ is the maximal Euclidean radius among the truncation half-spheres bounding the universal cover of M. The ratio $\frac{\mathrm{Area}(\mathbb{E}^2/\Lambda)}{\bar{r}^2}$ turns out to be well defined, and we show how it depends only on the dihedral angles of τ.

If cusp self-tangencies happen we may have to stop inflating the cusp neighborhood before reaching the maximal boundary collar. In this situation the volume Vol_c of a maximal cusp neighborhood does not depend on d and is given by

$$\mathrm{Vol}_c = k \cdot \frac{Area_c}{2\bar{r}^2}e^{-2d_{\max}-2h}$$

where the values of $\mathrm{Area}_c := \mathrm{Area}(\mathbb{E}^2/\Lambda)$ and $\bar{r}$ are computed using a cusp whose neighborhood has distance from ∂M equal to $d_{\max} + h$.

Section 2.5 summarizes all the previous sections in the following:

Proposition 1. *Let M be an orientable hyperbolic 3-manifold with nonempty geodesic boundary and $k \geqslant 1$ toric cusps. Then the portion $V(d)$ of the volume of M covered by a d-collar B_d of the boundary, with $d \in [0, d_{\max}]$, and a maximal cusp neighborhood U with disjoint and embedded interiors is given by*

$$V(d) = \begin{cases} \pi\Big(\sum_{i=1}^{b} g_i - b\Big)\Big(2d + \sinh(2d)\Big) + k \cdot \dfrac{Area_c}{2\bar{r}^2}e^{-2d} & \text{if } U \text{ is } d\text{-maximal} \\ \pi\Big(\sum_{i=1}^{b} g_i - b\Big)\Big(2d + \sinh(2d)\Big) + k \cdot \dfrac{Area_c}{2\bar{r}^2}e^{-2d_{\max}-2h} & \text{if } U \text{ is not } d\text{-maximal} \end{cases}$$

where $Area_c$ and $\bar{r}$ are computed using one of the cusps whose neighborhood has minimal distance equal to $d_{\max} + h$ from B_d.

In order to get the best volume estimate of this kind we are interested in finding the maximum of the function $V(d)$ on its domain of definition, which is an interval $I := [d_{\min}, d_{\max}]$ where $d_{\min}$ is the minimum among the values of $d \in [0, d_{\max}]$ such that there exists a d-maximal cusp neighborhood. We prove:

Theorem 2. *The function $V(d)$ has its maximum at $d = d_{\min}$ or at $d = d_{\max}$.*

The best volume estimate for the volume of M using a boundary collar and a maximal cusp neighborhood with disjoint and embedded interiors can then be achieved only at the combinatorially extremal configurations, namely those where we consider the maximal boundary collar or the cusp neighborhood with maximal volume.

We end the section with some explicit computations on the examples with $c \leqslant 4$. For these 32 manifolds one has $V'(d) > 0$, which implies that the configuration maximizing $V(d)$ is always that given by a maximal boundary collar and a cusp neighborhood tangent to it. For all the examples with $c \leqslant 4$ we numerically calculate $V(d_{\max})$, which turns out to cover a portion variable from the 47% to the 77% of the volume of the manifold (with a 60% on average).

In **Section 2.6** we generalize the previous results by allowing each boundary component and each cusp to vary its own neighborhood independently. More precisely, we consider a hyperbolic 3-manifold M with either nonempty compact geodesic boundary, or some toric cusps, or both, and we ask what is the optimal way (in the sense of volume maximization) of inserting in M boundary collars and/or cusp neighborhoods having disjoint embedded interiors. Boundary collars and cusp neighborhoods will be collectively called *peripheral components*, and their total volume denoted by V. It turns out that an optimal choice necessarily occurs in a combinatorially extremal configuration. In particular, if we say that a peripheral component is *maximal* if it is tangent to itself or to ∂M, we have that:

- If M has two peripheral components, then V can have a local maximum only if one component is maximal and the other one is maximal given the first. If the two peripheral components are both cusps or both collars of boundary components with the same genus, then it is better to maximize first the component that individually can be made bigger than the other one; if one component is a cusp and the other is a collar, this statement coincides with Theorem 2.

- If M has three peripheral components P_1, P_2 and P_3, then V can have a local maximum only in one of the following configurations:

  - one component is maximal, another component is maximal given the first one, and the last component is maximal given the other two;
  - each of P_1, P_2 and P_3 is tangent to the other two and certain modified volumes $\tilde{v}(P_1)$, $\tilde{v}(P_2)$ and $\tilde{v}(P_3)$ satisfy the strict triangular inequalities, where $\tilde{v}(P_i)$ coincides with the volume of P_i when P_i is a cusp and is equal to $\frac{1}{2}\frac{\partial Vol(B_d)}{\partial d}$ when P_i is a collar.

- If M has n peripheral components such that none of them is maximal and such that there are fewer than n tangencies between different components, then the configuration cannot be a local maximum for V.

A long-term goal would be to follow in the bounded case a similar path as that paved by D. Gabai, G. R. Meyerhoff and P. Milley in the case of closed hyperbolic 3-manifolds. One could try to inflate peripheral components even more, pushing them against each other, and hoping to find that this gives rise to a handle structure that is a somehow simple protoMom structure if the starting manifold M had a low-enough volume. Then one could try to classify such simple protoMom structures, identify which ones admit a complete hyperbolic metric in the complement of their boundary tori, and enumerate their low-volume Dehn fillings in order to find even in the class of hyperbolic 3-manifolds with both nonempty geodesic boundary and toric cusps those with minimum volume.

Chapter 1

ProtoMom structures and ideal triangulations

1.1 Basic definitions

Mom structures In this paragraph we recall some basic definitions introduced by D. Gabai, G. R. Meyerhoff and P. Milley in [10]. In particular we recall the definition of *Mom structure*, which is the crucial object used by them to prove the "smallest closed hyperbolic manifold" conjecture.

Definition 1.1.1. Let M be a compact connected 3-manifold and let $B \subseteq \partial M$ be a closed surface (which may be disconnected or empty). A *handle structure* Δ on (M, B) is a decomposition of M obtained in the following way:

- take $B \times [0,1] \subseteq M$ such that $B \times \{0\} = B$;
- add a finite number of 0-handles;
- attach finitely many 1- and 2-handles to $B \times \{1\}$ and the 0-handles.

We call $B \times [0,1]$ the *base* and $B \times [0,1] \cup$ 0-handles the *extended base* of Δ, and we say that Δ is a *handle structure based on* B.

The *valence* of a 1-handle is the number of times, counted with multiplicity, the various 2-handles run over it, and the *valence* of a 2-handle is the number of 1-handles, counted with multiplicity, it runs over. We call the 0-handles, 1-handles and 2-handles *balls*, *beams* and *plates* respectively. We call *islands* and *bridges* the intersections of the extended base with the beams and the plates respectively. We call *lakes* the closures of the connected components of the complement of islands and bridges in $B \times \{1\} \cup \partial$(0-handles). We say that Δ is *full* if each lake is a disc.

Definition 1.1.2. A *Mom-n structure* is a triple (M, T, Δ) where

- M is a compact connected 3-manifold such that ∂M is a union of tori;
- T is a preferred toric boundary component of M;
- Δ is a handle decomposition of M based on T and without 0-handles obtained as follows:
  - take $T \times [0, 1] \subseteq M$ such that $T \times \{0\} = T$;
  - attach n 1-handles to $T \times \{1\}$;
  - add n 2-handles of valence 3 in such a way that each 1-handle has valence at least 2.

ProtoMom structures In this paragraph we recall some basic definitions introduced by E. Pervova in [17]. The main object of the chapter is the following generalization of the notion of Mom structure given in the previous paragraph:

Definition 1.1.3. A *protoMom structure* is a triple (M, Σ, Δ) where

- M is a compact connected orientable 3-manifold such that ∂M is the disjoint union of a nonempty closed surface Σ without spherical components and some tori (which will be referred to as *lateral tori*);
- Δ is a handle decomposition of M based on Σ obtained as follows:
  - take $\Sigma \times [0, 1] \subset M$ such that $\Sigma \times \{0\} = \Sigma$;
  - attach finitely many 1-handles to $\Sigma \times \{1\}$;
  - add finitely many 2-handles of valence 3 in such a way that each 1-handle has valence at least 2.

Remark 1.1.4. Let (M, Σ, Δ) be a protoMom structure, and let n be the number of 2-handles of Δ. Then Δ contains a number $n - g + 1$ of 1-handles, where g is the sum of the genera of the connected components of Σ.

Remark 1.1.5. A Mom structure is a protoMom structure where Σ is a torus.

Definition 1.1.6. A *weak protoMom structure* is defined as a protoMom structure but with no restrictions on the valences of 1-handles (*i.e.*, 1-handles of valence 0 and 1 are allowed).

Definition 1.1.7. Let N be a 3-manifold with nonempty boundary and let (M, Σ, Δ) be a protoMom structure on a submanifold $M \subset N$. The triple (M, Σ, Δ) is called an *internal protoMom structure on* N if $N \setminus M$ consists of a collar of ∂N and possibly some solid tori.

Remark 1.1.8. If a protoMom structure (M, Σ, Δ) is internal on N, then N can be obtained from M by Dehn-filling some of the lateral tori of M.

Triangulations, special spines and graphs A compact 2-dimensional polyhedron P is called *special* if the following two conditions hold. First, the link of each point is homeomorphic to one of the following 1-dimensional polyhedra:

- a circle;
- a circle with a diameter;
- a circle with three radii.

Second, the components of the set of points with link of the first type (called *faces*) are open discs and the components of the set of points with link of the second type (called *edges*) are open segments.

The points with link of the third type (called *vertices*) and the points belonging to edges are called *singular*. The set $S(P)$ of all singular points of P is a 4-valent graph without circular components that we will refer to as the *singular graph* of P.

Definition 1.1.9. Let N be a 3-manifold with nonempty boundary, and let P be a special polyhedron embedded in the interior of N. We say that P is a *special spine* of N if $N \setminus P$ is an open collar of ∂N.

Definition 1.1.10. An *ideal triangulation* of a compact 3-manifold N with nonempty boundary is a realization of N as the gluing of a finite number of tetrahedra (along a complete system of simplicial pairings of their faces) with a regular open neighborhood of each vertex removed. In this way N is decomposed into *truncated tetrahedra*, and ∂N is triangulated by the truncation triangles.

There is a well-known duality between ideal triangulations of 3-manifolds with boundary and their special spines:

Proposition 1.1.11. *Let N be a compact 3-manifold with nonempty boundary. Then the set of ideal triangulations of N correspond bijectively to the set of special spines of N, where the spine corresponding to a triangulation is the 2-skeleton of the cellularization dual to the triangulation.*

1.2 From ideal triangulations to internal protoMom structures

1.2.1 The *standard procedure*

Fix a compact orientable 3-manifold N with boundary that consists of a (possibly disconnected) surface Σ without spherical components. Let τ be an arbitrary ideal triangulation of N, seen as a decomposition of N into truncated tetrahedra. Then τ gives rise to a number of internal protoMom structures on N, which are constructed in the following way.

Consider the 3-manifold M' obtained by removing from each tetrahedron of τ a small open ball whose closure is contained in the interior of that tetrahedron. The triangulation τ naturally induces on M' a general-based handle decomposition Δ' obtained by taking a collar of $T \cup \Sigma$ and thickening each edge of τ to a 1-handle and each face to a 2-handle. We will construct (in a non-unique way) a protoMom structure (M, Σ, Δ) with $M \subset M'$ and $\Delta \subset \Delta'$ (we will say that (M, Σ, Δ) is *τ-induced* [17]).

Let P_τ be the special spine of N dual to τ, and let $\Gamma := S(P_\tau)$ be its singular graph, which is a 4-valent graph. We will use in particular the bijection between the set of edges of Γ and the set of 2-handles of Δ'; with an abuse of terminology we will sometimes refer to an edge to mean the corresponding 2-handle or vice-versa, but this will never be confusing. Choose a maximal forest $T(\Gamma)$ in Γ (trees consisting of a single vertex are allowed) and remove the corresponding 2-handles of Δ', thus obtaining a manifold M'' whose boundary consists of T, Σ and a number of spheres, and with a handle decomposition Δ''. This procedure leading from (N, τ) to (M'', Δ'') will be referred to as *step (0)* or *operation (0)*.

Then apply, in the given order, the following operations (at each step we will denote by M_i and Δ_i the initial manifold and handle decomposition, and by M_f and Δ_f the final ones obtained by the operations listed):

(1) If Δ_i does not contain 1-handles of valence 0 or 1, choose a 2-handle α such that one of the following is true and remove it:

  I the sides of α lie on two distinct spherical components of the boundary;

  II one side of α lies on a spherical component and the other one on a toric component;

  III both sides of α lie on the same spherical component.

  Notice that this move changes the boundary of the manifold by removing a spherical component (cases I and II) or by replacing a spherical component with a toric one (case III).

(2) If Δ_i contains a 1-handle B of valence 1, cancel it together with the 2-handle A incident to it (we will call this operation *elementary collapse*). Notice that, if M_f is obtained from M_i by an elementary collapse, then M_f and M_i are homeomorphic, and in particular the boundary does not change.
  Repeat the process until no 1-handles of valence 1 are left.

(3) If Δ_i contains a 1-handle B of valence 0 but no 1-handles of valence 1, remove B. Notice that this operation changes the boundary of the manifold, decreasing the genus of one of its components by 1.
  Repeat the process until no 1-handles of valence 0 are left.

(4) If the boundary of the manifold does not consist only of Σ and some tori, restart from step (1) again.

Since operation (3) can only replace a number of toric boundary components with the same number of spherical ones, every cycle of operations (1)-(3) decreases the number of spherical boundary components, unless step (1)-III creates a torus around a 1-handle of valence 0 which is removed in the subsequent step (3). If this is the case, however, since the number of handles is decreased, after a finite number of the cycles (1)-(3) the number of spherical boundary components will decrease anyway. By induction we can then conclude that this procedure will end, *i.e.* the boundary of the resulting manifold will consist only of Σ and one or more tori:

Observation 1.2.1. *The procedure (0)-(4) described above will always have as a result a manifold whose boundary consists only of* Σ *and some tori.*

Consider now the following operation, which we call (1') since it is similar to operation (1):

(1') If Δ_i does not contain 1-handles of valence 0 or 1, either choose a 2-handle α such that the sides of α lie on two distinct toric boundary components and remove it, or do nothing.

Notice that this operation changes the boundary of the manifold by replacing two toric components by one of genus 2. Since we want to avoid to increase the genus of a boundary component without control, we allow the use of operation $(1')$ only if the subsequent application of operations (2) and (3) leads to a boundary that consists only of tori and Σ, *i.e.* only if the removal of a 2-handle with operation $(1')$ will, possibly after elementary collapses, let us remove one (and only one) 1-handle of valence 0 in step (3) (notice that the boundary of this 1-handle of valence 0 must intersect the genus-2 boundary component of the manifold). Then we can repeat the process, if we are allowed to.
We sum up this discussion introducing the following operation:

(5) Perform steps $(1')$, (2) and (3), in this order. If the boundary of the manifold does not consist only of Σ and some tori, undo this operation and stop the procedure. On the other hand, if the boundary of the manifold consists only of Σ and some tori, either stop the procedure or repeat this operation again.

Remark 1.2.2. Notice that we are not allowed to directly remove a 2-handle α whose sides lie one on a spherical boundary component and the other one on a genus-2 boundary component, but the same output can be achieved by removing α with steps (1)-(4) before creating the genus-2 boundary component with step (5).

Definition 1.2.3. A sequence of the steps (0)-(5) described above will be called a *standard procedure*.

Observation 1.2.4. *If (M, Δ) is obtained by a standard procedure from a triangulated compact orientable 3-manifold (N, τ) with boundary that consists of a nonempty surface Σ without spherical components, then (M, Σ, Δ) is a τ-induced protoMom structure internal on N.*

Remark 1.2.5. We can modify the standard procedure by allowing steps (1) and (1′) even if there are 1-handles of valence 0 or 1, and by making steps (2) and (3) not mandatory. We will call such a procedure a *weak standard procedure*. A triple (M, Σ, Δ) obtained by a *weak* standard procedure will then be a *weak* protoMom structure internal on N.

1.2.2 Chains and simple chains

Let Δ be a handle structure on a manifold M. Let us recall the following definition from the previous subsection:

Definition 1.2.6. An *elementary collapse* on a handle structure Δ is the removal of a 1-handle of valence 1 together with the 2-handle incident to it.

Definition 1.2.7. Given a manifold (M, Δ) with at least one 1-handle of valence 1, after a finite sequence of elementary collapses we will obtain a manifold (M', Δ') without 1-handles of valence 1. A *chain* of (M, Δ) is a connected component of the union of all handles of (M, Δ) not in (M', Δ') and of all 1-handles which have valence 0 in (M', Δ') but not in (M, Δ).

To *collapse* a chain C is to perform a finite sequence of elementary collapses in such a way that the 2-handles removed by them are exactly the 2-handles of C.

Remark 1.2.8. If (M', Δ') is obtained from (M, Δ) by an elementary collapse, then every chain of (M', Δ') is equal to or is strictly contained in a chain of (M, Δ).

Proposition 1.2.9. *Let C be a chain of (M, Δ) and let h_i be the number of i-handles of C. If C is involved in a standard procedure (*i.e.*, C is a chain of the manifold at the beginning of a step (2) of a standard procedure), then $h_1 - h_2 = 0$ or 1, which is exactly the number of 1-handles of valence 0 obtained by collapsing C.*

Proof. Since every elementary collapse removes exactly one 1-handle and one 2-handle, for every chain C the difference $h_1 - h_2$ is equal to the number of 1-handles of valence 0 created by the collapse of C (notice that by definition these 1-handles are contained in C).

Now let C be a chain involved in a standard procedure, and let k be number of 1-handles of valence 0 that the collapse of C creates. C intersects only one boundary component B of ∂M, and the collapse of C does not change ∂M up to homeomorphism. This implies that, after the collapse of C, the boundary component B' corresponding to B has k 1-handles of valence 0 but has to be homeomorphic to a sphere or a torus, since at the beginning of every step (2) of a standard procedure (*i.e.*, before collapsing chains) the manifold has only spherical and/or toric lateral boundary components. Thus k can be at most 1. □

Definition 1.2.10. A 1- or 2-handle of a chain has *c-valence* n if it intersects, respectively, n distinct 2- or 1-handles of that chain.
A *head* of a chain is a 1-handle with valence 1 (and thus c-valence 1).
A *bifurcation* of a chain is a 2-handle with c-valence 3 incident to three distinct 1-handles of valence at most 2.
A chain C is *simple* if its 1-handles have c-valence at most 2, and hence every 2-handle of C with c-valence 3 is a bifurcation.
We call *subchain* of a simple chain C every connected component of C with bifurcations removed.
A simple chain will be said to have *no crossings* if its fundamental group is trivial and *one crossing* if its fundamental group is $\mathbb{Z}$.

Proposition 1.2.11. *Simple chains involved in a standard procedure are of one of the following types:*

- *a chain with only one head and no crossings;*
- *a chain with two heads and no crossings;*
- *a chain with exactly one head and one crossing.*

Proof. Every simple chain C can be collapsed with the following procedure:

(1) First choose a head B_0 of C and perform an elementary collapse at it. Notice that this can create up to two new 1-handles of valence 1 (*e.g.*, two new 1-handles of valence 1 are created if and only if a bifurcation is removed).

(2) If the last elementary collapse has created at least one new 1-handle of valence 1, choose one of them and perform an elementary collapse at it. Repeat the process if possible, or move to the next step otherwise.

(3) If the last elementary collapse has created no new 1-handles of valence 1, collapse, among the 1-handles of valence 1 created so far that have not been already collapsed, the one that was created later. Notice that this operation is well-defined since, if the last elementary collapse created no new 1-handles of valence 1, all the non-collapsed 1-handles of valence 1 created so far have been created at different times. Then, if there is still at least one 1-handle of valence 1, restart from step (2).

In other words, we proceed collapsing the chain only through newly created 1-handles of valence 1; in particular, once we started to collapse a subchain, we have to collapse it entirely before collapsing anything else. Roughly speaking, the removal of a bifurcation "unlocks" to the collapse process two subchains of C (or a single subchain if it has both ends on that bifurcation), and the collapse of one of them can in turn "unlock" other subchains, and so on, thus creating different possible sequences $(C_i)_{i\in I}$ of subchains where every subchain C_i is "unlocked" by the collapse of the subchain C_{i-1}. Thus, in the procedure described above, when a bifurcation A is collapsed we have to entirely collapse a subchain C_1 unlocked by the removal of A together with all the sequences of subchains containing C_1 before we can start to collapse the other subchain C_2 unlocked by the removal of A.

Notice that, with the procedure (1)-(3) described above:

- if C has $h > 0$ heads distinct from B_0, each of them will be left, after the collapse of the whole chain C, as a 1-handle of valence 0;

- if a subchain starts and ends at the same bifurcation (*i.e.*, if its intersection with a bifurcation has two connected components), the collapse of this subchain will leave one of its two ending 1-handles with valence 0;

- if a sequence of subchains starts and ends at the same bifurcation A, the collapse of this sequence of subchains will leave one of its two 1-handles touching A with valence 0.

This can be summarized by saying that if a simple chain retracts onto a bouquet of $k \geqslant 1$ circles, its collapse will leave k 1-handles of valence 0.

During a standard procedure the lateral boundary of the manifold (*i.e.*, $\partial M_i \setminus \Sigma$) always consists only of spheres and/or tori as long as we are performing steps (1)-(4), and only of tori and/or a surface of genus 2 as long as we are performing step (5). Since elementary collapses do not change the boundary of the manifold, collapsing a single chain during the standard procedure can create at most one 1-handle of valence 0 (and this happens only after performing operation (1) in case III or operation (1′)). This means that simple chains can have at most one extra ($h = 1$) head or one crossing ($k = 1$), but not both. $\square$

Having seen how simple chains behave, we want now to investigate non-simple chains involved in a standard procedure. Let C be a chain of a manifold (M_i, Δ_i), and let (M_f, Δ_f) be the manifold obtained after collapsing C. Let T_i be the (only) boundary component of M_i with nonempty intersection with C, and let T_f be the corresponding boundary component of M_f. T_f must be a sphere or a torus (in particular, it can be the boundary of a sphere with a 1-handle of valence 0 attached). Imagine to undo the collapse process, restoring the chain C by a sequence of steps each consisting in attaching a new 1-handle B_j of valence 1 and a new 2-handle A_j of valence 3. Of course A_j must be attached to B_j with valence exactly 1; A_j must also be attached twice to other 1-handles of C and/or to 1-handles with nonempty intersection with T_f. All the possible cases are listed below, and for each of them we discuss how it changes the number of heads, crossings, bifurcations and valences of the chain or chains "growing back" into C:

(a) A_j is attached twice to T_f: a new chain, and thus a new head, is created.

(b) A_j is attached to T_f and to a head of the chain: that head is replaced by a new one, and so the total number of heads remains the same.

(c) A_j is attached to T_f and to a non-head 1-handle of the chain: a new head is created, and the valence of a 1-handle with c-valence $\geqslant 2$ is increased by one.

(d) A_j is attached twice to a head: a head is replaced with a new one, a new crossing is created and the valence of a 1-handle with c-valence $\geqslant 2$ is increased by one.

(e) A_j is attached twice to a non-head 1-handle: a new head and a new crossing are created, and the valence of a 1-handle with c-valence $\geqslant 2$ is increased by two.

(f) A_j is attached to two distinct 1-handles of the same chain: a new head and a new crossing are created, and the sum of the number of heads removed plus the increases in the valences of the 1-handles with c-valence $\geqslant 2$ is equal to 2.

(g) A_j is attached to two distinct chains: a new head is created, the number of chains is reduced by one, and the sum of the number of heads removed plus the increases in the valences of the 1-handles with c-valence $\geqslant 2$ is equal to 2.

Let h be the number of extra heads (*i.e.*, in excess of one) of a chain C, let x be the number of crossings of C, and let v be the sum of all the extra valences among the 1-handles of C (*i.e.*, $v := \sum_{val \geqslant 2}(val - 2)$). We can now prove the following:

Proposition 1.2.12. *Chains involved in the standard procedure have, with the above notation,* $h + x - v = 0$ *or* 1*, which is also the number of* 1*-handles of valence* 0 *that the collapse of* C *will create.*

Proof. The sum $h + x - v$ is invariant in cases (b)-(f) described above. Let k be the number of newly created chains during the "restoring procedure". Then, while restoring C, we have to attach exactly k new 2-handles as in case (a) (*i.e.*, 2-handles attached twice to T), thus creating the first head of the chain and $k - 1$ extra heads. Since at the end of the restoring procedure we must have a single chain C, this means that we have to attach a number of 2-handles as in case (g) (*i.e.*, 2-handles attached to two distinct chains) equal to $k - 1$: this creates $k - 1$ new heads (in addition to the $k - 1$ extra heads created in cases of type (a)), but the sum of the number of heads removed plus the increases in the valences of 1-handles with c-valence $\geqslant 2$ is exactly equal to $2k - 2$. □

1.2.3 ProtoMom subgraphs and the *reduced procedure*

ProtoMom subgraphs We introduce here the notion of *protoMom subgraph*, which is a slightly different variation of the notion of *Mom-subgraph* presented in section 3 of [17].

Let N be an orientable 3-manifold with boundary that consists of a compact surface Σ without spherical components, and let τ be an arbitrary ideal triangulation of N, seen as a decomposition of N into truncated tetrahedra. We will now associate (in a non-unique way) to every (possibly weak) protoMom structure (M, Σ, Δ) internal on N obtained from (N, τ) by a standard procedure a coloring on the 4-valent graph Γ corresponding to τ, in the following way:

- assign color t to all the edges in the maximal forest $T(\Gamma)$ chosen in step (0) and to all the edges corresponding to the 2-handles removed in cases I-II of steps (1). Notice that the result of a standard procedure remains the same if we choose, as the starting maximal forest, the set of all t-colored edges instead of $T(\Gamma)$.
- assign color d to all the edges corresponding to the 2-handles removed with steps (2) (*i.e.*, with elementary collapses).
- assign to the edges corresponding to the 2-handles removed in case III of steps (1) or in steps (1′) color z if one has to remove a 1-handle of valence 0 in the subsequent step (3), or color c otherwise. We make this distinction because we mean c-colored edges to be the ones whose removal creates a "true" toric boundary component, while the removal of a z-colored edge creates a "temporary genus" around a 1-handle of valence 0 which will be removed in the subsequent step (3), thus restoring the previous boundary component type.
- assign color f to all remaining edges, which correspond to the non-removed 2-handles.

Definition 1.2.13. Γ with a coloring as the one described above will be called a *protoMom subgraph* of M and denoted by Γ^c.

Notice that, given a protoMom-structure (M, Σ, Δ) internal on (N, τ), we have a different protoMom subgraph of M for every different way of obtaining M from (N, τ) with a standard procedure. In particular we have the following obvious fact:

Observation 1.2.14. *Let (M_1, Σ, Δ_1) and (M_2, Σ, Δ_2) be two protoMom-structures internal to N, and let Γ_1^c and Γ_2^c be two (of the distinct possible many) protoMom subgraphs associated to them. Then $M_1 = M_2$ if and only if $F(\Gamma_1^c) = F(\Gamma_2^c)$, where $F(\Gamma^c)$ denotes the set of f-colored edges of Γ^c.*

Remark 1.2.15. Given (N, τ) and Γ as above, there is by definition a surjective function from the set of standard procedures on (N, τ) to the set of protoMom subgraphs on Γ. This function is not injective in general, *e.g.* two standard procedures which differ from one another only for the order in which their handles are removed give rise to the same coloring of Γ.

We now want to describe the relation between the protoMom subgraphs we introduced and the two types of Mom-subgraphs described in section 3 of [17].

Remark 1.2.16. Let Γ^c be a protoMom subgraph. If we change the color of all d- and z-colored edges of Γ^c to f we obtain a *general Mom-subgraph* Γ^g in the sense of [17]. Moreover, if Γ^c has exactly one c-colored edge, Γ^g is a *minimal Mom-subgraph* in the sense of [17].

However, not all Mom-subgraphs can be obtained in this way. In general, we have the following:

Observation 1.2.17. *Let Γ^c be a protoMom subgraph, and let $T_1, \dots T_k$ be the connected components of the union of the t-colored edges of Γ. For each $i \in \{1, \dots, k\}$ let C_i be the set of c- or z-colored edges with both endpoints on T_i (notice that each c- or z-colored edge has both endpoints on the same tree by construction), and choose for each $i \in \{1, \dots, k\}$ an edge $e_i \in C_i$. If we set the color of $e_1, \dots, e_k$ to c and change the color of all the other c-, d- and z-colored edges of Γ^c to f we obtain a general Mom-subgraph Γ^g, and every general Mom-subgraph can be obtained in this way.*
Moreover, Γ^g is a minimal Mom-subgraph if and only if $k = 1$.

d-colored edges correspond to the 2-handles removed with elementary collapses, and z-colored edges correspond to the 2-handles whose removal leads, possibly after collapsing some chains, to the creation of a 1-handle of valence 0. Changing their color to f means to keep all the corresponding 2-handles, *i.e.* we are not collapsing any chain nor removing any 1-handle of valence 0. In other words, we are skipping entirely steps (2)-(3) of the standard procedure and thus we are not removing any 1-handle. Remark 1.2.5 let us conclude that:

Remark 1.2.18. Mom-subgraphs on the 4-valent graph dual to a triangulated manifold (N, τ) correspond bijectively to weak protoMom structures internal on N with no 1-handles removed. Such a structure is called *τ-maximal*, since it is not properly contained in any other τ-induced protoMom structure internal on N.

