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Coxeter groups: statistics and Kazhdan-Lusztig polynomials

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Coxeter groups: statistics and Kazhdan–Lusztig polynomials

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Contents

Introduction **i**

1 Coxeter groups and the symmetric group **1**
- 1.1 Coxeter groups 1
- 1.2 Finite Coxeter groups and affine Weyl groups 4
- 1.3 Combinatorial interpretation of finite Coxeter groups and affine Weyl groups 7
- 1.4 Statistics on permutation groups 10

2 Excedance statistics **15**
- 2.1 Statistics on signed permutation groups 16
  - 2.1.1 Colored excedance statistic 17
  - 2.1.2 Absolute excedance statistic 24
- 2.2 Statistics on the Coxeter group S_n^D 25
- 2.3 Signed excedance statistics on finite permutation groups 27
- 2.4 Excedance statistics on affine Weyl groups 33
  - 2.4.1 Colored excedance on the affine Weyl groups 34
  - 2.4.2 Absolute excedance for the affine Weyl groups 39
  - 2.4.3 Signed excedances in affine Weyl groups 43

3 Mahonian statistics **50**
- 3.1 Mahonian statistic on the affine permutation group 50
- 3.2 A bijection 54
- 3.3 An approach with n-cores and abaci models 56
- 3.4 Affine signed permutations 60

4 KL-polynomials of boolean elements **65**
- 4.1 R-polynomials and Kazhdan–Lusztig polynomials 65
- 4.2 Parabolic Kazhdan–Lusztig polynomials 67
- 4.3 Main results 69

4.4 Kazhdan–Lusztig polynomials of boolean signed permutation 83
4.5 Poincaré polynomials 87

5 KL polynomials of quasi-minuscule quotients 91
5.1 Properties of quasi-minuscule quotients of type B and D 91
5.2 Hasse diagrams 95
5.3 Kazhdan–Lusztig polynomials 101
5.4 Applications 107

Introduction

Permutation statistics were first introduced by Euler [Eul13] and then extensively studied by MacMahon [Mac15]. In the last decades much progress has been made, both in the discovery and the study of new statistics, and in extending these to other groups and arbitrary words with repeated letters.

MacMahon considered four different statistics for a permutation σ in the group of all permutation S_n of the set $\{1, \ldots, n\}$. The number of descents $\mathrm{des}(\sigma)$, the number of excedances $\mathrm{exc}(\sigma)$, the number of inversions $\mathrm{inv}(\sigma)$ and the major index $\mathrm{maj}(\sigma)$. Given a permutation $\sigma = [\sigma_1, \ldots, \sigma_n]$, with $\sigma(i) = \sigma_i$, we say that the pair $(i, j) \in \{1, \ldots, n\}^2$ is an inversion of σ if $i < j$ and $\sigma_i > \sigma_j$, that $i \in \{1, \ldots, n-1\}$ is a descent if $\sigma_i > \sigma_{i+1}$ and that $i \in \{1, \ldots, n\}$ is an excedance if $\sigma_i > i$. The major index is the sum of all the descents. In the literature all these statistics have been enumerated. The generating function of the descent statistic is given by the Eulerian polynomial. MacMahon showed that excedances are *equidistrubuted* with descents and that the number of inversions is equidistributed with the major index. Therefore the generating function of the excedance statistic is yet the Eulerian polynomial. Any statistic equidistributed with descents is said to be *Eulerian* and any statistic equidistributed with inversions is said to be *Mahonian*. Most of the permutation statistics found in the literature fall into one of these two categories. In the last decades, the joint distribution of more statistics have been computed (see e. g. [SW07] and the references cited there).

In combinatorics, the subset of all *derangements* of the set of permutations have received an increasing interest. It is the set of all permutations σ which have no fixed points, i. e. $\sigma(i) \neq i$ for all $i \in \{1, \ldots, n\}$. Although its enumeration can be computed by the classical inversion formula (see e. g. [Sta97, Section 2]), many authors have studied properties of its generating functions ([Bre90], [GR80], [Wac89]).

The group of classical permutations S_n is the most fundamental example of Coxeter groups. Coxeter groups are a class of groups, introduced by Coxeter

[Cox34] and defined by generators and certain relations, that arise in several areas of mathematics such as algebra, geometry as well as in physics (see e. g. [Hil82], [Dav08]).

From the Coxeter group point of view, given a permutation $\sigma \in S_n$ the number of inversions $\mathrm{inv}(\sigma)$ is the length of σ, denoted $l(\sigma)$, i. e. the minimum length of an expression of σ as product of generators. Moreover $\mathrm{des}(\sigma)$ is the number of generators s_i such that $l(\sigma s_i) < l(\sigma)$.

All finite Coxeter groups and a family of infinite Coxeter groups, the $ABCD$-family of affine Weyl groups, have a combinatorial interpretation in terms of generalized permutations (see e. g. the unified description of Eriksson [Eri94] or [BB05, Chapter 8]).

In the last years many authors have extended the previous statistics to all finite Coxeter groups (examples of works in this direction are the papers by Adin, Brenti and Roichman [ABR01], Brenti [Bre94], Clarke and Foata [CF94, CF95a, CF95c], Reiner [Rei93] and Steingrimsson [Ste94]) and in the group of affine permutations (see the paper by Clark and Ehrenborg [CE11]) .

In the first part of this thesis we define and study excedance statistics for all affine Weyl groups. We also introduce and study some new excedance statistics on finite Coxeter groups and a new major index for the affine Weyl groups.

In the second part of the thesis we study Kazhdan–Lusztig aspects of Coxeter groups. Kazhdan–Lusztig theory lies in different research areas of mathematics such as representation theory, algebraic geometry, Verma modules and combinatorics. Kazhdan–Lusztig theory originated in the paper [KL79] by Kazhdan and Lusztig. In this paper the authors introduced a new family of representations of Hecke algebras, which are strictly related to the Coxeter groups. Such representations are obtained using a family of polynomials, now called Kazhdan–Lusztig polynomials. Such polynomials have been shown to have several applications in different contexts (see e. g. [BL00], [Deo94], [GM88], [KL80]). Once these applications of Kazhdan–Lusztig polynomials had been found, there followed the problem of computing them. The main tools are fairly complicated recursive formulas appearing in [KL79] or the intersection cohomology of Schubert varieties ([KL80], [SSV98],[Zel83]). In the last thirty years many mathematicians have tried to find closed formulas, at least for small classes of elements in particular Coxeter groups (see e. g. [BW01][Boe88], [BS00], [Cas03], [CM06], [LS81], [Mar06], [SSV98] and also [Bre02a]).

In order to find a method for the computation of the dimensions of the intersection cohomology modules corresponding to Schubert varieties in G/P, where P is a parabolic subgroup of the Kac-Moody group G, in 1987 Deodhar ([Deo87]) introduced parabolic analogues of Kazhdan–Lusztig polynomials. These parabolic Kazhdan–Lusztig polynomials reduce to the ordinary ones for the trivial parabolic subgroup and are also related to them in other ways. Besides these connections the parabolic polynomials also play a direct role in several areas including the theories of generalized Verma modules ([CC87]), tilting modules ([Soe97b, Soe97a]) and Macdonald polynomials ([HHL05a, HHL$^+$05b]).

It is known that the coefficient of the monomial with highest possible degree in the classical and parabolic Kazhdan–Lusztig polynomials are the same (on pairs of elements for which the parabolic ones are defined). All these leading coefficients have important consequences: they appear in multiplication formulas for the Kazhdan-Lusztig basis elements of the Hecke algebra and are used in the construction of Kazhdan–Lusztig graphs and cells which in turn are used to construct Hecke algebra representations (see [KL79]). Moreover they control the recursive structure of the Kazhdan–Lusztig polynomials and their computation is not known to be any easier than that of the entire Kazhdan–Lusztig polynomials.

In Chapter 4 we show some techniques that allows to compute Kazhdan–Lusztig polynomials for a special class of elements, called boolean elements.

A special class of quotients is the class of quasi–minuscule quotients. They are defined (see e. g. [Ste01]) as a quotient $W \setminus W'$ where W' is the stabilizer of the dominant root in a finite orbit of roots (or conjugacy class of reflections) of W. Such quotients have the properties that in the case of finite and crystallographic groups there is a representation of a Lie algebra with Weyl group W whose weights consist of 0 and the orbit in question.

In the last part of this thesis we give a characterization of Kazhdan–Lusztig polynomials of quasi–minuscule quotients, showing that they are or monic monomials or 0.

We now describe the contents of the thesis.

In Chapter 1 we briefly give some basic preliminaries about Coxeter groups that are needed in the rest of the work. Moreover we give a combinatorial description of finite Coxeter groups and affine Weyl groups. In particular we introduce the classical statistics already known on the symmetric group.

In Chapter 2 we first describe the statistics that have been introduced in the group of signed permutations and focus our attention on the analogues of the descent statistics and major index. Then we define excedance statistics for the infinite families of finite and affine Coxeter groups. We give enumeration results for such statistics and bijections with the corresponding generalizations of the descents statistics. Such bijections shows that the excedance statistics are "eulerian" and thus justify our definition. Moreover we give signed-enumeration results for these statistics: the sign of an element is being given by the parity of its length.

In Chapter 3 we generalize the major index statistic to the classical affine Weyl groups. In 1968 Foata ([Foa68]) gave a bijection which proved combinatorially that the major index is mahonian. We extend this bijection to the group of affine permutations and prove that our definition of major index is mahonian. Moreover, we give a mahonian statistic which is strictly related to the abacus model of the group of the affine permutations (see e. g. [JK81]). Then, by using a recent work of Hanusa and Jones [HJ], which generalizes the abacus model to the other affine Weyl groups, we are able to extend our definition to these groups, and show that the resulting statistic is mahonian.

With Chapter 4 we start the second part of this thesis. Here we briefly give some basic preliminaries about Kazhdan–Lusztig polynomials and then we give a result which allows to compute combinatorially the Kazhdan–Lusztig polynomials of boolean elements in any Coxeter group whose Coxeter graph is a tree. This result extends one by Marietti ([Mar06]) which applies to Coxeter groups whose Coxeter graph is a path. Moreover, in the case of symmetric and hyperoctahedral groups we give a formula for the Kazhdan–Lusztig polynomials of boolean elements in terms of excedance statistics and other permutation statistics. As an application of our main result, we compute the Poincarè polynomials of all boolean elements of all finite Coxeter groups and essentially all affine Weyl groups.

In Chapter 5, which is due to the joint work with F. Brenti, we study the parabolic Kazhdan–Lusztig polynomials of the quasi-minuscule quotients of finite Weyl groups and we show that all such polynomials are either zero or a monic monomial.

Chapter 1

Coxeter groups and the symmetric group

Coxeter groups are defined in a simple way by generators and relations. They arise in several fields of mathematics. In this chapter we restrict our attention on definitions, notation and results that we will use in the rest of this work.

We let $\mathbb{P} := \{1, 2, 3, \dots\}$, $\mathbb{N} := \mathbb{P} \cup \{0\}$, $\mathbb{Z}$ be the set of integers and $\mathbb{Q}$ be the set of rational numbers. For $n \in \mathbb{N}$ we let $[n] := \{1, 2, \dots, n\}$ (with $[0] := \emptyset$) and for $x, y \in \mathbb{Z}$, $x \leq y$, we let $[x, y] := \{x, x + 1, \dots, y - 1, y\}$. The cardinality of a set A will be denoted by $|A|$. Given a statement P we will sometimes find it convenient to let

$$\chi(P) := \begin{cases} 1, & \text{if } P \text{ is true} \\ 0, & \text{if } P \text{ is false.} \end{cases}$$

1.1 Coxeter groups

Let $S = \{s_1, \dots, s_r\}$ be a finite set of cardinality r. A *Coxeter matrix* is a symmetric matrix $m : S \times S \to \mathbb{P} \cup \{\infty\}$ such that $m(s_i, s_j) = 1 \Leftrightarrow i = j$, for all $i, j \in [r]$.

Any Coxeter matrix uniquely determines a group W generated by S and with relations

$$(s_i s_j)^{m(s_i, s_j)} = \epsilon_W$$

for all $i, j \in [r]$ with $m(s_i, s_j) \neq \infty$, where ϵ_W denotes the unity of the group W. Every group W with such a presentation is called a *Coxeter group.* The pair (W, S) is a *Coxeter system* and S is a set of *Coxeter generators.* The cardinality $|S| = r$ is the *rank* of W.

The Coxeter matrix m of a Coxeter system (W, S) is encoded in its *Coxeter graph*. This is the labeled graph whose vertex set is S and the pair $\{s_i, s_j\}$ is an edge if and only if $m(s_i, s_j) \geq 3$ (∞ is allowed) and the corresponding label is $m(s_i, s_j)$ (labels equals to 3 are usually omitted). A Coxeter system whose Coxeter graph is connected is called *irreducible*.

By definition of Coxeter matrix, all generators are involutions. Hence any element $w \in W$ can be written as a product of generators

$$w = s_{i_1} s_{i_2} \cdots s_{i_j}, \qquad s_{i_j} \in S.$$

The *length* of w is

$$l(w) := \min\{j \in \mathbb{N} \,|\, w = s_{i_1} s_{i_2} \cdots s_{i_j} \text{ for some } s_{i_1} \cdots s_{i_j} \in S\}.$$

Any expression of w which is a product of exactly $l(w)$ elements of S is called a *reduced expression* for w. There is only one element of length zero, the identity ϵ_W.

Given two elements $u, v \in W$, $u \leq v$, we will denote with $l(u, v) := l(v) - l(u)$.

For all $u \in W$, we let

$$D_L(u) := \{s \in S | l(su) < l(u)\},$$
$$D_R(u) := \{s \in S | l(us) < l(u)\},$$

called the set of left and right descents of u. The set of all *reflections* of W is defined by

$$T(W) := \{wsw^{-1} | s \in S, w \in W\}.$$

In particular, $S \subseteq T(W)$ and the elements of S are called simple reflections. In the follows we will write T instead of $T(w)$ where no confusion arises.

The proof of the following result can be found in [BB05, Theorem 1.4.3].

Theorem 1.1.1 (Strong Exchange Property)**.** *Given a Coxeter system (W, S), let $w \in W$. Suppose that $w = s_{i_1} s_{i_2} \cdots s_{i_j}$, with $s_{i_h} \in S$ and let $t \in T$. If $l(tw) < l(w)$, then $tw = s_{i_1} \cdots s_{i_{h-1}} s_{i_{h+1}} \cdots s_{i_j}$ for some $h \in [j]$.*

As immediate consequence we have the following result.

Proposition 1.1.2. *Let (W, S) be a Coxeter system and $w \in W$. If $s \in D_L(w)$ (resp. $s \in D_L(w)$) then there exists a reduced expression $s_{i_1} \cdots s_{i_{l(w)}}$ of w with $s_{i_1} = s$ (resp. $s_{i_{l(w)}} = s$).*

Let $s, s' \in S$ and define $\alpha_{s,s'} := ss'ss'ss' \cdots$ the alternating word of length $m(s, s')$. Given a word w in the alphabet S let us call a *nil–move* the deletion of a subword of the form ss, and a *braid–move* the replacement of a factor $\alpha_{s,s'}$ by $\alpha_{s',s}$. The following result can be found in [BB05, Theorem 3.3.1].

Theorem 1.1.3 (Word Property)**.** *Let (W, S) be a Coxeter system and $w \in W$.*

- *Any expression $s_1 s_2 \cdots s_q$ for w can be transformed into a reduced expression for w by a sequence of nil–moves and braid–moves;*
- *every two reduced expressions for w can be connected via a sequence of braid–moves.*

We will always assume that a Coxeter group W is partially ordered by (strong) *Bruhat* order.

Definition 1.1.4. *Let $u, v \in W$ and suppose that there exist $t_1, \ldots, t_j \in T$ such that $v = t_j \cdots t_2 t_1 u$ and*

$$l(u) < l(t_1 u) < \cdots < l(t_j \cdots t_2 t_1 u).$$

Then $u \leq v$ with respect to the Bruhat order*.*

The following fundamental result can be found in [BB05, Theorem 2.2.2].

Theorem 1.1.5 (Subword Property)**.** *Let (W, S) be a Coxeter system and $u, v \in W$. Let $w = s_1 \cdots s_j$ be a reduced expression. Then $u \leq w$ if and only if there exists a reduced expression of u which is a subword of w, i. e. $u = s_{i_1} \cdots s_{i_k}$, with $1 \leq i_1 < i_2 < \cdots i_k \leq j$.*

With the usual notation of the posets, for all $u, v \in W$ we denote by $[u, v] := \{w \in W | u \leq w \leq v\}$ and call it an *interval* of P.

The *Bruhat graph* of W is the following direct graph. Take W as vertex set. For $u, v \in W$ put an arrow from u to v if and only if $l(u) < l(v)$ and $ut = v$ for some $t \in T$. Clearly $u < v$ if and only if there is a directed path in the Bruhat graph from u to v.

Let $J \subseteq S$. The subgroup of W generated by the set J is called the *parabolic subgroup* generated by J and it is denoted by W_J. The pair (W_J, J) is a Coxeter system with relations induced by (W, S). The set of minimal length representatives for the right cosets is defined by

$$W^J := \{w \in W | l(sw) > l(w) \text{ for all } s \in J\}.$$

Note that $W^{\emptyset} = W$ (here we use a notation different from that of [BB05], in which the same set is denoted by ${}^J W$) . The Bruhat order extends naturally on W^J. For all $u, v \in W^J$ we denote by $[u, v]^J := \{w \in W^J | u \leq w \leq v\}$.

The following result is well known (see e. g. [BB05, Proposition 2.4.4])

Proposition 1.1.6. *Let (W, S) be a Coxeter system and $J \subseteq S$. Then the following hold:*

1. *every* $w \in W$ *has a unique factorization* $w = w_J \cdot w^J$ *such that* $w_J \in W_J$ *and* $w^J \in W^J$*;*

2. *for this factorization,* $l(w) = l(w^J) + l(w_J)$*.*

We conclude this section by giving another result about parabolic quotients. Its proof can be found in [BB05, Theorem 2.5.5].

Theorem 1.1.7 (Chain Property)**.** *If* $u < v$ *in* W^J*, then there exist elements* $w_i \in W^J$*,* $l(w_i) = l(u) + i$*, for* $i \in [0, k]$ *such that* $u = w_0 < w_1 < \cdots < w_k = w$*.*

1.2 Finite Coxeter groups and affine Weyl groups

All finite Coxeter group are first classified by Coxeter [Cox35]. In Table 1.1 we list all finite Coxeter groups. The finite Coxeter groups for which $m(s, s') \in \{2, 3, 4, 6\}$ for all pairs of generators $s \neq s'$ are called *Weyl groups*, a name motivated by Lie theory.

Other Coxeter groups widely studied are the *affine Weyl groups*: they are infinite Coxeter groups which contain a normal abelian subgroup such that the corresponding quotient group is finite. In each case, the quotient group is itself a Coxeter group. The Coxeter graph of an affine Weyl group is obtained from the Coxeter graph of the associated Coxeter group by adding an additional vertex and one or two additional edges. In Table 1.2 we list all affine Weyl groups.

All the finite Coxeter groups and affine Weyl groups have been enumerated by their length function as shown by the following theorem, whose proof can be found for example in [BB05, Theorems 7.1.5 and 7.1.10]

Theorem 1.2.1. *Let* (W, S) *be a finite irreducible Coxeter system, and set* $n = |S|$*. Then there exist positive integers* $e_1, \ldots, e_n$*, called* exponents*, such that*

$$W(q) := \sum_{w \in W} q^{l(w)} = \prod_{i=1}^{n} [e_i + 1]_q.$$

If (W, S) *is an affine Weyl group and* $e_1, \ldots, e_n$ *are the exponents of the corresponding finite group, then*

$$W(q) = \prod_{i=1}^{n} \frac{[e_i + 1]_q}{1 - q^{e_i}}.$$

Here recall that $[n]_q$ is the q-analogue of n, more precisely, $[n]_q := 1 + q + \cdots + q^{n-1}$ and $[0]_q := 0$.

In Table 1.3 we report the exponents of all finite Coxeter groups.

Table 1.1: List of all finite Coxeter groups

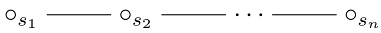
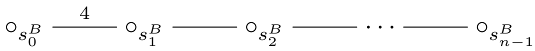
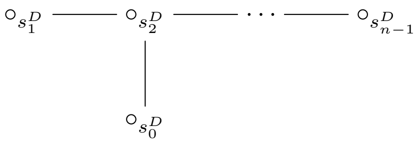
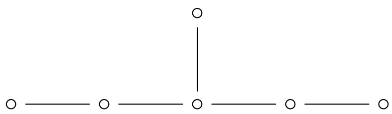
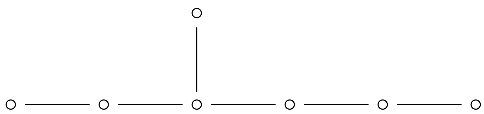
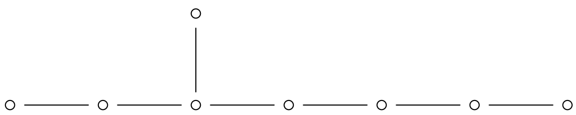
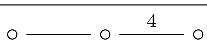
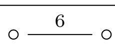
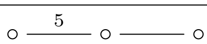
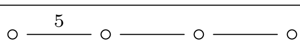
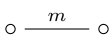
Name	Diagram	Order
$A_n (n \geq 1)$		$(n+1)!$
$B_n \equiv C_n (n \geq 2)$		$2^n n!$
$D_n (n \geq 4)$		$2^{n-1} n!$
E_6		$2^7 3^4 5$
E_7		$2^{10} 3^4 5\ 7$
E_8		$2^{14} 3^5 5^2 7$
F_4		1152
G_2		12
H_3		120
H_4		14400
$I_2(m)\ (m \geq 3)$		$2m$

Table 1.2: List of all affine Weyl groups

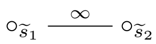
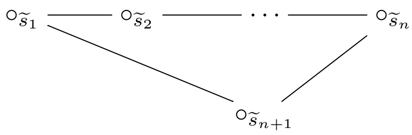
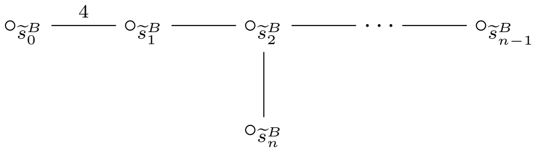
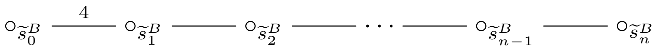
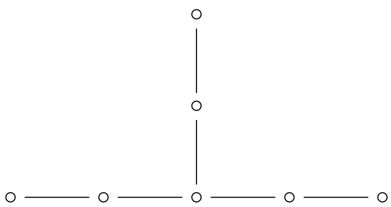
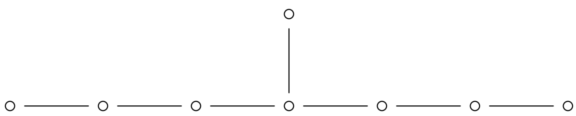
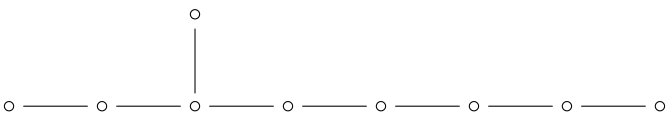
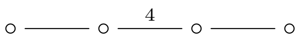
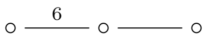
Name	Diagram
$\widetilde{A}_1$	
$\widetilde{A}_n$ $(n \geq 2))$	
$\widetilde{B}_n (n \geq 3)$	
$\widetilde{C}_n (n \geq 2)$	
$\widetilde{D}_n (n \geq 4)$	
$\widetilde{E}_6$	
$\widetilde{E}_7$	
$\widetilde{E}_8$	
$\widetilde{F}_4$	
$\widetilde{G}_2$	

Table 1.3: Exponents of finite Coxeter groups

Coxeter group	exponents
A_n $(n \geq 1)$	$1, 2, \ldots, n$
$B_n \equiv C_n (n \geq 2)$	$1, 3, 5, \ldots, 2n-1$
D_n $(n \geq 4)$	$1, 3, 5, \ldots, 2n-3, n-1$
E_6	$1, 4, 5, 7, 8, 11$
E_7	$1, 5, 7, 9, 11, 13, 17$
E_8	$1, 7, 11, 13, 17, 19, 23, 29$
F_4	$1, 5, 7, 11$
G_2	$1, 5$
H_3	$1, 5, 9$
H_4	$1, 11, 19, 29$
$I_2(m)$ $(m \geq 3)$	$1, m-1$

1.3 Combinatorial interpretation of finite Coxeter groups and affine Weyl groups

The most important example of Coxeter group is the group of all permutations of the set $\{1, \ldots, n\}$, denoted by S_n. Let $\sigma \in S_n$ we will write $\sigma = [\sigma_1, \ldots, \sigma_n]$ to denote $\sigma(i) = \sigma_i$. We refer to it as the *complete notation* of σ. Sometimes we will use the *disjoint cycle form* $\sigma = (c_1^1, \ldots, c_{j_1}^1) \cdots (c_1^k, \ldots, c_{j_l}^k)$ to denote that $\sigma(c_h^i) = c_{h+1}^i$, where the index $h+1$ is taken modulo j_i (omit all cycles with only one element). For example, if $\sigma = [6, 3, 4, 2, 5, 1, 8, 7]$ then we also write $\sigma = (7, 8)(2, 3, 4)(1, 6)$. Given two permutations $\sigma, \tau \in S_n$ we let $\sigma\tau := \sigma \circ \tau$ the composition of functions. So that, for example, $(1, 2, 3)(2, 4, 3) = (1, 2, 4)$.

The group S_n is a Coxeter group with generators the transpositions $s_i = (i, i+1)$ for $i = 1, \ldots, n-1$ and relations

$$\begin{aligned} (s_i s_i + 1)^3 &= e \text{ for } i = 1, \ldots, n-2; \\ (s_i s_j)^2 &= e \text{ for } |i - j| \geq 2; \\ s_i^2 &= e \text{ for } i = 1, \ldots, n-1. \end{aligned}$$

According to Table 1.1, the permutation group S_n is a Coxeter group of type A_{n-1}. The reflection set T of S_n is the set of all transposition

$$T = \{(a, b) | 1 \leq a < b \leq n\},$$

as immediately seen from the computation $w s_i w^{-1} = (w(i), w(i+1))$, for $w \in$

S_n.

All other finite Coxeter groups have a combinatorial interpretation in terms of bijections of subsets of $\mathbb{Z}$. Many of the facts that we will show here are part of the folklore of the subject. The first unified and comprehensive study of combinatorial descriptions of a large class of Coxeter groups, which includes the countable families of affine Weyl groups is given in the thesis of H. Eriksson [Eri94].

Fix any positive number n. We denote by S_n^B the group of all bijections σ of the set $[-n, n] \setminus \{0\}$ such that $\sigma(-i) = -\sigma(i)$ for all $i \in [-n, n] \setminus \{0\}$, with composition as the group operation. This group is usually known as the group of "signed permutations" on $[1, n]$, or as the hyperoctahedral group of rank n. We have that S_n^B is a Coxeter group of type B_n (see e. g. [BB05, Proposition 8.1.3]). If $\sigma \in S_n^B$ then we write $\sigma = [a_1, \ldots, a_n]$ to mean that $\sigma(i) = a_i$ for $i = 1, \ldots, n$. We refer to it as the window notation of σ. The set of all generators in S_n^B is given by $\{s_0^B, s_1^B, \ldots, s_{n-1}^B\}$, where $s_0^B = [-1, 2, \ldots, n]$ and $s_i^B = [1, \ldots, i-1, i+1, i, i+2, \ldots, n]$ for all $i \in [n-1]$.

We denote by S_n^D the group of all bijections $\sigma \in S_n^B$ such that

$$|\{i \in [1, n] : \sigma(i) < 0\}| \equiv 0 \mod 2$$

. It is a Coxeter group of type D_n (see e. g. [BB05, Proposition 8.2.3]). For example $[6, 1, -2, 5, -4, 3]$ and $[5, -6, -3, 1, -2, 4]$ are both permutations in S_6^B, but only the first is also in S_6^D. The set of all generators in S_n^D is given by $\{s_0^D, s_1^D, \ldots, s_{n-1}^D\}$, where $s_0^D = [-2, -1, \ldots, n]$ and $s_i^D = s_i^B$ for all $i \in [n-1]$.

Combinatorial interpretations of affine Weyl groups require infinite sets. We denote by $\widetilde{S}_n$ the group of all bijections $\pi : \mathbb{Z} \to \mathbb{Z}$ satisfying the two conditions

$$\pi(i+n) = \pi(i) + n \quad \text{for all } i \qquad \text{and} \qquad \sum_{i=1}^{n} (\pi(i) - i) = 0.$$

It is an affine Weyl group of type $\widetilde{A}_{n-1}$ (see e. g. [BB05, Proposition 8.3.3]). If $\pi \in \widetilde{S}_n$ then we write $\pi = [a_1, \ldots, a_n]$ to mean that $\pi(i) = a_i$ for $i = 1, \ldots, n$. We refer to it as the window notation of π. Moreover, sometimes we will write $\pi = (\mathbf{r}_\pi | \sigma_\pi)$, with $\mathbf{r}_\pi = (r_1, \ldots, r_n) \in \mathbb{Z}^n$, $\sigma_\pi \in S_n$ to mean that $\pi(i) = \sigma_\pi(i) + n r_i$. Note that any such pair $(\mathbf{r}|\sigma)$, with $\sum_{i=1}^n r_i = 0$ uniquely determines one element in $\widetilde{S}_n$. For example, the permutation $[12, -9, 10, 2, -5, 11] \in \widetilde{S}_6$ can be represented by $(1, -2, 1, 0, -1, 1|[6, 3, 4, 2, 1, 5])$. In this notation it is simple to see that given any two permutations $(r_1, \ldots, r_n|\rho), (t_1, \ldots, t_n|\tau) \in \widetilde{S}_n$ then

$$(r_1, \ldots, r_n|\rho) \circ (t_1, \ldots, t_n|\tau) = (r_{\tau(1)+t_1, \ldots, r_{\tau(n)}+t_n}). \tag{1.3.1}$$

The set of all generators in $\widetilde{S}_n$ is given by $\{\widetilde{s}_1, \cdots, \widetilde{s}_n, \widetilde{s}_{n+1}\}$, where $\widetilde{s}_i = [1, \ldots, i-1, i+1, i, i+2, \ldots, n+1]$ for all $i \in [n]$ and $\widetilde{s}_{n+1} = [0, 2, \ldots, n, n+2]$.

We denote by $\widetilde{S}_n^C$ the set of all bijections $\pi \in S(\mathbb{Z})$ such that $\pi(-i) = -\pi(i)$ and $\pi(i+(2n+1)) = \pi(i)+2n+1$ for all $i \in \mathbb{Z}$, with composition as the group operation. The group $\widetilde{S}_n^C$ is an affine Weyl group of type $\widetilde{C}$ (see e. g. [BB05, Proposition 8.4.3]). If $\pi \in \widetilde{S}_n^C$ then we write $\pi = [a_1, \ldots, a_n]$ to mean that $\pi(i) = a_i$ for $i = 1, \ldots, n$. We refer to it as the window notation of π. Moreover, sometimes we will write $\pi = (\mathbf{r}_\pi|\sigma_\pi)$, with $\mathbf{r}_\pi = (r_1, \ldots, r_n) \in \mathbb{Z}^n$, $\sigma_\pi \in S_n^B$ to mean that $\pi(i) = \sigma_\pi(i) + r_i(2n+1)$. Note that any such pair $(\mathbf{r}|\sigma)$ uniquely determines one element in $\widetilde{S}_n^C$. For example, the permutation $[12, -9, 10, 2, -5, 6] \in \widetilde{S}_6^C$ can be represented $(1, -1, 1, 0, 0, 0|[-1, 4, -3, 2, -5, 6])$. It is simple to see that equation (1.3.1) holds also for any pair $(r_1, \ldots, r_n|\rho), (t_1, \ldots, t_n|\tau) \in \widetilde{S}_n^C$. The set of all generators in $\widetilde{S}_n^C$ is given by $\{\widetilde{s}_0^C, \widetilde{s}_1^C, \cdots, \widetilde{s}_n^C\}$, where $\widetilde{s}_0^C = [-1, 2, \ldots, n]$, $\widetilde{s}_i^C = [1, \ldots, i-1, i+1, i, i+2, \ldots, n+1]$ for all $i \in [n-1]$ and $\widetilde{s}_n^C = [1, 2, \ldots, n-1, n+1]$.

We denote by $\widetilde{S}_n^B$ the subgroup of $\widetilde{S}_n^C$ of all bijections π such that $|\{i \leq n | \pi(i) > n\}| \equiv 0 \mod 2$. The group $\widetilde{S}_n^B$ is well defined and it is an affine Weyl group of type $\widetilde{B}_n$ (see e. g. [BB05, Proposition 8.5.3]). Note that $\pi = (\mathbf{r}_\pi|\sigma_\pi) \in \widetilde{S}_n^C$ is in $\widetilde{S}_n^B$ if and only if

$$\sum_{i=1}^{n} r_i \equiv 0 \mod 2, \tag{1.3.2}$$

as can be proven by inspection. In the following we say that the vector $\mathbf{r} \in Z^n$ is *even* if it satisfies (1.3.2), we say that $\mathbf{r}$ is *odd* otherwise. The set of all generators in $\widetilde{S}_n^B$ is given by $\{\widetilde{s}_0^B, \widetilde{s}_1^B, \cdots, \widetilde{s}_n^B\}$, where $\widetilde{s}_i^B = \widetilde{s}_i^C$ for all $i \in [0, n-1]$ and $\widetilde{s}_n^B = [1, 2, \ldots, n-2, n+1, n+2]$.

Finally, we denote by $\widetilde{S}_n^D$ the subgroup of $\widetilde{S}_n^B$ of all bijections π such that

$$|\{i > 0 | \pi(i) < 0\}| \equiv 0 \mod 2.$$

The group $\widetilde{S}_n^D$ is well defined and it is an affine Weyl group of type $\widetilde{D}_n$ (see e. g. [BB05, Proposition 8.6.3]). Note that $\pi = (\mathbf{r}_\pi|\sigma_\pi) \in \widetilde{S}_n^B$ is in $\widetilde{S}_n^D$ if and only if

$$\sigma_\pi \in S_n^D. \tag{1.3.3}$$

The set of all generators of $\widetilde{S}_n^D$ is given by $\{\widetilde{s}_0^D, \widetilde{s}_1^D, \cdots, \widetilde{s}_n^D\}$, where $\widetilde{s}_0^D = [-2, -1, 3, \ldots, n]$ and $\widetilde{s}_i^D = \widetilde{s}_i^B$ for all $i \in [n]$.

1.4 Statistics on permutation groups

Many statistics have been defined on the permutation group. Given $\sigma \in S_n$ and, in general, for all $(\sigma_1, \dots, \sigma_n) \in \mathbb{Z}$ we define the *descent set* by

$$\mathrm{Des}(\sigma) := \{i \in [n-1] | \sigma_i > \sigma_{i+1}\}.$$

The *descent number* of σ is

$$\mathrm{des}(\sigma) := |\,\mathrm{Des}(\sigma)|$$

and the *major index* of σ is

$$\mathrm{maj}(\sigma) := \sum_{i \in \mathrm{Des}(\sigma)} i.$$

For example, if $\sigma = [6, 4, 2, 5, 3, 1] \in S_6$ then $\mathrm{Des}(\sigma) = \{1, 2, 4, 5\}$, $\mathrm{des}(\sigma) = 4$ and $\mathrm{maj}(\sigma) = 12$.

A number $i \in [n]$ for which $\sigma_i > i$ is called an *excedance*. We denote by $\mathrm{Exc}(\sigma)$ the set of all the excedances and by

$$\mathrm{exc}(\sigma) = |\,\mathrm{Exc}(\sigma)|.$$

A pair $(i, j) \in [n] \times [n]$ is an *inversion* if and only if $i < j$ and $\sigma_i > \sigma_j$. We set

$$\mathrm{inv}(\sigma)$$

be the number of inversions.

For example, if $\sigma = [6, 4, 2, 5, 3, 1]$ then $\mathrm{inv}(\sigma) = 12$, $\mathrm{Exc}(\sigma) = \{1, 2, 4\}$ and $\mathrm{exc}(\sigma) = 3$.

All the previous statistics can be easily extended to any sequence of integers in $\mathbb{Z}$ in an obvious way. In this case we will write again des, maj, inv and exc to denote them in the general contest. For example, given σ be the sequence $(4, -3, 2, -9, 7)$ then $\mathrm{des}(\sigma) = 2$, $\mathrm{maj}(\sigma) = 4$, $\mathrm{inv}(\sigma) = 5$ and $\mathrm{exc}(\sigma) = 2$.

In this general contest we will also consider the *negative statistic* defined for all $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbb{Z}^n$ by

$$\mathrm{neg}(\sigma_1, \dots, \sigma_n) = |\{i \in [n] | \sigma_i < 0\}|.$$

The number of inversions of a permutation σ is strictly related to the length of σ, as shown by the following result, whose proof can be found for example in [BB05, Propositions 1.5.2 and 1.5.3].

Proposition 1.4.1. *Let $\sigma \in S_n$. Then*

$$l(\sigma) = \mathrm{inv}(\sigma),$$
$$D_R(\sigma) = \{s_i | i \in \mathrm{Des}(\sigma)\}.$$

The number of permutations in S_n with h descents is denoted by $A(n, h+1)$ and it is called an *Eulerian number.* The generating function

$$A_n(q) := \sum_{\sigma \in S_n} q^{1+\mathrm{des}(\sigma)} = \sum_{i=1}^{n} A(n,h)q^h$$

is called the *Eulerian polynomial.* In what follows we will find it convenient to use the following notation

$$A_n^*(q) := \begin{cases} 1 & \text{if } n = 0; \\ \frac{1}{q}A_n(q) & \text{otherwise.} \end{cases}$$

It is a well-known result that the descent statistic and the excedance statistic are equidistributed on S_n, i. e.

$$A_n^*(q) = \sum_{\sigma \in S_n} q^{\mathrm{des}(\sigma)} = \sum_{\sigma \in S_n} q^{\mathrm{exc}(\sigma)}.$$

Any statistic defined on S_n equidistributed with the descents is called *Eulerian.* Therefore the excedance statistic is eulerian. A bijective proof of the previous identity is the following: let $\sigma \in S_n$ and write its cycle notation such that each cycle has its minimum element in the first place and then order all cycles in decreasing order of their minimum elements. Delete all brackets in this notation and let τ be the unique permutation associate to the remaining sequence read in reverse order (now we are considering the complete notation). It is a routine to check that $\mathrm{exc}(\sigma) = \mathrm{des}(\tau)$ and that the map above constructed is a bijection. For example, start from $\sigma = [6, 4, 2, 5, 3, 1] = (2, 4, 5, 3)(1, 6)$. Then $\mathrm{exc}(\sigma) = 3$, $\tau = [6, 1, 3, 5, 4, 2]$ and $\mathrm{des}(\tau) = 3$.

The Eulerian polynomials have the following generating function (see e. g. [Sta97, Exercise 3.81.c])

$$\sum_{n \geq 0} A_n(q) \frac{z^n}{n!} = \frac{(1-q)e^{qz}}{e^{qz} - qe^z}. \tag{1.4.4}$$

As shown in Proposition 1.4.1 it is easy to have an interpretation of all descents in Coxeter theoretic sense. This allows us to extend the definition of eulerian statistic to other finite Coxeter groups and affine Weyl groups.

For all $j \in [n-1]$, let $t_j := s_j s_{j-1} \cdots s_2 s_1$. Explicitly

$$t_j = [j+1, 1, 2, \cdots, j, j+2, \ldots, n]. \tag{1.4.5}$$

It is possible to verify the following well-known property: for any $\sigma \in S_n$ there exist unique integers $k_1, \ldots, k_{n-1}$, with $0 \leq k_i \leq i$ for $i = 1, \ldots, n-1$, such that

$$\sigma = t_{n-1}^{k_{n-1}} \cdots t_2^{k_2} t_1^{k_1}. \tag{1.4.6}$$

In this notation, it is not hard to see that

$$\mathrm{maj}(\sigma) = \sum_{i=1}^{n-1} k_i.$$

It is a well-known result of MacMahon that the major index and the inversion statistic are equidistributed in S_n, i. e.

$$W(q) = \sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)} = \prod_{i=1}^{n} [i]_q.$$

Any statistic defined on S_n equidistributed with the inversion statistic is called *Mahonian.* Therefore the major index is Mahonian.

The definition of mahonian statistic can be extended to finite Coxeter and affine Weyl groups: for each one of these groups it was given a definition of inversion statistic equidistributed with the length function. For more information about inversion statistic in these groups, see [BB05, Chapter 8].

A bijective proof of the previous identity was first given by Foata [Foa68].

The map $\phi : S_n \to S_n$ such that $\mathrm{maj}(\sigma) = \mathrm{inv}(\phi(\sigma))$ for any $\sigma \in S_n$ is defined by the following algorithm.

Let $\sigma = [\sigma_1, \dots, \sigma_n]$.

(i) Define $\pi_1 := \sigma_1$; assume that π_k has been defined for some k with $1 \leq k \leq n$;

(ii) if the last element of π_k is greater than σ_{k+1} (resp. smaller), split the sequence π_k after each letter greater (resp. smaller) than σ_{k+1};

(iii) in each component of such splitted sequence move the last element to the beginning; obtain π_{k+1} by joining these components and by adding σ_{k+1} to the end;

(iv) increase k; if $k = n$ then $\phi(\sigma) = \pi_n$, otherwise return to (ii).

For example, if $\sigma = [6, 4, 2, 3, 1, 5]$ then $\pi_1 = [6]$; $\pi_2 = [6, 4]$; $\pi_3 = [6, 4, 2]$; $\pi_4 = [2, 6, 4, 3]$; $\pi_5 = [2, 6, 4, 3, 1]$ and $\phi(\sigma) = \pi_6 = [2, 4, 6, 3, 1, 5]$. In [Foa68, Theorem 4.3] the following result is shown.

Theorem 1.4.2. *The map* $\phi : S_n \to S_n$ *is a bijection such that* $\mathrm{maj}(\sigma) = \mathrm{inv}(\phi(\sigma))$ *for any* $\sigma \in S_n$.

In [FS78] it is possible to find an algorithm which describes the inverse function ϕ^{-1}.

In the last decades, several studies have been done to compute joined distribution of the previous statistics. Some of the first famous results are due to

Carlitz and Gessel who studied the joined distribution of the major index and the descent statistic in S_n.

