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Mathematical analysis of models of non-homogeneous fluids and of hyperbolic equations with low regularity coefficients

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MATHEMATICAL ANALYSIS OF MODELS OF NON-HOMOGENEOUS FLUIDS AND OF HYPERBOLIC EQUATIONS WITH LOW REGULARITY COEFFICIENTS

(Analyse mathématique des modèles de fluids non-homogènes et d'équations hyperboliques à coefficients peu réguliers)

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Non si arriva ad una meta se non per ripartire
[. . .] portando nel cuore
quella ricchezza di cose e di persone che si è vissuta
Anonymous

Wichtig ist, dass man nicht aufhört zu fragen
(The important thing is not to stop questioning)
A. Einstein

Abstract

The present thesis is devoted to the study both of strictly hyperbolic operators with low regularity coefficients and of the density-dependent incompressible Euler system.

On the one hand, we show a priori estimates for a second order strictly hyperbolic operator whose highest order coefficients satisfy a log-Zygmund continuity condition in time and a log-Lipschitz continuity condition with respect to space. Such an estimate involves a time increasing loss of derivatives. Nevertheless, this is enough to recover well-posedness for the associated Cauchy problem in the space H^∞ (for suitably smooth second order coefficients).

In a first time, we consider a complete operator in space dimension 1, whose first order coefficients were assumed Hölder continuous and that of order 0 only bounded. Then, we deal with the general case of any space dimension, focusing on a homogeneous second order operator: the step to higher dimension requires a really different approach.

On the other hand, we consider the density-dependent incompressible Euler system.

We show its well-posedness in endpoint Besov spaces embedded in the class of globally Lipschitz functions, producing also lower bounds for the lifespan of the solution in terms of initial data only.

This having been done, we prove persistence of geometric structures, such as striated and conormal regularity, for solutions to this system.

In contrast with the classical case of constant density, even in dimension 2 the vorticity is not transported by the velocity field. Hence, a priori one can expect to get only local in time results. For the same reason, we also have to dismiss the vortex patch structure.

Littlewood-Paley theory and paradifferential calculus allow us to handle these two different problems. A new version of paradifferential calculus, depending on a paramter $\gamma \geq 1$, is also needed in dealing with hyperbolic operators with nonregular coefficients.

The general framework is that of Besov spaces, which includes in particular Sobolev and Hölder sets. Intermediate classes of functions, of logaritmic type, come into play as well.

Keywords

Littlewood-Paley theory, paradifferential calculus with parameters, Besov spaces, strictly hyperbolic operator, log Zygmund continuity, log-Lipschitz continuity, loss of derivatives, logarithmic Sobolev spaces, incompressible Euler system, variable density, lifespan, vortex patches, striated and conormal regularity.

Résumé

Cette thèse est consacrée à l'étude des opérateurs strictement hyperboliques à coefficients peu réguliers, aussi bien qu'à l'étude du système d'Euler incompressible à densité variable.

Dans la première partie, on montre des estimations a priori pour des opérateurs strictement hyperboliques dont les coefficients d'ordre le plus grand satisfont une condition de continuité log-Zygmund par rapport au temps et une condition de continuité log-Lipschitz par rapport à la variable d'espace. Ces estimations comportent une perte de dérivées qui croît en temps. Toutefois, elles sont suffisantes pour avoir encore le caractère bien posé du problème de Cauchy associé dans l'espace H^∞ (pour des coefficients du deuxième ordre ayant assez de régularité).

Dans un premier temps, on considère un opérateur complet en dimension d'espace égale à 1, dont les coefficients du premier ordre sont supposés hölderiens et celui d'ordre 0 seulement borné. Après, on traite le cas général en dimension d'espace quelconque, en se restreignant à un opérateur de deuxième ordre homogène: le passage à la dimension plus grande exige une approche vraiment différente.

Dans la deuxième partie de la thèse, on considère le système d'Euler incompressible à densité variable.

On montre son caractère bien posé dans des espaces de Besov limites, qui s'injectent dans la classe des fonctions globalement lipschitziennes, et on établit aussi des bornes inférieures pour le temps de vie de la solution ne dépendant que des données initiales.

Cela fait, on prouve la persistance des structures géometriques, comme la régularité stratifiée et conormale, pour les solutions de ce système.

À la différence du cas classique de densité constante, même en dimension 2 le tourbillon n'est pas transporté par le champ de vitesses. Donc, a priori on peut s'attendre à obtenir seulement des résultats locaux en temps. Pour la même raison, il faut aussi laisser tomber la structure des poches de tourbillon.

La théorie de Littlewood-Paley et le calcul paradifférentiel nous permettent d'aborder ces deux différents problèmes. En plus, on a besoin aussi d'une nouvelle version du calcul paradifférentiel, qui dépend d'un paramètre $\gamma \geq 1$, pour traiter les opérateurs à coefficients peu réguliers.

Le cadre fonctionnel adopté est celui des espaces de Besov, qui comprend en particulier les ensembles de Sobolev et de Hölder. Des classes intermédiaires de fonctions, de type logarithmique, entrent, elles aussi, en jeu.

Mots clés

Théorie de Littlewood-Paley, calcul paradifférentiel avec paramètres, espaces de Besov, opérateur strictement hyperbolique, continuité log-Zygmund, continuité log-Lipschitz, perte de dérivées, espaces de Sobolev logarithmiques, système d'Euler incompressible, densité variable, temps de vie, poches de tourbillon, régularité stratifiée et conormale.

Contents

Introduction

The present thesis is devoted to the analysis of some kinds of partial differential equations arising from the study of physical models. In particular, it developed into two different directions: the study of strictly hyperbolic operators with low regularity coefficients and that of the density-dependent incompressible Euler system.

On the one hand, we studied the Cauchy problem for a general wave operator, whose second order coefficients were supposed to be non-Lipschitz. In contrast to the Lipschitz case, one can't expect to get well-posedness in any Sobolev space H^s any more. Nevertheless, well-posedness in the space H^∞ still holds true, but with a finite loss of derivatives, which is proved to be, in a certain sense, necessary.

A wide number of works are devoted to this topic under different hypothesis. The general idea is to compensate the loss of regularity in time with suitable hypothesis with respect to the space variable. So, the first situation to consider is when the coefficients depend only on time: in this case, one can prove energy estimates with (in general) a constant loss of derivatives.

When coefficients depend also on x, instead, the loss is (in general) linearly increasing in time. In particular this is also our case, in which we combined log-Zygmund and log-Lipschitz continuity conditions: we imposed the coefficients to satisfy the former one in time and the latter one in space, uniformly with respect to the other variable.

On the other hand, we considered the density-dependent incompressible Euler system, which describes the dynamics of a non-homogeneous inviscid incompressible fluid.

The classical case, in which the density is supposed to be constant, was deeply studied. Main research interests were questions such as well-posedness in Besov spaces embedded in the class $\mathcal{C}^{0,1}$ of globally Lipschitz functions, continuation criteria and global in time existence, propagation of geometric structures. Our purpose was to extend (or invalidate) previous results to the more realistic situation of variable density. In particular, in a first time we investigated well-posedness in endpoint Besov spaces, and then we considered propagation of striated and conormal regularity.

The main difference with respect to the classical system is that, this time, even in the two-dimensional case, one can expect to get only local in time existence results. As a matter of fact, the vorticity is no more transported by the flow associated to the velocity field (which was the key to the proof of the global existence issue for homogeneous fluids), because of the presence of a density term in its equation. The global in time existence issue is still an open problem.

The techniques we used to handle the two different problems were mostly based on Fourier Analysis. In particular, an extensive use of Littlewood-Paley theory and of paradifferential calculus, as introduced by J.-M. Bony in the famous paper [8], was needed.

The main idea is to define a dyadic partition of unity in the phase space, thanks to suitable smooth, compactly supported functions:

$$\chi(\xi) + \sum_{j\geq 1}\varphi_j(\xi) \;\equiv\; 1 \qquad \forall\,\xi\in\mathbb{R}^N\,,$$

where χ and φ_j (for all j) belong to $\mathcal{C}^\infty_0(\mathbb{R}^N_\xi)$, with

$$\operatorname{supp}\chi\subset\{|\xi|\leq 2\} \qquad\text{and}\qquad \operatorname{supp}\varphi_j\subset\left\{C_1\,2^{j-1}\;\leq\;|\xi|\;\leq\;C_2\,2^{j+1}\right\}\,.$$

Now, given a $u \in \mathcal{S}'$, we can define[1] $\Delta_0 u := \chi(D)u$ and $\Delta_j u := \varphi_j(D)u$ for all $j \geq 1$. We get, in this way, the *Littlewood-Paley decomposition* of a tempered distribution u:

$$u = \sum_{j=0}^{+\infty} \Delta_j u \qquad \text{in } \mathcal{S}'.$$

By Paley-Wiener theorem, each dyadic item $\Delta_j u$ is a smooth function. Moreover, thanks to spectral localization, integrability properties of $\Delta_j u$ are strictly linked with those of their derivatives $D^\alpha \Delta_j u$ (see lemma 1.2 below). These facts are fundamental and widely used in the analysis of partial differential equations.

Using Littlewood-Paley decomposition, one can define what a Besov space is. For all $s \in \mathbb{R}$ and all $(p,r) \in [1,+\infty]^2$, the *non-homogeneous Besov space* $B^s_{p,r}$ is defined as the set of tempered distributions u for which the quantity

$$(1) \qquad \|u\|_{B^s_{p,r}} := \left\| \left(2^{js} \|\Delta_j u\|_{L^p}\right)_{j\in\mathbb{N}} \right\|_{\ell^r} < +\infty .$$

These functional spaces extend Sobolev and Hölder classes: it turns out that $H^s \equiv B^s_{2,2}$ for all $s \in \mathbb{R}$ and $\mathcal{C}^\omega \equiv B^\omega_{\infty,\infty}$ for all $\omega \in]0,1[$.

Properties of Besov spaces were deeply studied (see e.g. [2] and [51]) and they are now a classical topic. Nevertheless, in paper [23] the authors considered Sobolev spaces of logarithmic type and gave a dyadic characterization of them. Inspired by this fact, we defined the class of logarithmic Besov spaces, putting, in definition (1), the new weight $2^{js}(1+j)^\alpha$ (for some $\alpha \in \mathbb{R}$) instead of the single exponential term. We also proved that they enjoy properties analogous to the classical Besov spaces.

Using Littlewood-Paley decomposition again, one can write the *Bony's decomposition* (see paper [8]) of the product of two tempered distributions:

$$(2) \qquad u\,v = T_u v + T_v u + R(u,v),$$

where we have defined the paraproduct and remainder operators respectively as

$$T_u v := \sum_j S_{j-1}u\,\Delta_j v \qquad \text{and} \qquad R(u,v) := \sum_j \sum_{|k-j|\leq 1} \Delta_j u\,\Delta_k v .$$

These operators have nice continuity properties on the class of Besov spaces. Moreover, paraproduct plays an important role also in nonlinear analysis (see for instance the paralinearization theorem 1.33 below). We will make an extensive use of decomposition (2) throughout all this manuscript. However, paraproduct is nothing else than an example of paradifferential operator, associated to a function which depends only on the space variable x.

More in general, we can associate a paradifferential operator to every symbol $a(x,\xi)$ which is smooth with respect to ξ, only locally bounded in x and its ξ-derivatives satisfy particular growth conditions (see e.g. [51]). First of all, fixed a suitable cut-off function ψ, one can smooth out a with respect to x, defining a classical symbol $\sigma_a(x,\xi)$ strictly related to it. Then one can define the paradifferential operator associated to a, still denoted by T_a, as the paradifferential operator associated to this classical symbol, i.e. $\sigma_a(x,D_x)$. On the one hand, the whole construction is independent of the cut-off function ψ, up to lower order terms. On the other hand, one can make it depend on a parameter $\gamma \geq 1$, as done in e.g. [50] and [52]. This simple change came into play in a crucial way in the study of strictly hyperbolic operators with low regularity coefficients (see chapter 3), because it allows a more refined analysis.

After this brief overview about the theoretical tools we needed in our analysis, let us explain better the two different guidelines of our research work.

[1] Throughout we agree that $f(D)$ stands for the pseudo-differential operator $u \mapsto \mathcal{F}^{-1}(f\,\mathcal{F}u)$.

Strictly hyperbolic operators with low-regularity coefficients

Consider a second order strictly hyperbolic operator L defined in a strip $[0,T] \times \mathbb{R}^N$, for some $T > 0$ and any dimension $N \geq 1$:

$$Lu \;:=\; \partial_t^2 u \,-\, \sum_{j,k=1}^{N} \partial_j \left(a_{jk}(t,x)\, \partial_k u\right) \tag{3}$$

and assume that there exist two constants $0 < \lambda_0 \leq \Lambda_0$ such that

$$\lambda_0\, |\xi|^2 \;\leq\; \sum_{j,k=1}^{N} a_{jk}(t,x)\, \xi_j\, \xi_k \;\leq\; \Lambda_0\, |\xi|^2$$

for all $(t,x) \in [0,T] \times \mathbb{R}^N$ and all $\xi \in \mathbb{R}^N$. The inequality on the left is the condition of strict hyperbolicity, while that on the right is a boundedness requirement on the coefficients of our operator.

It is well-known (see e.g. [42] or [53]) that, if coefficients a_{jk} are Lipschitz continuous with respect to t and only measurable and bounded in x, then the Cauchy problem for L is well-posed in $H^1 \times L^2$. Hence, if a_{jk} are $\mathcal{C}^\infty$ and bounded with all their derivatives with respect to the space variable, one can recover well-posedness in $H^{s+1} \times H^s$ for all $s \in \mathbb{R}$. Moreover, for all $s \in \mathbb{R}$ one gets (for a constant C_s depending only on s) the energy estimate

$$\sup_{0\leq t\leq T}\left(\|u(t,\cdot)\|_{H^{s+1}} \,+\, \|\partial_t u(t,\cdot)\|_{H^s}\right) \;\leq \tag{4}$$
$$\leq\; C_s\left(\|u(0,\cdot)\|_{H^{s+1}} + \|\partial_t u(0,\cdot)\|_{H^s} + \int_0^T \|Lu(t,\cdot)\|_{H^s}\, dt\right)$$

for all $u \in \mathcal{C}([0,T]; H^{s+1}(\mathbb{R}^N)) \cap \mathcal{C}^1([0,T]; H^s(\mathbb{R}^N))$ such that $Lu \in L^1([0,T]; H^s(\mathbb{R}^N))$.

In particular, estimate (4) still holds for every $u \in \mathcal{C}^2([0,T]; H^\infty(\mathbb{R}^N))$, and this implies that the Cauchy problem for L is well-posed in H^∞ *with no loss of derivatives.*

If the Lipschitz continuity (in time) hypothesis is not fulfilled, then (4) is no more true. Nevertheless, one can still recover H^∞ well-posedness, but this time with a loss of derivatives in the energy estimate. This loss can not be avoided, as shown in paper [16]. As a matter of fact, the authors proved there that, if the regularity of the coefficients is measured by a modulus of continuity, then every modulus of continuity worse than the Lipschitz one always entails a loss of derivatives.

The first case to consider is when the a_{jk}'s depend only on time: so, by Fourier transform one can pass to the phase space, in which the problem becomes an ordinary differential equation. In paper [18], Colombini, De Giorgi and Spagnolo assumed a log-Lipschitz integral continuity condition, while Tarama (see [56]) considered the more general class of (integral) log-Zygmund functions. In both the previous hypothesis, one can get an energy estimate *with a constant loss of derivatives*: there exists a constant $\delta > 0$ such that, for all $s \in \mathbb{R}$, the inequality

$$\sup_{0\leq t\leq T}\left(\|u(t,\cdot)\|_{H^{s+1-\delta}} \,+\, \|\partial_t u(t,\cdot)\|_{H^{s-\delta}}\right) \;\leq \tag{5}$$
$$\leq\; C_s\left(\|u(0,\cdot)\|_{H^{s+1}} + \|\partial_t u(0,\cdot)\|_{H^s} + \int_0^T \|Lu(t,\cdot)\|_{H^s}\, dt\right)$$

holds true for all $u \in \mathcal{C}^2([0,T]; H^\infty(\mathbb{R}^N))$, for some constant C_s depending only on s. The original idea of the work of Colombini, De Giorgi and Spagnolo was smoothing out coefficients using a convolution kernel, and then linking the approximation parameter (say) ε with the dual variable ξ: they got, in this way, a different approximation in different zones of the phase space. The improvement of Tarama, instead, was obtained defining a new energy, which involves also first derivatives of the smoothed coefficients, in such a way to compensate lower regularity.

The case of dependence both in time and in space variables was considered by Colombini and Lerner in paper [22]: they assumed the a_{jk}'s to satisfy a pointwise isotropic log-Lipschitz condition. They studied the related Cauchy problem, and they found an energy estimate *with a loss of derivatives increasing in time*: for all $s \in]0, 1/4]$, there exist positive constants β and C_s and a time $T^* \in]0, T]$ such that

$$\begin{aligned} (6) \qquad & \sup_{0 \le t \le T^*} \left(\|u(t,\cdot)\|_{H^{-s+1-\beta t}} + \|\partial_t u(t,\cdot)\|_{H^{-s-\beta t}} \right) \le \\ & \qquad \le C_s \left(\|u(0,\cdot)\|_{H^{-s+1}} + \|\partial_t u(0,\cdot)\|_{H^{-s}} + \int_0^{T^*} \|Lu(t,\cdot)\|_{H^{-s-\beta t}}\, dt \right) \end{aligned}$$

for all $u \in \mathcal{C}^2([0,T]; H^\infty(\mathbb{R}^N))$. Due to the dependence of the a_{jk}'s on x, it was no more possible to perform a Fourier transform to pass in the phase space. To overcome this problem, they took advantage of the Littlewood-Paley decomposition: they defined a localized energy for each localized component $\Delta_\nu u$ of the solution u, and then they performed a weighed summation over ν to define a total energy. Again, they smoothed out coefficients in time and they linked the approximation parameter to ν, which exactly corresponds to get different approximation in different regions of the phase space: recall that $|\xi| \sim 2^\nu$ on the spectrum of $\Delta_\nu u$. A quite hard work was required to control the operator norm (over L^2) of commutator terms $[\Delta_\nu, a_{jk}]$, coming from the equation for the localized part $\Delta_\nu u$.

More recently (see paper [19]), Colombini and Del Santo imposed a pointwise log-Zygmund condition with respect to time and a pointwise log-Lipschitz condition with respect to space, uniforlmy with respect to the other variable. These hypothesis read in the following way: there exists a constant K_0 such that, for all $\tau > 0$ and all $y \in \mathbb{R}^N \setminus \{0\}$, one has

$$(7) \qquad \sup_{(t,x)} |a_{jk}(t+\tau,x) + a_{jk}(t-\tau,x) - 2a_{jk}(t,x)| \le K_0\, \tau \, \log\left(1 + \frac{1}{\tau}\right)$$

$$(8) \qquad \sup_{(t,x)} |a_{jk}(t,x+y) - a_{jk}(t,x)| \le K_0\, |y| \, \log\left(1 + \frac{1}{|y|}\right).$$

Again, they decomposed the energy in localized parts, even if each of these items were defined in a new way, following the original idea of Tarama to control the bad behaviour in time of the coefficients. Moreover, the regularization of the coefficients by a convolution kernel was performed with respect to both time and space. They obtained an energy estimate analogous to (6) (and so a well-posedness issue in the space H^∞ for coefficients a_{jk} smooth enough with respect to x), but only in the case of one space dimension, i.e. $N = 1$: it wasn't so clear how to define a Tarama-like energy (which was somehow necessary) in higher dimensions.

In a first time, in paper [21] in collaboration with Colombini, we extended the result of [19] to the Cauchy problem (again in dimension $N = 1$) for a complete second order strictly hyperbolic operator

$$Pu := \partial_t^2 u - \partial_x \left(a(t,x)\, \partial_x u\right) + b_0(t,x)\, \partial_t u + b_1(t,x)\, \partial_x u + c(t,x)\, u\,,$$

where, in addition to hypothesis (7) and (8), we assumed also b_0 and b_1 to belong to $L^\infty([0,T]; \mathcal{C}^\omega)$ (for some $\omega \in]0,1[$) and $c \in L^\infty([0,T] \times \mathbb{R})$.
We came back to the main ideas of the work of Colombini and Del Santo. In particular, the energy associated to u was defined in the same way, and we handled highest order terms as they did. Again, we obtained an energy estimate of the same kind of (6): as one can expect, the presence of lower order terms involves no substantial problems in getting it. Nevertheless, Hölder regularity of coefficients of first order terms comes into play in the analysis of commutator terms $[\Delta_\nu, b_j]$

(for $j = 1\,,\,2$), and it entails a constraint on the Sobolev index s for which inequality (6) holds true (see theorem 2.1).

Recently, in [20] with Colombini, Del Santo and Métivier, we considered operator (3) under hypothesis (7) and (8) in any space dimension $N \geq 1$. Let us point out that we focused on a homogeneous operator to make our computations not too complicated, but the same technique actually works also for complete second order operators. We managed to get an energy estimate analogous to (6) (this time for any $s \in]0,1[$), which entails the well-posedness issue in H^∞ (for a_{jk} of class $\mathcal{C}_b^\infty$ in space).
To get the improvement, we resorted to a new energy: as already pointed out, Tarama's energy doesn't admit a straightforward generalization in higher space dimension. So, we came back to the original definition of Colombini and Del Santo again, but this time we replaced multiplication by functions $a(t,x)$ with low regularity modulus of continuity, by action of paradifferential operators T_a associated to them (as briefly explained above). Let us also point out that this construction already involves a smoothing effect with respect to the space variables, so that it was enough to perform a mollification of the coefficients only in time.
Nevertheless, positivity hypothesis on a (required for defining a strictly hyperbolic problem) doesn't translate, in general, to positivity of the corresponding operator, which is fundamental in obtaining energy estimates. So, we had to take advantage of paradifferential calculus depending on a parameter $\gamma \geq 1$, as developed by Métivier (see [50]) and by Métivier and Zumbrun (see paper [52]). This tool allowed us to perform a more refined analysis: in particular, we could define a paraproduct operator starting from high enough frequencies, so that it is a positive operator, if the corresponding symbol is.
We had also to deal with a different class of Sobolev spaces, of logarithmic type, already considered by Colombini and Métivier in [23]. This comes from the fact that the action of paradifferential operators associated to log-Lipschitz (in x) and log-Zygmund (in t) symbols, such those we considered in our strictly hyperoblic problem, involves a logarithmic loss of regularity.

Density-dependent incompressible Euler system

The density-dependent incompressible Euler system

$$(9) \qquad \begin{cases} \partial_t\rho \,+\, u\cdot\nabla\rho \,=\, 0 \\ \rho\,(\partial_t u \,+\, u\cdot\nabla u) \,+\, \nabla\Pi \,=\, \rho\, f \\ \operatorname{div} u \,=\, 0 \end{cases}$$

describes the evolution of a non-homogeneous incompressible fluid under the action of a body force $f = f(t,x) \in \mathbb{R}^N$. The function $\rho(t,x) \in \mathbb{R}_+$ represents the density of the fluid, $u(t,x) \in \mathbb{R}^N$ its velocity field and $\Pi(t,x) \in \mathbb{R}$ its pressure. The term $\nabla\Pi$ can be also seen as the Lagrangian multiplier associated to the divergence-free constraint over the velocity.