The reduced procedure A variation of the standard procedure, which we will call the *reduced procedure*, is the following:

1. Remove from each tetrahedron of the triangulation τ of N a small open ball to obtain (M', Δ') as in the step (0) of a standard procedure;

2. Choose a maximal forest $T(\Gamma)$ in Γ (trees consisting of a single point are allowed) such that the lateral boundary of the manifold (M'', Δ''), obtained by removing the 2-handles corresponding to $T(\Gamma)$, consists of a number k of spheres $S_1, \ldots, S_k$ and such that for each S_i there exists a 2-handle α_i with both sides on S_i (notice that one such $T(\Gamma)$ always exists, for example by taking a maximal tree as $T(\Gamma)$); then color the edges of $T(\Gamma)$ by t.

3. For each S_i remove a 2-handle α_i with both sides on S_i (this creates a lateral torus T_i), and color the corresponding edge by c.

4. Remove a number $h \geqslant 0$ of 2-handles α_j $(j = 1, \ldots, h)$ with both sides on the same lateral torus T_{i_j} $(i_j \in \{1, \ldots, k\})$, and color the corresponding edges by z. These z-colored edges want to represent the 2-handles whose removal increases the genus of a boundary component by one but only temporarily, *i.e.* for each z-colored handle a 1-handle of valence 0 will be later removed, thus decreasing the genus of that boundary component to its previous value.

5. If there are 1-handles of valence 1, collapse the corresponding chains, and color all the edges corresponding to the collapsed chains by d.

6. If there are 1-handles of valence 0, remove them, and color all remaining edges of Γ by f.

A reduced procedure on (N, τ) gives as a result a manifold $M \subset N$ with a handle structure Δ induced by τ, and a complete coloring Γ^c of the graph Γ dual to τ. Of course (M, Δ) is not necessarily a protoMom structure. However, a sufficient condition is the following:

Remark 1.2.19. (M, Σ, Δ) obtained with a reduced procedure is a protoMom structure internal to N and with k lateral tori if:

- performing steps 1, 2, 3 and 5 (*i.e.*, collapsing the chains created with steps 2 and 3 but not the ones created with step 4) creates no 1-handles of valence 0 (indeed, if this is not the case, one of the k boundary component would be a sphere and not a torus).

- for each lateral torus T_i, if in step 4 a number n of 2-handles with both sides on T_i are removed, then, after collapsing all newly created chains in step 5, exactly a number n of 1-handles of valence 0 are removed in step 6 (indeed, if this is not the case, one of the k boundary component would be a genus $g \geqslant 2$ surface and not a torus).

The main result of this section is the following:

Theorem 1.2.20. *Let* (M, Σ, Δ) *be a protoMom structure internal on* (N, τ) *and let* Γ^c *be one of the protoMom subgraph associated to it. Then the pair* $((M, \Sigma, \Delta), \Gamma^c)$ *can be obtained from* (N, τ) *by a standard procedure if and only if it can be obtained from* (N, τ) *by a reduced procedure.*

Proof. The idea of the proof is to "modify" the standard procedure into the reduced one through some "intermediate procedures", and in particular a procedure which does all and only the moves done in the standard one but in a different order, collapsing all chains and removing all 1-handles of valence 0 in a single, not-repeated, final "cleanup" step. Let us be more precise now.

Suppose first that Γ^c is obtained by a standard procedure. Consider the following move, which is similar to step (1) of a standard procedure:

(1_A) Choose a 2-handle α such that one of the following is true and remove it:

  I the sides of α lie on two distinct boundary components ∂_1 and ∂_2 of genera h_1 and h_2, where h_i is the number of 1-handles of valence 0 which have a nonempty intersection with the new boundary component ∂_i' homeomorphic to ∂_i obtained after the collapse of all the chains whose heads have a nonempty intersection with ∂_i (roughly speaking, ∂_i is the boundary of a sphere with h_i 1-handles of valence 0 attached);

  II the sides of α lie on two distinct boundary components ∂_1 and ∂_2 of genus h_1 and $1 + h_2$;

III both sides of α lie on the same boundary component ∂_1 of genus h_1.

Notice that operation (1) of the standard procedure can be expressed as "If Δ_i does not contain 1-handles of valence 0 or 1, then perform move (1_A)" (*i.e.*, at the beginning of each step (1) of a standard procedure we always have $h_i = 0$).

Given that, we will not change the resulting protoMom subgraph if we use, instead of the standard procedure, a similar procedure where cycles of steps (1)-(3) are replaced by cycles of steps (1_A)-(2) followed by a single step (3).

In a similar way, consider now the move (1_B) which is obtained from move (1_A) by replacing case III with the following one:

III both sides of α lie on the same boundary component ∂_1 of genus h_1 and α is not contained in a chain.

Consider now the "intermediate procedure" which is similar to the standard one but where cycles of steps (1)-(3) are replaced by cycles of step (1_B), followed by a single step (2) and then followed by a single step (3). Remember that each elementary collapse affects only a single boundary component and does not change the topology of the manifold; the difference between the definitions of moves (1_A) and (1_B) assures that with the latter we do not remove 2-handles which are already "virtually removed", since they belong to a chain that will be collapsed in the subsequent step (2). Since every 2-handle which belongs to a chain at any time during a standard procedure will still belong to some chain after performing all operations (1_B), we will not change the resulting protoMom subgraph if we use this "intermediate procedure" instead of the standard one.

Starting from this intermediate procedure we will now construct a reduced procedure which has the same protoMom subgraph as a result, thus proving the "only if" part of the thesis. Choose as a maximal forest to be colored by t the maximal forest chosen with the standard procedure plus all the 2-handles which are removed in cases I-II of steps (1). Then color by c all the 2-handles removed in case III of steps (1) such that no 1-handles of valence 0 is removed in the subsequent step (3), and color by z all the other 2-handles which are removed in case III of steps (1) and all other 2-handles which are removed in steps ($1'$). As we proved above, we then can collapse all chains at once (color the corresponding edges by d), and finally remove all 1-handles of valence 0. This shows that Γ^c can be obtained by a reduced procedure too.

The converse is also true since, if Γ^c is obtained by a reduced procedure, then it can be obtained by a standard procedure in the following way (every step below represents a cyclic performance of operations (1)-(3) or ($1'$)-(3) of the standard procedure):

- first remove all t-colored edges as the starting maximal forest, and then collapse all eventually-created chains;
- remove a c-colored edge, and collapse all eventually-created chains (notice that 1-handles of valence 0 cannot be created by these collapses); repeat this process for all c-colored edges;
- remove a z-colored edge, collapse all eventually-created chains, and remove the newly created 1-handle of valence 0; repeat this process for all z-colored edges.

□

Remark 1.2.21. The standard procedure constructed from a reduced one in the last paragraph of the previous proof gives a bijective correspondence between z-colored edges (which are removed during steps (1) and steps ($1'$)) and 1-handles of valence 0 (which are removed in subsequent steps (3)).

Conjecture 1.2.22. *Every protoMom M has at least one subgraph with the following property: every z-colored 2-handle A can be chosen in a such a way that the corresponding 1-handle of valence 0 is incident to A.*

Moreover, every other protoMom subgraph of M can be changed into one with this property by a finite sequence of color-switching between a z-edge and a d-edge such that their corresponding 2-handles are incident to a common 1-handle.

Partial proof. If the removal of A creates a 1-handle of valence 0 there is nothing to prove. So we can suppose that A is such that its removal creates no 1-handles of valence 0, but at least one 1-handle B' of valence 1 (and thus at least a chain C containing B'). Let us first consider the case when A passes along B' only once, and let A' be the other 2-handle incident to B'. The manifold obtained by removing A is then homeomorphic to the manifold obtained by removing A': if we remove A as a z-colored 2-handle and then A' with an elementary collapse, or if we remove A' as a z-colored 2-handle and A with an elementary collapse, we obtain the same manifold up to homeomorphism, but we change the protoMom subgraph Γ^c by switching colors between a z- and a d-colored edge. Repeating the process we can homeomorphically "move the hole" (*i.e.*, the color z) along the 2-handles of a chain as long as they have in common a 1-handle of valence 2.

The conjecture is then true if all chains are simple. To be more precise, a 2-handle A whose removal will lead, possibly after the collapse of newly created simple chains, to the creation of a 1-handle of valence 0 is of one of the following types:

1. A passes three times along a single 1-handle B of valence 3;
2. A passes two times along a 1-handle B of valence 2 and once along a distinct 1-handle B' (if B' has valence 2 this will create a chain whose collapse do not create any other 1-handles of valence 0);
3. A is such that its removal creates exactly one 1-handle B' of valence 1, belonging to a simple chain with one head and one crossing;
4. A is such that its removal creates two distinct 1-handles of valence 1 belonging to two distinct simple chains (one with one head and one crossing, and the other with one head and no crossings);
5. A is such that its removal creates two distinct 1-handles of valence 1 belonging to the same simple chain (with two heads and no crossings);
6. A is such that its removal creates three distinct 1-handles of valence 1, and the corresponding simple chains are as in the previous two cases plus one more chain with one head and no crossings.

The statement is obviously true in cases 1 and 2. In case 3, since "moving the hole" across bifurcations can be done homeomorphically too, we can then "move the hole" up to one of the bifurcations of the crossing and reduce to case 6. The same holds for case 4. In case 5, if the two-headed created chain has a bifurcation we reduce to case 6, while if it is composed by a single subchain than the statement holds.

So we are left with case 6. Notice that the union of A and the two or three simple chains created by its removal has fundamental group $\mathbb{Z}$, so there is a non-trivial "loop of handles" (made by the 1- and 2-handles intersecting a chosen non-trivial loop in the handle structure), and this "loop of handles" can be chosen with a minimal number of handles (it will have an equal number of 1- and 2-handles, and there exists a non-trivial loop crossing exactly once each of them). Possibly after "moving the hole" along the chain, we can assume A is contained in this "loop of handles". Then the removal of A will create only simple chains without crossings, and exactly one of them will be with two heads B and B'. If we start collapsing the two-headed chain from B', then B will be left as a 1-handle of valence 0.

To conclude the proof of the conjecture we are left with the following:

- Do we need to "move the hole" across 1-handles of valence > 2? And if so, can it be done in some way?
- Prove the conjecture if A passes 2 or 3 times along B' in the non-simple case.

1.3 From internal protoMom structures to ideal triangulations

In this section we will prove the following result, which was stated and proved in [17] (we provide here a slightly different proof because it will be later needed to prove Theorem 1.4.3):

Theorem 1.3.1. *Every full weak protoMom structure (M, Σ, Δ) can be obtained via a standard (or reduced) procedure from an ideal triangulation of a compact connected orientable 3-manifold N such that ∂N is homeomorphic to Σ.*

Induced triangulations of lateral tori Since a full weak protoMom structure (M, Σ, Δ) contains no 1-handles of valence 0, each lateral torus T_i of M admits a natural decompositions into discs of the following 3 types:

- lakes contained in $\Sigma \times \{1\}$ (which are discs thanks to fullness);
- strips of the form $\ell \times I$, where ℓ is an arc in ∂D^2 for some 1-handle $D^2 \times I$;
- hexagons of the form $D^2 \times \{*\}$, where $* \in \partial I$ for some 2-handle $D^2 \times I$.

This decomposition induces a natural triangulation of T_i obtained by compressing each lake to a point and each strip $\ell \times [0,1]$ to the arc $\{*\} \times [0,1]$ with $* \in \ell$. We denote this triangulation by $\tau'(T_i)$.

Moves on triangulations of surfaces We introduce now three moves on triangulations of surfaces [17], which will be used to prove Theorem 1.3.1. Let S be a closed surface (for our purpose S will usually be a sphere or a torus ∂H, where H is a ball or a solid torus) and let τ be a triangulation of S.

Definition 1.3.2. Let v be a vertex of τ, let e' and e'' be two distinct edges incident to v, and let v' and v'' be the other endpoints of e' and e'' (which are not necessarily distinct from v). The *move (s1)* consists in cutting S along $e' \cup v \cup e''$ and filling the resulting square with two triangles with a common edge joining v' and v'' (see Figure 1.1). We will say that the move (s1) *is performed at v along e', e''*.

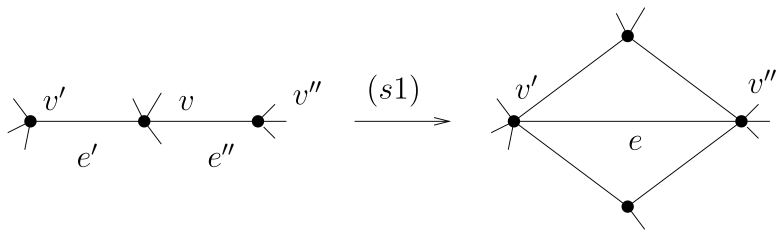

Figure 1.1: The move (s1)

Remark 1.3.3. Performing the move (s1) at v along e', e'' can be viewed as gluing to S a 2-dimensional triangle δ via a piecewise-linear identification of two of its edges with $e' \cup e''$.

Definition 1.3.4. Let e be an edge of τ adjacent to two distinct faces α and β. The *move (s2)* consists in replacing e with the other diagonal e' of the square $\alpha \cup \beta$ (see Figure 1.2). We will say that the move (s2) *is performed at* e.

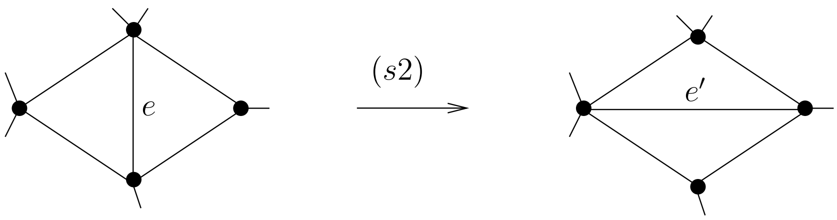

Figure 1.2: The move (s2)

Remark 1.3.5. Suppose $S = \partial H$, where H is a handlebody. If we glue to H a tetrahedron δ via a piecewise-linear identification of two of its faces with $\alpha \cup \beta$, then the surface $S' := \partial(H \cup \delta)$ is obtained from S by performing the move (s2) at e.

Definition 1.3.6. Let v be a vertex of τ incident to exactly 3 distinct triangles α_1, α_2 and α_3. The *move (s3)* consists in removing v and the three edges incident to it, thus merging α_1, α_2 and α_3 into a single triangle) (see Figure 1.3). We will say that the move (s3) *is performed at* v.

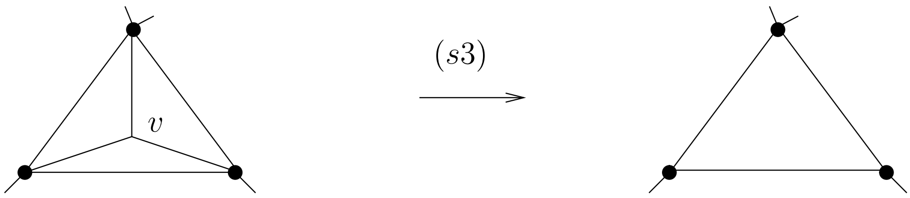

Figure 1.3: The move (s3)

Remark 1.3.7. Suppose $S = \partial H$, where H is a handlebody. If one glues to H a tetrahedron δ via a piecewise-linear identification of three of its faces with $\alpha_1 \cup \alpha_2 \cup \alpha_3$, then the surface $S' := \partial(H \cup \delta)$ is obtained from S by performing the move (s3) at v.

From 2-dimensional triangulations to 3-dimensional ones We want to give here a different proof of Lemma 1.3.8 and Lemma 1.3.9 (stated in [17], section 2.2) trying to minimize the number of tetrahedra of the triangulation constructed in the proof, since that will be needed to prove Theorem 1.4.3 of the next subsection.

Lemma 1.3.8. *Let τ be a "loose" triangulation of a two-dimensional sphere $S = \partial H$, where H is a 3-ball. Then there exists a triangulation τ' of H such that $\tau'|_{\partial H} = \tau$, and $\tau'^{(0)} = \tau^{(0)}$.*

Proof. The main idea for constructing τ' is to choose a vertex v of τ and take cones over all triangles not containing v, but for this purpose v must have a "loose" star with nonempty complement and all the k edges departing from v must have distinct endpoints $v_1 \neq \cdots \neq v_k \neq v$.

(1) If τ contains a vertex v of valence 1, then the complement of its star is always nonempty and the edge departing from v has an endpoint $v_1 \neq v$, so we can take cones from v.

(2) If τ contains a vertex v of valence 2, we have one of the situations shown in Figure 1.4. In case (a) we can take cones over v. In case (b) we can perform the move (s2) at e and then take cones over v; this will add an extra tetrahedron to τ' in the sense of Remark 1.3.5. In case (c) we have at least one vertex of valence 1 and so we can take cones from it.

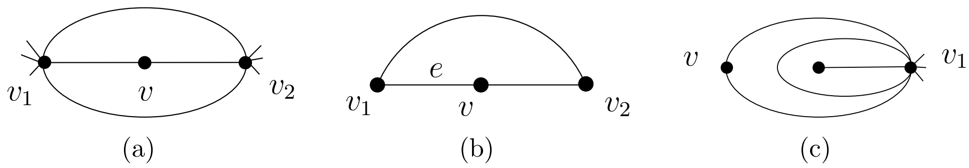

Figure 1.4: A vertex v of valence 2 in a triangulated sphere

(3) Suppose now that all the vertices of τ have valence at least 3. We want to prove that there exists a vertex v such that all edges departing from it have distinct endpoints which are also $\neq v$ (so we can take cones from v).

In fact, if there is no such vertex, every vertex v of τ has a "circle edge" (an edge with both endpoints on v) or is contained in at least one "2-edge circle" (which means there are two distinct edges departing from v and ending on the same v', their union giving an embedded circle). Note that, since we are assuming τ has no vertices of valence 1, the existence of a "circle edge" implies that there is also at least a "2-edge circle".

A "2-edge circle" divides the sphere into two triangulated discs, both of which may or may not contain other "2-edge circles". Take an innermost "2-edge circle" (*i.e.*, a "2-edge circle" such that a disc D bounded by it does not contain any other "2-edge circle", and thus no "circle edges" too). Since every vertex has a "circle edge" or belongs to a "2-edge circle", this disc D does not contain any other vertex, and thus D cannot be a triangulated disc, which contradicts our hypothesis.

So τ has a vertex v of valence at least 3 with all edges departing from it having distinct endpoints. Since the complement of its star is always nonempty we can take cones over v. $\square$

The following lemma will be the key ingredient in the proof of Theorem 1.3.1:

Lemma 1.3.9. *Let τ be a "loose" triangulation of a two-dimensional torus $T = \partial H$, where H is a solid torus. Then there exists a triangulation τ' of H such that $\tau'|_{\partial H} = \tau$, and $\tau'^{(0)} = \tau^{(0)}$.*

Proof. Consider the set L of all embedded cycles in $\tau^{(1)}$ trivial in H but nontrivial in T.

(1) Assume first that there exists a cycle $\ell \in L$ of length $k \geqslant 3$. Add to T along ℓ a k-gon Π triangulated by choosing one of its vertices and taking cones over all vertices not incident to the chosen one. Cutting along Π we obtain a triangulated sphere bounding a ball, to which we can apply Lemma 1.3.8. The desired conclusion follows by gluing back the two copies of Π.

(2) Assume now that all the cycles in L consist of at most two edges. Then, up to symmetries, τ is as in one of the cases shown in Figure 1.6, or can be obtained by one of them by a finite sequence of the moves shown in Figure 1.5, each one adding a vertex of valence 1 or 2 and two triangles. We will handle these "simple" cases by hand, constructing for each one a triangulation with the wanted properties. Let us be more precise now.

In case (a) τ is isomorphic to the triangulation of the boundary of the solid torus induced by the triangulation of the latter with a single tetrahedron. Cases (c) and (d) can be reduced to case (a) after performing move (s3),

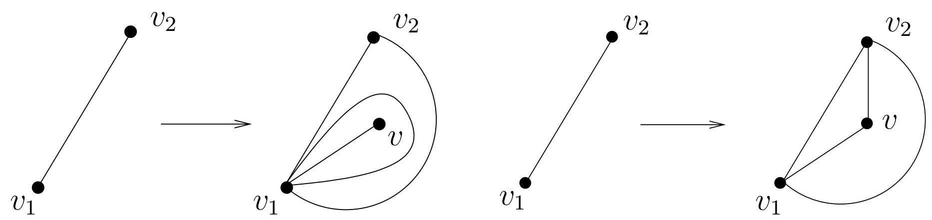

Figure 1.5: Adding a vertex of valence 1 or 2 (v_1 and v_2 may coincide)

respectively at v and at v_1 and v_2. Performing move (s2) at e reduces case (e) and case (f) to case (b), and case (b) to case (c). Since moves (s2) and (s3) each add one tetrahedron, the triangulation τ of H we found consists of 1 (case a), 2 (case c), 3 (cases b and d) or 4 (cases e and f) tetrahedra. $\square$

Proof of Theorem 1.3.1 Denote the lateral tori of (M, Σ, Δ) by $T_1, \ldots, T_k$ and consider their induced triangulations $\tau'(T_1), \ldots, \tau'(T_k)$ described at the beginning of this section. For each $i = 1, \ldots, k$ let H_i be the solid torus bounded by T_i in N. Lemma 1.3.9 applied to $T_i = \partial H_i$ gives a triangulation τ_i of H_i such that $\tau_i|_{\partial H_i} = \tau'(T_i)$ and all the vertices of τ_i are contained in ∂H_i. If we glue each H_i to M along a simplicial homeomorphism between $(T_i, \tau'(T_i))$ and $(\partial H_i, \tau_i|_{\partial H_i})$ we obtain a manifold N with boundary consisting only of Σ. It follows directly from the construction that the triangulations τ_i, $i = 1, \ldots, k$, yield an ideal triangulation τ of N and that the protoMom structure (M, Σ, Δ) can be obtained from τ via a standard procedure (or, equivalently, a reduced one). $\square$

An explicit procedure We want to end this section by describing explicitly a reduced procedure on the triangulated manifold (N, τ) constructed in the proof of Theorem 1.3.1 that produces as a result the weak protoMom structure (M, Σ, Δ) we started from.

Every tetrahedron δ of τ belongs to a torus H_i, in the sense that we have $\mathrm{Int}(\delta) \subset H_i$ for some $i = 1, \ldots, k$. The explicit construction of the triangulation of each H_i done in the proofs of Lemma 1.3.9 and Lemma 1.3.8 involves moves of types (s2) and (s3), the insertion of some polygons and taking cones over some triangles. With the notation used in Remark 1.3.5 and in Remark 1.3.7 we start coloring Γ in the following way:

(Δ) Color by f all the 2-handles of Δ, including in particular all the handles

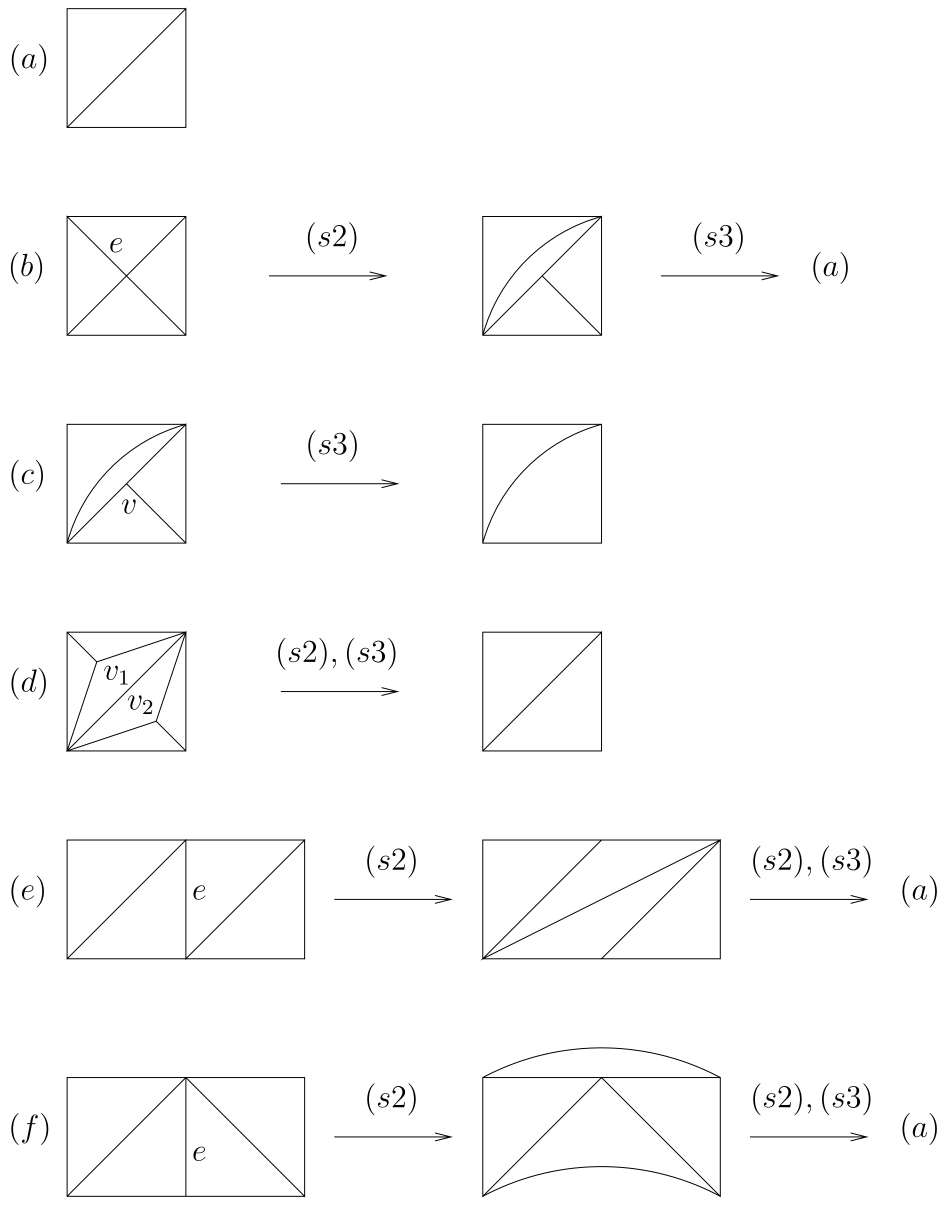

Figure 1.6: Torus triangulations with nontrivial cycles of length $\leqslant 2$

corresponding to the faces of $\tau'(T_i)$, $i = 1, \ldots, k$.

(s2) When a move (s2) is performed, color by t one of the two faces of δ other than α and β and color by d the other one. The 1-handle corresponding to the new edge inserted with δ will be part of a chain and will be collapsed together with this d-colored 2-handle.

(s3) When a move (s3) is performed, color by t the face of δ other than α_1, α_2 and α_3.

We now want to color the 2-handles corresponding to the new faces introduced when inserting a polygon and taking cones. Let Π be the polygon inserted in the proof of Lemma 1.3.9 in order to create a situation as in Lemma 1.3.8, let v be the preferred vertex with nonempty star complement, and let $S = \partial H$ be the triangulated sphere over whose triangles we take cones (notice that the faces of the triangulation τ of S may come from the torus of Lemma 1.3.9, from the polygon Π and/or from the new faces of the tetrahedra added with moves of types (s2) and (s3)). Then proceed as follows:

- Take a maximal tree on the 4-valent graph $\Gamma^H \subset \Gamma$ dual to the triangulation τ' of H, and color its edges by t.
- Let D be the triangulated disc obtained from S by removing v and the interiors of all the edges and faces it belongs to. Each t-colored edge of Γ^H corresponds to a face of τ' with exactly one boundary edge on D (the other two edges, which are not necessarily distinct, are in the interior of H): if we remove all these boundary edges we get a triangulation τ^D of D with a number $|\tau'^{(0)}| - 1$ of vertices, the same number of edges and exactly one face (which is a disc). This implies that all the handles corresponding to the vertices and edges of τ^D not contained in ∂D can be removed with a sequence of elementary collapses. Color by d all such 2-handles.
- Choose one of the triangles of Π, color by c the corresponding 2-handle, and color by d the 2-handles corresponding to the other triangles of Π.

For each H_i such that none of the tetrahedra belonging to it has a c-colored face (*e.g.*, the one-tetrahedron torus of case (2)-(a) in the proof of Lemma 1.3.9), choose a non-colored face connecting two (not necessarily distinct) tetrahedra belonging to H_i and color it by c (notice that one such face exists by construction). Color other eventual non-colored faces by d.