Given two variables q, t we define an operator $\delta_t : \mathbb{Q}[q,t] \to \mathbb{Q}[q,t]$ by

$$\delta_t(P(q,t)) = \frac{P(q,qt) - P(q,t)}{qt - t}$$

for all $P \in \mathbb{Q}[q,t]$. Note that $\delta_t(q^n) = 0$ and

$$\delta_t(t^n) = [n]_q t^{n-1}$$

for all $n \in \mathbb{N}$, and

$$\delta_t(A(q,t)B(q,t)) = \delta_t(A(q,t))B(q,t) + A(q,qt)\delta_t(B(q,t))$$

for all $A, B \in \mathbb{Q}[q,t]$. Also,

$$\delta_t(P)(1,t) = \frac{d}{dt}(P(1,t))$$

for all $P \in \mathbb{Q}[q,t]$. For $n \in \mathbb{P}$ we let

$$\overline{A}_n(t,q) := \sum_{\sigma \in S_n} t^{\mathrm{des}(\sigma)} q^{\mathrm{maj}(\sigma)}$$

and $\overline{A}_0(t,q) := 1$. For example, $\overline{A}_3(t,q) = 1 + 2tq^2 + 2tq + t^2q^3$. The following two results are due to Carlitz [Car75] and Gessel [Ges77], and proofs of them can also be found in [Gar80].

Theorem 1.4.3. *Let $n \in \mathbb{P}$. Then*

$$\overline{A}_n(t,q) = (1 + tq[n-1]_q)\overline{A}_{n-1}(t,q) + tq(1-t)\delta_t(\overline{A}_{n-1}(t,q)).$$

Theorem 1.4.4. *Let $n \in \mathbb{P}$. Then*

$$\sum_{r \geq 0} [r+1]_q^n t^r = \frac{\overline{A}_n(t,q)}{\prod_{i=0}^n (1 - tq^i)}$$

in $\mathbb{Z}[q][[t]]$.

In the follows we will also consider a subset of the permutations. We say that a permutation $\sigma \in S_n$ is a *derangement* if and only if for all $i \in [n]$ then $\sigma(i) \neq i$. We denote with DS_n the subset of all derangements. The previous statistics have been studied also for this subset. Here we give only some results for the excedance statistic which we will use in the following. Brenti [Bre90] defined the derangement polynomials (ot type A) by

$$d_n(q) := \sum_{\sigma \in SD_n} q^{\mathrm{exc}(\sigma)}, \qquad n \geq 1, \tag{1.4.7}$$

and $d_0(q) = 1$. It has been shown that $d_n(q)$ is symmetric and unimodal for $n \geq 1$ (see [Bre90, Corollary 1]) and that it has only real zeros (Canfield, unpublished). The following formula is given in [CTZ09, Theorem 1.1] and it is derived from [Bre90].

Theorem 1.4.5. *For $n \geq 0$,*

$$d_n(q) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} A_k^*(q).$$

The concept of derangements can be easily extended to any bijection: given a bijection $p : S \to S$ we say that an element $s \in S$ is a *fixed point* if $p(s) = s$ and we denote by $\mathrm{fix}(p)$ the number of all fixed points of p. The map p is a derangement if and only if $\mathrm{fix}(p) = 0$. If we denote with DS the set of all derangements of S, the *Principle of Inclusion–Exclusion* (see e. g. [Sta97, Sections 3.7, 3.8]) allows us to compute the generating function of a statistic stat on DS by knowing its value on S: more precisely, if

$$F_S(q) = \sum_{s \in S} q^{\mathrm{stat}(s)}$$

is the generating function of stat on S, then the Principle of Inclusion–Exclusion gives the following result

$$\sum_{s \in DS} q^{\mathrm{stat}(s)} = \sum_{T \subseteq S} (-1)^{|S \setminus T|} F_T(q).$$

Moreover, if we suppose that the generating function of the statistic stat depends only on the cardinality of the set $T \subseteq S$ (this is the case of all excedance statistics which we will study) then it reduces to

$$\sum_{s \in DS} q^{\mathrm{stat}(s)} = \sum_{i=0}^{|S|} \binom{|S|}{i} (-1)^{|S|-i} F(i), \tag{1.4.8}$$

where $F(i)$ is the generating function of stat in any subset T of S having $|T| = i$.

Chapter 2

Excedance statistics in signed and affine permutation groups

An increasing number of enumerative results true for S_n have been generalized to the hyperoctahedral group. Several definitions of "major index" statistic have been introduced for S_n^B (see e. g. [CF95b, CF95d, Rei93, Ste94]) but the first example which generalize MacMahon's result is given by the *flag-major index* first introduced in [AR01]. After, Foata posed the problem to extend the bivariate distribution of the descent number and major index to S_n^B and it was solved in [ABR01].

Generalizing the excedance statistic and the corresponding results has been much more difficult and has occupied a number of mathematicians for a number of years (see e. g. [BG06, Bre94, CTZ09, FH09, FH11]). The first definition of excedance in the hyperoctahedral group can be found in [Bre94]: here Brenti introduced a statistic equidistributed with the Coxeter descents of the group. A good study of it can be found in [CTZ09]. Two other definitions of excedance are given in [BG06]: here Bagno and Garber introduced these statistics for the family of all colored permutation groups (i. e. all the wreath products $\mathbb{Z}_r^n \rtimes S_n$, with $\mathbb{Z}_r$ the cyclic group of order r) which includes the hyperoctahedral group isomorphic to $\mathbb{Z}_2^n \rtimes S_n$.

Part of this chapter is in [Mon12a].

2.1 Statistics on signed permutation groups

Now we recall some statistics introduced in the last decades in the hyperoctahedral group.

For $i \in [n-1]$ let $t_i^B = s_i^B s_{i-1}^B \cdots s_0^B$. Explicitly,

$$t_i^B = [-i-1, 1, 2, \cdots, i, i+2, \cdots, n]$$

for $i \in [n-1]$. It is not hard to show that for any $\sigma \in S_n^B$ there exist unique integers $k_0, \ldots, k_{n-1}$, with $0 \leq k_i \leq 2i+1$ for $i \in [0, n-1]$ such that

$$\sigma = t_{n-1}^{k_{n-1}} \cdots t_2^{k_2} t_1^{k_1} t_0^{k_0}.$$

Such decomposition is similar to that in (1.4.6) for S_n.

For this reason, in 2001, Adin and Roichman [AR01] defined the *flag-major index* of σ by

$$\operatorname{fmaj}(\sigma) := \sum_{i=0}^{n-1} k_i \tag{2.1.1}$$

and shown the following result.

Theorem 2.1.1. *Let $\sigma = [\sigma_1, \ldots, \sigma_n] \in B_n$. Then*

$$\operatorname{fmaj}(\sigma) = 2\operatorname{maj}(\sigma) + \operatorname{neg}(\sigma).$$

The generating function of flag-major index is

$$\sum_{\sigma \in B_n} q^{\operatorname{fmaj}(\sigma)} = \prod_{i=1}^{n} [2i]_q. \tag{2.1.2}$$

In the same way, Adin and Roichman defined in their paper ([AR01]) the *flag-descent number* fdes_B as follows: for $\sigma = [\sigma_1, \ldots, \sigma_n] \in S_n^B$ let

$$\operatorname{fdes}_B(\pi) := 2\operatorname{des}(\sigma) + \chi(\sigma_1 > 0).$$

It is simple to verify that $\operatorname{fdes}(\sigma) = \operatorname{des}(\sigma(-n), \ldots, \sigma(-1), \sigma(1), \ldots, \sigma(n))$. In [ABR01, Theorem 4.4] it is proven that

$$\sum_{\sigma \in S_n^B} q^{\operatorname{fdes}(\sigma)} = (1+q)^n A_n^*(q). \tag{2.1.3}$$

The two definitions of flag-major index and flag-descent number are consistent as shown by the following result (see [ABR01]) which extends the Carlitz's identity of Theorem 1.4.4.

Theorem 2.1.2. *For all $n \geq 1$*

$$\sum_{k \geq 0} [k+1]_q^n t^k = \frac{\sum_{\sigma \in B_n} t^{\mathrm{fdes}(\sigma)} q^{\mathrm{fmaj}(\sigma)}}{(1-t)\prod_{i=1}^{n}(1-t^2q^{2i})}.$$

Generalizing the excedance statistic to S_n^B has been much more difficult. The first example in this direction is given in [Bre94]. Here Brenti introduced the so called type B excedance as follows.

Definition 2.1.3. *Given $\sigma \in S_n^B$, an element $i \in [1, n]$ is a type B excedance of σ if $\sigma(i) = -i$ or $\sigma(|\sigma(i)|) > \sigma(i)$.*

Such statistic has the same distribution as the descent statistic in the Coxeter group theoretic sense (see [Bre94] for more details). For this reason we denote this excedance number of a signed permutation $\sigma \in S_n^B$ by $\mathrm{exc}^{\mathrm{cox}}(\sigma)$. For a good exposition about the excedance of type B and *derangements* with respect to these statistics see the paper of Chen, Tang and Zhao [CTZ09].

Before introducing the excedance statistics which we will study, we recall some other statistics on the hyperoctahedral group which will be useful in the next chapters.

Given $\sigma \in S_n^B$ we let

$$\mathrm{inv}(\sigma) := |\{(i,j) \in \{1,\dots,n\}^2 | i < j, \sigma(i) > \sigma(j)\}|, \tag{2.1.4}$$

$$N_1(\sigma) := \mathrm{neg}(\sigma) = |\{i \in \{1,\dots,n\} | \sigma(i) < 0\}| \text{ and} \tag{2.1.5}$$

$$N_2(\sigma) := |\{(i,j) \in \{1,\dots,n\}^2 | i < j, \sigma(i) + \sigma(j) < 0\}|. \tag{2.1.6}$$

The sum of the previous statistic is mahonian, as shown in the following result, whose proof can be found in [BB05, Propositions 8.1.1 and 8.1.2].

Proposition 2.1.4. *Let $\sigma \in S_n^B$. Then $l(\sigma) = \mathrm{inv}(\sigma) + N_1(\sigma) + N_2(\sigma)$, and $D_R(\sigma) = \{s_i^B | \sigma(i) > \sigma(i+1)\}$, where $\sigma(0) := 0$.*

The analogous result for S_n^D is the following (see [BB05, Propositions 8.2.1 and 8.2.2] for a proof).

Proposition 2.1.5. *Let $\sigma \in S_n^D$. Then $l(\sigma) = \mathrm{inv}(\sigma) + N_2(\sigma)$ and $D(\sigma) = \{s_i^D | \sigma(i) > \sigma(i+1)\}$, where $\sigma(0) := -\sigma(2)$.*

2.1.1 Colored excedance statistic

In the last years other generalizations of the excedance statistic was given.

In [BG06] Bagno and Garber introduced a definition of excedance on the set of signed permutations, called *colored excedance*.

Definition 2.1.6. *For any* $\sigma \in S_n^B$ *we set*

$$\mathrm{exc}^{\mathrm{col}}(\sigma) := 2\,\mathrm{exc}(\sigma) + \mathrm{neg}(\sigma).$$

For example, if $\sigma = [4, 3, -1, -5, 2]$, then $\mathrm{neg}(\sigma) = 2$, $\mathrm{exc}(\sigma) = 2$ and $\mathrm{exc}^{\mathrm{col}}(\sigma) = 6$. Colored excedance statistic is also noted as *flag-excedance statistic* (see e. g. [Fir04], [FH09], [FH11]). In [BG06, Lemma 3.5] is given the following result which motivates the coefficient 2 appearing in the above definition.

Lemma 2.1.7. *Let* $\sigma \in S_n^B$ *and consider the* color order *on* $[-n, n] \setminus \{0\}$.

$$-1 < -2 < \cdots < -n < 1 < 2 < \cdots n.$$

Then

$$\mathrm{exc}^{\mathrm{col}}(\sigma) = |\{i \in [-n, n] \setminus \{0\} | \sigma(i) > i\}|$$

where the comparison is with respect to the color order.

For example, if we write $\sigma = (4, 3, -1, -5, 2)$ in an extended form:

$$\begin{pmatrix} -1 & -2 & -3 & -4 & -5 & 1 & 2 & 3 & 4 & 5 \\ -4 & -3 & 1 & 5 & -2 & 4 & 3 & -1 & -5 & 2 \end{pmatrix}$$

then we have the six excedances corresponding to columns $-1, -2, -3, -4, 1, 2$.

Let $B_n(q)$ be the generating function of the colored descent statistic on S_n^B, i. e.

$$B_n(q) := \sum_{\sigma \in S_n^B} q^{\mathrm{exc}^{\mathrm{col}}(\sigma)}.$$

Proposition 2.1.8. *Let* $n \in \mathbb{P}$. *Then*

$$\sum_{\sigma \in S_n^B} q^{2\,\mathrm{exc}(\sigma)} t^{\mathrm{neg}(\sigma)} = A_n^*\left(\frac{q^2 + t}{1 + t}\right)(1 + t)^n.$$

In particular,

$$B_n(q) = A_n^*(q)(1 + q)^n. \tag{2.1.7}$$

Proof. Let $\sigma \in S_n^B$ be a signed permutation and let $|\sigma|$ be the corresponding permutation in S_n. Then the window notations of σ and $|\sigma|$ differs in $\mathrm{neg}(\sigma)$ entries. Note that by putting a negative sign in any entry i such that $|\sigma_i| > i$ the number of (classical) excedances decrements and obviously the statistic neg increases. If the entry i is such that $|\sigma_i| < i$, then any variation of sign preserves the number of classical excedances. Let h be the number of excedances of $|\sigma|$. With previous remarks we have

$$\begin{aligned}\sum_{\sigma\in S_n^B} q^{2\,\mathrm{exc}(\sigma)}t^{\mathrm{neg}(\sigma)} =&\sum_{h=0}^{n-1}\sum_{\substack{|\sigma|\in S_n\\ \mathrm{exc}(|\sigma|)=h}}\Big(\sum_{k_1=0}^{h}\binom{h}{k_1}q^{2k_1}t^{h-k_1}\Big)\Big(\sum_{k_2=0}^{n-h}\binom{n-h}{k_2}t^{n-h-k_2}\Big)\\ =&\sum_{h=0}^{n} A(n,h)(q^2+t)^h(1+t)^{n-h}\\ =&A_n^*\left(\frac{q^2+t}{1+t}\right)(1+t)^n,\end{aligned}$$

where the indices k_1, k_2 denote the number of elements i with $\sigma(i) > 0$ in the set of all indices with $|\sigma(i)| > i$ and $|\sigma(i)| \le i$, respectively.

The last part of the statement follows by setting $t = q$. □

Note that equation (2.1.7) can also be computed by [FH11, Theorem 5.3].

We now give an analogue formula for derangements in S_n^B. Let $d_n^B(q)$ be the generating function of the colored excedances on the set DS_n^B, i. e.

$$d_n^B(q) := \sum_{\sigma\in DS_n^B} q^{\mathrm{exc}^{\mathrm{col}}(\sigma)}.$$

Proposition 2.1.9. *Let $n \in \mathbb{P}$. Then*

$$\sum_{\sigma\in DS_n^B} q^{2\,\mathrm{exc}(\sigma)}t^{\mathrm{neg}(\sigma)} = t^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{t+1}{t}\right)^k d_k\left(\frac{q^2+t}{t+1}\right).$$

In particular,

$$d_n^B(q) = q^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{q+1}{q}\right)^k d_k(q).$$

Proof. In this proof, let $\overline{DS}_n^B$ be denote the set of all permutations σ in S_n^B with $|\sigma(i)| \ne i$ for all $i \in [1, n]$. By using the same arguments in the proof of Proposition 2.1.8, we have

$$\begin{aligned}\sum_{\sigma\in \overline{DS}_n^B} q^{2\,\mathrm{exc}(\sigma)}t^{\mathrm{neg}(\sigma)} =&\sum_{h=0}^{n-1}\sum_{\substack{\sigma\in DS_n\\ \mathrm{exc}(\sigma)=h}}\Big(\sum_{k_1=0}^{h}\binom{h}{k_1}q^{2k_1}t^{h-k_1}\Big)\Big(\sum_{k_2=0}^{n-h}\binom{n-h}{k_2}t^{n-h-k_2}\Big)\\ =&\sum_{h=0}^{n-1}[q^h]d_n(q)(q^2+t)^h(t+1)^{n-h}\\ =&d_n\left(\frac{q^2+t}{t+1}\right)(t+1)^n,\end{aligned}$$

where $[q^h]d_n(q)$ denotes the coefficient of q^h in $d_n(q)$. Now we consider all permutations in $DS_n^B \setminus \overline{DS}_n^B$. It is clear that for any permutation σ in this set there exists at least one index i such that $\sigma(i) = -i$. By choosing all possible sets of such indices, we get

$$\begin{aligned}\sum_{\sigma\in DS_n^B} q^{2\operatorname{exc}(\sigma)}t^{\operatorname{neg}(\sigma)} =& \sum_{k=0}^{n}\binom{n}{k}\sum_{\sigma\in \overline{DS}_k^B} q^{2\operatorname{exc}(\sigma)}t^{\operatorname{neg}(\sigma)}t^{n-k}\\ =& t^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{t+1}{t}\right)^k d_k\left(\frac{q^2+t}{t+1}\right)\end{aligned}$$

□

By Proposition 2.1.9, it is easy to compute the polynomials $d_n^B(q)$. Below are the polynomials $d_n^B(q)$ for $n \leq 4$.

$$\begin{aligned}d_1^B(q) =& q,\\ d_2^B(q) =& q + 3q^2 + q^3,\\ d_3^B(q) =& q + 7q^2 + 13q^3 + 7q^4 + q^5,\\ d_4^B(q) =& q + 15q^2 + 57q^3 + 87q^4 + 57q^5 + 15q^6 + q^7.\end{aligned}$$

All previous polynomials are symmetric. This property is true for all $n \geq 1$.

Proposition 2.1.10. *For all $n \geq 1$ the polynomial*

$$d_n^B(q) = \sum_{i=1}^{2n-1} a_i q^i$$

satisfies: $a_i = a_{2n-i}$ for all $i \in [1, n-1]$.

Proof. It suffices to prove that

$$q^{2n} d_n^B\left(\frac{1}{q}\right) = d_n^B(q).$$

By Proposition 2.1.9, we have

$$\begin{aligned}q^{2n} d_n^B\left(\frac{1}{q}\right) =& q^{2n}\frac{1}{q^n}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1+q^{-1}}{q^{-1}}\right)^k d_k\left(\frac{1}{q}\right)\\ =& q^n\sum_{k=0}^{n}\binom{n}{k}(1+q)^k d_k\left(\frac{1}{q}\right)\\ =& q^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1+q}{q}\right)^k q^k d_k\left(\frac{1}{q}\right) = d_n^B(q),\end{aligned}$$

where in the last equality we used $q^k d_k(\frac{1}{q}) = d_k(q)$, true since $d_n(q)$ is symmetric for $n \geq 1$ (see [Bre90], Corollary 1). □

Generating functions of the colored excedances on S_n^B and DS_n^B are strictly related one to the other as shown by the following easy formula

$$B_n(q) = \sum_{k=0}^{n} \binom{n}{k} d_k^B(q). \tag{2.1.8}$$

In fact, suppose that a signed permutation σ has $n-k$ fixed points. Then by removing the fixed points and reducing the remaining elements to $[-k,k] \setminus \{0\}$ by keeping the relative order, we get a derangement on $[-k,k] \setminus \{0\}$.

By (2.1.8) and [FH11, Theorem 5.2] we have the following result.

Corollary 2.1.11. *We have*

$$e^z \sum_{n\geq 0} d_n^B(q) \frac{z^n}{n!} = \sum_{n\geq 0} B_n(q) \frac{z^n}{n!} = \frac{q-1}{q} \frac{1}{1-q^{-1}e^{(q^2-1)z}}.$$

By (2.1.3) and Proposition 2.1.8 we have the following already known fact (see e. g. [FH11, Theorem 1.4]).

Corollary 2.1.12. *The flag descent numbers and the colored excedance numbers are equidistributed, i. e.*

$$\sum_{\sigma\in S_n^B} q^{\mathrm{fdes}(\sigma)} = \sum_{\sigma\in S_n^B} q^{\mathrm{exc}^{\mathrm{col}}(\sigma)}. \tag{2.1.9}$$

Next we give a combinatorial interpretation of identity (2.1.9), by giving a bijection from S_n^B onto itself such that the excedance number of a signed permutation is equal to the flag descent number of its image.

Combinatorial proof of Corollary 2.1.12. Let $\sigma \in S_n^B$ be a signed permutation and express it as a product of disjoint signed cycles (see e. g. Brenti [Bre94] and Chen [Che93]). We write each cycle with its minimum element first and then we order all the cycles ordered in decreasing (natural) order of their first elements. For example, if $\sigma = [2,3,-5,-1,4,6,7]$ then we can write σ in the cycle form $\sigma = (7)(6)(-5,4,-1,2,3)$. It is clear by definition that the number of colored excedances of σ is the sum of the colored excedances of each its cycles. We now give an algorithm than manipulate each cycle, by preserving the excedance statistic. Let ρ be a cycle of σ whose first element is negative. Then decompose ρ in more cycles, just writing a closed and an opened bracket before each negative element different from the first. For example, if ρ is the cycle $(-5,4,-1,2,3)$ then we consider the two (ordered) cycles $(-5,4),(-1,2,3)$. Since σ maps the rightmost element of each new cycle in a negative element, the excedence number is the same. Note that this operation is invertible: since all cycles are ordered by their first elements in decreasing order, we can recognize what cycles are obtained by previous algorithm.

We now consider all cycles without negative elements. We associate to all these cycles the sequence obtained just writing their elements without brackets. In our previous example, consider the cycles $(7)(6)$: we associate to them the ordered sequence $(7, 6)$, which we will consider as a unique cycle (distinct from $(6, 7)$).

Now we are able to define the map $\psi : S_n^B \to S_n^B$ such that $\mathrm{exc}^{\mathrm{col}}(\sigma) = \mathrm{fdes}(\psi(\sigma))$ for all $\sigma \in S_n^B$. Let $\sigma \in S_n^B$. Write σ in cycles as done in the previous algorithm. Then we write a sequence according to the follows rules: let n_c be the number of the cycles and let $\delta = 1$ if the first cycle has only positive elements, $\delta = 0$ otherwise. If $n_c + \delta$ is odd then we write all elements of the first cycle with negative signs and in the same order; if $n_c + \delta$ is even then we write all elements of the first cycle with positive signs and in the reverse order. In the next steps we write the elements of all other cycles with the opposite sign respect to the previous cycle: when the sign is positive we reverse the order of the elements in the cycle. We have therefore a sequence of n elements in $[-n, n] \setminus \{0\}$. Let σ' be the signed permutation associated to this sequence in window notation and set $\psi(\sigma) = \sigma'$. It is easy to verify that $\mathrm{exc}^{\mathrm{col}}(\sigma) = \mathrm{fdes}(\psi(\sigma))$. □

For example, by starting from the cycles $(7, 6)(-5, 4)(-1, 2, 3)$ we have the permutation $[6, 7, -5, -4, 3, 2, 1]$.
Therefore $\psi([2, 3, -5, -1, 4, 6, 7]) = [6, 7, -5, -4, 3, 2, 1]$. We have $\mathrm{exc}^{\mathrm{col}}(\sigma) = 6 = \mathrm{fdes}(\psi(\sigma))$. If $\sigma = [-2, 1, -3, 6, -4, 5]$, then we have $\sigma = (-2, 1)(-3)(-4, 6, 5)$, $\psi(\sigma) = [-2, -1, 3, -4, -6, -5]$ and $\mathrm{exc}^{\mathrm{col}}(\sigma) = \mathrm{fdes}(\sigma') = 5$.

Remark 2.1.13. *The previous algorithm used to create the bijection ψ, allows us to give another proof of Corollary 2.1.11. By writing each signed permutation $\sigma \in S_n^B$ in cycles as described in the previous algorithm, we have that the excedance number of σ is the sum of the excedances of its cycles. Therefore, we have to compute*

$$\sum_{m \geq 1} \sum_{l_1, \ldots, l_m \geq 1} \binom{l_1 + \cdots + l_m}{l_1, \ldots, l_m} \prod_{j=1}^{m} A^*_{l_j}(q^2)(q^{l-1} + q^l) \frac{z^{l_1 + \cdots + l_m}}{(l_1 + \cdots + l_m)!}.$$

The multinomial coefficient gives us the number of all partitions of $l_1 + \cdots + l_m$ in m blocks (i. e. the elements in the cycles); each factor $A^*_{l_j}(q^2)$ gives the statistic $\sum q^{2\,\mathrm{exc}(\sigma)}$ for all possible arrangements of the elements in the j-th block; $q^l + q^{l-1}$ gives the number of negative signs respectively when the first block has or not negative elements. Then, by applying (1.4.4), we get the same result of Corollary 2.1.11.

To conclude this section, we give a recursion formula for the generating function of the derangements. In [BGMS08, Proposition 5.1] there is a recurrence relation for the generating function of all signed permutations.

Proposition 2.1.14. *For $n \geq 2$ the generating function*

$$GFD_n^B(q,t) := \sum_{\sigma \in DS_n^B} q^{2\,\mathrm{exc}(\sigma)} t^{\mathrm{neg}(\sigma)}$$

satisfies the following recursion:

$$\begin{aligned} GFD_n^B(q,t) =& \big(q^2(n-1)+nt\big) GFD_{n-1}^B(q,t) + (q^2+t)\frac{1-q^2}{2q}\frac{\partial}{\partial q} GFD_{n-1}^B(q,t) \\ &+ (q^2+t)(n-1) GFD_{n-2}^B(q,t). \end{aligned}$$

Proof. Let $\sigma \in S_{n-1}^B$ and define $\sigma_i^{\pm} = [\sigma(1), \ldots, \sigma(i-1), \pm n, \sigma(i+1), \ldots, \sigma(n-1), \sigma(i)]$. It is clear that

$$\begin{aligned} \mathrm{exc}(\sigma_i^+) =& 1 + \mathrm{exc}(\sigma) - \chi(\sigma(i) > i) & \qquad \mathrm{neg}(\sigma_i^+) =& \mathrm{neg}(\sigma) \\ \mathrm{exc}(\sigma_i^-) =& \mathrm{exc}(\sigma) - \chi(\sigma(i) > i) & \qquad \mathrm{neg}(\sigma_i^+) =& \mathrm{neg}(\sigma) + 1. \end{aligned}$$

Therefore, we have

$$\begin{aligned} GFD_n^B(q,t) =& tGFD_{n-1}^B(q,t) + \sum_{i=1}^{n-1} \sum_{\sigma' \in DS_{n-2}^B} q^{2\,\mathrm{exc}(\sigma')} t^{\mathrm{neg}(\sigma')} (q^2+t) \\ &+ \sum_{i=1}^{n-1} \sum_{\sigma' \in DS_{n-1}^B} q^{2\,\mathrm{exc}(\sigma') - 2\chi(\sigma'(i)>i)} t^{\mathrm{neg}(\sigma')} (q^2+t), \end{aligned}$$

where $tGFD_{n-1}(q,t)$ corresponds to all elements $\sigma \in DS_n^B$ having a cycle $(-n)$, the first sum to all elements having a cycle $(i, \pm n)$ and the second sum to all other elements. Then,

$$\begin{aligned} GFD_n^B(q,t) =& tGFD_{n-1}^B(q,t) + (n-1)(q^2+t) GFD_{n-2}^B(q,t) + \\ & (q^2+t) \sum_{\sigma' \in DS_{n-1}^B} \Big(\mathrm{exc}(\sigma') q^{2\,\mathrm{exc}(\sigma')-2} t^{\mathrm{neg}(\sigma')} + \\ & + (n-1-\mathrm{exc}(\sigma')) q^{2\,\mathrm{exc}(\sigma')} t^{\mathrm{neg}(\sigma')} \Big) \\ =& tGFD_{n-1}^B(q,t) + (q^2+t)(n-1) GFD_{n-2}^B(q,t) + \\ & + (q^2+t)\Big(\frac{1-q^2}{2q}\frac{\partial}{\partial q} GFD_{n-1}^B(q,t) + (n-1) GFD_{n-1}^B(q,t) \Big). \end{aligned}$$

□

2.1.2 Absolute excedance statistic

In [BG06] another excedance statistic was introduced. Given $\sigma \in S_n^B$ we denote by $|\sigma|$ the permutation in S_n such that $|\sigma|(i) = |\sigma(i)|$ for all $i \in [1, n]$.

Definition 2.1.15. *For each $\sigma \in S_n^B$, the* absolute excedance number *of σ is the number*

$$\mathrm{exc}^{\mathrm{abs}}(\sigma) = \mathrm{exc}(|\sigma|) + \mathrm{neg}(\sigma).$$

It is easy to verify that

$$\sum_{\sigma \in S_n^B} q^{\mathrm{exc}(|\sigma|)} t^{\mathrm{neg}(\sigma)} = (1+t)^n A_n^*(q). \tag{2.1.10}$$

In particular, if we set $t = q$, by Proposition 2.1.8 we get

$$\sum_{\sigma \in S_n^B} q^{\mathrm{exc}^{\mathrm{abs}}(\sigma)} = \sum_{\sigma \in S_n^B} q^{\mathrm{exc}^{\mathrm{col}}(\sigma)}. \tag{2.1.11}$$

We give a bijection which prove (2.1.11) combinatorially. Let $\sigma \in S_n^B$ and define $\phi : S_n^B \to S_n^B$ by setting $\phi(\sigma)(i) = -\sigma(i)$ if $|\sigma(i)| > i$, and $\phi(\sigma)(i) = \sigma(i)$ otherwise. It is easy to verify that $\mathrm{exc}^{\mathrm{col}}(\sigma) = \mathrm{exc}^{\mathrm{abs}}(\phi(\sigma))$. For example, let $\sigma = [4, 3, -2, -6, 5, -1]$, then $\phi(\sigma) = [-4, -3, -2, 6, 5, -1]$ and $\mathrm{exc}^{\mathrm{col}}(\sigma) = 7 = \mathrm{exc}^{\mathrm{abs}}(\phi(\sigma))$. Note that $\phi(DS_n^B) = DS_n^B$, then it follows that

$$\sum_{\sigma \in DS_n^B} q^{\mathrm{exc}^{\mathrm{abs}}(\sigma)} = \sum_{\sigma \in DS_n^B} q^{\mathrm{exc}^{\mathrm{col}}(\sigma)}. \tag{2.1.12}$$

With the same techniques used in the previous section, it is possible to verify that

$$\sum_{\sigma \in DS_n^B} q^{\mathrm{exc}(|\sigma|)} t^{\mathrm{neg}(\sigma)} = \sum_{h=0}^{n} \binom{n}{h} d_h(q)(t+1)^h t^{n-h} \tag{2.1.13}$$

By (2.1.10) and (2.1.12), many of results given for the colored excedance can be extended for the absolute excedance. Here we summarize them.

Corollary 2.1.16. *Let $n \in \mathbb{P}$ then*

$$\sum_{\sigma \in DS_n^B} q^{\mathrm{exc}^{\mathrm{abs}}(\sigma)} = d_n^B(q) = q^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{q+1}{q}\right)^k d_k(q).$$

Moreover the previous polynomial is symmetric and the exponential generating function is

$$\sum_{n \geq 0} \sum_{\sigma \in DS_n^B} q^{\mathrm{exc}^{\mathrm{abs}}(\sigma)} \frac{z^n}{n!} = \frac{q-1}{q} \frac{e^{-z}}{1 - q^{-1} e^{(q^2-1)z}}.$$

Another immediate consequence is the following.

Corollary 2.1.17. *The flag descent numbers and the absolute excedance numbers are equidistributed, i. e.*

$$\sum_{\sigma\in S_n^B} q^{\mathrm{fdes}(\sigma)} = \sum_{\sigma\in S_n^B} q^{\mathrm{exc}^{\mathrm{abs}}(\sigma)}$$

and the map $\psi\circ\phi^{-1} : S_n^B \to S_n^B$ is a bijection which prove combinatorially this identity.

2.2 Statistics on the Coxeter group S_n^D

In 2003 Biagioli [Bia03] introduced a new descent statistic and a new major index for S_n^D in order to solve the problem of Foata for the group S_n^D. For any $\sigma = [\sigma_1, \ldots, \sigma_n] \in S_n^D$, Biagioli set

$$\mathrm{DDes}(\sigma) = \mathrm{Des}(\sigma) \biguplus \{-\sigma(i) - 1 | i \in \mathrm{neg}(\sigma)\} \setminus \{0\}.$$

For example, let $\sigma = [5, 3, -2, -1, 6, 4] \in S_6^D$. Then $\mathrm{Des}(\sigma) = \{1, 2, 5\}$ and $\mathrm{DDes}(\sigma) = \{1, 1, 2, 5\}$. Note that such set coincides with the set of classical descents if $\sigma \in S_n$.

Also set

$$\mathrm{ddes} = |\{\mathrm{DDes}\}|$$

and

$$\mathrm{dmaj}(\sigma) = \sum_{i\in\mathrm{DDes}(\sigma)} i. \tag{2.2.14}$$

The following results were proven by Biagioli (see [Bia03, Proposition 2.14 and Theorem 2.17]).

Proposition 2.2.1. *Let $n \in \mathbb{P}$. Then*

$$\sum_{\sigma\in S_n^D} q^{\mathrm{dmaj}(\sigma)} = \sum_{\sigma\in S_n^D} q^{l(\sigma)}.$$

Theorem 2.2.2. *Let $n \in \mathbb{P}$. Then*

$$\sum_{k\geq 0} [k+1]_q^n t^k = \frac{\sum_{\sigma\in S_n^D} t^{\mathrm{ddes}(\sigma)} q^{\mathrm{dmaj}(\sigma)}}{(1-t)(1-tq^n)\prod_{i=1}^{n-1}(1-t^2q^{2i})}$$

The previous result extends the Carlitz's identity to S_n^D.

Since S_n^D is a subgroup of S_n^B we can define the excedance statistics introduced in previous sections on it. In particular, since each permutation in S_n^B has an even number of negative elements in its window notation, it is simple to compute the generating function for the excedance statistics on S_n^D.

Corollary 2.2.3. *For all $n \in \mathbb{P}$ we get*

$$\sum_{\sigma \in S_n^D} q^{\mathrm{exc}^{\mathrm{col}}(\sigma)} = \frac{1}{2}\Big(A_n^*(q)(1+q)^n + A_n^*(-q)(1-q)^n\Big).$$

and

$$\begin{aligned}\sum_{\sigma \in DS_n^D} q^{\mathrm{exc}^{\mathrm{col}}(\sigma)} =& \frac{1}{2}\Big(d_n^B(q) + d_n^B(-q)\Big) \\ =& \frac{q^n}{2}\sum_{k=0}^{n}\binom{n}{k}\bigg(\Big(\frac{1+q}{q}\Big)^k d_k(q) + (-1)^{n-k}\Big(\frac{1-q}{q}\Big)^k d_k(-q)\bigg).\end{aligned}$$

It suffices to take only the even powers of q from the generating functions on S_n^B.

It is also simple to compute the absolute excedance statistic.

Corollary 2.2.4. *For all $n \in \mathbb{P}$ we get*

$$\sum_{\sigma \in S_n^D} q^{\mathrm{exc}^{\mathrm{abs}}(\sigma)} = \frac{1}{2}A_n^*(q)\Big((1+q)^n + (1-q)^n\Big)$$

and

$$\sum_{\sigma \in DS_n^B} q^{\mathrm{exc}^{\mathrm{abs}}(\sigma)} = \frac{1}{2}\sum_{h=0}^{n}\binom{n}{h}d_h(q)\Big((1+q)^h q^{n-h} + (1-q)^h(-q)^{n-h}\Big).$$

It suffices to take only even powers of the indeterminate t in (2.1.10) and in (2.1.13).

In order to compute the distribution of the excedance statistic of type B note that if $\sigma \in S_n^B$ is such that $|\sigma| \neq (1, 2, \ldots, n)$, let m be the greatest index such that $\sigma(m) \neq \pm m$ then $\mathrm{exc}^{\mathrm{cox}}(\sigma_m) = \mathrm{exc}^{\mathrm{cox}}(\sigma)$ where σ_m is obtained from σ just by changing the sign of m in its window notation. For example $\mathrm{exc}^{\mathrm{cox}}([1, 3, -4, 2, -5]) = \mathrm{exc}^{\mathrm{cox}}([1, 3, 4, 2, -5]) = 3$. Therefore, except for the elements with only cycles of length 1, the distribution of the excedance is the same for permutations with even negative elements or odd negative elements, i. e. the generating function on S_n^D is the half of that of S_n^B plus some terms corresponding to permutations with only cycles of length 1. If we denote with $B_n^{\mathrm{cox}}(q)$ the generating function $B_n^{\mathrm{cox}}(q) = \sum_{\sigma \in S_n^B} q^{\mathrm{exc}^{\mathrm{cox}}(\sigma)}$ and with $d_n^{\mathrm{cox}}(q)$ the generating function of the derangements with respect to the excedances of type B we have

Proposition 2.2.5. *For all $n \in \mathbb{P}$*

$$\sum_{\sigma \in S_n^D} q^{\mathrm{exc}^{\mathrm{cox}}(\sigma)} = \frac{B_n^{\mathrm{cox}}(q) - (q+1)^n}{2} + \frac{(1+q)^n + (1-q)^n}{2} = \frac{B_n^{\mathrm{cox}}(q) + (1-q)^n}{2}.$$

and

$$\sum_{\sigma \in DS_n^B} q^{\mathrm{exc}^{\mathrm{cox}}(\sigma)} = \frac{1}{2} d_n^{\mathrm{cox}}(q) + \frac{\epsilon(n)}{2} q^n,$$

where $\epsilon(n) = 1$ *if* n *is even,* $\epsilon(n) = 0$ *if* n *is odd.*

2.3 Signed excedance statistics on finite permutation groups

Carlitz's identity is an example of generating function joining two statistics on the same set. One of the classical objects of study on the permutation groups is the computation of the generating function of a statistic stat joined with the length statistic evaluated at -1. In this contest, we simply say that we enumerate the signed statistic.

Several signed-enumeration results over permutations are known. Recently, Sivasubramanian [Siv11] proved several results on signed-excedance enumeration of the set of classical permutations and its subset of all derangements. The main idea used by the author is to compute the signed generating functions of the statistics via determinants.

The purpose of this section is to extend the study of signed-excedance statistics done by Sivasubramanian to all finite and affine Weyl groups and to all derangements of such groups.

In [Man91] and [Siv11] the following result is showed.

Theorem 2.3.1. *For* $n \geq 1$*,*

$$\sum_{\sigma \in S_n} (-1)^{\mathrm{inv}(\sigma)} q^{\mathrm{exc}(\sigma)} = (1-q)^{n-1}.$$

In [MR03] and [Siv11] the following result is given.

Theorem 2.3.2. *For* $n \geq 2$*,*

$$\sum_{\sigma \in DS_n} (-1)^{\mathrm{inv}(\sigma)} q^{\mathrm{exc}(\sigma)} = (-1)^{n-1} q[n-1]_q.$$

Since the inversion statistic is equivalent to the length statistic on S_n (and the same is for the generalizations of the inversion statistic to other finite Coxeter and affine Weyl groups), it is possible to consider the analogous of the signed statistic also on other Coxeter groups.

In order to compute the generating functions of the signed excedances, we start with a useful result.

Lemma 2.3.3. *Let* $\sigma \in S_n^W$, *with* $W = B, D$. *Then*

$$l(\sigma) \equiv l(|\sigma|) + \mathrm{neg}(\sigma) \mod 2.$$

Proof. We argue the proof by induction on $l(\sigma)$. If $\mathrm{inv}(\sigma) = 0$ then $\sigma = [1, \dots, n]$ and the claim is trivial. Now let $l(\sigma) > 0$ and $i \in [0, n-1]$ (if $W = D$, $i \in [1, n]$) such that $l(\sigma) > l(s_i^W \sigma)$. For $i \neq 0$ we have by induction

$$\begin{aligned} l(\sigma) =& 1 + l(s_i^W \sigma) \equiv 1 + l(|s_i^W \sigma|) + \mathrm{neg}(s_i^W \sigma) \\ =& 1 + (l(|\sigma|) \pm 1) + \mathrm{neg}(s_i^W \sigma) \\ \equiv& l(|\sigma|) + \mathrm{neg}(\sigma) \end{aligned}$$

where the last step follows since s_i^W has an even number of negative signs in its window notation. If $i = 0$ then $l(|s_0^B \sigma|) = l(|\sigma|)$ and $\mathrm{neg}(s_0^B \sigma) = \mathrm{neg}(\sigma) \pm 1$ and the proof is similar. □

Now we are able to give the generating function of the signed colored excedances on the group S_n^B.

Theorem 2.3.4. *For* $n \geq 1$ *we have*

$$\sum_{\sigma \in S_n^B} (-1)^{l(\sigma)} q^{2\,\mathrm{exc}_A(\sigma)} t^{\mathrm{neg}(\sigma)} = (1 - q^2)^{n-1}(1 - t). \tag{2.3.15}$$

In particular,

$$\sum_{\sigma \in S_n^B} (-1)^{l(\sigma)} q^{\mathrm{exc}^{\mathrm{col}}(\sigma)} = (1 - q)^n (1 + q)^{n-1}.$$

Proof. We claim that the sum on the left-hand side of (2.3.15) is the determinant of the following $n \times n$ matrix

$$L_n = \begin{pmatrix} 1-t & q^2-t & q^2-t & \cdots & q^2-t \\ 1-t & 1-t & q^2-t & \cdots & q^2-t \\ 1-t & 1-t & 1-t & \cdots & q^2-t \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-t & 1-t & 1-t & \cdots & 1-t \end{pmatrix}.$$

Given a matrix $M = (m_{i,j})_{i,j \leq n}$, $\det M = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} m_{i,\sigma(i)}$. The signed permutations in S_n^B differ from the permutations in S_n only by the signs of their evaluations at $1, \dots, n$. In particular, a negative sign at place i in σ changes the parity of the length (by Lemma 2.3.3) and deletes the excedance if $\sigma(i) > i$. This justifies the summand $-t$ in each entry of L_n. Moreover if $\sigma(i) > i$ we have a contribution of q^2. It is routine to compute the determinant of L_n. □

Corollary 2.3.5. *For the set of all derangements in S_n^B we have*

$$\sum_{\sigma\in DS_n^B}(-1)^{l(\sigma)}q^{2\,\mathrm{exc}_A(\sigma)}t^{\mathrm{neg}(\sigma)}=(-1)^{n-1}(q^2[n-1]_{q^2}-t[n]_{q^2}). \qquad (2.3.16)$$

In particular,

$$\sum_{\sigma\in DS_n^B}(-1)^{l(\sigma)}q^{\mathrm{exc}^{\mathrm{col}}(\sigma)}=(-1)^n q[-q]_{2n-1}.$$

Proof. The argument is similar to the proof of Theorem 2.3.4. Namely, it suffices to compute the determinant of the following matrix

$$\begin{pmatrix} -t & q^2-t & q^2-t & \cdots & q^2-t \\ 1-t & -t & q^2-t & \cdots & q^2-t \\ 1-t & 1-t & -t & \cdots & q^2-t \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-t & 1-t & 1-t & \cdots & -t \end{pmatrix}.$$

□

Note that (2.3.15) and (2.3.16) are linear in t. This fact can also be seen directly as follows: let σ be a permutation with at least two negative elements in its window notation. Let a_1, a_2 be the minimal (in absolute value) negative elements. Then the permutation σ' whose window notation is obtained from that of σ by swapping a_1, a_2 has $\mathrm{exc}^{\mathrm{col}}(\sigma')=\mathrm{exc}^{\mathrm{col}}(\sigma)$ and $l(\sigma')=-l(\sigma)$, so their contribution vanish.