We assume that the space variable x belongs to the whole $\mathbb{R}^N$, with $N \geq 2$.

The case in which the fluid is supposed to be homogeneous, i.e. $\rho \equiv \overline{\rho}$ (strictly positive) constant and the system reads

$$(10) \qquad \begin{cases} \partial_t u \,+\, u\cdot\nabla u \,+\, \nabla\Pi \,=\, 0 \\ \operatorname{div} u \,=\, 0\,, \end{cases}$$

was deeply studied and there is a broad literature devoted to it.

In contrast, not so many work were devoted to the study of the non-homogeneous case. First results for equations (9) in smooth bounded domains of $\mathbb{R}^2$ or $\mathbb{R}^3$ were obtained by Beirão da Veiga and Valli for Hölder continuous initial data (see papers [5], [6] and [4]). The Sobolev spaces framework was considered instead by Valli and Zajączowski in [59], and by Itoh and Tani in [46]. In paper [45], Itoh studied instead the evolution in the whole $\mathbb{R}^3$ for initial data $(\nabla\rho_0, u_0) \in H^2 \times H^3$, and Danchin (see [27]) extended the results to any dimension $N \geq 2$ and to any Sobolev space with high enough regularity index.

In the same paper [27], Danchin considered also the case of data in the endpoint Besov space $B^{1+N/2}_{2,1}$. Before, Zhou (see [63]) had proved well-posedness for system (9) in spaces $B^{1+N/p}_{p,1}$ for any $1 < p < +\infty$, but he had to assume the initial density ρ_0 to be a small perturbation of a constant state.

Let us note that, in the case of the whole $\mathbb{R}^N$, the hypothesis on the adopted framework almost always entailed a L^2 condition over the velocity field, and if not (as e.g. in the last mentioned work of Zhou), additional smallness assumptions over the density term were required.

Let us also point out that all the well-posedness results stated for system (9) are only local in time, even in the two-dimensional case. As a matter of fact, as already remarked, for $N = 2$ the vorticity equation reads

$$\partial_t\omega \,+\, u\cdot\nabla\omega \,+\, \nabla\left(\frac{1}{\rho}\right)\wedge\nabla\Pi \,=\, 0\,, \tag{11}$$

and so one can't get conservation of Lebesgue norms, which was the key to the global in time existence issue, due to the presence of the density term.

Recently (see paper [28]), Danchin was able to prove the well-posedness result for (9) in any Besov space $B^s_{p,r}$, with $1 < p < +\infty$, embedded in the set $\mathcal{C}^{0,1}$ of the globally Lipschitz functions. As a matter of fact, our system is essentially a coupling of two transport equations by the velocity field u: so, no gain of smoothness may be expected during the time evolution, while preserving regularity requires u to be at least locally Lipschitz with respect to the space variable. Hence, Danchin proved that the functional framework suitable for (9) is the same as that for which equations (10) are well-posed. Moreover, he obtained his results for any initial density state, with no smallness, or closeness to a positive constant, requirements on it. However, he had to assume the velocity field u to belong to L^2 to handle the pressure term. As a matter of fact, in the non-constant density case $\nabla\Pi$ satisfies an elliptic equation (in divergence form) with low regularity coefficients,

$$-\,\mathrm{div}\,\left(a\,\nabla\Pi\right) \,=\, \mathrm{div}\,F\,, \tag{12}$$

(here we set $a := 1/\rho$) and it can be solved independently of a only in the energy space L^2. Let us point out that the control on the L^2 (or in general L^p) norm of $\nabla\Pi$ was needed also to bound its Besov norm.

Requiring the initial velocity $u_0 \in L^2$, however, is somehow restrictive: in the two-dimensional case this condition implies the vorticity to have average 0 over $\mathbb{R}^2$, and this fact precludes us from considering, for instance, vortex patches structures. Therefore, Danchin also proved well-posedness (in any dimension $N \geq 2$) for data in the space $B^s_{p,r} \hookrightarrow \mathcal{C}^{0,1}$, with $2 \leq p \leq 4$. So, no finite energy hypothesis were formulated, even if the previous assumption allows us to recover $\nabla\Pi \in L^2$ again. In particular, this result applies (thanks to Biot-Savart law) to any suitably smooth velocity field whose vorticity is compactly supported.

In the same paper, Danchin also proved a continuation criterion in the same spirit of the well-known result by Beale, Kato and Majda (see the famous paper [3]). The condition to extend solutions beyond T is

$$\int_0^T\left(\|\nabla u(t)\|_{L^\infty} \,+\, \|\nabla\Pi(t)\|_{B^{s-1}_{p,r}}\right)dt \,<\, +\infty\,,$$

and, in the case $s > 1 + N/p$, it is possible to replace ∇u with the vorticity Ω.

Finally, Danchin also tackled the case of the spaces $B^s_{\infty,r} \hookrightarrow \mathcal{C}^{0,1}$, but requiring moreover u_0 to belong to L^p (for some $1 < p < +\infty$) and ρ_0 to be close (in $B^s_{\infty,r}$ norm) to a constant state

$\overline{\rho}$. Under these additional assumptions, the equation for the pressure term can be rewritten (as already done by Zhou in paper [63]) in the form

$$-\,\overline{a}\,\Delta\Pi \;=\; \operatorname{div} F \,+\, \operatorname{div}\,\left((a-\overline{a})\nabla\Pi\right)\,,$$

with $\overline{a} := 1/\overline{\rho}$. So, using standard L^p estimates for Laplace operator (which hold for all $1 < p < +\infty$) and the smallness hypothesis on the density, one can absorb last term of the right-hand side of the previous equation.

In paper [29] in collaboration with Danchin, we focused on this last case: we managed to extend the well-posedness result in $B^s_{\infty,r}$ without assuming any smallness condition on the initial density. Let us point out that this framework is quite interesting, as it includes also the particular case of Hölder spaces of the type $\mathcal{C}^{1,\alpha}$ and the endpoint Besov space $B^1_{\infty,1}$, which is the largest one embedded in $\mathcal{C}^{0,1}$, and so the largest one in which one can expect to recover well-posedness for system (9). Of course, we still had to assume u_0 to belong to the energy space L^2, in order to assure the existence of the solution $\nabla\Pi$ to (12): we recall again that this equation can be solved independetely of a only in L^2. Now, the improvement with respect to the previous result was due to the different method used to handle the pressure term, which actually works for all $p \in [1,+\infty]$. In particular, we separated $\nabla\Pi$ into low and high frequencies, using Littlewood-Paley decomposition. Low frequencies could be controlled by the Lebesgue norm; high frequencies, instead, could be controlled in terms of $\Delta\Pi$, which satisfies the equation

$$-\,\Delta\Pi \;=\; \nabla\left(\log a\right)\cdot\nabla\Pi \,+\, \frac{1}{a}\,\operatorname{div}\left(f + u\cdot\nabla u\right)\,.$$

First term of the previous relation is of lower order: so, it can be absorbed interpolating between the L^2 estimate and the higher regularity estimates for the Laplace operator. We were also able to state a continuation criterion analogous to that of Danchin for the case $B^s_{p,r}$, $1 < p < +\infty$. Moreover, for the same reasons explained above, we considered also infinite energy data: in this case, vorticity (which was a fundamental quantity in the constant density case) comes into play by Biot-Savart law. We still assumed integrability properties for u_0 and its vorticity Ω_0, in order to assure the pressure term to belong to L^2, a requirement we could not bypass. Also under these hypothesis, we got well-posedness for equations (9). In particular, this result applies (as in the analogous case considered by Danchin) to any velocity field with suitably smooth compactly supported vorticity.

As already pointed out before, all the results we got were local in time. Nevertheless, in our paper we were able to give an estimate on the lifespan of the solutions. We first showed that in any space dimension, if the initial velocity is of order ε (with ε small enough), then the existence time is at least of order ε^{-1}. In this case, no restriction on the non-homogeneity are needed: the result is a straightforward consequence of rescaling properties for equations (9). Next, taking advantage of equality (11) and of more refined estimates for transport equation (established recently by Vishik in [60] and then generalized by T. Hmidi and S. Keraani in [41]), we showed that the lifespan of the solution tends to infinity if $\rho_0 - 1$ goes to 0. More precisely, if

$$\|\rho_0\,-\,1\|_{B^1_{\infty,1}}\;=\;\varepsilon \qquad\qquad \text{and} \qquad\qquad \|\omega_0\|_{B^0_{\infty,1}}\,+\,\|u_0\|_{L^2}\;=\;U_0$$

with ε small enough, then the lifespan is at least of order $U_0^{-1}\log(\log\varepsilon^{-1})$.

This having been done, in [35] we studied the problem of propagations of striated and conormal regularity for solutions to (9), in any dimension $N \geq 2$. We considered the initial velocity u_0 and the initial vorticity Ω_0 to be in some Lebesgue spaces, in order to guarantee, once again, $\nabla\Pi \in L^2$. We also supposed Ω_0 to have regularity properties of geometric type. Moreover, we required the initial density ρ_0 to be bounded with its gradient and to satisfy geometric assumptions analogous to those for Ω_0. Under these hypothesis, we proved existence (obviously, local in time) and uniqueness of a solution to (9), and propagation of these geometric structures. Let us point out

that no explicit Lipschitz continuity hypothesis over the velocity field were formulated. This property follows from striated regularity for the vorticity, and it works as in the homogeneous case. As a matter of fact, proving it relies only on Biot-Savart law, hence nothing changes with respect to the classical instance: in particular, no further hypothesis on the density term were needed. Let us point out that we also obtained propagation of geometric structures to the velocity field and to the pressure term.
Moreover, in the same work, we gave an estimate from below for the lifespan of the solution in terms of inital data only, in any dimension $N \geq 2$. Let us recall that, in the classical case of constant density, it was given (up to a multiplicative constant) by

$$T_{cl} \;:=\; \left(\|\Omega_0\|_{L^q\cap L^\infty}\;\;\log\left(e+\frac{\|\Omega_0\|_{\mathcal{C}^\varepsilon_{X_0}}}{\|\Omega_0\|_{L^q\cap L^\infty}}\right)\right)^{-1}$$

(see paper [26]). For the non-homogeneous system, instead, we got that the lifespan is given by

$$T_{nh} \;:=\; \left(V'(0)\left(1+\|\nabla\rho_0\|_{L^\infty}\right)^{3+\delta}\left(1+R_0+\Gamma_0^{7/3}\right)\right)^{-1}\,,$$

where the exponent $\delta > 1$ came out in the estimates for the pressure term, the quantities R_0 and Γ_0 are related to the geometric properties of the initial data and we defined

$$V'(0) \;:=\; \|u_0\|_{L^p}+\|\Omega_0\|_{L^q\cap L^\infty}+\|\Omega_0\|_{\mathcal{C}^\varepsilon_{X_0}} \;\geq\; c\,\|\Omega_0\|_{L^q\cap L^\infty}\;\log\left(e+\frac{\|\Omega_0\|_{\mathcal{C}^\varepsilon_{X_0}}}{\|\Omega_0\|_{L^q\cap L^\infty}}\right).$$

Hence, up to multiplication by a constant, $T_{nh} \;\leq\; T_{cl}$. Let us point out that we made the logarithmic dependence disappear in estimating the Lipschitz norm of the velocity field, to simplify our computations, but maybe this is a quite rough result.
Finally, in the physical case $N = 2$ or 3, we refined our result on conormal regularity: if the initial hypersurface Σ_0 is also connected, then it defines a bounded domain $D_0 \subset \mathbb{R}^N$ of which it is the boundary, and this property propagates in time (as the flow of the solution is a diffeomorphism). By analogy with the structure of vortex patches, we wanted to investigate the dynamics into the bounded domain. Obviously, even in dimension $N = 2$, even if the initial vorticity is a vortex patch, we cannot expect to propagate this property, because of the presence of the density term in equation (11). Nevertheless, we proved that, if initial density and vorticity are Hölder continuous inside the domain D_0 (in addition to satisfy global hyposthesis in order to assure persistence of conormal properties), than their regularity is preserved in time evolution. The main difficulty was to prove that Hölder continuity propagates also to the velocity field and the pressure term: in last analysis, we had to prove these two quantities to be regular enough at the boundary of the domain D_0 transported by the flow. Now, the required smoothness was ensured by the previously proved conormal regularity.

The thesis is structured in the following way.

In the first part, we will present the Fourier Analysis tools we needed in our study. We will expound, in a quite complete way, the classical Littlewood-Paley theory. We will recall the definition and the basic properties of the classical non-homogeneous Besov spaces, and we will quote also some fundamental results on paradifferential calculus. For doing this, we will strictly follow the presentation given in [2], and, as these results are classical, we will omit their proofs.

Then, we will pass to consider logarithmic Besov spaces, and we will show that previous properties hold true (up to suitable slight modifications) also for this class. This time, we will give here all the details.

In the last part of this chapter, we will explain also the main ideas of paradifferential calculus depending on parameters, mainly following the presentation of [52], and we will quote some results we will need in the sequel.

Then, we will consider the problem of second order strictly hyperbolic operators with low regularity coefficients.

In chapter 2 we will analyse the case of a complete operator in one space dimension. This chapter contains the results proved in paper [21] in collaboration with Colombini.

In chapter 3, instead, we will present the issues got in [20] with Colombini, Del Santo and Métivier. We will extend previous result in any space dimension, but only for a homogeneous operator, i.e. without lower order terms.

In each of these sections, we will introduce also some additional tools. In particular, we will analyse properties of functions with low regularity modulus of continuity and of the corresponding paradifferential operators.

Finally, the last part of the thesis is devoted to the study of the density-dependent incompressible Euler equations. Chapter 4 is devoted to well-posedness issues in endpoint Besov spaces embedded in the space of globally Lipschitz functions. We will provide also a continuation criterion and a lower bound for the lifespan of the solutions. This chapter is based on paper [29] in collaboration with Danchin.

Then, we will consider the problem of propagation of geometric structures for this system. We will focus on striated and conormal regularity, and in propagation of Hölder continuity in the iterior of a bounded domain of $\mathbb{R}^2$ or $\mathbb{R}^3$ (but still assuming global hypothesis, as briefly explained above). This will be the matter of chapter 5, and it contains the results proved in paper [35].

Introduction

Cette thèse est consacrée à l'analyse de quelques modèles d'équations différentielles à dérivées partielles qui naissent de l'étude des models physiques. En particulier, elle se développe dans deux directions différentes: l'étude des opérateurs hyperboliques à coefficients peu réguliers et celle du système d'Euler incompressible à densité variable.

Dans la première partie, on a étudié le problème de Cauchy pour un opérateur des ondes général, dont les coefficients du deuxième ordre étaient supposés non-Lipschitz. Au contraire du cas Lipschitz, on ne peut pas s'attendre à avoir encore le caractère bien posé dans n'importe quel espace de Sobolev H^s. Toutefois, le caractère bien posé dans l'espace H^∞ est encore vrai, mais avec une perte de dérivées, qui a été prouvée d'être, dans un certain sense, nécessaire.

Beaucoup de travaux sont dédiés à ce sujet, sous de différentes hypothèses. L'idée générale est celle de compenser la perte de régularité en temps avec des hypothèses convenables par rapport à la variable d'espace. Donc,la première situation à cosindérer est quand les coefficients ne dépendent que du temps: dans ce cas, on peut prouver des estimations de l'énergie avec (en général) une perte constante de dérivées.

Par contre, quand les coefficients dépendent aussi de x, la perte est (en général) linéairement croissante dans le temps. En particulier, c'est aussi notre cas, où nous avons mélangé les conditions log-Zygmund et log-Lipschitz: nous avons imposé que les coefficients satisfont la première en temps et l'autre dans la variable d'espace, uniformément par rapport à l'autre variable.

Dans la deuxième partie de la thèse, on a considéré le système d'Euler incompressible à densité variable, qui décrit la dynamique d'un fluide non-visqueux, incompressible et non-homogène.

Le cas classique, où la densité est supposée constante, a été étudié à fond. Parmi les questions principales d'intérêt, il y avait le caractère bien posé dans des espaces de Besov contenus dans la classe $\mathcal{C}^{0,1}$ des fonctions globalement lipschitziennes, des critères de prolongement et l'existence globale en temps, la propagation des structures géométriques. Notre but était d'élargir (ou d'invalider) les résultats précédents au cas, bien plus réaliste, de densité variable. En particulier, dans un premier temps on a examiné le caractère bien posé dans des espaces de Besov limites, et après on a considéré la propagation de la régularité stratifiée et conormale.

La différence principale avec le système classique est que, cette fois, même dans le cas de dimension 2, on peut s'attendre seulement des résultats d'existence locale en temps. En fait, le tourbillon n'est plus transporté par le flot associé au champs de vitesses (qui était la clé pour la preuve de l'existence globale pour des fluides homogènes), à cause de la présence d'un terme dépendant de la densité dans son équation.

Les techniques utilisées pour traiter ces deux problèmes différents étaient essentiellement basées sur l'Analyse de Fourier. En particulier, c'était nécessaire un vaste emploi de la théorie de Littlewood-Paley et du calcul paradifférentiel, comme présenté par J.-M. Bony dans le célèbre article [8].

L'idée principale est de définir une partition de l'unitée dyadique dans l'espace des phases, grâce à des fonctions convenablement régulières et à support compact:

$$\chi(\xi) + \sum_{j\geq 1}\varphi_j(\xi) \equiv 1 \qquad \forall\, \xi\in\mathbb{R}^N\,,$$

où χ et φ_j (pour tout j) appartiennent à $\mathcal{C}^\infty_0(\mathbb{R}^N_\xi)$, avec

$$\operatorname{supp}\chi \subset \{|\xi|\leq 2\} \qquad \text{et} \qquad \operatorname{supp}\varphi_j \subset \left\{C_1\, 2^{j-1} \leq |\xi| \leq C_2\, 2^{j+1}\right\}\,.$$

Alors, donnée une $u\in\mathcal{S}'$, on peut définir[2] $\Delta_0 u := \chi(D)u$ et $\Delta_j u := \varphi_j(D)u$ pour tout $j\geq 1$.

[2]Dans tout le manuscrit on convient que $f(D)$ est l'opérateur pseudodifférentiel $u\mapsto\mathcal{F}^{-1}(f\,\mathcal{F}u)$.

De cette façon, on a la *décomposition de Littlewood-Paley* d'une distribution temperée u:

$$u = \sum_{j=0}^{+\infty} \Delta_j u \qquad \text{dans } \mathcal{S}'.$$

Grâce au théorème de Paley-Wiener, chaque unité dyadique $\Delta_j u$ est une fonction lisse. En plus, grâce à la localisation spectrale, les propriétés d'integrabilité de $\Delta_j u$ sont strictement liées avec celles de leurs dérivées $D^\alpha \Delta_j u$ (voir aussi le lemme 1.2 en dessous). Ces faits sont fondamentals et largement utilisés dans l'analyse des équations différentielles à dérivées partielles.

En utilisant la décomposition de Littlewood-Paley, on peut définir ce qu'un espace de Besov est. Pour tout $s \in \mathbb{R}$ and tout $(p,r) \in [1,+\infty]^2$, *l'espace de Besov non-homogène* $B^s_{p,r}$ est défini comme l'ensemble des distributions temperées u pour lesquelles la quantité

$$\|u\|_{B^s_{p,r}} := \left\| \left(2^{js} \, \|\Delta_j u\|_{L^p} \right)_{j \in \mathbb{N}} \right\|_{\ell^r} < +\infty. \tag{13}$$

Ces espaces fonctionels agrandissent les classes de Sobolev et de Hölder: on a $H^s \equiv B^s_{2,2}$ pour tout $s \in \mathbb{R}$ et $\mathcal{C}^\omega \equiv B^\omega_{\infty,\infty}$ pour n'importe quel $\omega \in]0,1[$.

Les propriétés des espaces de Besov ont été étudiées à fond (voir par exemple [2] et [51]) et ils sont maintenant un sujet classique. De toute façon, dans l'article [23] les auteurs ont considéré des espaces de Sobolev de type logarithmique et ils ont donné une leur caractérisation dyadique. Inspiré par ce fait, nous avons défini la classe des espaces de Besov logarithmiques, en rempleçant le terme exponentiel par le nouveau poids $2^{js}(1+j)^\alpha$ (pour quelque $\alpha \in \mathbb{R}$). Nous avons aussi prouvé qu'ils jouissent des propriétés analogues à celles des espaces de Besov classiques.

En utlisant encore la décomposition de Littlewood-Paley, on peut écrire la *décomposition de Bony* (voir l'article [8]) d'un produit de deux distributions temperées:

$$u\,v = T_u v + T_v u + R(u,v), \tag{14}$$

où on a défini les opérateurs de paraproduit et de reste respectivement comme

$$T_u v := \sum_j S_{j-1} u \, \Delta_j v \qquad \text{et} \qquad R(u,v) := \sum_j \sum_{|k-j| \leq 1} \Delta_j u \, \Delta_k v.$$

Ces opérateurs ont d'agréables propriétés de continuité sur la classe des espaces de Besov. De plus, le paraproduit joue un rôle important aussi dans l'analyse non-linéaire (voir par exemple le théorème de paraliéairisation 1.33 en dessous). Nous allons faire un large emploi de la décomposition (14)dans tout le manuscrit. Cependant, le paraproduit n'est qu'un seul exemple d'opérateur paradifférentiel, associé à une fonction qui dépend seulement de la variable d'espace x.

Plus en général, on peut associer un opérateur paradifférentiel à tout symbol $a(x,\xi)$ lisse par rapport à ξ, seulement localement borné en x et dont les dérivées en ξ satisfont de particulières conditions de croissance (voir par exemple [51]). Avant tout, ayant fixé une convenable fonction de coupage ψ, on peut régulariser a par rapport à la variable x: on obtient ainsi un symbole classique $\sigma_a(x,\xi)$ strictement relié à a. Après, on peut définir l'opérateur paradifférentiel associé à a, indiqué par T_a, comme l'opérateur pseudodifférentiel associé au symbol classique, c'est-à-dire $\sigma_a(x,D_x)$. D'un côté, la construction entière est indépendante de la fonction de coupure ψ, à moin de termes d'ordre inférieur. De l'autre côté, on peut faire la dépendre d'un paramètre $\gamma \geq 1$, comme fait dans [50] et [52] par exemple. Ce très simple changement va entrer en jeu d'une façon essentielle dans l'étude des opérateurs hyperboliques à coefficients peu réguliers (voir le chapitre 3), parce qu'il permet une analyse plus raffinée.

Après cette brève présentation à l'égard des outils théoriques dont on aura besoin dans notre analyse, on va expliquer plus en détail les deux différentes directions principales de notre travail de recherche.