This gives a complete coloring of Γ, and by construction it is the protoMom subgraph associated to the reduced procedure that takes all the t-colored edges as the maximal forest $T(\Gamma)$ in step 2 and the c-colored edges as the $\alpha_1, \dots, \alpha_k$ in step 3.
Notice that in the procedure constructed above there are no z-colored edges, and thus no 1-handles of valence 0 have to be removed in step 6:

Observation 1.3.10. *Let (M, Σ, Δ) be a full weak protoMom structure internal on a compact connected orientable 3-manifold N such that ∂N is homeomorphic to Σ. Then there exists an ideal triangulation τ of N such that (M, Σ, Δ) is τ-induced and the associated protoMom subgraph has no z-colored edges.*

1.4 Complexity estimates

In this section we give a precise estimate on the number of tetrahedra in a triangulation giving rise to a protoMom structure. We do this by estimating the number of tetrahedra employed in constructing the manifold N and its triangulation τ in the proof of Theorem 1.3.1 of the previous section.

Lemma 1.4.1. *Let τ be a triangulation of a two-dimensional sphere $S = \partial H$, where H is a 3-ball. Then there exists a triangulation τ' of H such that $\tau'|_{\partial H} = \tau$, $\tau'^{(0)} = \tau^{(0)}$ and $|\tau'^{(3)}| \leq |\tau^{(2)}|$. Moreover, if $|\tau^{(2)}| \neq 2$, then $|\tau'^{(3)}| < |\tau^{(2)}|$.*

Proof. This follows easily from the proof of Lemma 1.3.8: in case (2)-(b) of that proof we start with two triangles and end up with a triangulation of H with two tetrahedra, while in every other case taking cones from a vertex v (of valence 1 or more) give a triangulation of H with $|\tau^{(2)}| - \mathrm{val}(v) < |\tau^{(2)}|$ tetrahedra. $\square$

Lemma 1.4.2. *Let τ be a triangulation of a two-dimensional torus $T = \partial H$, where H is a solid torus. Then there exists a triangulation τ' of H such that $\tau'|_{\partial H} = \tau$, $\tau'^{(0)} = \tau^{(0)}$ and $|\tau'^{(3)}| \leq 2|\tau^{(2)}|$. Moreover, if $|\tau^{(2)}| \neq 2$, then $|\tau'^{(3)}| \leq 2|\tau^{(2)}| - 4$.*

Proof. If all embedded cycles in $\tau^{(1)}$ trivial in H but nontrivial in T consist of at most two edges, we are in one of the cases shown in Figure 1.6. The result follows because, as shown in the proof of Lemma 1.3.9, the following holds:

	(a)	(b)	(c)	(d)	(e)	(f)
$\lvert\tau^{(2)}\rvert =$	2	4	4	6	4	4
$\lvert\tau'^{(3)}\rvert =$	1	3	2	3	4	4

Suppose now that in the set L of embedded cycle in $\tau^{(1)}$ trivial in H but nontrivial in T there exists a cycle ℓ of length $k \geqslant 3$. Adding a triangulated k-gon Π to T along ℓ (as shown in the proof of Lemma 1.3.9) leads us to a triangulation τ'' of a sphere S' (homeomorphic to one of the boundary components of the regular neighborhood of $T \cup \Pi$ in H with the induced triangulation) with $|\tau''^{(2)}| = |\tau^{(2)}| + 2(k-2)$. There remains to estimate k.

Since ℓ is embedded in $\tau^{(1)}$ it cannot pass twice through the same vertex (*i.e.*, it cannot self-intersect) and it cannot contain "circle edges" nor vertices of valence 1. Moreover, if ℓ contains two edges belonging to the same triangle, we can an embedded cycle $\ell' \in L$ with length $k-1$, as shown in Figure 1.7. Notice that, if ℓ has length 3 and contains two edges belonging to the same triangle, then the cycle ℓ' will result in having length $2 < 3$.

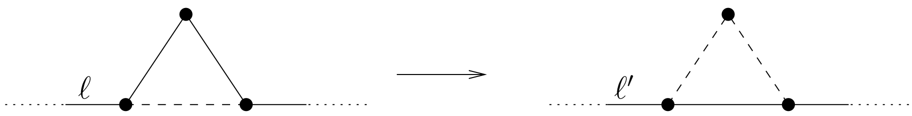

Figure 1.7: Finding a shorter nontrivial cycle

We choose as ℓ a cycle of minimal length k in the set $C \subset L$ of all embedded nontrivial cycles with length $\geqslant 3$. Suppose first that we are not in the "special case" noted above, *i.e.* that ℓ is not a length-3 cycle with two edges belonging to the same triangle; then minimality implies that ℓ contains at most one edge for each triangle of τ. Since every edge in ℓ bounds two distinct triangles (ℓ does not contain vertices of valence 1) we have that k is at most half the number of triangles of τ (which is always an even number).

Suppose now we are in the "special case", *i.e.* that the cycle ℓ has length 3 but contains two edges belonging to the same triangle. We want to show that the inequality $k \leq \frac{|\tau^{(2)}|}{2}$ holds in this case too (we of course have $k = 3$

here), but without using minimality. For the inequality $3 \leq \frac{|\tau^{(2)}|}{2}$ not to hold we must have $|\tau^{(2)}| < 6$. But, since $|\tau^{(1)}| = \frac{3|\tau^{(2)}|}{2}$ and $|\tau^{(0)}| \geqslant 3$, we have $0 = |\tau^{(0)}| - |\tau^{(1)}| + |\tau^{(2)}| \equiv |\tau^{(0)}| - \frac{|\tau^{(2)}|}{2}$ and then $|\tau^{(2)}| = 2|\tau^{(0)}| \geqslant 6$, a contradiction. So the inequality $k \leq \frac{|\tau^{(2)}|}{2}$ holds in the "special case" too.

In fact the inequality $k \leq \frac{|\tau^{(2)}|}{2}$ is the best one in the general case, since we do have examples where equality holds (as shown in Figure 1.8).

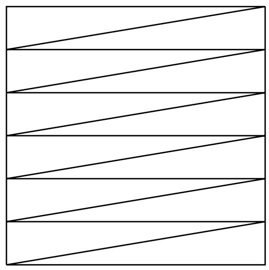

Figure 1.8: A cycle $\ell \in L$ with minimal length $k = \frac{|\tau^{(2)}|}{2}$

So we have established the inequality $|\tau''^{(2)}| \leq 2|\tau^{(2)}| - 4$ when $C \neq \emptyset$ too. Since $|\tau'^{(3)}| \leq |\tau''^{(2)}|$ by Lemma 1.4.1, we have the desired result. $\square$

The main result of this section is the following:

Theorem 1.4.3. *Every full protoMom structure (M, Σ, Δ) arises from an ideal triangulation of some manifold N such that ∂N is homeomorphic to Σ and $c(N) \leqslant 4n$, where n in the number of 2-handles of Δ.*

Proof. Consider the induced triangulations $\tau(T_i)$, $i = 1, \ldots, k$, on the k lateral tori T_i of M described at the beginning of the previous section. Since the triangles of $\tau \equiv \tau(T_1) \cup \cdots \cup \tau(T_k)$ arise only from the two faces of each of the n 2-handles of Δ, we have that $|\tau^{(2)}| \equiv \sum_i |\tau(T_i)^{(2)}| = 2n$.

If, as in the proof of Theorem 1.3.1, we glue triangulated solid tori (H_i, τ_i') to M along simplicial homeomorphisms between $(T_i, \tau(T_i))$ and $(\partial H_i, \tau_i'|_{\partial H_i})$, we obtain a manifold N with boundary consisting only of Σ. Lemma 1.4.2 implies that for each $i \in \{1, \ldots, k\}$ we have $|\tau_i^{'(3)}| \leq 2|\tau(T_i)^{(2)}|$, and thus N is obtained from M by adding $\sum_i |\tau_i^{'(3)}| \leq 2 \sum_i |\tau(T_i)^{(2)}| = 4n$ tetrahedra. $\square$

1.5 Moves on protoMom structures

In this section we want to find a set of local moves that relate to each other any two protoMom structures internal on the same manifold N.

Description of the C-moves The elementary collapse is an example of a local move that transforms a weak protoMom structure into another weak protoMom structure. On the other hand, if a weak protoMom structure (M, Σ, Δ) is obtained from a weak protoMom structure (M', Σ, Δ') by the elementary collapse removing a 1-handle B of valence 1 together with the 2-handle A incident to it, then restoring back B and A is a local move that transforms (M, Σ, Δ) into (M', Σ, Δ'). This move is an example of the C^{+1}-*move* described in [17]:

Definition 1.5.1. Let (M, Σ, Δ) be a weak protoMom structure internal on a compact connected orientable 3-manifold N with $\partial N = \Sigma$. Let $\ell \subset (\partial M \setminus \Sigma)$ be an embedded arc such that:

- the endpoints of ℓ are contained in the union of the islands;
- ℓ intersects the union of the 1-handles along precisely two arcs and the union of the lakes along precisely one arc;
- ℓ is disjoint from the 2-handles of Σ.

Then the C^{+1}-*move along* ℓ consists in first adding a 1-handle B contained in N, parallel to ∂M and with bases in the immediate vicinity of the endpoints of ℓ, then completing ℓ to a closed curve ℓ' that passes precisely once along B and is disjoint from any 2-handles, and finally gluing a new 2-handle A along ℓ'.

Remark 1.5.2. Obviously a C^{+1}-move along ℓ is invertible and its inverse is the elementary collapse which removes B and A. We will call an elementary collapse a C^{-1}-*move* and we will refer to any $C^{\pm 1}$-move simply as to a C-*move*.

Remark 1.5.3. If (M, Σ, Δ) is a weak protoMom structure obtained by a triangulated manifold (N, τ) in such a way that no 1-handle of valence 0 has been removed and that the associated protoMom subgraph contains at least one d-colored edge, then there exists at least one C^{+1}-move transforming

(M, Σ, Δ) into another weak protoMom structure internal on N.
Notice however that this is not necessary true if a 1-handle of valence 0 was removed during the procedure leading to (M, Σ, Δ).

Remark 1.5.4. If two weak τ-induced protoMom structures can be obtained one from the other by a C-move, then there exists a protoMom subgraph associated to one of them and one to the other such that they differ only for a color switching of an edge between colors d and f.

Description of the M-moves Let Γ^c be a protoMom subgraph of a weak protoMom structure (M, Σ, Δ) with no d- and z-colored edges and exactly one c-colored edge, and suppose that the 2-handle corresponding to the c-colored edge is the face A of a tetrahedron with at least one non-removed face A'. If we add A to (M, Σ, Δ) and then remove A' we obtain another weak protoMom structure. This local move is an example of the *M-move* described in [17]:

Definition 1.5.5. Let (M, Σ, Δ) be a weak protoMom structure internal on a compact connected orientable 3-manifold N with $\partial N = \Sigma$. Let B be a 1-handle of Δ with valence at least 1, and let A be a 2-handle incident to B. Let $\ell \subset \partial M$ be a closed embedded curve such that:

- ℓ is disjoint from the 2-handles of Δ;
- ℓ passes exactly 3 times along 1-handles (counting with multiplicity);
- ℓ passes along ∂B, and the attaching curve of the 2-handles different from A do not separate ℓ from the attaching curve of A on ∂B;
- ℓ bounds a disc in the complement of M in N.

Then the *M-move* at (B, A) along ℓ consists in removing the 2-handle A and gluing another 2-handle A' along ℓ instead.

Remark 1.5.6. An M-move performed at (B, A) along ℓ is invertible and its inverse is the M-move performed at (B, A') along ℓ_A, where ℓ_A is the attaching curve of A.

Notice that the result of an M-move is not necessarily a weak protoMom structure. We have the following definition and observation [17]:

Definition 1.5.7. A C- or M-move is *admissible* if it transforms a weak protoMom structure into a weak protoMom structure.

Observation 1.5.8. *Every C-move is admissible. An M-move is admissible if and only if after its application the boundary still consists of tori. Obviously an M-move which is the inverse of an admissible M-move is admissible.*

Remark 1.5.9. If two weak τ-induced protoMom structures can be obtained one from the other by an M-move, then there exists a protoMom subgraph associated to one of them and one to the other such that they differ only for a color switching between a f-colored edge and a non-f-colored edge incident to it.

It follows from Remark 1.2.16 that a protoMom subgraph associated to a τ-maximal protoMom structure has no edges of color d or z. The admissible M-moves for τ-maximal protoMom structures were described in [17] and correspond to the following moves on their protoMom subgraphs:

($m1$) If a c-colored edge e shares a vertex v with a f-colored edge e' with both endpoints on the same tree of $T(\Gamma)$, switch the colors of e and e' (see Figure 1.9).

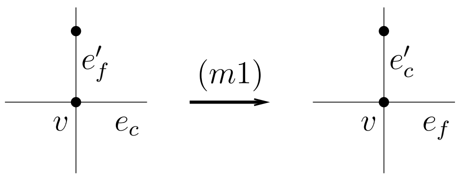

Figure 1.9: The move $(m1)$

($m2$) If a t-colored edge e shares a vertex v with a f- or c-colored edge e' and if e lies along a path in $T(\Gamma)$ joining the endpoints of e', switch the colors of e and e' (see Figure 1.10). If we will need to distinguish between the situations where e' had color f or c we will use the notation $(m2_f)$ move or $(m2_c)$ move respectively.

($\tilde{m}2$) This move is simply a composition of moves of type (m2): if a t-colored edge e lies along a path in $T(\Gamma)$ joining the endpoints of a f- or c-colored edge e', switch the colors of e and e'.

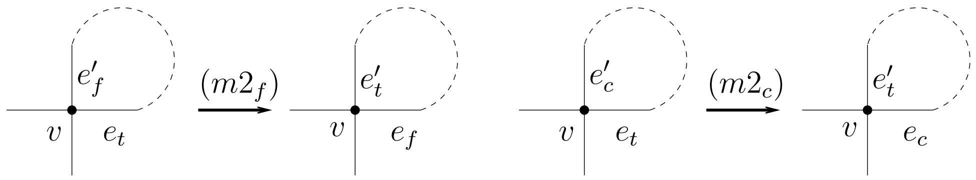

Figure 1.10: The move $(m2)$

$(m2')$ Let e be a f-colored edge with endpoints on two distinct trees of $T(\Gamma)$ and let e' be a t-colored edge incident to e at v. Suppose that e' separates the vertex v from each path in $T(\Gamma)$ joining the endpoints of a c-colored edge (notice however that there is exactly one such path for every tree of $T(\Gamma)$). Then switch the colors of e and e' (see Figure 1.11).

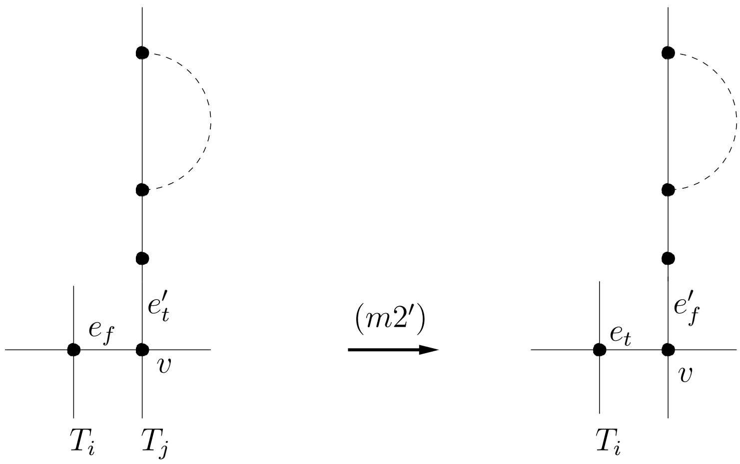

Figure 1.11: The move $(m2')$

$(m3)$ If a f-colored edge e with endpoints on two distinct trees of $T(\Gamma)$ shares a vertex v with a c-colored edge e', assign color t to e and color f to e' (see Figure 1.12).

$(\bar{m}3)$ Let e be a t-colored edge and let e' be a f-colored edge incident to it at v with both endpoints on the same connected component of $T(\Gamma)\setminus\mathrm{Int}(e)$. Suppose that e separates the vertex v from each path in $T(\Gamma)$ joining the endpoints of a c-colored edge. Then assign color f to e and color c to e' (see Figure 1.13).

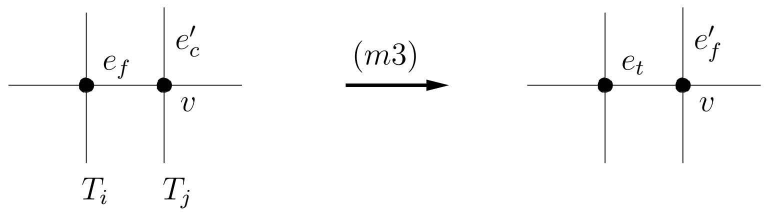

Figure 1.12: The move $(m3)$

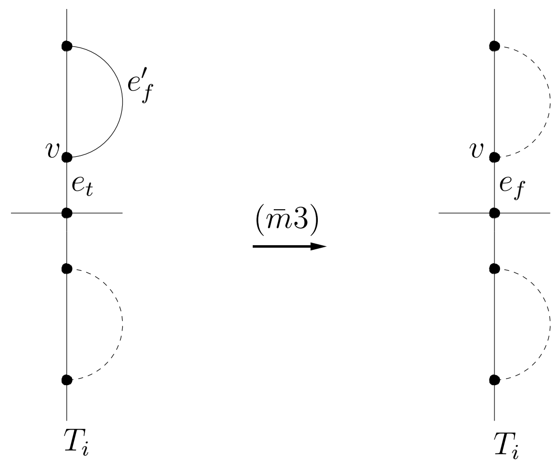

Figure 1.13: The move $(\bar{m}3)$

Remark 1.5.10. Moves $(m1)$, $(m2)$, $(\tilde{m}2)$, $(m2')$, $(m3)$ and $(\bar{m}3)$ described above are called *admissible moves* of protoMom subgraphs, since they correspond to admissible M-moves on τ-maximal protoMom structures. More precisely:

$(m1)$ By assigning color f to e we substitute a toric boundary component with a spherical one, but by assigning color c to e' we turn that boundary component back to a torus.

$(m2_f)$ By assigning color f to e we create a new spherical boundary component, but by assigning color t to e' we merge it back with a toric one.

$(m2_c)$ This move leaves the protoMom structure unchanged.

$(m2')$ By assigning color f to e' we separate from a toric boundary component a spherical one, but by assigning color t to e we merge it back with another toric boundary component.

($m3$) By assigning color f to e' we turn a toric boundary component into a spherical one, but by assigning color t to e we merge it with a toric one, thus reducing the number of boundary components by one.

($\bar{m}3$) By assigning color f to e we cut from a toric boundary component a spherical one, but by assigning color c to e' we turn it into a toric one, thus increasing the number of boundary components by one.

Remark 1.5.11. The inverse of an M-move of type ($m1$), ($m2_f$), ($m2_c$) or ($m2'$) is an M-move of the same type. The inverse of an M-move of type ($m3$) is an M-move of type ($\bar{m}3$), and vice versa.

Relating protoMom structures by the moves We first recall here, with different wording, two of the the main results of [17]:

Proposition 1.5.12. *Let Γ be a connected 4-valent graph. Then every protoMom subgraph Γ^c of Γ without d- and z-colored edges can be transformed into a protoMom subgraph with exactly one c-colored edge via a sequence of admissible moves.*

Proposition 1.5.13. *Let Γ be a connected 4-valent graph. Then any two protoMom subgraphs Γ_1^c and Γ_2^c of Γ without d- and z-colored edges and with exactly one c-colored edge can be transformed one into the other via a sequence of admissible moves.*

Remark 1.5.14. In terms of manifolds Proposition 1.5.12 means that every τ-maximal weak protoMom structure can be transformed into a τ-maximal weak protoMom structure with a single lateral torus via a sequence of admissible M-moves, and Proposition 1.5.13 means that every two τ-maximal weak protoMom structures with a single lateral torus are related by a sequence of admissible M-moves.

A way to transform a full weak protoMom structure into a τ-maximal one follows easily from Observation 1.3.10 and the procedure explicitly constructed in that paragraph:

Lemma 1.5.15. *Let (M, Σ, Δ) be a full weak protoMom structure internal on a 3-manifold N. Then there exists a sequence of C-moves transforming (M, Σ, Δ) into a τ-maximal weak protoMom structure, where τ is the triangulation constructed in the proof of Theorem 1.3.1.*

Remark 1.5.16. Given an ideal triangulation τ of a 3-manifold N, not every full weak protoMom structure (M, Σ, Δ) internal on N can be transformed into a τ-maximal one with the use of C-moves only. For example, suppose that the thickening of τ contains a 2-handle A attached three times to a 1-handle B of valence 3 as shown in Figure 1.14: if A and B are not included

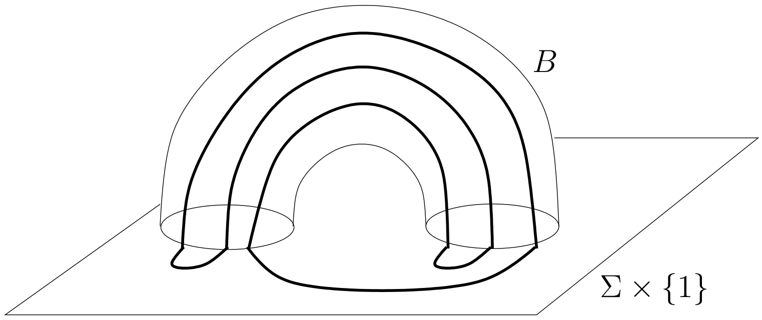

Figure 1.14: The attaching curve of the 2-handle A

in Δ, than any protoMom subgraph associated to (M, Σ, Δ) has A as a z-colored edge, and thus C-moves cannot transform (M, Σ, Δ) into a structure including B. However, a move that inserts B and A will be a local admissible move and, in this eaxample, we can use it together with some C-moves to recover a τ-maximal weak protoMom structure. Unfortunately, this move is not sufficient in general: when we add a 1-handle of valence 0, in order for the move to be admissible, we have to decrease the genus of that boundary component, and if we are not in the special case of the example we have to restore a c-colored 2-handle too, which will be later removed again when a z-colored 2-handle is restored; and both of these moves (adding a 1-handle of valence 0 together with a c-colored handle, and adding a z-colored handle while removing a c-colored one) are not local.

We end the chapter by recalling the proof of the main result of [17]:

Theorem 1.5.17. *Let N be a compact connected orientable 3-manifold such that ∂N is a nonempty surface without spherical components. Then any two* *full weak protoMom structures internal on N are related by a sequence of admissible M- and C-moves.*

Proof. Let (M_i, Σ, Δ_i), $i \in \{1, 2\}$, be two full weak protoMom structures internal on N. By Lemma 1.5.15 (M_i, Σ, Δ_i) can be transformed into a τ_i-maximal weak protoMom structure $(M'_i, \Sigma, \Delta'_i)$, where τ_i is a suitable triangulation of N, and by Proposition 1.5.12, Proposition 1.5.13 and Remark 1.5.14 $(M'_i, \Sigma, \Delta'_i)$ can be transformed into any other τ_i-maximal weak protoMom structure by a sequence of admissible M-moves. We are then left to find a sequence of admissible moves that let us transform a τ_1-maximal weak protoMom structure into a τ_2-maximal weak protoMom structure.

Since τ_1 and τ_2 are ideal triangulations of the same 3-manifold N and have more than one tetrahedron, they are related by a finite sequence of $(2 \to 3)$-moves and/or $(3 \to 2)$-moves (which are one the inverse of the other; see Figure 1.15). Then we can suppose that τ_2 can be obtained from τ_1 by a single $(2 \to 3)$- or $(3 \to 2)$-move (the conclusion then follows by induction).

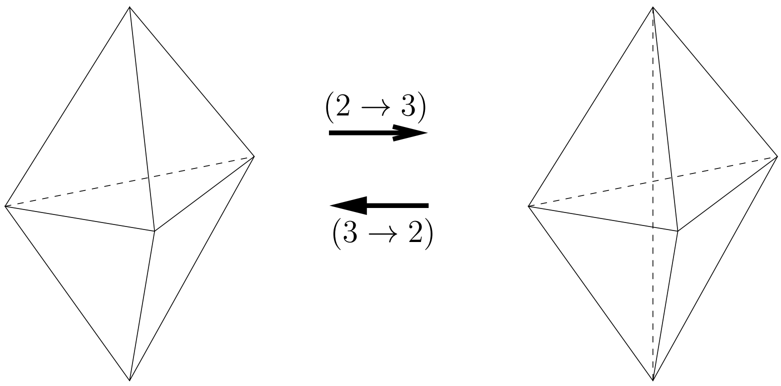

Figure 1.15: The $(2 \to 3)$-move and its inverse

In the first case, let α be the 2-handle corresponding to the triangle destroyed by the $(2 \to 3)$-move, and let $(M''_1, \Sigma, \Delta''_1)$ be any τ_1-maximal weak protoMom structure not containing α (*e.g.*, one with α colored by t). $(M''_1, \Sigma, \Delta''_1)$ can be viewed as a τ_2-induced weak protoMom structure, which can be transformed into a τ_2-maximal one by a C-move (*e.g.*, inserting the handles corresponding to the edge and to one of the triangles created by the move).

In the other case, let B, α_1, α_2 and α_3 be the 1-handle and the 2-handles corresponding to the edge and the triangles destroyed by the $(3 \to 2)$-move, and let $(M_1'', \Sigma, \Delta_1'')$ be any τ_1-maximal weak protoMom structure not containing α_1 and α_2 (*e.g.*, one with α_1 and α_2 colored by t). The C-move that consists in collapsing the 1-handle B of valence 1 together with α_3 transforms $(M_1'', \Sigma, \Delta_1'')$ into a non-maximal τ_1-induced weak protoMom structure that can be viewed as a τ_2-maximal weak protoMom structure. $\square$

Remark 1.5.18. Notice that in particular any two non-weak protoMom structures internal on the same N are related by a sequence of admissible M- and C-moves, although in general such a sequence passes along intermediate *weak* protoMom structures. The problem of finding a set of combinatorial moves relating any two non-weak protoMom structures internal on the same N that are admissibile in the sense of transforming non-weak protoMom structures into non-weak protoMom structures remains open.

Chapter 2

Some volume estimates in the mixed case

2.1 Notation

2.1.1 Introduction

Let us fix for the rest of this chapter an orientable hyperbolic 3-manifold M with nonempty compact geodesic boundary and with one or more toric cusps. We want to find a lower estimate for the volume of M using the volumes of a collar of the boundary and of a neighborhood of the cusps with disjoint and embedded interiors.

For our computations we will often use a geometric triangulation of M given by geometric tetrahedra which can have both ideal and truncated vertices. An ideal tetrahedron which can have all, some or none of its vertices truncated will be called a *(hyper)ideal* tetrahedron. For the rest of the chapter let τ be a fixed triangulation of M consisting of geometric (hyper)ideal tetrahedra.

We will concentrate our attention and explicit computations on the simplest examples, the 32 hyperbolic manifolds of complexity $c \leq 4$ with both geodesic boundary and toric cusps. A geometric triangulation for each of these 32 manifolds can be found in the census [18] or in the Appendix. Most sections of this chapter will end with some exemplifications of its main results in the case of these low-complexity manifolds. The main empiric facts about these simple manifolds, as well as all the relevant computations that will be carried over throughout the chapter, are summarized in the Appendix.

2.1.2 Realization of a tetrahedron

We will often need to realize in the half-space model a tetrahedron with at least one ideal vertex, in order to be more explicit with our computations. Although the first parts of the next section do not need to realize such a tetrahedron, for easiness of reference we prefer to introduce here what will be the *usual notation* in dealing with a realized tetrahedron in the half-space model. This notation, which will be extensively used in most of this chapter, is quite heavy, but can be easily followed with the aid of Figure 2.1.

Take a tetrahedron Δ with at least one ideal vertex V_0. In the half-space model, suppose that $V_0 = \infty$, and let V_1, V_2 and V_3 be the intersections of the three edges of Δ departing from V_0 with the plane $\{z = 0\}$; notice that, even if V_1, V_2 and V_3 are usually not "true" vertices of Δ, they bijectively correspond to the vertices of Δ other than V_0. The vertical plane Π_i passing through V_j and V_k, where $\{i, j, k\} = \{1, 2, 3\}$, contains the *vertical face* of the tetrahedron Δ opposite to the vertex corresponding to V_i. The remaining face of Δ (the *horizontal face*, which is the face opposite to V_0) lies in a half-sphere S_f with center O and radius R. A vertex corresponding to $V_i \neq V_0$ can be truncated by a truncation half-sphere S_i with center V_i and radius r_i. The dihedral angle along the vertical truncated edge $\overline{V_iV_0}$ will be called θ_i, while α_i will denote the dihedral angle between Π_i and S_f.

Notice that, if $V \neq V_0$ is an ideal vertex of Δ corresponding to V_i, then $V = V_i$ and V_i can be considered a truncated vertex with a truncation sphere of radius $r_i = 0$.