Since the group S_n^D is the subgroup of S_n^B of all permutations σ with $\mathrm{neg}(\sigma)$ even, it is immediate to compute the signed generating functions for S_n^D from the previous results.

Corollary 2.3.6. *For all $n \geq 4$ we have*

$$\sum_{\sigma\in S_n^D}(-1)^{l(\sigma)}q^{\mathrm{exc}^{\mathrm{col}}(\sigma)}t^{\mathrm{neg}(\sigma)}=(1-q^2)^{n-1}.$$

and

$$\sum_{\sigma\in DS_n^D}(-1)^{l(\sigma)}q^{\mathrm{exc}^{\mathrm{col}}(\sigma)}t^{\mathrm{neg}(\sigma)}=(-1)^{n-1}(q^2[n-1]_{q^2}).$$

We now obtain the signed generating function of the absolute excedance statistic.

Theorem 2.3.7. *For all $n \geq 2$ we have*

$$\sum_{\sigma\in S_n^B}(-1)^{l(\sigma)}q^{\mathrm{exc}_A(|\sigma|)}t^{\mathrm{neg}(\sigma)}=(1-t)^n(1-q)^{n-1}.$$

In particular,

$$\sum_{\sigma\in S_n^B}(-1)^{l(\sigma)}q^{\mathrm{exc}^{\mathrm{abs}}(\sigma)}=(1-q)^{2n-1}.$$

Furthermore

$$\sum_{\sigma\in DS_n^B}(-1)^{l(\sigma)}q^{\mathrm{exc}^{\mathrm{abs}}(\sigma)}=(-1)^n\frac{(2q-q^2)^n-q}{1-q} \tag{2.3.17}$$

Proof. The two generating functions are given by the determinants of

$$\begin{pmatrix} 1-t & q-qt & q-qt & \cdots & q-qt \\ 1-t & 1-t & q-qt & \cdots & q-qt \\ 1-t & 1-t & 1-t & \cdots & q-qt \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-t & 1-t & 1-t & \cdots & 1-t \end{pmatrix}$$

and

$$\begin{pmatrix} -q & q-q^2 & q-q^2 & \cdots & q-q^2 \\ 1-q & -q & q-q^2 & \cdots & q-q^2 \\ 1-q & 1-q & -q & \cdots & q-q^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-q & 1-q & 1-q & \cdots & -q \end{pmatrix}$$

(For the second part it is also possible to use the formula (1.4.8)). □

It is not too hard to see that

$$\frac{1}{q^n}\left(\frac{(2q-q^2)^n-q}{1-q}+q[n-1]_q\right)=\sum_{r=0}^{n-1}(-q)^r\sum_{j=r+1}^{n}\binom{n}{j}\binom{j-1}{r} \tag{2.3.18}$$

The right-hand side of (2.3.18) is a polynomial in $-q$ whose coefficients are the elements of the sequence A118801 in [OEI]. In [SW10] it is shown that the coefficient of $(-q)^r$ is equal to the number of all words in the alphabet $\{0,1,2\}$ of length n with r zeros and which do not end with the suffix $(0,2^i)$ for some $i\geq 0$ (if $r=0$ then the word cannot be 2^n). Thus gives a combinatorial interpretation of all signless coefficients in (2.3.17).

By taking the even powers of t in Theorem 2.3.7 we obtain the following corollary.

Corollary 2.3.8. *We have that*

$$\sum_{\sigma\in S_n^D}(-1)^{l(\sigma)}q^{\mathrm{exc}^{\mathrm{abs}}(\sigma)}t^{\mathrm{neg}(\sigma)}=\frac{(1-t)^n+(1+t)^n}{2}(1-q)^{n-1},$$

and

$$\sum_{\sigma\in DS_n^D}(-1)^{l(\sigma)}q^{\mathrm{exc}^{\mathrm{abs}}(\sigma)}=(-1)^n\frac{(2-q)^nq^n+q^{2n}-2q}{2(1-q)}.$$

In particular,

$$\sum_{\sigma\in S_n^D}(-1)^{l(\sigma)}q^{\mathrm{exc}^{\mathrm{abs}}(\sigma)}=\frac{(1-q)^n+(1+q)^n}{2}(1-q)^{n-1}.$$

The generating function for the signed-excedance of type B we consider the Coxeter excedance for the signed permutations. We have the following result.

Theorem 2.3.9. *For $n\geq 1$ we have*

$$\sum_{\sigma\in S_n^B}(-1)^{l(\sigma)}q^{\mathrm{exc}^{\mathrm{cox}}(\sigma)}t^{\mathrm{neg}(\sigma)}=(1-q)^{n-1}-qt^n(1-q)^{n-1},$$

and

$$\sum_{\sigma\in DS_n^B}(-1)^{l(\sigma)}q^{\mathrm{exc}^{\mathrm{cox}}(\sigma)}t^{\mathrm{neg}(\sigma)}=\frac{q(t-tq-1)^n-(-q)^n}{q-1}.$$

In particular,

$$\sum_{\sigma\in S_n^B}(-1)^{l(\sigma)}q^{\mathrm{exc}^{\mathrm{cox}}(\sigma)}=(1-q)^n, \tag{2.3.19}$$

and

$$\sum_{\sigma\in DS_n^B}(-1)^{l(\sigma)}q^{\mathrm{exc}^{\mathrm{cox}}(\sigma)}=(-q)^n. \tag{2.3.20}$$

Proof. In this case we cannot apply the technique involving determinants. Let $\sigma\in S_n^B$ be a signed permutation. We write σ in cycle notation, $\sigma=(c_{1,1},\ldots,c_{1,l_1})\cdots(c_{k,1},\ldots,c_{k,l_k})$ with $\sigma(|c_{i,j}|)=c_{i,j+1}$ for $i\leq k$ and $j<l_i$ and $\sigma(|c_{i,l_i}|)=c_{i,1}$. For example, if $\sigma=[6,-2,3,-1,-4,5]$, then we write $\sigma=(-1,6,5,-4)(-2)(3)$. Then the number of excedances of type B of σ is equal to the ascents in each cycle (i. e. all the pairs $c_{i,j}<c_{i,j+1}$ or $c_{i,l_i}<c_{i,1}$) plus the number of cycles of length one with a negative element. The length of σ is equal modulo 2 to the number of negative elements (by Lemma 2.3.3) plus the lengths of all cycles decreased by 1 (as for the classical permutations).

Now fix $k\in[1,n]$ and let $a_1<\cdots<a_k$ be k distincts element in $[1,n]$. Let $\{b_1,\ldots,b_{n-k}\}$ be the complementary set of $\{a_1,\ldots,a_k\}$ in $[1,n]$, $b_1<\cdots<b_{n-k}$. Consider all permutations $\sigma\in S_n^B$ with $|\sigma(i)|\neq i$ for all $i\in[1,n]$ and with the elements in their window notation given by $-a_1,\ldots,-a_k,b_1,\ldots,b_{n-k}$ in any order. By considerating on cycle structure, it is easy to verify that the signed enumerator of the excedances of type B on such subset of S_n^B is equal to the enumerator of the signed excedances on the set of classical derangements (for this purpose use the bijection which maps a_i in i and b_i in $k+i$ and note

that it preserves the number of excedances since there is no cycle of length 1 by assumption). By Theorem 2.3.2 we have that

$$\sum_{\substack{\sigma\in S_n^B\\ \sigma(i)\neq\pm i\forall i}}(-1)^{l(\sigma)}q^{\mathrm{exc}^{\mathrm{cox}}(\sigma)}t^{\mathrm{neg}(\sigma)}=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-1}q[n-1]_q(-t)^k$$
$$=-\,q[n-1]_q(t-1)^n$$

For $\sigma\in S_n^B$. Let $h(\sigma)$ be the number of elements i in $[1,n]$ such that $\sigma(i)\neq\pm i$. Then we have that

$$\sum_{\sigma\in S_n^B}(-1)^{l(\sigma)}q^{\mathrm{exc}^{\mathrm{cox}}(\sigma)}t^{\mathrm{neg}(\sigma)}=\sum_{h=0}^{n}\sum_{\sigma\in S_n^B;h(\sigma)=h}(-1)^{\mathrm{inv}(\sigma)}q^{\mathrm{exc}^{\mathrm{cox}}(\sigma)}t^{\mathrm{neg}(\sigma)}$$
$$=(1-qt)^n+\sum_{h=1}^{n}\binom{n}{h}(-q[h-1]_q)(t-1)^h(1-qt)^{n-h}$$
$$=(1-qt)^n-\frac{q}{q-1}\sum_{h=1}^{n}\binom{n}{h}(q^{h-1}-1)(t-1)^h(1-qt)^{n-h}$$
$$=(1-qt)^n-\frac{1}{q-1}\big((q(t-1)+1-qt)^n-(1-qt)^n\big)+$$
$$+\frac{q}{q-1}\big((t-1+1-qt)^n-(1-qt)^n\big)$$
$$=(1-q)^{n-1}-qt^n(1-q)^{n-1}.$$

(Note that the factors $(1-qt)$ in the second line of the previous equation are determined by thel cycles of length 1 and each negative sign in them increases the length and the number of excedances).

We now consider the set of all derangements in S_n^B. By the previous remark, we have only to substitute $(1-qt)$ with $(-qt)$ in the previous formula. So we have

$$\sum_{\sigma\in DS_n^B}(-1)^{l(\sigma)}q^{\mathrm{exc}^{\mathrm{cox}}(\sigma)}t^{\mathrm{neg}(\sigma)}=(-qt)^n+\sum_{h=1}^{n}\binom{n}{h}(-q[h-1]_q)(t-1)^h(-qt)^{n-h}$$
$$=(-qt)^n-\frac{q}{q-1}\sum_{h=1}^{n}\binom{n}{h}(q^{h-1}-1)(t-1)^h(-qt)^{n-h}$$
$$=(-qt)^n-\frac{1}{q-1}\big((q(t-1)-qt)^n-(-qt)^n\big)$$
$$+\frac{q}{q-1}\big((t-1-qt)^n-(-qt)^n\big)$$
$$=\frac{q(t-tq-1)^n-(-q)^n}{q-1}.$$

□

Note that (2.3.19) and (2.3.20) can also be computed combinatorially as follows: let $\sigma\in S_n^B$ and suppose there exists i such that $|\sigma(i)|\neq i$. Suppose that such i is minimal. Then let σ' be the unique permutation given by changing the sign of i in the window notation of σ. It is easy to check that $\mathrm{exc}^{\mathrm{cox}}(\sigma') = \mathrm{exc}^{\mathrm{cox}}(\sigma)$ and $l(\sigma') \equiv 1 + l(\sigma) \bmod 2$. Therefore, the generating function of the signed-excedances of type B can be computed on the set of all permutations $\pi \in S_n^B$ such that $|\pi(i)| = i$ for all $i \in [1, n]$ which is $(1-q)^n$ (or $(-q)^n$ in the case of derangements).

By taking even powers of t in the previous result we obtain the signed generating function for S_n^D and its derangement set.

Corollary 2.3.10. *We have*

$$\sum_{\sigma \in S_n^D} (-1)^{l(\sigma)} q^{\mathrm{exc}^{\mathrm{cox}}(\sigma)} t^{\mathrm{neg}(\sigma)} = \begin{cases} (1-q)^{n-1} - qt^n(1-q)^{n-1} & \text{if } n \text{ is even} \\ (1-q)^{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

In particular,

$$\sum_{\sigma \in S_n^D} (-1)^{l(\sigma)} q^{\mathrm{exc}^{\mathrm{cox}}(\sigma)} = (1-q)^{2\lfloor \frac{n}{2} \rfloor}$$

where $\lfloor a \rfloor$ denotes the integer part of a (a floor of a) and

$$\sum_{\sigma \in DS_n^B} (-1)^{l(\sigma)} q^{\mathrm{exc}^{\mathrm{cox}}(\sigma)} = \frac{1}{2} \frac{q(q-2)^n + (q-2)(-q)^n}{q-1}.$$

2.4 Excedance statistics on affine Weyl groups

A natural question that we can pose is to extend the definition of excedance statistic also to all numerable families of affine Weyl groups. In this section we recall some results already known for the group $\widetilde{S}_n$ and we extend the colored and absolute excedance statistics to $\widetilde{S}_n^B$, $\widetilde{S}_n^C$ and $\widetilde{S}_n^D$.

In [CE11] Clark and Ehrenborg introduced the concept of excedance in $\widetilde{S}_n$, by defining the following statistic. For any $\pi \in \widetilde{S}_n$, the number of excedances of π is

$$\mathrm{exc}(\pi) = \sum_{i=1}^{n} \left| \left\lceil \frac{\pi(i) - i}{n} \right\rceil \right|, \tag{2.4.21}$$

where $\lceil a \rceil$ denotes the smallest integer greater than or equal to a. If π maps $[1, n]$ in itself, then π can be naturally identified with a classical permutation and in this case the previous definition agrees with the classical one. In the same paper the authors showed that if $\pi = (\mathbf{r}_\pi, \sigma_\pi) \in \widetilde{S}_n$ then

$$\mathrm{exc}(\pi) = \|\mathbf{r}_\pi - \mathbf{p}_{\sigma_\pi}\|, \tag{2.4.22}$$

where $\mathbf{p}_{\sigma_\pi}$ is the vector in $\mathbb{N}^n$ whose i-th entry equals -1 if i is an excedance of $\sigma_\pi \in S_n$ and 0 otherwise. The symbol $\|\cdot\|$ denotes the norm 1 in $\mathbb{R}^n$.

The following result is given in [CE11, Theorem 6.5].

Theorem 2.4.1. *The generating function for affine excedances is*

$$\sum_{\pi\in\widetilde{S}_n} q^{\mathrm{exc}(\pi)} = \frac{1}{(1-q^2)^{n-1}}\sum_{k=0}^{n-1} A(n,k+1)\sum_{i=0}^{n-1-k}\binom{n-1-k}{i}\binom{n-1+k}{n-1-i} q^{2i+k}.$$

In the follows we give the definitions of the affine excedance for the other infinite families of affine Weyl groups and compute their generating functions.

2.4.1 Colored excedance on the affine Weyl groups

We start by defining a colored excedance for the group $\widetilde{S}_n^C$.

Definition 2.4.2. *The colored affine excedance for any permutation $\pi\in\widetilde{S}_n^C$ is*

$$\mathrm{exc}^{\mathrm{col}}(\pi) = 2\sum_{i=1}^{n}\left|\left\lceil\frac{\pi(i)-i}{2n+1}\right\rceil\right| + \mathrm{neg}(\sigma_\pi).$$

We recall that σ_π is the element in S_n^B defined by $\sigma_\pi(i) := \pi(i) \mod 2n+1$. For example, let $\pi\in\widetilde{S}_5^C$ be the permutation given by $\pi = [-3,-1,4,2,16]$ in window notation. Then $\sigma_\pi = (-3,-1,4,2,5)$ and $\mathrm{exc}^{\mathrm{col}}(\pi) = 2(0+0+1+0+1)+2 = 6$.

It is easy to verify that if π is a bijection on $[-n,n]$ then this definition agrees with Definition 2.1.6. In the follows we identify each element $\pi\in\widetilde{S}_n^C$ with the pair $(\mathbf{r}_\pi,\sigma_\pi)$ as defined in Section 1.3.

Lemma 2.4.3. *For all $\pi\in\widetilde{S}_n^C$, let $\mathbf{p}_{\sigma_\pi}\in\{-1,0\}^n$ be the vector with $p_{\sigma_\pi}(i) = -1$ if $\sigma_\pi(i) > i$ and 0 otherwise. Then*

$$\mathrm{exc}^{\mathrm{col}}(\pi) = 2\|\mathbf{r}_\pi - \mathbf{p}_{\sigma_\pi}\| + \mathrm{neg}(\sigma_\pi)$$

where $\|\cdot\|$ denotes the norm 1 in the Euclidian space $\mathbb{R}^n$.

Proof. It suffices to note that

$$\left|\left\lceil\frac{\pi(i)-i}{2n+1}\right\rceil\right| = \left|\left\lceil\frac{(2n+1)r_i+\sigma_\pi(i)-i}{2n+1}\right\rceil\right| = \begin{cases} |r_i| & \text{if } i\geq\sigma_\pi(i),\\ |r_i+1| & \text{if } i<\sigma_\pi(i).\end{cases}$$

By summing over all i the lemma follows. □

Here we give a useful easy result.

Lemma 2.4.4. *Let $a,b\in\mathbb{N}$ then*

$$\sum_{\mathbf{r}\in\mathbb{Z}^a\times(\mathbb{Z}^*)^b} q^{\|\mathbf{r}\|} = \frac{2^b q^b(1+q)^a}{(1-q)^{a+b}}$$

Proof. It suffices to compute $(1+2q+2q^2+\dots)^a(2q+2q^2+\dots)^b$. □

Theorem 2.4.5. *The generating function for the colored affine excedances of $\widetilde{S}_n^C$ is*

$$GF^{\text{col}}_{\widetilde{C}_n}(q) := \sum_{\pi\in\widetilde{S}_n^C} q^{\text{exc}^{\text{col}}(\pi)} = n!\left(\frac{1+q^2}{1-q}\right)^n.$$

Proof. By Lemma 2.4.3 we have that

$$GF^{\text{col}}_{\widetilde{C}_n}(q) = \sum_{\sigma\in S_n^B} q^{\text{neg}(\sigma)}\Big(\sum_{\mathbf{r}\in\mathbb{Z}^n} q^{2\|\mathbf{r}-\mathbf{p}_\sigma\|}\Big).$$

Since for any fixed σ, $\|\mathbf{r}-\mathbf{p}_\sigma\|$ assumes the same values of $\|\mathbf{r}\|$ when $\mathbf{r}$ runs overl $\mathbb{Z}^n$, we can conclude by Lemma 2.4.4 that

$$GF^{\text{col}}_{\widetilde{C}_n}(q) = \sum_{\sigma\in S_n^B} q^{\text{neg}(\sigma)}\left(\frac{1+q^2}{1-q^2}\right)^n = n!\left(\frac{1+q^2}{1-q}\right)^n.$$

□

It is also possible to compute the generating function for the derangements.

Theorem 2.4.6. *The generating function for the colored affine excedances of $\widetilde{DS}_n^C$ is*

$$GFD^{\text{col}}_{\widetilde{C}_n} := \left(\frac{1+q^2}{1-q}\right)^n \sum_{i=0}^{n}\binom{n}{i}\left(\frac{q(1+q)}{1+q^2}\right)^i d_{n-i}(1)$$

Proof. By Lemma 2.4.3 we have that

$$GFD^{\text{col}}_{\widetilde{C}_n} = \sum_{h=0}^{n}\sum_{\substack{\sigma\in S_n^B\\ \text{fix}(\sigma)=h}} q^{\text{neg}(\sigma)} \sum_{\mathbf{r}\in\mathbb{Z}^{n-h}\times(\mathbb{Z}^*)^h} q^{2\|\mathbf{r}-\mathbf{p}_\sigma\|},$$

Note that $\mathbf{r}$ ranges in $\mathbb{Z}^{n-h}\times(\mathbb{Z}^*)^h$ since $(\mathbf{r},\sigma)$ is a derangement and therefore $r_i\neq 0$ for all indices such that $\sigma(i)=i$.

Note that for any fixed σ, $\|\mathbf{r}-\mathbf{p}_\sigma\|$ assumes the same values of $\|\mathbf{r}\|$ when $\mathbf{r}$ runs over $\mathbb{Z}^{n-h}\times(\mathbb{Z}^*)^h$ (note that when $\mathbf{p}_\sigma(i)\neq 0$ then r_i could be zero). Then we conclude by Lemma 2.4.4

$$GFD^{\text{col}}_{\widetilde{C}_n} = \sum_{h=0}^{n}\sum_{\substack{\sigma\in S_n^B\\ \text{fix}(\sigma)=h}} q^{\text{neg}(\sigma)}\left(\frac{2^h q^{2h}(1+q^2)^{n-h}}{(1-q^2)^n}\right).$$

Now we have to compute the coefficient $\sum_{\sigma\in S_n^B,\text{fix}(\sigma)=h} q^{\text{neg}(\sigma)}$. We first choose h elements which are fixed by σ. Then we distinguish the indices $i\in[1,n]$ such that $\sigma(i)=-i$ from the others. Then

$$\sum_{\substack{\sigma\in S_n^B\\ \text{fix}(\sigma)=h}} q^{\text{neg}(\sigma)} = \binom{n}{h}\sum_{k=0}^{n-h}\binom{n-h}{k}\sum_{\sigma\in DS_{n-h-k}} q^k(1+q)^{n-h-k}.$$

Therefore we complete as follows.

$$\begin{aligned} GFD^{\mathrm{col}}_{\widetilde{C}_n} &= \left(\frac{1+q^2}{1-q}\right)^n \sum_{h=0}^{n} \binom{n}{h} \sum_{k=0}^{n-h} \binom{n-h}{k} d_{n-h-k}(1) \frac{q^k}{(1+q)^{h+k}} \frac{2^h q^{2h}}{(1+q^2)^h} \\ &= \left(\frac{1+q^2}{1-q}\right)^n \sum_{i=0}^{n} \sum_{h=0}^{i} \left(\frac{q}{1+q}\right)^i d_{n-i}(1) \binom{n}{h} \binom{n-h}{i-h} \left(\frac{2q}{1+q^2}\right)^h \\ &= \left(\frac{1+q^2}{1-q}\right)^n \sum_{i=0}^{n} \binom{n}{i} \left(\frac{q}{1+q}\right)^i d_{n-i}(1) \sum_{h=0}^{i} \binom{i}{h} \left(\frac{2q}{1+q^2}\right)^h \\ &= \left(\frac{1+q^2}{1-q}\right)^n \sum_{i=0}^{n} \binom{n}{i} \left(\frac{q}{1+q}\right)^i d_{n-i}(1) \left(\frac{2q}{1+q^2}+1\right)^i, \end{aligned}$$

where in the second equation we introduce the index $i = h + k$. □

We now study the statistic on the group $\widetilde{S}^B_n$. Since $\widetilde{S}^B_n$ is a subgroup of $\widetilde{S}^C_n$ the previous definition can be extended to this group.

Theorem 2.4.7. *The generating function for the colored affine excedances of $\widetilde{S}^B_n$ is*

$$GF^{\mathrm{col}}_{\widetilde{B}_n}(q) := \sum_{\pi \in \widetilde{S}^C_n} q^{\mathrm{exc}^{\mathrm{col}}(\pi)} = \frac{n!}{2}\left(\frac{1+q^2}{1-q}\right)^n + \left(\frac{(1+q)^2(1-q)}{1+q^2}\right)^n A^*_n\left(\frac{q-1}{q+1}\right).$$

Proof. By definition for any fixed $\sigma \in S^B_n$ we have that

$$\sum_{\mathbf{r} \in Z^n \text{ even}} q^{2\|\mathbf{r}-\mathbf{p}_\sigma\|} = \sum_{\mathbf{r}'} q^{2\|r'\|} \tag{2.4.23}$$

where the second sum is taken on all even (resp. odd) elements $\mathbf{r}' \in \mathbb{Z}^n$ if $\mathbf{p}_\sigma$ is even (resp. odd). In this proof we denote with $R(1) = \sum_{\mathbf{r} \text{ even}} q^{2\|\mathbf{r}\|}$ and with $R(-1) = \sum_{\mathbf{r} \text{ odd}} q^{2\|r\|}$. By Lemma 2.4.3 and (1.3.2) we can write

$$GF^{\mathrm{col}}_{\widetilde{B}_n}(q) = \sum_{h=0}^{n} \sum_{\substack{\sigma \in S_n \\ \mathrm{exc}(\sigma)=h}} \Big(\sum_{k_1=0}^{h} \binom{h}{k_1} q^{h-k_1} R((-1)^{k_1}) \Big) \Big(\sum_{k_2=0}^{n-h} \binom{n-h}{k_2} q^{n-h-k_2} \Big). \tag{2.4.24}$$

We now justify the previous expression: we obtain all permutations in S^B_n by adding negative signs to all permutations in S_n. We consider a permutation $\sigma \in S_n$ with h excedances (in the classical sense). Each negative sign applied to elements which are excedances of σ, decreases the value $\mathrm{exc}_A(\sigma)$. The index k_1 denotes the number of all positive signs in set of all excedances of σ. The factor $R((-1)^{k_1})$ follows by (2.4.23). The monomials q^{h-k_1} and q^{n-h-k_2} are the contributions of $q^{\mathrm{neg}(\sigma)}$ in the definition of $GF^{\mathrm{col}}_{\widetilde{B}_n}(q)$.

Note that in the first product of (2.4.24) all the even (resp. odd) powers of q have the same coefficient $R((-1)^{k_1})$. Therefore we can write:

$$\begin{aligned}GF^{\text{col}}_{\widetilde{B}_n}(q) =&\sum_{h=0}^{n} A(n,h)(1+q)^{n-h}\\ &\cdot\left(R(1)\frac{(q+1)^h+(q-1)^h}{2}+R(-1)\frac{(q+1)^h-(q-1)^h}{2}\right)\\ =&n!(1+q)^n\frac{R(1)+R(-1)}{2}+(1+q)^n A_n^*\left(\frac{q-1}{q+1}\right)\frac{R(1)-R(-1)}{2}.\end{aligned}$$

But we easily compute $R(1), R(-1)$ by Lemma 2.4.4 and therefore we can conclude

$$GF^{\text{col}}_{\widetilde{B}_n}(q) = \frac{n!}{2}\left(\frac{1+q^2}{1-q}\right)^n + \left(\frac{(1+q)^2(1-q)}{1+q^2}\right)^n A_n^*\left(\frac{q-1}{q+1}\right).$$

□

The formula for the derangements is not very elegant.

Theorem 2.4.8. *The generating function for the colored affine excedences of the derangements of* $\widetilde{S}_n^B$ *is*

$$\begin{aligned}GFD^{\text{col}}_{\widetilde{B}_n}(q) :=& \sum_{\sigma\in\widetilde{DS}_n^B} q^{\text{exc}^{\text{col}}(\sigma)}\\ =&\frac{(1+q^2)^n}{2(1-q)^n}\sum_{a=0}^{n}\binom{n}{a}\left(\frac{(1+q)q}{(1+q^2)}\right)^a d_{n-a}(1)\\ &+\frac{(1+q)^{2n}(1-q)^n}{2(1+q^2)^n}\sum_{a=0}^{n}\binom{n}{a}\left(\frac{(1-2q-q^2)q}{(1+q)^2(1-q)}\right)^a d_{n-a}\left(\frac{q-1}{q+1}\right).\end{aligned}$$

Sketch of the proof. By using the same techniques of Theorem 2.4.7, we can write

$$\begin{aligned}GFD^{\text{col}}_{\widetilde{B}_n}(q) =& \sum_{a=0}^{n}\binom{n}{a}\sum_{h=0}^{n-a}\sum_{\substack{\sigma\in D_{n-a}\\ \text{exc}(\sigma)=h}}\Big(\sum_{k_0=0}^{a}\binom{a}{k_0}q^{a-k_0}\sum_{k_1=0}^{h}\binom{h}{k_1}q^{h-k_1}R_{k_0}((-1)^{k_1})\Big)\\ &\Big(\sum_{k_1=0}^{n-a-h}\binom{n-a-h}{k_2}q^{k_2}\Big),\end{aligned}$$

where $R_{k_0}(1)$ and $R_{k_0}(-1)$ denote the even powers and the odd powers respectively of the formal series in Lemma 2.4.4, with $b = k_0$ and $a = n - k_0$.

The index a denotes the fixed elements of a permutation $\sigma \in S_n$, the index h denotes the excedances of σ, k_0 are the positive signs in the set of fixed elements of σ, k_1 are the positive signs in the set of excedances of σ, and k_2 the negative signs in all other positions. The result follows by algebraic computation. □

Finally, we consider the group $\widetilde{S}_n^D$. By (1.3.3) we know that the colored affine excedance statistic in $\widetilde{S}_n^D$ has only monomials in q with even exponents. Therefore, the following results are trivial.

Corollary 2.4.9. *The generating function for the colored affine excedences of $\widetilde{S}_n^D$ is*

$$GF^{\mathrm{col}}_{\widetilde{D}_n}(q) := \sum_{\pi \in \widetilde{S}_n^D} q^{\mathrm{exc}^{\mathrm{col}}(\sigma)} = \frac{GF^{\mathrm{col}}_{\widetilde{B}_n}(q) + GF^{\mathrm{col}}_{\widetilde{B}_n}(-q)}{2}.$$

The generating function for the colored affine excedences of the derangements of $\widetilde{S}_n^D$ is

$$GFD^{\mathrm{col}}_{\widetilde{D}_n}(q) := \sum_{\pi \in D\widetilde{S}_n^D} q^{\mathrm{exc}^{\mathrm{col}}(\sigma)} = \frac{GFD^{\mathrm{col}}_{\widetilde{B}_n}(q) + GFD^{\mathrm{col}}_{\widetilde{B}_n}(-q)}{2}.$$

We note that in $\widetilde{S}_n^C$ there are more permutations with zero excedances than just the identity. In order to have that the only permutation with zero excedances is the trivial permutation then it is possible to change the definition as follows.

Definition 2.4.10. *Let $\pi \in \widetilde{S}_n^C$. Then the variant colored affine excedance of π is*

$$\mathrm{exc}^{\mathrm{col}\,2}(\pi) = \sum_{i=1}^{n} \left| \left\lceil \frac{\pi(i) - i}{2n+1} \right\rceil \right| + \mathrm{exc}_A(\sigma_\pi) + \mathrm{neg}(\sigma_\pi).$$

For example, let $\pi \in \widetilde{S}_5^C$ be the permutation given by $\pi = [-3, -1, 4, 2, 16]$ in window notation. Then $\sigma_\pi = (-3, -1, 4, 2, 5)$ and $\mathrm{exc}^{\mathrm{col}\,2}(\pi) = (0 + 0 + 1 + 0 + 1) + 1 + 2 = 5$.

It is easy to verify that with this definition the only permutation with zero excedances is the identity. The generating functions can be computed with the same techniques of the previous results, therefore we omit the proofs.

Theorem 2.4.11. *The generating function for the variant colored affine excedances of $\widetilde{S}_n^C$ is*

$$GF^{\mathrm{col}\,2}_{\widetilde{C}_n}(q) := \sum_{\pi \in \widetilde{S}_n^C} q^{\mathrm{exc}^{\mathrm{col}\,2}(\sigma)} = \left(\frac{(1+q)^2}{1-q} \right)^n A_n^* \left(\frac{2q}{1+q} \right);$$

the generating function on its derangement set is

$$\begin{aligned} GFD^{\mathrm{col}\,2}_{\widetilde{C}_n}(q) :&= \sum_{\pi \in \widetilde{DS}_n^C} q^{\mathrm{exc}^{\mathrm{col}\,2}(\sigma)} \\ &= \left(\frac{(1+q)^2}{1-q} \right)^n \sum_{a=0}^{n} \binom{n}{a} \left(\frac{q(3+q)}{(1+q)^2} \right)^a d_{n-a} \left(\frac{2q}{1+q} \right). \end{aligned}$$

The analogous formulas for the group $\widetilde{S}_n^B$ is the following.

Theorem 2.4.12. *The generating function for the variant colored affine excedances of $\widetilde{S}_n^B$ is*

$$GF_{\widetilde{B}_n}^{\mathrm{col}\,2}(q) := \sum_{\pi \in \widetilde{S}_n^C} q^{\mathrm{exc}^{\mathrm{col}\,2}(\sigma)} = \frac{1}{2}\left(\frac{(1+q)^2}{1-q}\right)^n A_n^*\left(\frac{2q}{1+q}\right) + \frac{(1-q)^n}{2};$$

the generating function on its derangement set is

$$GFD_{\widetilde{B}_n}^{\mathrm{col}\,2}(q) := \sum_{\pi \in \widetilde{DS}_n^C} q^{\mathrm{exc}^{\mathrm{col}\,2}(\sigma)} = \frac{1}{2}\left(\frac{q(3+q)}{1-q}\right)^n + \frac{(-q)^n}{2}$$
$$+ \frac{1}{2}\left(\frac{(1+q)^2}{1-q}\right)^n \sum_{a=0}^{n-1} \binom{n}{a} \left(\frac{q(3+q)}{(1+q)^2}\right)^a d_{n-a}\left(\frac{2q}{1+q}\right).$$

The analogue of Lemma 2.4.3 is the following identity for any $\pi = (\mathbf{r}_\pi, \sigma_\pi) \in \widetilde{S}_n^C$:

$$\mathrm{exc}^{\mathrm{col}\,2}(\pi) = \|\mathbf{r}_\pi - \mathbf{p}_{\sigma_\pi}\| + \mathrm{exc}_A(\sigma_\pi) + \mathrm{neg}(\sigma_\pi).$$

If $\mathbf{r} \in \mathbb{Z}^n$ is even then it is obvious that $\|\mathbf{r} - \mathbf{p}_{\sigma_\pi}\| + \mathrm{exc}_A(\sigma_\pi)$ is an even number. Therefore the generating functions on $\widetilde{S}_n^D$ and its derangements subset are even formal power series.

Corollary 2.4.13. *The generating function for the variant colored affine excedances of $\widetilde{S}_n^D$ is*

$$GF_{\widetilde{D}_n}^{\mathrm{col}\,2}(q) := \sum_{\pi \in \widetilde{S}_n^D} q^{\mathrm{exc}^{\mathrm{col}\,2}(\sigma)} = \frac{1}{2}\Big(GF_{\widetilde{B}_n}^{\mathrm{col}\,2}(q) + GF_{\widetilde{B}_n}^{\mathrm{col}\,2}(-q)\Big);$$

the generating function on its derangement set is

$$GFD_{\widetilde{D}_n}^{\mathrm{col}\,2}(q) := \sum_{\pi \in \widetilde{DS}_n^D} q^{\mathrm{exc}^{\mathrm{col}\,2}(\sigma)} = \frac{1}{2}\Big(GFD_{\widetilde{B}_n}^{\mathrm{col}\,2}(q) + GFD_{\widetilde{B}_n}^{\mathrm{col}\,2}(-q)\Big)$$

2.4.2 Absolute excedance for the affine Weyl groups

In this section we introduce the analogue of the absolute excedance for the affine Weyl groups. We follows the same ideas in the previous section.

Definition 2.4.14. *The absolute affine excedance for any permutation $\pi \in \widetilde{S}_n^C$ is*

$$\mathrm{exc}^{\mathrm{abs}}(\pi) = \sum_{i=1}^{n} \left| \left\lceil \frac{|\pi(i)| - i}{2n+1} \right\rceil \right| + \mathrm{neg}(\sigma_\pi),$$

For example, let $\pi \in \widetilde{S}_5^C$ be the permutation given by $\pi = [-3, -1, 4, 2, 16]$ in window notation. Then $\sigma_\pi = (-3, -1, 4, 2, 5)$ and $\text{exc}^{\text{abs}}(\pi) = (1 + 0 + 1 + 0 + 1) + 2 = 5$.

We now want to have a result as in Lemma 2.4.3 that allows us to compute easily the value of the absolute affine excedance. There is no explicit simple formula but we note that for $i \in [1, n]$

$$\left| \left\lceil \frac{|\pi(i)| - i}{2n+1} \right\rceil \right| = \begin{cases} r_i + \chi(\sigma_\pi(i) > i) & \text{if } r_i > 0 \\ \chi(|\sigma_\pi(i)| > i) & \text{if } r_i = 0 \\ |r_i| + \chi(\sigma_\pi(i) < -i) & \text{if } r_i < 0 \end{cases} \tag{2.4.25}$$

where $\pi = (\mathbf{r}_\pi, \sigma_\pi)$ is as usual. In particular, if r_i runs over $\mathbb{Z}$, the previous expression assumes all values in $\{1, 1, 2, 2, 3, 3, \dots\}$ if $|\sigma_\pi(i)| > i$ and in $\{0, 1, 1, 2, 2, 3, 3, \dots\}$ otherwise. Moreover, if i is such that $|\sigma_\pi(i)| > i$ and r_i runs over $\mathbb{Z}$ and it is always even or always odd then the previous expression assumes all values in $\mathbb{P}$.

Now we can describe the generating functions.

Theorem 2.4.15. *The generating function for the absolute affine excedences of $\widetilde{S}_n^C$ is*

$$GF^{\text{abs}}_{\widetilde{C}_n}(q) := \sum_{\pi \in \widetilde{S}_n^C} q^{\text{exc}^{\text{abs}}(\pi)} = \left(\frac{(1+q)^2}{1-q} \right)^n A_n^*\left(\frac{2q}{1+q} \right).$$

Proof. By (2.4.25), we have

$$GF^{\text{abs}}_{\widetilde{C}_n}(q) = \sum_{h=0}^{n-1} \sum_{\substack{\sigma \in S_n \\ \text{exc}(\sigma)=h}} \sum_{k=0}^{n} \binom{n}{k} q^k \sum_{\mathbf{r} \in (\mathbb{Z}^*)^h \times \mathbb{Z}^{n-h}} q^{\|\mathbf{r}\|},$$

where h denotes the classical excedances of the permutation σ and k denotes all possible negative signs to have permutations in S_n^B. Now apply Lemma 2.4.4 and conclude that

$$\begin{aligned} GF^{\text{abs}}_{\widetilde{C}_n}(q) &= \sum_{h=0}^{n-1} \sum_{\substack{\sigma \in S_n \\ \text{exc}(\sigma)=h}} (q+1)^n \frac{2^h q^h (1+q)^{n-h}}{(1-q)^n} \\ &= \left(\frac{(1+q)^2}{1-q} \right)^n A_n^*\left(\frac{2q}{1+q} \right). \end{aligned}$$

□

Theorem 2.4.16. *The generating function for the absolute affine excedences* *of the derangements of $\widetilde{S}_n^C$ is*

$$\begin{aligned} GFD^{\text{abs}}_{\widetilde{C}_n}(q) :&= \sum_{\pi \in D\widetilde{S}_n^C} q^{\text{exc}^{\text{abs}}(\pi)} \\ &= \left(\frac{(1+q)^2}{(1-q)}\right)^n \sum_{a=0}^{n} \binom{n}{a} \left(\frac{q(3+q)}{(1+q)^2}\right)^a d_{n-a}\left(\frac{2q}{1-q}\right). \end{aligned}$$

Sketch of the proof. Just compute

$$\sum_{a=0}^{n} \binom{n}{a} \sum_{h=0}^{n-a} \sum_{\substack{\sigma \in D_{n-a} \\ \text{exc}(\sigma)=h}} (q+1)^{n-a} \sum_{k=0}^{a} \binom{a}{k} q^{a-k} \sum_{\mathbf{r} \in (\mathbb{Z}^*)^{h+k} \times \mathbb{Z}^{n-h-k}} q^{\|\mathbf{r}\|}.$$

□

Theorem 2.4.17. *The generating function for the colored affine excedances of $\widetilde{S}_n^B$ is*

$$\begin{aligned} GF^{\text{abs}}_{\widetilde{B}_n}(q) &:= \sum_{\pi \in \widetilde{S}_n^B} q^{\text{exc}^{\text{abs}}(\pi)} \\ &= \frac{1}{2}\left(\frac{(1+q)^2}{1-q}\right)^n A_n^*\left(\frac{2q}{1+q}\right) + \frac{(1-q)^n}{2}. \end{aligned}$$

Proof. We have to consider two cases: first suppose that $\sigma \in S_n$ is a classical permutation with at least one excedance. Fix an index corresponding to one such excedance; for example, let suppose it is 1. Then we choose all possible signs in σ and choose all vectors $\mathbf{r}$ in such a way that $(r_2, \ldots, r_n)$ are taken freely and r_1 determines the parity of $\mathbf{r}$. Then when we sum $q^{\text{exc}^{\text{abs}}(\mathbf{r},\sigma)}$ on all possible values of $\mathbf{r}$ we have the same result of

$$\sum_{\mathbf{r} \in \mathbb{P} \times (\mathbb{Z}^*)^{h-1} \times \mathbb{Z}^{n-h}} q^{\|\mathbf{r}\|}$$

as justified by the comments immediately after (2.4.25). Otherwise suppose that σ is the identity, then we have to sum the contributions of (even) $\mathbf{r}$ and of all possible signs. Then we have

$$GF^{\text{abs}}_{\widetilde{B}_n}(q) = \sum_{h=1}^{n} \sum_{\substack{\sigma \in S_n \\ \text{exc}(\sigma)=h}} (q+1)^n \sum_{\mathbf{r} \in \mathbb{P} \times (\mathbb{Z}^*)^{h-1} \times \mathbb{Z}^{n-h}} q^{\|\mathbf{r}\|} + (q+1)^n \sum_{r \in \mathbb{Z}^n \text{ even}} q^{\|\mathbf{r}\|}.$$

The result follows after algebraic manipulation and by Lemma 2.4.4. □

Theorem 2.4.18. *The generating function for the absolute affine excedences of the derangements of $\widetilde{S}_n^B$ is*

$$\begin{aligned} GFD^{\mathrm{abs}}_{\widetilde{B}_n}(q) &:= \sum_{\pi \in D\widetilde{S}^B_n} q^{\mathrm{exc}^{\mathrm{col}}(\pi)} \\ &= \frac{1}{2}\left(\frac{(1+q)^{2n}}{(1-q)^n}\sum_{a=0}^{n-2}\binom{n}{a}\left(\frac{q(3+q)}{(1+q)^2}\right)^a d_{n-a}\left(\frac{2q}{1+q}\right)\right) \\ &+ \frac{(q(3+q))^n}{2(1-q)^n} + \frac{(-q)^n}{2}. \end{aligned}$$

Sketch of the proof. With the same arguments of Theorem 2.4.17, it suffices to compute

$$\sum_{a=0}^{n-2}\binom{n}{a}\sum_{h=1}^{n-a-1}\sum_{\substack{\sigma\in D_{n-a}\\ \mathrm{exc}(\sigma)=h}}(q+1)^{n-a}\sum_{k=0}^{a}\binom{a}{k}q^{a-k}\sum_{\mathbf{r}\in\mathbb{P}\times(\mathbb{Z}^*)^{h+k-1}\times\mathbb{Z}^{n-h-k}} q^{\|\mathbf{r}\|}$$
$$+\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sum_{\mathbf{r}\in(\mathbb{Z}^*)^k\times\mathbb{Z}^{n-k},\ \mathrm{even}} q^{\|\mathbf{r}\|}.$$

□

Note that for $\widetilde{S}^C_n$ and $\widetilde{S}^B_n$ the generating functions of the affine absolute excedances and the variant affine colored excedances are the same. Then we ask if there exists a bijection that shows these identities in a combinatorial way. Given $\pi = (\mathbf{r}_\pi, \sigma_\pi) \in \widetilde{S}^C_n$ and $i \in [1,n]$, we say that i satisfies the condition $\overline{C}$ with respect to π if both $r_i \geq 0$ and $\sigma(i) > i$ or both $r_i \leq 0$ and $\sigma(i) < -i$. We define the map

$$\widetilde{\phi} : \widetilde{S}^C_n \to \widetilde{S}^C_n$$

by setting for all $(\mathbf{r},\sigma) \in \widetilde{S}^C_n$ $\widetilde{\phi}((\mathbf{r},\sigma)) = ((\mathbf{r}',\sigma'))$ with $r'_i = -r_i$ and $\sigma'(i) = -\sigma(i)$ if i satisfies condition $\overline{C}$ with respect to π and $r'_i = r_i$ and $\sigma'(i) = \sigma(i)$ otherwise. It is easy to verify by the analogue of Lemma 2.4.3 and by (2.4.25) that $\mathrm{exc}^{\mathrm{col}\,2}(\pi) = \mathrm{exc}^{\mathrm{abs}}(\widetilde{\phi}(\pi))$; moreover, $\widetilde{\phi}_{|S^B_n} \simeq \phi$ up to a natural identification, where ϕ is defined immediately after (2.1.11). For example, let $\pi = [13,-14,12,4,16] \in \widetilde{S}^C_5$. Then $\widetilde{\phi}(\pi) = [-13,14,12,4,16]$ and $\mathrm{exc}^{\mathrm{col}\,2}(\pi) = 7 = \mathrm{exc}^{\mathrm{abs}}(\widetilde{\phi}(\pi))$.