Opérateurs strictement hyperboliques à coefficients peu réguliers

On considère un opérateur strictement hyperbolique du deuxième ordre L sur une bande $[0,T] \times \mathbb{R}^N$, pour quelque $T > 0$ et toute diménsion $N \geq 1$:

$$Lu \;:=\; \partial_t^2 u \;-\; \sum_{j,k=1}^{N} \partial_j \left(a_{jk}(t,x)\,\partial_k u\right)\,, \tag{15}$$

et on suppose qu'il y a deux constants $0 < \lambda_0 \leq \Lambda_0$ telles que

$$\lambda_0\,|\xi|^2 \;\leq\; \sum_{j,k=1}^{N} a_{jk}(t,x)\,\xi_j\,\xi_k \;\leq\; \Lambda_0\,|\xi|^2$$

pour n'importe quel $(t,x) \in [0,T] \times \mathbb{R}^N$ et tout $\xi \in \mathbb{R}^N$. L'inégalité à gauche est la condition de stricte hyperbolicité, tandis que celle à droite dit que les coefficients de l'opérateur sont bornés.

C'est bien connu (voir par exemple [42] ou [53]) que, si les coefficients a_{jk} sont lipschitziens par rapport à t et seulement mesurable et bornés en x, alors le problème de Cauchy pour L a un caractère bien posé dans l'espace $H^1 \times L^2$. Donc, si les a_{jk} sont $\mathcal{C}^\infty$ et bornés avec toutes leurs dérivées par rapport à la variable d'espace, on peut retrouver le caractère bien posé en $H^{s+1} \times H^s$ pour tout $s \in \mathbb{R}$. De plus, pour n'importe quel $s \in \mathbb{R}$ on obtient (pour une certaine constante C_s dépendent seulement de s) l'estimation de l'énergie

$$\begin{aligned}\sup_{0\leq t\leq T}\left(\|u(t,\cdot)\|_{H^{s+1}} \;+\|\partial_t u(t,\cdot)\|_{H^s}\right) \;\leq \\ \leq\; C_s\left(\|u(0,\cdot)\|_{H^{s+1}}+\|\partial_t u(0,\cdot)\|_{H^s}+\int_0^T \|Lu(t,\cdot)\|_{H^s}\,dt\right)\end{aligned} \tag{16}$$

pour toute $u \in \mathcal{C}([0,T];H^{s+1}(\mathbb{R}^N)) \cap \mathcal{C}^1([0,T];H^s(\mathbb{R}^N))$ telle que $Lu \in L^1([0,T];H^s(\mathbb{R}^N))$.

En particulier, l'estimation (16) est vraie pour toute $u \in \mathcal{C}^2([0,T];H^\infty(\mathbb{R}^N))$, et ça implique que le problème de Cauchy pour L est bien posé dans H^∞ *avec aucune perte de dérivées.*

Si l'hypothèse de continuité Lipschitz (en temps) n'est pas satisfaite, alors (16) n'est plus vraie. Cependant, on peut encore retrouver le caractère bien posé dans H^∞, mais cette fois avec une perte de dérivées dans l'estimation de l'énergie. Cette perte ne peut pas être évitée, comme il est prouvé dans l'article [16]. En fait, les auteurs y prouvent que, si la régularité des coefficients est mesurée par un module de continuité, alors chaque module de continuité pire que le lipschitzien comporte toujours une perte de dérivées.

Le premier cas à considérer est quand les a_{jk} dépendent seulement du temps: alors, en utilisant la transformée de Fourier, on peut passer à l'espace des phases, où le problème devient une équation différentielle ordinaire. Dans l'article [18], Colombini, De Giorgi et Spagnolo ont supposé une condition log-Lipschitz intégrale, tandis que Tarama (voir [56]) a considéré la classe plus générale des fonctions log-Zygmund (intégrales). Sous toutes les deux hypothèses, on obtient une estimation de l'énergie *avec une perte constante de dérivées*: il y a une constante $\delta > 0$ telle que, pour n'importe quel $s \in \mathbb{R}$, on a l'inégalité

$$\begin{aligned}\sup_{0\leq t\leq T}\left(\|u(t,\cdot)\|_{H^{s+1-\delta}} \;+\|\partial_t u(t,\cdot)\|_{H^{s-\delta}}\right) \;\leq \\ \leq\; C_s\left(\|u(0,\cdot)\|_{H^{s+1}}+\|\partial_t u(0,\cdot)\|_{H^s}+\int_0^T \|Lu(t,\cdot)\|_{H^s}\,dt\right)\end{aligned} \tag{17}$$

pour toute $u \in \mathcal{C}^2([0,T];H^\infty(\mathbb{R}^N))$, pour une certaine constante C_s qui dépende seulement de s. L'idée originelle de Colombini, De Giorgi et Spagnolo était de régulariser les coefficients en utilisant un noyau de convolution, et après de relier le paramètre d'approximation ε avec la variable duale ξ: ainsi, ils ont exécuté une différente approximation dans de différentes zones de l'espace des phases. L'amélioration de Tarama, plutôt, a été obtenue en définissant une nouvelle énergie, qui concerne aussi les premières dérivées des coefficients lissés, de façon de compenser la pire régularité.

Le cas de dépendence en temps aussi bien qu'en espace a été considéré par Colombini et Lerner dans l'article [22]: ils ont supposé que les a_{jk} satisfaisaient une condition log-Lipschitz ponctuelle dans toutes les variables. Ils ont donc étudié le problème de Cauchy connexe, et ils ont trouvé une estimation de l'énergie *avec une perte de dérivées qui croît en temps*: pour chaque $s \in]0, 1/4]$, il y a des constantes positives β et C_s et un temps $T^* \in]0, T]$ tels que

$$(18) \quad \sup_{0 \leq t \leq T^*} \left(\|u(t, \cdot)\|_{H^{-s+1-\beta t}} + \|\partial_t u(t, \cdot)\|_{H^{-s-\beta t}} \right) \leq \\ \leq C_s \left(\|u(0, \cdot)\|_{H^{-s+1}} + \|\partial_t u(0, \cdot)\|_{H^{-s}} + \int_0^{T^*} \|Lu(t, \cdot)\|_{H^{-s-\beta t}} \, dt \right)$$

pour n'importe quelle $u \in \mathcal{C}^2([0, T]; H^\infty(\mathbb{R}^N))$. À cause de la dépendence des a_{jk} de x, maintenant ce n'était plus possible d'utiliser la transformée de Fourier pour passer à l'espace des phases. Pour surmonter ce problème, ils ont profité de la décomposition de Littlewood-Paley: ils ont défini une énergie localisée pour chaque composante $\Delta_\nu u$ de la solution u, et après ils ont exécuté une somme pesée sur ν pour définir l'énergie totale. Encore une fois, ils ont régularisé les cofficients en temps et ils ont lié le paramètre d'approximation avec ν, fait qui correspond exactement à choisir une différente approximation dans de différentes zones de l'espace des phases: il faut se rappeler que $|\xi| \sim 2^\nu$ sur le spectre de $\Delta_\nu u$. Un travail assez dur était exigé pour contrôler la norm (sur l'espace L^2) des opérateurs de commutation $[\Delta_\nu, a_{jk}]$ venant de l'équation pour la partie localisée $\Delta_\nu u$.

Plus récemment (voir l'article [19]) Colombini et Del Santo ont imposé une condition log-Zygmund ponctuelle par rapport au temps et une log-Lipschitz ponctuelle par rapport à l'espace, uniformément par rapport à l'autre variable. Ces hypthèses se traduisent de la façon suivante: il y a une constant K_0 telle que, pour n'importe quel $\tau > 0$ et quel $y \in \mathbb{R}^N \setminus \{0\}$, on a

$$(19) \qquad \sup_{(t,x)} |a_{jk}(t + \tau, x) + a_{jk}(t - \tau, x) - 2a_{jk}(t, x)| \leq K_0 \, \tau \, \log\left(1 + \frac{1}{\tau}\right)$$

$$(20) \qquad \sup_{(t,x)} |a_{jk}(t, x + y) - a_{jk}(t, x)| \leq K_0 \, |y| \, \log\left(1 + \frac{1}{|y|}\right).$$

Dans ce cas aussi, ils ont décomposé l'énergie dans des parties localisées, même si chacune d'elles était définie d'une nouvelle façon, en suivant l'idée originelle de Tarama, pour contrôler le mauvais comportement des coefficients par rapport au temps. De plus, la régularisation des coefficients par un noyau de convolution était exécutée et en temps, et en espace. Ainsi, ils ont obtenu un estimation de l'énergie analogue à (18) (et donc le caractère bien posé dans l'esapce H^∞ pour des cofficients a_{jk} assez réguliers par rapport à x), mais seulement dans le cas de dimension d'espace $N = 1$: en fait, ce n'était pas clair du tout comment définir une énergie de type Tarama (qui était, d'une certaine façon, nécessaire) en dimension plus grande.

Dans un premier temps, dans l'article [21] en collaboration avec Colombini, on a élargi le résultat de [19] au problème de Cauchy (encore en dimension $N = 1$) pour un opérateur hyperbolique du deuxième ordre complet

$$Pu := \partial_t^2 u - \partial_x (a(t, x) \, \partial_x u) + b_0(t, x) \, \partial_t u + b_1(t, x) \, \partial_x u + c(t, x) \, u ,$$

où, en plus des hypothèse (19) et (20), nous avons supposé aussi que b_0 et b_1 appartenaient à $L^\infty([0, T]; \mathcal{C}^\omega)$ (pour quelque $\omega \in]0, 1[$) et $c \in L^\infty([0, T] \times \mathbb{R})$.
Nous avons recouru aux idées principales du travail de Colombini et Del Santo. En particulier,

l'énergie associée à u était définie de la même façon, et on a traité les termes d'ordre le plus grand comme eux. Encore, on a trouvé une estimation de l'énergie du même type que (18): comme on peut s'attendre, la présence des termes d'ordre inférieur n'entraîne pas de problèmes considérables pour l'obtenir. Cependant, la régularité hölderienne des coefficients des termes du premier ordre entre en jeu dans l'analyse des commutateurs $[\Delta_\nu, b_j]$ (pour $j = 1\,,\,2$), et elle comporte une contrainte sur l'exposant de Sobolev s pour lequel (18) est vrai (voir aussi le théorème 2.1).

Dernièrement, dans [20] avec Colombini, Del Santo et Métivier, on a considéré l'opérateur (15) sous les hypothèses (19) and (20) dans n'importe quelle dimension $N \geq 1$. On fait remarquer que on s'est intéressé à un opérateur homogène pour n'alourdir pas trop nos calculs, mais la même technique marche en fait pour un opérateur du deuxième ordre complet. Nous sommes arrivés à trouver une estimation de l'énergie analogue à (18) (cette fois pour tout $s \in]0,1[$), et donc le caractère bien posé dans H^∞ (pour des a_{jk} de classe $\mathcal{C}^\infty_b$ dans l'espace).
Pour obtenir l'amélioration, nous sommes recourus à une nouvelle énergie: comme on vient de remarquer, la définition de Tarama n'admet pas une généralisation directe en dimension plus grande. Donc, on a fait appel encore à la définition originelle de Colombini et Del Santo, mais en remplaçant la multiplication par des fonctions $a(t,x)$ avec un module de continuité peu régulier par l'action des opérateurs paradifférentiels T_a, à elles associés (comme on a brièvement expliqué en dessus). On veut remarquer aussi que cette construction comporte déjà un effet de régularisation dans la variable d'espace, de façon que c'était suffisant d'exécuter la convolution avec un noyau lisse seulement par rapport au temps.
Toutefois, l'hypothèse de positivité sur a (exigée pour définir un problème strictement hyperbolique) ne se traduit pas, en général, dans la positivité de l'opérateur correspondant, qui est fondamentale dans les estimations de l'énergie. Ainsi, nous avons dû profiter du calcul paradifférentiel dépendant d'un paramètre $\gamma \geq 1$, comme développé par Métivier (voir [50]) et par Métivier et Zumbrun (voir l'article [52]). Cet outil nous permet d'exécuter une analyse plus fine: en particulier, on peut définir un opérateur de paraproduit à partir des frequences assez grandes, de façon qu'il soit un opérateur positif, si le symbole correspondant est positif.
On a dû aussi traiter avec de différentes classes d'espaces de Sobolev, de type logarithmique, déjà considérés par Colombini et Métivier dans [23]. Cela vient du fait que l'action des opérateurs paradifférentiels associés aux symboles log-Lipschitz (en x) et log-Zygmund (en t), tels que ceux qu'on considère dans notre problème strictement hyperbolique, comporte une perte de régularité logarithmique.

Système d'Euler incompressible à densité variable

Le système d'Euler incompressible à densité variable

$$(21)\qquad \begin{cases} \partial_t\rho \,+\, u\cdot\nabla\rho \,=\, 0 \\ \rho\,(\partial_t u \,+\, u\cdot\nabla u) \,+\, \nabla\Pi \,=\, \rho\, f \\ \operatorname{div} u \,=\, 0 \end{cases}$$

décrit l'évolution d'un fluide incompressible non-homogène et non-visqueux sous l'action d'une force externe $f = f(t,x) \in \mathbb{R}^N$. La fonction $\rho(t,x) \in \mathbb{R}_+$ représente la densité du fluide, $u(t,x) \in \mathbb{R}^N$ son champs de vitesses et $\Pi(t,x) \in \mathbb{R}$ sa pression. Le terme $\nabla\Pi$ peut être vu aussi comme le multiplicateur de Lagrange associé à la contrainte de divergence nulle sur la vitesse.

On suppose que la variable d'espace x appartient à l'espace entier $\mathbb{R}^N$, avec $N \geq 2$.

Le cas où le fluide est supposé homogène, c'est-à-dire $\rho \equiv \overline{\rho}$ constante (strictement positive) et le système devient

$$(22)\qquad \begin{cases} \partial_t u \,+\, u\cdot\nabla u \,+\, \nabla\Pi \,=\, 0 \\ \operatorname{div} u \,=\, 0\,, \end{cases}$$

a été étudié à fond et il y a une vaste littérature à lui consacrée.

Au contraire, pas beaucoup de travaux ont été consacrés à l'étude du cas non-homogène. De premiers résultats pour les équations (21) dans des domaines lisses de $\mathbb{R}^2$ ou $\mathbb{R}^3$ ont été obtenus par Beirão da Veiga et Valli pour des données initiales hölderiennes (voir les articles [5], [6] et [4]). Le cadre des espaces de Sobolev était considéré plutôt par Valli et Zajączowski dans [59], et par Itoh et Tani dans [46]. Dans l'article [45], Itoh a étudié l'évolution dans $\mathbb{R}^3$ entier, pour des données initiales $(\nabla\rho_0, u_0) \in H^2 \times H^3$, et Danchin (voir [27]) a agrandi les résultats à n'importe quelle dimension $N \geq 2$ et à tout espace de Sobolev avec un exposant de régularité assez grand.

Dans le même article [27], Danchin a considéré aussi le cas de données dans l'espaces de Besov limite $B^{1+N/2}_{2,1}$. Avant, Zhou (voir [63]) avait prouvé le caractère bien posé pour le système (21) dans les espaces $B^{1+N/p}_{p,1}$ pour tout $1 < p < +\infty$, mais en supposant aussi que la densité initiale ρ_0 était une petite perturbation d'un état constant.

Il faut noter que, dans le cas de $\mathbb{R}^N$ entier, l'hypothèse sur le cadre adopté implique presque toujours une condition L^2 sur le champs de vitesses, et si ce n'est pas le cas (comme, par exemple, dans le travail de Zhou qu'on vient de citer), des suppositions de petitesse sur la densité étaient demandées.

Il faut aussi remarquer que tous les résultats sur le caractère bien posé énoncé pour le système (21) sont seulement locaux en temps, même dans le cas deux-dimensionnel. En fait, comme on a déjà dit, pour $N = 2$ l'équation du tourbillon devient

$$\partial_t\omega \,+\, u\cdot\nabla\omega \,+\, \nabla\left(\frac{1}{\rho}\right)\wedge\nabla\Pi \,=\, 0\,, \tag{23}$$

donc on ne peut pas s'attendre la conservation des normes de Lebesgue, fait qui était la clé pour le résultat d'existence globale en temps, à cause du terme de densité.

Récemment (voir l'article [28]), Danchin a prouvé le résultat sur le caractère bien posé du système (21) dans tout espace de Besov $B^s_{p,r}$, avec $1 < p < +\infty$, contenu dans l'ensemble $\mathcal{C}^{0,1}$ des fonctions globalement lipschitiziennes. En fait, notre système est essentiellement un couplage de deux équations de transport par le champ de vitesses u: alors, aucun gain de régularité peut être attendu, tandis que, si on veut la préserver, il faut demander que u soit au moins localement lipschitzienne par rapport à la variable d'espace. Donc, Danchin a demontré que le cadre fonctionnel convenable pour l'étude de (21) est le même que pour les équations (22). De plus, il est arrivé à obtenir son résultat pour n'importe quel état de densité initial, sans de conditions addittionelles de petitesse. De toute façon, il a dû supposé que le champ de vitesses u appartenait à L^2 pour traiter le terme de pression. En fait, dans le cas de densité non constante, $\nabla\Pi$ satisfait une équation elliptique (en forme de divergence) à coefficients peu réguliers,

$$-\operatorname{div}\,(a\,\nabla\Pi) \,=\, \operatorname{div} F \tag{24}$$

(ici on a posé $a := 1/\rho$), et elle peut être résolue de façon indépendente de a seulement dans l'espace d'énergie L^2. Le contrôle sur la norme L^2 (ou, en général, L^p) de $\nabla\Pi$ était nécessaire aussi pour borner sa norme de Besov.

Toutefois, la condition $u_0 \in L^2$ est, d'une certaine manière, assez restrictive: dans le cas deux-dimensionnel, elle implique que le tourbillon doit avoir moyenne nulle sur $\mathbb{R}^2$, et ce fait nous empêche de considérer, par exemple, des structures de type poches de tourbillon. Donc, Danchin a prouvé aussi le caractère bien posé (dans n'importe quelle dimension $N \geq 2$) pour des données dans des espaces $B^s_{p,r} \hookrightarrow \mathcal{C}^{0,1}$, avec $2 \leq p \leq 4$. Aucune hypothèse d'énergie finite était formulée, même si la condition précédente nous permet de retrouver encore $\nabla\Pi \in L^2$. En particulier, ce résultat s'applique (grâce à la loi de Biot-Savart) à tout champ de vitesses assez lisse dont le tourbillon est à support compact.

Dans le même travail, Danchin a prouvé aussi un critère de continuation dans le même esprit de celui, bien connu, de Bealo, Kato et Majda (voir l'article [3]). La condition qui permet de prolonger les solutions au-delà d'un certain temps T est

$$\int_0^T \left(\|\nabla u(t)\|_{L^\infty} \,+\, \|\nabla\Pi(t)\|_{B^{s-1}_{p,r}} \right) dt \,<\, +\infty \,,$$

et, dans le cas $s > 1 + N/p$, c'est possible de remplacer ∇u avec le tourbillon Ω.

Finalement, Danchin a attaqué aussi le cas des espaces $B^s_{\infty,r} \hookrightarrow \mathcal{C}^{0,1}$, mais en demandant en plus que u_0 appartenait à L^p (pour quelques $1 < p < +\infty$) et que ρ_0 était proche (dans la norme $B^s_{\infty,r}$) à un état constant $\overline{\rho}$. Sous ces hypothèses supplémentaires, l'équation pour le terme de pression peut s'écrire (comme déjà fait dans l'article [63] de Zhou) dans la forme

$$-\,\overline{a}\,\Delta\Pi \,=\, \mathrm{div}\, F \,+\, \mathrm{div}\,\left((a-\overline{a})\nabla\Pi\right)\,,$$

avec $\overline{a} := 1/\overline{\rho}$. Alors, en utilisant les estimations L^p habituelles pour l'opérateur de Laplace (qui sont valables pour tout $1 < p < +\infty$) et l'hypothèse de petitesse sur la densité, on peut absorber le dernier terme du membre de droite de l'équation précédente.

Dans l'article [29] en collaboration avec Danchin, on s'est concentré sur ce dernier cas: on est arrivé à élargir le résultat sur le caractère bien posé dans $B^s_{\infty,r}$ sans supposer aucune condition de petitesse sur la densité initiale. Il faut remarquer que ce cadre fonctionnel est assez intéressant, car il comprend aussi les cas particuliers des espaces de Hölder du type $\mathcal{C}^{1,\alpha}$ et l'espaces de Besov limite $B^1_{\infty,1}$, qui est le plus grand contenu dans $\mathcal{C}^{0,1}$, et donc le plus grand où on peut s'attendre à retrouver le caractère bien posé pour le système (21). Bien sûr, on a dû supposer que u_0 était dans L^2, pour avoir l'existence d'une solution $\nabla\Pi$ de (24). L'amélioration par rapport au résultat précédent a été obtenue grâce à une différente méthode utilisée pour traiter le terme de pression, qui en fait marche pour n'importe quel $p \in [1, +\infty]$. En particulier, on a séparé $\nabla\Pi$ dans les basses et les hautes fréquences, en utilisant la décomposition de Littlewood-Paley. Les basses fréquences pouvaient être contrôlées par la norme de Lebesgue, tandis que les hautes étaient bornées par la norme de $\Delta\Pi$, qui satisfait l'équation

$$-\,\Delta\Pi \,=\, \nabla\left(\log a\right)\cdot\nabla\Pi \,+\, \frac{1}{a}\,\mathrm{div}\left(f + u\cdot\nabla u\right)\,.$$

Le premier terme de la rélation précédente est d'ordre inférieur, et donc il peut être absorbé en interpolant entre les estimations L^2 et celles, d'ordre plus grand, pour l'opérateur de Laplace. Nous étions aussi capables de formuler un critère de prolongement analogue à celui de Danchin pour le cas $B^s_{p,r}$, $1 < p < +\infty$. De plus, pour les mêmes raisons qu'on a expliquées tout à l'heure, on a considéré aussi des données ayant énergie infinie: dans ce cas, le tourbillon (qui était une quantité fondamentale dans le cas de densité constante) est entré en jeu via la loi de Biot-Savart. On a encore supposé des propriétés d'intégrabilité pour u_0 et son tourbillon Ω_0, afin d'assurer que le terme de pression appartenait à L^2, une condition qu'on n'est pas arrivé à éviter. Aussi sous ces hypothèses, nous avons trouvé le caractère bien posé pour les équations (21). En particulier, ce résultat s'applique (comme dans le cas analogue considéré par Danchin) à tout champ de vitesses dont le tourbillon est assez lisse et à support compact.

Comme on a déjà remarqué, tous les résultats obtenus étaient seulement locaux en temps. Cependant, on était aussi capable de donner une estimation pour le temps de vie des solutions. Dans un premier moment, on a montré que, dans n'importe quelle dimension d'espace, si la vitesse initiale est d'ordre ε (avec ε assez petit), alors le temps d'existence est au moins d'ordre ε^{-1}. Dans ce cas, aucune limitation sur la non-homogénéité est demandée: le résultat est une conséquence directe des propriétés de redimensionnement pour les équations (21). Après, grâce à l'égalité (23) et à des estimations plus précises pour l'équation de transport (récemment établies par Vishik dans [60] et généralisées par T. Hmidi et S. Keraani dans [41]), on a montré que le temps de vie de la solution tend à l'infini si $\rho_0 - 1$ devient proche de 0. Plus exactement, si

$$\|\rho_0 \,-\, 1\|_{B^1_{\infty,1}} \,=\, \varepsilon \qquad\qquad \text{et} \qquad\qquad \|\omega_0\|_{B^0_{\infty,1}} \,+\, \|u_0\|_{L^2} \,=\, U_0$$

avec ε assez petit, alors le temps de vie est au moins d'ordre $U_0^{-1}\log(\log\varepsilon^{-1})$.