2.1.3 The examples with $c \leq 4$

Census of the examples Through all the chapter we will concentrate our attention and explicit computations on the simplest examples, the 32 hyperbolic 3-manifolds of complexity $c \leqslant 4$ with both nonempty geodesic boundary and at least one toric cusp. The study of these manifolds was an important tool to achieve the results of this chapter. We will enumerate these 32 manifolds following the order they have in the census given in [18] (and listed for reference in the Appendix), and divide them into the following 7 subsets according to some similar characteristics:

- Group A (1 manifold): the only manifold with $c = 3$ (in [18] this is the last manifold on the list of manifolds with complexity 3).

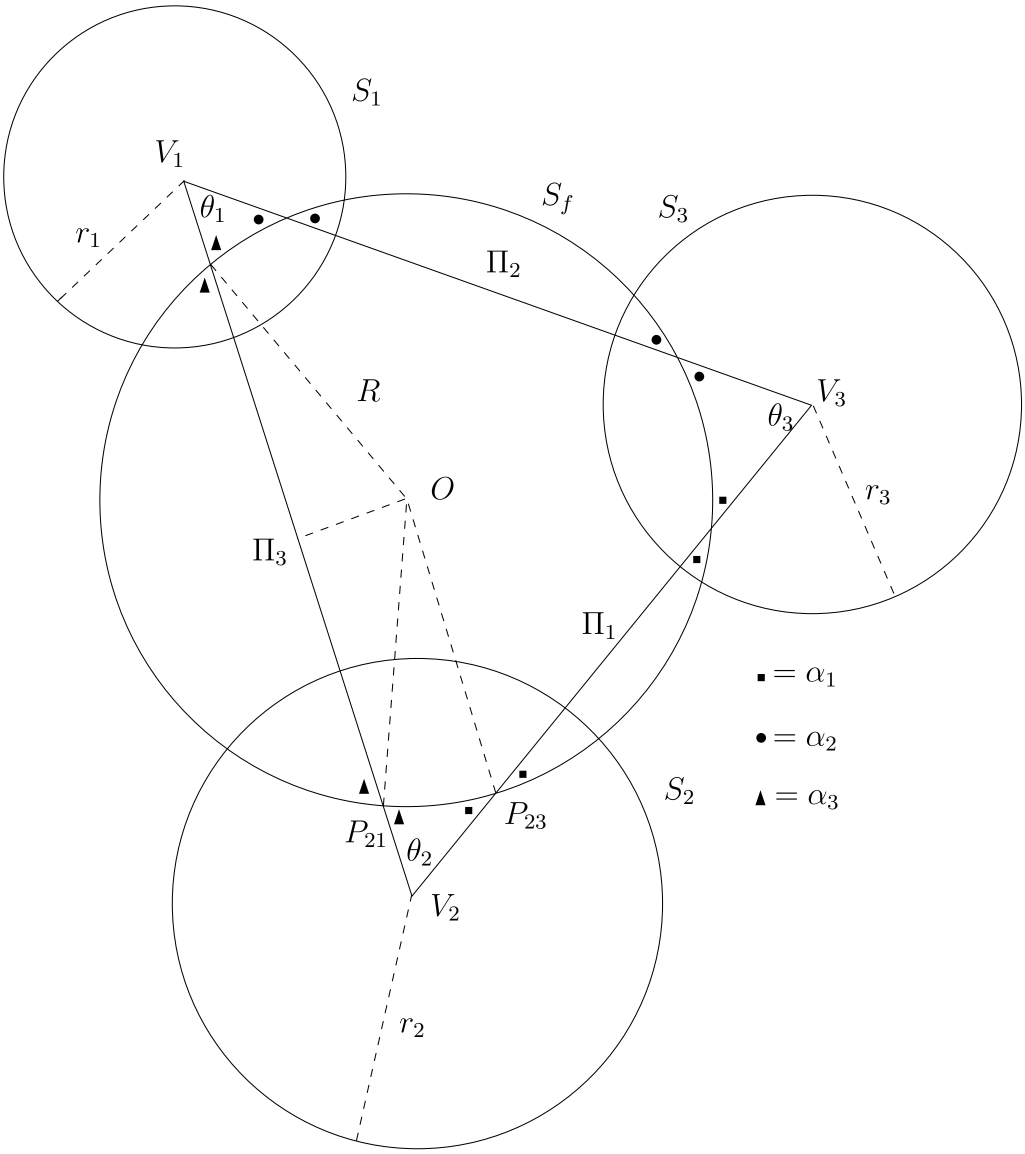

Figure 2.1: The *usual notation* for a tetrahedron with an ideal vertex $V_0 = \infty$ in the half-space model; the figure lies in the plane $\{z = 0\}$

- Group B (12 manifolds): all the manifolds with a genus-3 boundary.
- Groups C-F contain manifolds with $c = 4$, one cusp and a genus-2 boundary, which are subdivided into groups accordingly to similarities of the triangulation used to represent them in [18]:
  - Group C (2 manifolds): 2 ideal vertices, dihedral angles along edges with an ideal endpoint all equal to $\frac{\pi}{3}$.
  - Group D (2 manifolds): 2 ideal vertices, dihedral angles along edges with an ideal endpoint all equal to $\frac{\pi}{2}$ or $\frac{\pi}{4}$.
  - Group E (2 manifolds): 4 ideal vertices.
- Group F (12 manifolds): the remaining manifolds with 2 ideal vertices (*i.e.*, the ones with dihedral angles along edges with an ideal endpoint not all equal to $\frac{\pi}{2}$, $\frac{\pi}{4}$ or $\frac{\pi}{3}$).
- Group G (1 manifold): the only manifold with 2 cusps.

Additional symmetries These 32 manifolds with $c \leqslant 4$ have some symmetries, stated as empiric facts, that help in the calculations. The one that we will use more often is the following:

Fact 2.1.1. All the tetrahedra with ideal vertices appearing in the list for manifolds with $c \leqslant 4$ have exactly one ideal vertex and, with the usual notation summarized in Figure 2.1, have $\theta_1 = \theta_3 \leqslant \theta_2$ and $\alpha_1 = \alpha_3 \leqslant \alpha_2$.

From the end of the next section to the end of the whole chapter we will often specialize our formulae to the special case where $\theta_1 = \theta_3 \leqslant \theta_2$ and $\alpha_1 = \alpha_3 \leqslant \alpha_2$ (which is not necessary true in more general situations) as an intermediate step before treating the 32 examples directly.

2.2 Boundary collar

2.2.1 Volume of a boundary collar

Introduction and definitions For every $d \geqslant 0$ we define B_d to be the set of points whose distance from the boundary is at most d, namely we set

$$B_d := \{P \in M |\ d(P, \partial M) \leqslant d\}$$

Moreover, for every $P \in M$ we define $\Pi_\partial^\perp(P)$ to be the set of points in ∂M realizing its distance to the boundary (*i.e.*, the set of the shortest orthogonal projections of P onto ∂M), namely we set

$$\Pi_\partial^\perp(P) := \{Q \in \partial M |\ d(P, Q) = d(P, \partial M)\}$$

Notice that B_d can be seen as the union of all the geodesic arcs of length d orthogonal to ∂M.

Trivially, for $d = 0$ we have $B_0 = \partial M$ and $\Pi_\partial^\perp(P) = P$ for every $P \in B_0$. We can consider the choice of increasing values for d as if we were continuously inflating B_d, increasing its volume. If d is small enough then $\#\Pi_\partial^\perp(P) = 1$ for every $P \in B_d$, and so the volume of B_d can be computed only using d and the area of ∂M [6].

As we continue to inflate B_d, let $d_{\max}$ be the first value of d for which B_d touches itself:

$$d_{\max} := \inf\{d > 0 |\ \exists P \in B_d \text{ with } \#\Pi_\partial^\perp(P) > 1\}$$

Remark 2.2.1. $d_{\max}$ is the *only* value of d such that the set

$$\{P \in B_d |\ \#\Pi_\partial^\perp(P) > 1\}$$

has volume 0 but is nonempty. And thus we also have

$$d_{\max} = \sup\{d > 0 |\ \#\Pi_\partial^\perp(P) = 1\ \forall P \in B_d\}$$

If $d > d_{\max}$, the volume of $\{P \in B_d |\ \#\Pi_\partial^\perp(P) > 1\}$ will be positive, and thus the volume of B_d can no longer be computed using only d and the boundary area. Of course, for $d \to \infty$, we have $B_\infty = M$.

This introductory discussion is the reason behind the main definition of this section:

Definition 2.2.2. The set $B_d := \{P \in M |\ d(P, \partial M) \leqslant d\}$ is said to be a *d-collar of ∂M* (or simply a *boundary collar* of M) if $d \leqslant d_{\max}$.

Remark 2.2.3. Two equivalent definitions of a *d-collar* B_d of ∂M are the following ones:

- For $d > 0$, if for every P in $B_d^o := \{P \in M |\ d(P, \partial M) < d\}$ we have $\#\Pi_\partial^\perp(P) = 1$ then B_d is the closure of B_d^o (for $d = 0$ we set $B_0 := \partial M$ instead).

- If all the geodesic arcs of length d orthogonal to ∂M have disjoint interiors, then B_d is the union of all such arcs.

Definition 2.2.4. A d-collar B_d of ∂M is said to be a *maximal boundary collar* if $d = d_{\max}$. A maximal boundary collar will be denoted by $B_{d_{\max}}$.

Volume of a boundary d-collar The volume $V_\partial(d)$ of a d-collar of ∂M can be explicitly computed once d and the area $\mathcal{A}_\partial$ of the geodesic boundary of M are known [6]:

Proposition 2.2.5. *For $d \in [0, d_{\max}]$ the volume of a d-collar of the boundary is equal to*

$$V_\partial(d) = \mathcal{A}_\partial \cdot \frac{2d + \sinh(2d)}{4}$$

The following holds:

Proposition 2.2.6. *If ∂M is a connected orientable hyperbolic geodesic surface of genus g, then*

$$\mathcal{A}_\partial = -2\pi \cdot \chi(\partial M) = 4\pi(g-1)$$

More generally, if ∂M is an orientable hyperbolic geodesic surface with b connected components ∂M_i of genus g_i, $i = 1, \ldots, b$, we have

$$\mathcal{A}_\partial = -2\pi \cdot \sum_{i=1}^{b} \chi(\partial M_i) = 4\pi \sum_{i=1}^{b} (g_i - 1)$$

And thus

$$V_\partial(d) = \pi \cdot \left[\sum_{i=1}^{b} (g_i - 1)\right] \cdot \left[2d + \sinh(2d)\right]$$

Proof. Let us first suppose that ∂M is connected. Let τ' be an arbitrary geodesic triangulation of ∂M with f faces, $e = \frac{3f}{2}$ edges and v vertices. $\mathcal{A}_\partial$ is the sum of the areas of all the faces of τ', each one being equal to $\pi - \sum$ internal angles. Since the sum of the internal angles around every vertex must be 2π, the sum of all the internal angles of all the triangles of τ' is equal to $2\pi v$. Moreover, we have:

$$\chi(\partial M) = 2 - 2g = v - e + f \equiv v - \frac{f}{2}$$

And thus

$$\mathcal{A}_\partial = \pi \cdot f - 2\pi \cdot v = 2\pi\left(\frac{f}{2} - v\right) = -2\pi \cdot \chi(\partial M) = 2\pi(2g-2) = 4\pi(g-1)$$

If ∂M is not connected we can conclude by applying the same argument to every connected component of ∂M. $\square$

Triangulations of M and area of ∂M In the previous paragraph we recalled how $\mathcal{A}_\partial$ depends only on the genera g_i of the connected components of ∂M. In this section we show a connection of $\mathcal{A}_\partial$ and the genera g_i with an arbitrary triangulation of the whole manifold M.

Proposition 2.2.7. *Let M be an orientable hyperbolic 3-manifold with nonempty geodesic boundary and possibly some toric cusps, and let τ be a "loose" triangulation of M into n (hyper)ideal tetrahedra. Then the area of ∂M is*

$$\mathcal{A}_\partial = 4\pi(n-e)$$

where $e \leqslant n$ is the number of edges of τ in M (after identifications).

Thus

$$V_\partial(d) = \pi(n-e) \cdot \left[2d + \sinh(2d)\right]$$

We also have $n - e = \sum_i (g_i - 1)$, where g_i are the genera of the connected components of ∂M.

Proof. It suffices to prove that $\chi(\partial M) = 2e - 2n$. Recall that $M = \mathrm{int}\overline{M}$ where $\partial\overline{M}$ consists of ∂M and possibly some tori, whence $\chi(\partial M) = \chi(\partial\overline{M})$. τ induces a triangulation on $\partial\overline{M}$ with $4n$ triangles, $\frac{3}{2} \cdot 4n \equiv 6n$ edges and $2e$ vertices, and then

$$\chi(\partial M) = \chi(\partial\overline{M}) = 2e - 6n + 4n = 2e - 2n$$

$\square$

2.2.2 Width of a maximal boundary collar

In this subsection we prove the following characterization of $d_{\max}$:

Proposition 2.2.8. *Let M be a hyperbolic 3-manifold with geodesic boundary and toric cusps. Then $d_{\max}$ is equal to half the length of the shortest geodesic arc in M with both endpoints on ∂M and orthogonal to it.*

Proof. Consider in the half-space model the vertical half-plane

$$\Pi := \{(x, y, z)|\ x = 0, z > 0\}$$

, which is a hyperbolic plane. To find the set ∂K_d of points (x, y, z) with distance d from Π we have to solve the equation

$$d = 2\text{atanh}\left(\frac{x^2 + (z-t)^2}{x^2 + (z+t)^2}\right)^{\frac{1}{2}}$$

with $x^2 + z^2 = t^2$ [8]. First we notice that

$$\left(\tanh\frac{d}{2}\right)^2 = \frac{x^2 + (z-t)^2}{x^2 + (z+t)^2} \equiv \frac{x^2 + z^2 + t^2 - 2zt}{x^2 + z^2 + t^2 + 2zt} \equiv \frac{2t^2 - 2zt}{2t^2 + 2zt} \equiv \frac{t - z}{t + z},$$

and so

$$t = \frac{1 + \tanh^2\frac{d}{2}}{1 - \tanh^2\frac{d}{2}} \cdot z$$

We then have

$$x = \pm\sqrt{t^2 - z^2} = \pm\sqrt{\frac{(1 + \tanh^2\frac{d}{2})^2}{(1 - \tanh^2\frac{d}{2})^2} - 1} \cdot z =$$

$$= \pm\sqrt{\frac{4\tanh^2\frac{d}{2}}{(1 - \tanh^2\frac{d}{2})^2}} \cdot z = \pm\frac{2\tanh\frac{d}{2}}{1 - \tanh^2\frac{d}{2}} \cdot z$$

We can conclude that ∂K_d is made of two half-planes, given by $\partial K_d = \{(x, y, z)|\ z = \pm mx\}$ where

$$m = \frac{1 - \tanh^2\frac{d}{2}}{2\tanh\frac{d}{2}} \equiv \frac{2}{e^{2x} - e^{-2x}}$$

Since the model is conformal, each geodesic arc exiting orthogonally from Π is then orthogonal to ∂K_d too (*i.e.*, on the (x, z)-plane the projections of Π and ∂K_d are half-lines radial to the circle containing the geodesic arc).

A maximal collar of the boundary has a point P such that $P = \alpha_1 \cap \alpha_2$, where α_1 and α_2 are two geodesic arcs of length $d_{\max}$ orthogonal to the boundary. Moreover, α_1 and α_2 have the same tangent line (with opposite directions) in P. The union of α_1 and α_2 is then a geodesic arc α of length $2d_{\max}$ with both endpoints to the boundary and orthogonal to it. α is indeed the shortest geodesic arc from ∂M to ∂M and orthogonal to ∂M, since for $d < d_{\max}$ all geodesic arcs of length d and orthogonal to ∂M at one of their endpoints are disjoint from one another. $\square$

Corollary 2.2.9. *Let M be a hyperbolic 3-manifold with geodesic boundary and toric cusps, and let τ be a triangulation of M into (hyper)ideal geometric tetrahedra. If there exists a point $P \in B_{d_{\max}}$ such that $\Pi_\partial^\perp(P)$ intersects two distinct truncation triangles of the same tetrahedron $\Delta \in \tau$, then the maximal boundary collar of ∂M has width $d_{\max}$ equal to half the length of the shortest truncated edge of Δ.*

Proof. Using the same notation as in the proof of Proposition 2.2.8, we have $P \in \alpha \subset \Delta$. There is a geodesic plane containing α, and this plane must also contain the centers of the two truncation spheres since α is orthogonal to them (the tangent vector at the intersection point between α and a truncation sphere points exactly at the center of the sphere). So this plane is the plane of a face of Δ, where also one of the truncated edges lies. Easy euclidean geometry facts (*i.e.*, there is at most a unique semicircle with center on a line which is orthogonal to two given semicircles with center on the same line and external to each other) show that the two are the same, which means α is exactly the truncated edge of Δ joining these two truncation triangles, and its length is $2d_{\max}$. □

Remark 2.2.10. It follows from the proof of Corollary 2.2.9 that the set of the geodesic arcs in M with both endpoints on ∂M and orthogonal to it contains in particular all the finite-length truncated edges of any geometric (hyper)ideal triangulation τ of M. As we will recall in the next subsection, given a geometric triangulation τ, the lengths of its truncated edges can be completely computed using only the dihedral angles of the triangulation [8].

2.2.3 Explicit computations for $c \leqslant 4$

Boundary area The boundary of all the 32 manifolds with $c \leqslant 4$ is connected, compact and orientable, with genus either 3 (for manifolds in group B) or 2 (for the manifolds of all the other groups). It follows then, for $c \leqslant 4$, we have either $\mathcal{A}_\partial = 8\pi$ (group B) or $\mathcal{A}_\partial = 4\pi$ (groups A and C-G).

Width of the maximal boundary collar Thanks to the symmetries of the tetrahedra appearing in the list for manifolds with $c \leqslant 4$, Corollary 2.2.9 applies to all these 32 manifolds: for $c \leqslant 4$ then $d_{\max}$ is always equal to half the length of the shortest truncated edge of the given triangulation τ.

For $c \leqslant 4$ the boundary area $\mathcal{A}_\partial$ is equal to 4π or 8π, and so the number e of "glued edges", which we proved to be $e = c - \frac{\mathcal{A}_\partial}{4\pi}$, can at most be 3:

indeed it is equal to 2 for manifolds in groups A and B and it is equal to 3 for all the other manifolds. Moreover, at least one of these "glued edges" has an ideal endpoint: in fact, the number of "glued edges" with an ideal endpoint is equal to half the number of ideal vertices (before identifications), which is 2 for manifolds in groups E and G and exactly 1 for all the other manifolds. We can then conclude that for $c \leqslant 4$ there are 1 (for manifolds in groups A, B, E, and G) or 2 (for manifolds in groups C, D and F) truncated edges of finite length. We want now to compute these lengths.

General formulae for the length of edges Let us recall the formula for computing the length of a truncated edge using only the dihedral angles of the geometric triangulation τ [8]:

$$\cosh L(e_1) = \frac{C(e_1)}{\sqrt{d(V_{123})d(V_{156})}}$$

where

- $L(e_1)$ is the length of the truncated edge e_1 with endpoints V_{123} and V_{156}
- V_{ijk} is the common vertex of edges e_i, e_j and e_k
- γ_i is the dihedral angle along e_i
- $C(e_1) = \cos\gamma_1(\cos\gamma_3\cos\gamma_6+\cos\gamma_2\cos\gamma_5)+\cos\gamma_2\cos\gamma_6+\cos\gamma_3\cos\gamma_5+\cos\gamma_4\sin^2\gamma_1$
- $d(V_{123}) = 2\cos\gamma_1\cos\gamma_2\cos\gamma_3+\cos^2\gamma_1+\cos^2\gamma_2+\cos^2\gamma_3-1$

Let us consider a tetrahedron with at least one ideal vertex in the half-space model with the usual notation. If we specialize to the case of $\theta_1 = \theta_3 \leqslant \theta_2$ and $\alpha_1 = \alpha_3 \leqslant \alpha_2$, which is always true for the examples with $c \leqslant 4$, we have

- $C(\overline{V_j V_k}) = \cos\alpha_i(\cos\theta_j\cos\alpha_j+\cos\theta_k\cos\alpha_k)+\cos\alpha_j\cos\alpha_k+\cos\theta_j\cos\theta_k+\cos\theta_i\sin^2\alpha_i$
- $d(V_i) = 2\cos\theta_i\cos\alpha_j\cos\alpha_k+\cos^2\theta_i+\cos^2\alpha_j+\cos^2\alpha_k-1$

and thus

$$\begin{aligned} d(V_2) &= 2\cos\theta_2\cos^2\alpha_1 + \cos^2\theta_2 + 2\cos^2\alpha_1 - 1 \\ &= 2(1-2\cos^2\theta_1)\cos^2\alpha_1 + (1-2\cos^2\theta_1)^2 + 2\cos^2\alpha_1 - 1 \\ &= 4(\cos^2\alpha_1 - \cos^2\theta_1)(1-\cos^2\theta_1) \end{aligned}$$

$$d(V_1) \equiv d(V_3) = 2\cos\theta_1\cos\alpha_1\cos\alpha_2 + \cos^2\theta_1 + \cos^2\alpha_1 + \cos^2\alpha_2 - 1$$

$$\begin{aligned} C(\overline{V_1V_3}) &= 2\cos\alpha_2\cos\theta_1\cos\alpha_1 + \cos^2\alpha_1 + \cos^2\theta_1 + \cos\theta_2\sin^2\alpha_2 \\ &= 2\cos\theta_1\cos\alpha_1\cos\alpha_2 + \cos^2\alpha_1 - \cos^2\theta_1 - \cos^2\alpha_2 + \\ & 2\cos^2\theta_1\cos^2\alpha_2 + 1 \end{aligned}$$

$$\begin{aligned} C(\overline{V_1V_2}) &\equiv C(\overline{V_2V_3}) = \cos\alpha_1(\cos\theta_1\cos\alpha_1 + \cos\theta_2\cos\alpha_2) + \\ & + \cos\alpha_1\cos\alpha_2 + \cos\theta_1\cos\theta_2 + \cos\theta_1(1-\cos^2\alpha_1) = \\ &= 2\cos\theta_1\cos\alpha_1\cos\alpha_2 - 2\cos^2\theta_1\cos\alpha_1\cos\alpha_2 - 2\cos^3\theta_1 + 2\cos\theta_1 \\ &= 2\cos\theta_1(\cos\theta_1\cos\alpha_1\cos\alpha_2 + \cos\alpha_1\cos\alpha_2 - \cos^2\theta_1 + 1) \\ &= 2\cos\theta_1(1+\cos\theta_1)(\cos\alpha_1\cos\alpha_2 + 1 - \cos\theta_1) \end{aligned}$$

Explicit computations in the examples Let us move now to the explicit computations.

Manifolds in groups A, B, C, E and G have additional symmetries, which are $\theta_1 = \theta_3 = \theta_2 = \frac{\pi}{3}$ and $\alpha_1 = \alpha_3 = \alpha_2$. So the expression for the length of a truncated edge simplifies to

$$\begin{aligned} \cosh L(\overline{V_iV_j}) &= \frac{2\cdot\frac{1}{2}\left(1+\frac{1}{2}\right)\left(\cos^2\alpha_1 + 1 - \frac{1}{2}\right)}{4\left(\cos^2\alpha_1 - \frac{1}{4}\right)\left(1-\frac{1}{4}\right)} = \frac{\frac{3}{2}\left(\cos^2\alpha_1 + \frac{1}{2}\right)}{3\left(\cos^2\alpha_1 - \frac{1}{4}\right)} \\ &= \frac{2\cos^2\alpha_1 + 1}{4\cos^2\alpha_1 - 1} \equiv \frac{1}{2}\left(1 + \frac{3}{4\cos^2\alpha_1 - 1}\right) \end{aligned}$$

which gives us the following results:

- $L = 0.6420275..$ (group A)
- $L = 0.3860645..$ (group B)

- $L = 0.6802847..$ (manifold C.1)
- $L = 0.6876359..$ (manifold C.2)
- $L = 0.6931472..$ (groups E and G)

Manifolds in group C have also another truncated glued edge, which however comes only from edges belonging to tetrahedra without ideal vertices. To compute its length l we can use tetrahedron (1), where all the edges but $e := \overline{V_2V_3}$ glue together into the truncated edge we already computed. Let θ be the dihedral angle along e, α the one along the edge opposite to e, and γ the one along the other four edges (which have equal dihedral angles).Then

$$\begin{aligned} l &= \cosh^{-1}\left(\frac{2\cos\theta\cos^2\gamma + 2\cos^2\gamma + \cos\alpha\sin^2\theta}{2\cos\theta\cos^2\gamma + 2\cos^2\gamma + \cos^2\theta - 1}\right) \\ &= \cosh^{-1}\left(\frac{2\cos^2\gamma + \cos\alpha(1-\cos\theta)}{2\cos^2\gamma + \cos\theta - 1}\right) \equiv \cosh^{-1}\left(1 + \frac{1+\cos\alpha}{\frac{2\cos^2\gamma}{1-\cos\theta} - 1}\right) \end{aligned}$$

and thus $l = 1.8238367..$ (manifold C.1) or $l = 1.9777500..$ (manifold C.2), which in both cases is longer than the length previously computed.

Manifolds in group D have $\theta_1 = \theta_3 = \frac{\pi}{4}$ and $\theta_2 = \alpha_2 = \frac{\pi}{2}$, and so

$$\begin{aligned} L(\overline{V_1V_3}) &= \cosh^{-1}\left(\frac{0 + \cos^2\alpha_1 + \frac{1}{2} + 0}{0 + \frac{1}{2} + \cos^2\alpha_1 + 0 - 1}\right) = \cosh^{-1}\left(\frac{2\cos^2\alpha_1 + 1}{2\cos^2\alpha_1 - 1}\right) \\ &= \cosh^{-1}\left(1 + \frac{2}{2\cos^2\alpha_1 - 1}\right) = 1.9216484.. \end{aligned}$$

$$\begin{aligned} L(\overline{V_1V_2}) &= L(\overline{V_2V_3}) = \cosh^{-1}\left(\frac{\sqrt{2}\big(1+\frac{\sqrt{2}}{2}\big)\big(0+1-\frac{\sqrt{2}}{2}\big)}{\sqrt{\big(\cos^2\alpha_1 - \frac{1}{2}\big)\cdot 4\big(\cos^2\alpha_1 - \frac{1}{2}\big)\big(1-\frac{1}{2}\big)}}\right) \\ &= \cosh^{-1}\left(\frac{1-\frac{1}{2}}{\cos^2\alpha_1 - \frac{1}{2}}\right) = \cosh^{-1}\left(\frac{1}{2\cos^2\alpha_1 - 1}\right) = 0.6859633.. \end{aligned}$$

Manifolds in group F have no additional symmetries or equal dihedral angles, and so we compute $L(\overline{V_1V_2}) = L(\overline{V_2V_3})$ and $L(\overline{V_1V_3})$ directly from the general formulae above. It turns out that $L(\overline{V_1V_3})$ is always the shortest one, with $0.9392105.. < L(\overline{V_1V_3}) < 0.9865146..$ depending on the manifold.

The maximal width $d_{\max}$ of a boundary collar, which is half the length of the shortest truncated edge, is then

- $d_{\max} = 0.3210137..$ (group A)
- $d_{\max} = 0.1930322..$ (group B)
- $d_{\max} = 0.3401423..$ or $d_{\max} = 0.3438179..$ (group C)
- $d_{\max} = 0.3429816..$ (group D)
- $0.4696052.. < d_{\max} < 0.4932573..$ (group F)
- $d_{\max} = 0.3465736..$ (groups E and G)

2.3 Cusp self-tangencies

2.3.1 An explicit condition

Introduction Tangencies of two cusp neighborhoods and self-tangencies of a cusp neighborhood can happen inside a tetrahedron with one or more ideal vertices, but also inside tetrahedra with no ideal vertices. If a tetrahedron has two or more ideal vertices, their neighborhoods can touch themselves before they touch the boundary as we "inflate" them. However, this particular situation never happens for $c \leqslant 4$, since it can be empirically checked that, in the triangulations of the 32 manifolds with $c \leqslant 4$ that we are using, every tetrahedron has at most one ideal vertex.

A cusp neighborhood, while being inflated, may exit the tetrahedron Δ containing a corresponding ideal vertex before touching one of the truncated triangles of Δ, and (self)tangencies may thus happen inside another tetrahedron (for example one with a face in common with Δ, but possibly with no ideal vertices). In order to check if this can happen, we have to compute the radii of the truncation half-spheres in the half-space model with respect to the radius of the half-sphere where the face from which the cusp neighborhood may exit lies. Let us be more precise now.

Definition 2.3.1. Let $C_1, \ldots, C_k$ be the cusps of M and let U_i, $i = 1, \ldots, k$, be a neighborhood of C_i bounded by a horospherical cross-section. If the U_i's have equal volume and disjoint embedded interiors, then their union U will be called a *cusp neighborhood* of M. Notice that in the half-space model the neighborhood U of a single cusp C is given by a lattice Λ acting horizontally on a horoball $\{z \geqslant z_0\} \subset \mathbb{H}^3$ for some $z_0 > 0$.