Note that this bijection does not preserve the number of negative signs in the window notation of the permutations. In fact the generating functions on $\widetilde{S}^D_n$ of the affine absolute excedances will be different from the generating functions of the variant affine colored excedances.

Proposition 2.4.19. *The generating function for the affine absolute excedances* *of* $\widetilde{S}_n^D$ *is*

$$GF^{\mathrm{abs}}_{\widetilde{D}_n}(q) := \sum_{\pi \in \widetilde{S}_n^D} q^{\mathrm{exc}^{\mathrm{abs}}(\pi)} = \left(\frac{1}{2} + \frac{(1-q)^n}{2(1+q)^n}\right) GF^{\mathrm{abs}(q)}_{\widetilde{B}_n}.$$

The generating function for the affine absolute excedances of derangements on $\widetilde{S}_n^D$ *is*

$$GFD^{\mathrm{abs}}_{\widetilde{D}_n}(q) := \sum_{\pi \in \widetilde{DS}_n^D} q^{\mathrm{exc}^{\mathrm{abs}}(\pi)} = \frac{1}{2} GFD^{\mathrm{abs}}_{\widetilde{B}_n}(q) + \frac{1}{4}\left(\frac{q(q-3)}{1+q}\right)^n + \frac{q^n}{4} + \frac{(1+q)^n}{4} \sum_{a=0}^{n-1} \binom{n}{a} \left(\frac{q}{q+1}\right)^a d_{n-a}\left(\frac{2q}{1+q}\right).$$

The proof uses the same arguments of those of Theorems 2.4.15 and 2.4.16.

2.4.3 Signed excedances in affine Weyl groups

As done for the excedance statistics in finite Coxeter groups, we now enumerate the signed statistic for the affine groups. We start with the affine group $\widetilde{S}$.

Lemma 2.4.20. *Let* $\pi = (\mathbf{r}_\pi, \sigma_\pi) \in \widetilde{S}_n$. *Then*

$$l(\pi) \equiv l(\sigma_\pi) \mod 2$$

Proof. We prove the claim by induction on the length of π. If $l(\pi) = 0$ then $\pi = \sigma_\pi$ and the result is trivial. Now we consider $\widetilde{s}_i\pi$, $i \in [n]$, such that $l(\widetilde{s}_i\pi) > l(\pi)$. By induction we have

$$l(\widetilde{s}_i\pi) = 1 + l(\pi) \equiv 1 + l(\sigma_\pi) \equiv l(\sigma_{\widetilde{s}_i\pi}).$$

since $\sigma_{\widetilde{s}_i\pi}$ is the product of a transposition and σ_π. □

Theorem 2.4.21. *We have that*

$$\sum_{\pi \in \widetilde{S}_n} (-1)^{l(\pi)} q^{\mathrm{exc}(\pi)} = \frac{1}{(1-q^2)^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} \sum_{i=0}^{n-1-k} \binom{n-1-k}{i} \binom{n-1+k}{n-1-i} (-q)^{2i+k}$$

Proof. We first prove that

$$\sum_{\pi \in \widetilde{S}_n} (-1)^{l(\pi)} q^{\mathrm{exc}(\pi)} = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \sum_{\mathbf{r} \in L_n} q^{\|\mathbf{r} - \mathbf{p_k}\|}$$

where $\mathbf{p_k} = (\underbrace{-1, \ldots, -1}_{k \text{ times}}, 0, \ldots, 0) \in \mathbb{R}^n$ and $L_n = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid \sum_{i=1}^n x_i = 0\}$. Let $\rho \in S_n$ be a permutation with k excedances and consider the set of all bijections $\pi \in \widetilde{S}_n$ with $\sigma_\pi = \rho$. By permuting the coordinates of the vector $\mathbf{r}$, we have that

$$\sum_{\mathbf{r}\in L_n} q^{\|\mathbf{r}-\mathbf{p}_\sigma\|} = \sum_{\mathbf{r}\in L_n} q^{\|\mathbf{r}-\mathbf{p}_\mathbf{k}\|}.$$

By Lemma 2.4.20 and (2.4.22) we have that

$$\begin{aligned}\sum_{\pi\in\widetilde{S}_n} (-1)^{l(\pi)} q^{\mathrm{exc}(\pi)} &= \sum_{\sigma\in S_n}\sum_{\mathbf{r}\in L_n} (-1)^{l(\sigma)} q^{\mathrm{exc}(\mathbf{r},\sigma)} \\ &= \sum_{\sigma\in S_n} (-1)^{l(\sigma)} \sum_{\mathbf{r}\in L_n} q^{\|\mathbf{r}-\mathbf{p}_\sigma\|} \\ &= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \sum_{\mathbf{r}\in L_n} q^{\|\mathbf{r}-\mathbf{p}_\mathbf{k}\|}\end{aligned}$$

where the last equation follows from Theorem 2.3.1.

In [CE11] it is shown that

$$\sum_{\mathbf{r}\in L_n} q^{\|\mathbf{r}-\mathbf{p}_\mathbf{k}\|} = \frac{1}{(1-q^2)^{n-1}} \sum_{i=0}^{n-1-k} \binom{n-1-k}{i}\binom{n-1+k}{n-1-i} q^{2i+k}.$$

The result follows. ☐

One may prove that for all $n \geq 1$ the following identity holds

$$\sum_{\pi\in\widetilde{S}_n} (-1)^{l(\pi)} q^{\mathrm{exc}(\pi)} = \frac{\sum_{k=0}^{n-1} \binom{n-1}{k}^2 (-q)^k}{(1+q)^{n-1}}. \qquad (2.4.26)$$

For a proof use Theorem 2.4.21, simplify the denominators and then apply the Gosper–Zeilberger algorithm to both side of the new identity[1].

The right-hand side of (2.4.26) is the generating function of coordination sequences for A_{n-1} (see e. g. [BG97]) in the indeterminate $-q$. Recall that given a graph G with a root O the coordination sequence is given by $(S(0), S(1), S(2), \dots)$ where $S(i)$ denotes the number of vertices in G having distance i from O. The graph here considered is the vector star of the root system A_{n-1}, i. e. the graph obtained by the rule that a lattice point has all other lattice points as neighbours that can be reached by a root vector (for more details see [CS97]). Therefore we have

Proposition 2.4.22. *For all $n \geq 2$ the sequence of signless coefficients of $\sum_{\pi\in\widetilde{S}_n} (-1)^{l(\pi)} q^{\mathrm{exc}(\pi)}$ is the coordination sequence for A_{n-1}.*

[1]The author thanks prof. C. Krattenthaler for him hints for proving the identity.

The formula for the set of all derangements in $\widetilde{S}_n$ is not very elegant. It can be proved from Theorem 2.4.21 by the Principle of Inclusion–Exclusion.

Corollary 2.4.23. *For all $n \geq 1$ we have that*

$$\sum_{\pi \in \widetilde{DS}_n} (-1)^{l(\pi)} q^{\mathrm{exc}(\pi)} = \sum_{h=0}^{n} (-1)^{n-i} \binom{n}{h} \frac{1}{(1+q)^{h-1}} \sum_{k=0}^{h-1} \binom{h-1}{k}^2 (-q)^k$$

We now consider the other affine Weyl groups. With the same techniques used in the proof of Lemma 2.4.20 one can show the following result, whose proof we omit.

Lemma 2.4.24. *Let $\pi \in \widetilde{S}_n^W$, with $W = B, C, D$. Then*

$$l(\pi) \equiv l(\sigma_\pi) \bmod 2$$

With this lemma it is possible to prove the following result.

Theorem 2.4.25. *For all $n \geq 2$ we have that*

$$\sum_{\sigma \in \widetilde{S}_n^B} (-1)^{\mathrm{inv}(\sigma)} q^{\mathrm{exc}^{\mathrm{col}}(\sigma)} = \sum_{\sigma \in \widetilde{S}_n^C} (-1)^{\mathrm{inv}(\sigma)} q^{\mathrm{exc}^{\mathrm{col}}(\sigma)} = \sum_{\sigma \in \widetilde{S}_n^D} (-1)^{\mathrm{inv}(\sigma)} q^{\mathrm{exc}^{\mathrm{col}}(\sigma)} = 0$$

Proof. Let $W = B$, C, or D. By Lemmas 2.4.3 and 2.4.24, we have that

$$\begin{aligned} SGF_n^W(t,q) &:= \sum_{\pi \in \widetilde{S}_n^W} (-1)^{l(\pi)} q^{2\|\mathbf{r}_\pi - \mathbf{p}_{\sigma_\pi}\|} t^{\mathrm{neg}(\sigma_\pi)} \\ &= \left(\sum_{\sigma \in S_n^W} (-1)^{l(\sigma)} t^{\mathrm{neg}(\sigma)} \right) \left(\sum_{\mathbf{r} \in \mathbb{Z}^n} q^{2\|\mathbf{r} - \mathbf{p}_\sigma\|} \right) \end{aligned}$$

(where we recall that $S_n^C = S_n^B$.) The first factor in the last expression is always 0 whenever $n \geq 2$: in fact left multiplication by $\widetilde{s}_1^W$ gives an involution on S_n^W which preserves the statistic neg and changes the sign of the statistic $(-1)^l$. The result follows. □

It is possible to compute $SGF_1^W(t,q)$: although $\widetilde{S}_1^W$ is not defined, it could be thought of as the set of all elements $\pi \in \widetilde{S}_2^W$ with $\pi(1) = 1$ and $SGF_1^W(t,q)$ is used in computing the analogous formulas for the derangement set. Namely, one has

$$\begin{aligned} SGF_1^C(t,q) &= (1-t)\frac{1+q^2}{1-q^2} \\ SGF_1^B(t,q) &= (1-t)\frac{1+q^4}{1-q^4} \\ SGF_1^D(t,q) &= \frac{1+q^4}{1-q^4}. \end{aligned}$$

By the Principle of Inclusion–Exclusion, we obtain immediately the following result.

Corollary 2.4.26. *We have that*

$$\sum_{\pi\in\widetilde{D}S_n^C}(-1)^{l(\pi)}q^{\mathrm{exc}^{\mathrm{col}}(\pi)} = (-1)^n+(-1)^{n-1}n\frac{1+q^2}{1-q}$$

and

$$\begin{aligned}\sum_{\pi\in\widetilde{D}S_n^B}(-1)^{l(\pi)}q^{\mathrm{exc}^{\mathrm{col}}(\pi)} =&(-1)^n+(-1)^{n-1}n\frac{1+q^4}{(1+q)(1+q^2)}\\ \sum_{\pi\in\widetilde{D}S_n^D}(-1)^{l(\pi)}q^{\mathrm{exc}^{\mathrm{col}}(\pi)} =&(-1)^n+(-1)^{n-1}n\frac{1+q^4}{1-q^4}\end{aligned}$$

We now enumerate the signed variant colored excedance statistic.

Theorem 2.4.27. *We have that*

$$\sum_{\pi\in\widetilde{S}_n^C}(-1)^{l(\pi)}q^{\mathrm{exc}^{\mathrm{col}\,2}(\pi)} = (1+q)^n.$$

and

$$\begin{aligned}\sum_{\pi\in\widetilde{S}_n^B}(-1)^{l(\pi)}q^{\mathrm{exc}^{\mathrm{col}\,2}(\pi)} =&\frac{(1+q)^{n+1}+(1-q)^{n+1}}{2(1+q)}\\ \sum_{\pi\in\widetilde{S}_n^D}(-1)^{l(\pi)}q^{\mathrm{exc}^{\mathrm{col}\,2}(\pi)} =&\frac{(1+q)^{n+1}+(1-q)^{n+1}}{2(1-q^2)}\end{aligned}$$

Proof. We first compute the following generating function

$$SGF_n^W(q,s,t) = \sum_{\pi\in\widetilde{S}_n^C}(-1)^{l(\pi)}q^{\mathrm{exc}_A(\sigma_\pi)}t^{\mathrm{neg}(\sigma_\pi)}s^{\|\mathbf{r}_\pi-\mathbf{p}_{\sigma_\pi}\|},$$

for $W=C,B,D$. By Theorem 2.3.4 and Lemma 2.4.4 we have that

$$\begin{aligned}SGF_n^C(q,s,t) =& \sum_{\sigma\in S_n^B}(-1)^{l(\sigma)}q^{\mathrm{exc}_A(\sigma)}t^{\mathrm{neg}(\sigma)}\sum_{\mathbf{r}\in\mathbb{Z}^n}s^{\|\mathbf{r}-\mathbf{p}_\sigma\|}\\ =&(1-q)^{n-1}(1-t)\left(\frac{1+s}{1-s}\right)^n.\end{aligned}$$

By setting $t=s=q$ the first part of the result follows.

We now consider the group $\widetilde{S}_n^B$. Since the vector $\mathbf{r}$ is even, we have that

$\mathbf{r}-\mathbf{p}_\sigma$ is even if and only if $\mathrm{exc}_A(\sigma)$ is an even number. Therefore

$$\begin{aligned}
SGF_n^B(q,s,t) =& \sum_{\sigma\in S_n^B,\ \mathrm{exc}_A(\sigma)\text{ even}} (-1)^{l(\sigma)} q^{\mathrm{exc}_A(\sigma)} t^{\mathrm{neg}(\sigma)} \sum_{\mathbf{r}\text{ even}} s^{\|\mathbf{r}\|} \\
&+ \sum_{\sigma\in S_n^B,\ \mathrm{exc}_A(\sigma)\text{ odd}} (-1)^{l(\sigma)} q^{\mathrm{exc}_A(\sigma)} t^{\mathrm{neg}(\sigma)} \sum_{\mathbf{r}\text{ odd}} s^{\|\mathbf{r}\|} \\
=&(1-t)\frac{(1-q)^{n-1}+(1+q)^{n-1}}{2}\frac{1}{2}\left(\left(\frac{1+s}{1-s}\right)^n+\left(\frac{1-s}{1+s}\right)^n\right) \\
&+(1-t)\frac{(1-q)^{n-1}-(1+q)^{n-1}}{2}\frac{1}{2}\left(\left(\frac{1+s}{1-s}\right)^n-\left(\frac{1-s}{1+s}\right)^n\right) \\
=&\frac{1-t}{2}\left((1-q)^{n-1}\left(\frac{1+s}{1-s}\right)^n+(1+q)^{n-1}\left(\frac{1-s}{1+s}\right)^n\right).
\end{aligned}$$

and the second identity in the claim follows by setting $s=t=q$. For the last identity, take only the even powers of t in $SGF_n^B(q,s,t)$. □

By the Principle of Inclusion–Exclusion we obtain the signed generating functions on the sets of derangements.

Corollary 2.4.28. *We have that*

$$\sum_{\pi\in\widetilde{D}S_n^C} (-1)^{l(\pi)} q^{\mathrm{exc}^{\mathrm{col}\,2}(\pi)} = q^n$$

and

$$\sum_{\pi\in\widetilde{D}S_n^B} (-1)^{l(\pi)} q^{\mathrm{exc}^{\mathrm{col}\,2}(\pi)} =\frac{(-1)^n}{2(1+q)}\left((1+q)(-q)^n+(1-q)q^n+2q\right)$$

$$\sum_{\pi\in\widetilde{D}S_n^B} (-1)^{l(\pi)} q^{\mathrm{exc}^{\mathrm{col}\,2}(\pi)} =\frac{(-1)^n}{2(1-q^2)}\left((1+q)(-q)^n+(1-q)q^n-2q^2\right)$$

Finally, we enumerate the signed absolute excedance statistic.

Theorem 2.4.29. *We have that*

$$\sum_{\pi\in\widetilde{S}_n^C} (-1)^{l(\pi)q^{\mathrm{exc}^{\mathrm{abs}}(\pi)}} = (1+q)(1-q)^{n-1}.$$

and

$$\sum_{\pi\in\widetilde{S}_n^B} (-1)^{l(\pi)q^{\mathrm{exc}^{\mathrm{abs}}(\pi)}} = \frac{(1+q)(1-q)^{n-1}}{2}+\frac{(1-q)^{2n}}{2(1+q)^n}$$

$$\sum_{\pi\in\widetilde{S}_n^D} (-1)^{l(\pi)q^{\mathrm{exc}^{\mathrm{abs}}(\pi)}} = \frac{(1-q)^{n-1}}{2}+\frac{(1+q)^{n+1}}{4(1-q)}+\frac{(1-q)^{2n}}{4(1+q)^n}.$$

Proof. The set of all permutations $\pi = (\mathbf{r}_\pi, \sigma_\pi) \in \widetilde{S}_n^C$ can be obtained by taking all pairs $(\mathbf{r}, \sigma)$, with $\mathbf{r} \in \mathbb{Z}^n$ and $\sigma \in S_n^B$. All signed permutations $\sigma \in S_n^B$ can be obtained by choosing all permutations in S_n with exactly h excedances (h ranges from 0 to $n-1$) and then changing all possible signs in their window notation. By (2.4.25) we therefore have that

$$\sum_{\pi \in \widetilde{S}_n^C} (-1)^{l(\pi)} q^{\mathrm{exc}^{\mathrm{abs}}(\pi)} = \sum_{h=0}^{n-1} \sum_{\substack{\sigma \in S_n \\ \mathrm{exc}(\sigma)=h}} \sum_{k=0}^{n} \binom{n}{k} (-q)^k (-1)^{l(\sigma)} \sum_{\mathbf{r} \in (\mathbb{Z}^*)^h \times \mathbb{Z}^{n-h}} q^{\|\mathbf{r}\|},$$

where we have used Lemmas 2.3.3 and 2.4.24. Hence

$$\begin{aligned}\sum_{\pi \in \widetilde{S}_n^C} (-1)^{l(\pi)} q^{\mathrm{exc}^{\mathrm{abs}}(\pi)} &= \sum_{h=0}^{n-1} \sum_{\sigma \in S_n, \mathrm{exc}(\sigma)=h} (1-q)^n (-1)^{l(\sigma)} \frac{2^h q^h (1+q)^{n-h}}{(1-q)^n} \\ &= (1+q)^n \sum_{h=0}^{n-1} \sum_{\sigma \in S_n, \mathrm{exc}(\sigma)=h} (-1)^{l(\sigma)} \left(\frac{2q}{1+q}\right)^h \\ &= (1+q)^n \left(1 - \frac{2q}{1+q}\right)^{n-1},\end{aligned}$$

where the last identity follows from Theorem 2.3.1. The result follows.

We now consider the group $\widetilde{S}_n^B$. We recall that $(\mathbf{r}, \sigma) \in \widetilde{S}_n^C$ is also in $\widetilde{S}_n^B$ if and only if $\mathbf{r}$ is even. Suppose that there exists $i \in [1, n-1]$ such that $|\sigma(i)| > i$. Then $\mathbf{r}$ can be chosen arbitrarily on all the entries $j \neq i$ and then set choose r_i to be even or odd as necessary. By (2.4.25), we have that

$$\mathbb{P} = \{r_i + \chi(\sigma_\pi(i) > i) | r_i > 0\} \cup \{\chi(|\sigma_\pi(i)| > i) | r_i = 0\} \cup \{|r_i| + \chi(\sigma_\pi(i) < -i) | r_i < 0\},$$

where r_i ranges over all even (resp. odd) integers.

Therefore

$$\begin{aligned}&\sum_{\pi \in \widetilde{S}_n^B} (-1)^{l(\pi)} q^{\mathrm{exc}^{\mathrm{abs}}(\pi)} = \\ &\quad \sum_{h=1}^{n-1} \sum_{\sigma \in S_n, \mathrm{exc}(\sigma)=h} \sum_{k=0}^{n} \binom{n}{k} (-q)^k (-1)^{l(\sigma)} \sum_{\mathbf{r} \in \mathbb{P} \times (\mathbb{Z}^*)^{h-1} \times \mathbb{Z}^{n-h}} q^{\|\mathbf{r}\|} \\ &\quad + \sum_{k=0}^{n} \binom{n}{k} (-q)^k \sum_{\mathbf{r} \in \mathbb{Z}^n \text{ even}} q^{\|\mathbf{r}\|},\end{aligned}$$

where the last summand is due to all permutations $\sigma \in S_n^B$ such that $\mathrm{exc}(|\sigma|) = 0$. Now the computation is the same as before.

For the group $\widetilde{S}_n^D$ we have to consider only permutations in $\widetilde{S}_n^B$ with an even number of negative signs. Since negative signs contribute in the signed generating function only for the factor $(1-q)^n$ it is easy to verify that

$$\sum_{\pi \in \widetilde{S}_n^D} (-1)^{l(\pi)} q^{\mathrm{exc}^{\mathrm{abs}}(\pi)} = \frac{(1-q)^n + (1+q)^n}{2(1-q)^n} \sum_{\pi \in \widetilde{S}_n^B} (-1)^{l(\pi)} q^{\mathrm{exc}^{\mathrm{abs}}(\pi)}.$$

□

Corollary 2.4.30. *We have that*

$$\sum_{\pi \in \widetilde{D}S_n^C} (-1)^{l(\pi)} q^{\mathrm{exc}^{\mathrm{abs}}(\pi)} = (-1)^n \frac{q^n(1+q) - 2q}{1-q}$$

$$\sum_{\pi \in \widetilde{D}S_n^B} (-1)^{l(\pi)} q^{\mathrm{exc}^{\mathrm{abs}}(\pi)} = (-1)^n \left(\frac{q^n(1+q) - 2q}{2(1-q)} + \frac{q^n(3-q)^n}{2(1+q)^n} \right)$$

$$\sum_{\pi \in \widetilde{D}S_n^D} (-1)^{l(\pi)} q^{\mathrm{exc}^{\mathrm{abs}}(\pi)} =$$

$$(-1)^n \left(\frac{q^n - 1}{2(1-q)} + \frac{(1+q)((-q)^n - 1)}{4(1-q)} + \frac{q^n(3-q)^n}{4(1+q)^n} + \frac{3}{4} \right).$$

Chapter 3

Mahonian statistics on affine Weyl groups

Always moved by the Carlitz's identity, in this chapter we consider the affine Weyl groups $\widetilde{A}_n, \widetilde{B}_n, \widetilde{C}_n$ and $\widetilde{D}_n$ and introduce new Mahonian statistics, which coincides with the classical statistic on the subgroup S_n inside them. We have already seen that in literature there exist generalizations of major index to the groups S_n^B and S_n^D (see Sections 2.1 and 2.2), but there is no result in this direction for affine Weyl groups.

3.1 Mahonian statistic on the affine permutation group

As done for S_n (see (1.4.5)), we let $t_i := \widetilde{s}_i\widetilde{s}_{i-1}\cdots\widetilde{s}_1$ for all $i \in [n-1]$ and let $u_i := \widetilde{s}_{i+1}\cdots\widetilde{s}_{i+n-2}\widetilde{s}_{i+n-1}\widetilde{s}_{i+n-2}\cdots\widetilde{s}_i$, for all $i \in [n-1]$ where all indices are taken modulo n. In particular, $t_i = [i+1, 1, 2, \dots, i-1, i+1, \dots, n]$ and $u_i = [1, 2, \dots, i-1, i+n, i+1-n, i+2, \dots, n]$. We have the following result.

Lemma 3.1.1. *For all $i, j \in [n]$ we have $u_iu_j = u_ju_i$. Moreover, for any permutation $\pi \in \widetilde{S}_n$ there exist unique nonnegative integers $h_1, \dots, h_{n-1}, k_1, \dots, k_{n-1}$ and $c \le n-1$, with $0 \le h_i \le i+1$ for $i = 1, \dots, n$ such that*

$$\pi = t_{n-1}^{h_{n-1}} \cdots t_2^{h_2} t_1^{h_1} u_{n-1}^{k_{n-1}} u_{n-2}^{k_{n-2}} \cdots u_1^{k_1} (t_{n-1}^{-1})^c. \tag{3.1.1}$$

The exponent c satisfies the following further condition: let i be the smallest integer such that $k_i = 0$ (we set $k_n = 0$), then $c < i$.

Analogously we can factorize $\pi = t_{n-1}^c u_{n-1}^{k_{n-1}} u_{n-2}^{k_{n-2}} \cdots u_1^{k_1} t_{n-1}^{h_{n-1}} \cdots t_2^{h_2} t_1^{h_1}$ by choosing other (unique) integers with the same rules.

Proof. Since $u_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0|[1, \ldots, n])$, by (1.3.1) it is trivial to check that u_i, u_j commute for all $i, j \in [n-1]$. Let $\pi = (r_1, \ldots, r_n|\pi') \in \widetilde{S}_n$. Compute first the sequence $s(r) = (s_1^r, \ldots, s_n^r)$, where $s_i^r = \sum_{j=1}^{i} r_i$. Note that $s_n^r = 0$. If $(s_1^r, \ldots, s_n^r) \in \mathbb{N}^n$ then set $c = 0$, otherwise consider the shifted sequence $r^{(1)} = (r_n, r_1, \ldots, r_{n-1})$ and compute $s(r^{(1)})$. Set $c = 1$. Continue this procedure until the sequence $s(r^{(i)})$ has only nonnegative integers, for some $i < n$. Note that there exists at least one such i. Then $c = i$. Consider now $\pi(t_{n-1})^c = (r'_1, \ldots, r'_n|\pi'')$. For all $i = 1, \ldots, n-1$ set $k_i = \sum_{j=1}^{i} r'_j \geq 0$. Note that the condition on c in the statement is satisfied. Finally, use the factorization of $\pi'' \in S_n$ as in (1.4.6) to determine the exponents $h_1, \ldots, h_{n-1}$. It is a routine to check that $t_{n-1}^c u_{n-1}^{k_{n-1}} u_{n-2}^{k_{n-2}} \cdots u_1^{k_1} t_{n-1}^{h_{n-1}} \cdots t_2^{h_2} t_1^{h_1}$ is a factorization of π. The fundamental condition on c ensures that such factorization is unique. In fact, any factor u_i multiplied on the left of a permutation $(r_1, \ldots, r_n|\pi')$, with $s(r) \in \mathbb{N}^n$, transforms the sequence r in a sequence r' with $s(r') \in \mathbb{N}^n$ and c is minimal with the property $s(r^{(c)}) \in \mathbb{N}^n$.

For the last part of the statement, use the same techniques of the previous algorithm but write $\pi = (r_1, \ldots, r_n|\pi')$ where $\pi(i) = \pi'_i + nr_{\pi'_i}$. □

Note that if $\pi \in \widetilde{S}_n$ is also a permutation in S_n (i. e. $\pi([n]) = [n]$) then both factorizations in Lemma 3.1.1 return the factorization given in (1.4.6). Now we explain the algorithm in the previous proof with an example. Let $\pi = [-6, 4, 15, -5, 2, 11] \in \widetilde{S}_n$. Then $\pi = (-2, 0, 2, -1, 0, 1|[6, 4, 3, 1, 2, 5])$. Now compute the sequence $s(r) = (-2, -2, 0, -1, -1, 0)$. Since $s(r) \notin \mathbb{N}^n$ then we have to shift the sequence $r = (-2, 0, 2, -1, 0, 1)$. After four steps we have $r^{(4)} = (2, -1, 0, 1, -2, 0)$ and $s(r^{(4)}) = (2, 1, 1, 2, 0, 0)$. Consider now $\pi'(t_{n-1})^4 = [15, -5, 2, 11, -6, 4]$, then by classical arguments we have $[3, 1, 2, 5, 6, 4] = t_5^2 t_4^3 t_2$. Therefore, $\pi = t_5^2 t_4^3 t_2 u_4^2 u_3 u_2 u_1^2 (t_5^{-1})^4$.

We can also factorize π as follows: $\pi = t_5^4 u_1^2 u_2^2 u_3^3 u_4 t_5^3 t_4^2 t_2^2 t_1$. We left all details to the reader.

Now we have everything needed to define the Mahonian statistic, which we will call $\widetilde{A}$-flag major index.

Definition 3.1.2. *Let $\pi \in \widetilde{S}_n$ and suppose that $\pi = t_{n-1}^{h_{n-1}} \cdots t_2^{h_2} t_1^{h_1} u_{n-1}^{k_{n-1}} u_{n-2}^{k_{n-2}} \cdots u_1^{k_1} (t_{n-1}^{-1})^c$, as in Lemma 3.1.1. Then the* $\widetilde{A}$-flag major index *is defined by*

$$\mathrm{fmaj}_{\widetilde{A}}(\pi) = c + \sum_{i=1}^{n-1} (i+1)(k_i - \chi(i \leq c)) + h_i. \qquad (3.1.2)$$

Note that if $\pi \in \widetilde{S}_n$ is such that $\pi([n]) = [n]$ then $\mathrm{fmaj}_{\widetilde{A}}(\pi) = \mathrm{maj}(\pi_{|[n]})$. Let

$\pi = [-6, 4, 15, -5, 2, 11]$. Then we have shown that $\pi = t_5^2 t_4^3 t_2 u_4^2 u_3 u_2 u_1^2 (t_5^{-1})^4$. Therefore $\text{fmaj}_{\widetilde{A}}(\pi) = 4 + (2 + 0 + 0 + 5 + 0) + (2 + 3 + 0 + 1 + 0) = 17$.

It is also possible to give a similar definition by using the second factorization in Lemma 3.1.1. In this case the equation (3.1.2) is the same.

We have the following result.

Proposition 3.1.3. *The $\widetilde{A}$-flag major index defined in (3.1.2) is Mahonian, i.e.*

$$\sum_{\pi \in \widetilde{S}_n} q^{\text{fmaj}_{\widetilde{A}}(\pi)} = \prod_{i=1}^{n-1} \frac{[i+1]_q}{1-q^i} = \sum_{\pi \in \widetilde{S}_n} q^{l(\pi)}$$

Proof. Lemma 3.1.1 give a bijection between elements in $\widetilde{S}_n$ and triples $(c,\{r_1, \ldots, r_{n-1}\},\sigma)$ where $\sigma \in S_n$, $c < n$ and $\{r_1, \ldots, r_{n-1}\} \in \mathbb{P}^c \times \mathbb{N}^{n-1-c}$. Since σ is independent from c and $\{r_1, \ldots, r_{n-1}\}$ we can compute as follows

$$\begin{aligned}
\sum_{\pi \in \widetilde{S}_n} q^{\text{fmaj}_{\widetilde{A}}(\pi)} &= \left(\sum_{c=0}^{n-1} q^c \prod_{i=1}^{n-1} \sum_{k_i \geq \chi(i \leq c)} (q^{i+1})^{k_i - \chi(i \leq c)} \right) \left(\sum_{\sigma \in S_n} q^{\text{maj}(\sigma)} \right) \\
&= \frac{1 + q + \cdots + q^{n-1}}{\prod_{i=1}^{n-1} (1 - q^{i+1})} \prod_{j=1}^{n-1} [j+1]_q \\
&= \prod_{i=1}^{n-1} \frac{[i+1]_q}{(1-q^i)}
\end{aligned}$$

where in the first row we use the fact that $\sum_{i=1}^{n-1} h_i = \text{maj}(\sigma)$ by (1.4.6) (the coefficients h_i are as in (3.1.2)), in the second equation we use the first part of Theorem 1.2.1. Compare the result with the second part of Theorem 1.2.1 to complete the proof. □

Proposition 3.1.3 shows that the $\widetilde{A}$-flag major index above introduced is Mahonian and we know that it generalizes the classical major index on the subset of the classical permutations. We now investigate on an analogue of the Carlitz's identity in Theorem 1.4.4. For this purpose we need to introduce an analogue of the descent statistic: such statistic will be not coincide with the descent statistic in the Coxeter group theoretic sense. Indeed this is not true for the flag descent used in Theorem 2.1.2.

The main idea to define a new descent statistic, which we will call $\widetilde{A}$-flag descent, is the following: in S_n the major index is a weighted sum of the descents of the permutations by definition. Similarly, we want to define such descent statistic by deleting all possible weights. Therefore, we will define the $\widetilde{A}$-flag descents by deleting all coefficients in (3.1.2).

Definition 3.1.4. *Let* $\pi \in \widetilde{S}_n$. *Suppose that* $\pi = t_{n-1}^{h_{n-1}} \cdots t_2^{h_2} t_1^{h_1} u_{n-1}^{k_{n-1}} u_{n-2}^{k_{n-2}} \cdots u_1^{k_1} (t_{n-1}^{-1})^c.$, *as in Lemma 3.1.1. Then define the* $\widetilde{A}$-flag descent *statistic by*

$$\begin{aligned} \mathrm{fdes}_{\widetilde{A}}(\pi) &= c + \left(\sum_{i=1}^{n-1} k_i - \chi(i \leq c) \right) + \mathrm{des}(t_{n-1}^{h_{n-1}} \cdots t_1^{h_1}) && (3.1.3) \\ &= \left(\sum_{i=1}^{n-1} k_i \right) + \mathrm{des}(t_{n-1}^{h_{n-1}} \cdots t_1^{h_1}). \end{aligned}$$

The previous definition can be thought as follows. Let $r = (r_1, \ldots, r_n)$ be a sequence of integers such that $\sum_{i=0}^{n} r_i = 0$. We call a *move* of r each operation that subtract 1 to a positive integer r_i and add 1 to the number on its right (if $i = n$ then add 1 to r_1): for example, if $r = (4, 3, -1, -2, -7, 3)$ a move can transform r in $(3, 4, -1, -2, -7, 3)$ or $(4, 2, 0, -2, -7, 3)$ or also in $(5, 3, -1, -2, -7, 2)$. Then the flag descents of the sequence r is the number of moves which transform r in the sequence $(0, \ldots, 0)$. The $\widetilde{A}$-flag descents of a permutation $\pi = (r_1, \ldots, r_n | \pi')$ are therefore the flag descents of $(r_1, \ldots, r_n)$ plus the classical descents of the permutation $\pi' t_{n-1}^c$, where c is determined by the algorithm in the proof of Lemma 3.1.1.

For example, let $\pi = [-6, 4, 15, -5, 2, 11] = (-2, 0, 2, -1, 0, 1 | [6, 4, 3, 1, 2, 5])$. Then $c = 4$, $\mathrm{des}([3, 1, 2, 5, 6, 4]) = 2$ and $\mathrm{fdes}_{\widetilde{A}}(\pi) = 6 + 2 = 8$.

By labeling each one of these $\widetilde{A}$-flag descents it is possible to define the $\widetilde{A}$-flag major index of an affine permutation as the sum of the labels of all descents, as in the definition of major index of the classical permutations.

Note that there exists a dual definition of $\widetilde{A}$-flag descents, by using the other factorization in Lemma 3.1.1: in this case given $\pi \in \widetilde{S}_n$, the moves are computed on the sequence $(r'_1, \ldots, r'_n)$ such that $\pi(i) = \pi'(i) + n r'_{\pi'(i)}$, with $\pi' \in S_n$.

Here we give the analogue of the Carlitz's identity.

Proposition 3.1.5. *We have*

$$\sum_{k \geq 0} [k+1]_q^n t^k = \frac{(1 - tq^n) \sum_{\pi \in \widetilde{S}_n} q^{\mathrm{fmaj}_{\widetilde{A}}(\pi)} t^{\mathrm{fdes}_{\widetilde{A}}(\pi)}}{(1 - (tq)^n)(1 - t)}$$

Proof. With the same arguments of the proof of Proposition 3.1.3, we compute

$$\sum_{\pi\in\widetilde{S}_n} q^{\mathrm{fmaj}_{\widetilde{A}}(\pi)} t^{\mathrm{fdes}_{\widetilde{A}}(\pi)} =$$

$$\begin{aligned}
&\left(\sum_{c=0}^{n-1} q^c t^c \prod_{i=1}^{n-1} \sum_{k_i\geq\chi(i\leq c)} (tq^{i+1})^{k_i-\chi(i\leq c)}\right)\left(\sum_{\sigma\in S_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{des}(\sigma)}\right)\\
=&\frac{1+tq+\cdots+t^{n-1}q^{n-1}}{\prod_{i=1}^{n-1}(1-tq^{i+1})}\prod_{i=0}^{n-1}(1-tq^i)\sum_{k\geq 0}[k+1]_q^n t^k\\
=&\frac{(1-(tq)^n)(1-t)}{(1-tq^n)}\sum_{k\geq 0}[k+1]_q^n t^k
\end{aligned}$$

where in the second equation we use Theorem 1.4.4. □

3.2 A bijection

Proposition 3.1.3 says that the inversion statistic and the $\widetilde{A}$-flag major index are equidistributed in $\widetilde{S}_n$. In this section we want to construct a bijection $\psi : \widetilde{S}_n \to \widetilde{S}_n$ such that $\mathrm{fmaj}_{\widetilde{A}}(\pi) = l(\psi(\pi))$. For this purpose we recall other well-known bijections which will be used to define ψ.

Denote with $\widetilde{S}_n^I$ the subset of all permutations $\sigma \in \widetilde{S}_n$ such that $\sigma_1 < \sigma_2 < \cdots < \sigma_n$. In the Coxeter group theoretic sense $\widetilde{S}_n^I$ is the parabolic quotient generated by $\{\widetilde{s}_1, \ldots, \widetilde{s}_{n-1}\}$. Let $\mathcal{P}_{n-1}$ be the set of all partitions of length $\leq n-1$. Define the map $\mathrm{Inv} : \widetilde{S}_n^I \to \mathcal{P}_{n-1}$ as follows: let $\pi \in \widetilde{S}_n^I$, then $\mathrm{Inv}(\pi) = (p_1, \ldots, p_{n-1})$ where $p_i = |\{j > n | \pi(j) < \pi(n+1-i)\}|$.

For example, let $\pi = [-4, -2, 1, 5, 6, 15] \in \widetilde{S}_6$. Then $\mathrm{Inv}(\pi) = (9, 2, 2, 0, 0)$. Note that the sum of the elements in $\mathrm{Inv}(\pi)$ is equal to $l(\pi)$ (it is exactly $\mathrm{inv}(\pi)$, defined in [BB05, Section 8.3]). In [BB95, Theorem 4.4] we have the following result.

Theorem 3.2.1. *The map* $\mathrm{Inv} : \widetilde{S}_n^I \to \mathcal{P}_{n-1}$ *is a bijection.*

The inverse map Inv^{-1} is described in [BB95] as follows. For $i \in [2, n]$ and $\pi \in \widetilde{S}_n^I$ let

$$E_i([\pi_1, \ldots, \pi_n]) := [\pi_1, \ldots, \pi_{j-1}, \pi_j - k, \pi_{j+1}, \ldots, \pi_{i-1}, \pi_i + k, \pi_{i+1}, \ldots, \pi_n]$$

where

$$k := \min\{h \in [n-1] | h + \pi_i \notin \{\pi_{i+1}, \ldots, \pi_n\} (\mathrm{mod}\, n), h + \pi_i < \pi_{i+1}\}$$

and j is the unique element of $[i-1]$ such that $a_j \equiv a_i + k (\mathrm{mod}\, n)$. If $\lambda = \lambda_1 \geq \cdots \geq \lambda_{n-1}$ is a partition in $\mathcal{P}_{n-1}$ then $\mathrm{Inv}^{-1}(\lambda) = E_2^{\lambda_{n-1}} E_3^{\lambda_{n-2}} \cdots E_n^{\lambda_1}([1, 2, \ldots, n])$.

For example, let $\lambda = (2,1,1,1,0) \in \mathcal{P}$. Then

$$\begin{aligned}\mathrm{Inv}^{-1}(\lambda) &= E_3E_4E_5E_6^2([1,2,3,4,5,6]) \\ &= E_3E_4E_5E_6([0,2,3,4,5,7]) \\ &= E_3E_4E_5([0,1,3,4,5,8]) \\ &= E_3E_4([-1,1,3,4,6,8]) \\ &= E_3([-2,1,3,5,6,8]) \\ &= [-3,1,4,5,6,8]\end{aligned}$$

Now we have introduced all preliminaries which are needed to define a bijection of $\widetilde{S}_n$ which maps the $\widetilde{A}$-flag major index statistic in the affine inversion statistic.