Cela fait, dans [35] on a étudié le problème de la propagation de la régularité stratifiée et conormale pour les solutions de (21), dans toute dimension $N \geq 2$. On a supposé que la vitesse initiale u_0 et son tourbillon Ω_0 étaient dans quelques espaces de Lebesgue, afin d'assurer, encore une fois, $\nabla\Pi \in L^2$. On a supposé aussi des propriétés de type géometrique pour Ω_0. De plus, on a demandé que la densité initiale ρ_0 était bornée avec son gradient et qu'elle satisfaisait des conditions géomotriques analogues à celles pour Ω_0. Sous ces hypothèses, on est arrivé à prouver l'existence (bien sûr, localement en temps) et l'unicité des solutions de (21), et la persistance de ces structures géometriques. Il faut remarquer qu'on n'a pas demandé expressément que le champ de vitesses était lipschitizien. Cette propriété suit de la régularité stratifiée pour le tourbillon, et elle est vraie comme dans le cas homogène. En fait, sa preuve repose seulement sur la loi de Biot-Savart, et donc rien ne change par rapport au cas classique: en particulier, aucune condition en plus était demandée. Il faut dire aussi que les structures géometriques se propagent au champ de vitesses et au terme de pression.
Dans le même travail, on a donné une estimation du dessous pour le temps de vie de la solution en termes seulement des données initiales, dans toute dimension $N \geq 2$. Dans le cas de densité constante, il faut se rappeler qu'il était donné (à moins de multiplication par des constantes) par

$$T_{cl} \,:=\, \left(\|\Omega_0\|_{L^q\cap L^\infty}\;\log\left(e+\frac{\|\Omega_0\|_{\mathcal{C}^\varepsilon_{X_0}}}{\|\Omega_0\|_{L^q\cap L^\infty}}\right)\right)^{-1}$$

(voir l'article [26]). Dans le cas non-homogène, au contraire, nous avons trouvé qu'il est borné par

$$T_{nh} \,:=\, \left(V'(0)\,(1+\|\nabla\rho_0\|_{L^\infty})^{3+\delta}\left(1+R_0+\Gamma_0^{7/3}\right)\right)^{-1}\,,$$

où l'exposant $\delta > 1$ vient des estimations pour le terme de pression, les quantités R_0 et Γ_0 sont liées aux propriétés géometriques des données initiales et on a défini

$$V'(0) \,:=\, \|u_0\|_{L^p}+\|\Omega_0\|_{L^q\cap L^\infty}+\|\Omega_0\|_{\mathcal{C}^\varepsilon_{X_0}} \,\geq\, c\,\|\Omega_0\|_{L^q\cap L^\infty}\;\log\left(e+\frac{\|\Omega_0\|_{\mathcal{C}^\varepsilon_{X_0}}}{\|\Omega_0\|_{L^q\cap L^\infty}}\right).$$

Alors, à moins de multiplication par des constantes, $T_{nh} \leq T_{cl}$. Nous voulons remarquer qu'on a laissé tomber la dépendence logarithmique dans l'estimation de la norme Lipschitz du champ de vitesses pour simplifier les calculs, mais peut-être ce résultat est assez approximatif.
Finalement, dans les cas physiques $N = 2$ ou 3, on est arrivé à raffiner notre résultat sur la régularité conormale: si l'hypersurface initiale Σ_0 est aussi connèxe, alors elle définit un domaine borné $D_0 \subset \mathbb{R}^N$ dont elle est le contour, et cette propriété se propage en temps (car le flot de la solution est un difféomorphisme). Par analogie avec la structure des poches de tourbillon, nous voulions étudier la dynamique dans le domaine borné. Évidemment, même en dimension $N = 2$, même si la donnée initiale est un poche de tourbillon, on ne peut pas s'attendre à conserver cette propriété, à cause de la presence du terme de densité dans l'équation (23). Cependant, nous avons prouvé que, si la densité et le tourbillon initials sont hölderiens à l'intérieur du domaine D_0 (et ils satisfont des hypothèses globales pour garantir la persistance des propriétés conormales), alors leur régularité est préservée dans l'évolution temporelle. La difficulté la plus grande était de prouver que la continuité hölderienne se propageait aussi au champ de vitesses et au terme de pression: enfin, on devait prouver que ces deux quantités étaient assez régulières à la frontière du domaine D_0 transporté par le flot. Maintenant, ça était garanti par la régularité conormale qu'on venait de prouver.

La thèse est structurée de la façon suivante.

Dans la première partie, on va présenter les outils de l'Analyse de Fourier dont on aura besoin dans notre étude. On donne un cadre assez général de la théorie classique de Littlewood-Paley. On va rappeler la definition et les propriétés principales des espaces de Besov non-homogènes, et on va citer aussi quelques résultats fondamentaux à l'égard du calcul paradifférentiel. Pour faire ça, on va suivre fidèlement la présèntation donnée dans [2], et, comme ces résultats sont classiques, on va omettre leurs preuves.

Après, on va considérer les espaces de Besov logarithmiques, et on va montrer que les propriétés précédentes sont encore valables (à moins des changements appropriés) aussi pour cette classe. Cette fois, on va donner tous les détails des preuves.

Dans la dernière partie de ce chapitre, on va expliquer les idées principales du calcul paradifférentiel à paramètre, en suivant principalement la présentation de [52], et on va citer quelques résultats dont on aura besoin dans la suite.

Après, on va considérer le problème des opérateurs du deuxième ordre strictement hyperboliques à coefficients peu réguliers.

Dans le chapitre 2 on va analyser le cas d'un opérateur complet en dimension d'espace égale à un. Ce chapitre contient les résultats prouvés dans l'article [21] en collaboration avec Colombini.

Dans le chapitre 3, plutôt, on va présenter les résultats obtenus dans [20] avec Colombini, Del Santo et Métivier. On élargit le précédent au cas de n'importe quelle dimension d'espace, mais (pour simplifier les calculs) seulement pour un opérateur homogène, c'est-à-dire sans les termes d'ordre inférieur.

Dans chaqu'une de ces sections, on va introduire aussi quelques outils en plus que ceux qu'on a présentés dans le chapitre 1. En particulier, on va analyser les propriétés des fonctions ayant un module de continuité peu régulier et celles des opérateurs paradifférentiels correspodants.

Finalement, la dernière partie de la thése est dévouée à l'étude des équations d'Euler incompressibles à densité variable. Le chapitre 4 est consacré à la preuve du caractère bien posé dans des espaces de Besov limites, contenus dans l'espace des fonctions globalement lipschitziennes. On va aussi donner un critère de prolongement et une borne du dessous pour le temps de vie des solutions. Ce chapitre est basé sur l'article [29] en collaboration avec Danchin.

Après, on va considérer le problème de la propagation des structures géometriques pour ce système. On va se concentrer sur la régularité stratifiée et celle conormale, et sur la propagation de la continuité hölderienne à l'intérieur d'un domaine borné de $\mathbb{R}^2$ ou $\mathbb{R}^3$ (comme expliqué avant). Ceci est le sujet du chapitre 5, et il contient les résultats prouvés dans l'article [35].

Part I

Fourier Analysis methods for Partial Differential Equations

Chapter 1

Littlewood-Paley Theory

This chapter is devoted to the presentation of the main tools, mostly based on Fourier Analysis, we will need in the study of some partial differential equations.

The basic idea is to split a tempered distribution into a sum of smooth functions, whose Fourier transform is compactly supported in a ball or an annulus and which have, due to this spectral localization, very nice properties. This will be explained in the first section.
Then, taking advantage of the previous decomposition, we will introduce the class of (non-homogeneous) Besov spaces and we will recall its main properties.
Next section will be devoted to the classical paradifferential calculus. In particular, we will analyse the product of two tempered distributions using the well-known paraproduct decomposition, introduced first by J.-M. Bony in the paper [8]. Moreover, we will consider also composition of Besov functions by a smooth one.

Unless otherwise specified, one can find the proof of all the results quoted up to now in chapter 2 of [2] (see also [51], chapter 4).

In section 1.4, instead we will introduce a new class of Besov spaces, in a certain sense of logarithmic type. We will see that most of the classical results holds true also in this new setting. As this part is not classical, we will give also the details of the proofs.

Finally, last section of this chapter is devoted to paradifferential calculus depending on a parameter $\gamma \geq 1$ and to more general classes of paradifferential operators. These topics have been already introduced in paper [52], appendix B (see also [50]): we will essentially follow it in our presentation. However, we will allow symbols to have also a logarithmic growth, and we will analyse their action on the class of logarithmic Besov spaces, introduced before.

1.1 Littlewood-Paley decomposition

Let us define the so called *Littlewood-Paley decomposition*, based on a dyadic partition of unity with respect to the Fourier variable.

So, fix a smooth radial function χ supported in (say) the ball $B(0,8/5)$, equal to 1 in a neighborhood of $B(0,5/4)$ and such that $r \mapsto \chi(r\,e)$ is nonincreasing over $\mathbb{R}_+$ for all unitary vector $e \in \mathbb{R}^N$. Moreover, set $\varphi(\xi) = \chi(\xi) - \chi(2\xi)$: obviously, its support is contained in the annulus $\mathcal{C} = \left\{\xi \in \mathbb{R}^N \mid 5/8 \leq |\xi| \leq 8/5\right\}$.

Now we quote some fundamental properties, which are easy to verify. First of all,

$$\forall\, \xi \in \mathbb{R}^N, \quad \chi(\xi) + \sum_{j\geq 1} \varphi\left(2^{-j}\xi\right) = 1 \qquad \text{and} \qquad \forall\, \xi \in \mathbb{R}^N \backslash \{0\}, \quad \sum_{j\in\mathbb{Z}} \varphi\left(2^{-j}\xi\right) = 1\,.$$

Moreover, if we define the annulus $\widetilde{\mathcal{C}} = \mathcal{C} + B(0,2/5)$, we have $2^j\widetilde{\mathcal{C}} \cap 2^k\mathcal{C} = \emptyset$ for all $|j-k| \geq 3$, and we have also

$$|j-k| \geq 2 \implies \operatorname{supp}\varphi\left(2^{-j}\cdot\right) \cap \operatorname{supp}\varphi\left(2^{-k}\cdot\right) = \emptyset$$
$$j \geq 2 \implies \operatorname{supp}\chi \cap \operatorname{supp}\varphi\left(2^{-j}\cdot\right) = \emptyset\,.$$

For convenience, set $g = \mathcal{F}^{-1}\chi$ and $h = \mathcal{F}^{-1}\varphi$. The dyadic blocks $(\Delta_j)_{j\in\mathbb{Z}}$ are defined by[1]

$$\Delta_j := 0 \ \text{ if } \ j \leq -1, \quad \Delta_0 := \chi(D) = g * \cdot \quad \text{and} \quad \Delta_j := \varphi(2^{-j}D) = 2^{jN}\, h\left(2^j\cdot\right) * \cdot \ \text{ if } \ j \geq 0\,,$$

while the low frequency cut-off operator is defined as

$$S_j u := \chi(2^{-j+1}D) = \sum_{k \leq j-1} \Delta_k = 2^{(j-1)N}\, g\left(2^{j-1}\cdot\right) * \cdot \quad \text{for} \quad j \geq 1\,.$$

The above defined operators map L^p into L^p (for all $p \in [1,+\infty]$) continuously, with norms independent of the indeces j and p. Moreover, formally we have the decomposition $\mathrm{Id} = \sum_j \Delta_j$, which makes sense in $\mathcal{S}'$, as next result says.

Proposition 1.1. *For any $u \in \mathcal{S}'$, one has $u = \lim_{j\to+\infty} S_j u$ in $\mathcal{S}'$.*

For the sequel, it's important to understand properties of spectrally localized functions[2]: hence, the following two lemmas will be fundamental. The former one describes, by the so called *Bernstein's inequalities*, the way derivatives take effect on such a class of functions, while the latter concerns the action of Fourier multipliers.

Lemma 1.2. *Let $0 < r < R$. A constant C exists so that, for any nonnegative integer k, any couple (p,q) in $[1,+\infty]^2$ with $p \leq q$ and any function $u \in L^p$, we have, for all $\lambda > 0$,*

$$\operatorname{supp}\widehat{u} \subset B(0,\lambda R) \implies \|\nabla^k u\|_{L^q} \leq C^{k+1}\,\lambda^{k+N\left(\frac{1}{p}-\frac{1}{q}\right)}\,\|u\|_{L^p}$$
$$\operatorname{supp}\widehat{u} \subset \{\xi \in \mathbb{R}^N \,|\, r\lambda \leq |\xi| \leq R\lambda\} \implies C^{-k-1}\,\lambda^k\,\|u\|_{L^p} \leq \|\nabla^k u\|_{L^p} \leq C^{k+1}\,\lambda^k\,\|u\|_{L^p}\,.$$

Lemma 1.3. *Let $\mathcal{C}$ be an annulus, $m \in \mathbb{R}$ and σ be a smooth function on $\mathbb{R}^N\backslash\{0\}$ such that, for any $\alpha \in \mathbb{N}^N$, there exists a constant C_α for which*

$$\forall\, \xi \in \mathbb{R}^N\,, \qquad |\partial^\alpha \sigma(\xi)| \leq C_\alpha\,|\xi|^{m-|\alpha|}\,.$$

Then, there exists a constant C, depending only on the C_α for $|\alpha| \leq N+2$, such that, for any $p \in [1,+\infty]$, any $\lambda > 0$ and any function $u \in L^p$ spectrally supported in $\lambda\,\mathcal{C}$, we have

$$\|\sigma(D)u\|_{L^p} \leq C\,\lambda^m\,\|u\|_{L^p}\,.$$

1.2 Non-homogeneous Besov spaces

Using the Littlewood-Paley decomposition, one can define what a Besov space $B^s_{p,r}$ is.

Definition 1.4. Let u be a tempered distribution, s a real number, and $1 \leq p,r \leq +\infty$. The non-homogeneous Besov space $B^s_{p,r}$ is defined as the subset of tempered distributions u for which

$$\|u\|_{B^s_{p,r}} := \left\|\left(2^{js}\,\|\Delta_j u\|_{L^p}\right)_{j\in\mathbb{N}}\right\|_{\ell^r} < +\infty\,.$$

[1] Throughout we agree that $f(D)$ stands for the pseudo-differential operator $u \mapsto \mathcal{F}^{-1}(f\,\mathcal{F}u)$.
[2] Recall that the *spectrum* of a tempered distribution is the support of its Fourier transform

From the above definition, it is easy to show that for all $s \in \mathbb{R}$, the Besov space $B^s_{2,2}$ coincides with the non-homogeneous Sobolev space H^s, while for all $s \in \mathbb{R}_+ \backslash \mathbb{N}$, the space $B^s_{\infty,\infty}$ is actually the Hölder space $\mathcal{C}^s$ (see e.g. [51], chapter 4, for the proof of these two facts).

If $s \in \mathbb{N}$, instead, we set $\mathcal{C}^s_* := B^s_{\infty,\infty}$, to distinguish it from the space $\mathcal{C}^s$ of the differentiable functions with continuous partial derivatives up to the order s. Moreover, the strict inclusion $\mathcal{C}^s_b \hookrightarrow \mathcal{C}^s_*$ holds, where $\mathcal{C}^s_b$ denotes the subset of $\mathcal{C}^s$ functions bounded with all their derivatives up to the order s.

If $s < 0$, we define the "negative Hölder space" $\mathcal{C}^s$ as the Besov space $B^s_{\infty,\infty}$.

Finally, let us also point out that for any $k \in \mathbb{N}$ and $p \in [1,+\infty]$, we have the following chain of continuous embeddings:

$$B^k_{p,1} \hookrightarrow W^{k,p} \hookrightarrow B^k_{p,\infty}\,,$$

where $W^{k,p}$ denotes the set of L^p functions with derivatives up to order k in L^p.

Besov spaces have many nice properties which will be recalled in a while. For the time being, let us just mention that if the condition

$$s > 1 + \frac{N}{p} \qquad \text{or} \qquad s = 1 + \frac{N}{p} \quad \text{and} \quad r = 1 \tag{1.1}$$

holds true, then $B^s_{p,r}$ is an algebra continuously embedded in the set $\mathcal{C}^{0,1}$ of bounded Lipschitz functions, and that (by Bernstein's inequalities) the gradient operator maps $B^s_{p,r}$ in $B^{s-1}_{p,r}$.

First of all, let us show that definition 1.4 is independent of the choice of the cut-off functions defining the Littlewood-Paley decomposition.

Lemma 1.5. *Let $\mathcal{C} \subset \mathbb{R}^N$ be an annulus, $s \in \mathbb{R}$ and $(p,r) \in [1,+\infty]^2$. Let $\left(u_j\right)_{j\in\mathbb{N}}$ be a sequence of smooth functions such that*

$$\operatorname{supp}\widehat{u}_j \,\subset\, 2^j\,\mathcal{C} \qquad \textit{and} \qquad \left\|\left(2^{js}\,\|u_j\|_{L^p}\right)_{j\in\mathbb{N}}\right\|_{\ell^r} \,<\, +\infty\,.$$

Then $u := \sum_{j\in\mathbb{N}} u_j$ belongs to $B^s_{p,r}$ and

$$\|u\|_{B^s_{p,r}} \,\leq\, C_s\,\left\|\left(2^{js}\,\|u_j\|_{L^p}\right)_{j\in\mathbb{N}}\right\|_{\ell^r}\,.$$

Bernstein's inequalities immediately imply the following embedding result.

Proposition 1.6. *The space $B^{s_1}_{p_1,r_1}$ is continuously embedded in the space $B^{s_2}_{p_2,r_2}$ for all indices satisfying $1 \leq p_1 \leq p_2 \leq +\infty$ and*

$$s_2 \,<\, s_1 - N\left(\frac{1}{p_1}-\frac{1}{p_2}\right) \qquad \textit{or} \qquad s_2 \,=\, s_1 - N\left(\frac{1}{p_1}-\frac{1}{p_2}\right) \;\; \textit{and} \;\; 1 \leq r_1 \leq r_2 \leq +\infty\,.$$

The following statement is of great importance for proving existence results for partial differential equations in the Besov spaces framework.

Theorem 1.7. *The set $B^s_{p,r}$ is a Banach space.*
Moreover, it satisfies the Fatou property: if $\left(u_j\right)_{j\in\mathbb{N}}$ is a bounded sequence of $B^s_{p,r}$, then there exist an element $u \in B^s_{p,r}$ and a subsequence $\left(u_{\psi(j)}\right)_{j\in\mathbb{N}}$ such that

$$\lim_{j\to+\infty} u_{\psi(j)} \,=\, u \;\; \textit{in} \;\; \mathcal{S}' \qquad \textit{and} \qquad \|u\|_{B^s_{p,r}} \,\leq\, C\,\liminf_{j\to+\infty}\left\|u_{\psi(j)}\right\|_{B^s_{p,r}}\,.$$

Let us also quote a density result.

Lemma 1.8. *Let $r < +\infty$. For all $u \in B^s_{p,r}$ we have*

$$\lim_{j\to+\infty}\|u \,-\, S_j u\|_{B^s_{p,r}} \,=\, 0\,.$$

Proposition 1.9. *Fix p and r finite.*
Then the space of test functions $\mathcal{D}(\mathbb{R}^N)$ is densely embedded in $B^s_{p,r}(\mathbb{R}^N)$.

Remark 1.10. If $r=+\infty$, instead, the closure of $\mathcal{D}(\mathbb{R}^N)$ for the $B^s_{p,r}$ norm is the subset of tempered distributions $u\in\mathcal{S}'$ such that $\lim_{j\to+\infty} 2^{js}\,\|\Delta_j u\|_{L^p}\,=\,0$.

We introduce now the following notation:

- for any index $q\in[1,+\infty]$, we denote with q' its conjugate exponent, i.e. $q'\in[1,+\infty]$ is defined by the relation $(1/q)\,+\,(1/q')\,=\,1$;
- the symbol $(\,\cdot\,,\,\cdot\,)_{L^2}$ will indicate the scalar product in L^2.

Let us recall also some duality properties of Besov spaces.

Proposition 1.11. *For all $s\in\mathbb{R}$ and $(p,r)\in[1,+\infty]^2$, the map*

$$\begin{array}{ccl} B^s_{p,r}\times B^{-s}_{p',r'} & \longrightarrow & \mathbb{R} \\ (u\,,\,\phi) & \longmapsto & \langle u,\phi\rangle\,:=\,\displaystyle\sum_{|j-k|\leq 1}(\Delta_j u,\Delta_k\phi)_{L^2} \end{array}$$

defines a continuous bilinear functional on the space $B^s_{p,r}\times B^{-s}_{p',r'}$. Moreover, for all $u\in\mathcal{S}'$

$$\|u\|_{B^s_{p,r}}\,\leq\,C\sup_{\phi\in\mathcal{S},\|\phi\|_{B^{-s}_{p',r'}}\leq 1}\langle u,\phi\rangle\,.$$

More generally, the space $B^{-s}_{p',r'}$ can be identified with the dual space of the completion of $\mathcal{D}$ for the $B^s_{p,r}$ norm.

Now let us consider the action of Fourier multipliers on non-homogeneous Besov spaces. First of all, a definition is in order.

Definition 1.12. A smooth function $f\,:\,\mathbb{R}^N\longrightarrow\mathbb{R}$ is said to be a *S^m-multiplier* if, for all multi-index $\alpha\in\mathbb{N}^N$, there exists a constant C_α such that

$$\forall\,\xi\in\mathbb{R}^N\,,\qquad|\partial^\alpha f(\xi)|\,\leq\,C_\alpha\,\left(1+|\xi|\right)^{m-|\alpha|}\,.$$

Proposition 1.13. *Let $m\in\mathbb{R}$ and f be a S^m-multiplier.*
Then for all $s\in\mathbb{R}$ and $(p,r)\in[1,+\infty]^2$ the operator $f(D)$ maps $B^s_{p,r}$ into $B^{s-m}_{p,r}$ continuously.

Next statement considers instead the case of homogeneous (away from the origin) multipliers: it will be useful in part III.

Proposition 1.14. *Let $F\,:\,\mathbb{R}^N\,\to\,\mathbb{R}$ be a smooth function. Let us suppose also that F is homogeneous of degree $m\in\mathbb{R}$ away from a neighborhood of the origin: there exists a real number $\varrho>0$ such that*

$$\forall\,|\xi|\geq\varrho\,,\;\forall\,\lambda>0\,,\qquad F(\lambda\xi)\,=\,\lambda^m\,F(\xi)\,.$$

Then for all $(p,r)\in[1,+\infty]^2$ and all $s\in\mathbb{R}$, the operator $F(D)$ maps $B^s_{p,r}$ in $B^{s-m}_{p,r}$.

Remark 1.15. Let $\mathcal{P}$ be the Leray projector over divergence free vector fields and $\mathcal{Q}:=\mathrm{Id}-\mathcal{P}$. Recall that, in Fourier variables, for all vector fields u we have

$$\widehat{\mathcal{Q}u}(\xi)\,=\,\frac{\xi}{|\xi|^2}\,\xi\cdot\widehat{u}(\xi)\,.$$

Therefore, both $(\mathrm{Id}\,-\,\Delta_0)\mathcal{P}$ and $(\mathrm{Id}\,-\,\Delta_0)\mathcal{Q}$ satisfy the assumptions of the above proposition with $m=0$, hence they are self-maps on $B^s_{p,r}$ *for any $s\in\mathbb{R}$ and $1\leq p,r\leq+\infty$.*

Now, let us state a characterization of Besov spaces with negative index of regularity in terms of the low frequencies cut-off operators.