Definition 2.3.2. By *inflating* a cusp neighborhood $\{z \geqslant z_i\}$ we mean to replace it with another one of the form $\{z \geqslant z_j\}$ with $z_j < z_i$. We will stop inflating a cusp neighborhood when it touches itself, another cusp neighborhood or a d-collar of the boundary; we will call this a *maximal* cusp neighborhood. A maximal cusp neighborhood will be called *d-maximal* if it is tangent to a d-collar of the boundary (and, in particular, 0-maximal if it is tangent to ∂M).

Explicit condition With the usual notation in the half-space model, let P_{ij} denote the first intersection point with the "horizontal sphere" S_f we encounter along the Euclidean segment $\overline{V_iV_j}$ from V_i to V_j (see Figure 2.1). We obviously have the following relation between lengths of Euclidean segments: $\overline{V_iV_j} = \overline{V_iP_{ij}} + \overline{P_{ij}P_{ji}} + \overline{P_{ji}V_j}$. It is easy to see that $\overline{P_{ij}P_{ji}} = 2R\sin\alpha_k$, where $\{i,j,k\} = \{1,2,3\}$, while for the other segments we have the following:

Lemma 2.3.3. *With the usual notation,* $\overline{V_iP_{ij}} = R\frac{\cos\alpha_j + \cos(\alpha_k+\theta_i)}{\sin\theta_i}$

Proof. Considering the quadrilateral $OP_{ik}V_iP_{ij}$ (see Figure 2.2) we can compute its angle in O to be $\phi := \pi - (\alpha_j + \alpha_k + \theta_i)$. Since the triangle $OP_{ik}P_{ij}$

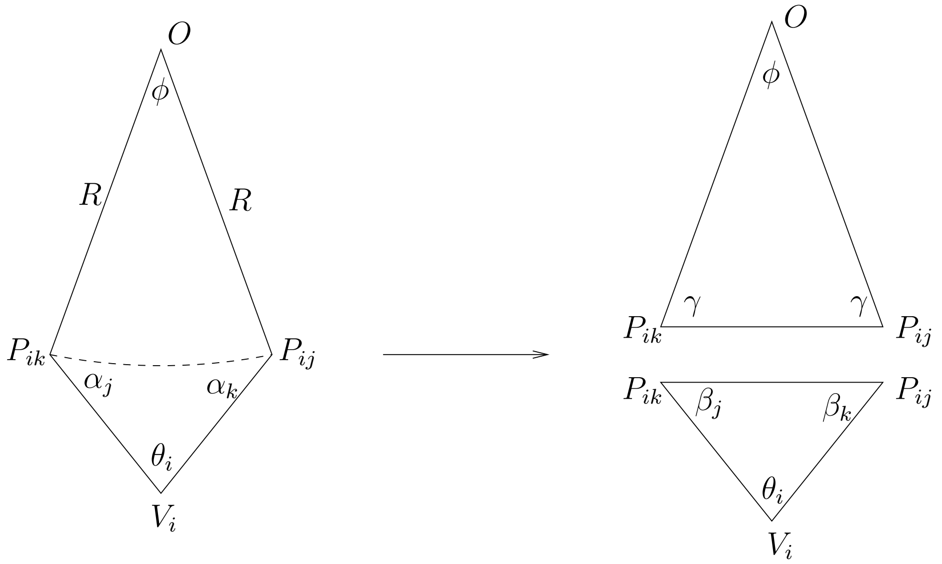

Figure 2.2: A detail from Figure 2.1

is isosceles (with two angles equal to $\gamma := \frac{\alpha_j+\alpha_k+\theta_i}{2}$) we have

$$\overline{P_{ij}P_{ik}} = 2 \cdot R \sin\left(\frac{\pi - (\alpha_j + \alpha_k + \theta_i)}{2}\right) = 2R\cos\left(\frac{\alpha_j + \alpha_k + \theta_i}{2}\right).$$

In the triangle $P_{ik}V_iP_{ij}$ the angle at P_{ik} is equal to

$$\beta_j := \frac{\pi}{2} + \alpha_j - \gamma = \frac{\pi + \alpha_j - \alpha_k - \theta_i}{2}$$

and similarly the angle at P_{ij} is equal to $\beta_k := \frac{\pi+\alpha_k-\alpha_j-\theta_i}{2}$. The sine theorem applied to the triangle $P_{ik}V_iP_{ij}$ then gives

$$\frac{\overline{V_iP_{ij}}}{\sin\left(\frac{\pi+\alpha_j-\alpha_k-\theta_i}{2}\right)} = \frac{2R\cos\left(\frac{\alpha_j+\alpha_k+\theta_i}{2}\right)}{\sin\theta_i}$$

hence

$$\begin{aligned}
\overline{V_iP_{ij}} &= \frac{2R}{\sin\theta_i}\cos\left(\frac{\alpha_j + \alpha_k + \theta_i}{2}\right)\sin\left(\frac{\pi + \alpha_j - \alpha_k - \theta_i}{2}\right) \\
&= \frac{2R}{\sin\theta_i}\cos\left(\frac{\alpha_j + \alpha_k + \theta_i}{2}\right)\sin\left(\frac{\pi + \alpha_j - \alpha_k - \theta_i}{2}\right) \\
&= \frac{2R}{\sin\theta_i}\frac{\cos\alpha_j + \cos(\alpha_k + \theta_i)}{2} = R\frac{\cos\alpha_j + \cos(\alpha_k + \theta_i)}{\sin\theta_i}
\end{aligned}$$

□

Notice that if V_i is an ideal vertex we have $\alpha_j + \alpha_k + \theta_i = \pi$, which means $\cos(\alpha_k + \theta_i) = -\cos\alpha_j$, and hence $\overline{V_iP_{ij}} = 0$ as expected (V_i lies on S_f).

We have now all the elements to compute the Euclidean length of the edges $\overline{V_iV_j}$, which will be needed later:

Proposition 2.3.4. *With the usual notation, we have*

$$\overline{V_iV_j} = R\left(\frac{\cos\alpha_i}{\sin\theta_j} + \frac{\cos\alpha_j}{\sin\theta_i} + \frac{\cos\alpha_k\sin\theta_k}{\sin\theta_i\sin\theta_j}\right)$$

Proof. We have already seen that $\overline{V_iV_j} = \overline{V_iP_{ij}} + \overline{P_{ij}P_{ji}} + \overline{P_{ji}V_j}$, where $\overline{P_{ij}P_{ji}} = 2R\sin\alpha_k$ and

$$\overline{V_iP_{ij}} = R\frac{\cos\alpha_j + \cos(\alpha_k + \theta_i)}{\sin\theta_i}$$

Putting all this together we obtain

$$\frac{\overline{V_iV_j}}{R} = \frac{\cos\alpha_j + \cos(\alpha_k+\theta_i)}{\sin\theta_i} + 2\sin\alpha_k + \frac{\cos\alpha_i + \cos(\alpha_k+\theta_j)}{\sin\theta_j} =$$

$$= \frac{1}{\sin\theta_i\sin\theta_j}\cdot\Big(\cos\alpha_j\sin\theta_j + \cos(\alpha_k+\theta_i)\sin\theta_j + 2\sin\alpha_k\sin\theta_i\sin\theta_j +$$

$$+\cos\alpha_i\sin\theta_i + \cos(\alpha_k+\theta_j)\sin\theta_i\Big)$$

We want to simplify this long formula piece by piece, using the sum-to-product and product-to-sum trigonometric identities. We start with

$$\cos(\alpha_k+\theta_i)\sin\theta_j + \cos(\alpha_k+\theta_j)\sin\theta_i =$$

$$= \frac{\sin(\alpha_k+\theta_i+\theta_j) + \sin(-\alpha_k-\theta_i+\theta_j)}{2} +$$

$$+ \frac{\sin(\alpha_k+\theta_j+\theta_i) + \sin(-\alpha_k-\theta_j+\theta_i)}{2} =$$

$$= \sin(\alpha_k+\theta_i+\theta_j) + \frac{1}{2}\cdot 2\sin(\frac{-2\alpha_k}{2})\cos(\frac{2\theta_j-2\theta_i}{2}) =$$

$$= \sin(\alpha_k+\theta_i+\theta_j) - \sin\alpha_k\cos(\theta_i-\theta_j)$$

We move now to

$$2\sin\alpha_k\sin\theta_i\sin\theta_j = 2\sin\alpha_k\frac{\cos(\theta_i-\theta_j)-\cos(\theta_i+\theta_j)}{2} =$$

$$= \sin\alpha_k\cos(\theta_i-\theta_j) - \sin\alpha_k\cos(\theta_i+\theta_j)$$

Adding these two pieces together we obtain

$$\sin(\alpha_k+\theta_i+\theta_j) - \sin\alpha_k\cos(\theta_i+\theta_j) =$$

$$= \sin(\alpha_k+\theta_i+\theta_j) - \frac{\sin(\alpha_k+\theta_i+\theta_j)+\sin(\alpha_k-\theta_i-\theta_j)}{2} =$$

$$= \frac{1}{2}\sin(\alpha_k+\theta_i+\theta_j) + \frac{1}{2}\sin(-\alpha_k+\theta_i+\theta_j) = \sin(\theta_i+\theta_j)\cos\alpha_k$$

And then, since $\theta_i+\theta_j = \pi-\theta_k$, we can conclude that

$$\frac{\overline{V_iV_j}}{R} = \frac{\cos\alpha_j}{\sin\theta_i} + \frac{\cos\alpha_i}{\sin\theta_j} + \frac{\sin(\theta_i+\theta_j)\cos\alpha_k}{\sin\theta_i\sin\theta_j} =$$

$$= \frac{\cos\alpha_j}{\sin\theta_i} + \frac{\cos\alpha_i}{\sin\theta_j} + \frac{\sin\theta_k\cos\alpha_k}{\sin\theta_i\sin\theta_j}$$

$\square$

Lemma 2.3.5. *With the usual notation we have*

$$r_i^2 = \frac{R^2}{\sin^2\theta_i}(\cos^2\alpha_j + \cos^2\alpha_k + \cos^2\theta_i + 2\cos\alpha_j\cos\alpha_k\cos\theta_i - 1).$$

Proof. Applying the cosine theorem to the triangles OV_iP_{ij} and OV_iP_{ik} we have (see Figure 2.2)

$$\overline{OV_i}^2 = R^2 + \overline{V_iP_{ij}}^2 - 2R\overline{V_iP_{ij}}\cos\left(\frac{\pi}{2}+\alpha_k\right) = R^2 + \overline{V_iP_{ij}}^2 + 2R\overline{V_iP_{ij}}\sin\alpha_k$$

$$\overline{OV_i}^2 = R^2 + \overline{V_iP_{ik}}^2 - 2R\overline{V_iP_{ik}}\cos\left(\frac{\pi}{2}+\alpha_j\right) = R^2 + \overline{V_iP_{ik}}^2 + 2R\overline{V_iP_{ik}}\sin\alpha_j$$

and hence

$$\begin{aligned}
r_i^2 &= \overline{OV_i}^2 - R^2 = \overline{V_iP_{ij}}(\overline{V_iP_{ij}} + 2R\sin\alpha_k) \\
&= \frac{R^2}{\sin^2\theta_i}(\cos\alpha_j + \cos\alpha_k\cos\theta_i - \sin\alpha_k\sin\theta_i)\cdot \\
&\quad \cdot(\cos\alpha_j + \cos\alpha_k\cos\theta_i + \sin\alpha_k\sin\theta_i) \\
&= \frac{R^2}{\sin^2\theta_i}\left((\cos\alpha_j + \cos\alpha_k\cos\theta_i)^2 - \sin^2\alpha_k\sin^2\theta_i\right) \\
&= \frac{R^2}{\sin^2\theta_i}\left(\cos^2\alpha_j + \cos^2\alpha_k\cos^2\theta_i + 2\cos\alpha_j\cos\alpha_k\cos\theta_i + \right. \\
&\quad \left. -(1-\cos^2\alpha_k)\sin^2\theta_i\right) \\
&= = \frac{R^2}{\sin^2\theta_i}(\cos^2\alpha_j + \cos^2\alpha_k + \cos^2\theta_i + 2\cos\alpha_j\cos\alpha_k\cos\theta_i - 1)
\end{aligned}$$

□

Notice that if V_i is ideal we obtain $r_i = 0$ as expected.

In order to check if our inflating cusp neighborhood can exit Δ before touching its truncated triangles we have to compare $r := \max\{r_1, r_2, r_3\}$ with R:

Theorem 2.3.6. *With the above notation, $r \leqslant R$ if and only if, for all $\{i,j,k\} = \{1,2,3\}$, one has $T^-_{j,k} \leqslant \cos\theta_i \leqslant T^+_{j,k}$, where*

$$T^{\pm}_{j,k} := \frac{-\cos\alpha_j\cos\alpha_k \pm \sqrt{\cos^2\alpha_j\cos^2\alpha_k + 2\sin^2\alpha_j + 2\sin^2\alpha_k}}{2}$$

Proof. $r \leqslant R \Leftrightarrow r^2 \leqslant R^2 \Leftrightarrow$

$$\Leftrightarrow \cos^2\alpha_j + \cos^2\alpha_k + \cos^2\theta_i + 2\cos\alpha_j\cos\alpha_k\cos\theta_i - 1 \leqslant \sin^2\theta_i \Leftrightarrow$$

$$\Leftrightarrow 2\cos^2\theta_i + 2\cos\alpha_j\cos\alpha_k\cos\theta_i + \cos^2\alpha_j + \cos^2\alpha_k - 2 \leqslant 0$$

and the conclusion follows. □

Notice in particular that, if the condition of the previous Theorem is satisfied, there are no cusp self-tangencies.

Additional symmetries We want now to specialize the results of the previous paragraph to the case where, with the usual notation, we have $\theta_1 = \theta_3 \leqslant \theta_2$ and $\alpha_1 = \alpha_3 \leqslant \alpha_2$ (which is always the case for manifolds with $c \leqslant 4$), simplifying the condition found in Theorem 2.3.6.

Corollary 2.3.7. *If, with the usual notation, we have $\theta_1 = \theta_3 \leqslant \theta_2$ and $\alpha_1 = \alpha_3 \leqslant \alpha_2$ (which is always the case for $c \leqslant 4$), then*

$$r_2^2 = R^2\left(\frac{\cos^2\alpha_1}{\cos^2\theta_1} - 1\right)$$

Proof. Using the usual symmetries and the obvious $\theta_1 + \theta_2 + \theta_3 = \pi$, we can further simplify the formula for r_2^2 as follows:

$$\begin{aligned}
r_2^2 &= \frac{R^2}{\sin^2\theta_2}(2\cos^2\alpha_1 - \sin^2\theta_2 + 2\cos^2\alpha_1\cos\theta_2) \\
&= \frac{R^2}{\sin^2(\pi-2\theta_1)}2\cos^2\alpha_1\big(1+\cos(\pi-2\theta_1)\big) + \\
&\quad -R^2\frac{R^2}{\sin^2(2\theta_1)}2\cos^2\alpha_1\big(1-\cos(2\theta_1)\big) - R^2 = \\
&= \frac{R^2}{2\sin^2\theta_1\cos^2\theta_1}\cos^2\alpha_1(2\sin^2\theta_1) - R^2 = R^2\left(\frac{\cos^2\alpha_1}{\cos^2\theta_1} - 1\right)
\end{aligned}$$

□

Observation 2.3.8. *For $c \leqslant 4$ we always have $\alpha_1 \leqslant \theta_1 \leqslant \frac{\pi}{2}$, and so the value of r_2^2 given by the formula is, as wanted, always positive. Moreover, this lemma can be seen as a proof of the fact that, if $\theta_1 = \theta_3 \leqslant \theta_2$ and $\alpha_1 = \alpha_3 \leqslant \alpha_2$, then $\alpha_1 \leqslant \theta_1 \leqslant \frac{\pi}{2}$.*

Corollary 2.3.9. *If, with the usual notation, we have $\theta_1 = \theta_3 \leqslant \theta_2$ and $\alpha_1 = \alpha_3 \leqslant \alpha_2$ (which is always the case for $c \leqslant 4$), then*

$$r_2 < R \Leftrightarrow \cos\theta_2 < \sin^2\alpha_1$$

or, equivalently,

$$r_2 \geqslant R \Leftrightarrow \theta_2 \leqslant \frac{\pi}{2} \text{ and } \alpha_1 \geqslant \arcsin\sqrt{\cos\theta_2}$$

Proof.

$$r_2 < R \Leftrightarrow \frac{\cos^2\alpha_1}{\cos^2\theta_1} < 2 \Leftrightarrow \cos^2\alpha_1 < 2\cos^2\theta_1 \equiv \cos(2\theta_1) + 1 \Leftrightarrow$$

$$\Leftrightarrow 0 < \cos(\pi - \theta_2) + \sin^2\alpha_1 \Leftrightarrow \cos\theta_2 < \sin^2\alpha_1$$

□

2.3.2 Explicit computations for $c \leqslant 4$

All the manifolds in groups A, B, C, E and G have $\theta_1 = \theta_2 = \theta_3 = \frac{\pi}{3}$. Thus $r_2 \equiv r_1 < R \Leftrightarrow \alpha_1 > \frac{\pi}{4}$, but this never happens in those manifolds. Since $r \geqslant R$, for those manifolds the maximal cusp neighborhood is always tangent to the boundary (*i.e.*, no cusp self-tangencies happen while we inflate it).

All the 12 manifolds in group F have $\theta_2 < 1.273$, and so $\arcsin\sqrt{\cos\theta_2} > \arcsin\sqrt{\cos(1.273)} = 0.572..$; the first 9 manifolds in this group have a smaller α_1, and the last 3 manifolds have $\theta_2 < 1.089$ and $\alpha_1 < 0.602 < \arcsin\sqrt{\cos(1.089)} = 0.7487...$ So the maximal cusp neighborhood is always tangent to the boundary (*i.e.*, is 0-maximal) even for group F.

The two manifolds in group D require a little more work. They have $\theta_2 = \alpha_2 = \frac{\pi}{2}$ (and hence $\theta_1 = \theta_3 = \frac{\pi}{4}$), and easy calculations show that $r_1^2 = r_2^2 = r_3^2 = R^2(1 - 2\sin^2\alpha_1) < R^2$ (and hence $r = R \cdot 0.896349488..$). Thus in case D the maximal cusp neighborhood exits tetrahedra (2) and (3) (*i.e.*, the ones with an ideal vertex) passing through the face opposite to the ideal vertex; these faces are glued to face 3 of, respectively, tetrahedron (0) and (1). We have to check if the two neighborhoods touch each other (on a common face of tetrahedra (0) and (1), for symmetry reasons) before they touch the boundary.

Since situations for tetrahedra (2)-(0) and (3)-(1) are completely symmetric, we concentrate only on the first pair. Consider tetrahedron (2) in the half-space model with the usual notation. We want to concretely describe tetrahedron (0) in the same model, glued to tetrahedron (2) by the common face lying on S_f. Let Δ_i denote tetrahedron (i), and rename the vertices of Δ_0 to match their gluing rule with vertices of Δ_2: V_1, V_2, V_3, and V_0 the remaining one (*i.e.*, the one opposite to the face contained in S_f). Let F_i be the face of Δ_0 opposite to V_i, lying on a geodesic half-sphere with center O_i and radius b_i, and let r_i be the radius of the truncation sphere centered at V_i. Dihedral angles in Δ_0 are very symmetric: they are all equal to $0.4173085058.. =: \gamma$, except for $\frac{\pi}{2}$ along $\overline{V_2V_3}$ and 2γ along $\overline{V_2V_0}$. Since F_2 is orthogonal to S_f and must contain the truncated edge $\overline{V_1V_3}$ (a half-circle of radius R in the model), F_2 is contained in the vertical plane Π_2.

Thank to the symmetry of Δ_2 and Δ_0, we can assume $C_0 = (0,0)$, $C_1 = (x \geqslant 0, 0)$, $C_3 = (0, -x)$, $b_1 = b_2 =: b$, $r_1 = r_3$, and $V_2 = (y, -y)$ for some $y \in \mathbb{R}$. We then need to solve the following equations:

$$x^2 = b^2 + R^2 - 2bR\cos\left(\frac{\pi}{2} - \gamma\right) \text{ [dihedral angle between } F_1 \text{ and } F_0]$$

$$2x^2 = 2b^2 - 2b^2\cos(\pi - 2\gamma) \text{ [dihedral angle between } F_1 \text{ and } F_3]$$

$$\frac{\sqrt{2}}{2}x = b\cos\gamma \text{ [dihedral angle between } F_1 \text{ and } F_2]$$

The difference between the first two equations gives

$$b^2\cos(2\gamma) = R^2 - 2bR\sin\gamma \Leftrightarrow b^2(2\cos^2\gamma - 1) + 2bR\sin\gamma - R^2 = 0$$

whose solutions are

$$\begin{aligned} b_{1,2} &= \frac{-R\sin\gamma \pm \sqrt{R^2\sin^2\gamma + R^2(\cos^2\gamma - \sin^2\gamma)}}{2\cos^2\gamma - 1} \\ &= R \cdot \frac{-\sin\gamma \pm \cos\gamma}{(\cos\gamma - \sin\gamma)(\cos\gamma + \sin\gamma)} = \frac{R}{\sin\gamma \pm \cos\gamma} \end{aligned}$$

The only positive solution is $b = \frac{R}{\sin\gamma + \cos\gamma} = R \cdot 0.757871716..$, and so $x = \sqrt{2}b\cos\gamma = R \cdot 0.979814571...$

Truncation half-spheres orthogonal to F_2 must have centers on the line $\overline{V_1V_3}$, and for symmetry reasons we have that $V_1 = (z,z)$ for some $z \geqslant 0$, $V_3 = (-z,-z)$ and $V_4 = (0,0)$. We then have the following equations:

$$\begin{array}{rcll} 2z^2 & = & R^2 + r_1^2 & [S_1 \text{ orthogonal to } F_0] \\ (x-z)^2 + z^2 & = & b^2 + r_1^2 & [S_1 \text{ orthogonal to } F_1] \\ 2y^2 & = & R^2 + r_2^2 & [S_2 \text{ orthogonal to } F_0] \\ (x-y)^2 + y^2 & = & b^2 + r_2^2 & [S_2 \text{ orthogonal to } F_1] \\ x^2 & = & b^2 + r_4^2 & [S_4 \text{ orthogonal to } F_1] \end{array}$$

Solutions are easily computed to be $r_1 = r_2 = r_3 = R\frac{\sqrt{3}}{2}(1 + \tan\gamma) = R \cdot 1.249976176..$, and $r_4 = R\sqrt{\frac{\cos\gamma - \sin\gamma}{\cos\gamma + \sin\gamma}} = R \cdot 0.621020978...$

For our purpose it is sufficient to observe that $b < R \cdot 0.896349488..$, and so the cusp neighborhood cannot exit Δ_0 before touching the truncation triangles of Δ_2. Thus we do not have self-tangency in this case too.

Summing up the whole discussion, we have proven the following:

Proposition 2.3.10. *For $c \leqslant 4$ a maximal neighborhood of a cusp is always 0-maximal (*i.e.*, it is always tangent to the boundary of the manifold and never to itself or to a neighborhood of another cusp).*

2.4 Volume of a cusp neighborhood

Cusp volume inside a tetrahedron With the usual notation in the half-space model, let Δ be a tetrahedron with exactly one ideal vertex $V_0 = \infty$, and let $\mathcal{A}_\Delta$ be the Euclidean area of the triangle $V_1V_2V_3$. For $z_0 \geqslant \max\{r, R\}$, the volume $\mathrm{Vol}_\Delta(z_0)$ of the intersection between Δ and the cusp neighborhood $\{z \geqslant z_0\}$ is

$$\mathrm{Vol}_\Delta(z_0) = \int_{z_0}^{\infty} \frac{\mathcal{A}_\Delta}{z^3}\, dz = \frac{\mathcal{A}_\Delta}{2z_0^2}$$

$\mathcal{A}_\Delta$ can be computed using the formula for $\overline{V_iV_j}$ we found in Proposition 2.3.4:

Theorem 2.4.1. *With the usual notation, we have*

$$\mathcal{A}_\Delta = R^2 \frac{(\sum_{i=1}^{3} \cos\alpha_i \sin\theta_i)^2}{2\sin\theta_1 \sin\theta_2 \sin\theta_3}$$

Proof. From $\mathcal{A}_\Delta = \frac{1}{2}\overline{V_iV_j} \cdot \overline{V_iV_k} \sin\theta_i$ and $\frac{\overline{V_iV_j}}{\sin\theta_k} = \frac{\overline{V_iV_k}}{\sin\theta_j}$ we have

$$\begin{aligned}
\mathcal{A}_\Delta &= \frac{1}{2}\overline{V_iV_j}^2 \cdot \frac{\sin\theta_j \sin\theta_i}{\sin\theta_k} \\
&= R^2\Big(\frac{\cos\alpha_i}{\sin\theta_j} + \frac{\cos\alpha_j}{\sin\theta_i} + \frac{\cos\alpha_k \sin\theta_k}{\sin\theta_i \sin\theta_j}\Big)^2 \cdot \frac{\sin\theta_j \sin\theta_i}{2\sin\theta_k} \\
&= R^2 \frac{(\cos\alpha_i \sin\theta_i + \cos\alpha_j \sin\theta_j + \cos\alpha_k \sin\theta_k)^2}{\sin\theta_i \sin\theta_j \cdot 2\sin\theta_k}
\end{aligned}$$

□

Additional symmetries We want now to specialize the result of the previous paragraph to the case where, with the usual notation, we have $\theta_1 = \theta_3 \leqslant \theta_2$ and $\alpha_1 = \alpha_3 \leqslant \alpha_2$ (which is always the case for manifolds with $c \leqslant 4$):

Corollary 2.4.2. *If, with the usual notation, we have $\theta_1 = \theta_3 \leqslant \theta_2$ and $\alpha_1 = \alpha_3 \leqslant \alpha_2$, then*

$$\begin{aligned}
\overline{V_1V_2} = \overline{V_2V_3} &= R\frac{\cos\alpha_1 + \cos\alpha_2 \cos\theta_1}{\sin\theta_1 \cos\theta_1} \\
\overline{V_1V_3} &= 2R\frac{\cos\alpha_1 + \cos\alpha_2 \cos\theta_1}{\sin\theta_1} \\
\mathcal{A}_\Delta &= R^2\frac{(\cos\alpha_1 + \cos\alpha_2 \cos\theta_1)^2}{\sin\theta_1 \cos\theta_1}
\end{aligned}$$

Proof. Thank to the additional symmetries we can simplify the formulae for the lengths of the edges of the Euclidean triangle $V_1V_2V_3$ in the following way:

$$\begin{aligned}
\overline{V_1V_2} &= \overline{V_2V_3} = R\left(\frac{\cos\alpha_1}{\sin\theta_2} + \frac{\cos\alpha_2}{\sin\theta_1} + \frac{\cos\alpha_1 \sin\theta_1}{\sin\theta_1 \sin\theta_2}\right) \\
&= R\left(2\frac{\cos\alpha_1}{\sin(\pi - 2\theta_1)} + \frac{\cos\alpha_2}{\sin\theta_1}\right) \\
&= R\left(2\frac{\cos\alpha_1}{2\sin\theta_1 \cos\theta_1} + \frac{\cos\alpha_2}{\sin\theta_1}\right) \\
&= R\frac{\cos\alpha_1 + \cos\alpha_2 \cos\theta_1}{\sin\theta_1 \cos\theta_1}
\end{aligned}$$

$$\begin{aligned}\overline{V_1V_3} &= \overline{V_1P_{13}} + \overline{P_{13}P_{31}} + \overline{P_{31}V_3} \\ &= 2R\frac{\cos\alpha_1 + \cos\alpha_2\cos\theta_1 - \sin\alpha_2\sin\theta_1}{\sin\theta_1} + 2R\sin\alpha_2 \\ &= 2R\frac{\cos\alpha_1 + \cos\alpha_2\cos\theta_1}{\sin\theta_1}\end{aligned}$$

And so, since the triangle $V_1V_2V_3$ is isosceles, we have

$$\mathcal{A}_\Delta = \frac{1}{2}\overline{V_1V_3}\cdot\tan\theta_1\frac{\overline{V_1V_3}}{2} = R^2\frac{(\cos\alpha_1 + \cos\alpha_2\cos\theta_1)^2}{\sin\theta_1\cos\theta_1}$$

□

Volume of a d-maximal neighborhood of a cusp Let M be a hyperbolic 3-manifold with geodesic boundary and at least one toric cusp C, and let τ be a triangulation of M into (hyper)ideal geometric tetrahedra. τ induces a triangulation τ' on the torus $T \subset \partial\overline{M}$ relative to the cusp C. Fix a maximal tree Γ on the graph dual to τ', and fix a tetrahedron Δ_1 of M with at least one ideal vertex V_0 relative to the cusp C. Realize Δ_1 in the half space model with $V_0 = \infty$ as usual, and then follow the gluings encoded by Γ to realize the other tetrahedra Δ_i of τ that induced τ'. Notice that in this way we have realized a gluing of a number $n = \tau'^{(2)}$ of tetrahedra, possibly with some repetitions (*e.g.*, a tetrahedron with two ideal vertices relative to C will appear twice). With this construction a cusp neighborhood U of C is of the form $\{z \geqslant h\}$ with the same h for each Δ_i, and the volume of U is equal to the volume of $\{z \geqslant h\} \cap (\cup_i\Delta_i)$.