Let $\pi = (r_1, \ldots, r_n|\pi') \in \widetilde{S}_n$ be an affine permutation. Suppose that $\pi = t_{n-1}^{h_{n-1}} \cdots t_2^{h_2} t_1^{h_1} u_{n-1}^{k_{n-1}} u_{n-2}^{k_{n-2}} \cdots u_1^{k_1} (t_{n-1}^{-1})^c$ as in Lemma 3.1.1. Let $k'_i = k_i - \chi(i \leq c)$ for all $i \in [n-1]$ and

$$\lambda_\pi = (nk_{n-1}+k_1+\cdots+k_{n-2}, k'_1+\cdots+k'_{n-2}, k'_2+\cdots+k'_{n-2}, \ldots, k'_{n-3}+k'_{n-2}, k'_{n-2}). \tag{3.2.4}$$

Theorem 3.2.2. *The map $\psi : \widetilde{S}_n \to \widetilde{S}_n$ defined by*

$$\psi(\pi) = \mathrm{Inv}^{-1}(\lambda_\pi) \circ \phi(\pi' t_{n-1}^c). \tag{3.2.5}$$

is a bijection such that $\mathrm{fmaj}_{\widetilde{A}}(\pi) = l(\psi(\pi))$ *for all* $\pi \in \widetilde{S}_n$.

Proof. First of all, note that λ is an element in $\mathcal{P}_{n-1}$, so the definition is consistent. We now explicit the inverse ψ^{-1}. Let $\pi \in \widetilde{S}_n$. Then there is exactly one permutation $\sigma \in S_n$ such that $\sigma\pi(1) < \sigma\pi(2) < \cdots < \sigma\pi(n)$. Therefore $\sigma\pi \in \widetilde{S}_n^I$ and $\lambda = \mathrm{Inv}(\sigma\pi)$ is a partition in $\mathcal{P}_{n-1}$. Suppose that $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ then set $c \in [0, n-1]$, such that $c \equiv \lambda_1 - \lambda_2 (\mathrm{mod}\, n)$; moreover, set $k_{n-1} = (\lambda_1 - c - \lambda_2)/n + \chi(c = n-1)$ and $k_i = \lambda_{i+1} - \lambda_{i+2} + \chi(i \leq c)$ for all $i \in [n-2]$ (where we use the convention $\lambda_n = 0$). Then it is easy to check that the map $\psi' : \widetilde{S}_n \to \widetilde{S}_n$ given by $\psi'(\pi) = \sigma^{-1} u_{n-1}^{k_{n-1}} u_{n-2}^{k_{n-2}} \cdots u_1^{k_1} (t_{n-1}^{-1})^c$, where $c, k_1, \ldots, k_{n-1}$ and σ are defined as above, is the inverse map of ψ.

Now we have to prove the second fact in the statement. First note that for any element $\pi \in \widetilde{S}_n$, if $\sigma \in S_n$ is such that $\sigma\pi \in \widetilde{S}_n^I$, then $l(\pi) = l(\sigma\pi) + \mathrm{inv}(\sigma^{-1})$: this can be proven or directly by enumerating all pairs determining inversions, or by [BB05, Proposition 2.4.4] which uses properties of parabolic quotients of Coxeter groups.

Therefore,

$$l(\psi(\pi)) = l(\mathrm{Inv}^{-1}(\lambda)) + \mathrm{inv}(\phi(\pi' t_{n-1}^c))$$

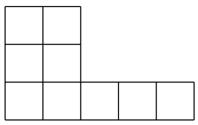
By Theorem 1.4.2 $\mathrm{inv}(\phi(\pi' t_{n-1}^c)) = \mathrm{maj}(\pi' t_{n-1}^c)$ and by Theorem 3.2.1 we have that $l(\mathrm{Inv}^{-1}(\lambda)) = \lambda_1 + \cdots + \lambda_{n-1}$. By (3.2.4), $\lambda_1 + \cdots + \lambda_{n-1} = c + \sum_{i=1}^{n-1}(i+1)(k_j - \chi(i \leq j))$ and the statement follows by (3.1.2). □

For example, let $\pi = [-6, 4, 15, -5, 2, 11] \in \widetilde{S}_6$. We have already shown that $\mathrm{fmaj}_{\widetilde{A}}(\pi) = 17$. We have $c = 4$, $\lambda = (6, 2, 1, 1, 1)$ and $\mathrm{Inv}^{-1}(\lambda) = [-7, 2, 3, 4, 7, 12]$. Moreover, $\pi' t_{n-1}^4 = [3, 1, 2, 5, 6, 4]$ and $\phi([3, 1, 2, 5, 6, 4]) = [5, 1, 3, 2, 6, 4]$. Therefore, $\psi(\pi) = [7, -7, 3, 2, 12, 4]$ and $l(\psi(\pi)) = 17$.

3.3 An approach with n-cores and abaci models

The main idea behind the definition of $\widetilde{A}$-flag major index is to transform a sequence of n integers whose sum is zero in positive integers and sum them with certain weights. Thus we can try to have other statistics in the same way just starting from other sequences which characterize affine permutations. For this purpose, now we introduce the well-known concepts of n-cores and abacus model from which we compute sequences of n integers whose sum is zero.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition. The *Young diagram* of λ is the set of the points $(i, j) \in \mathbb{N} \times \mathbb{N}$ such that $1 \leq i \leq \lambda_j$. We draw the diagram so that each point (i, j) is represented by the unit cell southwest of the point. For example, the diagram of $(5, 2, 2)$ is



For partitions λ, μ, we write $\mu \subset \lambda$ whenever the diagram of μ is contained within the diagram of λ, or equivalently, $\mu_i \leq \lambda_i$ for all i. When $\mu \subset \lambda$, we may define the *skew diagram* $\lambda \setminus \mu$ to be the set theoretic difference $\lambda - \mu$. A *connected skew diagram* is one with at least one cell such that for each pair of cells (c_1, c_2) there is a sequence $(d_1 = c_1, d_2, \ldots, d_n = c_2)$ of cells of the diagram with d_i, d_{i+1} having one common side for all $i < n$. A *ribbon* is a connected skew diagram containing no 2×2 subdiagram. We say that a ribbon R is a removable ribbon of λ if $\lambda - R$ is again a partition. An n-core is a partition having no removable ribbon of length n.

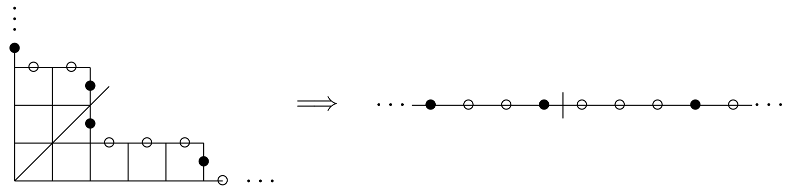
Associate to each cell x of a diagram of a partition λ, the *content* of x defined by $c(x) = i - j$ where the cell x lies in row j and column i. We also consider the *residue* of x, defined as the content of x modulo n.

Let

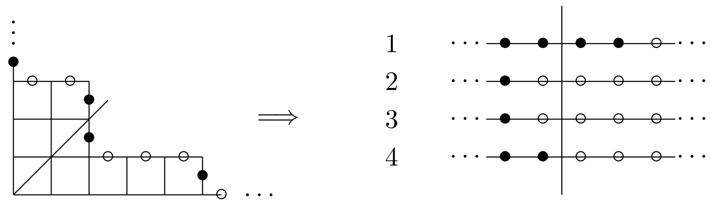
$$\mathcal{C} : \widetilde{S}_n^I \to n - \text{core partitions} \tag{3.3.6}$$

be the map defined recursively as follows. Associate the empty partition with the identity in $\widetilde{S}_n^I$; say $\mathcal{C}(w) = \lambda$ and $\mathrm{inv}_{\widetilde{A}}(s_i w) > \mathrm{inv}_{\widetilde{A}}(w)$, then $\mathcal{C}(s_i w)$ is obtained from the diagram of λ by adding all cells with residue i such that $\mathcal{C}(s_i w)$ is again a diagram of a partition. It is known (see e. g., [JK81],[LM05],[AB12]) that $\mathcal{C}$ is a bijection between $\widetilde{S}_n^I$ and the set of all n-core partitions.

Consider the diagram of a partition λ in the $\mathbb{N} \times \mathbb{N}$ plane. Walk in unit steps along the boundary of λ and place a bead ($\bullet$) on each vertical step and a spacer ($\circ$) on each horizontal step. Then straighten the boundary to have a doubly infinite sequence of beads and spacers. For example, we construct the sequence for $(5, 2, 2)$ as follows.



Define the *content* of a bead or spacer to be the content of the diagonal immediately southeast. The *abacus associated to* λ is the binary string of λ with beads and spacers indexed by their content. Swapping a bead and a spacer with the same residue is equivalent to adding or removing a n-ribbon of λ. Divide the abacus into n sequences, each containing all beads and spacer of the same residue. Removing a n-ribbon from the boundary of λ is equivalent to moving a bead left along its sequence. Therefore a n-core is characterized by sequences with all beads on the left and all spacers on the right. For $n = 4$ and $\lambda = (5, 2, 2)$, which is a 4-core partition, we have the following sequences.



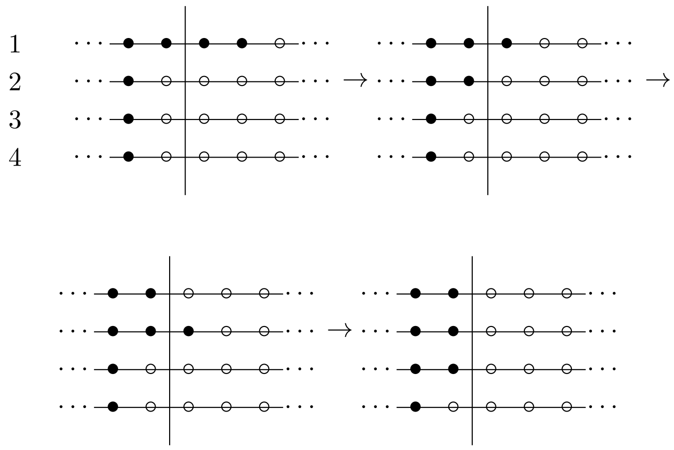
For $i = 1, \ldots, n$ the *length* of the sequence with content i is given by the number of its beads with positive content minus the number of its spacer with nonpositive content (note that at most one of the two numbers is non-zero). In the previous example, $\lambda = (5, 2, 2)$, $n = 4$, the lengths of sequences with contents $1, 2, 3, 4$ are respectively, $2, -1, -1, 0$. Note that the sum of all lengths is equal to zero. In fact, all beads with positive content are exactly all beads on southeast side of the main diagonal $D : i = j$ in the diagram; vice versa all spacers with nonpositive content are those on northwest side of D. Both their cardinalities are trivially equal to the number of cells of the diagram intersected by D.

This property of the lengths is necessary to define a new statistic similar to the $\widetilde{A}$-flag major index.

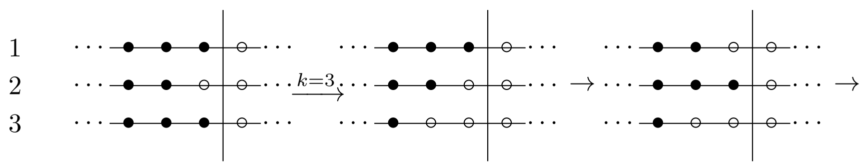
Remark 3.3.1. *Note that by moving the first sequence of the abacus after the n-th sequence and shifting all beads and all spacer of one step to the left (i. e. by decreasing its length of one unit) we have another abacus of the same partition. We say that the contents of the new abacus are $2, \dots, n+1$. Therefore for any partition λ it is possible to associate an abacus whose sequences have contents $k, k+1, \dots, k+n-1$ for all $k \in \mathbb{Z}$.*

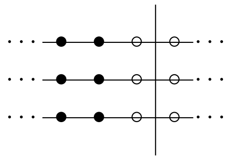
In the following we adopt the important assumption that in all abaci *all sequences with content n will be depicted with length decreased by 1.* Let λ be a n-core partition. Then there is a $k \in \mathbb{N}$ such that the abacus of λ with contents $k, \dots, k+n-1$ can be transformed in the abacus of the empty partition by swapping beads of any sequence with spacers of sequences immediately below.

For example, the abacus of the partition $\lambda = (5, 2, 2)$ and $n = 4$ can be transformed as follows (choose $k = 1$)



The abacus of $(3, 1)$ and $n = 3$, instead can be transformed as follows (choose $k = 3$).





Let k be the minimal index with this property. Then for all $i \in [2, n]$ define a *descent move of weight* i to be a swap of a bead with positive content of the $i-1$-th sequence with a spacer of the i-th sequence necessary to transform the abacus of λ in the abacus of the empty partition. By doing all these steps by the top sequence to the lowest, there could be swaps between beads with content 0 and a spacers below: in this case we say that it is a *descent move of weight* 1. For example, for $\lambda = (5, 2, 2)$ and $n = 4$ (see above) the weights of descent moves are $2, 2, 3$; for $\lambda = (3, 1)$ and $n = 3$ the weights are $1, 1$.

Now we have all the necessary to compute the analogue of $\widetilde{A}$-flag descent statistic and $\widetilde{A}$-flag major index.

Let $\pi \in \widetilde{S}_n$. By [BB05, Proposition 2.4.4], π can be uniquely factorized in $\pi = \pi'\sigma$, with $\pi' \in \widetilde{S}_n^I$ and $\sigma \in S_n$. Then define the *abacus descent* of π to be $\mathrm{ades}(\pi) := \mathrm{des}(\sigma) + |\{\text{descent moves of } \mathcal{C}(\pi')\}|$. Note that $\mathrm{des}(\sigma)$ is equivalent to all descents in the sequence $[\pi_1, \ldots, \pi_n]$.

Define the *abacus major index* of π to be

$$\mathrm{amaj}(\pi) := \mathrm{maj}(\sigma) + \sum_{d \in \{\text{descent moves of } \mathcal{C}(\pi')\}} w(d)$$

where $w(d)$ denotes the weight of the descent move d. With the same argument used in the proofs of Propositions 3.1.3 and 3.1.5 it is possible to prove the following result.

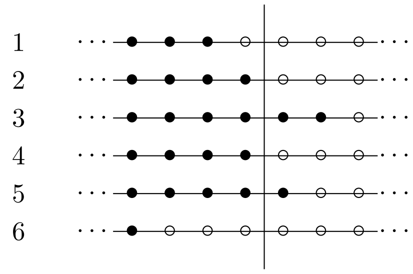
Proposition 3.3.2. *The statistic* amaj *defined on* $\widetilde{S}_n$ *is Mahonian. Moreover, the following identity holds*

$$\sum_{k \geq 0} [k+1]_q^n t^k = \frac{(1 - tq^n) \sum_{\pi \in \widetilde{S}_n} q^{\mathrm{amaj}_{\widetilde{A}}(\pi)} t^{\mathrm{ades}_{\widetilde{A}}(\pi)}}{(1 - (tq)^n)(1 - t)}$$

We omit the details.

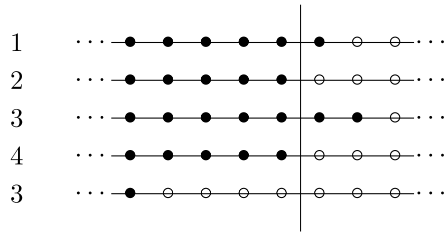
For example, consider the affine permutation $\pi = [-6, 4, 15, -5, 2, 11]$. Then, obviously, $\pi = \pi'\sigma$, with $\pi' = [-6, -5, 2, 4, 11, 15]$ and $\sigma = [1, 4, 6, 2, 3, 5]$. It is possible to check that $\mathcal{C}(\pi') = (9, 6, 5, 2, 2, 2, 2)$. The corresponding abacus is

(recall that n-th sequence is shifted of one step to the left)



Choose $k = 2$. The weights of all descent moves are $3, 3, 4, 4, 5, 5, 5, 1$. Therefore, $\mathrm{ades}(\pi) = \mathrm{ades}(\pi') + \mathrm{des}(\sigma) = 8 + 1 = 9$ and $\mathrm{amaj}(\pi) = \mathrm{amaj}(\pi') + \mathrm{maj}(\sigma) = 30 + 3 = 33$.

If $\pi = [-10, 2, 4, 6, 13]$, then $\pi \in \widetilde{S}_5^I$. We have $\mathcal{C}(\pi) = (8, 4, 3, 2, 2, 2, 2, 1, 1, 1, 1)$, with abacus



With $k = 1$, we have that the weights of all descent moves are $2, 3, 4, 4, 4, 5, 5, 5$ and so $\mathrm{ades}(\pi) = 8$ and $\mathrm{amaj}(\pi) = 32$.

3.4 Affine signed permutations

In [HJ], Hanusa and Jones extend the abacus model for other affine Weyl groups. In the following we will denote by $\widetilde{W}_n$ be one of the groups $\widetilde{B}_n$, $\widetilde{C}_n$ or $\widetilde{D}_n$.

We say that a signed abacus is a diagram containing $2n$ sequences labeled by $-n, \ldots, -1, 1, \ldots, n$. The i-th sequence contains entries labeled by the integers $mN + i$, for each *level* $m \in Z$. As done for abaci of $\widetilde{S}_n$, we draw a signed abacus so that each sequence is horizontal, oriented with $-\infty$ to the left and ∞ to the right, with labels increasing from the top to the bottom. Each entry may contain a bead or a spacer. The linear order defined by the labels $mN + i$ (for $m \in Z$, $|i| \in [n]$) is called the *reading order* of the abacus. Following [HJ], we say that a bead b is *active* if there exist spacers (on any sequence) that occur prior b in reading order. Otherwise we say that b is *inactive*. A sequence is called *flush* if no bead on the sequence is preceded in reading order by a spacer on that same sequence. The abacus is *flush* if all sequences are flush. The abacus is *balanced* if

- there is at least one bead on every sequence;
- the sum of the labels of the rightmost beads on sequences i and $-i$ is 0 for all $i \in [n]$.

The second condition is different from the original one given in [HJ]. This is justified by a result in Proposition 3.4.1. We say that the abacus is *even* if there exists an even number of spacers lying before N in the reading order.

Proposition 3.4.1. *Let $\pi \in \widetilde{C}_n$ and define $\mathcal{A}(\pi)$ to be the flush abacus whose rightmost beads in each sequence are elements with labels in $\{\pi(1), \ldots, \pi(n), \pi(-1), \ldots, \pi(-n)\}$. Then $\mathcal{A}$ is a bijection from the parabolic quotient $\widetilde{C}_n/C_n$ to the set of balanced flush abaci. The same map define a bijection from $\widetilde{B}_n/B_n$ and the set of even balanced flush abaci and from $\widetilde{D}_n/D_n$ and the set of even balanced flush abaci.*

The statement and the proof are essentially the same of [HJ, Lemma 3.6] (the authors use different sets of generators of the classical subgroups B_n and D_n).

We now introduce a definition of abacus descents and abacus major index for the affine Weyl groups $\widetilde{C}_n$, $\widetilde{B}_n$ and $\widetilde{D}_n$.

Let $\pi \in \widetilde{C}_n$. Then π can be decomposed as $\pi = \pi'\sigma$, with $\pi' \in \widetilde{C}_n/C$ and $\sigma \in B_n$. For this purpose, it suffices to consider the elements $\{|\pi_1|, \ldots, |\pi_n|\}$ and put the elements in the increasing order. Thus we get the sequence of π', while σ is the only (signed) permutation which transforms π' in π (by right multiplication).

Let $\pi' \in \widetilde{C}_n/C_n$ and choose as canonical representant the only permutation with $0 < \pi'_1 < \cdots < \pi'_n$.

Consider the abacus of π' and define a *move* each action on the abacus which swaps the rightmost bead of the sequence i with the leftmost spacer of the sequence $-i$, for all $|i| \in [n]$. Note that by applying a move, we get again a balanced abacus. We say that the *weight* of such move is $2|i| - 1$, exactly the difference of the labels of the sequences minus 1 (remember that there is no sequence labeled by 0).

We define the *abacus major index* of π' as twice the sum of the weights of the minimal moves necessary to transform the abacus of π' in the abacus of the identity, minus the sum of the elements in $\{2i - 1 | \pi'(i) \in [-n, -1] \bmod 2n - 1\}$. We denote such statistic with $\operatorname{amaj}(\pi')$.

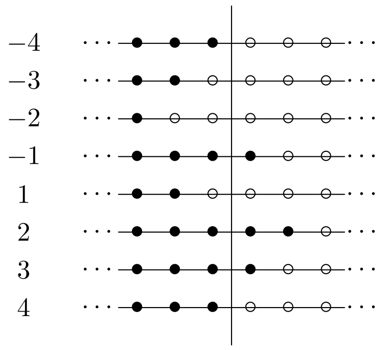
Now we define the abacus major index of $\pi = \pi'\sigma$, with $\pi' \in \widetilde{C}_n/C_n$ and $\sigma \in B_n$ by

$$\operatorname{amaj}(\pi) = \operatorname{amaj}(\pi') + \operatorname{fmaj}(\sigma).$$

Similarly it is possible to define an analogue of abacus descents as follows: if π' is in $\widetilde{C}_n/C_n$ then the abacus descents of π' is given by the number $\text{ades}(\pi')$ defined by the sum of the absolute value of the levels of the rightmost beads in all sequences (equivalently twice the number of minimal moves necessary to transform the abacus of π' in the abacus of the identity), and for any $\pi = \pi'\sigma$, with $\pi' \in \widetilde{C}_n/C_n$ and $\sigma \in B_n$ set

$$\text{ades}(\pi) = \text{ades}(\pi') + \text{fdes}(\sigma).$$

We now give an example. Consider in $\widetilde{C}_4/C_4$ the permutation $\pi' = [4, 8, 12, 20]$. The corresponding abacus is given by



Note that the rightmost beads in all sequences have label in $\{\pm 4, \pm 12, \pm 20, \pm 8\}$. The moves which transform the previous abacus in that of identity permutation have weights 1,3,3,5. Moreover the set $\{2i - 1 | \pi'(i) \in [-n, -1] \bmod 2n - 1\}$ reduces to $\{1\}$. Therefore the abacus major index of π' is $\text{amaj}(\pi') = 2(1 + 3 + 3 + 5) - 1 = 23$. The number of abacus descents is given by $\text{ades}(\pi') = 2 \cdot 4 = 8$.

Now consider $\pi = [-4, 12, 8, -20] = \pi'[-1, 3, 2, -4] \in \widetilde{C}_4$. Then $\text{amaj}(\pi) = 23 + 6 = 29$ and $\text{ades}(\pi) = 8 + 3 = 11$.

With the above definitions we get the following result.

Proposition 3.4.2. *The statistic abacus major index* amaj *is Mahonian. Moreover the following identity holds*

$$\sum_{k \geq 0} [k+1]_q^n t^n = \frac{\prod_{i=1}^n (1 - t^2 q^{2i-1}) \sum_{\pi \in \widetilde{C}_n} t^{\text{ades}(\pi)} q^{\text{amaj}(\pi)}}{(1-t) \prod_{i=1}^n (1 - t^2 q^{2i})(1 + t^2 q^{4i-2})}.$$

Proof. We first prove that amaj is Mahonian. We have

$$\begin{aligned}\sum_{\pi\in\widetilde{C}_n} q^{\mathrm{amaj}(\pi)} &= \left(\sum_{\pi'\in\widetilde{C}_n/C_n} q^{\mathrm{amaj}(\pi')}\right)\left(\sum_{\sigma\in B_n} q^{\mathrm{fmaj}(\sigma)}\right)\\ &= \left(\prod_{i=1}^{n}\sum_{j_i\geq 0}(q^{2i-1})^{j_i}\right)\prod_{i=1}^{n}[2i]_q\\ &= \prod_{i=1}^{n}\frac{[2i]_q}{1-q^{2i-1}}\end{aligned}$$

where in the second equation we use (2.1.2). Compare the result with Theorem 1.2.1. Thus the first part is proven. The proof of the last part is similar. Distinguish the case when $\pi'(i) \bmod N$ is in $\{-1,\dots,-n\}$ or not, for all $i = 1,\dots,n$. By Theorem 2.1.2 we have

$$\begin{aligned}\sum_{\pi\in\widetilde{C}_n} t^{\mathrm{ades}(\pi)}q^{\mathrm{amaj}(\pi)} =& \left(\sum_{\pi'\in\widetilde{C}_n/C_n} t^{\mathrm{ades}(\pi')}q^{\mathrm{amaj}(\pi')}\right)\left(\sum_{\sigma\in B_n} t^{\mathrm{fdes}(\sigma)}q^{\mathrm{fmaj}(\sigma)}\right)\\ =& \left(\prod_{i=1}^{n}\sum_{j_i\geq 0}(q^{2i-1})^{2j_i+1}t^{2j_i+2}+(q^{2i-1})^{2j_i}t^{2j_i}\right)\\ &\cdot(1-t)\prod_{i=1}^{n}(1-t^2q^{2i})\sum_{k\geq 0}[k+1]_q^n t^k\\ =&(1-t)\prod_{i=1}^{n}\frac{(1+t^2q^{2i-1})(1-t^2q^{2i})}{1-t^2q^{4i-2}}\sum_{k\geq 0}[k+1]_q^n t^k.\end{aligned}$$

□

The same idea could be used to introduce new statistics in the group $\widetilde{B}_n$.

Since for $\widetilde{B}_n$ the abaci are only even, we cannot use the same definition of amaj and ades used for $\widetilde{C}_n$. Consider an even balanced abacus and fill with beads and spacers the sequences labeled by $-n,\dots,-2$ (and therefore also those labeled by $2,\dots,n$). Now the sequence -1 can be filled only with bears such that the rightmost of them has level even or odd, according to the parity of the levels of the beads in the other sequences. Thus it suggests to change (in some sense to halve) the contributes of the sequences -1 and 1 to the definition of the abacus major index and of the abacus descents.

Therefore the *abacus major index* of $\pi' \in \widetilde{B}_n \setminus B_n$ $\mathrm{amaj}_{\widetilde{B}}(\pi')$ is defined by $\mathrm{amaj}_{\widetilde{C}}(\pi')$ minus the absolute value of the level of the rightmost bead in the sequence labeled by 1 (or equivalently -1). The number of abacus descents of π' is given by the sum of the absolute value of the levels of the rightmost beads in the sequences labeled by $-n, \dots, -1, 2 \dots, n$, or equivalently by $\mathrm{ades}_{\widetilde{C}}(\pi')$ minus the level of the rightmost bead in the sequence labeled by 1.

The above definitions can be extended to $\pi \in \widetilde{B}_n$, $\pi = \pi'\sigma$, with $\pi' \in \widetilde{B}_n \setminus B_n$ and $\sigma \in B_n$ by $\mathrm{amaj}(\pi) = \mathrm{amaj}(\pi') + \mathrm{fmaj}(\sigma)$ and $\mathrm{ades}(\pi) = \mathrm{ades}(\pi') + \mathrm{fdes}(\sigma)$.

Consider the signed permutation $\pi = [-4, 12, 8, -20] \in \widetilde{B}_4 \setminus B_4$. Then $\mathrm{amaj}_{\widetilde{B}}(\pi) = \mathrm{amaj}_{\widetilde{C}}(\pi) - 1 = 29 - 1 = 28$. Analogously, $\mathrm{ades}_{\widetilde{B}}(\pi) = \mathrm{ades}_{\widetilde{C}}(\pi) - 1 = 11 - 1) = 10$.

Proposition 3.4.3. *The statistic* abacus major index *defined on* $\widetilde{B}_n$ *is Mahonian. Moreover, the following identity holds.*

$$\sum_{k \geq 0} [k+1]_q^n t^n =$$

$$\frac{2(1-tq^2)\prod_{i=2}^n (1-t^4q^{4i-2}) \sum_{\pi \in \widetilde{B}_n} t^{\mathrm{ades}(\pi)} q^{\mathrm{amaj}(\pi)}}{\big((1+tq)\prod_{i=2}^n (1+t^2q^{2i-1})^2 + (1-tq)\prod_{i=2}^n (1-t^2q^{2i-1})^2\big)(1-t)\prod_{i=1}^n (1-t^2q^2)}$$

The proof is similar to that of Proposition 3.4.2.

Finally, it is possible to define an abacus major index also on the group $\widetilde{D}_n$. Start from an element $\pi \in \widetilde{D}_n \setminus D_n$ and consider its abacus according to Proposition 3.4.1. Then the abacus major index of π $\mathrm{amaj}(\pi)$ is is defined in the same way as the abacus major index in $\widetilde{B}_n$ with the only difference that the weights due to the sequences labeled by $\pm n$ are $n-1$ instead of $2n-1$. Then for any $\pi \in \widetilde{D}_n$, $\pi = \pi'\sigma$, with $\pi' \in \widetilde{D}_n \setminus D_n$ and $\sigma \in D_n$ define $\mathrm{amaj}(\pi) = \mathrm{amaj}(\pi') + \mathrm{dmaj}(\sigma)$, where $\mathrm{dmaj}(\sigma)$ is defined in (2.2.14).

It is possible to check that the abacus major index statistic is Mahonian.

Chapter 4

Kazhdan-Lusztig polynomials of boolean elements

In 1979 Kazhdan and Lusztig [KL79] defined, for every Coxeter group W, a family of polynomials with integer coefficients, indexed by pairs of elements of W. These polynomials, known as the Kazhdan-Lusztig polynomials of W are related to the algebraic geometry, topology of Schubert varieties and are important in representation theory. In order to prove the existence of these polynomials, Kazhdan and Lusztig used another family of polynomials, called R-polynomials, that arises from the multiplicative structure of the Hecke algebra associated to W.

In this chapter we defined such polynomials and compute that polynomials indexed by a family of elements, called boolean elements. Most of this chapter is in [Mon12b].

4.1 R-polynomials and Kazhdan–Lusztig polynomials

Kazhdan–Lusztig and R-polynomials can be defined in several equivalent ways, but here we use the more combinatorial one. We start with the following theorem which is also a definition of R-polynomials.

Theorem 4.1.1. *There is a unique family of polynomials* $\{R_{u,v}(q)\}_{u,v\in W} \subseteq \mathbb{Z}[q]$ *satisfying the following conditions:*

(i) $R_{u,v}(q) = 0$, *if* $u \not\leq v$;

(ii) $R_{u,v}(q) = 1$, *if* $u = v$;

(iii) *If* $s \in D_R(v)$, *then*

$$R_{u,v} = \begin{cases} R_{us,vs}(q) & \text{if } s \in D_R(u), \\ qR_{us,vs}(q) + (q-1)R_{u,vs}(q) & \text{if } s \notin D_R(u). \end{cases} \tag{4.1.1}$$

The proof of this theorem can be found for example in [Hum90, Sections 7.4 and 7.5], see also [KL79]. The polynomials introduced in the previous theorem are called *R-polynomials* of W. Theorem 4.1.1 can be used to compute the R-polynomials by induction of $l(v)$.

We now can introduce the definition of the Kazhdan–Lusztig polynomials. As done for the R-polynomials, we use a theorem to introduce them.

Theorem 4.1.2. *There is a unique family of polynomials* $\{P_{u,v}(q)\}_{u,v\in W} \subset \mathbb{Z}[q]$ *satisfying the following conditions:*

(i) $P_{u,v}(q) = 0$, *if* $u \not\leq v$;

(ii) $P_{u,v}(q) = 1$, *if* $u = v$;

(iii) $\deg(P_{u,v}(q)) \leq \frac{1}{2}(l(u,v) - 1)$, *if* $u < v$;

(iv) *If* $u \leq v$, *then*

$$q^{l(u,v)} P_{u,v}\left(\frac{1}{q}\right) = \sum_{a\in[u,v]} R_{u,a}(q)P_{a,v}(q). \tag{4.1.2}$$

A proof of Theorem 4.1.2 can be found in [Hum90, Sections 7.9 to 7.11], see also [KL79]. The polynomials introduced in the previous theorem are called *Kazhdan–Lusztig polynomials* of W.

In Theorem 4.1.2, (iii) gives an upper bound of the degree of $P_{u,v}(q)$. For $u, v \in W$, $u \leq v$, we let

$$\mu(u,v) := \begin{cases} [q^{\frac{1}{2}(l(u,v)-1)}](P_{u,v}) & \text{if } l(u,v) \text{ is odd}, \\ 0 & \textit{otherwise}. \end{cases} \tag{4.1.3}$$

The previous coefficients have many important applications in Kazhdan-Lusztig theory and representation theory. Computing the coefficient $\mu(u,v)$ is, in general, as difficult as obtaining the whole polynomial $P_{u,v}(q)$, for $u, v \in W$.

Here we give an important result which gives us a recursive formula to compute Kazhdan–Lusztig polynomials without the use of R-polynomials.

Theorem 4.1.3. *Let $u, v \in W$, $u \leq v$, and $s \in D_R(v)$. Then*

$$P_{u,v}(q) = q^{1-c}P_{us,vs}(q) + q^c P_{u,vs}(q) - \sum_{w \in W: s \in D_R(w)} q^{\frac{l(w,v)}{2}} \mu(w, vs) P_{u,w}(q) \tag{4.1.4}$$

where $c = 1$ if $s \in D_R(u)$ and $c = 0$ otherwise.

A proof of the previous result can be found in [Hum90, Section 7.11].

4.2 Parabolic Kazhdan–Lusztig polynomials

In 1987 Deodhar [Deo87] extended the definition of Kazhdan–Lusztig polynomial to parabolic Coxeter groups. His definition depends on one parameter x, which could be -1 or q. For our purposes, we do not give the original definition, which uses, as for the classical polynomials, the multiplicative structure of the Hecke algebra associated, but we only give the analogous recursion formula of Theorem 4.1.3.

For $J \subseteq S$, $x \in \{-1, q\}$ and $u, v \in W^J$ we denote by $P^{J,x}_{u,v}(q)$ the parabolic Kazhdan–Lusztig polynomials in W^J of type x.

For $u, v \in W^J$ let $\mu_{J,q}(u, v)$ be the coefficient of $q^{\frac{1}{2}(l(u,v)-1)}$ in $P^{J,q}_{u,v}(q)$ (so $\mu_{J,q}(u, v) = 0$ when $l(v) - l(u)$ is even). It is well known that if $u, v \in W^J$ then $\mu_{J,q}(u, v) = \mu(u, v)$, the coefficient of $q^{\frac{1}{2}(l(u,v)-1)}$ in $P_{u,v}(q)$ (see Corollary 4.2.3 below). The following result is due to Deodhar, and we refer the reader to [Deo87] for its proof.

Proposition 4.2.1. *Let (W, S) be a Coxeter system, $J \subseteq S$, and $u, v \in W^J$, $u \leq v$. Then for each $s \in D_R(v)$ we have that*

$$P^{J,q}_{u,v}(q) = \widetilde{P}_{u,v} - \widetilde{M}_{u,v} \tag{4.2.5}$$

where

$$\widetilde{P}_{u,v} = \begin{cases} P^{J,q}_{us,vs}(q) + qP^{J,q}_{u,vs}(q) & \text{if } us < u; \\ qP^{J,q}_{us,vs}(q) + P^{J,q}_{u,vs}(q) & \text{if } u < us \in W^J; \\ 0 & \text{if } u < us \notin W^J. \end{cases}$$

and

$$\widetilde{M}_{u,v} = \sum_{u \leq w < vs: ws < w} \mu(w, vs) q^{\frac{l(w,v)}{2}} P^{J,q}_{u,w}(q).$$

The parabolic Kazhdan–Lusztig polynomials are related to their ordinary counterparts in several ways, including the following one, which may be taken as their definition in most cases.

Proposition 4.2.2. *Let (W, S) be a Coxeter system, $J \subseteq S$ and $u, v \in W^J$. Then we have that*

$$P^{J,q}_{u,v}(q) = \sum_{w \in W_J} (-1)^{l(w)} P_{wu,v}(q).$$

Moreover, if W_J is finite and $w_0(J)$ is its longest element, then

$$P^{J,-1}_{u,v}(q) = P_{w_0(J)u, w_0(J)v}(q).$$

A proof of this result can be found in [Deo87, Proposition 3.4 and Remark 3.8]. By 1.1.6 the degree of $P_{wu,v}(q)$ in Proposition 4.2.2 is less than $\frac{1}{2}(l(u,v)-1)$ except in the case $w = \epsilon_W$. Therefore we have

Corollary 4.2.3. *For any $J \subseteq S$ and $u, v \in W^J$ we have*

$$\mu_{J,q}(u, v) = \mu(u, v).$$

The following result is probably known, but for lack of an adequate reference we provide its proof here.

Proposition 4.2.4. *Let (W, S) a Coxeter system and $J \subseteq S$. Let $u, v \in W^J$ and $s \in D_R(v)$.*

a) If $us \notin W^J$ then $P^{J,q}_{u,v}(q) = 0$;

b) if $us \in W^J$ then $P^{J,q}_{us,v}(q) = P^{J,q}_{u,v}(q)$;

c) if $\mu(u, v) \neq 0$ then $D_R(v) \subseteq D_R(u)$ and $D_L(v) \subseteq D_L(u)$.

Proof. If $us \notin W^J$ then by Proposition 4.2.1 we have

$$P^{J,q}_{u,v}(q) = - \sum_{u \leq w < vs; ws < w} \mu(w, vs) q^{\frac{l(w,v)}{2}} P^{J,q}_{u,w}(q).$$

The sum may be empty or we can apply induction on $l(v) - l(u)$ and have $P^{J,q}_{u,w}(q) = 0$. In both cases $P^{J,q}_{u,v}(q) = 0$. For *b*) use the same arguments as in the proof of [BB05, Proposition 5.1.8]. For the first part of *c*) use *a*), *b*) and the property that $P^{J,q}_{u,v}(q)$ has maximal degree. For the second part of *c*) use the identity $P_{u,v}(q) = P_{u^{-1},v^{-1}}(q)$ (see [BB05, Exercise 5.12]) and Corollary 4.2.3. □

In the rest of this chapter we will consider parabolic Kazhdan–Lusztig polynomials of type q. Therefore we will write $P^J_{u,v}$ instead of $P^{J,q}_{u,v}$.

Let (W, S) be any Coxeter system and t be a reflection in W. Following Marietti ([Mar02], [Mar06] and [Mar10]), we say that t is a *boolean reflection* if it admits a *boolean expression*, which is, by definition, a reduced expression of the form $s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_1$ with $s_k \in S$, for all $k \in \{1, \dots n\}$ and $s_i \neq s_j$ if $i \neq j$. We say that $u \in W$ is a *boolean element* if u is smaller than a boolean reflection in the Bruhat order. Let $\overline{v}$ be a reduced word of a boolean element and $s \in S$. We denote by $\overline{v}(s)$ the number of occurrences of s in $\overline{v}$.

Given a Coxeter system (W, S), we say that W is a *tree-Coxeter group* if its Coxeter graph is a tree.

4.3 Main results

In this section we give some preliminary lemmas which are needed to prove the main theorem of this chapter. For any generator $s_i \in S$ we set $S^i := S \setminus \{s_i\}$ and we denote by $\mathrm{com}(s_i)$ the subset of S^i of all elements commuting with s_i.

Lemma 4.3.1. *Let $u, v \in W^J$ such that $s_i u, s_i v \in W^J_{S^i}$ (i. e. there exist reduced words for u, v starting with s_i and with no other occurrence of s_i). Then*

$$P^J_{u,v} = P^{J \cap \mathrm{com}(s_i)}_{s_i u, s_i v}.$$

Proof. The statement is trivial if $l(v) = 1$. Suppose that $l(v) > 1$. Then there exists $s_j \in D_R(v)$, $j \neq i$. Note that for any $w \in W$ with $s_i w \in W_{S^i}$ we have that $D_L(w) \subseteq \{s_i\} \cup \mathrm{com}(s_i)$, more precisely $D_L(w) = \{s_i\} \cup (D_L(s_i w) \cap \mathrm{com}(s_i))$. Therefore $u s_j \in W^J$ if and only if $s_i u s_j \in W^{J \cap \mathrm{com}(s_i)}$. In this case, by Proposition 4.2.1 we have

$$\begin{aligned} P^J_{u,v} =& q^c P^J_{us_j, vs_j} + q^{1-c} P^J_{u, vs_j} - \sum_{\substack{u \leq w \leq vs_j \\ ws_j < w}} \mu(w, vs_j) q^{\frac{l(w, vs_j)}{2}} P^J_{u,w} \\ =& q^c P^{J \cap \mathrm{com}(s_i)}_{s_i us_j, s_i vs_j} + q^{1-c} P^{J \cap \mathrm{com}(s_i)}_{s_i u, s_i vs_j} + \\ & - \sum_{\substack{s_i u \leq s_i w \leq s_i vs_j \\ s_i ws_j < s_i w}} \mu(s_i w, s_i vs_j) q^{\frac{l(s_i w, s_i vs_j)}{2}} P^{J \cap \mathrm{com}(s_i)}_{s_i u, s_i w} \\ =& P^{J \cap \mathrm{com}(s_i)}_{s_i u, s_i v} \end{aligned}$$

by induction, where c is 0 or 1. The equalities hold since the map from $[u, v]^J$ to $[s_i u, s_i v]^{J \cap \mathrm{com}(s_i)}$ given by left-multiplication by s_i is an isomorphism of posets. □

Lemma 4.3.2. *Let $u, v \in W^J$ be such that $u, s_i v \in W_{S^i}$ (i. e. there is no occurrence of s_i in any reduced expression of u and $s_i v$). Then*

$$P^J_{u,v} = \begin{cases} P^J_{u, s_i v} & \text{if } s_i v \in W^J \\ 0 & \text{otherwise} \end{cases}$$

Proof. If $l(v) = 1$ there is nothing to prove. Let we suppose $l(v) > 1$ and let $s_j \in D_R(v)$, $s_j \neq s_i$. If $us_j \notin W^J$ the claim is trivial by Proposition 4.2.4. Therefore we may assume $us_j \in W^J$.

Suppose that $s_i v \in W^J$. Then by Proposition 4.2.1 we get

$$\begin{aligned} P^J_{u,v} =& q^c P^J_{us_j,vs_j} + q^{1-c} P^J_{u,vs_j} - \sum_{\substack{u \leq w \leq vs_j \\ ws_j < w}} \mu(w, vs_j) q^{\frac{l(w,vs_j)}{2}} P^J_{u,w} \\ =& q^c P^J_{us_j,s_i vs_j} + q^{1-c} P^J_{u,s_i vs_j} - \sum_{\substack{u \leq s_i w \leq s_i vs_j \\ s_i ws_j < s_i w}} \mu(s_i w, s_i vs_j) q^{\frac{l(s_i w, s_i vs_j)}{2}} P^J_{u,s_i w} \\ =& P^J_{u,s_i v} \end{aligned}$$

where c is 0 or 1. The equalities hold by induction on $l(vs_j)$: if $w \in W_{S^i}$ then $\mu(w, vs_j)$ is 0 since by induction either $P^J_{w,vs_j} = 0$ or $P^J_{w,vs_j} = P^J_{w,s_i vs_j}$ and therefore P^J_{w,vs_j} does not have the maximum degree. Otherwise, if $s_i w \notin W^J$ then $P^J_{u,w} = 0$ by induction, else $P^J_{u,w} = P^J_{u,s_i w}$ and $\mu(w, vs_j) = \mu(s_i w, s_i vs_j)$ by Lemma 4.3.1 and Corollary 4.2.3.