Proposition 1.16. *There exists a constant C for which the following result holds. Let $s<0$, $(p,r)\in[1,+\infty]^2$ and $u\in\mathcal{S}'$.*

Then $u\in B^s_{p,r}$ if and only if the sequence $\left(2^{js}\,\|S_ju\|_{L^p}\right)_{j\in\mathbb{N}}\in\ell^r$. Moreover,

$$C^{-|s|+1}\,\|u\|_{B^s_{p,r}}\;\leq\;\left\|\left(2^{js}\,\|S_ju\|_{L^p}\right)_{j\in\mathbb{N}}\right\|_{\ell^r}\;\leq\;C\,\left(1+\frac{1}{|s|}\right)\,\|u\|_{B^s_{p,r}}\,.$$

Finally, let us conclude this section with a fundamental interpolation result.

Theorem 1.17. *There exists a constant C such that, for any two real numbers $s_1<s_2$, any $\theta\in]0,1[$ and any $(p,r)\in[1,+\infty]^2$, one has*

$$\begin{aligned}
\|u\|_{B^{\theta s_1+(1-\theta)s_2}_{p,r}} &\leq \|u\|^{\theta}_{B^{s_1}_{p,r}}\,\|u\|^{1-\theta}_{B^{s_2}_{p,r}}\\
\|u\|_{B^{\theta s_1+(1-\theta)s_2}_{p,1}} &\leq \frac{C}{s_2-s_1}\left(\frac{1}{\theta}+\frac{1}{1-\theta}\right)\|u\|^{\theta}_{B^{s_1}_{p,\infty}}\,\|u\|^{1-\theta}_{B^{s_2}_{p,\infty}}\,.
\end{aligned}$$

1.2.1 Time-dependent Besov spaces

Littlewood-Paley decomposition plays a fundamental role in the analysis of partial differential equations in the Besov spaces framework.

The standard procedure lies in writing the equation for each localized part of the solution, then estimating it in some space $L^\varrho([0,T];L^p)$ using classical results for smooth solutions and finally performing a weighted ℓ^r summation.
In this way, however, one doesn't get estimates in the space $L^\varrho([0,T];B^s_{p,r})$, because the time integration comes before the summation. So, the following definition gains relevance.

Definition 1.18. Given a $T>0$, $s\in\mathbb{R}$ and $(p,r)\in[1,+\infty]^2$, we define the space $\widetilde{L}^\varrho_T(B^s_{p,r})$ as the subset of tempered distributions u over $]0,T[\times\mathbb{R}^N$ such that

$$\|u\|_{\widetilde{L}^\varrho_T(B^s_{p,r})}\;:=\;\left\|\left(2^{js}\,\|\Delta_ju\|_{L^\varrho_T(L^p)}\right)_{j\in\mathbb{N}}\right\|_{\ell^r}\;<\;+\infty\,.$$

The previous definition was introduced first in paper [15] in the Sobolev spaces framework, and then in [14] for the general Besov classe.

The relation between this space and the classical $L^\varrho_T(B^s_{p,r}):=L^\varrho([0,T];B^s_{p,r})$ can be easily got by Minkowski's inequality: one has

$$\begin{cases}
\|u\|_{\widetilde{L}^\varrho_T(B^s_{p,r})}\;\leq\;\|u\|_{L^\varrho_T(B^s_{p,r})} & \text{if}\quad \varrho\leq r\\
\|u\|_{\widetilde{L}^\varrho_T(B^s_{p,r})}\;\geq\;\|u\|_{L^\varrho_T(B^s_{p,r})} & \text{if}\quad \varrho\geq r\,.
\end{cases}$$

Paradifferential calculus results, such as (for instance) continuity of composition and of paraproduct and remainder operators, hold true also for this class of Besov spaces. One has to pay attention only to the time exponent ϱ, which follows the rules of Hölder inequality. For instance, as we will see, we have

$$\|u\,v\|_{\widetilde{L}^\varrho_T(B^s_{p,r})}\;\leq\;C\left(\|u\|_{L^{\varrho_1}_T(L^\infty)}\,\|v\|_{\widetilde{L}^{\varrho_2}_T(B^s_{p,r})}\;+\;\|u\|_{\widetilde{L}^{\varrho_3}_T(B^s_{p,r})}\,\|v\|_{L^{\varrho_4}_T(L^\infty)}\right)\,,$$

with $1/\varrho\,=\,1/\varrho_1\,+\,1/\varrho_2\,=\,1/\varrho_3\,+\,1/\varrho_4$.

1.2.2 Homogeneous Besov spaces

For completeness, let us spend a few words on homogeneous Besov spaces. First of all, let us define a subclass of the space $\mathcal{S}'(\mathbb{R}^N)$.

Definition 1.19. We define $\mathcal{S}'_h(\mathbb{R}^N)$ to be the space of tempered distributions u such that, for all $\theta \in \mathcal{D}(\mathbb{R}^N)$, one has

$$\lim_{\lambda \to +\infty} \|\theta(\lambda D)\, u\|_{L^\infty} \;=\; 0\,. \tag{1.2}$$

This requirement is actually a condition on low frequencies only: as a matter of fact, the following proposition holds true.

Proposition 1.20. *$u \in \mathcal{S}'_h$ if and only if there exists $\theta \in \mathcal{D}(\mathbb{R}^N)$ which satisfies relation* (1.2) *and moreover $\theta(0) \neq 0$.*

For instance, it's easy to verify the following claims.

- Let $u \in \mathcal{S}'$; if $\widehat{u}$ is locally integrable near 0, then $u \in \mathcal{S}'_h$.
  In particular, one has the inclusion $\mathcal{E}'(\mathbb{R}^N) \subset \mathcal{S}'_h(\mathbb{R}^N)$.
- Let u be a tempered distribution such that $\theta(D)u \in L^p$, for some $p \in [1, +\infty[$ and some function $\theta \in \mathcal{D}$ with $\theta(0) \neq 0$. Then $u \in \mathcal{S}'_h$.
- Let P be a nonzero polynomial. Then $P \notin \mathcal{S}'_h$, but $e^{i(\cdot,\eta)}P \in \mathcal{S}'_h$ for all $\eta \in \mathbb{R}^N \backslash \{0\}$.
  In particular, $\mathcal{S}'_h$ is not a closed subset of $\mathcal{S}'$ for the weak-$*$ topology.

Now, keeping in mind that $\sum_{j \in \mathbb{Z}} \varphi\left(2^{-j}\xi\right) \equiv 1$ for all $\xi \in \mathbb{R}^N \backslash \{0\}$, we define, for all $j \in \mathbb{Z}$, the homogeneous dyadic blocks

$$\dot{\Delta}_j u \;:=\; \varphi\left(2^{-j}\, D\right) u \;=\; 2^{jN}\; h\left(2^j\, \cdot\right) \,*\, u\,,$$

and the homogeneous low frequencies cut-off operators

$$\dot{S}_j u \;=\; \sum_{k \leq j-1} \dot{\Delta}_k\,.$$

So, formally, one can write $\mathrm{Id} \;=\; \sum_{j \in \mathbb{Z}} \dot{\Delta}_j$.

Now, we are ready to define the homogeneous Besov space $\dot{B}^s_{p,r}$.

Definition 1.21. Let $s \in \mathbb{R}$ and $(p, r) \in [1, +\infty]^2$. The space $\dot{B}^s_{p,r}$ is defined as the subset of distributions $u \in \mathcal{S}'_h$ such that

$$\|u\|_{\dot{B}^s_{p,r}} \;:=\; \left\| \left(2^{js} \left\|\dot{\Delta}_j u\right\|_{L^p}\right)_{j \in \mathbb{Z}} \right\|_{\ell^r} \;<\; +\infty\,.$$

The space $\dot{B}^s_{p,r}$, endowed with $\|\cdot\|_{\dot{B}^s_{p,r}}$, is a normed space, but it is not complete, in general.

Theorem 1.22. *Let $s \in \mathbb{R}$ and $(p, r) \in [1, +\infty]^2$ such that*

$$s \;<\; \frac{N}{p} \qquad \text{or} \qquad s \;=\; \frac{N}{p} \quad \text{and} \quad r \;=\; 1\,.$$

Then the space $\dot{B}^s_{p,r}$, endowed with the norm $\|\cdot\|_{\dot{B}^s_{p,r}}$, is complete.
Moreover, in the case $s < N/p$ it satisfies also the Fatou's property.

Note that if the condition on indices in previous theorem is not verified, $\dot{B}^s_{p,r}$ is no longer a Banach space: the problem lies in convergence for low frequencies. Let us spend a few words about that.

We start from the remark that $\dot{\Delta}_j f = 0$ for all $j \in \mathbb{Z}$ if and only if f is a polynomial. As a matter of fact, thanks to Fourier transform it's quite easy to see that the condition $\dot{\Delta}_j f = 0$ for all $j \in \mathbb{Z}$ is equivalent to require $\operatorname{supp}\widehat{f} \subset \{0\}$.

Now, as for all $k \in \mathbb{Z}$ the identity $\mathrm{Id} = S_k + \sum_{j \geq k} \dot{\Delta}_j$ holds (note that S_k is the non-homogeneous low frequencies cut-off operator), we have

$$f = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j f \quad \text{in} \quad \mathcal{S}' \qquad \Longleftrightarrow \qquad \lim_{k \to -\infty} S_k f = 0 \quad \text{in} \quad \mathcal{S}' .$$

Next lemma explains us the meaning to give to condition on the right-hand side.

Lemma 1.23. *For all tempered distribution $f \in \mathcal{S}'(\mathbb{R}^N)$, there exist an integer $n \in \mathbb{N}$ and a family of polynomials $(P_k(f))_{k \in \mathbb{Z}}$, of degrees $d_k \leq n$, such that*

$$\lim_{k \to -\infty} \Big(S_k f - P_k(f) \Big) = 0$$

uniformly on all compact subsets of $\mathbb{R}^N$ and in the $\mathcal{S}'$ topology.

Therefore, the equality $f = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j f$ means that

$$\lim_{k \to -\infty} \left(\sum_{j \geq k} \dot{\Delta}_j f - P_k(f) \right) = f \qquad \text{in} \qquad L^\infty_{loc}(\mathbb{R}^N) \cap \mathcal{S}'(\mathbb{R}^N) .$$

See also [9] and the references therein for a more complete treatment of this construction. However, previous arguments lead to another definition of the homogeneous Besov spaces, modulo polynomials, in such a way to get a Banach space independentely of the regularity indices. It turns out that the two definitions coincide in the case $s < N/p$ or $s = N/p$ and $r = 1$.

Let us finally remark that most of the results stated for the non-homogenous framework are still true also in this case. Moreover, one can characterize homogeneous Besov spaces in terms of the heat flow or of finite differences. We refer to [2] for the details of these and other properties of spaces $\dot{B}^s_{p,r}$.

1.3 Non-homogeneous paradifferential calculus

This section is devoted to the study of the action of some operators on non-homogeneous Besov spaces. In particular, we are going to focus on the product of two tempered distributions and on left and right composition of a smooth function with a Besov one. In section 1.5, instead, we will introduce more general paradifferential operators.

For the proofs of all the results quoted here, we refer again to [2], chapter 2.

1.3.1 Bony's paraproduct decomposition

Given two tempered distributions u and v, formally one has $u\,v = \sum_{j,k} \Delta_j u\, \Delta_k v$. Now, due to the spectral localization of cut-off operators, we can write the following *Bony's decomposition*:

$$u\,v = T_u v + T_v u + R(u,v)\,, \tag{1.3}$$

where we have defined

$$T_u v := \sum_j S_{j-1} u\, \Delta_j v \qquad \text{and} \qquad R(u,v) := \sum_j \sum_{|k-j| \leq 1} \Delta_j u\, \Delta_k v\,.$$

The above operator T is called "paraproduct", whereas R is called "remainder".
Sometimes we will also write

$$u\,v\,=\,T_u v\,+\,T'_v u \qquad \text{with} \qquad T'_v u\,:=\,\sum_j S_{j+2}v\,\Delta_j u\,.$$

Let us immediately note that the generic term $S_{j-1}u\;\Delta_j v$ is spectrally supported in the dyadic annulus $2^j\widetilde{\mathcal{C}}$, while $\Delta_j u\,\Delta_k v$ is spectrally localized in a ball of radius proportional to 2^j.

Let us now recall some continuity properties of paraproduct and remainder operators on non-homogeneous Besov spaces.

Theorem 1.24. *(i) For any $(s,p,r)\in\mathbb{R}\times[1,+\infty]^2$ and $t>0$, the paraproduct operator T maps $L^\infty\times B^s_{p,r}$ in $B^s_{p,r}$, and $B^{-t}_{\infty,r_1}\times B^s_{p,r_2}$ in $B^{s-t}_{p,q}$, with $1/q\,:=\,\min\{1\,,\,1/r_1\,+\,1/r_2\}$. Moreover, the following estimates hold true:*

$$\|T_u v\|_{B^s_{p,r}}\,\leq\,C\,\|u\|_{L^\infty}\|\nabla v\|_{B^{s-1}_{p,r}} \qquad \text{and} \qquad \|T_u v\|_{B^{s-t}_{p,q}}\,\leq\,C\,\|u\|_{B^{-t}_{\infty,r_1}}\|\nabla v\|_{B^{s-1}_{p,r_2}}\,.$$

(ii) For any (s_1,p_1,r_1) and (s_2,p_2,r_2) in $\mathbb{R}\times[1,\infty]^2$ such that $s_1+s_2\geq 0$, $1/p:=1/p_1+1/p_2\leq 1$ and $1/r:=1/r_1+1/r_2\leq 1$ the remainder operator R maps $B^{s_1}_{p_1,r_1}\times B^{s_2}_{p_2,r_2}$ in $B^{s_1+s_2}_{p,r}$, and one has:

$$\|R(u,v)\|_{B^{s_1+s_2}_{p,r}}\,\leq\,\frac{C^{s_1+s_2+1}}{s_1+s_2}\,\|u\|_{B^{s_1}_{p_1,r_1}}\,\|v\|_{B^{s_2}_{p_2,r_2}} \qquad \textit{if} \quad s_1+s_2\,>\,0$$

$$\|R(u,v)\|_{B^0_{p,\infty}}\,\leq\,C^{s_1+s_2+1}\,\|u\|_{B^{s_1}_{p_1,r_1}}\,\|v\|_{B^{s_2}_{p_2,r_2}} \qquad \textit{if} \quad s_1+s_2\,=\,0\,,\quad r=1\,.$$

Remark 1.25. Actually, under the assumptions of the above proposition, one can prove more accurate estimates for the paraproduct operator: for all $k\in\mathbb{N}$,

$$\|T_u v\|_{B^s_{p,r}}\,\leq\,C\,\|u\|_{L^\infty}\|\nabla^k v\|_{B^{s-k}_{p,r}} \qquad \text{and} \qquad \|T_u v\|_{B^{s-t}_{p,q}}\,\leq\,C\,\|u\|_{B^{-t}_{\infty,r_1}}\|\nabla^k v\|_{B^{s-k}_{p,r_2}}\,.$$

Let us also quote a lemma, which continuity properties for the remainder operator are based on, and which will turn out to be useful in the applications.

Lemma 1.26. *Let $\mathcal{B}$ be a ball of $\mathbb{R}^N$, $s>0$ and $(p,r)\in[1,+\infty]^2$. Let $(u_j)_{j\in\mathbb{N}}$ be a sequence of smooth functions such that*

$$\operatorname{supp}\widehat{u}_j\,\subset\,2^j\,\mathcal{B} \qquad \textit{and} \qquad \left\|\left(2^{js}\,\|u_j\|_{L^p}\right)_{j\in\mathbb{N}}\right\|_{\ell^r}\,<\,+\infty\,.$$

Then $u:=\sum_{j\in\mathbb{N}}u_j$ belongs to $B^s_{p,r}$ and

$$\|u\|_{B^s_{p,r}}\,\leq\,C_s\,\left\|\left(2^{js}\,\|u_j\|_{L^p}\right)_{j\in\mathbb{N}}\right\|_{\ell^r}\,.$$

Combining the theorem 1.24 with Bony's decomposition (1.3), we easily get the following "tame estimate":

Corollary 1.27. *Let a be a bounded function such that $\nabla a\in B^{s-1}_{p,r}$ for some $s>0$ and $(p,r)\in[1,+\infty]^2$. Then for any $b\in B^s_{p,r}\cap L^\infty$ we have $ab\in B^s_{p,r}\cap L^\infty$ and there exists a constant C, depending only on N, p and s, such that*

$$\|ab\|_{B^s_{p,r}}\leq C\Big(\|a\|_{L^\infty}\|b\|_{B^s_{p,r}}+\|b\|_{L^\infty}\|\nabla a\|_{B^{s-1}_{p,r}}\Big).$$

In applications, we will often have to handle compositions between a paraproduct operator and a Fourier multiplier. We already know how each of them acts on the Besov class; now, we want to focus on their commutator operator. Before doing this, however, let us quote a preliminary result.

Lemma 1.28. *Let $\theta \in \mathcal{C}^1(\mathbb{R}^N)$ such that the function $(1+|\cdot|)\,\widehat{\theta} \in L^1$.*

Then there exists a constant C such that, for any Lipschitz function f and any $u \in L^p$ with $p \in [1,+\infty]$, for all $\lambda > 0$ one has

$$\left\|\left[\theta(\lambda^{-1}D)\,,\,f\right]\,u\right\|_{L^p} \leq C\,\lambda^{-1}\,\|\nabla f\|_{L^\infty}\,\|u\|_{L^p}\,.$$

In particular, if we take $\theta = \varphi$ and $\lambda = 2^j$, this lemma may be interpreted as a gain of one derivative by commutation between the localization operator Δ_j and the multiplication by a Lipschitz function. It will be interesting to compare this result with what we will get in chapter 2, where we will assume f to be non-Lipschitz (see lemma 2.3).

Now let us state commutator estimates between a paraproduct operator and a Fourier multiplier, as announced.

Lemma 1.29. *Let $m \in \mathbb{R}$, $R > 0$ and $f \in \mathcal{C}^\infty(\mathbb{R}^N)$ be a homogeneous smooth function of degree m out of the ball $B(0,R)$.*
Moreover, let $\sigma \in [0,1[\,,\ s \in \mathbb{R}$ and $(p,r) \in [1,+\infty]^2$.

Then, there exists a constant C (depending only on R, σ and N) such that

$$\left\|\left[T_u, f(D)\right]v\right\|_{B^{s-m+\sigma}_{p,r}} \leq C\,\|\nabla u\|_{B^{\sigma-1}_{\infty,\infty}}\,\|v\|_{B^s_{p,r}}\,. \tag{1.4}$$

In the limit case $\sigma = 1$, the previous estimate is still true with $\|\nabla u\|_{L^\infty}$ in the place of $\|\nabla u\|_{B^{\sigma-1}_{\infty,\infty}}$.

1.3.2 The paralinearization theorem

In this paragraph, we want to investigate the effect of composition by smooth functions on Besov spaces. We will focus on left composition, while we refer to chapter 5 for some properties of right composition. Let us state a first fundamental result (whose proof can be found e.g. in [2]) for the general situation.

Theorem 1.30. *Let $f \in \mathcal{C}^\infty(\mathbb{R})$ such that $f(0) = 0$, $s > 0$ and $(p,r) \in [1,+\infty]^2$.*

If $u \in L^\infty \cap B^s_{p,r}$, then so does $f \circ u$ and moreover

$$\|f \circ u\|_{B^s_{p,r}} \leq C\,\|u\|_{B^s_{p,r}}\,,$$

for a constant C depending only on s, f' and $\|u\|_{L^\infty}$.

We can state another result (see [28], section 2, for its proof), which is strictly related to the previous one.

Proposition 1.31. *Let $I \subset \mathbb{R}$ be an open interval and $f : I \longrightarrow \mathbb{R}$ be a smooth function.*

Then, for all compact subset $J \subset I$, $s > 0$ and $(p,r) \in [1,+\infty]^2$, there exists a constant C such that, for all functions u valued in J and with gradient $\nabla u \in B^{s-1}_{p,r}$, we have that also $\nabla(f \circ u) \in B^{s-1}_{p,r}$ and

$$\|\nabla\,(f \circ u)\|_{B^{s-1}_{p,r}} \leq C\,\|\nabla u\|_{B^{s-1}_{p,r}}\,.$$

In the case $f \in \mathcal{C}^\infty_b(\mathbb{R})$, theorem 1.30 can be a little bit improved (see again [2] for te proof).

Theorem 1.32. *Let $f \in \mathcal{C}^\infty_b(\mathbb{R})$ such that $f(0) = 0$, $s > 0$ and $(p,r) \in [1,+\infty]^2$. Let us take a $u \in B^s_{p,r}$ such that $\nabla u \in B^{-1}_{\infty,\infty}$.*

Then also $f \circ u \in B^s_{p,r}$ and there exists a constant C, depending only on s, f and $\|\nabla u\|_{B^{-1}_{\infty,\infty}}$, for which

$$\|f \circ u\|_{B^s_{p,r}} \leq C\,\|u\|_{B^s_{p,r}}\,.$$

If u is more regular, one can expect to get more informations on $f \circ u$. The paralinearization theorem says that, up to a remainder term which turns out to be more regular than u, one can write the composition $f \circ u$ as a paraproduct involving $f' \circ u$ and u. See also chapter 2 of [2] for more details.

Theorem 1.33. *Let s, $\varrho > 0$, with $\varrho \notin \mathbb{N}$. Let also $p \in [1,+\infty]$, $1 \leq r_1 \leq r_2 \leq +\infty$ and set $1/r := \min\{1\,,\,1/r_1 + 1/r_2\}$. Finally, let $f \in \mathcal{C}^\infty(\mathbb{R})$.*

Then, for any function $u \in B^s_{p,r_1} \cap B^\varrho_{\infty,r_2}$ there exists a constant C, which depends only on f'' and $\|u\|_{L^\infty}$, such that

$$\left\|f \circ u - T_{f'\circ u}u\right\|_{B^{s+\varrho}_{p,r}} \leq C\,\|u\|_{B^\varrho_{\infty,r_2}}\,\|u\|_{B^s_{p,r_1}}\,.$$

1.4 Logarithmic Besov spaces

As pointed out at the beginning of section 1.2, classical Sobolev spaces can be characterized via dyadic decomposition: for all $s \in \mathbb{R}$ there exists a constant $C_s > 0$ such that

$$\frac{1}{C_s}\sum_{\nu=0}^{+\infty} 2^{2\nu s}\,\|\Delta_\nu u\|^2_{L^2} \leq \|u\|^2_{H^s} \leq C_s \sum_{\nu=0}^{+\infty} 2^{2\nu s}\,\|\Delta_\nu u\|^2_{L^2}\,. \tag{1.5}$$

In other words, the H^s norm of a tempered distribution u is equivalent to the ℓ^2 norm of the sequence $\left(2^{s\nu}\,\|\Delta_\nu u\|_{L^2}\right)_{\nu\in\mathbb{N}}$. Now, one may ask what we get if, in the sequence, we put weights different to the exponential term $2^{s\nu}$. Before answering this question, we introduce some definitions. For the details of the presentation, we refer also to [23], section 3.