Each Δ_i has an associated area $\mathcal{A}_{\Delta_i}$ (*i.e.*, the Euclidean area of the projection of Δ_i onto $\{z = 0\}$). We can sum up these areas to get the *cusp area*

$$\text{Area}_c := \sum_{i=1}^{n}\mathcal{A}_{\Delta_i}$$

Let $r_{\max}$ and $R_{\max}$ be the maximum value respectively of r and R among the n realized tetrahedra Δ_i. Let B_d be a d-collar of the boundary, with $d \leqslant d_{\max}$. Notice that for $d \geqslant 0$ the plane $\{z = e^d r_{\max}\}$ has distance d from the plane $\{z = r_{\max}\}$: if $e^d r_{\max} \geqslant R_{\max}$ then $\{z \geqslant e^d r_{\max}\}$ is a d-maximal cusp neighborhood of C. It follows then that:

Proposition 2.4.3. *With the usual notation, if $e^d r_{\max} \geqslant R_{\max}$ then the volume $Vol_c(d)$ of a d-maximal cusp neighborhood is*

$$Vol_c(d) = \frac{Area_c}{2r_{\max}^2} e^{-2d}$$

To get rid of the condition $re^d \geqslant R$, which may not be true in general, we have to proceed as follows. Instead of realizing only n tetrahedra as before, choose a universal cover $p : \tilde{M} \subset \mathbb{H}^3 \to M$ such that the cusp torus T lifts to ∞ (*e.g.*, the universal cover extending the realized Δ_i's). The cusp C is then given by a lattice Λ acting horizontally on $\mathbb{H}^3$. Let $\bar{r}$ be the maximal Euclidean radius among the half-spheres with center in $\{z = 0\}$ which bound $\tilde{M}$ (notice that, in particular, we have $\bar{r} \geqslant r_{\max}$): $\bar{r}$ is well-defined, since the maximum has to be taken over a Λ-equivariant family of half-spheres. Then we can generalize the previous result to the following:

Proposition 2.4.4. *With the above notation, the volume $Vol_c(d)$ of a d-maximal cusp neighborhood is*

$$Vol_c(d) = \frac{Area_c}{2\bar{r}^2} e^{-2d}$$

Remark 2.4.5. Notice that, as expected, the value of $\mathrm{Vol}_c(d)$ does not depend on the universal cover chosen: in fact, two different universal covers differ by the composition of a horizontal Euclidean isometry and a dilation, and both preserve the ratio $\frac{\mathrm{Area}_c}{\bar{r}^2}$. In particular, if $\bar{r} = r_{\max}$, we can use Lemma 2.3.5 and Theorem 2.4.1 to explicit the fact that $\frac{\mathrm{Area}_c}{\bar{r}^2}$ depends only on the dihedral angles of the triangulation.

Multiple cusps If M has $k > 1$ cusps we consider as a cusp neighborhood the union of a cusp neighborhood of equal volume for each cusp, requiring that they all have disjoint and embedded interiors. Proposition 2.4.4 than generalizes to:

Proposition 2.4.6. *With the usual notation, the volume $Vol_c(d)$ of a d-maximal cusp neighborhood is*

$$Vol_c(d) = k \cdot \frac{Area_c}{2\bar{r}^2} e^{-2d}$$

where k is the number of cusps of M and the values of $Area_c$ and $\bar{r}^2$ are computed using a cusp whose neighborhood is tangent to the d-collar of ∂M.

Cusp (self)tangencies Let us now consider the case where there are cusp (self)tangencies. If, for $d < d_{\max}$, we have to stop inflating the cusp neighborhood before reaching the d-collar, we can increase the width of the collar up to touching the cusp neighborhood or up to $d_{\max}$. In the first case $\mathrm{Vol}_c(d) = k\frac{\mathrm{Area}_c}{2\bar{r}^2}e^{-2d}$ as before; in the second case $\mathrm{Vol}_c(d_{\max}) < k\frac{\mathrm{Area}_c}{2\bar{r}^2}e^{-2d_{\max}}$ and the maximal cusp neighborhood is of the form $\{z \geqslant e^{d_{\max}+h} \cdot \bar{r}\}$ for some $h > 0$. Notice that h does not depend on d. This concludes the proof of the following result about the volume of a self-tangent cusp neighborhood:

Proposition 2.4.7. *With the usual notation, the volume Vol_c of a maximal cusp neighborhood U that is not d-maximal is*

$$Vol_c = k \cdot \frac{Area_c}{2\bar{r}^2}e^{-2d_{\max}-2h}$$

where the values of $Area_c$ and $\bar{r}$ are computed using a cusp whose neighborhood has distance from ∂M equal to $d_{\max} + h$.
However, if $h \leqslant 0$, then U is d'-maximal for some $d' \in (d, d_{\max}]$.

2.5 Volume estimates

2.5.1 The general formula

In this section we sum up the results of the previous sections in order to analyze what portion $V(d)$ of the volume of M can be covered with a d-collar B_d of the boundary and a maximal cusp neighborhood U with disjoint and embedded interiors. Putting together Proposition 2.2.5, Proposition 2.4.6 and Proposition 2.4.7 we have the following:

Proposition 2.5.1. *Let M be an orientable hyperbolic 3-manifold with non-empty geodesic boundary and $k \geqslant 1$ toric cusps. Then the portion $V(d)$ of the volume of M covered by a d-collar B_d of the boundary, $d \in [0, d_{\max}]$, and a maximal cusp neighborhood U with disjoint and embedded interiors is given by*

$$V(d) = \begin{cases} \mathcal{A}_\partial \dfrac{2d + \sinh(2d)}{4} + k \cdot \dfrac{Area_c}{2\bar{r}^2}e^{-2d} & \text{if } U \text{ is } d\text{-maximal} \\ \mathcal{A}_\partial \dfrac{2d + \sinh(2d)}{4} + k \cdot \dfrac{Area_c}{2\bar{r}^2}e^{-2d_{\max}-2h} & \text{if } U \text{ is not } d\text{-maximal} \end{cases}$$

where

- $\mathcal{A}_\partial = 4\pi\sum_{i=1}^{b}(g_i - 1)$ *is the area of* ∂M *and depends only on the genera* g_i *of the* b *connected components of* ∂M
- $Area_c$ *and* $\bar{r}$ *are computed using one of the cusps whose neighborhood has minimal distance from* B_d
- $h > -d_{\max}$ *is such that the minimal distance between* U *and* ∂M *is* $d_{\max} + h$

2.5.2 The maximum of $V(d)$

We want now to find the maximum of the function $V(d)$ in its domain $I \subset [0, d_{\max}]$. Let $d_{\min}$ be the minimum among the values of $d \in [0, d_{\max}]$ such that there exists a d-maximal cusp neighborhood (notice that $d_{\min} = 0$ if and only if there are no self-tangencies for any cusp neighborhood). The main result of this chapter is the following:

Theorem 2.5.2. *The function* $V(d)$ *has its maximum in* $d = d_{\min}$ *or in* $d = d_{\max}$.

Proof. If the cusp neighborhood U is maximal but not d-maximal we can inflate the boundary collar B_d up to $d_{\max}$ or up to a $d' \in (d, d_{\max})$ such that U is d'-maximal.

We can then suppose that U is d-maximal. Hence $V(d)$ is given by

$$V(d) = \mathcal{A}_\partial \frac{2d + \sinh(2d)}{4} + k \cdot \frac{Area_c}{2\bar{r}^2} e^{-2d}$$

The derivatives of $V(d)$ are

$$V'(d) = \mathcal{A}_\partial \frac{1 + \cosh(2d)}{2} - k \cdot \frac{Area_c}{\bar{r}^2} e^{-2d}$$

$$V''(d) = \mathcal{A}_\partial \cdot \sinh(2d) + 2k \cdot \frac{Area_c}{\bar{r}^2} e^{-2d}$$

Since $\sinh(2d) \geqslant 0$ for $d \geqslant 0$ and $e^{-2d} \geqslant 1$, we have $V''(d) > 0$ in its domain, and thus $V'(d)$ is strictly increasing in d. This implies that $V'(d) = 0$ for at most one $\overline{d} \in (0, d_{\max})$, and so $V(d)$ has at most one internal extremal point $\overline{d}$, which will surely be a minimum. The maximum of $V(d)$ must then be in $\partial I = \{d_{\min}, d_{\max}\}$. $\square$

Remark 2.5.3. The best volume estimate for the volume of M using a boundary collar and a maximal cusp neighborhood with disjoint and embedded interiors can then be achieved only at the combinatorially extremal configurations, *i.e.* the ones where we take the maximal boundary collar or the cusp neighborhood with maximum volume.

In the particular situation where a 0-maximal cusp neighborhood exists, the proof of Theorem 2.5.2 implies that, if $V'(0) \geqslant 0$, then the best volume estimate is given by $V(d_{\max})$. This can be easily generalized to the following sufficient condition:

Lemma 2.5.4. *If, with the usual notation,*

$$\mathcal{A}_\partial \cdot \bar{r}^2 \geqslant Area_c$$

then $d_{\max}$ *is a global maximum for* $V(d)$.

This condition, which is much easier to check, will be sufficient for the explicit computations on the examples with $c \leqslant 4$ we are going to make in the next subsection.

2.5.3 Explicit computations for $c \leqslant 4$

This subsection is devoted to prove the following:

Theorem 2.5.5. *For $c \leqslant 4$ the best estimate for the volume of M given by a d-collar of ∂M and a maximal cusp neighborhood with disjoint and embedded interiors is always $V(d_{\max})$,* i.e. *the one using the maximal boundary collar.*

Proof. The proof will be carried out by checking the condition of Lemma 2.5.4 in the examples with $c \leqslant 4$. Remember that the number k of cusps is 2 for the manifold in group G and 1 for the manifolds in all the other groups.

All manifolds in groups A, B, C, E and G have $\theta_1 = \theta_2 = \theta_3 = \frac{\pi}{3}$ and $\alpha_1 = \alpha_2 = \alpha_3$, so $r^2 = R^2(4\cos^2\alpha_1 - 1)$ and

$$\mathcal{A}_\Delta = \frac{4}{\sqrt{3}} R^2 \cos^2\alpha_1 \cdot \big(1 + \frac{1}{2}\big)^2 = 3\sqrt{3} R^2 \cos^2\alpha_1$$

Hence

$$\mathcal{A}_\partial \cdot r^2 \geqslant k \cdot Area_c \Leftrightarrow 4\cos^2\alpha_1 - 1 \geqslant 3\sqrt{3}\cos^2\alpha_1 \cdot \frac{\#ideal.vertices}{\mathcal{A}_\partial}$$

If we set $z := 4\pi\frac{\#ideal.vertices}{\mathcal{A}_\partial}$, we have $z = 1$ (group B), $z = 2$ (groups A and C) or $z = 4$ (groups E and G), and then

$$\mathcal{A}_\partial \cdot r^2 \geqslant k \cdot Area_c \Leftrightarrow 16\pi\cos^2\alpha_1 - 4\pi \geqslant 3z\sqrt{3}\cos^2\alpha_1 \Leftrightarrow$$

$$\Leftrightarrow \cos^2\alpha_1 \geqslant \frac{4\pi}{16\pi - 3z\sqrt{3}} \Leftrightarrow$$

$$\Leftrightarrow \alpha_1 \leqslant \arccos\sqrt{\frac{4\pi}{16\pi - 3z\sqrt{3}}}$$

which is equivalent to $\alpha_1 \leqslant 1.0145..$ (for $z = 1$), $\alpha_1 \leqslant 0.975..$ (for $z = 2$) and $\alpha_1 \leqslant 0.859..$ (for $z = 4$). These conditions ca be easily checked to be true in all the examples of groups A, B, C, E and G. Hence $V(d_{\max})$ is the best estimate in these cases.

Manifolds in group D have $\theta_2 = 2\theta_1 = \alpha_2 = \frac{\pi}{2}$, and hence we have $r^2 = R^2(2\cos^2\alpha_1 - 1)$ and $\mathcal{A}_\Delta = 2R^2\cos^2\alpha_1$. Then

$$\mathcal{A}_\partial \cdot r^2 \geqslant Area_c \Leftrightarrow 2\cos^2\alpha_1 - 1 \geqslant 2\cos^2\alpha_1 \cdot \frac{2}{4\pi} \Leftrightarrow$$

$$\Leftrightarrow \cos^2\alpha_1 \geqslant \frac{\pi}{2\pi - 1} \Leftrightarrow \alpha_1 \leqslant 0.690..$$

which is true in all examples of group D. $V(d_{\max})$ is the best estimate in this case too.

Group F requires a little more work, since dihedral angles vary (slightly) between the manifolds of the group. The condition $\mathcal{A}_\partial \cdot r^2 \geqslant Area_c$ is satisfied if

$$r_2^2 \geqslant \frac{2\mathcal{A}_\Delta}{4\pi} \Leftrightarrow f(\theta_1, \alpha_1, \alpha_2) \geqslant 0 \text{ , where}$$

$$\begin{aligned} f(\theta_1, \alpha_1, \alpha_2) \quad &:= \quad 2\pi\sin\theta_1\cos\theta_1\left(\frac{\cos^2\alpha_1}{\cos^2\theta_1} - 1\right) - (\cos\alpha_1 + \cos\alpha_2\cos\theta_1)^2 \\ &= \quad 2\pi\tan\theta_1\cos^2\alpha_1 - \pi\sin(2\theta_1) - (\cos\alpha_1 + \cos\alpha_2\cos\theta_1)^2 \end{aligned}$$

We observe that

$$\begin{aligned}
\frac{\partial f(\theta_1,\alpha_1,\alpha_2)}{\partial \cos\alpha_2} &= -2(\cos\alpha_1+\cos\alpha_2\cos\theta_1)\cos\theta_1<0\\
\frac{\partial f(\theta_1,\alpha_1,\alpha_2)}{\partial \cos\alpha_1} &= 2\pi\frac{\sin\theta_1}{\cos\theta_1}\cdot 2\cos\alpha_1-2(\cos\alpha_1+\cos\alpha_2\cos\theta_1)\\
&= 2\cos\theta_1(\frac{2\pi\tan\theta_1-1}{\cos\theta_1}\cos\alpha_1-\cos\alpha_2)\\
&> 2\cos\theta_1\cos\alpha_1(\frac{2\pi\tan\theta_1-1}{\cos\theta_1}-1)
\end{aligned}$$

since $\alpha_1<\alpha_2<\frac{\pi}{2}$. And thus, since $0.93<\theta_1<\frac{\pi}{2}$,

$$\tan\theta_1>1\Rightarrow\frac{2\pi\tan\theta_1-1}{\cos\theta_1}>2\pi-1>1\Rightarrow\frac{\partial f}{\partial\cos\alpha_1}>0$$

$$\frac{\partial f(\theta_1,\alpha_1,\alpha_2)}{\partial\theta_1}=2\pi\frac{\cos^2\alpha_1}{\cos^2\theta_1}-2\pi\cos(2\theta_1)+2(\cos\alpha_1+\cos\alpha_2\cos\theta_1)\cos\alpha_2\sin\theta_1$$

$$>2\pi(1+\tan^2\theta_1)\cos^2\alpha_1-2\pi(\cos^2\theta_1-\sin^2\theta_1)>2\pi(\cos^2\alpha_1-\cos^2\theta_1)>0$$

since $\alpha_1<\theta_1<\frac{\pi}{3}$.

f is strictly increasing in both $\cos\alpha_1$ and θ_1, and strictly decreasing in $\cos\alpha_2$. Since $\theta_1>0.93$, $\alpha_1<0.61$ and $\alpha_2>0.74$, we can conclude that

$$f(\theta_1,\alpha_1,\alpha_2)>f(0.93,0.61,0.74)=1.0585..>0.$$

So even for the manifolds in group F the best estimate is $V(d_{\max})$. □

2.5.4 Explicit volume estimates for $c\leqslant 4$

We want now to explicitly compute, for the 32 manifolds with $c\leqslant 4$, the best lower estimate $V_{\max}$ for the volume of M given by a d-collar of ∂M and a maximal cusp neighborhood with disjoint and embedded interiors. Since we already proved that for $c\leqslant 4$ the best estimate is given by the maximal boundary collar and a $d_{\max}$-maximal cusp neighborhood, we have to compute

$$V_{\max}=\mathcal{A}_\partial\frac{2d_{\max}+\sinh(2d_{\max})}{4}+k\cdot\frac{Area_c}{2\bar{r}^2}e^{-2d_{\max}}$$

We already saw that for $c \leqslant 4$ we have $r_2^2 \geqslant r_1^2 = r_3^2$, and that a 0-maximal cusp neighborhood is always tangent to one of the truncation triangles of a tetrahedron with an ideal vertex (this condition was harder to check only for manifolds in group D, where the 0-maximal cusp neighborhood is not entirely contained in the union of the tetrahedra with one ideal vertex). Hence

$$\bar{r}^2 = r_{\max}^2 = r_2^2 = R^2 \left(\frac{\cos^2 \alpha_1}{\cos^2 \theta_1} - 1 \right)$$

which gives

- $r_{\max}^2 = R^2 \cdot\ 2.1029693..$ (group A)
- $r_{\max}^2 = R^2 \cdot\ 2.6066415..$ (group B)
- $r_{\max}^2 = R^2 \cdot\ 2.0257779..$ or $r_{\max}^2 = R^2 \cdot\ 2.0110325..$ (group C)
- $r_{\max}^2 = R^2 \cdot\ 0.8034424..$ (group D)
- $R^2 \cdot\ 1.3041296.. < r_{\max}^2 < R^2 \cdot\ 1.5341237..$ (group F)
- $r_{\max}^2 = 2R^2$ (groups E and G)

We also have to compute $k \cdot Area_c$, which is the sum, over all ideal vertices, of the areas $\mathcal{A}_\Delta$, each one being equal to

$$\mathcal{A}_\Delta = R^2 \frac{(\cos \alpha_1 + \cos \alpha_2 \cos \theta_1)^2}{\sin \theta_1 \cos \theta_1} = 2R^2 \frac{(\cos \alpha_1 + \cos \alpha_2 \cos \theta_1)^2}{\sin \theta_2}$$

which gives

- $\mathcal{A}_\Delta = R^2 \cdot\ 4.0308754..$ (group A)
- $\mathcal{A}_\Delta = R^2 \cdot\ 4.6851648..$ (group B)
- $\mathcal{A}_\Delta = R^2 \cdot\ 3.9306008..$ o R $\cdot$ 3.9114459.. (group C)
- $\mathcal{A}_\Delta = R^2 \cdot\ 1.8034424..$ (group D)
- $\mathcal{A}_\Delta = R^2 \cdot\ 3.8971144$ (group E)
- $R^2 \cdot\ 2.8499561.. < \mathcal{A}_\Delta < R^2 \cdot\ 3.2863178..$ (group F)
- $2\mathcal{A}_\Delta = R^2 \cdot\ 3.8971144$ (group G)

We have now all the needed data to compute the maximal volume given by a maximal collar of ∂M and a disjoint cusp neighborhood:

- $V_{\max} = 4.6797503..$ (group A)
- $V_{\max} = 5.5230060..$ (group B)
- $V_{\max} = 4.9344073..$ o $4.9837952..$ (group C)
- $V_{\max} = 5.0482681..$ (group D)
- $6.7732928.. < V_{\max} < 7.1336172..$ (group F)
- $V_{\max} = 5.5080591$ (groups E and G)

These lower estimates count around the 60% of the total volume on average, from a minimum of 47% for manifolds in group B to a maximum of 77% for some of the manifolds in group F.

2.6 Peripheral volume

2.6.1 Introduction

So far we have considered a boundary collar that has the same width for each connected component of ∂M, and a cusp neighborhood consisting of an equal-volume neighborhood of each of the cusps of M. We want now to vary independently the collar for each boundary component and the neighborhood of each cusp, trying to generalize Theorem 2.5.2 and Remark 2.5.3. More precisely, we want to address the following question:

Question 2.6.1. Let M be an orientable hyperbolic 3-manifold with either nonempty compact geodesic boundary, or some toric cusps, or both. What is the optimal way (in the sense of volume maximization) of inserting in M boundary collars and/or cusp neighborhoods having disjoint embedded interiors?

Definition 2.6.2. We will use the term *peripheral component* to refer to either a collar of a boundary component or to a horospherical neighborhood of a single cusp of M. The volume of a peripheral component P will be denoted by $v(P)$. A peripheral component P will be said to be *maximal* if $v(P)$ is maximal regardless of the other peripheral components (*i.e.*, if P is tangent to itself or to ∂M).

Let us fix the notation we will use in this section. M will be an orientable hyperbolic 3-manifold with either nonempty compact geodesic boundary, or some toric cusps, or both. We will denote by $\Sigma_1, \ldots, \Sigma_b$ the components of ∂M and by $\mathcal{A}(\Sigma_j)$ their areas. $T_1, \ldots, T_k$ will be the toric boundary components of the compactification of M. For $d \geqslant 0$ we will denote by $B_j(d)$ the d-collar of Σ_j, and for $v > 0$ we will denote by $C_i(v)$ the horospherical cusp neighborhood at T_i with volume v. Whenever distinct peripheral components are mentioned we will assume they all have disjoint and embedded interiors.

2.6.2 Two peripheral components

From Proposition 2.4.4 and Remark 2.4.5 it follows that:

Lemma 2.6.3. *Suppose that M has both geodesic boundary and cusps. Then, with the notation used in Proposition 2.4.4, the cusp torus T_i has a well-defined area $\mathcal{A}_j(T_i) := \frac{Area_c}{\bar{r}_j^2}$ relative to the boundary component Σ_j, where $\bar{r}_j$ is the maximal Euclidean radius of a half-sphere with center in $\{z = 0\}$ that bounds $\tilde{M}$ and projects in M to Σ_j.*

The volume of a cusp neighborhood $C_i(v)$ tangent to a boundary collar $B_j(d)$ will then be equal to $v = \frac{\mathcal{A}_j(T_i)}{2} \cdot e^{-2d}$.

The relation between the volumes of two tangent cusp neighborhoods is given by the following:

Proposition 2.6.4. *If $C_{i_1}(v_1)$ and $C_{i_2}(v_2)$ vary while remaining tangent to each other, then the product $v_1 \cdot v_2$ remains constant.*

Proof. Suppose that in some universal cover contained in $\mathbb{H}^3$ the cusp neighborhood $C_{i_1}(v_1)$ is the quotient of the horoball $\{z \geqslant z_{i_1}\}$ acted on by a lattice Λ_{i_1}. If the cusp changes so that in M its boundary moves of some small distance $d \in \mathbb{R}$ (with $d < 0$ meaning that the cusp is shrinking) then it becomes the quotient under Λ_{i_1} of $\{z \geqslant z'\}$ with

$$\int_{z'}^{z_{i_1}} \frac{1}{t}\, dt = d$$

whence $z' = z_{i_1} e^{-d}$. The cusp volume v_1 then changes from

$$\mathrm{Area}_c \cdot \int_{z_{i_1}}^{+\infty} \frac{1}{t^3}\, dt = \frac{\mathrm{Area}_c}{2} \cdot z_{i_1}^{-2}$$

to $\frac{\mathrm{Area}_c}{2} \cdot z^{-2}$, namely it changes by a factor e^{2d}. During a simultaneous variation of v_1 and v_2 with $C_{i_1}(v_1)$ and $C_{i_2}(v_2)$ remaining tangent to each other, the boundary of $C_{i_2}(v_2)$ moves by a distance $-d$. The calculations already carried out show that then its volume varies by a factor e^{-2d}, and the conclusion follows. □

An extension of Theorem 2.5.2 is the following:

Proposition 2.6.5. *Let M have two peripheral components P_1 and P_2. Then $v(P_1)+v(P_2)$ can have a local maximum only if, up to reordering the indices, P_1 is chosen so that $v(P_1)$ is maximal regardless of P_2, and then P_2 is chosen so that $v(P_2)$ is maximal given P_1.*

Proof. If, up to reordering the indices, P_1 is maximal but P_2 is not maximal nor tangent to P_1, then we can inflate P_2 until $v(P_2)$ is maximal given P_1. If P_1 is not maximal and P_2 is not tangent to P_1 then we can inflate P_1, and the configuration can thus not be a local maximum for $v(P_1) + v(P_2)$. So we are left with the situation where P_1 and P_2 are tangent to each other but none of them is maximal.

If the two peripheral components are a cusp and a collar, then the conclusion follows from Theorem 2.5.2.

If P_1 and P_2 and both cusps, then Proposition 2.6.4 implies that $v(P_2) = \frac{v_0^2}{v(P_1)}$ for some $v_0 > 0$. If we set $v_1 := v(P_1)$ and $V := v(P_1) + v(P_2)$, we have that $V(v_1) = v_1 + \frac{v_0^2}{v_1}$, and thus $V''(v_1) = 2 \cdot \frac{v_0^2}{v_1^3} > 0$. So the convex function V can have a local maximum only at the boundary of its domain of definition, *i.e.* only if P_1 is maximal or if $v(P_1)$ is minimal, which is equivalent to P_2 being maximal.

If P_1 and P_2 and both boundary collars with width respectively equal to d_1 and d_2, then their volume $V := v(P_1) + v(P_2)$ can be expressed as

$$V(d_1) = \frac{\mathcal{A}(\Sigma_1)}{4} \cdot (2d_1 + \sinh(2d_1)) + \frac{\mathcal{A}(\Sigma_2)}{4} \cdot (2(2d_0 - d_1) + \sinh(2(2d_0 - d_1)))$$

where $2d_0 > 0$ is the distance between Σ_1 and Σ_2. Since $0 \leqslant d_1 \leqslant 2d_0$, we have

$$V''(d_1) = \mathcal{A}(\Sigma_1) \cdot \sinh(2d_1) + \mathcal{A}(\Sigma_2) \cdot \sinh(2(2d_0 - d_1)) > 0$$

Then V can have a local maximum only for $d_1 = 0$ or $d_1 = 2d_0$, *i.e.* only if one of the collars is maximal. □

Corollary 2.6.6. *Suppose that M has two cusps C_1 and C_2 and no geodesic boundary, and for $\{i,j\} = \{1,2\}$ let $v_i^{\max}$ be the maximal volume that C_i can attain regardless of C_j. If $v_1^{\max} \geqslant v_2^{\max}$ then the maximum of $v(C_1) + v(C_2)$ is attained by maximizing first C_1 and then C_2 given C_1.*

Proof. Taking v_1 as a variable to parameterize $C_1(v_1) \cup C_2(v_2)$ tangent to each other, we have $v_2 = \frac{v_0^2}{v_1}$ for some $v_0 > 0$, and v_1 varies in $[v_1^{\min}, v_1^{\max}]$ where $v_1^{\min} = \frac{v_0^2}{v_2^{\max}}$. We want to maximize $V(v_1) := v_1 + \frac{v_0^2}{v_1}$. Since $V'(v_1) = 1 - \frac{v_0^2}{v_1^2}$ and $V''(v_1) = \frac{2v_0^2}{v_1^3} > 0$, the convex function $V(v_1)$ has its maximum at $\partial[v_1^{\min}, v_1^{\max}]$ and its minimum at $v_1 = v_0$.

If $v_1^{\min} > v_0$ then the maximum for $v_1 + \frac{v_0^2}{v_1}$ is obviously attained at $v_1^{\max}$.

On the other hand, if $v_1^{\min} \leqslant v_0$ then we have $v_2^{\max} = \frac{v_0^2}{v_1^{\min}} \geqslant v_0$. Since the function $v \mapsto v + \frac{v_0^2}{v}$ is increasing on $[v_0, +\infty)$, the inequalities $v_1^{\max} \geqslant v_2^{\max} \geqslant v_0$ imply

$$v_1^{\max} + \frac{v_0^2}{v_1^{\max}} \geqslant v_2^{\max} + \frac{v_0^2}{v_2^{\max}} = \frac{v_0^2}{v_1^{\min}} + v_1^{\min}$$

□

Corollary 2.6.7. *Suppose that M has two geodesic boundary components Σ_1 and Σ_2 of the same genus g and no cusps. For $\{i,j\} = \{1,2\}$ let $v_i^{\max}$ be the maximal volume that B_i can attain regardless of B_j. If $v_1^{\max} \geqslant v_2^{\max}$ then the maximum of $v(B_1) + v(B_2)$ is attained by maximizing first B_1 and then B_2 given B_1.*

Proof. Taking d_1 as a variable to parameterize $B_1(d_1) \cup B_2(d_2)$ tangent to each other, we want to maximize the function

$$V(d_1) := \pi(g-1) \cdot (2d_1 + \sinh(2d_1) + 2(2d_0 - d_1) + \sinh(2(2d_0 - d_1)))$$

where $2d_0 > 0$ is the distance between Σ_1 and Σ_2. For $\{i,j\} = \{1,2\}$ let $d_i^{\max}$ be the maximal width of the collar B_i (*i.e.*, the width corresponding to $v_i^{\max}$) and set $d_i^{\min} := 2d_0 - d_j^{\max}$. Notice that $v_1^{\max} \geqslant v_2^{\max}$ if and only if $d_1^{\max} \geqslant d_2^{\max}$.