Finally, if $s_i v \notin W^J$ choose s_j such that $s_i vs_j \notin W^J$ (it is always possible except in the case $v = s_i s_j$, but then the claim is trivial). Then by induction

$$P^J_{u,v} = - \sum_{\substack{u \leq w \leq vs_j \\ ws_j < w}} \mu(w, vs_j) q^{\frac{l(w,vs_j)}{2}} P^J_{u,w}.$$

Fix $w \in W^J$ with $u \leq w \leq vs_j$ and $ws_j < w$. We prove that $\mu(w, vs_j) P^J_{u,w} = 0$. If $w \in W_{S^i}$ then $\mu(w, vs_j) = 0$ by induction. Otherwise, if $s_i w \in W_{S^i}$ then by Lemma 4.3.1 we have $\mu(w, vs_j) = \mu(s_i w, s_i vs_j)$. Now, if $s_i w \notin W^J$ then by induction $P^J_{u,w} = 0$, else both $s_i vs_j \notin W^J$ and $s_i w \in W^J$ imply that $D_L(s_i vs_j) \not\subseteq D_L(s_i w)$ and by *c*) of Proposition 4.2.4) we have $\mu(s_i w, s_i vs_j) = 0$. □

We now introduce a family of numbers which we will use in the next section. The *Catalan triangle* is a triangle of numbers formed in the same manner as Pascal's triangle, except that no number may appear on the left of the first element (see [OEI, sequence A008313]).

1				
1				
1	1			
2	1			
2	3	1		
5	4	1		
5	9	5	1	
14	14	6	1	
14	28	20	7	1
42	48	27	8	1

Let $h \geq 1$. We set

$$f_h(q) = \sum_{i=0}^{[\frac{h}{2}]} C(h,i) q^{[\frac{h}{2}]-i}$$

where $[h]$ denotes the integer part of h and $C(h,i)$ is the i-th number in the h-th row (here we start the enumeration from 0). For example $f_4(q) = 2q^2 + 3q + 1$; $f_7(q) = 14q^3 + 14q^2 + 6q + 1$. We denote by $\mu(f_h(q))$ the coefficient of $q^{\frac{h}{2}}$ in $f_h(q)$. Therefore $\mu(f_h(q)) = 0$ if h is odd. Then we have the following easy result, whose proof is omitted.

Lemma 4.3.3. *For all $h \geq 0$,*

$$f_h(q)(1+q) - \mu(f_h(q)) q^{\frac{h}{2}+1} = f_{h+1}(q).$$

Note that in the first column we find the classical Catalan numbers (see [OEI, sequence A008313] for details).

Let (W,S) be a tree-Coxeter group. Let $t = s_{i_1} \cdots s_{i_{n-1}} s_{i_n} s_{i_{n-1}} \cdots s_{i_1}$ be a boolean reflection. Consider the Coxeter graph G and represent it as a rooted tree with root the vertex corresponding to the generator s_{i_n}. In this paper all the roots will be depicted on the right of their graphs. In Figure 4.1 we give the Coxeter graph of the affine Weyl group $\widetilde{D}_{11}$.

According to such rooted graph we say that s_j is on the right (respectively on the left) of s_i if and only if there exists an edge joining them and the only path from s_i to s_n contains s_j.

Let w be a word in the alphabet S and $s \in S$. We denote by $w(s)$ the number of occurrences of s in w. Let $u, v \in W$ be such that $u, v \leq t$. Let $\overline{u}, \overline{v}$ be the unique reduced expressions of u, v satisfying the following properties

- $\overline{v}$ is a subword of $s_1 \cdots, s_{n-1} s_n s_{n-1} \cdots s_1$ and if i is such that $\overline{v}(s_i) = 1$ and $\overline{v}(s_j) = 0$, where s_j is the only element on the right of s_i, then we choose the subword with s_i in the leftmost admissible position;

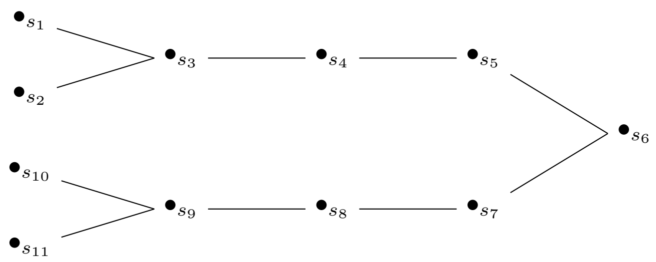


Figure 4.1: The Coxeter graph of $\widetilde{D}_{11}$ with root s_6, corresponding to the reflection $t = s_1 s_2 \cdots s_5 s_{10} s_{11} s_9 s_8 s_7 s_6 s_7 s_8 s_9 s_{11} s_{10} s_5 \dots s_2 s_1$.

- $\overline{u}$ is a subword of $\overline{v}$ and if i is such that $\overline{u}(s_i) = 1$ and $\overline{u}(s_j) = 0$, we apply the same above rule.

Here we give an example. Let $t = s_1 s_2 \cdots s_5 s_{10} s_{11} s_9 s_8 s_7 s_6 s_7 s_8 s_9 s_{11} s_{10} s_5 \dots \dots s_2 s_1$ in $\widetilde{D}_{11}$, see Figure 4.1. Let $v = s_4 s_5 s_{10} s_{11} s_6 s_7 s_8 s_9 s_5 s_4 s_2 s_1$ and $u = s_8 s_6 s_1$ then $\overline{v} = s_1 s_2 s_4 s_5 s_{10} s_{11} s_6 s_7 s_8 s_9 s_5 s_4$ and $\overline{u} = s_1 s_6 s_8$.

Now we give a graphical representation of the pair $(\overline{v}, \overline{u})$. We start from the rooted tree of the Coxeter graph and we substitute for each vertex a table with one column and two rows. In the first row we write $\overline{v}(s_j)$ (s_j is the element associated to the vertex); in the second row we write $\overline{u}(s_j)$. In the case $\overline{v}(s_j) = 1$, it is possible that s_j is on the left or on the right of s_n (the root) as subword of t. We distinguish the two cases by writing 1_l if s_j is on the left of s_n, and 1_r otherwise. By convention we write 1_l in the root s_n if $\overline{v}(s_n) \neq 0$. We apply the same rule to the second row. Moreover, in the first row, we use capital letter R instead of r if the second row of the column to the right does not contain 0.

We mark the column corresponding to s_j with $\circ$ if $j \in J$ and with $\times$ if $j \notin J$. Finally, if a vertex s_j has only one vertex on the left then we write the two corresponding columns in same table. In Figure 4.2 we give the graphical representation of the pair $(\overline{v}, \overline{u})$ in $\widetilde{D}_{11}$, with $J = \{s_5, s_7\}$.

In the sequel a symbol $*$ denotes the possibility to have arbitrary entries in the cell. A symbol such as $\not{1}_l$, $\not{0}$, etc. means that the value in the cell is not 1_l, 0, etc. Moreover we will be interested in subdiagrams of such representations, i. e. diagrams obtained by deleting one or more columns. Since the order of the tables from top to bottom is not important (while the order from left to right

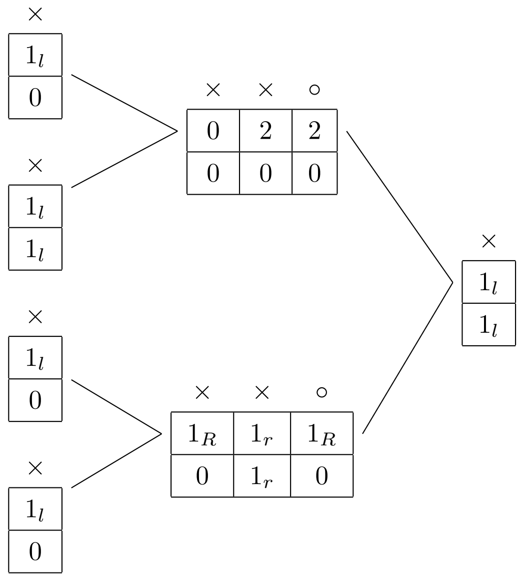
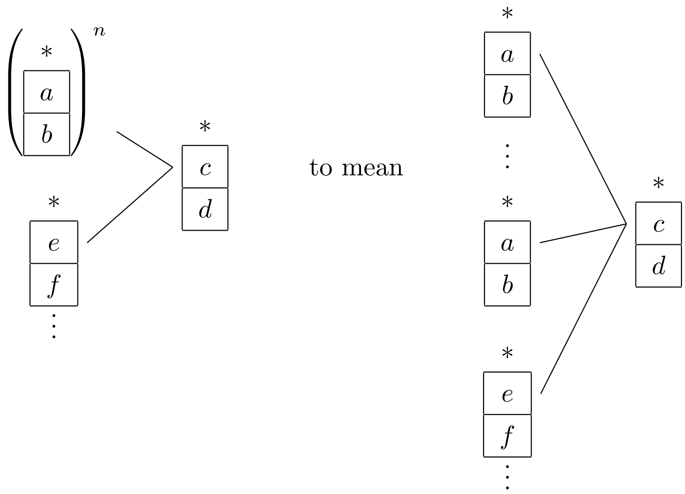


Figure 4.2: Diagram of ($\overline{v} = s_1s_2s_4s_5s_{11}s_{10}s_6s_7s_8s_9s_5s_4, \overline{u}s_1s_6s_8$) in $\widetilde{D}_{11}$.

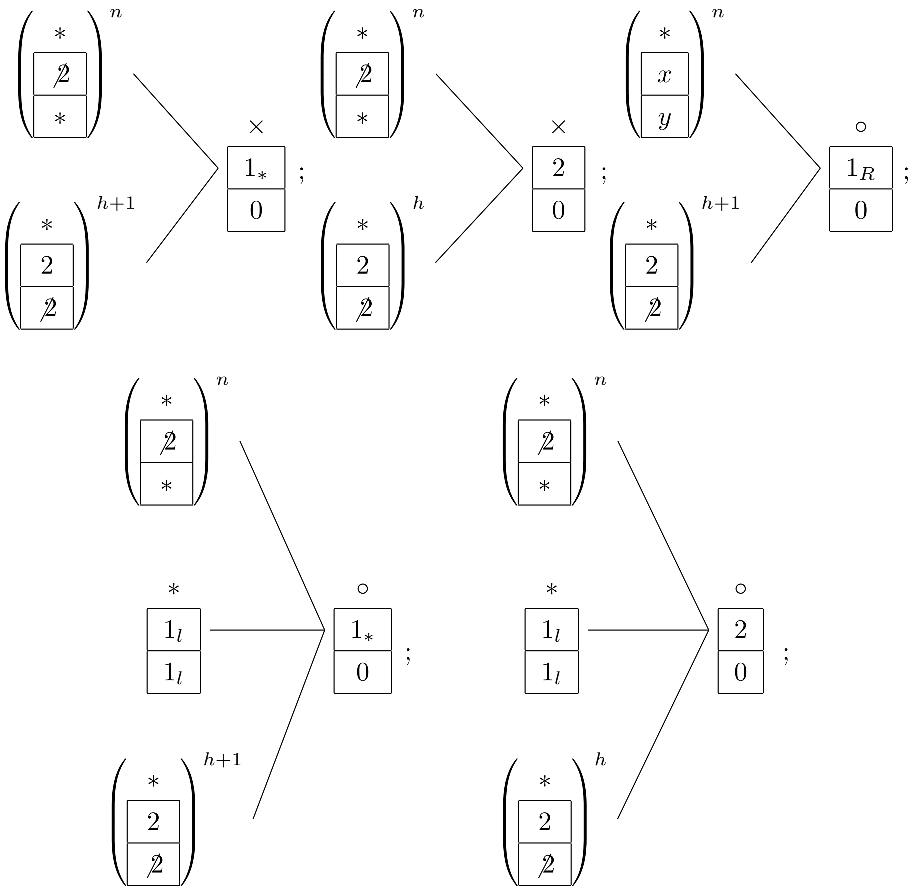
is fundamental), we use the following notation



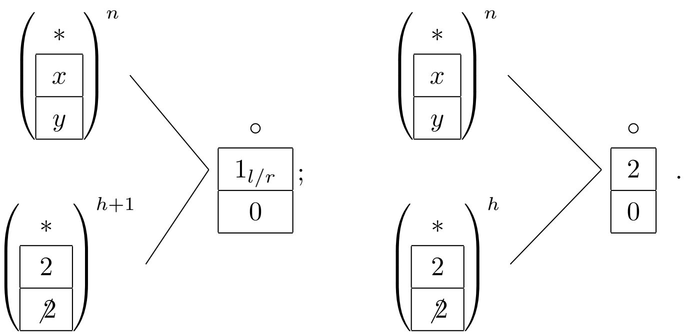
(4.3.6)

where the column with entries a, b is repeated n times. Now we give all the definitions necessary to Theorem 4.3.4.

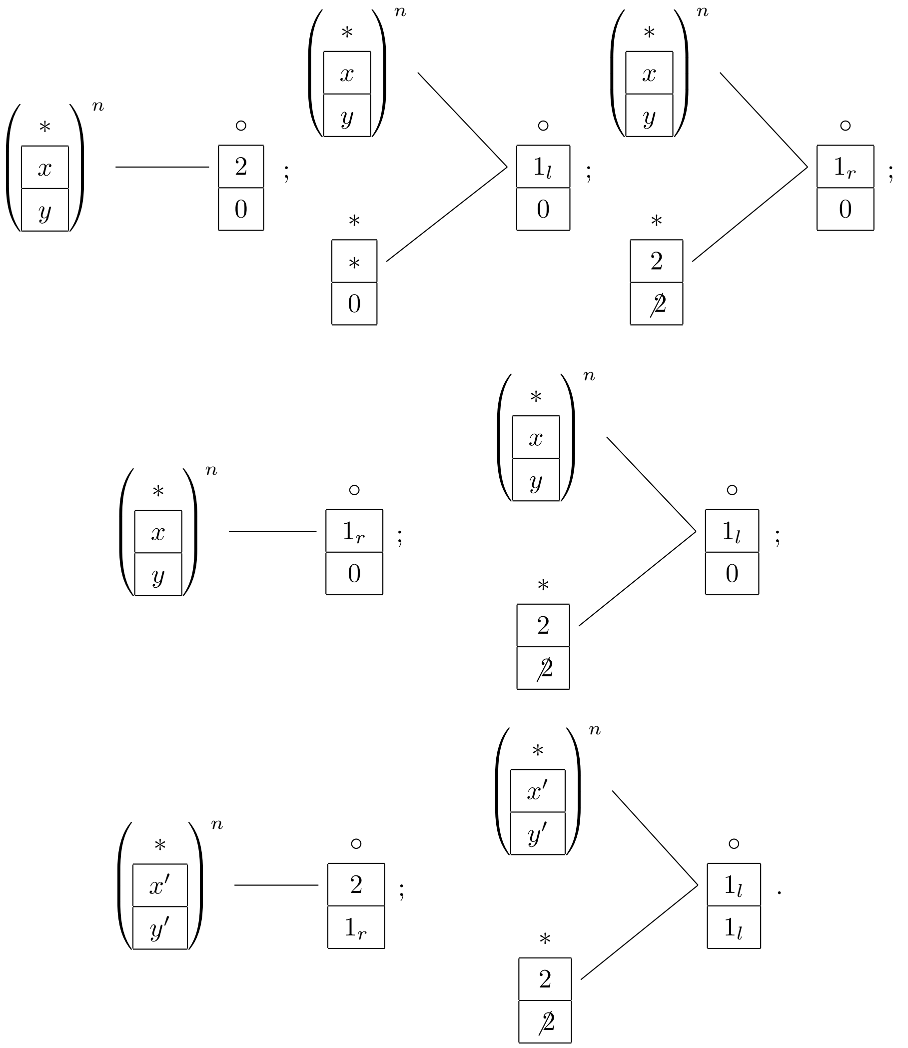
Given a pair $(\overline{v}, \overline{u})$ in W, we let $a_h(\overline{u}, \overline{v})$ be the number of subdiagrams in the diagram of $(\overline{u}, \overline{v})$ of one of the following type:



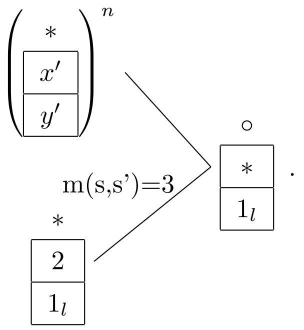
We define $b_h(\overline{u}, \overline{v})$ be the number of subdiagrams in the diagram of $(\overline{u}, \overline{v})$ of one of the following type:



We set $c(\overline{u}, \overline{v})$ be the number of subdiagrams in the diagram of $(\overline{u}, \overline{v})$ of one of the following type:



Finally, we set $c'(\overline{u},\overline{v})$ be the number of subdiagrams of the diagram of $(\overline{u},\overline{v})$ of the following type:

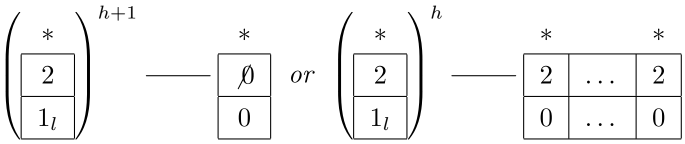


In all previous diagrams n is an arbitrary non–negative integer and $(x, y) \in P_1$, $(x', y') \in P_1 \cup P_2$ with $P_1 = \{(1_l, 0), (1_r, 0), (1_r, 1_r), (2, 1_r)\}$, $P_2 = \{(1_R, 0), (1_R, 1_r), (2, 0)\}$. In each diagram (x, y), (x', y'), $(\not{2}, *)$ or $(2, \not{2})$ are not necessarily the same pair for all $n \geq 0$ (or $h \geq 0$) columns. We can now state the main result of this work.

Theorem 4.3.4. *Let $J \subseteq S$, $u, v \in W^J$ and set $\overline{c}(\overline{u}, \overline{v}) = c(\overline{u}, \overline{v}) + c'(\overline{u}, \overline{v})$. We have*

$$P^J_{u,v}(q) = \begin{cases} \prod_{h \geq 1} f^{a_h}_{h+1}(f_{h+1} - 1)^{b_h} & \text{if } \overline{c}(\overline{u}, \overline{v}) = 0 \\ 0 & \text{otherwise} \end{cases}$$

Corollary 4.3.5. *Let $J \subseteq S$, $u, v \in W^J$ with $l(v) - l(u) \geq 3$ odd. Then $\mu(u, v) \neq 0$ if and only if the entries in each column of the diagram of $(\overline{u}, \overline{v})$ are equal except for exactly one subdiagram which is*



In this case $\mu(u, v) = C([\frac{h+1}{2}])$, the $[\frac{h+1}{2}]$-th Catalan number.

Proof. If in the diagram of $(\overline{u}, \overline{v})$ there are more than one subdiagram with the properties in the statement, then by Theorem 4.3.4 $P^J_{u,v}$ is the product of at least two factors. Since $l(v) - l(u)$ is equal to the sum of the differences between top row and bottom row entries, we have that the degree of $P^J_{u,v}$ is at most $\frac{l(v)-l(u)-2}{2}$. The last part of the statement follows by properties of $f_h(q)$. □

We now prove Theorem 4.3.4.

Proof. In this proof we use formula in Proposition 4.2.1 We argue by induction on $l(v)$. If $l(v) = 1$ then $P^J_{u,v} = 1$, since $u \leq v$ and the result is trivial. Now suppose $l(v) \geq 2$. Let C be one of the leftmost columns in the diagram. The entries of C can be filled by several values.

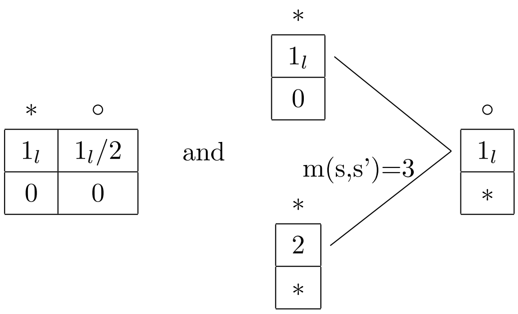
We first consider the case that C contains the pair $(1_r, 0)$ or $(1_R, 0)$. Let $s \in S$ be the element corresponding to C. Then $s \in D_R(v)$ and $us \not\leq vs$ because $s \leq us$ but $s \not\leq vs$. Moreover, since $s \not\leq vs$ we have that $w \not\leq vs$ and $P^J_{w,vs} = 0$ for any w such that $ws < w$. By Proposition 4.2.1 we have that $\widetilde{M}_{u,v} = 0$ and $P^J_{u,v} = P^J_{u,vs}$. The statement follows because in all the subdiagrams of type a_h (respectively b_h, c, c') we can delete the column C and have again a subdiagram of type a_h (respectively b_h, c, c').

If C contains the pair $(1_R, 1_r)$ or $(1_r, 1_r)$ then $u \not\leq vs$ and therefore $P^J_{u,v} = P^J_{us,vs}$ by Proposition 4.2.1. The statement follows.

Now suppose that C contains $(1_l, 1_l)$. By Lemma 4.3.1 $P^J_{u,v} = P^{J\cap \mathrm{com}(s)}_{su,sv}$. Since C is a column on the left, $|C(s)| = 1$. Therefore the Kazhdan–Lusztig polynomial associated to the diagram is the same of that associated to the diagram without the column C and with the column on the right marked with $\times$. Apply induction hypotheses and note that for any subdiagram of type a_h ($h \geq 0$) it is possible to remove one leftmost column with entries $(1_l, 1_l)$ having again a diagram of type a_h. Moreover note that this agrees with the assumption $(1_l, 1_l) \notin P_1 \cup P_2$.

If C contains $(2, 2)$ then $u \not\leq vs$ and by Proposition 4.2.1 $P^J_{u,v} = P^J_{us,vs}$. We are in the case $(1_l, 1_l)$. As before, it is possible to remove a column with entries $(2, 2)$ from a diagram of type a_h without change its type and the assumption $(2, 2) \notin P_1 \cup P_2$ ensures that such entries are not in any subdiagram of type b_h, c or c'. The claim follows by induction.

If C contains $(1_l, 0)$ then by Lemma 4.3.2 $P^J_{u,v} = P^J_{u,sv}$ except in the case $sv \notin W^J$. Then we have to exclude



These diagrams are included in $c'(\overline{u}, \overline{v})$, and in the summands $1, 2, 5$ and 6 of $c(\overline{u}, \overline{v})$. If C contains $(2, 1_r)$ use the same arguments above to have $P^J_{u,v} = P^J_{us,vs}$ and come back in the case $(1_l, 0)$.

Now suppose that C contains $(2, 0)$ and the bottom entry in the column on the right is non-zero. By Theorem 1.1.3, this assumption implies $s \notin D_L(us)$. Therefore $us \not\leq vs$. Moreover, there is no $w \in W^J$ with $u \leq w < vs$ and $ws < w$: in fact $ws < w \leq vs$ implies that the only occurrence of s in the word $\overline{w}$ is on the first place (since the same is for the word $\overline{vs}$); therefore $s \in D_L(w) \cap D_R(w)$ and thus if we denote with t the element on the right of s then $w(t) = 0$ but this implies that $u \not\leq w$, impossible. By Proposition 4.2.1 we have $P^J_{u,v} = P^J_{u,vs}$. Then, by previous arguments, we are in the case $(1_l, 0)$ and this agrees with the assumption $(2, 0) \in P_2$.

If C contains $(2, 1_l)$ and the second entry in the column on the right is non-zero, then $us \notin W^J$ if and only if the diagram is such as in $c'(\overline{u}, \overline{v})$ (it is an easy consequence of Theorem 1.1.3). Otherwise $P^J_{u,v} = P^J_{u,vs}$ since, as before, there is no $w \in W^J$ with $u \leq w < vs$ and $ws < w$. Then we come back to the case $(1_l, 1_l)$.

Finally we have to consider the cases $(2, 1_l)$ or $(2, 0)$ with the second entry in the column on the right equal to 0. By Proposition 4.2.4, they can be treated as the same case. Note that in the definition of diagrams of type a_h, b_h, c or c' there is no difference in both cases. Therefore we assume that C contains $(2, 1_l)$.

For the diagram

$$\left(\begin{array}{c} * \\ \boxed{2} \\ \boxed{1_l} \end{array}\right)^n \text{———} \begin{array}{c} * \\ \boxed{1_*} \\ \boxed{0} \end{array} \tag{4.3.7}$$

the corresponding Kazhdan–Lusztig polynomial is $P^J_{u,v} = f_n - \alpha$, where $\alpha = 1$ when there are $\circ$ and 1_l on the rightmost column and $\alpha = 0$ otherwise. To show this, note that $P^J_{u,vs}$ is represented by a diagram with a leftmost column having entries equal to $(1_l, 1_l)$. By induction, the polynomial is equal to $P^J_{su,svs}$, whose diagram is as in (4.3.7) but with $n-1$ instead of n. The polynomial $P^J_{us,vs}$ is represented by a diagram with a leftmost column with $(1_l, 0)$ and by induction $P^J_{us,vs} = P^J_{us,svs}$. Finally, by Corollary 4.3.5 and by induction $\mu(w, vs) \neq 0$ only if the diagram of w coincides with the diagram of v in all other columns not depicted in (4.3.7). Apply Proposition 4.2.1 and have $P^J_{u,v} = f_{n-1}(q) - \alpha + q f_{n-1}(q) - \mu(f_{n-1}(q)) q^{\frac{n-1}{2}}$. By Lemma 4.3.3 we get $P^J_{u,v} = f_n - \alpha$ (note that if $n = 1$, $f_1 - 1 = 0$ and this agrees with the 3^{rd} and 5^{th} summands in $c(\overline{u}, \overline{v})$).

For the last subcase,

$$\left(\begin{array}{c} * \\ \boxed{2} \\ \boxed{1_l} \end{array}\right)^n \text{———} \begin{array}{c} * \\ \boxed{2} \\ \boxed{0} \end{array} \tag{4.3.8}$$

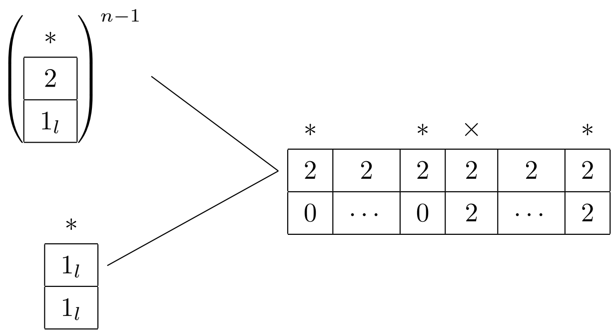
the analysis is a bit harder. Let's assume that on the right of this diagram there is a sequence of m columns

$$\begin{array}{|c|c|c|c|} \multicolumn{1}{c}{*} & \multicolumn{1}{c}{*} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{*} \\ \hline 2 & 2 & \cdots & 2 \\ \hline 0 & 0 & \cdots & 0 \\ \hline \end{array} \tag{4.3.9}$$

ending with a column whose entries are not $(2, 0)$ or with a column corresponding to a vertex of degree greater than 2. Suppose that exactly k of these columns have a $\circ$ and the other $m - k$ have a $\times$. Let $\overline{P}^J_{u,v}$ be the Kazhdan–Lusztig polynomial corresponding to the diagram of $(\overline{u}, \overline{v})$ after deleting the subdiagrams depicted in (4.3.8) and (4.3.9). Use the same techniques above and by induction we have

$$P^J_{u,v} = (f_n(q) - \alpha)q^k(1+q)^{m-k}\overline{P}^J_{u,v} + f_n(q)q^{k+1}(1+q)^{m-k}\overline{P}^J_{u,v} - \widetilde{M}_{u,v}$$

where $\alpha = 1$ if there is a $\circ$ on the rightmost column of (4.3.8) and $\alpha = 0$ otherwise, and $\widetilde{M}_{u,v}$ is the sum in Proposition 4.2.1. Note that by induction and Corollary 4.3.5 $\mu(w, vs) \neq 0$ only if the diagram of w coincides with that of v in all the columns not depicted in (4.3.8) and (4.3.9). More precisely, for any such w, the diagram of $(\overline{w}, \overline{vs})$ is of the form



and in all other columns the top entries are equal to the bottom entries. Therefore $\widetilde{M}_{u,v}$ is

$$\overline{P}^J_{u,v}\mu(f_n(q))\big(q^{\frac{n-2}{2}+k}(q+1)^{m-k-1}+q^{\frac{n-2}{2}+k+1}(q+1)^{m-k-2}+\cdots+q^{\frac{n-2}{2}+m-1}+q^{\frac{n-2}{2}+m}\big)$$

if n is even and 0 if n is odd. In this formula the powers of q include both the contributions of $q^{\frac{l(w,vs)}{2}}$ and of $P^J_{u,w}$. In the case n even, $n \geq 4$, we have

$$\begin{aligned}\widetilde{M}_{u,v} =& \overline{P}^J_{u,v}\mu(f_n(q))\big(q^{\frac{n-2}{2}+k}((q+1)^{m-k} - q^{m-k}) + q^{\frac{n-2}{2}+m}\big)\\ =& \overline{P}^J_{u,v}\mu(f_n(q))(q+1)^{m-k}q^{\frac{n-2}{2}+k}\end{aligned}$$

and therefore

$$\begin{aligned}P^J_{u,v} =& \overline{P}^J_{u,v}q^k(1+q)^{m-k}\big(f_n(q) - \alpha + qf_n(q) - \mu(f_n(q))q^{\frac{n-2}{2}}\big)\\ =& P^J_{u,v}q^k(1+q)^{m-k}(f_{n+1}(q) - \alpha)\end{aligned}$$

by Lemma 4.3.3. Analogously, if n is odd, $n \geq 3$, we have

$$P^J_{u,v} = \overline{P}_{u,v}q^k(1+q)^{m-k}\big(f_n(q) - \alpha + qf_n(q)\big) = \overline{P}_{u,v}q^k(1+q)^{m-k}(f_{n+1}(q) - \alpha).$$

The cases $n = 1$ and $n = 2$ are similar (note that $f_1(q) - \alpha = 0$ if $\alpha = 1$). Thus the proof is completed. □

In the case of the classical Kazhdan–Lusztig polynomials, Theorem 4.3.4 becomes much simpler.

Corollary 4.3.6. *Let W be a tree-Coxeter group and $u, v \in W$ be boolean elements. Then $P_{u,v}(q) = \prod_{h \geq 1} f_{h+1}^{a_h}$, where a_h is defined before Theorem 4.3.4.*

Proof. Just note that if $J = \emptyset$ then $b_h(\overline{u}, \overline{v}) = c(\overline{u}, \overline{v}) = 0$, $h \geq 1$. □

For example, the Kazhdan–Lusztig polynomial of the pair (u, v) depicted in Figure 4.2 is $P^J_{u,v} = f_2(q) - 1 = q$, since $a_h = 0$ for all $h \geq 0$, $b_1 = 1$ and $b_h = 0$ for all $h \neq 1$.

Remark 4.3.7. *Theorem 4.3.4 implies result in [Mar10, Theorem 5.2].*

We give the following easy consequence of Theorem 4.3.4 which proves, in the case of boolean elements, a conjecture of Brenti (private communication).

Corollary 4.3.8. *Let $I \subseteq J$ and $u, v \in W^J$. Then*

$$P^J_{u,v}(q) \leq P^I_{u,v}(q)$$

in the coefficientwise comparison.

Proof. Let $s \in J \backslash I$. The corresponding column of the diagram of $P^J_{u,v}$ is marked by $\circ$ and the same column in the diagram of $P^I_{u,v}$ is marked by $\times$. Consider a subdiagram of type a_h of the diagram of $P^I_{u,v}$. It is possible that the same subdagram in the diagram of $P^J_{u,v}$ is of type b_h or c. The vice versa is not possible. The claim follows by Theorem 4.3.4. □

Now we consider the case of $\widetilde{A}_n$ for $n \geq 2$ ($\widetilde{A}_1$ is a tree-Coxeter group). The Coxeter diagram of $\widetilde{A}_n$ is a cycle, therefore we cannot apply Theorem 4.3.4. However we use the same arguments of its proof to have an analogue result. Consider a boolean reflection t in $\widetilde{A}_n$ of length $2n+1$. Then it is easy to check that $t = s_{i+1}s_{i+2} \cdots s_n s_0 \cdots s_{i-1} s_i s_{i-1} \cdots s_0 s_n \cdots s_{i+2} s_{i+1}$ for some $i \in [0, n]$ (the indices are modulo $n+1$). For any pair $(u, v) \in W^2$, $u \leq v \leq t$ we depict a diagram whose rightmost column contains $(\overline{u}(s_i), \overline{v}(s_i))$. The leftmost column contains $(\overline{u}(s_{i+1}), \overline{v}(s_{i+1}))$ and the other columns are defined by following the cyclic Coxeter diagram of $\widetilde{A}_n$. See Figure 4.3 for an example.

In the follows we assume that $\overline{v}(s_j) \neq 0$ for all $j = 0, \ldots, n$. In fact, otherwise v can be identified as an element in A_n and we can apply Theorem 4.3.4.

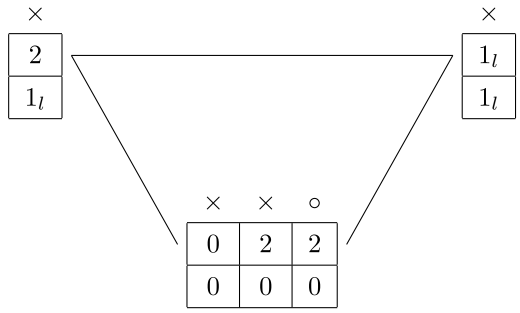
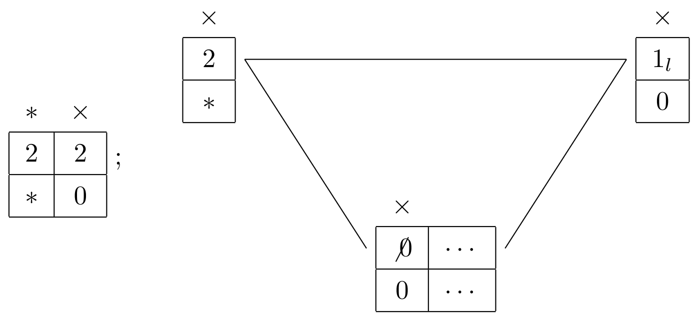
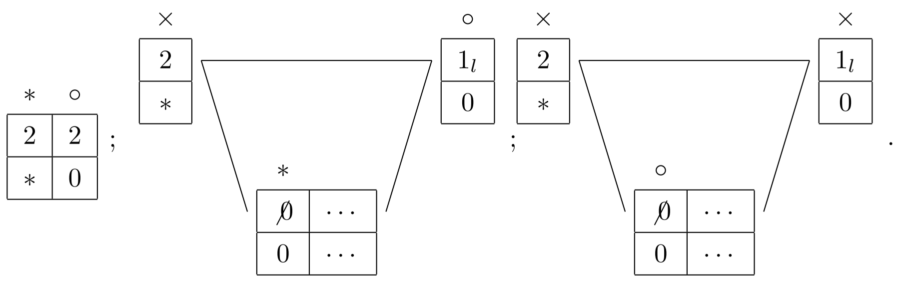


Figure 4.3: Diagram of $(\overline{v} = s_0s_2s_3s_4s_3s_2s_0\overline{u} = s_0s_4)$ in $\widetilde{A}_4$, with boolean reflection $t = s_0s_1s_2s_3s_4s_3s_2s_1s_0$ and $J = \{s_3\}$.

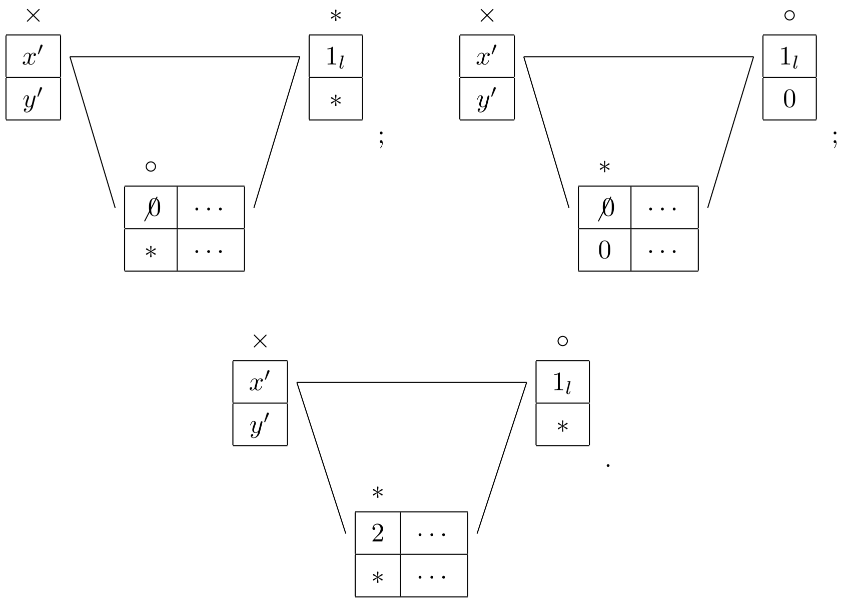
We define $a(\overline{u}, \overline{v})$ be the number of subdiagrams of the type



We set $b(\overline{u}, \overline{v})$ be the number of subdiagrams of the type



Finally, we set $c''(\overline{u},\overline{v})$ be the number of the subdiagrams of the type



where $(x,y),(x',y') \in \{(1_l,0),(1_r,0),(1_r,1_r),(2,1_r)\}$. Moreover (x',y') could be $(2,0)$ (respectively $(2,1_l)$) if there is a non-zero entry (resp. exactly one non-zero entry with a $\circ$) in the second row of one of the two columns on the right of the first column.

Theorem 4.3.9. *Let $u,v \in \widetilde{A}_n$ boolean elements. Then*

$$P^J_{u,v} = \begin{cases} q^{b(\overline{u},\overline{v})}(1+q)^{a(\overline{u},\overline{v})} & \text{if } c(\overline{u},\overline{v}) + c''(\overline{u},\overline{v}) = 0 \\ 0 & \text{otherwise} \end{cases}$$

The proof is the same as in Theorem 4.3.4. Delete the leftmost column if it contains $(1_*,0),(1_*,1_*),(2,2)$ by using Lemmas 4.3.2 and 4.3.1. If it contains the pair $(2,\not 2)$ then consider the cases with the second entries of both column on the right be zero and non-zero. In the first case apply Proposition 4.2.1 and note that $\widetilde{M}_{u,v} = 0$. We left to the reader all details. Note that $c'(\overline{u},\overline{v})$ does not appear in the statement.

Remark 4.3.10. *For the classical Kazhdan–Lusztig polynomials, Theorem 4.3.9 reduces to [Mar06, Theorem 4.4].*

4.4 Kazhdan–Lusztig polynomials of boolean signed permutation

In this section we consider the combinatorial interpretation of the finite Coxeter groups A_n, B_n and D_n as (signed) permutations and restate Theorem 4.3.4 by using statistics of such permutations.

Given a (signed) permutation π, if $\pi(i) > i$ we say that $\pi(i)$ is a *top excedance* and i is a *bottom excedance* of π . We know that a combinatorial interpretation of the group A_n is the permutation group S_{n+1} and that the set of all reflections in S_{n+1} is given by transpositions (i, j), with $i < j \leq n+1$. Any such transposition admits $s_i s_{i+1} \cdots s_{j-2} s_{j-1} s_{j-2} \cdots s_{i+1} s_i$ as reduced expression. So every reflection in the symmetric group is boolean and an element π is boolean if and only if it is smaller than the top transposition $(1, n+1)$, i. e. π admits a reduced expression which is a subword of $s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_1$.

Lemma 4.4.1. *Let $\pi \in S_{n+1}$. Then π is a boolean element if and only if $|\pi(\{1, \ldots, i\}) \cap \{1, \ldots, i\}| \geq i - 1$ for all $i \leq n$.*

Moreover, if $\overline{\pi}$ is the reduced expression of π, subword of $s_1 \cdots s_n \cdots s_1$, then $\overline{\pi}(s_i) = 1_l$ if $i+1$ is a top excedance of π; $\overline{\pi}(s_i) = 1_r$ if $i+1$ is a top excedance of π^{-1}; $\overline{\pi}(s_i) = 2$ if and only if $\pi(i+1) = i+1$ and $\pi(\{1, \ldots, i\}) \neq \{1, \ldots, i\}$; $\overline{\pi}(s_i) = 0$ if and only if $\pi(\{1 \ldots, i\}) = \{1 \ldots, i\}$.

Proof. We prove the first part of the statement by induction on n. If $n = 1$ there is nothing to prove. Suppose that $n \geq 2$. Now π is the product of s_1 (on the left, right or both) with a boolean element generated by $s_2, \ldots, s_n$. Multiplying by s_1 on the right is equivalent to exchanging the first and the second elements in the window notation of π; multiplying by s_1 on the left is equivalent to exchanging the elements 1 and 2 in the window notation of π. It is easy to see that $|\pi(\{1, \ldots, i\}) \cap \{1, \ldots, i\}|$ does not change for $i \geq 2$ and that the claim is always true for $i = 1$. The result follows by induction.

Vice versa, let $\pi \in S_{n+1}$ as in the statement. If $\pi(1) = 1$ then π can be identified with a permutation in S_n (in the following we will say that $\pi \in S_n$) and the claim is true by induction. Now suppose that $\pi(1) \neq 1$. If $\pi(2) = 1$ then $\pi s_1 \in S_n$ so we can apply induction. If $\pi(1) = 2$ then $s_i \pi \in S_n$ and we can apply induction again. Then we have to consider the case $\pi(1) \neq 1, 2$ and $\pi(2) \neq 1$. Since $|\pi(\{1, 2\}) \cap \{1, 2\}| \geq 1$ it forces to $\pi(2) = 2$ and then $s_1 \pi s_1 \in S_n$. The claim follows.