Let us set $\Pi(D) := \log(2 + |D|)$, i.e. its symbol is $\pi(\xi) := \log(2 + |\xi|)$.

Definition 1.34. For all $\alpha \in \mathbb{R}$, we define the space $H^{s+\alpha\log}$ as the space $\Pi^{-\alpha}H^s$, i.e.

$$f \in H^{s+\alpha\log} \quad\Longleftrightarrow\quad \Pi^\alpha f \in H^s \quad\Longleftrightarrow\quad \pi^\alpha(\xi)\left(1+|\xi|^2\right)^{s/2}\widehat{f}(\xi) \in L^2\,.$$

From the definition, it's obvious that the following inclusions hold for $s_1 > s_2$, $\alpha_1 > \alpha_2 > 0$:

$$H^{s_1+\alpha_1\log} \hookrightarrow H^{s_1+\alpha_2\log} \hookrightarrow H^{s_1} \hookrightarrow H^{s_1-\alpha_2\log} \hookrightarrow H^{s_1-\alpha_1\log} \hookrightarrow H^{s_2}\,. \tag{1.6}$$

We have the following dyadic characterization of these spaces (see [51], proposition 4.1.11).

Proposition 1.35. *Let s, $\alpha \in \mathbb{R}$. A tempered distribution u belongs to the space $H^{s+\alpha\log}$ if and only if:*

(i) for all $k \geq 0$, $\Delta_k u \in L^2(\mathbb{R}^N)$;

(ii) set $\delta_k := 2^{ks}(1+k)^\alpha\,\|\Delta_k u\|_{L^2}$ for all $k \in \mathbb{N}$, the sequence $(\delta_k)_k$ belongs to $\ell^2(\mathbb{N})$.

Moreover, $\|u\|_{H^{s+\alpha\log}} \sim \|(\delta_k)_k\|_{\ell^2}$.

Hence, this proposition generalizes property (1.5).

This new class of Sobolev spaces, which are in a certain sense of logarithmic type, will come into play in analysis of strictly hyperbolic operators with low regularity coefficients. However, inspired by their dyadic characterization, we want to define the more general class of "logarithmic Besov spaces".

Definition 1.36. Let s and α be real numbers, and $1 \leq p, r \leq +\infty$. The *non-homogeneous logarithmic Besov space* $B^{s+\alpha\log}_{p,r}$ is defined as the subset of tempered distributions u for which

$$\|u\|_{B^{s+\alpha\log}_{p,r}} := \left\|\left(2^{js}(1+j)^\alpha\,\|\Delta_j u\|_{L^p}\right)_{j\in\mathbb{N}}\right\|_{\ell^r} < +\infty\,.$$

Now, we want to investigate some basic properties of this new class of functions. We will see that most of the results stated in the classical case hold also for them. The proofs can be obtained from the previous ones with little modifications; anyway, we want to give here the most of the details.

1.4.1 General properties

First of all, let us show that definition 1.36 is independent of the choice of the cut-off functions defining the Littlewood-Paley decomposition.

Lemma 1.37. *Let $\mathcal{C} \subset \mathbb{R}^N$ be a ring, $(s,\alpha) \in \mathbb{R}^2$ and $(p,r) \in [1,+\infty]^2$. Let $(u_j)_{j\in\mathbb{N}}$ be a sequence of smooth functions such that*

$$\operatorname{supp}\widehat{u}_j \subset 2^j\,\mathcal{C} \qquad \text{and} \qquad \left\|\left(2^{js}\,(1+j)^\alpha\,\|u_j\|_{L^p}\right)_{j\in\mathbb{N}}\right\|_{\ell^r} < +\infty\,.$$

Then $u := \sum_{j\in\mathbb{N}} u_j$ belongs to $B^{s+\alpha\log}_{p,r}$ and

$$\|u\|_{B^{s+\alpha\log}_{p,r}} \leq C_{s,\alpha}\left\|\left(2^{js}\,(1+j)^\alpha\,\|u_j\|_{L^p}\right)_{j\in\mathbb{N}}\right\|_{\ell^r}\,.$$

Proof. By spectral localization, we gather that there exists a $n_0 \in \mathbb{N}$ such that $\Delta_k u_j = 0$ for all $|k-j| > n_0$. Therefore

$$\|\Delta_k u\|_{L^p} \leq \sum_{|j-k|\leq n_0}\|\Delta_k u_j\|_{L^p} \leq C\sum_{|j-k|\leq n_0}\|u_j\|_{L^p}\,.$$

From these relations it immediately follows that

$$2^{ks}\,(1+k)^\alpha\,\|\Delta_k u\|_{L^p} \leq C\sum_{|j-k|\leq n_0}2^{(k-j)s}\,\frac{(1+k)^\alpha}{(1+j)^\alpha}\,2^{js}\,(1+j)^\alpha\,\|u_j\|_{L^p}\,.$$

Now, as very often in the sequel, we use the fact that $(1+k)/(1+j) \leq 1+|k-j|$. Hence,

$$2^{ks}\,(1+k)^\alpha\,\|\Delta_k u\|_{L^p} \leq C\,(\theta*\delta)_k\,,$$

where we have set (here $\mathcal{I}_A$ denote the characteristic function of the set A)

$$\theta_h := 2^{hs}\,(1+h)^\alpha\,\mathcal{I}_{[0,n_0]}(h) \quad \text{and} \quad \delta_j := 2^{js}\,(1+j)^\alpha\,\|u_j\|_{L^p}\,.$$

Passing to the ℓ^r norm and applying Young's inequality for convolutions complete the proof. $\square$

Logarithmic Besov spaces are intermediate classes of functions between the classical ones. As a matter of facts, a chain of embeddings analogous to (1.6) still holds. Let us recall that, in the classical case, a loss of regularity is needed to lower the summation index: $B^s_{p,\infty} \hookrightarrow B^{s-\varepsilon}_{p,1}$ for all $\varepsilon > 0$. It's very easy to see that, in fact, only a logarithmic loss is required:

$$B^s_{p,\infty} \hookrightarrow B^{s-\alpha\log}_{p,1} \qquad \forall\,\alpha > 1\,.$$

Therefore, proposition 1.6 admits the following generalization:

Proposition 1.38. *The space $B^{s_1+\alpha_1\log}_{p_1,r_1}$ is continuously embedded in the space $B^{s_2+\alpha_2\log}_{p_2,r_2}$ whenever $1 \leq p_1 \leq p_2 \leq +\infty$ and one of the following conditions holds true:*

- $s_2 < s_1 - N\,(1/p_1 - 1/p_2)$
- $s_2 = s_1 - N\,(1/p_1 - 1/p_2)$ *and* $\alpha_1 - \alpha_2 > 1$
- $s_2 = s_1 - N\,(1/p_1 - 1/p_2)\,,\ \alpha_2 \leq \alpha_1$ *and* $1 \leq r_1 \leq r_2 \leq +\infty\,.$

Let us now state a simple lemma, which turns out to be important in the sequel.

Lemma 1.39. *Let $1 \leq r < +\infty$. For all $u \in B^{s+\alpha\log}_{p,r}$, we have*

$$\lim_{j\to+\infty}\|u - S_j u\|_{B^{s+\alpha\log}_{p,r}} = 0\,.$$

Proof. Let us take a $u \in B^{s+\alpha\log}_{p,r}$, with $r < +\infty$. Obviously, we have that

$$\lim_{j\to+\infty} \sum_{k\geq j} 2^{ksr}\,(1+k)^{r\alpha}\,\|\Delta_k u\|_{L^p} \;=\; 0\,.$$

Then, the thesis follows observing that $u - S_j u \;=\; \sum_{k\geq j} \Delta_k u$. $\square$

As one can expect, Fatou's property holds true also for the new class of Besov spaces.

Theorem 1.40. *Let $(s,\alpha) \in \mathbb{R}^2$ and $(p,r) \in [1,+\infty]^2$.*

The set $B^{s+\alpha\log}_{p,r}$ satisfies the Fatou property: if $\left(u_j\right)_{j\in\mathbb{N}}$ is a bounded sequence in $B^{s+\alpha\log}_{p,r}$, then there exist an element $u \in B^{s+\alpha\log}_{p,r}$ and a subsequence $\left(u_{\psi(j)}\right)_{j\in\mathbb{N}}$ such that

$$\lim_{j\to+\infty} u_{\psi(j)} \;=\; u \;\; \textit{in} \;\; \mathcal{S}' \qquad \textit{and} \qquad \|u\|_{B^{s+\alpha\log}_{p,r}} \;\leq\; C\, \liminf_{j\to+\infty} \left\|u_{\psi(j)}\right\|_{B^{s+\alpha\log}_{p,r}}\,.$$

Proof. By Bernstein inequalities, for all $n \in \mathbb{N}$, the sequence $\left(\Delta_n u_j\right)_{j\in\mathbb{N}}$ is bounded in $L^p \cap L^\infty$. Therefore, Cantor's diagonal process assures that there exist a subsequence $\left(u_{\psi(j)}\right)_j$ and a sequence $\left(\widetilde{u}_n\right)_n$ of $\mathcal{C}^\infty$ functions whose Fourier transform is supported in the ring $2^n\mathcal{C}$, such that, for all $j \in \mathbb{N}$ and all $f \in \mathcal{S}$, one has

$$\lim_{j\to+\infty} \langle \Delta_n u_{\psi(j)}, f\rangle \;=\; \langle \widetilde{u}_n, f\rangle \qquad \text{and} \qquad \|\widetilde{u}_n\|_{L^p} \;\leq\; \liminf_{j\to+\infty} \left\|\Delta_n u_{\psi(j)}\right\|_{L^p}\,.$$

Now, let us consider the sequence

$$\left(\left(2^{ns}\,(1+n)^\alpha\,\left\|\Delta_n u_{\psi(j)}\right\|_{L^p}\right)_n\right)_{j\in\mathbb{N}}\;:$$

it is obviously bounded in ℓ^r (because $\left(u_j\right)_j \subset B^{s+\alpha\log}_{p,r}$ is bounded). Hence, there exists a $\left(c_n\right)_n \in \ell^r$ which (up to an extraction) it converges to in ℓ^r for the weak-$*$ topology: for all $\left(\delta_n\right)_n \subset \mathbb{R}_+$ such that $\delta_n \neq 0$ only for a finite number of indices, one has

$$\lim_{j\to+\infty} \sum_{n\in\mathbb{N}} 2^{ns}\,(1+n)^\alpha\,\left\|\Delta_n u_{\psi(j)}\right\|_{L^p}\,\delta_n \;=\; \sum_{n\in\mathbb{N}} c_n\,\delta_n\,.$$

Moreover, we have that

$$\|\left(c_n\right)_n\|_{\ell^r} \;\leq\; \liminf_{j\to+\infty} \left\|u_{\psi(j)}\right\|_{B^{s+\alpha\log}_{p,r}}\,.$$

Passing to the limit, we get that the sequence $\left(2^{ns}\,(1+n)^\alpha\,\|\widetilde{u}_n\|\right)_n \in \ell^r$. Therefore, lemma 1.37 guarantees that $u := \sum_n \widetilde{u}_n \in B^{s+\alpha\log}_{p,r}$. By spectral localization, we obtain also that there exists a $n_0 \in \mathbb{N}$ such that, for all $n < n_0$ and all $f \in \mathcal{S}$,

$$\langle \sum_{m=n}^{n_0} \Delta_m u\,,\, f\rangle_{\mathcal{S}'\times\mathcal{S}} \;=\; \langle \sum_{m=n}^{n_0} \sum_{|m-h|\leq 1} \Delta_m \widetilde{u}_h\,,\, f\rangle_{\mathcal{S}'\times\mathcal{S}}\,.$$

Then, due to the definition of $\widetilde{u}_h$, we have

$$\sum_{m=n}^{n_0} \Delta_m u \;=\; \lim_{j\to+\infty} \sum_{m=n}^{n_0} \Delta_m u_{\psi(j)} \quad \text{in} \;\; \mathcal{S}'\,.$$

We apply previous equality for $n = 0$. Moreover, lemma 1.39 implies $(\mathrm{Id} - S_k)u_{\psi(j)} \longrightarrow_{k\to+\infty} 0$ in $B^{s+\alpha\log}_{p,r}$ (or in $B^{s-1}_{p,\infty}$ if $r = +\infty$) uniformly with respect to j. From these facts we gather

$$u \;=\; \lim_{k\to+\infty} \sum_{m=0}^{k} \Delta_m u \;\equiv\; \lim_{j\to+\infty} u_{\psi(j)} \quad \text{in} \;\; \mathcal{S}'\,,$$

and this completes the proof of the theorem. $\square$

Theorem 1.41. *The set* $\left(B^{s+\alpha\log}_{p,r}\,,\,\|\cdot\|_{B^{s+\alpha\log}_{p,r}}\right)$ *is a Banach space.*

Proof. It is quite easy to see that $\left(B^{s+\alpha\log}_{p,r}\,,\,\|\cdot\|_{B^{s+\alpha\log}_{p,r}}\right)$ is a normed space. Let us show that it is also complete.

So, take a Cauchy sequence $(u_n)_{n\in\mathbb{N}}\subset B^{s+\alpha\log}_{p,r}$. In particular, it is bounded: hence, by theorem 1.40 there exist a $u\in B^{s+\alpha\log}_{p,r}$ and a subsequence $\left(u_{\psi(n)}\right)_{n\in\mathbb{N}}$ which converges to u in $\mathcal{S}'$. Obviously, also $\left(u_{\psi(n)}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence: for any $\varepsilon>0$, there exists an index n_ε such that, for all $m\geq n\geq n_\varepsilon$, we have

$$\left\|u_{\psi(m)}\,-\,u_{\psi(n)}\right\|_{B^{s+\alpha\log}_{p,r}}\,<\,\varepsilon\,.$$

In particular, we infer that $\left(u_{\psi(m)}-u_{\psi(n)}\right)_m$ is bounded in $B^{s+\alpha\log}_{p,r}$. Then, thanks to the Fatou property (note that this time it's not necessary to pass to another extraction: the whole $\left(u_{\psi(m)}\right)_m$ converges to u in $\mathcal{S}'$), we have

$$\left\|u_{\psi(n)}\,-\,u\right\|_{B^{s+\alpha\log}_{p,r}}\,\leq\,C\,\liminf_{m\to+\infty}\left\|u_{\psi(n)}\,-\,u_{\psi(m)}\right\|_{B^{s+\alpha\log}_{p,r}}\,\leq\,C\,\varepsilon\,,$$

i.e. $\left(u_{\psi(n)}\right)_{n\in\mathbb{N}}$ converges to u also in $B^{s+\alpha\log}_{p,r}$.

Now, the whole $(u_n)_n$ must have limit u in $B^{s+\alpha\log}_{p,r}$, because it is a Cauchy sequence in this space. The theorem is proved. □

Let us quote a density result, analogous to proposition 1.9.

Proposition 1.42. *Fix p and r finite.*

Then the space of test functions $\mathcal{D}(\mathbb{R}^N)$ is densely embedded in $B^{s+\alpha\log}_{p,r}(\mathbb{R}^N)$.

Proof. Let us fix a $\varepsilon>0$. By lemma 1.39, there exists a $k>0$ such that

$$\|u\,-\,S_ku\|_{B^{s+\alpha\log}_{p,r}}\,<\,\varepsilon\,.$$

Now, let us take a cut-off function $\theta\in\mathcal{D}$ supported in the ball $B(0,2)$, such that $0\leq\theta\leq1$ and equal to 1 on the ball $B(0,1)$. For all $R>0$, we define $\theta_R(\cdot)\,:=\,\theta(\cdot/R)$.

If we set $h=\max\{0,[s]+2\}$ (where $[s]$ denotes the biggest integer smaller than or equal to s), for all $j\geq1$ Bernstein's inequalities give us

$$2^{js}\,(1+j)^\alpha\,\left\|\Delta_j\left(\theta_RS_ku\,-\,S_ku\right)\right\|_{L^p}\,\leq\,C\,2^{-j}\,(1+j)^\alpha\,\left\|D^h\left(\theta_RS_ku\,-\,S_ku\right)\right\|_{L^p}\,.$$

Therefore, taking the ℓ^r norm we gather

$$\|\theta_RS_ku\,-\,S_ku\|_{B^{s+\alpha\log}_{p,r}}\,\leq\,C\left(\left\|D^h\left(\theta_RS_ku\,-\,S_ku\right)\right\|_{L^p}\,+\,\|\theta_RS_ku\,-\,S_ku\|_{L^p}\right)\,.$$

Now, as $p<+\infty$, Leibniz rule and Lebesgue's dominated convergence theorem ensure us that

$$\|\theta_RS_ku\,-\,S_ku\|_{B^{s+\alpha\log}_{p,r}}\,<\,\varepsilon$$

for some $R>0$ big enough. As $S_ku\in\mathcal{C}^\infty$, this concludes the proof. □

1.4.2 Duality, multipliers, interpolation

Let us now investigate duality properties of logarithmic Besov spaces. Also in this case, it's not hard to generalize the corresponding classical result.

Proposition 1.43. *For all s, $\alpha \in \mathbb{R}$ and $(p,r) \in [1,+\infty]^2$, the map*

$$\begin{array}{rcl} B^{s+\alpha\log}_{p,r} \times B^{-s-\alpha\log}_{p',r'} & \longrightarrow & \mathbb{R} \\ (u,\phi) & \longmapsto & \langle u,\phi\rangle := \displaystyle\sum_{|j-k|\leq 1} (\Delta_j u, \Delta_k \phi)_{L^2} \end{array}$$

defines a continuous bilinear functional on the space $B^{s+\alpha\log}_{p,r} \times B^{-s-\alpha\log}_{p',r'}$.
Moreover, let us denote with $\mathcal{S}^{-s,-\alpha}_{p',r'}$ the set of $\phi \in \mathcal{S}$ such that $\|\phi\|_{B^{-s-\alpha\log}_{p',r'}} \leq 1$. Then, for all $u \in \mathcal{S}'$, one has

$$\|u\|_{B^{s+\alpha}_{p,r}} \leq C \sup_{\phi\in\mathcal{S}^{-s,-\alpha}_{p',r'}} \langle u,\phi\rangle\,.$$

Proof. For $|j-k| \leq 1$, we have

$$|\langle \Delta_j u, \Delta_k \phi\rangle| \leq \|\Delta_j u\|_{L^p} \|\Delta_k \phi\|_{L^{p'}} 2^{(j-k)s} \frac{(1+j)^\alpha}{(1+k)^\alpha} \omega_{k,j}\,,$$

where we have defined

$$\omega_{k,j} := 2^{(k-j)s} \frac{(1+k)^\alpha}{(1+j)^\alpha}\,.$$

As done in the end of the proof to lemma 1.37, it's easy to see that $\omega_{k,j}$ can be bounded in terms of the difference $|k-j|$ only; hence $|\omega_{k,j}| \leq C$. Therefore, by Hölder inequality we get

$$|\langle u,\phi\rangle| \leq C \|u\|_{B^{s+\alpha\log}_{p,r}} \|\phi\|_{B^{-s-\alpha\log}_{p',r'}}\,.$$

Now, let us call $\ell^{r'}_n$ the set of sequences $(b_j)_j$ such that $b_j = 0$ for all $j > n$ and $\left\|(b_j)_j\right\|_{\ell^{r'}} \leq 1$. Then we have

$$\begin{aligned} \|u\|_{B^{s+\alpha\log}_{p,r}} &= \sup_{n\in\mathbb{N}} \left\|\left(\mathcal{I}_{[0,n]}(j)\; 2^{js}(1+j)^\alpha \|\Delta_j u\|_{L^p}\right)_{j\in\mathbb{N}}\right\|_{\ell^r} \\ &= \sup_{n\in\mathbb{N}} \sup_{(b_j)_j\in\ell^{r'}_n} \sum_{j\leq n} b_j\, 2^{js}(1+j)^\alpha \|\Delta_j u\|_{L^p}\,. \end{aligned}$$

By duality for Lebesgue spaces and density of $\mathcal{S} \subset L^{p'}$, we know that for any fixed $\varepsilon > 0$, for all $j \geq 0$ there exists a function $\phi_j \in \mathcal{S}$, $\|\phi_j\|_{L^{p'}} \leq 1$, for which

$$\|\Delta_j u\|_{L^p} \leq \int_{\mathbb{R}^N} \Delta_j u(x)\,\phi_j(x)\,dx + \varepsilon \frac{2^{-js}(1+j)^{-\alpha}}{(1+|b_j|)(1+j^2)}\,.$$

We have to notice that

$$\int_{\mathbb{R}^N} \Delta_j u(x)\,\phi_j(x)\,dx = \mathcal{F}(\Delta_j u\;\phi_j)_{|\xi=0} = \int_{\mathbb{R}^N_\eta} \varphi(2^{-j}\eta)\,\widehat{u}(\eta)\,\widehat{\phi}_j(\eta)\,d\eta\,;$$

so it's enough to consider the frequencies of ϕ_j "caught" by operator Δ_j:

$$\int_{\mathbb{R}^N} \Delta_j u(x)\,\phi_j(x)\,dx = \int_{\mathbb{R}^N} \Delta_j u(x)\;(\Delta_{j-1}+\Delta_j+\Delta_{j+1})\,\phi_j(x)\,dx\,.$$

Let us now define

$$\Phi_n := \sum_{j\leq n} b_j\, 2^{js}\,(1+j)^\alpha\,\Delta_j\phi_j\,.$$

It's quite obvious that $\Phi_n \in \mathcal{S}^{-s,-\alpha}_{p',r'}$ for all n, and (by lemma 1.37) that $\|\Phi_n\|_{B^{-s-\alpha\log}_{p',r'}} \leq C$ independentely of n. Then we have

$$\left\|\left(\mathcal{I}_{[0,n]}(j)\; 2^{js}\,(1+j)^\alpha\,\|\Delta_j u\|_{L^p}\right)_{j\in\mathbb{N}}\right\|_{\ell^r} \leq \langle u,\Phi_n\rangle + \varepsilon$$

for all $n\in\mathbb{N}$, from which we infer the result. $\square$

Now we want to consider the action of Fourier multipliers on non-homogeneous logarithmic Besov spaces. First of all, we have to give a more general definition of symbols.

Definition 1.44. A smooth function $f:\mathbb{R}^N\longrightarrow\mathbb{R}$ is said to be a $S^{m+\delta\log}$-*multiplier* if, for all multi-index $\nu\in\mathbb{N}^N$, there exists a constant C_ν such that

$$\forall\,\xi\in\mathbb{R}^N\,,\qquad |\partial^\nu f(\xi)| \leq C_\nu\,(1+|\xi|)^{m-|\nu|}\,\log^\delta(1+|\xi|)\,.$$

Proposition 1.45. *Let $m,\delta\in\mathbb{R}$ and f be a $S^{m+\delta\log}$-multiplier.*

Then for all real numbers s and α and all $(p,r)\in[1,+\infty]^2$, the operator $f(D)$ maps $B^{s+\alpha\log}_{p,r}$ into $B^{(s-m)+(\alpha-\delta)\log}_{p,r}$ continuously.