The derivatives of $V(d_1)$ are

$$\begin{aligned} V'(d_1) &= 2\pi(g-1)(\sinh(2d_1) - \sinh(2(2d_0 - d_1))) \\ V''(d_1) &= 4\pi(g-1)(\sinh(2d_1) + \sinh(2(2d_0 - d_1))) > 0 \end{aligned}$$

and then $V(d_1)$ has its maximum at $\partial[d_1^{\min}, d_1^{\max}]$ and its minimum at $d_1 = d_0$.

If $d_1^{\min} \geqslant d_0$ then the maximum for $V(d_1)$ is obviously attained at $d_1^{\max}$.

On the other hand, if $d_1^{\min} < d_0$ then we have $d_2^{\max} > d_0$. Since the function $V(d_1)$ is increasing for $d_1 \geqslant d_0$, the inequalities $d_1^{\max} \geqslant d_2^{\max} \geqslant d_0$ imply

$$V(d_1^{\max}) \geqslant V(d_2^{\max}) = V(d_1^{\min})$$

□

2.6.3 The general case

Proposition 2.6.8. *Let M have n peripheral components $P_1, \ldots, P_n$ and construct a graph with vertices $P_1, \ldots, P_n$ and edges joining peripheral components that are tangent to each other. Suppose that this graph has a connected component Γ that is a tree and such that none of its vertex P_i is maximal. Then the configuration cannot be a local maximum for $v(P_1) + \ldots + v(P_n)$.*

Proof. Under the stated assumptions we can locally deform the peripheral components corresponding to the vertices of Γ using one parameter that can be both increased and decreased. Up to reordering the indices, we can assume that P_1 is a vertex of Γ.

If all the vertices of Γ are cusps, choose the volume v_1 of P_1 as a deformation parameter, and let $v_1^{(0)}$ denote the initial volume of P_1. The volume of a peripheral component P_j tangent to P_1 then varies as $\frac{c_j}{v_1}$ for some $c_j > 0$, and the volume of a P_h tangent to P_j varies as $\frac{cost}{\frac{c_j}{v_1}} \equiv c_h \cdot v_1$ for some $c_h > 0$. We can then see that each v_i in Γ varies either as $c_i \cdot v_1$ or as $\frac{c_i}{v_1}$ (depending on the parity of the distance between P_i and P_1 along Γ) for some $c_i > 0$. Since each v_i is a convex function of v_1 and since v_1 can be both increased and decreased, the sum of all the volumes of the peripheral components in Γ is a convex function of v_1 that cannot have a local maximum at $v_1^{(0)}$.

Suppose now that in Γ there exists at least one boundary collar component P_1 and choose its width d_1 as a deformation parameter. Denote by $d_1^{(0)}$ the initial width of the collar P_1. A collar component P_j' tangent to P_1 obviously varies its width as $2d_j' - d_1$, where $2d_j'$ is the distance between Σ_j and Σ_1. Proposition 2.4.3 implies that a cusp P_i' tangent to P_1 varies its volume as $c_i' \cdot e^{-2d_1}$ for some $c_i' > 0$, and for Proposition 2.6.4 a cusp P_i'' tangent to P_i' varies its volume as $c_i'' \cdot e^{2d_1}$ for some $c_i'' > 0$. It follows by induction on the number of edges in Γ one needs to travel through in passing from $B_1(d_1)$ to any $B_j(d_j)$ or $C_i(v_i)$ that each d_j in Γ varies as either $c_j + d_1$ or as $c_j - d_1$ for some $c_j \in \mathbb{R}$, and each v_i in Γ varies as either $c_i \cdot e^{-2d_1}$ or as $c_i \cdot e^{2d_1}$ for some $c_i > 0$. Since $v(B_j(d_j))$ is a convex function of d_j, then $v(P_h)$ is a convex function of d_1 for each P_h in Γ. The sum of all the volumes of the peripheral components in Γ is then a convex function of d_1 and thus it cannot have a local maximum at $dv_1^{(0)}$. □

The previous proposition can be easily generalized to the following:

Proposition 2.6.9. *Let M have n peripheral components $P_1, \ldots, P_n$ such that no P_j is maximal. If there are fewer than n tangencies between different P_j's, then the configuration cannot be a local maximum for $v(P_1) + \ldots + v(P_n)$.*

2.6.4 Experimental facts

Imposing all the collars of the boundary components to have the same width, and all the toric cusps to have the same volume, would not typically be the most efficient way to maximize the peripheral volume: the proofs of Corollary 2.6.6 and of Corollary 2.6.7 show in fact that the peripheral volume $V := v(P_1) + v(P_2)$ of two peripheral components P_1 and P_2 that are both cusps or both collars of boundary components with the same genus attains its *minimum* at the configuration where $v(P_1) = v(P_2)$.

One may ask if the explicit volume estimates carried out in Subsection 2.5.4 for the 32 examples with $c \leqslant 4$ can be improved by allowing each peripheral component to vary independently from the other ones. In [7] a census was carried out of all the 5,192 hyperbolic 3-manifolds with nonempty compact geodesic boundary that can be triangulated using up to 4 tetrahedra. It turns out that the geodesic boundary is always connected, and that for the 32 manifolds with at least one toric cusp only one (the manifold in group G) has two cusps. The computations we already carried out show that, if we let vary independently the two cusp neighborhoods of the manifold in group G, none of them can become self-tangent or tangent to the other before becoming tangent to the boundary: hence, even for the manifold in group G, allowing each peripheral component to vary independently has, for symmetry reasons, the same effect (with regard to volume maximization) as imposing all the toric cusps to have the same volume. In other words, the manifolds with complexity $c \leqslant 4$ are too simple for their peripheral volume function to benefit from the generalization introduced in this subsection.

We conclude this discussion with the obvious generalization of Proposition 2.3.10:

Proposition 2.6.10. *For all the 5,192 hyperbolic* 3*-manifolds with nonempty compact geodesic boundary and complexity* $c \leqslant 4$ *the largest peripheral volume is obtained by using the maximal boundary collar.*

2.6.5 Three peripheral components

In this subsection we take a deeper insight at the case of three peripheral components, improving the description given by the general results of the previous subsection. A similar punctual analysis is perhaps possible also for four or more peripheral components, but we will not carry it out here. Before stating the main result of this subsection we have to define a certain modified volume:

Definition 2.6.11. The *modified volume* $\widetilde{v}$ of a boundary collar $B_j(d)$ is

$$\widetilde{v}(B_j(d)) = \frac{\mathcal{A}(\Sigma_j)}{4} \cdot (1 + \cosh(2d))$$

The *modified volume* $\widetilde{v}$ of a cusp $C_i(v_i)$ is set to be equal to v_i.

Notice that for a boundary collar $B_j(d)$ we have $\widetilde{v}(B_j(d)) = \frac{1}{2}\frac{\partial v(B_j(d))}{\partial d}$ and that the modified volume $\widetilde{v}(B_j(d))$ is a strictly increasing function of $v(B_j(d))$. The rest of this subsection will be devoted to prove the following:

Theorem 2.6.12. *Let M have three peripheral components P_1, P_2, and P_3. Then $v(P_1) + v(P_2) + v(P_3)$ can have a local maximum only in one of the following configurations:*

- *Up to reordering the indices, first P_1 is chosen so that $v(P_1)$ is maximal regardless of P_2 and P_3, next P_2 is chosen so that $v(P_2)$ is maximal given P_1, and last P_3 is chosen so that $v(P_3)$ is maximal given P_1 and P_2;*
- *Each of P_1, P_2 and P_3 is tangent to the other two, and the modified volumes $\widetilde{v}(P_1)$, $\widetilde{v}(P_2)$ and $\widetilde{v}(P_3)$ satisfy the strict triangular inequalities.*

Moreover, if in the latter case no P_j is individually maximal, then the configuration indeed gives a local maximum for $v(P_1) + v(P_2) + v(P_3)$.

Proof. We already know from Proposition 2.6.9 that at a local maximum for $v(P_1) + v(P_2) + v(P_3)$ either some P_j is maximal or each P_j is tangent to each other P_i. In the former case, suppose that P_1 is maximal. If P_2 or P_3 is maximal given P_1 then up to switching P_2 and P_3 we have a configuration as described in the first item of the statement. Otherwise P_2 and P_3 are tangent to each other but not to themselves or to P_1, and it follows from Proposition 2.6.5 that the configuration cannot locally maximize the volume.

We are left to deal with the configuration in which each P_j is tangent to each other P_i but it is not individually maximal. Notice that in particular this implies that each P_j can be individually both inflated (since it is not tangent to itself) or shrinked (since a collar with width 0 would imply that some other peripheral component is maximal). In this case the local deformation we can perform is as follows:

- We can inflate P_1 and shrink P_2 and P_3 so that they stay tangent to P_1;

- We can shrink P_1, in which case we can further deform P_2 and P_3 in such a way that they stay tangent to each other. But Proposition 2.6.5 implies that along this deformation we cannot have a local maximum except at the extrema, namely when either P_2 or P_3 is tangent to P_1.

To analyze exactly how the volume behaves under this deformation we need to distinguish four cases, accordingly to the number of collars and cusps among the three peripheral components P_1, P_2 and P_3.

Case I: Three cusps Let $C_1(v_1^{(0)})$, $C_2(v_2^{(0)})$ and $C_3(v_3^{(0)})$ be the initial cusps with indices chosen so that $v_2^{(0)} \geqslant v_3^{(0)}$. We then let v_1 vary in a neighborhood of $v_1^{(0)}$ and note that, according to the above description of the deformation, for $v_1 \leqslant v_1^{(0)}$ the total deformed volume is given by v_1 plus

$$\max\left\{ \frac{v_1^{(0)} \cdot v_2^{(0)}}{v_1} + \frac{v_3^{(0)}}{v_1^{(0)}} \cdot v_1,\ \frac{v_2^{(0)}}{v_1^{(0)}} \cdot v_1 + \frac{v_1^{(0)} \cdot v_3^{(0)}}{v_1} \right\},$$

but the assumption $v_2^{(0)} \geqslant v_3^{(0)}$ readily implies that the maximum is given by the first expression. Therefore near $v_1^{(0)}$ the total deformed volume is

$$V(v_1) = \begin{cases} v_1 + \dfrac{v_1^{(0)} \cdot v_2^{(0)}}{v_1} + \dfrac{v_3^{(0)}}{v_1^{(0)}} \cdot v_1 & \text{for} \quad v_1 \leqslant v_1^{(0)} \\ \\ v_1 + \dfrac{v_1^{(0)} \cdot v_2^{(0)}}{v_1} + \dfrac{v_1^{(0)} \cdot v_3^{(0)}}{v_1} & \text{for} \quad v_1 \geqslant v_1^{(0)} \end{cases}$$

and its derivatives are

$$V'(v_1) = \begin{cases} 1 - \dfrac{v_1^{(0)} \cdot v_2^{(0)}}{v_1^2} + \dfrac{v_3^{(0)}}{v_1^{(0)}} & \text{for} \quad v_1 \leqslant v_1^{(0)} \\ \\ 1 - \dfrac{v_1^{(0)} \cdot v_2^{(0)}}{v_1^2} - \dfrac{v_1^{(0)} \cdot v_3^{(0)}}{v_1^2} & \text{for} \quad v_1 \geqslant v_1^{(0)} \end{cases}$$

$$V''(v_1) = \begin{cases} \dfrac{2v_1^{(0)} \cdot v_2^{(0)}}{v_1^3} & \text{for} \quad v_1 \leqslant v_1^{(0)} \\ \\ \dfrac{2v_1^{(0)} \cdot (v_2^{(0)} + v_3^{(0)})}{v_1^3} & \text{for} \quad v_1 \geqslant v_1^{(0)} \end{cases}$$

Since

$$V''_-(v_1^{(0)}) = \frac{2v_2^{(0)}}{\big(v_1^{(0)}\big)^2} > 0 \quad \text{and} \quad V''_+(v_1^{(0)}) = \frac{2(v_2^{(0)} + v_3^{(0)})}{\big(v_1^{(0)}\big)^2} > 0$$

then V has a local maximum at $v_1^{(0)}$ only if

$$V'_-(v_1^{(0)}) = 1 - \frac{v_2^{(0)}}{v_1^{(0)}} + \frac{v_3^{(0)}}{v_1^{(0)}} > 0 \quad \text{and} \quad V'_+(v_1^{(0)}) = 1 - \frac{v_2^{(0)}}{v_1^{(0)}} - \frac{v_3^{(0)}}{v_1^{(0)}} < 0$$

which are equivalent to the strict triangular inequalities

$$v_2^{(0)} - v_3^{(0)} < v_1^{(0)} < v_2^{(0)} + v_3^{(0)}$$

(notice that $v_2^{(0)} - v_3^{(0)} \geqslant 0$ by assumption).

Case II: One collar and two cusps We denote by Σ the boundary component and for $i = 2, 3$ by $\mathcal{A}(T_i)$ the areas relative to Σ of the tori T_2 and T_3, with indices chosen so that $\mathcal{A}(T_2) \geqslant \mathcal{A}(T_3)$. Suppose that the initial peripheral components are $U_1(d_1^{(0)})$, $C_2(v_2^{(0)})$ and $C_3(v_3^{(0)})$, with volumes

$$v_1^{(0)} = \frac{\mathcal{A}(\Sigma)}{4} \cdot (2d_1^{(0)} + \sinh(2d_1^{(0)})), \qquad v_i^{(0)} = \frac{\mathcal{A}(T_i)}{2} \cdot e^{-2d_1^{(0)}}$$

(notice that in particular we have $v_2^{(0)} \geqslant v_3^{(0)}$). Using d_1 to parameterize the deformation we have for $d_1 < d_1^{(0)}$ that the deformed total volume is given by $\frac{\mathcal{A}(\Sigma)}{4} \cdot (2d_1 + \sinh(2d_1))$ plus

$$\max\left\{\frac{\mathcal{A}(T_2)}{2} \cdot e^{-2d_1} + \frac{\mathcal{A}(T_3)}{2} \cdot e^{-2(2d_1^{(0)} - d_1)}, \right.$$
$$\left. \frac{\mathcal{A}(T_2)}{2} \cdot e^{-2(2d_1^{(0)} - d_1)} + \frac{\mathcal{A}(T_3)}{2} \cdot e^{-2d_1}\right\}$$

and the first expression prevails thanks to the assumption $\mathcal{A}(T_2) \geqslant \mathcal{A}(T_3)$.The deformed total volume is therefore

$$V(d_1) = \begin{cases} \frac{\mathcal{A}(\Sigma)}{4} \cdot (2d_1 + \sinh(2d_1)) + \frac{\mathcal{A}(T_2)}{2} \cdot e^{-2d_1} + \frac{\mathcal{A}(T_3)}{2} \cdot e^{2d_1 - 4d_1^{(0)}} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad d_1 \leqslant d_1^{(0)} \\ \frac{\mathcal{A}(\Sigma)}{4} \cdot (2d_1 + \sinh(2d_1)) + \frac{\mathcal{A}(T_2)}{2} \cdot e^{-2d_1} + \frac{\mathcal{A}(T_3)}{2} \cdot e^{-2d_1} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad d_1 \geqslant d_1^{(0)} \end{cases}$$

and we easily have

$$V_-'(d_1^{(0)}) = 2(\widetilde{v}_1^{(0)} - v_2^{(0)} + v_3^{(0)}), \qquad V_+'(d_1^{(0)}) = 2(\widetilde{v}_1^{(0)} - v_2^{(0)} - v_3^{(0)}).$$

Since $V_\pm''(d_1^{(0)}) > 0$, the conclusion follows precisely as in Case I.

Case III: Two collars and one cusp Let the peripheral components be $C_1(v_1^{(0)})$, $U_2(d_2^{(0)})$ and $U_3(d_3^{(0)})$, whence

$$v_1^{(0)} = \frac{\mathcal{A}_2(T_1)}{2} \cdot e^{-2d_2^{(0)}} = \frac{\mathcal{A}_3(T_1)}{2} \cdot e^{-2d_3^{(0)}},$$

and choose indices so that $\widetilde{v}_2^{(0)} \geqslant \widetilde{v}_3^{(0)}$. For the sake of brevity we now set

$$f_j(t) = \frac{\mathcal{A}(\Sigma_j)}{4}(2t + \sinh(2t))$$

so that $v(B_j(d_j)) = f_j(d_j)$, and $f_j'(d_j^{(0)}) = 2\widetilde{v}_j^{(0)}$. We deform the configuration using the parameter v_1, to do which we define the functions

$$d_j(v_1) = -\frac{1}{2}\log\frac{2v_1}{\mathcal{A}_j(T_1)},$$

noting that for $v_1 \geqslant v_1^{(0)}$ the peripheral configuration is given by

$$C_1(v_1), B_2(d_2(v_1)), B_3(d_3(v_1)).$$

For $v_1 < v_1^{(0)}$, on the contrary, we have one of the following:

- $d_2 = d_2(v_1)$ and $d_3 = d_3^{(0)} + d_2^{(0)} - d_2(v_1)$,
- $d_3 = d_3(v_1)$ and $d_2 = d_2^{(0)} + d_3^{(0)} - d_3(v_1)$.

The total deformed volume for $v_1 \leqslant v_1^{(0)}$ is then v_1 plus

$$\max\Big\{f_2(d_2(v_1)) + f_3(d_3^{(0)} + d_2^{(0)} - d_2(v_1)),\\ f_2(d_2^{(0)} + d_3^{(0)} - d_3(v_1)) + f_3(d_3(v_1))\Big\}$$

and the first expression prevails thanks to our assumption, because at the point $v_1 = v_1^0$ it has the same value as the second expression but smaller first derivative. Therefore

$$V(v_1) = \begin{cases} v_1 + f_2(d_2(v_1)) + f_3(d_3^{(0)} + d_2^{(0)} - d_2(v_1)) & \text{for} \quad v_1 \leqslant v_1^{(0)} \\ \\ v_1 + f_2(d_2(v_1)) + f_3(d_3(v_1)) & \text{for} \quad v_1 \geqslant v_1^{(0)} \end{cases}$$

whence

$$V_-'(v_1^{(0)}) = 1 - \frac{\widetilde{v}_2^{(0)}}{v_1^{(0)}} + \frac{\widetilde{v}_3^{(0)}}{v_1^{(0)}}, \qquad V_+'(v_1^{(0)}) = 1 - \frac{\widetilde{v}_2^{(0)}}{v_1^{(0)}} - \frac{\widetilde{v}_3^{(0)}}{v_1^{(0)}}$$

and precisely as in the previous two cases we conclude that we have a local maximum if and only if $v_1^{(0)}$, $\widetilde{v}_2^{(0)}$ and $\widetilde{v}_3^{(0)}$ satisfy the strict triangular inequalities.

Case IV: Three collars Let the boundary components be Σ_1, Σ_2 and Σ_3, and the initial configuration be $U_1(d_1^{(0)})$, $U_2(d_2^{(0)})$ and $U_3(d_3^{(0)})$ with $\widetilde{v}_2^{(0)} \geqslant \widetilde{v}_3^{(0)}$. Setting

$$f_j(t) = \frac{\mathcal{A}(\Sigma_j)}{4}(2t + \sinh(2t))$$

and using d_1 to parameterize the deformation we have that the deformed volume is given by

$$V(d_1) = \begin{cases} f_1(d_1) + f_2(d_2^{(0)} + d_1^{(0)} - d_1) + f_3(d_3^{(0)} - d_1^{(0)} + d_1) \\ \hfill \text{for} \quad d_1 \leqslant d_1^{(0)} \\ \\ f_1(d_1) + f_2(d_2^{(0)} + d_1^{(0)} - d_1) + f_3(d_3^{(0)} + d_1^{(0)} - d_1) \\ \hfill \text{for} \quad d_1 \geqslant d_1^{(0)} \end{cases}$$

This gives

$$V'_-(d_1^{(0)}) = 2(\widetilde{v}_1^{(0)} - \widetilde{v}_2^{(0)} + \widetilde{v}_3^{(0)}), \qquad V'_+(d_1^{(0)}) = 2(\widetilde{v}_1^{(0)} - \widetilde{v}_2^{(0)} - \widetilde{v}_3^{(0)})$$

and the conclusion is once again the same. $\square$

Appendix
The 32 manifolds with $c \leqslant 4$

Tables of empiric facts

The following table collects the main empiric facts regarding the 32 examples with $c \leqslant 4$ (the values displayed here are approximated at the second decimal for simplicity):

Group	A	B	C	D	E	F	G
#	1	12	2	2	2	12	1
c	3	4	4	4	4	4	4
∂-genus	2	3	2	2	2	2	2
# cusps	1	1	1	1	1	1	2
# edges	2	2	3	3	3	3	3
# tr.edges	1	1	2	2	1	2	1
# ideal.v	2	2	2	2	4	2	4
θ_2	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$1.08 \sim 1.28$	$\frac{\pi}{3}$
$\theta_1 = \theta_3$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi-\theta_2}{2}$	$\frac{\pi}{3}$
α_2	0.49	0.32	0.52	$\frac{\pi}{2}$	0.52	$0.74 \sim 1.11$	0.52
$\alpha_1 = \alpha_3$	0.49	0.32	0.52	0.32	0.52	$0.44 \sim 0.61$	0.52

The following table collects the results of some of the calculations carried out for the 32 examples with $c \leqslant 4$ (the values displayed here are approximated at the second decimal for simplicity):

Group	A	B	C	D	E	F	G
$\mathcal{A}_\partial$	4π	8π	4π	4π	4π	4π	4π
$d_{\max}$	0.32	0.19	0.34	0.34	0.35	$0.47 \sim 0.49$	0.35
$\bar{r}^2/R^2$	2.10	2.61	2.03, 2.01	0.80	2	$1.30 \sim 1.53$	2
$\mathcal{A}_\Delta/R^2$	4.03	4.69	3.93, 3.91	1.80	3.90	$2.85 \sim 3.29$	3.90/2
$V_{\max}$	4.68	5.52	4.93, 4.98	5.05	5.51	$6.77 \sim 7.13$	5.51
V_{tot}	7.80	11.81	8.45, 8.67	8.68	9.13	$9.29 \sim 9.77$	9.13
$\frac{V_{\max}}{V_{tot}}\%$	60%	47%	58%	58%	60%	$69\% \sim 77\%$	60%

Census notation

A census of the 32 hyperbolic 3-manifolds of complexity $c \leqslant 4$ with both nonempty geodesic boundary and at least one toric cusp can be found in [18]. In particular, the only manifold with $c = 3$ (*i.e.*, the one of group A) is the last manifold in the list `http://www.dm.unipi.it/pages/petronio/` `/public_html/files/3D/Geo_Bd/census_3.snp`, while the other 31 are the ones in the list `http://www.dm.unipi.it/pages/petronio/public_html/` `/files/3D/Geo_Bd/census_4_cusp.snp`.

The format in which these 32 manifolds and their triangulations are presented is the following one (we use the manifold with $c = 3$ as an example)

```
3
   0    0    1    2
 1230 3012 2103 3120
   0    2    2    2
 2103 0132 0321 0213
   0    1    1    1
 3120 0132 0321 0213
Angles :
N.0: 0.5538708696 0.5538708696 0.5538708696 0.5538708696 0.5538708696 0.5538708696
N.1: 0.4933266815 0.4933266815 0.4933266815 1.0471975512 1.0471975512 1.0471975512
N.2: 0.4933266815 0.4933266815 0.4933266815 1.0471975512 1.0471975512 1.0471975512
Volume: 7.7976368803  (3.1697912726, 2.3139228039, 2.3139228039)
Boundary: T^(2) T^(1)
Pi: <a, b, c|bcb^(-1)ac^(-1)b^(-1)a^(-2)b^(-1)c^(-1)ba^(2)bca^(-1)>
Homology: Z+Z+Z
-------------------
```

where there are, in order:

- an integer c, which is the number of tetrahedra of the given triangulation (these tetrahedra will be named $(0), \ldots, (c-1)$);
- a matrix with c rows (one for each tetrahedron $(0), \ldots, (c-1)$) and 4 columns (one for each of the vertices $0, \ldots, 3$ of a tetrahedron), whose entries represent the gluings of the triangulation;
- each (n, j)-entry of the matrix has a top integer m and a bottom line of 4 integers $v_0 v_1 v_2 v_3$ (which is a permutation of 0123), meaning that

the face of tetrahedron (n) opposite to vertex j is glued to the face of tetrahedron (m) opposite to vertex v_j via a gluing that identifies, for $i \neq j$, each vertex i of (n) with the vertex v_i of (m);

- a matrix with c rows (one for each tetrahedron $(0), \ldots, (c-1)$) and 6 columns (one for each of the edges of a tetrahedron, ordered as follows: $\overline{13}$, $\overline{23}$, $\overline{12}$, $\overline{02}$, $\overline{01}$, $\overline{03}$), whose (n, j)-entries are the dihedral angles along the j-th edge of tetrahedron (n); when considering a tetrahedron with at least one ideal vertex in the half-space model, dihedral angles listed here are ordered, with the usual notation, as α_2, α_1, α_3, θ_2, θ_1, θ_3;

- four lines with additional information (volume of the manifold as the sum of the volumes of the c tetrahedra, boundary type, presentation of Π_1, homology).

- a final line of separation (`--------------------`).

The complete census of these 32 manifolds is listed below.

List of manifolds

We recall here the census [18] of the manifolds we used as examples, *i.e* the 32 manifolds of complexity $c \leqslant 4$ with both nonempty geodesic boundary and one or more toric cusps.