Now we prove the second part. Fix an index $i \leq n$. Let $\overline{\pi}'$ be the subword of $\overline{\pi}$ with only letters $s_{i+1}, \ldots, s_n$. Then $s_i \overline{\pi}'(i) = i+1$. If we multiply $s_i \overline{\pi}'$ by s_j,

$j < i$, on the left or on the right, then the element $i+1$ may be moved on the left in the window notation. Therefore $i+1$ is a top excedance of π if $\overline{\pi}(s_i) = 1_l$. Analogously, $\overline{\pi}' s_i(i+1) = i$ and if we multiply $s_i\overline{\pi}'$ by s_j, $j < i$, on the left or on the right, the element in the $i+1$-th place in the window notation may be replaced with an element smaller than i. Therefore $i+1$ is a top excedance of π^{-1}. The third case is similar since $s_i\overline{\pi}' s_i(i+1) = i+1$. The last case is trivial. □

Given $\pi \in S_{n+1}$, we define the following sets.

$$\begin{aligned}\mathrm{Exc}(\pi) =& \{i \in [n] | i+1 \text{ is a top excedance for } \pi\};\\ \mathrm{Fix}(\pi) =& \{i \in [n] | \pi([i]) = [i]\};\\ \mathrm{NFix}(\pi) =& \{i \in [n] \setminus \mathrm{Fix}(\pi) | \pi(i+1) = i+1\}.\end{aligned}$$

Then by Theorem 4.3.4 and Lemma 4.4.1 we have

Corollary 4.4.2. *Let $J \subseteq \{s_1, \dots, s_n\}$ and $\pi, \rho \in S^J_{n+1}$ be two boolean permutations of $[n+1]$ such that $\pi \leq \rho$ in the Bruhat order. Then the parabolic Kazhdan–Lusztig polynomial $P^J_{\pi,\rho}$ is zero if and only if there exists an index $i \leq n$ such that one of the following condition is satisfied (we identify each $s_i \in J$ with the index i)*

- $i \in \mathrm{Exc}(\rho) \cap \mathrm{Fix}(\pi)$ *and* $i+1 \in J \cap \mathrm{NFix}(\rho)$*;*
- $i, i+1 \in \mathrm{Exc}(\rho) \cap \mathrm{Fix}(\pi)$ *and* $i+1 \in J$*;*
- $i \in \mathrm{Exc}(\rho^{-1}) \cap J$*,* $i, i+1 \in \mathrm{Fix}(\pi)$*, and* $i-1 \notin \mathrm{Exc}(\pi) \cap \mathrm{Exc}(\rho)$*;*
- $i, i+1 \in \mathrm{NFix}(\rho) \cap \mathrm{Exc}(\pi^{-1})$ *or* $i, i+1 \in \mathrm{NFix}(\rho) \cap \mathrm{Exc}(\pi)$ *and* $i+1 \in J$*;*
- $i, i+1 \in \mathrm{NFix}(\rho)$*,* $|\{i, i+1\} \cap \mathrm{Exc}(\pi^{-1})| = 1$*,* $|\{i, i+1\} \cap \mathrm{Fix}(\pi)| = 1$ *and* $i+1 \in J$*.*

In all other cases, let

$$A_{\pi,\rho} = \{i \in [n] | i, i+1 \in \mathrm{NFix}(\rho), i+1 \in \mathrm{Fix}(\pi)\}. \tag{4.4.10}$$

Then

$$P^J_{\pi,\rho} = q^{|A_{\pi,\rho} \cap J|}(1+q)^{|A_{\pi,\rho} \cap (S \setminus J)|}$$

For example, let $\pi, \rho \in S_{10}$ defined by $\pi = [2, 1, 3, 6, 4, 7, 5, 8, 9, 10]$ and $\rho = [4, 2, 3, 10, 5, 6, 7, 8, 1, 9]$. By Lemma 4.4.1 we have that π, ρ are boolean elements. Since the descents of $\pi^{-1} = [2, 1, 3, 5, 7, 4, 6, 8, 9, 10]$ are $1, 5$ and the descents of $\rho^{-1} = [9, 2, 3, 1, 5, 6, 7, 8, 10, 4]$ are $1, 3, 9$, then π, ρ are both in S_{10}^J for all J such that $J \cap \{s_1, s_3, s_5, s_9\} = \emptyset$. By [BB05, Theorem 2.1.5] we get $\pi \leq \rho$ and finally by Corollary 4.4.2 we have $P_{\pi,\rho}^J = 0$, if and only if $J \cap \{s_4, s_6, s_8\} \neq \emptyset$. In fact $\mathrm{Exc}(\pi) = \{1, 5\}$, $\mathrm{Exc}(\pi^{-1}) = \{1, 4, 6\}$, $\mathrm{Exc}(\rho) = \{3, 9\}$, $\mathrm{Exc}(\rho^{-1}) = \{8, 9\}$, $\mathrm{Fix}(\pi) = \{2, 3, 7, 8, 9\}$, $\mathrm{Fix}(\rho) = \emptyset$, $\mathrm{NFix}(\pi) = \emptyset$ and $\mathrm{NFix}(\rho) = \{2, 3, 5, 6, 7, 8\}$. For $J = \{s_2, s_4\} \equiv \{2, 4\}$ we have $P_{\pi,\rho}^J = q(q + 1)$.

For the ordinary Kazhdan–Lusztig polynomials Corollary 4.4.2 becomes

Corollary 4.4.3. *Let $\pi, \rho \in S_{n+1}$ be two boolean permutations of $[n + 1]$ such that $\pi \leq \rho$ in the Bruhat order. Then $P_{\pi,\rho} = (1 + q)^{|A_{\pi,\rho}|}$, where $A_{\pi,\rho}$ is defined in (4.4.10).*

Now we consider the Coxeter group B_n and its combinatorial representation S_n^B. It is easy to check that there are two boolean reflections in S_n^B which are maximal in the Bruhat order: they are $s_0 s_1 \cdots s_{n-1} \cdots s_1 s_0$ and $s_{n-1} \cdots s_1 s_0 s_1 \cdots s_{n-1}$ (recall that s_0 is the transposition $(1, -1)$ and s_i is the product $(i, i + 1)(-i, -i + 1)$ in disjoint cycle notation, see Table 1.1). In fact, given any boolean word t, if there is a letter s_1 between two occurrences of s_0 then move both elements s_0 to the beginning and to the end of t (it is possible since s_0 commutes with all other elements) and then manipulate the remaining letters as a subword in S_n; if s_0 is between two occurrences of s_1 (and therefore there is only one occurrence of s_0) then necessarily there is no occurrence of s_{i+1} between the two letters s_i for all $i \geq 1$, otherwise t is not a reduced word.

Lemma 4.4.4. *Let $t_1, t_2 \in S_n^B$, $t_1 = s_0 \cdots s_{n-1} \cdots s_0$, $t_2 = s_{n-1} \cdots s_0 \cdots s_{n-1}$. Let $\pi \in S_n^B$. Then π is a boolean element $\pi \leq t_1$ if and only if $||\pi([i])| \cap [i]| \geq i - 1$ for all $i \leq n$ and the only negative elements in the window notation of π may be the first entry or the element -1.*

Moreover, in this case, if $\overline{\pi}$ is the reduced word of π, which is a subword of t_1, then $\overline{\pi}(s_i) = 1_l$ if $i + 1$ is a top excedence of π (if $i = 0$ then the window notation of π has only one negative entry which is -1); $\overline{\pi}(s_i) = 1_r$ if $i + 1$ is a top excedance of π^{-1} (if $i = 0$ then the window notation of π has only one negative entry in the first place); $\overline{\pi}(s_i) = 2$ if and only if $\pi(i + 1) = \pi(i + 1)$ and $\pi([i + 1, n]) \neq [i + 1, n]$ (if $i = 0$ then there are exactly two negative entries in the window notation of π); $\overline{\pi}(s_i) = 0$ if and only if $\pi([i + 1, n]) = [i + 1, n]$ (if $i = 0$ then there is no negative element in the window notation of π).

The permutation π is a boolean element $\pi \leq t_2$ if and only if $||\pi([i])| \cap [i]| \geq i - 1$ and the only negative entry in the window notation of π (if it exists) is the element $-m - 1$ or the element in the $m + 1$-th entry, if $\pi(i) = i$ for all $i \leq m$ and $\pi(m + 1) \neq m + 1$.

Moreover, in this case, if $\overline{\pi}$ is a reduced word of π, which is a subword of

t_2 then for all $i \geq 1$, $\overline{\pi}(s_i) = 1_l$ if i is a bottom excedence of π^{-1}; $\overline{\pi}(s_i) = 1_r$ if $i+1$ is a bottom excedance of π; $\overline{\pi}(s_i) = 2$ if and only if $\pi(i) = \pi(i)$ and $\pi([i+1,n]) \neq [i+1,n]$; $\overline{\pi}(s_i) = 0$ if and only if $\pi([i+1,n]) = [i+1,n]$.

The proof is essentially the same of that of Lemma 4.4.1. We give the Corollary of Theorem 4.3.4 only for ordinary Kazhdan–Lusztig polynomials. The parabolic case could be done as in Corollary 4.4.2.

Let $\pi \in S_n^B$. We set

$$\mathrm{Fix}(\pi) = \{i \in [0,n-1] | \pi([i+1,n]) = [i+1,n]\}$$
$$\mathrm{NFix}(\pi) = \{i \in [n-1] \setminus \mathrm{Fix}(\pi) | \pi(i) = i\} \cup \{0 \text{ if } |\{\pi(i) < 0\}| = 2\}$$

Corollary 4.4.5. *Let $\pi, \rho \in S_n^B$ two boolean elements such that $\pi \leq \rho$ in the Bruhat order. Then the Kazhdan–Lusztig polynomial $P_{\pi,\rho}$ is given by*

$$P_{\pi,\rho} = \begin{cases} (1+q)^{B_{\pi,\rho}} & \text{if } \pi \leq \rho \leq t_1 \\ (1+q)^{B'_{\pi,\rho}} & \text{if } \pi \leq \rho \leq t_2 \end{cases}$$

where $B_{\pi,\rho} = \{i \in [0,n-1] | i, i+1 \in \mathrm{NFix}(\rho), i+1 \in \mathrm{Fix}(\pi)\}$, $B'_{\pi,\rho} = \{i \in [0,n-1] | i, i+1 \in \mathrm{NFix}(\rho), i \in \mathrm{Fix}(\pi)\}$.

Now we consider the Coxeter group D_n and its combinatorial representation S_n^D. It is easy to check that the unique boolean reflection of length $2n-1$ is $s_0 s_1 s_2 \cdots s_{n-1} s_{n-2} \cdots s_2 s_1 s_0$: in fact, let t any boolean reflection with the same length. Then any reduced word of t contain both occurrences of s_0 or s_1 outside the occurrences (maybe only one) of s_2. Then move, by commutativity, these occurrences to the leftmost and rightmost place. The central part can be identified with an element of S_n and we can conclude easily.

Lemma 4.4.6. *Let $\pi \in S_n^D$. Then π is a boolean element if and only if $|\{|\pi([i])|\} \cap [i]| \geq i-1$ for all $i \leq n$ and the only negative elements in the window notation are in the first two columns and in the entries containing $-1, -2$ (if the first two entries are not $\pm 1, \pm 2$ then these have the same sign).*

Let $\overline{\pi}$ be a reduced word of π, subword of $s_0 s_1 \cdots s_{n-1} \cdots s_1 s_0$, and let $i \geq 2$. Then $\overline{\pi}(s_i) = 1_l$ if $i+1$ is a top excedance of π; $\overline{\pi}(s_i) = 1_r$ if $i+1$ is a top excedance of π^{-1}; $\overline{\pi}(s_i) = 2$ if $\pi(i+1) = i+1$ and $\pi([i+1,n]) \neq [i+1,n]$; $\overline{\pi}(s_i) = 0$ if $\pi([i+1,n]) = [i+1,n]$. If $i \leq 1$ then there is an occurrence of s_1 on the right if $\pi(1) \geq 3$ or $\pi(2) \leq -3$; there is an occurrence of s_1 on the left if $\pi^{-1}(1) \geq 3$ or $\pi^{-1}(2) \leq -3$ or $\pi(1,2) \in \{(2,1),(-1,-2)\}$; there is an occurrence of s_0 on the right if $\pi(1) \leq -3$ or $\pi(2) \leq -3$; there is an occurrence of s_0 on the left if $\pi^{-1}(1) \leq -3$ or $\pi^{-1}(2) \leq -3$ or $\pi(1,2) \in \{(-2,-1),(-1,-2)\}$.

Corollary 4.4.7. *Let $\pi, \rho \in S_n^D$ be two boolean elements such that $\pi \leq \rho$. Then the Kazhdan–Lusztig polynomial $P_{\pi,\rho}$ is given by*

$$P_{\pi,\rho}(q) = (1+q)^{D_{\pi,\rho}}(1+2q)^{D'_{\pi,\rho}}$$

where $D_{\pi,\rho}$ is the number of indices i such that $\rho(i) = i$, $\rho(i+1) = i+1$, $\rho([i+2,n]) \neq [i+2,n]$ and $\pi([i+2,n]) = [i+2,n]$ incremented by 1 if $\rho(1) = 1, \rho(2) < 2, \rho(3) \neq 3$ and $\pi((1,2)) \neq (-1,-2)$ or $\rho^{-1}(2) < -2, |\rho^{-1}(1)| = 2$ and $\pi([3,n]) = [3,n]$ or $|\rho^{-1}(1)| > 2, \rho(2) \in \{-n,\dots,-3,-1,1\}$ and $\pi([3,n]) = [3,n]$; $D'_{\pi,\rho}$ is 1 if $\rho(1) = 1$, $\rho(2) < 2$, $\rho(3) = 3$ and $\pi([3,n]) = [3,n]$ and $D'_{\pi,\rho} = 0$ in all other cases.

4.5 Poincaré polynomials

Given $v \in W$, let $F_v(q) = \sum_{u \leq v} q^{l(v)} P_{u,v}$. It is well known that, if W is any finite Coxeter or affine Weyl group, $F_v(q)$ is the intersection homology Poincaré polynomial of the Schubert variety indexed by v (see [KL80]). In this section we compute the Poincaré polynomial for any boolean element in a Coxeter group whose Coxeter graph is a tree with at most one vertex having more than two adjacent vertices (such groups include all classical finite Coxeter and affine Weyl groups except $\widetilde{A}_n$ and $\widetilde{D}_n$).

Let $v \in W$ be a boolean element and consider the diagram of $(\epsilon_W, \overline{v})$. For convenience we will not depict the second row of each column which is always 0 and we omit all symbols $\times$. We will call it the diagram of v.

Let v be a boolean element and let s be the element of S associated to one of the leftmost vertices in the diagram of v. We set $F^{\setminus}_{v,s} = \sum q^{l(v)} P_{u,v}$ where the sum runs over all elements $u \leq v$ such that $\overline{u}(s) \neq 0$ and $F^0_{v,s} = \sum q^{l(v)} P_{u,v}$ where the sum runs over all elements $u \leq v$ such that $\overline{u}(s) = 0$.

Now consider a diagram d. Delete all entries equal to 0 and delete all edges whose left vertex is not a cell containing 2. Let $d_1, \dots, d_k$ be the remaining connected components. We refer to them as the *essential components* of d.

Lemma 4.5.1. *Let $v \in W$ be a boolean element and let d be the diagram of $\overline{v}$. Let $d_1, \dots, d_k$ be the essential components of the diagram d and $v_1, \dots, v_k$ be the boolean reflections corresponding to $d_1, \dots, d_k$. Then*

$$F_v(q) = \prod_{i=1}^{k} F_{v_i}(q).$$

Proof. We use induction on $l(v)$. If $l(v) = 1$ there is nothing to prove. Now let $l(v) > 1$ and let $d_1, \dots, d_k$ be the essential components of d associated to v. Let

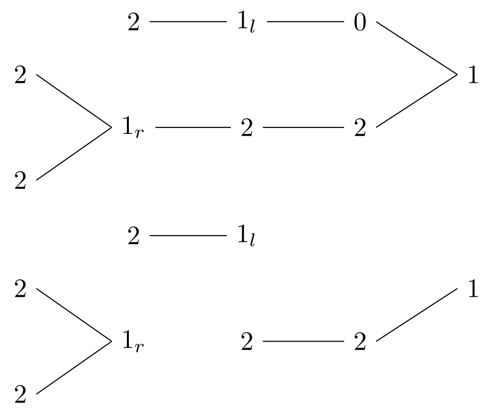


Figure 4.4: A diagram and its essential components.

s be the element associated to one of the leftmost vertex of v and let d_1 be the essential component containing such vertex. In this proof we denote by $\overline{F}_v(q)$ the polynomial corresponding to the diagram $d \setminus d_1$ (known by induction) and by $\widehat{F}_v(q)$ the polynomial corresponding to d_1. If $\overline{v}(s) = 1$ then by Theorem 4.3.4 or by Lemmas 4.3.2 and 4.3.1 and recursion in Proposition 4.2.1 we have that

$$F_v(q) = (1+q)\overline{F}_v(q), \quad F^{\setminus}_{v,s}(q) = q\overline{F}_v(q), \quad F^{0}_{v,s}(q) = \overline{F}_v(q).$$

If $\overline{v}(s) = 2$ then we can assume that d_1 starts with

$$(2)^h \text{ ------ } * \, ,$$

for $h \geq 1$ (otherwise choose another element in S). The previous diagram is depicted according to the same conventions used for example in (4.3.6).

Denote by s' the only element on the right of s. By Theorem 4.3.4 and by induction we have

$$\begin{aligned} F_v(q) =& (1+q)^{2h} {F'}^{\setminus}_{v,s'}(q) + (1+q)^h f_{h-\delta} {F'}^{0}_{v,s'}(q) \\ =& (1+q)^{2h} \widehat{F'}^{\setminus}_{v,s'}(q)\overline{F}_v(q) + (1+q)^h f_{h-\delta} \widehat{F'}^{0}_{v,s'}(q)\overline{F}_v(q) \\ =& \widehat{F'}_v(q)\overline{F}_v(q), \end{aligned}$$

where F'_v is the polynomial associated to d after deleting all the vertices $(2)^h$, and δ is determined uniquely by v (and Theorem 4.3.4). The first factor $(1+q)^{2h}$ denotes the possibility to have all pairs $(2,0)$, $(2,1_l)$, $(2,1_r)$ and $(2,2)$ in the diagram of $(\overline{u}, \overline{v})$ in all h leftmost columns; the second factor $(1+q)^h$ denotes the possibility to have only the pairs $(2,0)$ and $(2,1_l)$. Similar formulas can be computed for $F^{\setminus}_{v,s}(q)$ and $F^{0}_{v,s}$. Therefore we can apply the induction (it is possible that more superscripts \ or 0 are necessary; the proof does not change). □

Lemma 4.5.1 shows that it is simple to compute $F_v(q)$ for any boolean elements v by knowing $F_t(q)$ for all boolean reflection t.

Lemma 4.5.2. *Let $v \in W$ be a boolean reflection and suppose that its diagram d has one leftmost vertex s such that if $(2)^h$ —— $*$ is a subdiagram of d containing s, then necessarily $h = 1$. Then*

$$F_v(q) = (1+q)^2 F'_v(q)$$

where $F'_v(q)$ is the polynomial associated to diagram d after deleting the vertex s.

Proof. By Proposition 4.2.1, it is easy to check that

$$F_v(q) = (1+q)^2 {F'}^{\backslash}_{v,s} + (1+q){F'}^{0}_{v,s}(q)(1+q) = (1+q)^2 F'_v(q)$$

where the last factor $(1+q)$ is due to the contribution of $\begin{array}{|c|c|}\hline 2 & 2 \\ \hline * & 0 \\ \hline \end{array}$ in the Kazhdan–Lusztig polynomial $P_{u,v}(q)$ according to Theorem 4.3.4. □

As corollary of Lemmas 4.5.1 and 4.5.2 we have the following result due to Marietti [Mar06, Theorem 8.1]

Corollary 4.5.3. *Let $v \in S_{n+1}$ be a boolean element. Let t be the boolean reflection $s_1 \cdots s_n \cdots s_1$ with s_i be the transposition $(i, i+1)$. Let $\overline{v}$ be the reduced word of v subword of t. Then*

$$F_v(q) = (1+q)^{l(v)-2a(v)}(1+q+q^2)^{a(v)},$$

where $a(v)$ is the number of patterns $(2, 1_*)$ in $\overline{v}$.

By Lemmas 4.5.1 and 4.5.2 its proof reduces to compute $F_{s_1 s_2 s_1}(q) = (q^2 + q + 1)(1+q)$ and $F_{s_1} = (1+q)$ in S_3.

To prove the next result we have to compute the polynomials $F^{\backslash}_{v,s}(q)$ and $F^{0}_{v,s}(q)$ with v associated to the diagram 2 —— 2 —— $\ldots$ —— 1 with i vertices. Let $s \in S$ be the element corresponding to the first vertex and let $s' \in S$ be the element associated to the second vertex. If $i = 2$ then by direct computation we have

$$F^{\backslash}_{v,s}(q) = q(1+q)^2 \quad F^{0}_{v,s}(q) = (1+q).$$

By induction it is easy to compute that

$$\begin{aligned} F^{\backslash}_{v,s} =&(2q+q^2){F'}^{\backslash}_{v,s'}(q) + q{F'}^{0}_{v,s'}(q)(1+q) = q(1+q)^{2i-2} \\ F^{0}_{v,s} =&{F'}^{\backslash}_{v,s'}(q) + {F'}^{0}_{v,s'}(q)(1+q) = (1+q)^{2i-3}, \end{aligned} \qquad (4.5.11)$$

where $F'_v(q)$ denotes, as usual, the polynomial associated to the diagram without the first vertex. Similarly, let v be the boolean reflection corresponding to the diagram

with $i+1$ vertices. Then

$$F^{\backslash}_{v,s} = q(1+q)^{2i} \text{ and } F^{0}_{v,s} = (1+q)^{2i-1}. \qquad (4.5.12)$$

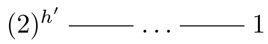
Proposition 4.5.4. *Let W be a Coxeter group such that its Coxeter graph is a tree and all vertices except at most one have degree less than* 3*. Denote with w such exceptional vertex. Let $v \in W$ be a boolean element. Then*

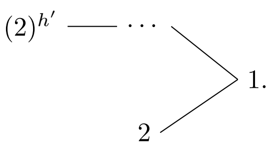
$$F_v(q) = (1+q+q^2)^{k-1}\big(q(1+q)^{h+1} + f_h(q)\big)(1+q)^{l(v)-2k-h-2},$$

where k is the number of essential components of the diagram d of v with at least two vertices and h is the number of entries equal to 2 *in the adjacent cells of w (also consider the cell on the right).*

The formula is also true when there is no vertex of degree greater than 2: in this case let w be any vertex of degree 2.

Proof. By Lemma 4.5.1 it suffices to compute the polynomial associated to the only non-trivial component. By Lemma 4.5.2 it suffices to consider only the following two cases.





In the first case we compute

$$\begin{aligned} F_v(q) =&(1+q)^{2h'}{F'}^{\backslash}_{v,s'}(q) + (1+q)^{h'} f_h {F'}^{0}_{v,s'}(q) \\ =&(1+q)^{h'+2i-3}\big(q(1+q)^{h+1} + f_h\big) \qquad \text{by (4.5.11)}, \end{aligned}$$

where $F'_v(q)$ is the polynomial associate to the diagram without the h' leftmost cells and i is an integer. The second case is similar; use (4.5.12). □

Chapter 5

Kazhdan–Lusztig polynomials of quasi-minuscule parabolic quotients

The purpose of this chapter is to study the Kazhdan–Lusztig polynomials for quasi-minuscule parabolic quotients. Given an affine Weyl system (W, S) let $s \in S$ and let $\overline{W}$ be the Coxeter group generated by $S \setminus \{s\}$; let $s' \in S$ be such that $(ss')^2 \neq 1$ and $J = S \setminus \{s, s'\}$. Then $\overline{W}^J$ is the *quasi-minuscule* parabolic quotient of $(\overline{W}, S\setminus\{s\})$. In this chapter, we will consider parabolic quotients W^J with (W, S) a Coxeter system of type B_n or D_n and J the maximal parabolic subgroup $s_1^W, \ldots, s_n^W \setminus \{s_{n-2}^W\}$ (W is B or D respectively). Kazhdan–Lusztig polynomials in all other cases of quasi-minuscule parabolic quotients of Coxeter systems of type A_n, B_n, D_n have already been computed in the literature (see e. g. [Bre02b],[Bre09]).

5.1 Properties of quasi-minuscule quotients of type B and D

Let $J_i := \{s_0, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{n-1}\}$, for $i \in \{0, \ldots, n-1\}$. Then it is clear from Proposition 2.1.4 that $v \in (S_n^B)^{J_i}$ if and only if

$$v^{-1}(0) < v^{-1}(1) < \cdots < v^{-1}(i); v^{-1}(i+1) < \cdots < v^{-1}(n) \tag{5.1.1}$$

where $v^{-1}(0) := 0$.

For any $i \in [0, n]$, we denote by L_i the set of all increasing sequences of length $n-i$ in $\{-n, \ldots, -1, 1, \ldots, n\}$ such that for any $j \in [n]$, $\{-j, j\}$ is not a subsequence. Given $\lambda \in L_i$ we denote with $|\lambda|$ the set with the absolute values of all elements in λ. Therefore $|\lambda|$ has $n-i$ distinct elements.

We define a map $\Lambda_B^i : (S_n^B)^{J_i} \to L_i$ such that for any $v \in (S_n^B)^{J_i}$

$$\Lambda_B^i(v) := (v^{-1}(i+1), \ldots, v^{-1}(n)). \tag{5.1.2}$$

In the rest of the chapter we sometimes will write s_j instead of s_j^B or s_j^D for any $j \in [0, n-1]$ since there is no risk of confusion.

Proposition 5.1.1. *The map $\Lambda_B^i : (S_n^B)^{J_i} \to (S_n^B)^{J_i}$ defined by (5.1.2) is a bijection. Furthermore, given $u, v \in (S_n^B)^{J_i}$ and $j \in \{0, \ldots, n-1\}$, the following properties are true:*

(P1) $v \cdot s_j \in (S_n^B)^{J_n}$ iff exactly one between j and $j+1$ is in $|\Lambda_B^i(v)|$ or both j and $j+1$ appear in $\Lambda_B^i(v)$ with different signs;

(P2) $s_j \in D_R(v)$ iff $-j \notin \Lambda_B^i(v)$ and $-j-1 \in \Lambda_B^i(v)$ or $j \in \Lambda_B^i(v)$ and $j+1 \notin |\Lambda_B^i(v)|$; in particular, in this case $j+1, -j \notin \Lambda_B^i(v)$;

(P3) $u \leq v$ iff $\Lambda_B^i(u) \geq \Lambda_B^i(v)$, i. e. $u^{-1}(j) \geq v^{-1}(j)$ for all $j \in [i+1, n]$.

Proof. It is obvious that the map Λ_B^i is a bijection. We now prove $(P1)$. Let $v \in (S_n^B)^{J_i}$. By (5.1.1) the product $v \cdot s_j$ is in $(S_n^B)^{J_i}$ if and only if $0 < s_j(v^{-1}(1)) < \cdots < s_j(v^{-1}(i))$ and $s_j(v^{-1}(i+1)) < \cdots < s_j(v^{-1}(n))$. Since s_j $(j > 0)$ swaps j and $j+1$ and s_0 sends -1 into 1 and vice versa, $(P1)$ follows.

Fix $v \in (S_n^B)^{J_i}$ and $j \in [0, n-1]$ such that $l(vs_j) < l(v)$. It is well known that given a Coxeter system (W, S) and $J \subseteq S$, for any $w \in W^J$ and $s \in S$ such that $ws < w$ in W then $ws \in W^J$. Therefore v, s_j satisfy conditions in $(P1)$.

We first consider the case $j = 0$. By $(P1)$ we know that $1 \in |\Lambda_B^i(v)|$. Since s_0 acts on the window notation of v^{-1} by inverting the sign of 1, we have that $\mathrm{inv}(v^{-1}) = \mathrm{inv}(s_0 v^{-1})$ and $N_2(v^{-1}) = N_2(s_0 v^{-1})$. Moreover $N_1(v^{-1}) = N_1(s_0 v^{-1}) + 1$ if and only if $-1 \in \Lambda_B^i(v)$ (see (2.1.4) for the definition of the previous statistics). Therefore $(P2)$ follows by Proposition 2.1.4. We now suppose that $j > 0$. By $(P1)$ we have three subcases.

If exactly one between j and $j+1$ is in $|\Lambda_B^i(v)|$ and its sign is negative then by condition (5.1.1) we have $\mathrm{inv}(v^{-1}) = \mathrm{inv}(s_j v^{-1})$ and $N_1(v^{-1}) = N_1(s_j v^{-1})$. Moreover $N_2(v^{-1}) = N_2(s_j v^{-1}) + 1$ if and only if $-j-1 \in \Lambda_B^i(v)$.

If exactly one between j and $j+1$ is in $|\Lambda_B^i(v)|$ and its sign is positive then $N_1(v^{-1}) = N_1(s_j v^{-1})$ and $N_2(v^{-1}) = N_2(s_j v^{-1})$. Moreover $\mathrm{inv}(v^{-1}) = \mathrm{inv}(s_j v^{-1}) + 1$ if and only if $j \in \Lambda_B^i(v)$.

If both j and $j+1$ are in $\Lambda_B^i(v)$ with different signs then $\mathrm{inv}(v^{-1}) = \mathrm{inv}(s_j v^{-1})$ and $N_1(v^{-1}) = N_1(s_j v^{-1})$. Moreover $N_2(v^{-1}) = N_2(s_j v^{-1}) + 1$ if and only if $-j-1 \in \Lambda_B^i(v)$.

In all cases $(P2)$ follows by Proposition 2.1.4.

We now prove $(P3)$. We fix $u, v \in (S_n^B)^{J_i}$ with $u \leq v$ and $l(v) = l(u)+1$. We prove that $\Lambda_B^i(u) \geq \Lambda_B^i(v)$ by induction on $l(v)$. If $l(v) = 1$ then it is obvious since $v = s_i$, $u = e$, $\Lambda_B^i(u) = (i+1, i+2, \ldots, n)$ and $\Lambda_B^i(v) = (i, i+2, \ldots, n)$ if $i > 0$ or $\Lambda_B^i(v) = (-1, 2, \ldots, n)$ if $i = 0$. We now suppose that $l(v) > 1$. Let s_j be such that $vs_j < v$. By Theorem 1.1.5 we have two possibilities: $v = us_j$ or $us_j < u$ and $us_j < vs_j$. In the following we discuss only the case $j > 0$; the case $j = 0$ is similar. If $v = us_j$ then by $(P2)$ $-j, j+1 \notin \Lambda_B^i(v)$. Therefore we get $\Lambda_B^i(vs_j)$ from $\Lambda_B^i(v)$ by replacing $-j-1$ with $-j$ and j with $j+1$. If $us_j < u$ then by induction $\Lambda_B^i(us_j) \geq \Lambda_B^i(vs_j)$; by replacing $j+1$ with j and $-j$ with $-j-1$ we get $\Lambda_B^i(u)$ and $\Lambda_B^i(v)$ by $(P2)$, and the inequality holds. By Theorem 1.1.7 the assertion is true for all $u \leq v$.

We now prove that if $\Lambda_B^i(u) \geq \Lambda_B^i(v)$ then $u \leq v$. Let $u, v \in (S_n^B)^{J_i}$ such that $\Lambda_B^i(u) \geq \Lambda_B^i(v)$. If $l(v) = 0$ then there is nothing to prove. Now suppose that $l(v) > 0$. Let $j \in [0, n-1]$ be the biggest index such that $s_j \in D_R(v)$. If $s_j \in D_R(u)$ then $\Lambda_B^i(us_j) \geq \Lambda_B^i(vs_j)$ (again we use (P2) and replace $-j-1$ and j with $-j$ and $j+1$) and by induction $us_j \leq vs_j$; therefore $u \leq v$. If $s_j \notin D_R(u)$ then we prove that $\Lambda_B^i(u) \geq \Lambda_B^i(vs_j)$ and therefore we conclude by induction that $u \leq vs_j \leq v$. Since we obtain $\Lambda_B^i(vs_j)$ from $\Lambda_B^i(v)$ by replacing $-j-1$ with $-j$ and j with $j+1$ then our claim could be false only if $-j-1 \in \Lambda_B^i(u)$ and $-j-1 \in \Lambda_B^i(v)$ are in the same position or if $j \in \Lambda_B^i(u)$ and $j \in \Lambda_B^i(v)$ are in the same position. Since we know by assumption that $s_j \notin D_R(u)$ then by (P2) we have only two cases to investigate: $-j-1, -j \in \Lambda_B^i(u)$ and $j, j+1 \in \Lambda_B^i(u)$. The condition $\Lambda_B^i(u) \geq \Lambda_B^i(v)$ implies that $-j-1, -j \in \Lambda_B^i(v)$ or $j, j+1 \in \Lambda_B^i(v)$ (since at least one of the positions of $-j-1$ and j is fixed). But then $s_j \notin D_R(v)$, so we get a contradiction. Therefore $\Lambda(u) \geq \Lambda(vs_j)$ and the proof is completed. □

Since Λ_B^i is a bijection for all $i \in \{0, \ldots, n-1\}$ it follows that the cardinality of $(S_n^B)^{J_i}$ is $\binom{n}{i} 2^{n-i}$.

Now, for the sake of completeness, we give a formula that allows us to compute the length of an element $v \in (S_n^B)^{J_i}$ via $\Lambda_B^i(v)$.

Proposition 5.1.2. *Let $i \in \{0, \ldots, n-1\}$ and $v \in (S_n^B)^{J_i}$. Let we order all* *elements of* $\Lambda^i_B(v)$ *as* $\mu = (\mu_1, \ldots, \mu_k)$ *such that* $|\mu_1| < |\mu_2| < \cdots < |\mu_k|$. *Then*

$$l(v) = \frac{n(n+1) - i(i+1)}{2} - \sum_{j=1}^{k} (\mu_j + j\epsilon_j) \tag{5.1.3}$$

where $\epsilon_i = 1$ *if* $\mu_i < 0$ *and* $\epsilon_i = 0$ *otherwise.*

Proof. We prove the statement by induction on $l(v)$. If $l(v) = 0$ then $\Lambda^i_B(v) = (i+1, \ldots, n)$ and the assertion follows. Let now we suppose that $l(v) > 0$. Then there exists $s_j \in D_R(v)$. If $j = 0$ then $\Lambda^i_B(v)$ is obtained from $\Lambda^i_B(vs_0)$ by replacing 1 with -1. Therefore, the RHS of (5.1.3) decreases of one unit. If $j > 0$ then $\Lambda^i_B(v)$ is obtained from $\Lambda^i_B(vs_j)$ by replacing $j+1$ and $-j$ with j and $-j-1$ (when they are both in $\Lambda^i_B(vs_j)$ we increase the index of corresponding $\epsilon_h \neq 0$). In all cases the RHS of (5.1.3) increases of one unit. By induction the thesis follows. □

We now consider the group S^D_n.

We use again the symbol J_i to denote $\{s^D_0, \cdots s^D_{n-1}\} \setminus \{s^D_i\}$, for $i \in [0, n-1]$, since there is no risk of confusion. It is clear from Proposition 2.1.5 that $v \in (S^D_n)^{J_i}$ if and only if

$$v^{-1}(0) < v^{-1}(1) < \cdots < v^{-1}(i); \quad v^{-1}(i+1) < \cdots v^{-1}(n) \tag{5.1.4}$$

where $v^{-1}(0) := -v^{-1}(2)$. Given $v \in (S^D_n)^{J_i}$ we associate the increasing sequence

$$\Lambda^i_D(v) := (v^{-1}(i+1), \ldots, v^{-1}(n)), \tag{5.1.5}$$

in L_i. Since the sign of $v^{-1}(1)$ is uniquely determined by all signs in $\Lambda^i_D(v)$, then it is obvious that $\Lambda^i_D(v)$ is a bijection. With the same techniques used in Proposition 5.1.1 it is possible to prove the following result.

Proposition 5.1.3. *The map* Λ^i_D *defined in (5.1.5) is a bijection between* $(S^D_n)^{J_i}$ *and* L_i. *Furthermore, given* $u, v \in (S^D_n)^{J_i}$ *and* $j \in \{1, \ldots, n-1\}$ *the following properties are true:*

(PD1) $v \cdot s_j \in (S^D_n)^{J_n}$ *iff exactly one between* j *and* $j+1$ *is in* $|\Lambda^i_D(v)|$ *or both* j *and* $j+1$ *are in* $\Lambda^i_D(v)$ *with different signs;* $v \cdot s_0 \in (S^D_n)^{J_n}$ *iff exactly one between* 1 *and* 2 *is in* $|\Lambda^i_D(v)|$ *or both* 1 *and* 2 *are in* $\Lambda^i_D(v)$ *with the same sign;*

(PD2) $s_j \in D_R(v)$ *iff* $-j \notin \Lambda^i_D(v)$ *and* $-j-1 \in \Lambda^i_D(v)$ *or* $j \in \Lambda^i_D(v)$ *and* $j+1 \notin |\Lambda^i_D(v)|$; *in particular, in this case* $j+1, -j \notin \Lambda^i_D(v)$; $s_0 \in D_R(v)$ *iff* $2 \notin \Lambda^i_D(v)$ *and at least one between* $-1, -2$ *is in* $\Lambda^i_D(v)$;

(PD3) if $u \leq v$ then $\Lambda^i_D(u) \geq \Lambda^i_D(v)$.

The most significant differences between the proof of the previous proposition and that of Proposition 5.1.1 are computations with index 0. Note that property $(PD3)$ is not an equivalence. In fact, let $u = [2, 4, 1, 3]$, $v = [-2, 4, -1, 3]$ in $(S^4_D)^{J_2}$. By Proposition 2.1.5 it is easy to verify that $l(u) = l(v) = 3$. Therefore u, v are incomparable but $\Lambda^i_D(v) = [-1, 3] < [1, 3] = \Lambda^i_D(u)$.

Since Λ^i_D is a bijection, we have that the cardinality of $(S^D_n)^{J_i}$ is $\binom{n}{i}2^{n-i}$.

5.2 Hasse diagrams

In this section we study $(S^B_n)^{J_{n-2}}$ for all $n \geq 2$ and $(S^D_n)^{J_{n-2}}$ for all $n \geq 4$ and we construct their Hasse diagrams. We do not draw these diagrams in traditional ways (i. e. lower elements on the bottom and higher elements on the top) but we rotate them clockwise, in such a way that we have something like a table.

Hasse diagram of $(S^B_n)^{J_{n-2}}$ is given in Figure 5.1, with the minimun on the bottom and the maximum on the right. We label each row of the diagram from 1 to $2n - 1$ from the top to the bottom and analogously we label each column of diagram from 1 to $2n - 1$ from the left to the right. Therefore, each vertex is identified by its coordinates with respect to these labels. We define the main diagonal of the diagram in Figure 5.1 the line with row and column coordinates equals (note that the main diagonal has no vertex).

Given an edge e, we denote with $l(e)$ the label of e (if it exists). Let $u, v \in (S^B_n)^{J_{n-2}}$ be the vertices of e, $u \leq v$, if e has a label then $v = us_{l(e)}$ (i. e. $u \leq v$ in the weak order). Note that any edge crossing the main diagonal has no label, because there is no such $s_{l(e)}$ (in the literature it is said that the elements corresponding to its vertices are not comparable in the weak order, thought they are comparable in the Bruhat order).

Proposition 5.2.1. *Diagram in Figure 5.1 is the Hasse diagram of $(S^B_n)^{J_{n-2}}$.*

Proof. For any vertex of the diagram with coordinates (i, j) in Figure 5.1 associate the set $\{i - n - \alpha_i, -j + n + \alpha_j\}$ where α_i (resp. α_j) is 0 if $i > n$ (resp. $j > n$) and 1 otherwise. We first prove that the absolute values of the elements in any such set are distinct.

Fix $i, j \in \{1, \ldots, 2n - 1\}$ such that there is a vertex in the i-th row and j-th column. Then $i + j \leq 2n$. Suppose that $i - n - \alpha_i = -j + n + \alpha_j$. If $i + j = 2n$ then $i \leq n - 1$ or $j \leq n - 1$, since there is no vertex in the main diagonal. It forces that $\alpha_i + \alpha_j = 1$ and therefore we have that $i + j < 2n + \alpha_i + \alpha_j$.

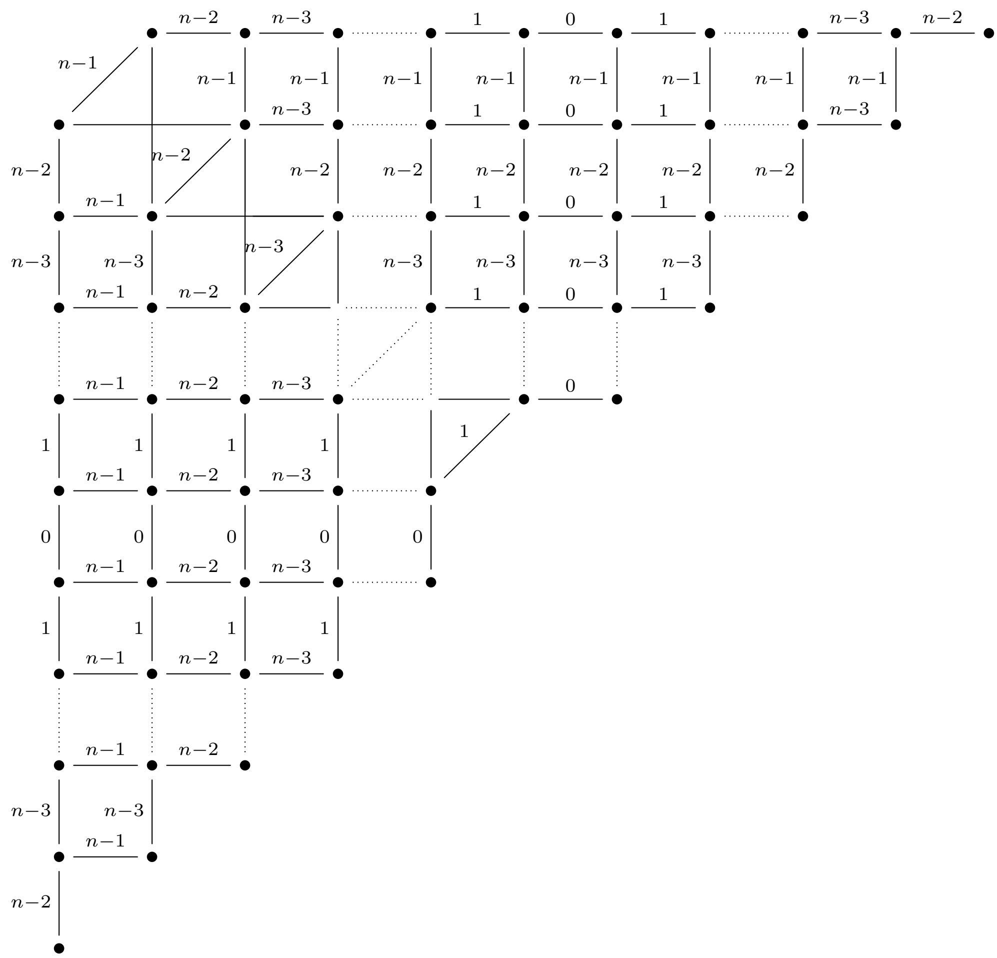


Figure 5.1: Rotate Hasse diagram of $(S_n^B)^{J_{n-2}}$.