Proof. According to lemma 1.37, it's enough to prove that, for all $j\geq 0$,

$$2^{(s-m)j}\,(1+j)^{\alpha-\delta}\,\|f(D)\,\Delta_j u\|_{L^p} \leq C\,2^{js}\,(1+j)^\alpha\,\|\Delta_j u\|_{L^p}\,.$$

Let us consider first low frequencies. Let us take a $\theta\in\mathcal{D}(\mathbb{R}^N)$ such that $\theta\equiv 1$ in a neighborhood of $\operatorname{supp}\chi$: passing to the phase space, it's easy to see that $f(D)\,\Delta_0 u = (\theta f)(D)\Delta_0 u$. As $\mathcal{F}^{-1}(\theta f)\in L^1$, Young's inequality for convolutions gives us the desired estimate for $j=0$.

Now, let us focus on high frequencies: for all $j\geq 1$ we can write

$$f(D)\,\Delta_j u = F_j * u\,,\qquad \text{with}\qquad F_j = \mathcal{F}^{-1}\left(f(\xi)\,\varphi(2^{-j}\xi)\right)\,.$$

For all $M\in\mathbb{N}$, we have:

$$\begin{aligned}(1+|x|^2)^M\,F_j(x) &= \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N_\xi} e^{ix\cdot\xi}\,(\mathrm{Id}-\Delta_\xi)^M\; f(\xi)\,\varphi(2^{-j}\xi)\,d\xi\\ &= \sum_{|\beta|+|\gamma|\leq 2M}\frac{C_{\beta,\gamma}}{(2\pi)^N}\,2^{-j|\gamma|}\int_{\mathbb{R}^N_\xi} e^{ix\cdot\xi}\,\partial^\beta f(\xi)\,\partial^\gamma\varphi(2^{-j}\xi)\,d\xi\,.\end{aligned}$$

In fact, the integration is not performed on the whole $\mathbb{R}^N$, but only on the support of $\varphi(2^{-j}\cdot)$, where we have

$$\begin{aligned}\left|\partial^\beta f(\xi)\,\partial^\gamma\varphi(2^{-j}\xi)\right| &\leq C_{\beta,\gamma}\,(1+|\xi|)^{m-|\beta|}\,\log^\delta(1+|\xi|)\,\Big|_{|\xi|\sim 2^j}\\ &\leq C_{\beta,\gamma}\,2^{j(m-|\beta|)}\,(1+j)^\delta\,.\end{aligned}$$

Therefore, one gathers

$$(1+|x|^2)^M\,|F_j(x)| \leq C_M\,2^{jm}\,(1+j)^\delta\,,$$

which implies that, for M big enough, $F_j\in L^1(\mathbb{R}^N_x)$ and $\|F_j\|_{L^1}\leq C\,2^{mj}\,(1+j)^\delta$. Young's inequality for convolution leads to the thesis. $\square$

Let us now state two simple interpolation inequalities.

Theorem 1.46. *Take real numbers $s_1 \leq s \leq s_2$ and $\alpha_1 \leq \alpha \leq \alpha_2$, and $(p,r) \in [1,+\infty]^2$. Let $\theta \in]0,1[$ be such that*

$$s \,=\, \theta s_1 \,+\, (1-\theta)s_2 \qquad \text{and} \qquad \alpha \,=\, \theta\alpha_1 \,+\, (1-\theta)\alpha_2\,.$$

Then there exists a constant C, depending only on s_1, s_2 and θ, such that the following inequalities hold:

$$\begin{aligned} \|u\|_{B^{s+\alpha\log}_{p,r}} &\leq \|u\|^{\theta}_{B^{s_1+\alpha_1\log}_{p,r}}\,\|u\|^{1-\theta}_{B^{s_2+\alpha_2\log}_{p,r}} \\ \|u\|_{B^{s+\alpha\log}_{p,1}} &\leq C\,\|u\|^{\theta}_{B^{s_1+\alpha_1\log}_{p,\infty}}\,\|u\|^{1-\theta}_{B^{s_2+\alpha_2\log}_{p,\infty}}\,. \end{aligned}$$

Proof. The former estimate immediately follows from Hölder inequality.

Let us focus on the proof of the latter one. Keeping in mind the definition of Besov norms, for all $j \geq 0$ we can write:

$$2^{js}\,(1+j)^{\alpha}\,\|\Delta_j u\|_{L^p} \;\leq\; \begin{cases} 2^{j(1-\theta)(s_2-s_1)}\,(1+j)^{(1-\theta)(\alpha_2-\alpha_1)}\,\|u\|_{B^{s_1+\alpha_1\log}_{p,\infty}} \\ 2^{-j\theta(s_2-s_1)}\,(1+j)^{-\theta(\alpha_2-\alpha_1)}\,\|u\|_{B^{s_2+\alpha_2\log}_{p,\infty}}\,. \end{cases}$$

Therefore, for all $k \in \mathbb{N}$ we get

$$\begin{aligned} \|u\|_{B^{s+\alpha\log}_{p,1}} \;\leq\; & \|u\|_{B^{s_1+\alpha_1\log}_{p,\infty}} \sum_{j\leq k} 2^{j(1-\theta)(s_2-s_1)}\,(1+j)^{(1-\theta)(\alpha_2-\alpha_1)} \;+ \\ & +\, \|u\|_{B^{s_2+\alpha_2\log}_{p,\infty}} \sum_{j>k} 2^{-j\theta(s_2-s_1)}\,(1+j)^{-\theta(\alpha_2-\alpha_1)}\,. \end{aligned} \tag{1.7}$$

Obviously, we have

$$\sum_{j>k} 2^{-j\theta(s_2-s_1)}\,(1+j)^{-\theta(\alpha_2-\alpha_1)} \;\leq\; \sum_{j>k} 2^{-j\theta(s_2-s_1)} \;=\; \frac{2^{-k\theta(s_2-s_1)}}{1-2^{-\theta(s_2-s_1)}}\,,$$

while for all $\varepsilon > 0$ we can write

$$\sum_{j\leq k} 2^{j(1-\theta)(s_2-s_1)}\,(1+j)^{(1-\theta)(\alpha_2-\alpha_1)} \;\leq\; \sum_{j\leq k} 2^{j(1-\theta)(s_2-s_1+\varepsilon)} \;=\; \frac{2^{k(1-\theta)(s_2-s_1+\varepsilon)}}{2^{(1-\theta)(s_2-s_1+\varepsilon)}-1}\,.$$

So, (1.7) becomes

$$\|u\|_{B^{s+\alpha\log}_{p,1}} \;\leq\; \|u\|_{B^{s_1+\alpha_1\log}_{p,\infty}}\,\frac{2^{k(1-\theta)(s_2-s_1+\varepsilon)}}{2^{(1-\theta)(s_2-s_1+\varepsilon)}-1} \;+\; \|u\|_{B^{s_2+\alpha_2\log}_{p,\infty}}\,\frac{2^{-k\theta(s_2-s_1)}}{1-2^{-\theta(s_2-s_1)}}\,.$$

Now, taking $k \geq 1$ large enough such that

$$2^{(k-1)(s_2-s_1)} \;\leq\; \frac{\|u\|_{B^{s_2+\alpha_2\log}_{p,\infty}}}{\|u\|_{B^{s_1+\alpha_1\log}_{p,\infty}}} \;\leq\; 2^{k(s_2-s_1)}$$

and $\varepsilon = k^{-2}$, for instance, completes the proof. □

Remark 1.47. Let us remark that a wider set of results of this kind can be easily proved, slightly modifying the previous argument. For instance, one may allow the interpolation parameter pertaining to α to be different from that pertaining to s.

1.4.3 Paraproducts

Let us end this section generalizing theorem 1.24 to logarithmic Besov spaces framework. First of all, let us state a characterization of logarithmic Besov spaces in terms of the low frequencies cut-off operators.

Proposition 1.48. *Let $s<0$ and $\alpha\in\mathbb{R}$, or $s=0$ and $\alpha<-1$. Let also $(p,r)\in[1,+\infty]^2$. Finally, let $u\in\mathcal{S}'$ given.*

Then $u\in B^{s+\alpha\log}_{p,r}$ if and only if the sequence $\left(2^{js}(1+j)^{\alpha}\|S_j u\|_{L^p}\right)_{j\in\mathbb{N}}\in\ell^r$. Moreover,

$$C\,\|u\|_{B^s_{p,r}}\ \le\ \left\|\left(2^{js}(1+j)^{\alpha}\|S_j u\|_{L^p}\right)_{j\in\mathbb{N}}\right\|_{\ell^r}\ \le\ \widetilde{C}\,\|u\|_{B^s_{p,r}}$$

for some constants C, $\widetilde{C}$ depending only on s and α.

Proof. From the definitions, we have $\Delta_j=S_{j+1}-S_j$. So we can write:

$$\begin{aligned}
2^{js}(1+j)^{\alpha}\left\|\Delta_j u\right\|_{L^p} &\le 2^{js}(1+j)^{\alpha}\left(\left\|S_{j+1}u\right\|_{L^p}+\left\|S_j u\right\|_{L^p}\right)\\
&\le 2^{(j+1)s}(2+j)^{\alpha}\left\|S_{j+1}u\right\|_{L^p}\frac{(1+j)^{\alpha}}{(2+j)^{\alpha}}\,2^{-s}+2^{js}(1+j)^{\alpha}\left\|S_j u\right\|_{L^p}\,.
\end{aligned}$$

By Minowski's inequality, we get the first part of the thesis.

On the other hand, using the definition of the operator S_j, we have

$$\begin{aligned}
2^{js}(1+j)^{\alpha}\left\|S_j u\right\|_{L^p} &\le 2^{js}(1+j)^{\alpha}\sum_{k\le j-1}\left\|\Delta_k u\right\|_{L^p}\\
&\le \sum_{k\le j-1}2^{(j-k)s}\frac{(1+j)^{\alpha}}{(1+k)^{\alpha}}\,2^{ks}(1+k)^{\alpha}\left\|\Delta_k u\right\|_{L^p}\\
&\le C\,(\theta*\delta)_j\,,
\end{aligned}$$

where we have argued as in proving lemma 1.37, setting

$$\theta_h:=2^{hs}(1+h)^{\alpha}\qquad\text{and}\qquad\delta_k:=2^{ks}(1+k)^{\alpha}\left\|\Delta_k u\right\|_{L^p}\,.$$

By the made hypothesis on s and α, the sequence $(\theta_h)_h\in\ell^1$; hence, Young's inequality for convolution allows us to conclude. □

Let us now analyse the action of paraproduct operator.

Theorem 1.49. *Let s and $\alpha\in\mathbb{R}$; let also $t>0$ and $\beta\in\mathbb{R}$, or $t=0$ and $\beta<-1$. Finally, let (p,r,r_1,r_2) belong to $[1,+\infty]^4$.*

The paraproduct operator T maps $L^\infty\times B^{s+\alpha\log}_{p,r}$ in $B^{s+\alpha\log}_{p,r}$, and $B^{-t+\beta\log}_{\infty,r_2}\times B^{s+\alpha\log}_{p,r_1}$ in $B^{(s-t)+(\alpha+\beta)\log}_{p,q}$, with $1/q:=\min\{1\,,\,1/r_1+1/r_2\}$. Moreover, the following estimates hold:

$$\begin{aligned}
\left\|T_u v\right\|_{B^{s+\alpha\log}_{p,r}} &\le C\,\|u\|_{L^\infty}\,\|\nabla v\|_{B^{(s-1)+\alpha\log}_{p,r}}\\
\left\|T_u v\right\|_{B^{(s-t)+(\alpha+\beta)\log}_{p,q}} &\le C\,\|u\|_{B^{-t+\beta\log}_{\infty,r_2}}\,\|\nabla v\|_{B^{(s-1)+\alpha\log}_{p,r_1}}\,.
\end{aligned}$$

Proof. As already remarked before theorem 1.24, the generic term $S_{j-1}u\,\Delta_j v$ is spectrally supported in the ring $2^j\widetilde{\mathcal{C}}$. Hence, thanks to lemma 1.37, it's enough to estimate its L^p norm. Then, applying proposition 1.48 gives us the thesis. □

Before going on, let us prove the analogous of lemma 1.26.

Lemma 1.50. *Let $\mathcal{B}$ be a ball of $\mathbb{R}^N$, the couple (p,r) belong to $[1,+\infty]^2$. Let $s>0$ and $\alpha\in\mathbb{R}$, or $s=0$ and $\alpha<-1$. Let $(u_j)_{j\in\mathbb{N}}$ be a sequence of smooth functions such that*

$$\operatorname{supp}\widehat{u}_j\,\subset\,2^j\mathcal{B}\qquad\text{and}\qquad\left(2^{js}\,(1+j)^\alpha\,\|u_j\|_{L^p}\right)_{j\in\mathbb{N}}\,\in\,\ell^r\,.$$

Then the function $u\,:=\,\sum_{j\in\mathbb{N}}u_j$ belongs to the space $B^{s+\alpha\log}_{p,r}$. Moreover, there exists a constant C, depending only on s and α, such that

$$\|u\|_{B^{s+\alpha\log}_{p,r}}\,\leq\,C\,\left\|\left(2^{js}\,(1+j)^\alpha\,\|u_j\|_{L^p}\right)_{j\in\mathbb{N}}\right\|_{\ell^r}\,.$$

Proof. We have to estimate $\|\Delta_k u\|_{L^p}\,\leq\,\sum_j\|\Delta_k u_j\|_{L^p}$.

From our hypothesis on the support of each $\widehat{u}_j$, we infer that there exists an index $n_0\in\mathbb{N}$ such that $\Delta_k u_j\equiv 0$ for all $k>j+n_0$. Therefore, arguing as already done in previous proofs,

$$\begin{aligned}2^{ks}\,(1+k)^\alpha\,\|\Delta_k u\|_{L^p}\;&\leq\;\sum_{j\geq k-n_0}2^{(k-j)s}\,\frac{(1+k)^\alpha}{(1+j)^\alpha}\,(1+j)^\alpha\,\|u_j\|_{L^p}\\&\leq\;\sum_{j\geq k-n_0}2^{(k-j)s}\,(1+|k-j|)^\alpha\,(1+j)^\alpha\,\|u_j\|_{L^p}\,.\end{aligned}$$

So we can conclude thanks to Young's inequality for convolutions (recall hypothesis over the indices s and α). □

Now we are ready to prove continuity properties of the remainder operator.

Theorem 1.51. *Let $(s,t,\alpha,\beta)\in\mathbb{R}^4$ and $(p_1,p_2,r_1,r_2)\in[1,+\infty]^4$ be such that*

$$\frac{1}{p}\,:=\,\frac{1}{p_1}\,+\,\frac{1}{p_2}\,\leq\,1\qquad\text{and}\qquad\frac{1}{r}\,:=\,\frac{1}{r_1}\,+\,\frac{1}{r_2}\,\leq\,1\,.$$

(i) If $s+t>0$, or $s+t=0$ and $\alpha+\beta<-1$, for any $(u,v)\in B^{s+\alpha\log}_{p_1,r_1}\times B^{t+\beta\log}_{p_2,r_2}$ we have

$$\|R(u,v)\|_{B^{(s+t)+(\alpha+\beta)\log}_{p,r}}\,\leq\,C\,\|u\|_{B^{s+\alpha\log}_{p_1,r_1}}\,\|v\|_{B^{t+\beta\log}_{p_2,r_2}}\,.$$

(ii) If $s+t=0$, $\alpha+\beta\geq-1$ and $r=1$, for any $(u,v)\in B^{s+\alpha\log}_{p_1,r_1}\times B^{t+\beta\log}_{p_2,r_2}$ we have

$$\|R(u,v)\|_{B^{(\alpha+\beta)\log}_{p,\infty}}\,\leq\,C\,\|u\|_{B^{s+\alpha\log}_{p_1,r_1}}\,\|v\|_{B^{t+\beta\log}_{p_2,r_2}}\,.$$

Proof. (i) We can write $R(u,v)\,=\,\sum_j R_j$, where we have set

$$R_j\,:=\,\sum_{|h-j|\leq1}\Delta_j u\,\Delta_h v\,.$$

As already pointed out before theorem 1.24, each R_j is spectrally localized on a ball of radius proportional to 2^j. Hence, from lemma 1.50 and Hölder's inequality we immediately infer first estimate.

(ii) In the second case, lemma 1.50 doesn't apply; nevertheless, we can control the norm of the remainder term if the index of summation is $+\infty$. As a matter of facts, we use the following inequality, which holds true for all $k\geq0$:

$$(1+k)^{\alpha+\beta}\,\|\Delta_k R(u,v)\|_{L^p}\,\leq\,C\sum_{j\geq k-n_0}(1+j)^\alpha\,\|\Delta_j u\|_{L^{p_1}}\,(1+j)^\beta\,\|\Delta_j v\|_{L^{p_2}}\,,$$

where, for simplicity, instead of the full R_j, we have considered only the term $\Delta_j u\,\Delta_j v$, the other ones being similar to it.

□

1.5 Paradifferential calculus with parameters

Paraproduct operator is the simplest example of paradifferential operator. The aim of this section is to introduce a more general paradifferential calculus (see e.g. [51], chapters 5 and 6). For convenience, we allow it to depend on some parameter $\gamma \geq 1$: this apparently harmless fact will come into play in a crucial way in chapter 3.

Here we will give only the main definitions and the basic properties: we refer to paper [50] and, in particular, to paper [52], appendix B, for a complete and detailed presentation of this topic. Moreover, in chapter 3 we will analyse in detail the action of paradifferential operators associated to low regularity symbols on the class of logarithmic Sobolev spaces.

1.5.1 New classes of symbols

Fix a parameter $\gamma \geq 1$ and take a cut-off function $\psi \in \mathcal{C}^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ which verifies the following properties:

- there exist $0 < \varepsilon_1 < \varepsilon_2 < 1$ such that

$$\psi(\eta, \xi) \;=\; \begin{cases} 1 & \text{for} \quad |\eta| \leq \varepsilon_1 \, (\gamma + |\xi|) \\ 0 & \text{for} \quad |\eta| \geq \varepsilon_2 \, (\gamma + |\xi|) \,; \end{cases}$$

- for all $(\beta, \alpha) \in \mathbb{N}^N \times \mathbb{N}^N$, there exists a constant $C_{\beta,\alpha}$ such that

$$\left|\partial_\eta^\beta \partial_\xi^\alpha \psi(\eta, \xi)\right| \;\leq\; C_{\beta,\alpha} \, (\gamma + |\xi|)^{-|\alpha|-|\beta|} \,.$$

We will call such a function an "admissible cut-off".

For example, if $\gamma = 1$, one can take

$$\psi(\eta, \xi) \;\equiv\; \psi_0(\eta, \xi) \;:=\; \sum_{k=1}^{+\infty} \chi_{k-1}(\eta) \, \varphi_k(\xi) \,,$$

where χ and φ are the localization (in phase space) functions associated to a Littlewood-Paley decomposition: see e.g. ex. 5.1.5 of [51]. Similarly, if $\gamma > 1$ it is possible to find a suitable integer $\mu \geq 0$ such that

$$\psi_\mu(\eta, \xi) \;:=\; \chi_\mu(\eta) \, \chi_\mu(\xi) \;+\; \sum_{k=\mu+1}^{+\infty} \chi_{k-1}(\eta) \, \varphi_k(\xi) \tag{1.8}$$

is an admissible cut-off function.

Remark 1.52. Let us immediately point out that we can also define a dyadic decomposition depending on the parameter γ. First of all, we set

$$\Lambda(\xi, \gamma) \;:=\; \left(\gamma^2 \;+\; |\xi|^2\right)^{1/2} \,.$$

Then, taken the usual smooth function χ associated to a Littlewood-Paley decomposition, we define

$$\chi_\nu(\xi, \gamma) \;:=\; \chi\left(2^{-\nu} \Lambda(\xi, \gamma)\right) \,, \quad S_\nu^\gamma \;:=\; \chi_\nu(D_x, \gamma) \,, \quad \Delta_\nu^\gamma \;:=\; S_{\nu+1}^\gamma - S_\nu^\gamma \,.$$

The usual properties of the support of the localization functions still hold, and for all fixed $\gamma \geq 1$ and all tempered distributions u, we have

$$u \;=\; \sum_{\nu=0}^{+\infty} \Delta_\nu^\gamma \, u \qquad \text{in} \quad \mathcal{S}' \,.$$

Moreover, we can introduce logarithmic Besov spaces using the new localization operators S_ν^γ, Δ_ν^γ. For the details see section B.1 of [52]. What is important to retain is that, once we fix $\gamma \geq 1$, the previous construction is equivalent to the classical one, and one can still recover previous results. For instance, if we define the space $H_\gamma^{s+\alpha\log}$ as the set of tempered distributions for which

$$\|u\|^2_{H_\gamma^{s+\alpha\log}} := \int_{\mathbb{R}^N_\xi} \Lambda^{2s}(\xi,\gamma)\,\log^{2\alpha}(1+\gamma+|\xi|)\,|\widehat{u}(\xi)|^2\,d\xi \;<\; +\infty\,, \tag{1.9}$$

for every fixed $\gamma \geq 1$ it coincides with $H^{s+\alpha\log}$, the respective norms are equivalent and the characterization given by proposition 1.35 still holds true.

Let us come back to the cut-off function ψ introduced above. Thanks to it, we can define more general paradifferential operators, associated to low regularity functions: let us explain how.

Define the function G^ψ as the inverse Fourier transform of ψ with respect to the variable η:

$$G^\psi(x,\xi) := \left(\mathcal{F}_\eta^{-1}\psi\right)(x,\xi)\,.$$

The following properties hold true (see lemma 5.1.7 of [51] for the proof).

Lemma 1.53. *For all* $(\beta,\alpha) \in \mathbb{N}^N \times \mathbb{N}^N$,

$$\left\|\partial_x^\beta\partial_\xi^\alpha G^\psi(\cdot,\xi)\right\|_{L^1(\mathbb{R}^N_x)} \;\leq\; C_{\beta,\alpha}\,(\gamma+|\xi|)^{-|\alpha|+|\beta|}\,, \tag{1.10}$$

$$\left\||\cdot|\log\left(2+\frac{1}{|\cdot|}\right)\partial_x^\beta\partial_\xi^\alpha G^\psi(\cdot,\xi)\right\|_{L^1(\mathbb{R}^N_x)} \;\leq\; C_{\beta,\alpha}\,(\gamma+|\xi|)^{-|\alpha|+|\beta|-1}\,\log(1+\gamma+|\xi|). \tag{1.11}$$

Thanks to G, we can smooth out a symbol a in the x variable and then define the paradifferential operator associated to a as the classical pseudodifferential operator associated to this smooth function.

First of all, let us define the new calss of symbols we are dealing with, which actually includes the space of Fourier multipliers $S^{m+\delta\log}$ introduced in definition 1.44. In what follows, we take a subspace $\mathcal{X}$ of L^∞. This is convenient for our analysis, but definitions and some other basic properties actually make sense also for a general $\mathcal{X} \subset \mathcal{S}'$.

Definition 1.54. Let $\mathcal{X} \subset L^\infty$ be a Banach space, and m and δ be two given real numbers.