Group A (1 manifold)

```
4
   1    1    1    2
 0213 2103 1023 1230
   0    0    0    3
 0213 2103 1023 1230
   0    3    3    3
 3012 2103 1230 1302
   1    2    2    2
 3012 2103 2031 3012
Angles :
N.  0:   1.0471975512   1.0471975512   0.3189729464   0.3189729464   0.3189729464   1.0471975512
N.  1:   1.0471975512   1.0471975512   0.3189729464   0.3189729464   0.3189729464   1.0471975512
N.  2:   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024
N.  3:   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024
Volume: 11.8126805424  (2.4471636914, 2.4471636914, 3.4591765798, 3.4591765798)
Boundary: T^(1) T^(3)
Pi: <a, b, c, d|bd^(-1)b^(-1)dc^(-1)d^(-1)ca^(-1)cab^(-1)c^(-1)ac^(-1)dcd^(-1)bda^(-1)>
Homology: Z+Z+Z+Z
-------------------
```

Group B (12 manifolds)

```
4
   1    1    1    2
 0213 2103 1023 1230
   0    0    0    3
 0213 2103 1023 1230
   0    3    3    3
 3012 2103 1230 1302
   1    2    2    2
 3012 2103 2031 3012
Angles :
N.  0:   1.0471975512   1.0471975512   0.3189729464   0.3189729464   0.3189729464   1.0471975512
N.  1:   1.0471975512   1.0471975512   0.3189729464   0.3189729464   0.3189729464   1.0471975512
N.  2:   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024
N.  3:   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024
Volume: 11.8126805424  (2.4471636914, 2.4471636914, 3.4591765798, 3.4591765798)
Boundary: T^(1) T^(3)
Pi: <a, b, c, d|bd^(-1)b^(-1)dc^(-1)d^(-1)ca^(-1)cab^(-1)c^(-1)ac^(-1)dcd^(-1)bda^(-1)>
Homology: Z+Z+Z+Z
-------------------
```

```
4
   0    0    1    1
 1230 3012 2103 2031
   0    0    2    3
 2103 1302 2103 3120
   1    3    3    3
 2103 0132 0321 0213
   1    2    2    2
 3120 0132 0321 0213
Angles :
N.  0:   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024
N.  1:   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024
N.  2:   0.3189729464   0.3189729464   0.3189729464   1.0471975512   1.0471975512   1.0471975512
N.  3:   0.3189729464   0.3189729464   0.3189729464   1.0471975512   1.0471975512   1.0471975512
Volume: 11.8126805424  (3.4591765798, 3.4591765798, 2.4471636914, 2.4471636914)
Boundary: T^(3) T^(1)
Pi: <a, b, c, d|dc^(-1)a^(2)ba^(-1)bcdcb^(-1)d^(-1)bc^(-1)d^(-1)c^(-1)b^(-1)ab^(-1)a^(-2)c>
Homology: Z+Z+Z+Z
-------------------
4
   1    1    1    2
 0213 2103 1023 1230
   0    0    0    3
 0213 2103 1023 1230
   0    3    3    3
 3012 2103 1230 0132
   1    2    2    2
 3012 2103 0132 3012
Angles :
N.  0:   1.0471975512   1.0471975512   0.3189729466   0.3189729466   0.3189729466   1.0471975512
N.  1:   1.0471975512   1.0471975512   0.3189729466   0.3189729466   0.3189729466   1.0471975512
N.  2:   0.3641123023   0.3641123023   0.3641123023   0.3641123023   0.3641123023   0.3641123023
N.  3:   0.3641123023   0.3641123023   0.3641123023   0.3641123023   0.3641123023   0.3641123023
Volume: 11.8126805424  (2.4471636913, 2.4471636913, 3.4591765799, 3.4591765799)
Boundary: T^(1) T^(3)
Pi: <a, b, c, d|dc^(-2)ac^(-1)d^(2)b^(-2)abd^(-2)ca^(-1)c^(2)d^(-1)ba^(-1)>
Homology: Z+Z+Z+Z
-------------------
4
   0    0    1    1
 1230 3012 2103 2031
   0    0    2    3
 2103 1302 2103 2310
   1    3    3    3
 2103 0132 0321 0213
   1    2    2    2
 3201 0132 0321 0213
Angles :
N.  0:   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024
N.  1:   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024
N.  2:   0.3189729464   0.3189729464   0.3189729464   1.0471975512   1.0471975512   1.0471975512
N.  3:   0.3189729464   0.3189729464   0.3189729464   1.0471975512   1.0471975512   1.0471975512
Volume: 11.8126805424  (3.4591765798, 3.4591765798, 2.4471636914, 2.4471636914)
Boundary: T^(3) T^(1)
Pi: <a, b, c, d|dc^(-1)a^(2)ba^(-1)bcdb^(-1)c^(-1)d^(-1)cbd^(-1)c^(-1)b^(-1)ab^(-1)a^(-2)c>
Homology: Z+Z+Z+Z
-------------------
```

```
4
   0    0    1    1
 1230 3012 2103 3201
   0    0    2    3
 2103 2310 2103 3120
   1    3    3    3
 2103 0132 0321 0213
   1    2    2    2
 3120 0132 0321 0213
Angles :
N.  0:   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024
N.  1:   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024
N.  2:   0.3189729464   0.3189729464   0.3189729464   1.0471975512   1.0471975512   1.0471975512
N.  3:   0.3189729464   0.3189729464   0.3189729464   1.0471975512   1.0471975512   1.0471975512
Volume: 11.8126805424  (3.4591765798, 3.4591765798, 2.4471636914, 2.4471636914)
Boundary: T^(3) T^(1)
Pi: <a, b, c, d|dc^(-1)a^(-1)bcdcb^(-2)a^(-2)d^(-1)a^(2)b^(2)c^(-1)d^(-1)c^(-1)b^(-1)ac>
Homology: Z+Z+Z+Z
-------------------
4
   0    0    1    1
 1230 3012 2103 3201
   0    0    2    3
 2103 2310 2103 2310
   1    3    3    3
 2103 0132 0321 0213
   1    2    2    2
 3201 0132 0321 0213
Angles :
N.  0:   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024
N.  1:   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024
N.  2:   0.3189729464   0.3189729464   0.3189729464   1.0471975512   1.0471975512   1.0471975512
N.  3:   0.3189729464   0.3189729464   0.3189729464   1.0471975512   1.0471975512   1.0471975512
Volume: 11.8126805424  (3.4591765798, 3.4591765798, 2.4471636914, 2.4471636914)
Boundary: T^(3) T^(1)
Pi: <a, b, c, d|dc^(-1)a^(-1)bcdb^(-2)a^(-2)c^(-1)d^(-1)ca^(2)b^(2)d^(-1)c^(-1)b^(-1)ac>
Homology: Z+Z+Z+Z
-------------------
4
   0    0    1    1
 1230 3012 1302 3201
   0    0    2    3
 2031 2310 2103 2310
   1    3    3    3
 2103 0132 0321 0213
   1    2    2    2
 3201 0132 0321 0213
Angles :
N.  0:   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024
N.  1:   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024
N.  2:   0.3189729464   0.3189729464   0.3189729464   1.0471975512   1.0471975512   1.0471975512
N.  3:   0.3189729464   0.3189729464   0.3189729464   1.0471975512   1.0471975512   1.0471975512
Volume: 11.8126805424  (3.4591765798, 3.4591765798, 2.4471636914, 2.4471636914)
Boundary: T^(3) T^(1)
Pi: <a, b, c, d|dc^(-1)a^(-1)b^(2)cdb^(-1)a^(-2)c^(-1)d^(-1)ca^(2)bd^(-1)c^(-1)b^(-2)ac>
Homology: Z+Z+Z+Z
-------------------
```

```
4
   1    1    1    2
 0321 2103 1023 1230
   0    0    0    3
 0321 2103 1023 3120
   0    3    3    3
 3012 0132 0321 0213
   1    2    2    2
 3120 0132 0321 0213
Angles :
N.  0:   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024
N.  1:   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024
N.  2:   0.3189729464   0.3189729464   0.3189729464   1.0471975512   1.0471975512   1.0471975512
N.  3:   0.3189729464   0.3189729464   0.3189729464   1.0471975512   1.0471975512   1.0471975512
Volume: 11.8126805424  (3.4591765798, 3.4591765798, 2.4471636914, 2.4471636914)
Boundary: T^(3) T^(1)
Pi: <a, b, c, d|dab^(-1)ca^(-1)cb^(-1)acd^(-1)bd^(-1)b^(-1)dc^(-1)a^(-1)bc^(-1)ac^(-1)ba^(-1)>
Homology: Z+Z+Z+Z
-------------------
4
   1    1    1    2
 0213 2103 1023 1230
   0    0    0    3
 0213 2103 1023 1230
   0    3    3    3
 3012 2103 2031 0132
   1    2    2    2
 3012 2103 0132 1302
Angles :
N.  0:   1.0471975512   1.0471975512   0.3189729464   0.3189729464   0.3189729464   1.0471975512
N.  1:   1.0471975512   1.0471975512   0.3189729464   0.3189729464   0.3189729464   1.0471975512
N.  2:   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024
N.  3:   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024
Volume: 11.8126805424  (2.4471636914, 2.4471636914, 3.4591765798, 3.4591765798)
Boundary: T^(1) T^(3)
Pi: <a, b, c, d|d^(2)c^(-2)ac^(-1)db^(-2)abd^(-1)ca^(-1)c^(2)d^(-2)ba^(-1)>
Homology: Z+Z+Z+Z
-------------------
4
   0    0    1    1
 1230 3012 1302 3201
   0    0    2    3
 2031 2310 2103 3120
   1    3    3    3
 2103 0132 0321 0213
   1    2    2    2
 3120 0132 0321 0213
Angles :
N.  0:   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024
N.  1:   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024   0.3641123024
N.  2:   0.3189729464   0.3189729464   0.3189729464   1.0471975512   1.0471975512   1.0471975512
N.  3:   0.3189729464   0.3189729464   0.3189729464   1.0471975512   1.0471975512   1.0471975512
Volume: 11.8126805424  (3.4591765798, 3.4591765798, 2.4471636914, 2.4471636914)
Boundary: T^(3) T^(1)
Pi: <a, b, c, d|dc^(-1)a^(-1)b^(2)cdcb^(-1)a^(-2)d^(-1)a^(2)bc^(-1)d^(-1)c^(-1)b^(-2)ac>
Homology: Z+Z+Z+Z
-------------------
```

```
4
   1    1    1    2
 0321 2103 3120 1230
   0    0    0    3
 0321 2103 3120 3120
   0    3    3    3
 3012 0132 0321 0213
   1    2    2    2
 3120 0132 0321 0213
Angles :
N.  0:   0.3641122970   0.3641122970   0.3641122970   0.3641122970   0.3641122970   0.3641122970
N.  1:   0.3641122970   0.3641122970   0.3641122970   0.3641122970   0.3641122970   0.3641122970
N.  2:   0.3189729434   0.3189729434   0.3189729434   1.0471975362   1.0471975362   1.0471975362
N.  3:   0.3189729434   0.3189729434   0.3189729434   1.0471975362   1.0471975362   1.0471975362
Volume: 11.8126810072  (3.4591765861, 3.4591765861, 2.4471639175, 2.4471639175)
Boundary: T^(3) T^(1)
Pi: <a, b, c, d|dab^(-1)cb^(-1)aca^(-1)cd^(-1)bd^(-1)b^(-1)dc^(-1)ac^(-1)a^(-1)bc^(-1)ba^(-1)>
Homology: Z+Z+Z+Z
-------------------
4
   1    1    1    2
 0321 0132 3120 1230
   0    0    0    3
 0321 0132 3120 3120
   0    3    3    3
 3012 0132 0321 0213
   1    2    2    2
 3120 0132 0321 0213
Angles :
N.  0:   0.3641122966   0.3641122966   0.3641122966   0.3641122966   0.3641122966   0.3641122966
N.  1:   0.3641122966   0.3641122966   0.3641122966   0.3641122966   0.3641122966   0.3641122966
N.  2:   0.3189729430   0.3189729430   0.3189729430   1.0471975358   1.0471975358   1.0471975358
N.  3:   0.3189729430   0.3189729430   0.3189729430   1.0471975358   1.0471975358   1.0471975358
Volume: 11.8126810191  (3.4591765866, 3.4591765866, 2.4471639230, 2.4471639230)
Boundary: T^(3) T^(1)
Pi: <a, b, c, d|dab^(-1)aca^(-1)cd^(-1)bc^(-1)bd^(-1)b^(-1)cb^(-1)dc^(-1)ac^(-1)a^(-1)ba^(-1)>
Homology: Z+Z+Z+Z
-------------------
```

Group C (2 manifolds)

```
4
   0    0    1    1
 1230 3012 1302 2031
   0    0    2    3
 2031 1302 2103 3120
   1    3    3    3
 2103 0132 0321 0213
   1    2    2    2
 3120 0132 0321 0213
Angles :
N.  0:   2.3919116485   0.1665378952   0.1665378952   2.3919116485   0.1665378952   0.1665378952
N.  1:   0.4302348768   1.4993620101   0.4302348768   0.4302348768   0.7993453836   0.4302348768
N.  2:   0.5161248059   0.5161248059   0.5161248059   1.0471975512   1.0471975512   1.0471975512
N.  3:   0.5161248059   0.5161248059   0.5161248059   1.0471975512   1.0471975512   1.0471975512
Volume: 8.4466552626  (1.1884298766, 2.6755909824, 2.2913172018, 2.2913172018)
```

```
Boundary: T^(2) T^(1)
Pi: <a, b, c|cb^(-1)a^(3)b^(-1)c^(-1)b^(-1)a^(-1)c^(-1)abcba^(-3)b>
Homology: Z+Z+Z
-------------------
4
   0    0    1    1
 1230 3012 2103 0321
   0    0    2    3
 2103 0321 2103 3120
   1    3    3    3
 2103 0132 0321 0213
   1    2    2    2
 3120 0132 0321 0213
Angles :
N.  0:   0.1905721283   0.6432770029   2.0140270696   0.1905721283   0.2892990512   2.0140270696
N.  1:   0.4088676288   1.6118541650   0.4088676288   0.4088676288   0.8548230402   0.4088676288
N.  2:   0.5204080740   0.5204080740   0.5204080740   1.0471975512   1.0471975512   1.0471975512
N.  3:   0.5204080740   0.5204080740   0.5204080740   1.0471975512   1.0471975512   1.0471975512
Volume: 8.6707518039  (1.5179560783, 2.5789500207, 2.2869228525, 2.2869228525)
Boundary: T^(2) T^(1)
Pi: <a, b, c|bcb^(-1)a^(3)c^(-1)b^(-1)a^(-2)b^(-1)c^(-1)ba^(2)bca^(-3)>
Homology: Z+Z+Z
-------------------
```

Group D (2 manifolds)

```
4
   0    0    1    2
 1230 3012 2103 1230
   0    1    1    3
 2103 1230 3012 3120
   0    3    3    3
 3012 0132 0321 0213
   1    2    2    2
 3120 0132 0321 0213
Angles :
N.  0:   0.4173085058   0.8346170116   0.4173085058   0.4173085058   1.5707963268   0.4173085058
N.  1:   0.4173085058   0.4173085058   1.5707963268   0.4173085058   0.4173085058   0.8346170116
N.  2:   0.3188708094   0.3188708094   1.5707963268   0.7853981634   0.7853981634   1.5707963268
N.  3:   0.3188708094   0.3188708094   1.5707963268   0.7853981634   0.7853981634   1.5707963268
Volume: 8.6817371548  (2.6143188561, 2.6143188561, 1.7265497213, 1.7265497213)
The canonical decomposition consists of a two pyramids, and is obtained from this
triangulation by merging together some tetrahedra.
Boundary: T^(2) T^(1)
Pi: <a, b, c|ca^(2)b^(2)c^(-1)a^(-1)b^(-1)c^(-1)bacb^(-2)a^(-2)>
Homology: Z+Z+Z
-------------------
4
   0    0    1    2
 1230 3012 2103 1230
   0    1    1    3
 2103 3201 2310 3120
   0    3    3    3
 3012 0132 0321 0213
   1    2    2    2
 3120 0132 0321 0213
```

```
Angles :
N.  0:   0.4173085058   0.8346170116   0.4173085058   0.4173085058   1.5707963268   0.4173085058
N.  1:   0.4173085058   0.4173085058   1.5707963268   0.4173085058   0.4173085058   0.8346170116
N.  2:   0.3188708094   0.3188708094   1.5707963268   0.7853981634   0.7853981634   1.5707963268
N.  3:   0.3188708094   0.3188708094   1.5707963268   0.7853981634   0.7853981634   1.5707963268
Volume: 8.6817371548  (2.6143188561, 2.6143188561, 1.7265497213, 1.7265497213)
The canonical decomposition consists of two pyramids, and is obtained from this
triangulation by merging together some tetrahedra.
Boundary: T^(2) T^(1)
Pi: <a, b, c|ca^(2)bc^(-1)a^(-1)b^(-2)c^(-1)b^(2)acb^(-1)a^(-2)>
Homology: Z+Z+Z
-------------------
```

Group E (2 manifolds)

```
4
   1    1    2    3
 1230 2031 2103 2310
   0    0    2    3
 1302 3012 2310 3201
   0    1    3    3
 2103 3201 1230 1302
   0    1    2    2
 3201 2310 2031 3012
Angles :
N.  0:   0.5235987756   1.0471975512   1.0471975512   1.0471975512   0.5235987756   0.5235987756
N.  1:   1.0471975512   1.0471975512   0.5235987756   0.5235987756   0.5235987756   1.0471975512
N.  2:   0.5235987756   0.5235987756   0.5235987756   1.0471975512   1.0471975512   1.0471975512
N.  3:   1.0471975512   0.5235987756   1.0471975512   0.5235987756   1.0471975512   0.5235987756
Volume: 9.1344744577  (2.2836186144, 2.2836186144, 2.2836186144, 2.2836186144)
Boundary: T^(1) T^(2)
Pi: <a, b, c, d|bdcda^(-1)bc^(-1)bca^(-1), dad^(-1)a^(-1)>
Homology: Z+Z+Z
-------------------
4
   1    1    2    3
 1230 2031 2103 2310
   0    0    2    3
 1302 3012 2310 2031
   0    1    3    3
 2103 3201 1230 1302
   0    1    2    2
 3201 1302 2031 3012
Angles :
N.  0:   0.5235987680   1.0471975358   1.0471975358   1.0471975358   0.5235987680   0.5235987680
N.  1:   1.0471975358   1.0471975358   0.5235987680   0.5235987680   0.5235987680   1.0471975358
N.  2:   0.5235987680   0.5235987680   0.5235987680   1.0471975358   1.0471975358   1.0471975358
N.  3:   1.0471975358   0.5235987680   1.0471975358   0.5235987680   1.0471975358   0.5235987680
Volume: 9.1344754117  (2.2836188529, 2.2836188529, 2.2836188529, 2.2836188529)
Boundary: T^(1) T^(2)
Pi: <a, b, c, d|bdcb^(-1)ad^(-1)c^(-1)b^(-1)ca^(-1), dad^(-1)a^(-1)>
Homology: Z+Z+Z
-------------------
```

Group F (12 manifolds)

```
4
   0    0    1    2
 1230 3012 1302 1230
   0    1    1    3
 2031 1230 3012 2310
   0    3    3    3
 3012 0132 0321 0213
   1    2    2    2
 3201 0132 0321 0213
Angles :
N.  0:   0.5254197471   0.7334249582   0.5254197471   0.5254197471   1.1795651803   0.5254197471
N.  1:   1.4438483960   0.4150801503   0.4150801503   1.4438483960   0.4150801503   0.4150801503
N.  2:   0.4469401898   0.4469401898   1.1079616674   0.9346761772   0.9346761772   1.2722402991
N.  3:   0.4469401898   0.4469401898   1.1079616674   0.9346761772   0.9346761772   1.2722402991
Volume: 9.2854556585  (2.8244165990, 2.4095600064, 2.0257395265, 2.0257395265)
Boundary: T^(2) T^(1)
Pi: <a, b, c|cba^(2)c^(-1)a^(-1)b^(2)c^(-1)b^(-3)aca^(-2)>
Homology: Z+Z+Z
-------------------
4
   0    0    1    2
 1230 3012 3201 1230
   0    1    1    3
 2310 3201 2310 1230
   0    3    3    3
 3012 0132 0321 0213
   1    2    2    2
 3012 0132 0321 0213
Angles :
N.  0:   0.5254197471   0.7334249579   0.5254197471   0.5254197471   1.1795651803   0.5254197471
N.  1:   0.4150801505   1.4438483958   0.4150801505   0.4150801505   1.4438483958   0.4150801505
N.  2:   0.4469401897   0.4469401897   1.1079616677   0.9346761772   0.9346761772   1.2722402993
N.  3:   0.4469401897   0.4469401897   1.1079616677   0.9346761772   0.9346761772   1.2722402993
Volume: 9.2854556585  (2.8244165992, 2.4095600065, 2.0257395264, 2.0257395264)
Boundary: T^(2) T^(1)
Pi: <a, b, c|cba^(2)b^(2)c^(-1)a^(-1)c^(-1)b^(-1)acb^(-2)a^(-2)>
Homology: Z+Z+Z
-------------------
4
   0    0    1    2
 1230 3012 1302 1230
   0    1    1    3
 2031 1230 3012 1230
   0    3    3    3
 3012 0132 0321 0213
   1    2    2    2
 3012 0132 0321 0213
Angles :
N.  0:   0.4419055754   0.8483967875   1.1477047520   0.4419055754   0.5465982532   1.1477047520
N.  1:   0.5538531560   0.5538531560   1.0966724342   0.5538531560   0.5538531560   0.7247678542
N.  2:   0.4781488562   1.0213532908   0.4781488562   0.9579602985   1.2256720565   0.9579602985
N.  3:   0.4781488562   1.0213532908   0.4781488562   0.9579602985   1.2256720565   0.9579602985
Volume: 9.4924679767  (2.5246108797, 2.8426456311, 2.0626057329, 2.0626057329)
Boundary: T^(2) T^(1)
Pi: <a, b, c|a^(3)ba^(-1)bc^(2)bc^(-1)b^(-1)a^(-2)cb^(-1)c^(-2)b^(-1)>
```

```
Homology: Z+Z+Z
-------------------
4
   0    0    1    2
 1230 3012 1302 1230
   0    1    1    3
 2031 3201 2310 1230
   0    3    3    3
 3012 0132 0321 0213
   1    2    2    2
 3012 0132 0321 0213
Angles :
N.  0:   0.4419055733   0.8483967802   1.1477047587   0.4419055733   0.5465982519   1.1477047587
N.  1:   0.5538531559   0.5538531559   1.0966724352   0.5538531559   0.5538531559   0.7247678636
N.  2:   0.4781488554   1.0213532872   0.4781488554   0.9579602993   1.2256720550   0.9579602993
N.  3:   0.4781488554   1.0213532872   0.4781488554   0.9579602993   1.2256720550   0.9579602993
Volume: 9.4924679767  (2.5246108777, 2.8426456270, 2.0626057360, 2.0626057360)
Boundary: T^(2) T^(1)
Pi: <a, b, c|a^(3)ba^(-1)bcbc^(-2)b^(-1)a^(-2)c^(2)b^(-1)c^(-1)b^(-1)>
Homology: Z+Z+Z
-------------------
4
   0    0    1    2
 1230 3012 2103 1230
   0    1    1    3
 2103 1230 3012 1230
   0    3    3    3
 3012 0132 0321 0213
   1    2    2    2
 3012 0132 0321 0213
Angles :
N.  0:   1.1121364183   0.5273755870   0.5273755870   1.1121364183   0.5273755870   0.5273755870
N.  1:   1.1121364183   0.5273755870   0.5273755870   1.1121364183   0.5273755870   0.5273755870
N.  2:   0.9173198170   0.5160451529   0.5160451529   1.1734467587   0.9840729475   0.9840729475
N.  3:   0.9173198170   0.5160451529   0.5160451529   1.1734467587   0.9840729475   0.9840729475
Volume: 9.5638576273  (2.6817708997, 2.6817708997, 2.1001579139, 2.1001579139)
Boundary: T^(2) T^(1)
Pi: <a, b, c|cb^(-1)a^(3)cb^(2)acbc^(-2)a^(-1)b^(-2)c^(-1)a^(-3)>
Homology: Z+Z+Z
-------------------
4
   0    0    1    2
 1230 3012 1302 1230
   0    1    1    3
 2031 3201 2310 3120
   0    3    3    3
 3012 0132 0321 0213
   1    2    2    2
 3120 0132 0321 0213
Angles :
N.  0:   1.1121364184   0.5273755870   0.5273755870   1.1121364184   0.5273755870   0.5273755870
N.  1:   0.5273755870   1.1121364184   0.5273755870   0.5273755870   1.1121364184   0.5273755870
N.  2:   0.9173198167   0.5160451529   0.5160451529   1.1734467586   0.9840729475   0.9840729475
N.  3:   0.9173198167   0.5160451529   0.5160451529   1.1734467586   0.9840729475   0.9840729475
Volume: 9.5638576273  (2.6817708996, 2.6817708996, 2.1001579140, 2.1001579140)
Boundary: T^(2) T^(1)
Pi: <a, b, c|cba^(3)cacb^(-3)c^(-1)b^(2)c^(-1)a^(-1)c^(-1)a^(-3)>
```

```
Homology: Z+Z+Z
-------------------
4
   1    1    1    2
 0213 2103 1023 1230
   0    0    0    3
 0213 2103 1023 1230
   0    3    3    3
 3012 2103 1230 3012
   1    2    2    2
 3012 2103 1230 3012
Angles :
N.  0:   1.1734467389   0.9840729348   0.5160451461   0.9173198077   0.5160451461   0.9840729348
N.  1:   1.1734467389   0.9840729348   0.5160451461   0.9173198077   0.5160451461   0.9840729348
N.  2:   1.1121364004   0.5273755795   0.5273755795   1.1121364004   0.5273755795   0.5273755795
N.  3:   1.1121364004   0.5273755795   0.5273755795   1.1121364004   0.5273755795   0.5273755795
Volume: 9.5638581701  (2.1001581513, 2.1001581513, 2.6817709337, 2.6817709337)
Boundary: T^(1) T^(2)
Pi: <a, b, c|bc^(-1)b^(-1)c^(-1)abcacb^(-1)c^(-1)b^(-1)a^(-1)cba^(-1)>
Homology: Z+Z+Z
-------------------
4
   0    0    1    2
 1230 3012 1302 1230
   0    1    1    3
 2031 1230 3012 3120
   0    3    3    3
 3012 0132 0321 0213
   1    2    2    2
 3120 0132 0321 0213
Angles :
N.  0:   0.9802052213   0.5819371658   0.5819371658   0.9802052213   0.5819371658   0.5819371658
N.  1:   0.5571978721   0.9258445704   0.6269338213   0.5571978721   0.9258445704   0.8042663856
N.  2:   0.8334096691   0.5535267696   0.5535267696   1.1328085371   1.0043920582   1.0043920582
N.  3:   0.8334096691   0.5535267696   0.5535267696   1.1328085371   1.0043920582   1.0043920582
Volume: 9.6926928756  (2.7497274073, 2.7039857762, 2.1194898461, 2.1194898461)
Boundary: T^(2) T^(1)
Pi: <a, b, c|cb^(2)a^(3)cacb^(-3)c^(-1)bc^(-1)a^(-1)c^(-1)a^(-3)>
Homology: Z+Z+Z
-------------------
4
   0    0    1    2
 1230 3012 2103 1230
   0    1    1    3
 2103 3201 2310 1230
   0    3    3    3
 3012 0132 0321 0213
   1    2    2    2
 3012 0132 0321 0213
Angles :
N.  0:   0.9802052215   0.5819371658   0.5819371658   0.9802052215   0.5819371658   0.5819371658
N.  1:   0.9258445705   0.5571978721   0.6269338214   0.9258445705   0.5571978721   0.8042663855
N.  2:   0.8334096689   0.5535267696   0.5535267696   1.1328085370   1.0043920583   1.0043920583
N.  3:   0.8334096689   0.5535267696   0.5535267696   1.1328085370   1.0043920583   1.0043920583
Volume: 9.6926928756  (2.7497274071, 2.7039857761, 2.1194898462, 2.1194898462)
Boundary: T^(2) T^(1)
Pi: <a, b, c|cb^(-2)a^(3)cbacb^(2)c^(-2)a^(-1)b^(-1)c^(-1)a^(-3)>
```

```
Homology: Z+Z+Z
-------------------
4
   1    1    1    2
 0213 2103 1023 1230
   0    0    0    3
 0213 2103 1023 1230
   0    3    3    3
 3012 2103 2031 1302
   1    2    2    2
 3012 2103 2031 1302
Angles :
N.  0:   1.0886370447   1.0264778045   0.6017639619   0.7404718930   0.6017639619   1.0264778045
N.  1:   1.0886370447   1.0264778045   0.6017639619   0.7404718930   0.6017639619   1.0264778045
N.  2:   0.8180465355   0.6727152487   0.6326747405   0.8180465355   0.7650276897   0.6326747405
N.  3:   0.8180465355   0.6727152487   0.6326747405   0.8180465355   0.7650276897   0.6326747405
Volume: 9.7749394573  (2.1288385112, 2.1288385112, 2.7586312174, 2.7586312174)
Boundary: T^(1) T^(2)
Pi: <a, b, c|bc^(-2)ba^(-1)c^(-1)bcacb^(-1)c^(-1)b^(-1)cab^(-1)ca^(-1)>
Homology: Z+Z+Z
-------------------
4
   0    0    1    2
 1230 3012 2103 1230
   0    1    1    3
 2103 1230 3012 2310
   0    3    3    3
 3012 0132 0321 0213
   1    2    2    2
 3201 0132 0321 0213
Angles :
N.  0:   0.6326747404   0.7650276903   0.8180465343   0.6326747404   0.6727152475   0.8180465343
N.  1:   0.6326747404   0.8180465343   0.6727152475   0.6326747404   0.8180465343   0.7650276903
N.  2:   0.6017639627   0.7404718948   0.6017639627   1.0264778043   1.0886370451   1.0264778043
N.  3:   0.6017639627   0.7404718948   0.6017639627   1.0264778043   1.0886370451   1.0264778043
Volume: 9.7749394573  (2.7586312190, 2.7586312190, 2.1288385096, 2.1288385096)
Boundary: T^(2) T^(1)
Pi: <a, b, c|cbaca^(2)b^(2)c^(-2)a^(-1)b^(-3)a^(-2)>
Homology: Z+Z+Z
-------------------
4
   0    0    1    2
 1230 3012 3201 1230
   0    1    1    3
 2310 3201 2310 3120
   0    3    3    3
 3012 0132 0321 0213
   1    2    2    2
 3120 0132 0321 0213
Angles :
N.  0:   0.6326747333   0.7650276815   0.8180465246   0.6326747333   0.6727152396   0.8180465246
N.  1:   0.8180465246   0.6326747333   0.6727152396   0.8180465246   0.6326747333   0.7650276815
N.  2:   0.6017639563   0.7404718858   0.6017639563   1.0264777921   1.0886370299   1.0264777921
N.  3:   0.6017639563   0.7404718858   0.6017639563   1.0264777921   1.0886370299   1.0264777921
Volume: 9.7749399255  (2.7586312434, 2.7586312434, 2.1288387194, 2.1288387194)
Boundary: T^(2) T^(1)
Pi: <a, b, c|cba^(2)c^(-1)a^(-1)bc^(-1)a^(-1)bc^(-1)b^(-2)acb^(-1)aca^(-2)>
```

```
Homology: Z+Z+Z
-------------------
```

Group G (1 manifold)

```
4
   1    1    1    2
 0213 2103 1023 1230
   0    0    0    3
 0213 2103 1023 3120
   0    3    3    3
 3012 0132 0321 0213
   1    2    2    2
 3120 0132 0321 0213
Angles :
N.  0:   1.0471975358   1.0471975358   0.5235987680   0.5235987680   0.5235987680   1.0471975358
N.  1:   1.0471975358   1.0471975358   0.5235987680   0.5235987680   0.5235987680   1.0471975358
N.  2:   0.5235987680   0.5235987680   0.5235987680   1.0471975358   1.0471975358   1.0471975358
N.  3:   0.5235987680   0.5235987680   0.5235987680   1.0471975358   1.0471975358   1.0471975358
Volume: 9.1344754117  (2.2836188529, 2.2836188529, 2.2836188529, 2.2836188529)
Boundary: T^(1) T^(2) T^(1)
Pi: <a, b, c, d|bc^(-1)ab^(-1)ca^(-1), dacd^(-1)bd^(-1)b^(-1)dc^(-1)a^(-1)>
Homology: Z+Z+Z+Z
-------------------
```

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