It is a contradiction and then $i - n - \alpha_i \neq -j + n + \alpha_j$. Now suppose that $i - n - \alpha_i = j - n - \alpha_j$. Suppose for example that $i \leq j$, so $\alpha_i \geq \alpha_j$. Then $0 \leq \alpha_i - \alpha_j = i - j \leq 0$. Therefore $i = j$ but this is impossible since there is no vertex on the main diagonal. It follows that the absolute values of the elements in any set are distinct.

Now we claim that the sets associated to different vertices are different. Fix two vertices of the diagram with coordinates $(i, j) \neq (i', j') \in \{1, \ldots, 2n-1\}$. Suppose that the associated sets are equals. Then necessarily $i - n - \alpha_i = -j' + n + \alpha_{j'}$ and $i' - n - \alpha_{i'} = -j + n + \alpha_j$. But $i + j \leq 2n$ and therefore $i + j + i' + j' \leq 4n < 4n + 2 = n + \alpha_i + \alpha_j + \alpha_{i'} + \alpha_{j'}$. Then the assumption is impossible and all the sets are pairwise distinct.

It is a simple exercise to verify that the elements of the diagram are exactly $\binom{n}{2}2^2$ and therefore we have a bijection between the vertex set and L_{n-2}. The map Λ_B^{n-2} defines a bijection between L_{n-2} and $(S_n^B)^{J_{n-2}}$. Property $(P3)$ of Proposition 5.1.1 shows that the edges of the diagram give the correct Bruhat order and property $(P2)$ of the same proposition gives the labels of all edges.

□

Now we want to depict the Hasse diagram of $(S_n^D)^{J_{n-2}}$. In the proof of Proposition 5.2.1 we use property $(P3)$ of Proposition 5.1.1. Property $(PD3)$ in Proposition 5.1.3 is not an equivalence, therefore it is necessary to have a characterization of the Bruhat order in $(S_n^D)^{J_{n-2}}$.

Proposition 5.2.2. *With notation of Proposition 5.1.3, for all $u, v \in (S_n^D)^{J_{n-2}}$ $(n \geq 4)$ we have $u \leq v$ if and only if $\Lambda_D^{n-2}(u) \geq \Lambda_D^{n-2}(v)$ and if $1 \in \Lambda_D^{n-2}(u)$ and $-1 \in \Lambda_D^{n-2}(v)$ (or vice versa) then signs of the other elements in $\Lambda_D^{n-2}(u)$ and $\Lambda_D^{n-2}(v)$ are different, with the only exceptions the pairs associated to $\{(1, 2), (-2, 1)\}$ or $\{(-1, 2), (-2, -1)\}$.*

Proof. It will be useful for the proof to write the exceptional elements as product of generators. It is easy to check that $\Lambda_D^{n-2}(s_{n-2} \cdots s_2 s_0 s_{n-1} \cdots s_1) = (-2, 1)$; $\Lambda_D^{n-2}(s_{n-2} \cdots s_2 s_0 s_{n-1} \cdots s_2) = (-1, 2)$; $\Lambda_D^{n-2}(s_{n-2} \cdots s_1 s_0 s_{n-1} \cdots s_2) = (1, 2)$; $\Lambda_D^{n-2}(s_{n-2} \cdots s_1 s_0 s_{n-1} \cdots s_2 s_0) = (-2, -1)$. Then it is easy to establish the bruhat order between the previous elements.

Fix $u, v \in (S_n^D)^{J_{n-2}}$ with $u \leq v$ and $l(v) = l(u) + 1$. We prove that $\Lambda_D^{n-2}(u), \Lambda_D^{n-2}(v)$ satisfy the claim by induction on $l(v)$. If $l(v) = 1$ then it is trivial since $u = e$, $v = s_{n-2}$, $\Lambda_D^{n-2}(u) = (n-1, n)$ and $\Lambda_D^{n-2}(v) = (n-2, n)$. We now suppose that $l(v) > 1$. Let s_j be such that $vs_j < v$. Then by Theorem 1.1.5 we have two possibilities: $v = us_j$ or $us_j < u$ and $us_j < vs_j$. We suppose first that $v = us_j$ and $j \geq 1$. By $(PD2)$ we have $-j, j+1 \notin \Lambda_D^{n-2}(v)$. Therefore we get $\Lambda_D^{n-2}(vs_j)$ from $\Lambda_D^{n-2}(v)$ by replacing $-j-1$ with $-j$ and j with $j+1$. It follows that $\Lambda_D^{n-2}(vs_j) \geq \Lambda_D^{n-2}(v)$ and that $1, -1$ do not appear both in $\Lambda_D^{n-2}(v)$ or $\Lambda_D^{n-2}(vs_j)$ except in the case $j = 1$ and $\Lambda_D^{n-2}(v) = (-2, 1)$, $\Lambda_D^{n-2}(vs_1) = (-1, 2)$. In all cases the claim is true. We now suppose that $v = us_0$. By $(PD2)$ we have $1, 2 \notin \Lambda_D^{n-2}(v)$. We get $\Lambda_D^{n-2}(vs_j)$ from $\Lambda_D^{n-2}(v)$ by replacing -2 with 1 and -1 with 2. Therefore $\Lambda_D^{n-2}(vs_j) \geq \Lambda_D^{n-2}(v)$ and $1, -1$ do not appear both in $\Lambda_D^{n-2}(v)$ or $\Lambda_D^{n-2}(vs_0)$ except in the case $\Lambda_D^{n-2}(v) = (-2, -1)$, $\Lambda_D^{n-2}(vs_0) = (1, 2)$. In all cases the claim is true.

Now we suppose that $us_j < u$, with $j \geq 1$. Then by induction $\Lambda_D^{n-2}(us_j)$, $\Lambda_D^{n-2}(vs_j)$ satisfy the claim. If we replace $j+1$ with j and $-j$ with $-j-1$ (by $(PD2)$) we have $\Lambda_D^{n-2}(u)$ and $\Lambda_D^{n-2}(v)$ and the inequality yet holds. Moreover, if $j = 1$, since $-2 \notin \Lambda_D^{n-2}(us_1) \cup \Lambda_D^{n-2}(vs_1)$ by $(PD2)$, then $1 \notin \Lambda_D^{n-2}(u) \cup \Lambda_D^{n-2}(v)$ (if $j > 1$ it is simple to check the claim).

Finally, we suppose that $us_0 < u$. Again $\Lambda_D^{n-2}(us_0), \Lambda_D^{n-2}(vs_0)$ satisfy the claim. If we replace 1 with -2 and 2 with -1 is simple to verify the inequality and that $1 \notin \Lambda_D^{n-2}(u) \cup \Lambda_D^{n-2}(v)$ (here we use also $(PD2)$ with the assumptions $s_0 \in D_R(u) \cap D_R(v)$ and $u \leq v$).

By Theorem 1.1.7 the inequality is true for all $u \leq v$. Suppose that the second part of the assertion is false, then there exist $u \leq v$ such that $-1 \in \Lambda_D^{n-2}(v)$, $1 \in \Lambda_D^{n-2}(u)$ and the other elements have the same sign. Suppose for example that $\Lambda_D^{n-2}(u) = (a, 1)$, $\Lambda_D^{n-2}(v) = (b, -1)$, with $b < a < -1$ (note that by above discussion $\Lambda_D^{n-2}(u) \geq \Lambda_D^{n-2}(v)$). By previous proof, we know that $l(v) - l(u) \geq 2$. Choose $u, v \in (S_n^D)^{J_{n-2}}$ such that this difference is minimal. Then there exists $w \in (S_n^D)^{J_{n-2}}$, $u < w < v$. We set $\Lambda_D^{n-2}(w) = (x, y)$. By assumption of minimality, $x, y \notin \{-1, 1\}$. But from relation $u < w < v$ we have that $-1 \leq y \leq 1$ and this is impossible. Therefore the claim is true.

We now prove the vice versa. Let $u, v \in (S_n^D)^{J_{n-2}}$ such that $\Lambda_D^{n-2}(u)$, $\Lambda_D^{n-2}(v)$ satisfy the claim. If $l(v) = 0$ then there is nothing to prove. Suppose that $l(v) > 0$. Let s_j be the element in $D_R(v)$ with the greatest index j (note that by $(PD2)$ we have $j > 0$ except for $\Lambda_D^{n-2}(v) = (-2, -1)$). If $s_j \in D_R(u)$ then it is possible to check that $\Lambda_D^{n-2}(us_j), \Lambda_D^{n-2}(vs_j)$ satisfy the claim: it suffices to use the property $(PD2)$. Then by induction $us_j \leq vs_j$ and so $u \leq v$.

If $s_j \notin D_R(u)$ then we prove that $\Lambda_D^{n-2}(u), \Lambda_D^{n-2}(vs_j)$ satisfy the claim and therefore we conclude by induction that $u \leq vs_j \leq v$. We suppose first that $j \geq 1$. Since we have $\Lambda_D^{n-2}(vs_j)$ from $\Lambda_D^{n-2}(v)$ by replacing $-j-1$ with $-j$ and j with $j+1$ then the inequality of our claim could be not true only if $-j-1 \in \Lambda_D^{n-2}(u)$ and $-j-1 \in \Lambda_D^{n-2}(v)$ in the same position or if $j \in \Lambda_D^{n-2}(u)$

and $j \in \Lambda_D^{n-2}(v)$ in the same position. Since we know by assumption that $s_j \notin D_R(u)$ then by $(PD2)$ we have only two cases: $\Lambda_D^{n-2}(u) = (-j-1, -j)$ or $\Lambda_D^{n-2}(u) = (j, j+1)$. The condition $\Lambda_D^{n-2}(u) \geq \Lambda_D^{n-2}(v)$ implies that $\Lambda_D^{n-2}(v) = (-j-1, -j)$ or $\Lambda_D^{n-2}(v) = (j, j+1)$ (since the position of $-j-1$ or of j is fixed). But then $u = v$, so we get a contradiction. Therefore $\Lambda_D^{n-2}(u) \geq \Lambda_D^{n-2}(vs_j)$. Moreover, if $j = 1$ by minimality of j and $(PD2)$ we have $\Lambda_D^{n-2}(v) = (-2, 1)$. Then $\Lambda_D^{n-2}(vs_1) = (-1, 2)$ and the claim follows trivially except for all cases such that $\Lambda_D^{n-2}(u) = (a, b)$ with $a = -2$ or $b = 1$. But in this case $s_1 \in D_R(u)$ except for $\Lambda_D^{n-2}(u) = (-2, \pm 1)$. The claim follows.

Now we suppose $j = 0$ and therefore $\Lambda_D^{n-2}(v) = (-2, -1)$. Since $s_0 \notin D_R(u)$ we have that $\Lambda_D^{n-2}(u) = (-1, 2)$ or $\Lambda_D^{n-2}(u) = (a, b)$ with $1 \leq a < b$. The claim is trivial. This complete the proof. □

Diagram in Figure 5.2 is a projection of a 3-dimensional diagram. It is possible to give coordinates to each vertex of the diagram: label the rows from 1 to $2n-2$ from the top to the bottom; label the columns from 1 to $2n-2$ from the left to the right (note that vertices denoted with $\star$ and $\circ$ are in the same row or column); finally, we use a third coordinate which is -1 if the vertex is $\circ$, 1 if the vertex is $\star$, 0 otherwise. We define the main diagonal of the diagram in Figure 5.2 to be the line with row and column coordinates equals (note that the main diagonal has no vertex).

Proposition 5.2.3. *Diagram in Figure 5.2 is the Hasse diagram of* $(S_n^D)^{J_{n-2}}$.

Proof. We use the same arguments of Proposition 5.2.1.

For any number i, let β_i equals to 1 if $i < n$, 0 if $i = n$ and -1 if $i > n$. Now we associate to each vertex the set $\{i-n-\beta_i-z(1-|\beta_i|), -j+n+\beta_j+z(1-|\beta_i|)\}$ where i, j, z denote respectively the row, the column and the third coordinate of the vertex. Note that if $(i_1, j_1, z_1), (i_2, j_2, z_2)$ are coordinates of vertices in the Hasse diagram, then $i_1 - n - \beta_{i_1} - z_1 = i_2 - n - \beta_{i_2} - z_2$ if and only if $i_1 = i_2$: in fact the equation is equivalent to $i_1 - i_2 = \beta_{i_1} - \beta_{i_2} + z_1 - z_2$. Suppose that $i_1 \geq i_2$ then LHS is non-negative. Moreover $\beta_{i_1} - \beta_{i_2} \leq 0$ and if exactly one between z_1, z_2 is not zero, then $\beta_{i_1} - \beta_{i_2} = -1$. If z_1, z_2 are both zero, then $i_1 = i_2 = n$. Therefore, the equality holds only if $i_1 = i_2$. The other implication is trivial.

We claim that if $i_1 + i_2 \leq 2n-1$ then the equality $i_1 - n - \beta_{i_1} - z_1 = -(i_2 - n - \beta_{i_2} - z_2)$ never holds. In fact, the equality is the same of $i_1 + i_2 = 2n + \beta_{i_1} + \beta_{i_2} + z_1 + z_2$. Suppose for example that $i_1 \leq i_2$. We know that $i_1 + i_2 \leq 2n-1$ therefore, $\beta_{i_1} = 1$. Since $|\beta_{i_2} + z_2| \leq 1$, RHS of the last equation is at least $2n$ and LHS is at most $2n-1$.

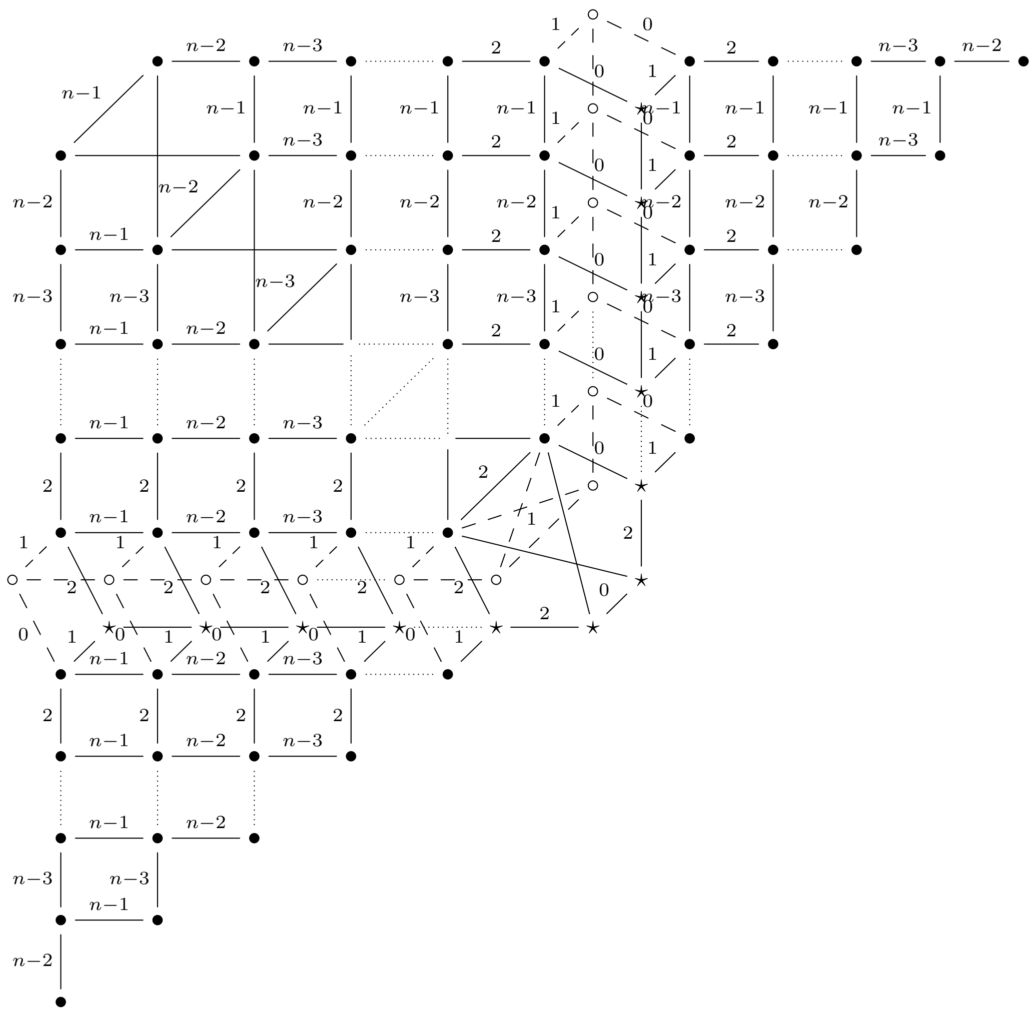


Figure 5.2: A 3-dim Hasse diagram of $(S_n^D)^{J_{n-2}}$. Vertices denoted by $\circ$ and dashed edges lie under the plane of the diagram; vertices denoted by $\star$ and edges with at least one $\star$-vertex lie over the plane of the diagram. To avoid confusion, we do not write labels of edges whose vertices have both third coordinate equal to 1 (respectively -1): they are the same labels of the edge in the corresponding columns or rows.

With the above discussion it is simple to verify that each set previous defined has always two elements and their absolute values are different and that two sets associated to different vertices are different.

It is a simple exercise to verify that the elements of the diagram are exactly $\binom{n}{2}2^2$ and therefore we have a bijection between the vertex set and L_{n-2}. The map Λ_D^{n-2} defines a bijection between L_{n-2} and $(S_n^D)^{J_{n-2}}$. Proposition 5.2.2 shows that the edges of the diagram give the correct Bruhat order and property $(PD2)$ gives the labels of all edges. □

5.3 Kazhdan–Lusztig polynomials

Throughout this section we denote with $P_{u,v}$ the Kazhdan–Lusztig polynomials associated to elements u, v of the quotients of the maximal parabolic subgroups, since there is no risk of confusion.

We describe the Kazhdan–Lusztig polynomials of $(S_n^B)^{J_{n-2}}$ and $(S_n^D)^{J_{n-2}}$ via their Hasse diagrams. To simplify our description we introduce some definitions. Let $v \in (S_n^B)^{J_{n-2}}$ and identify v with the corresponding vertex in the Hasse diagram. Suppose that the sum of the coordinates of v is strictly less than $2n$. The *square* of v is the set of all vertices in the Hasse diagram which are connected to v by paths with edges whose labels are in $D_R(v)$. If $D_R(v)$ has only one element, edges without labels are allowed (see Figure 5.3, (a),(b),(c)). Suppose now that the sum of coordinates of v is $2n$, then cardinality of $D_R(v)$ is 1. The *corner* of v is the set of all vertices depicted in Figure 5.3 (d), (e) and (f), where v is the rightmost and topmost vertex (note that only one configuration is possible for any fixed v). The left side (respectively the top side) of a corner is given by the two vertices on the left which are not in the top row (respectively the two vertices on the top which are not in the leftmost column).

Given a square of type (a), (b), or a corner of type (d), (e) as in Figure 5.3 we define the opposite square or the opposite corner to be the set of all vertices which are symmetric to any vertex of the square or corner with respect to the main diagonal of the diagram: for example, if a vertex of the square or corner has coordinates i, j then the vertex with coordinates j, i is in the opposite square or corner. The distance between a square or corner and its opposite is defined by the difference of row coordinates (or equivalent column coordinate) of their top rightmost vertices (for corner of type (e) in Figure 5.3 we set the distance with its opposite to be 2).

Lemma 5.3.1. *Let $u, v \in (S_n^B)^{J_{n-2}}$ be such that $u \leq v$ and their corresponding vertices are in the same row or column in the Hasse diagram of $(S_n^B)^{J_{n-2}}$. Then*

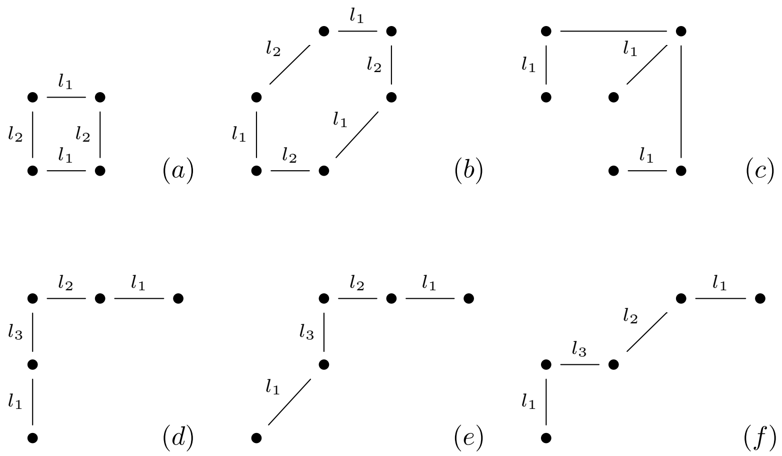


Figure 5.3: Squares (a), (b), (c) and corners (d), (e), (f) in the Hasse diagram of $(S_n^B)^{J_{n-2}}$.

$P_{u,v} \neq 0$ *if and only if* $u = v$ *or* u, v *are joined by an edge (i. e.* $l(v) - l(u) = 1$*).*

Proof. We argue the proof by induction on $l(v)$. If $l(v) = 0$ there is nothing to prove. Let $l(v) \geq 1$ and let $s \in D_R(v)$. If vs is not in the same row or column of v and u, then by Figure 5.1 and by assumption we have that $s \in D_R(u)$ (note that all edges in the same columns or row have the same labels). Therefore, by Proposition 4.2.1 we have $P_{u,v} = P_{us,vs}$ (note that other terms vanish since $u \not\leq vs$). The claim follows by induction. Now we suppose that u, v, vs are in the same row or column. If $us \notin (S_n^B)^{J_{n-2}}$ then by Proposition 4.2.4 $P_{u,v} = 0$ and this case is possible only if $l(v) - l(u) \geq 2$. Now suppose that $us \in (S_n^B)^{J_{n-2}}$. By the Hasse diagram, necessarily us, u, vs, v are in the same row or column. Now suppose that $us > u$, $us \neq v$; then $l(vs) - l(us) \geq 1$ (the equality holds only if $s = s_1$, see the Hasse diagram). If $l(vs) - l(us) > 1$ (therefore $l(v) - l(u) > 1$) then by Proposition 4.2.1 and by induction each term is zero. If $l(vs) - l(us) = 1$ then apply Proposition 4.2.1 and we have $P_{u,v} = qP_{us,vs} + P_{u,vs} - \mu(us, vs)qP_{us,u}$. By induction, $P_{u,v} = q + 0 - q = 0$. Finally, suppose that $us < u$ and $u \neq v$, therefore $l(v) - l(u) > 1$ since $vs > u$ by assumption. Then by Proposition 4.2.4, $P_{u,v} = P_{us,v}$. Since $l(v) - l(us) > l(v) - l(u) > 1$, by previous case $P_{us,v} = 0$. The vice versa of the claim is trivial. □

Proposition 5.3.2. *All Kazhdan–Lusztig polynomials in* $(S_n^B)^{J_{n-2}}$ *are either zero or a monic power of* q*. In particular, for* $u, v \in (S_n^B)^{J_{n-2}}$*,* $u \leq v$*,* $P_{u,v}$ *is non-trivial in the following cases:*

1. *u is in the square of v and* $P_{u,v} = 1$*;*

2. *u is in the left side of the corner of v and $P_{u,v} = q$;*

3. *u is in the top side of the corner of v and $P_{u,v} = 1$;*

4. *u is in the opposite square of v and $P_{u,v} = q^{d_v - 1}$;*

5. *u is in the left side of the opposite corner of v and $P_{u,v} = q^{d_v}$;*

6. *u is in the top side of the opposite corner of v and $P_{u,v} = q^{d_v - 1}$;*

where d_v denotes the distance between the square (or corner) of v and its opposite.

Proof. Let $u, v \in (S_n^B)^{J_{n-2}}$, $u \leq v$. By Proposition 4.2.4, we can assume $D_R(v) \subseteq D_R(u)$. We argue the proof by induction on $l(v)$. If $l(v) = 0$ there is nothing to prove. Then we suppose $l(v) \geq 1$ and the assertion true for smallest values. There are several cases to analyze. First note that for any $j \leq n-1$ there are two columns and two rows of edges with label j in the Hasse diagram (for $j = 0$ there are only one column and one row). Therefore, if $w \in (S_n^B)^{J_{n-2}}$ is such that $s_j \in D_R(w)$ then u is on the right of these columns or on the top of these rows. It follows that, if $s_i, s_j \in D_R(w)$, with $i \neq j$ then there are at most four vertices with the same property.

If u is in the square of v then $u = v$ or $l(v) - l(u) = 1$ (the last case is possible only for square of type (c) in Figure 5.3, here we use the assumption $D_R(v) \subseteq D_R(u)$), therefore $P_{u,v} = 1$. If v is in the first column then the assertion follows by Lemma 5.3.1. Now suppose that v is such that $|D_R(v)| = 2$. By previous remark there are only few possibilities. First case: u and v are in the same column or row and the result follows from Lemma 5.3.1. Second case: u is in the opposite square of v. Fix $s \in D_R(v)$ corresponding to a horizontal edge, and apply Proposition 4.2.1. Then $P_{u,v} = P_{us,vs} + qP_{u,vs} + \widetilde{M}_{u,v}$. We have that $P_{us,vs} = 0$ since us in not in the opposite square of vs or in the square of vs (see Hasse diagram). Moreover $P_{u,vs} = q^{d_{vs}-1} = q^{d_v - 2}$ (note that this identity is also true when vs has coordinates $(i, i+2)$ for some $i \leq n-3$). Finally, $\widetilde{M}_{u,v} = 0$ since by assumption if w is such that $\mu(w, vs) \neq 0$, $ws < w$ then $D_R(w) = D_R(vs) \cup \{s\}$ and it is impossible to find any such w. Third case: u, v are not in the same column or row and u is not in the opposite square of v. Fix $s \in D_R(v)$ and apply Proposition 4.2.1. By induction all summands are 0 (easy to check that for all $u \leq w < vs$, $s \in D_R(w)$ one between $P_{w,vs}$ and $P_{u,w}$ is zero).

Finally, let v be such that $|D_R(v)| = 1$ and let u be such that u, v are not in the same row or column. Then necessarily v has a corner. By previous arguments, is simple to check that if $P_{u,v} \neq 0$ then u is in the row immediately below v or in one column to the left of v or in another row below v (these last two conditions are possible only if v is above the main diagonal of the Hasse diagram). If u is in the row immediately below v then by induction $P_{us,vs} = 0$ since us is not in the square of vs; moreover $P_{u,vs} = 1$ only if u is in the left side of the corner of v (otherwise u is not in the square of vs). By induction there is no $u \leq w < vs$ with $ws < w$ and $\mu(w, vs) \neq 0$ and therefore the only non-trivial polynomial is $P_{u,v} = qP_{u,vs} = q$. If u is in one column to the left of v (and not in the row immediately below v), with us on the left of u, then $P_{us,vs} \neq 0$ or $P_{u,vs} \neq 0$ if and only if us or u are in the opposite square of vs. Easy to check from the Hasse diagram that this condition is equivalent to assume u in the top side of the opposite corner of v. By induction $P_{u,vs} = 0$, since u is not in the opposite corner of v and there is no $u \leq w < vs$ with $ws < w$ and $\mu(w, vs) \neq 0$. Therefore, the only non-trivial polynomial is $P_{u,v} = P_{us,vs} = q^{d_{vs}-1} = q^{d_v-1}$. If u is in another row below v (there exists only one such row by assumption), with us below u and not in the same column of v, then $P_{us,vs} = 0$ since us is not in the opposite square of vs and $P_{u,vs} \neq 0$ if and only if u is in the opposite square of vs, i. e. u is in the left side of the opposite corner of v. Since there is no $u \leq w < vs$ with $ws < w$ and $\mu(w, vs) \neq 0$ by induction, then the only non-trivial polynomial is $P_{u,v} = q \cdot q^{d_{vs}-1} = q^{d_v}$. □

Now we classify all Kazhdan–Lusztig polynomials of $(S_n^D)^{J_{n-2}}$. We define the square or the corner (of type (d) in Figure 5.3) of an element $v \in (S_n^D)^{J_{n-2}}$ as done for $(S_n^B)^{J_{n-2}}$ (note that the Hasse diagrams of both groups are different only in the n-th column and n-th row). In $(S_n^D)^{J_{n-2}}$ there are more non-trivial Kazhdan–Lusztig polynomials than $(S_n^B)^{J_{n-2}}$. It is necessary to introduce other definitions. Let $v \in (S_n^D)^{J_{n-2}}$ be an element of coordinates $(i, j, 0)$ with $j < i < n-1$ or $j > n+1$ and such that $|D_R(v)| = 2$. We define the *key* of v the square of the element u of coordinates $(i, 2n-j+1, 0)$ if $j > n+1$ and the square of $(2n-i-1, j, 0)$ if $j < i < n-1$. If $|D_R(v)| = 1$ we define the *corner key* the set of all vertices of coordinates $(i, 2n-j+\delta, z)$, $(i+1, 2n-j+1+\delta, z)$ and $(i+2+\delta, 2n-j+2-\delta, z)$, with $\delta = 0$ or $\delta = 1$, if $j > n+1$, or the corner of $(2n-i-2, j, 0)$, if $j < n$. In the last definition we set the corner key of $(n-2, n-1, 0)$ be the set of all vertices whose coordinates are $(n+1, n-2+\delta, 0)$ $(\delta = 0; 1)$ and we will consider these vertices as the left side of the corner. Examples of keys or corner keys are depicted in Figure 5.4.

By Figure 5.4 there is a natural definition of vertical or horizontal keys.

Proposition 5.3.3. *All Kazhdan–Lusztig polynomials in $(S_n^D)^{J_{n-2}}$ are either zero or a monic power of q. In particular, for $u, v \in (S_n^D)^{J_{n-2}}$, $u \leq v$, $P_{u,v}$ is*

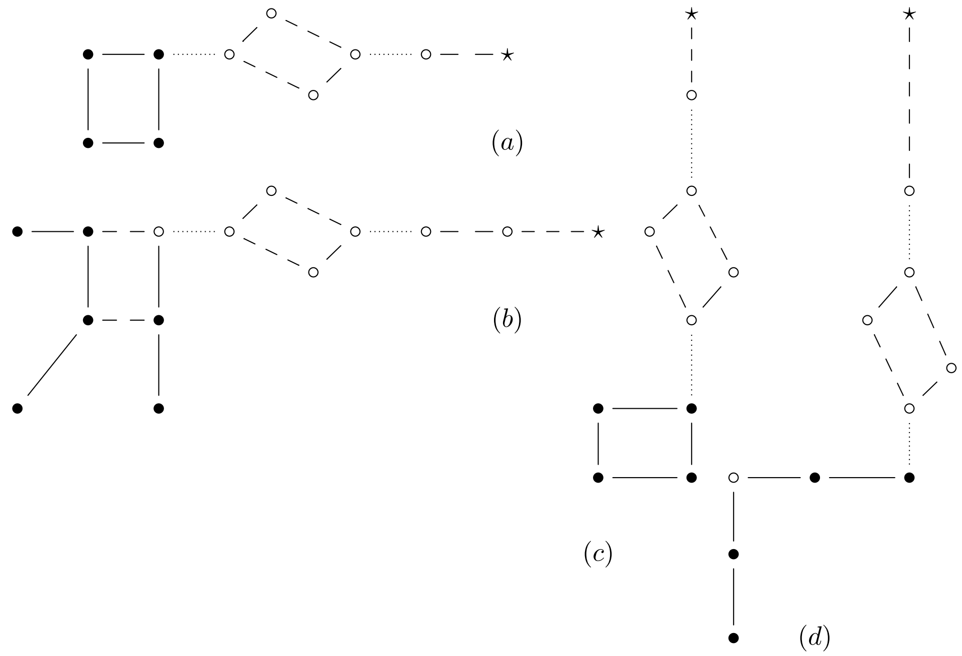


Figure 5.4: Examples of keys (a), (c) and corner keys (b), (d) of the vertex denoted by $\star$ in $(S_n^D)^{J_{n-2}}$. Only the vertices depicted with dots $\bullet$ are included in the definition.

non-trivial in the following cases:

1. *u is in the square of v and $P_{u,v} = 1$;*
2. *u is in the left side of the corner of v and $P_{u,v} = q$;*
3. *u is in the top side of the corner of v and $P_{u,v} = 1$;*
4. *u is in the opposite square of v, v has coordinates (i, j, k) with $i, j \leq n$ and $P_{u,v} = q^{d_v - 1}$;*
5. *u is in the key or corner key of v and $P_{u,v} = q^{d'_v + b}$ where $b = 1$ only if u is in the left side of a (vertical) corner key and $b = 0$ otherwise;*
6. *v has coordinates $(n-2, n+1, 0)$ and u has coordinates $(n-1, n-2, 0)$ or $(n, n-2, \pm 1)$ and $P_{u,v} = q$;*

where d_v denotes the distance between the square of v and its opposite and d'_v is the distance of v from the $n+1$-th column, if the key is horizontal, or from the $n-1$-th row, if the key is vertical.

Proof. As done in the proof of Proposition 5.3.2 we will consider only elements $u, v \in (S_n^D)^{J_{n-2}}$ with $D_R(v) \subseteq D_R(u)$ and we apply Proposition 4.2.4 to compute all polynomials. We first consider cases with u, v in the same row or the same column: if $u = v$ then obviously $P_{u,v} = 1$, otherwise we compute $P_{u,v}$ by induction on $l(v) \geq 1$. Let we suppose there exists $s \in D_R(v)$ such that $u < vs < v$ (i. e. vs is in the same row or column of u, v), then by Hasse diagram and by our assumption it is obvious that us is in the same row or column of u, v. We have $P_{us,vs} = 0$ since $D_R(vs) \not\subseteq D_R(us)$ (easy by Figure 5.2); $P_{u,vs} = q^{d'_{vs}} = q^{d'_v - 1}$ by induction and there is no w in the same row or column of u, v with $ws < w$ except u, v. Therefore, by equation (4.2.5) $P_{u,v} = q^{d'_v}$. Now we suppose there is no $s \in D_R(v)$ such that $u < vs < v$ (this happens only if v is immediately above the main diagonal of the Hasse diagram). Let $s \in D_R(v)$, then $u \not\leq vs$. Therefore $P_{u,v} = P_{us,vs}$. Note that $d'_{vs} = d'_v$, $d'_{us} = d'_u$. The claim follows by the previous case.

We will prove the other cases by induction on $l(v)$. If $l(v) = 0$ then there is nothing to prove. Then we suppose that $l(v) > 0$ and the assertion true for smallest values.

We claim that if $|D_R(v)| = 3$ then $P_{u,v} = 0$ for all $u \neq v$, $D_R(v) \subseteq D_R(u)$. In fact, let $s \in D_R(v) \setminus \{s_0, s_1\}$. By Hasse diagram $|D_R(vs)| = 3$ except the case with vs of coordinates $(n-2, n+1, 0)$. In this case (note that by assumption us is uniquely determined) $P_{us,vs} = 0$, $P_{u,vs} = \mu(u, vs) = q$ by induction and therefore $P_{u,v} = 0$. In all other cases all summands of $P_{u,v}$ in equation (4.2.5) are zero, by induction.

With the same arguments used in Proposition 5.3.2 and previous discussion, it is possible to check that if v is such that $|D_R(v)| = 2$ and $D_R(v) \neq \{s_0, s_1\}$ then $P_{u,v} \neq 0$ only if u is in the same row or column of v or u is in the opposite square of v, with coordinates of u, v all less or equal to n: the case with one coordinate greater than n uses a different induction hypotheses, but the techniques are essentially the same.

Now we consider the case with $D_R(v) = \{1, 2\}$. If we suppose $D_R(v) \subseteq D_R(u)$ and $u \neq v$, then v is uniquely determined (see the Hasse diagram) and has coordinates $(n-2, n+1, 0)$. In this case u is in the row labeled with $n-1$. By induction P_{us_2,vs_2} is always 0, $P_{u,vs_2} = 1$ only if u is in the square of vs_2. Moreover there is no w, with $ws_2 < w$, $\mu(w, vs_2) \neq 0$. Then the non-trivial polynomial is $P_{us_2s_0s_2,v} = q$.

Finally, we have to consider elements v with exactly one descent. If v is immediately above the main diagonal of the Hasse diagram then u is in the same row or column of v or in a row below v (and strictly to the left of v). The first case is already done. We study the last. Let $s \in D_R(v)$. By the shape of Hasse diagram, us and u are in a column to the left of v. Therefore $P_{us,vs}$ and $P_{u,vs}$ are non-zero only if u is in the column immediately to the left of vs (note that u is not in the same column of vs). In this case $P_{u,v} = qP_{u,vs} = q^{d'_v+1}$.

If v is at the end of its row, then we use the same arguments in the proof of Proposition 5.3.2 and we have the same results if u is in the left side of the corner of v and $P_{u,v} = 0$ if u is in the opposite corner of v (here induction hypotheses change but the techniques are the same). There are differences only if u is in the row immediately below v. In this case let $s \in D_R(v)$: $P_{us,vs} = 0$ since us is not in the row immediately below vs; $P_{u,vs} \neq 0$ only if u is in the key of vs. Moreover $D_R(vs)$ has already two elements different from s and therefore $\widetilde{M}_{u,v} = 0$. Then, we have only one non-trivial contribution and $P_{u,v} = q^{d'_{vs}+1} = q^{d'_v}$. □

With the same techniques used in the proof of Propositions 5.3.2 and 5.3.3 it is possible to prove the analogues results for $(S_n^B)^{J_{n-1}}$ and $(S_n^D)^{J_{n-1}}$: these quotients have been already studied in [Bre09], Theorems 4.2 and 4.3. It is not hard to check that the Hasse diagram of $(S_n^B)^{J_{n-1}}$ is the first column in Figure 5.1 with in addition two edges labeled by $n-1$ one joined to the initial vertex and the other to the final vertex; analogously, the Hasse diagram of $(S_n^B)^{J_{n-1}}$ is the first column in Figure 5.2 with in addition two edges labeled by $n-1$ joined respectively to the initial and final vertex. The equivalent of Proposition 5.3.2 is the case 1 of the same proposition; the equivalent of Proposition 5.3.3 is given by cases 1 and 5 of the same proposition (where d'_v changes in an obvious way).

5.4 Applications

In this section we give some consequences of our main results. We start with the following result which includes all quasi-minuscule parabolic quotients.

Corollary 5.4.1. *Let W^J be a quasi-minuscule parabolic quotient of (W, S) and $u, v \in W^J$, $u \leq v$. Then $P_{u,v}^{J,q}(q)$ is either zero or a monic power of q.*

Proof. If W is a Coxeter group of type A then the result follows directly from [Bre02b, Theorem 5.1]. If W is a Coxeter group of type B, C then the result follows from Propositions 5.3.2 and 5.3.3 and from [Bre09, Theorems 4.1, 4.2 and 4.3]. For all other finite exceptional Coxeter groups W the result follows from computer calculations. □

Corollary 5.4.2. *Let W^J be a quasi-minuscule parabolic quotient of (W,S) and $u,v \in W^J$, $u \leq v$. Then $\sum_{w \in W_J} (-1)^{l(w)} P_{wu,v}(q)$ is either zero or a monic power of q.*

Proof. This follows immediately from Corollary 5.4.1 and Proposition 4.2.2. □

Note that the exact power of q in Corollary 5.4.2 is explicitly determined in Propositions 5.3.2 and 5.3.3, [Bre09, Theorems 4.1,4.2 and 4.3] and in [Bre02b, Theorem 5.1].

One of the most celebrated conjectures about the Kazhdan–Lusztig polynomials is the so-called "combinatorial invariance conjecture" (see e.g. [BB05, p. 161] and the references cited there). This conjecture states that the Kazhdan–Lusztig polynomial $P_{u,v}(q)$ is determined only on the isomorphism class of $[u,v]$ as poset. For parabolic quotients it is not true in general as showed in Remark 5.4.4 below. For most of quasi-minuscule parabolic quotients we have the following result.

Corollary 5.4.3. *Let W^J be a quasi-minuscule parabolic quotient of (W,S), W of type A,B,D,E, and $u,v \in W^J$ be such that $J \subseteq D_R(u) \cap D_R(v)$. Then $P_{u,v}(q)$ depends only on*

$$\{x \in [u,v] | J \subseteq D_R(x)\}$$

as poset.

Proof. By proofs of Proposition 5.3.2, 5.3.3 and by [Bre09, Corollary 4.8] we have that if $u,v,w,z \in W^J$ with $[u,v]^J \cong [w,z]^J$ then $P^{J,q}_{u,v}(q) = P^{J,q}_{w,z}(q)$, if W is of type A,B,D. Computer calculations show us that it is also true if W is of type E_6, E_7, E_8. The same proof of [Bre09, Corollary 5.2] returns the claim. □

Remark 5.4.4. *It is not true in general that if $[u,v]^J \cong [w,z]^J$ then $P^{J,q}_{u,v}(q) = P^{J,q}_{w,z}(q)$. Let we consider W a Coxeter group of type F_4, with generators $\{s_0, s_1, s_2, s_3\}$, $(s_0s_1)^3 = 1$, $(s_1s_2)^4 = 1$ $(s_2s_3)^3 = 1$ and $(s_is_j)^2 = 1$ in all other cases. Let $J = \{s_0, s_1, s_2\}$.*

Let $v = s_3s_2s_1s_0s_2s_1s_3s_2$, $u = s_3s_2s_1s_0s_2$, $z = s_3s_2s_1s_0s_2s_3$, $w = s_3s_2s_1$. It is possible to show that $P^{J,q}_{u,v} = q$, $P^{J,q}_{w,z} = 0$ and $[u,v]^J \cong [w,z]^J$ and this interval is depicted in Figure 5.5.

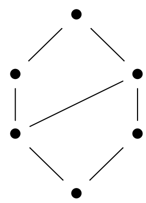


Figure 5.5: Hasse diagram of the intervals $[u, v]^J$, $[w, z]^J$ in F_4^J.

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