(i) We denote with $\Gamma^{m+\delta\log}(\mathcal{X})$ the space of functions $a(x,\xi,\gamma)$ which are locally bounded over $\mathbb{R}^N \times \mathbb{R}^N \times [1,+\infty[$ and of class $\mathcal{C}^\infty$ with respect to ξ, and which satisfy the following property: for all $\alpha \in \mathbb{N}^N$, there exists a $C_\alpha > 0$ such that, for all (ξ,γ),

$$\left\|\partial_\xi^\alpha a(\cdot,\xi,\gamma)\right\|_{\mathcal{X}} \;\leq\; C_\alpha\,(\gamma+|\xi|)^{m-|\alpha|}\,\log^\delta(1+\gamma+|\xi|)\,. \tag{1.12}$$

In a quite natural way, we can equip $\Gamma^{m+\delta\log}(\mathcal{X})$ with the seminorms

$$\|a\|_{(m,\delta,k)} := \sup_{|\alpha|\leq k}\;\sup_{\mathbb{R}^N_\xi\times[1,+\infty[}\left((\gamma+|\xi|)^{-m+|\alpha|}\,\log^{-\delta}(1+\gamma+|\xi|)\,\left\|\partial_\xi^\alpha a(\cdot,\xi,\gamma)\right\|_{\mathcal{X}}\right)\,. \tag{1.13}$$

(ii) $\Sigma^{m+\delta\log}(\mathcal{X})$ is the space of symbols $\sigma \in \Gamma^{m+\delta\log}(\mathcal{X})$ for which there exists a $0 < \epsilon < 1$ such that, for all (ξ,γ), the spectrum of the function $x \mapsto \sigma(x,\xi,\gamma)$ is contained in the ball $\{|\eta| \leq \epsilon\,(\gamma+|\xi|)\}$.

By spectral localization and Paley-Wiener theorem, a symbol $\sigma \in \Sigma^{m+\delta\log}(\mathcal{X})$ is smooth also in the x variable. So, we can define the subspaces $\Sigma^{m+\delta\log}_{\mu+\varrho\log}(\mathcal{X})$ (for μ and $\varrho \in \mathbb{R}$) of symbols σ which verify (1.12) and also, for all $\beta > 0$,

$$\left\|\partial_x^\beta\partial_\xi^\alpha\sigma(\cdot,\xi,\gamma)\right\|_{\mathcal{X}} \;\leq\; C_{\beta,\alpha}\,(\gamma+|\xi|)^{m-|\alpha|+|\beta|+\mu}\,\log^{\delta+\varrho}(1+\gamma+|\xi|)\,. \tag{1.14}$$

Now, given a symbol $a \in \Gamma^{m+\delta\log}(\mathcal{X})$, we can define

$$\sigma_a^{\psi}(x,\xi,\gamma) := (\psi(D_x,\xi)\,a)(x,\xi,\gamma) = \Big(G^{\psi}(\cdot,\xi) *_x a(\cdot,\xi,\gamma)\Big)(x)\,. \tag{1.15}$$

Proposition 1.55. *(i) For all m, $\delta \in \mathbb{R}$, the smoothing operator*

$$\mathcal{R}:\ a(x,\xi,\gamma) \mapsto \sigma_a^{\psi}(x,\xi,\gamma)$$

is bounded from $\Gamma^{m+\delta\log}(\mathcal{X})$ to $\Sigma^{m+\delta\log}(\mathcal{X})$.

(ii) If a is also differentiable with respect to x, with $\nabla_x a \in \mathcal{X}$, then $a - \sigma_a^{\psi} \in \Gamma^{(m-1)+\delta\log}(\mathcal{X})$.

(iii) In particular, if ψ_1 and ψ_2 are two admissible cut-off functions, then the difference of the two smoothed symbols, $\sigma_a^{\psi_1} - \sigma_a^{\psi_2}$, belongs to the space $\Sigma^{(m-1)+\delta\log}(\mathcal{X})$. Moreover, for all $k \in \mathbb{N}$ one has

$$\left\|\sigma_a^{\psi_1} - \sigma_a^{\psi_2}\right\|_{(m-1,\delta,k)} \le C_k\, \|\nabla_x a\|_{(m,\delta,k)}\,.$$

Remark 1.56. As we will see in a while, part (ii) of previous proposition says that the difference between the original symbol and the classical one associated to it is more regular. Part (iii), instead, infers that the whole construction is independent of the cut-off function fixed at the beginning. Nevertheless, we have to require at least $\mathcal{X} \subset W^{1,\infty}$.
Repeating the same steps of the proof (see e.g. proposition B.7 of [52]), it's easy to see that it's enough to consider symbols a which are $L^{\infty}(\mathbb{R}^N_x)$ and admitting a (even rough) modulus of continuity. Also under this assumption there is a gain of regularity (obviously no more of order 1, as in proposition 1.55, but logarithmic, for instance), which will allows us to recover previous properties. This will be always our case in chapter 3.

1.5.2 General paradifferential operators

As already mentioned, we can now define the paradifferential operator associated to a using the classical symbol corresponding to it:

$$T_a^{\psi}u(x) := \Big(\sigma_a^{\psi}(\,\cdot\,,D_x,\gamma)\,u\Big)(x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N_\xi} e^{ix\cdot\xi}\,\sigma_a^{\psi}(x,\xi,\gamma)\,\widehat{u}(\xi)\,d\xi\,. \tag{1.16}$$

For instance, if $a = a(x) \in L^{\infty}$ and if we take the cut-off function ψ_0, then T_a^{ψ} is actually the usual paraproduct operator. If we take ψ_μ as defined in (1.8), instead, we get a paraproduct operator which starts from high enough frequencies, which will be indicated with T_a^{μ} (see section 3.3 of [23]).

For convenience, we fix the regularity with respect to x: from now on, we will always work with $\mathcal{X} = L^{\infty}$. Therefore, we will miss it out in the notation.

Sometimes, additional regularity in x will be required. Following the presentation of [52], we will suppose $a(x,\xi,\gamma)$ to be $W^{1,\infty}$ with respect to the first variable. However, we always have to keep in mind remark 1.56. We refer to chapter 3 for the analysis of some particular classes of symbols.

Let us now study the action of general paradifferential operators on the class of logarithmic Besov spaces. First of all, a definition is in order.

Definition 1.57. We say that an operator P is of order $m + \delta\log$ if, for every $(s,\alpha) \in \mathbb{R}^2$ and every $(p,r) \in [1,+\infty]^2$, P maps $B^{s+\alpha\log}_{p,r}$ into $B^{(s-m)+(\alpha-\delta)\log}_{p,r}$ continuously.

The next fundamental result generalizes proposition B.9 of [52], which is stated for the Sobolev class. For simplicity, we temporarily drop out the dependence of symbols on γ in the notations.

Lemma 1.58. *For all $\sigma \in \Sigma^{m+\delta\log}$, the corresponding operator $\sigma(\,\cdot\,,D_x)$ is of order $m+\delta\log$.*

Proof. Let us take a $u \in B^{s+\alpha\log}_{p,r}$. We can write

$$\sigma(\,\cdot\,,D_x)\,u \;=\; \sum_{j\geq 0} \sigma(\,\cdot\,,D_x)\,\Delta_j u\,,$$

where each item $\sigma(\,\cdot\,,D_x)\Delta_j u$ is supported in a dyadic ring proportional to 2^j. As a matter of fact, on the phase space we have

$$\mathcal{F}_x\left(\sigma(\,\cdot\,,D_x)\Delta_j u\right)(\xi) \;=\; \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N_\zeta} \left(\mathcal{F}_x\sigma\right)(\xi-\zeta,\zeta)\,\varphi(2^{-j}\zeta)\,\widehat{u}(\zeta)\,d\zeta\,.$$

On the one hand, localization properties of $\mathcal{F}_x\sigma$ implies $(1-\epsilon)|\zeta|-\epsilon\gamma \leq |\xi| \leq (1+\epsilon)|\zeta|+\epsilon\gamma$. On the other hand, we have $|\zeta| \sim 2^j$, and this proves our claim.

Therefore, thanks to lemma 1.37, it's enough to prove that there exists a constant $C>0$ for which, for all $j \geq 0$,

$$2^{j(s-m)}\,(1+j)^{\alpha-\delta}\,\left\|\sigma(\,\cdot\,,D_x)\Delta_j u\right\|_{L^p} \;\leq\; C\,2^{js}\,(1+j)^{\alpha}\,\left\|\Delta_j u\right\|_{L^p}\,.$$

For all $j \geq 0$ we can write:

$$\sigma(x,D_x)\Delta_j u(x) \;=\; \int_{\mathbb{R}^N_y} K_j(x,x-y)\,u(y)\,dy\,,$$

where we have defined the kernel

$$K_j(x,z) \;:=\; \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N_\xi} e^{iz\cdot\xi}\,\sigma(x,\xi)\,\varphi(2^{-j}\xi)\,d\xi \;=\; \mathcal{F}^{-1}_\xi\left(\sigma(x,\,\cdot\,)\,\varphi(2^{-j}\,\cdot\,)\right)(z)$$

Now, arguing as in proposition 1.45 completes the proof. □

Lemma 1.58 immediately implies the following theorem, which describes the action of the new class of paradifferential operators.

Theorem 1.59. *Given a symbol $a \in \Gamma^{m+\delta\log}$, for any admissible cut-off function ψ, the operator T^ψ_a is of order $m+\delta\log$.*

As already remarked, the construction does not depends on the cut-off function ψ used at the beginning. Next result says that main features of a paradifferential operator depend only on its symbol, if it is regular enough.

Proposition 1.60. *If ψ_1 and ψ_2 are two admissible cut-off functions and $a \in \Gamma^{m+\delta\log}(W^{1,\infty})$, then the difference $T^{\psi_1}_a - T^{\psi_2}_a$ is of order $(m-1)+\delta\log$.*

Therefore, changing the cut-off function ψ doesn't change the paradifferential operator associated to a, up to lower order terms. So, in what follows we will miss out the dependence of σ_a and T_a on ψ.

1.5.3 Symbolic calculus

Symbolic calculus still holds true also for general paradifferential operators. For convenience, we restrict ourselves to logarithmic Sobolev spaces framework (i.e. $H^{s+\alpha\log} \equiv B^{s+\alpha\log}_{2,2}$).

First of all, let us quote two fundamental results (as usual, see e.g. [52], appendix B, for their proofs) about composition and adjoint operators.

Theorem 1.61. *Let $a \in \Gamma^{m+\delta\log}(W^{1,\infty})$ and $b \in \Gamma^{\mu+\varrho\log}(W^{1,\infty})$.*

Then $a\,b \in \Gamma^{(m+\mu)+(\delta+\varrho)\log}(W^{1,\infty})$. Moreover, the difference $T_a \circ T_b - T_{ab}$ is an operator of order $(m+\mu-1)+(\delta+\varrho)\log$.
If b is independent of x, then $T_a \circ T_b \equiv T_{ab}$.

Theorem 1.62. *Let $a \in \Gamma^{m+\delta\log}(W^{1,\infty})$. Let us denote with a^* the complex conjugate of the symbol a, and with $(T_a)^*$ the adjoint operator (over L^2) of T_a.*

Then the difference $(T_a)^* - T_{a^*}$ is an operator of order $(m-1)+\delta\log$.

The two previous theorems obviously extend to matrix valued symbols and operators.

Let us now state an estimate, which immediately follows from theorem 1.62 and from theorem B.19 of [52]. It will be of great importance in chapter 3.

Theorem 1.63. *Let $a \in \Gamma^{2m+2\delta\log}(W^{1,\infty})$ and suppose there exists a constant $\lambda > 0$ such that, for all (x,ξ), one has*

$$\mathrm{Re}\ a(x,\xi,\gamma) \ \geq \ \lambda\,(\gamma+|\xi|)^{2m+2\delta\log}.$$

Then there exist a constant $C>0$ and an index $k \in \mathbb{N}$ such that, for all $u \in H^{m+\delta\log}_\gamma$, the following estimate holds true:

$$\frac{\lambda}{2}\,\|u\|^2_{H^{m+\delta\log}_\gamma} \ \leq \ \mathrm{Re}\ (T_a u, u)_{L^2} \ + \ C\,\|\nabla_x a\|^2_{(m,k)}\ \|u\|^2_{H^{(m-1)+\delta\log}_\gamma}. \tag{1.17}$$

The constant C is uniformly bounded for symbols varying in a bounded set.

Here, $(\,\cdot\,,\,\cdot\,)_{L^2}$ denotes the scalar product in L^2, which extends by duality to the coupling $H^{s+\alpha\log} \times H^{-s-\alpha\log}$.

Part II

Hyperbolic Equations with low-regularity coefficients

Chapter 2

Non-Lipschitz coefficients: the one-dimensional case

In this chapter we obtain an energy estimate for a complete strictly hyperbolic operator over $\mathbb{R}_t \times \mathbb{R}_x$, whose second order coefficient satisfies a log-Zygmund continuity condition in the t variable, uniformly with respect to x, and a log-Lipschitz continuity condition in x, uniformly with respect to t. Moreover, we will suppose the coefficients of the first order part to be Hölder continuous and the coefficient of the 0-th order term to be only bounded.

Such a energy estimate allows to get the well-posedness of the Cauchy problem in the space H^∞ in the case the coefficients of the operator are smooth enough with respect to x.

In the next chapter, we will tackle the same problem, but in the more general case of several space variables. The reason why we decided to separate these two instances is that they require two really different approaches.

As explained below, in the one-dimensional case the Tarama's energy (introduced in [56] for coefficients depending only on time) admits a straightforward generalization. Combining it with the main ideas of paper [22] by Colombini and Lerner is enough to get energy estimates (see also paper [19], which deal with homogeneous operators).

Dealing with $x \in \mathbb{R}^N$, $N \geq 2$, instead, requires to pass from multiplication by functions to action by paradifferential operators associated to particular classes of symbols. So, the involved techniques are quite different to the preivous ones, even if the leading ideas are the same. Obviously, the same machinery works also for $N = 1$. Nevertheless, we decided to present these two cases separately: we think that, being simpler, the one-dimensional instance is a good introduction to the problem, and that in this way technical difficulties are better pointed out.

2.1 Introduction

Let us consider the second order operator

$$P = \partial_t^2 - \sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}(t)\partial_{x_j}) \tag{2.1}$$

and suppose that P is strictly hyperbolic with bounded coefficients, i.e. there exist two positive real numbers $\lambda_0 \leq \Lambda_0$ such that

$$\lambda_0\,|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(t)\,\xi_i\xi_j \leq \Lambda_0\,|\xi|^2 \tag{2.2}$$

for all $t \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$.

It is well-known (see e.g. [42] and [58]) that, if the coefficients a_{ij} are Lipschitz-continuous, then the following energy estimate holds for the operator P: for all $s \in \mathbb{R}$, there exists a constant $C_s > 0$ such that

$$\begin{aligned}(2.3)\qquad \sup_{t\in[0,T]} \left(\|u(t,\cdot)\|_{H^{s+1}} + \|\partial_t u(t,\cdot)\|_{H^s}\right) &\leq \\ &\leq C_s \left(\|u(0,\cdot)\|_{H^{s+1}} + \|\partial_t u(0,\cdot)\|_{H^s} + \int_0^T \|Pu(t,\cdot)\|_{H^s}\,dt\right)\end{aligned}$$

for every function $u \in \mathcal{C}^2([0,T]; H^\infty(\mathbb{R}^N))$.
In particular, the previous energy estimate implies that the Cauchy problem for (2.1) is well-posed in the space H^∞, with no loss of derivatives.

On the contrary, if the coefficients a_{ij} are not Lipschitz-continuous, then (2.3) is no more true in general, as it is shown by an example given by Colombini, De Giorgi and Spagnolo in paper [18]. Nevertheless, under suitable weaker regularity assumptions on the coefficients, one can recover the H^∞-well-posedness again, but this time from an energy estimate which involves a loss of derivatives.

A first result of this type was obtained in the quoted paper [18]. The authors supposed that a constant $C > 0$ exists such that, for all $\varepsilon \in]0,T]$,

$$(2.4)\qquad \int_0^{T-\varepsilon} |a_{ij}(t+\varepsilon) - a_{ij}(t)|\,dt \leq C\,\varepsilon\,\log\left(1 + \frac{1}{\varepsilon}\right).$$

The Fourier trasform with respect to x of the equation, together with the new "approximate energy technique" (i.e. the approximation of the coefficients is different in different zones of the phase space), enabled them to obtain the following energy estimate: there exist strictly positive constants K (independent of s) and C_s such that

$$\begin{aligned}(2.5)\qquad \sup_{t\in[0,T]} \left(\|u(t,\cdot)\|_{H^{s+1-K}} + \|\partial_t u(t,\cdot)\|_{H^{s-K}}\right) &\leq \\ &\leq C_s \left(\|u(0,\cdot)\|_{H^{s+1}} + \|\partial_t u(0,\cdot)\|_{H^s} + \int_0^T \|Pu(t,\cdot)\|_{H^s}\,dt\right)\end{aligned}$$

for all $u \in \mathcal{C}^2([0,T]; H^\infty(\mathbb{R}^n))$.

Considering again the case in which the coefficients of P depend only on the time variable, in the recent paper [56] (see also [62]) Tarama weakened the regularity hypothesis further, supposing a log-Zygmund type integral condition, i.e. that there exists a constant $C > 0$ such that, for all $\varepsilon \in]0,T/2]$,

$$(2.6)\qquad \int_\varepsilon^{T-\varepsilon} |a_{ij}(t+\varepsilon) + a_{ij}(t-\varepsilon) - 2a_{ij}(t)|\,dt \leq C\,\varepsilon\,\log\left(1 + \frac{1}{\varepsilon}\right).$$

Nevertheless, he was still able to prove the well-posedness to the Cauchy problem for (2.1) in the space H^∞: the improvement with respect to [18] was obtained introducing a new type of approximate energy, which involves the second derivatives of the approximating coefficients.

Much more difficulties arise if the operator P has coefficients depending both on the time variable t and on the space variables x. This case was considered by Colombini and Lerner in paper [22]. They supposed a pointwise isotropic log-Lipschitz regularity condition, i.e. that there exists $C > 0$ such that, for all $\varepsilon \in]0,T]$,

$$(2.7)\qquad \sup_{\substack{y,z\in[0,T]\times\mathbb{R}^N \\ |z|=\varepsilon}} |a_{ij}(y+z) - a_{ij}(y)| \leq C\,\varepsilon\,\log\left(1 + \frac{1}{\varepsilon}\right).$$

Because the coefficients of the operator P depend also on the space variables, here the Littlewood-Paley dyadic decomposition with respect to x takes the place of the Fourier trasform. Moreover it turns out to be, together with the approximate energy technique, the key tool to obtain the energy estimate: for all fixed $\theta \in]0, 1/4]$, there exist β, $C > 0$ and $T^* \in]0, T]$ such that

$$\begin{aligned} (2.8) \quad \sup_{t\in[0,T^*]} \left(\|u(t,\cdot)\|_{H^{-\theta+1-\beta t}} + \|\partial_t u(t,\cdot)\|_{H^{-\theta-\beta t}}\right) \leq \\ \leq C\left(\|u(0,\cdot)\|_{H^{-\theta+1}} + \|\partial_t u(0,\cdot)\|_{H^{-\theta}} + \int_0^T \|Pu(t,\cdot)\|_{H^{-\theta-\beta t}}\, dt\right) \end{aligned}$$

for all $u \in \mathcal{C}^2([0,T^*]; H^\infty(\mathbb{R}^N))$.
In this case, the loss of derivatives gets worse with the increasing of time.

In the recent paper [19], Colombini and Del Santo considered the case of one space variable (i.e. $N = 1$) and studied again the case of the coefficient a depending both on t and x, but under a special regularity condition: they mixed condition (2.6) together with (2.7). In particular, they supposed a to be log-Zygmund continuous with respect to t, uniformly with respect to x, and log-Lipschitz continuous with respect to x, uniformly with respect to t. The dyadic decomposition technique and the Tarama's approximate energy enabled them to obtain an estimate similar to (2.8).

The reason why they focused on the special instance $N = 1$ is that the case of several space variables needs some different and new ideas in the definition of the microlocal energy. In particular, to handle the problem one has to appeal to paradifferential calculus with parameters. We refer to the next chapter for the complete treatement of the more general case.

In the present chapter, instead, we are considering the case of the non-homogeneous operator

$$(2.9) \qquad Lu = \partial_t^2 u - \partial_x(a(t,x)\partial_x u) + b_0(t,x)\partial_t u + b_1(t,x)\partial_x u + c(t,x)u$$

in dimension $N = 1$. Here we assume the coefficient a to satisfy the same regularity assumptions as in [19] (see conditions (2.10) to (2.12) below). We will also suppose b_0, $b_1 \in L^\infty(\mathbb{R}_t; \mathcal{C}^\omega(\mathbb{R}_x))$ for some $\omega > 0$, where we have set $\mathcal{C}^\omega$ to be the space of ω-Hölder continuous functions, and c to be bounded on the whole $\mathbb{R}_t \times \mathbb{R}_x$. We will apply the Littlewood-Paley decomposition and the Tarama's approximate energy again to obtain an energy estimate with a loss of derivatives that depends on t, as in (2.8). As one can expect, the presence of lower order terms doesn't change the essence of the result.

One can find the estimate of the second order coefficient a in paper [19], however, for reader's convenience, we will give here all the details.

2.2 Main result

Let $a : \mathbb{R}^2 \to \mathbb{R}$ be a function such that, for positive constants $\lambda_0 \leq \Lambda_0$,

$$(2.10) \qquad \lambda_0 \leq a(t,x) \leq \Lambda_0\,.$$

These inequalities mean that operator L, as we will define in (2.15), is strictly hyperbolic with bounded coefficients. Let us assume also that a is log-Zygmund continuous with respect to t and log-Lipschitz continuous with respect to x, uniformly with respect to the other variable:

$$(2.11) \qquad \sup_{(t,x)} |a(t+\tau,x) + a(t-\tau,x) - 2\,a(t,x)| \leq C_0\,\tau\,\log\left(\frac{1}{\tau} + 1\right)$$

$$(2.12) \qquad \sup_{(t,x)} |a(t,x+y) - a(t,x)| \leq C_0\,y\,\log\left(\frac{1}{y} + 1\right).$$

Moreover, let

$$b_0\,,\, b_1 \;\in\; L^\infty(\mathbb{R}_t;\mathcal{C}^\omega(\mathbb{R}_x))\,, \tag{2.13}$$

for some real number $\omega > 0$, and

$$c \;\in\; L^\infty(\mathbb{R}_t \times \mathbb{R}_x)\;. \tag{2.14}$$

Theorem 2.1. *Let us consider, on the whole space $\mathbb{R}^2$, the complete second order operator*

$$Lu \;=\; \partial_t^2 u \;-\; \partial_x(a(t,x)\partial_x u) \;+\; b_0(t,x)\partial_t u \;+\; b_1(t,x)\partial_x u \;+\; c(t,x)u\;, \tag{2.15}$$

whose coefficients a, b_0, b_1 and c satisfy hypothesis (2.10)-(2.14).

Then, for all fixed

$$\theta \;\in\; \left]0, \min\left\{\frac{1}{2}, \frac{\omega}{1+\log 2}\right\}\right[\;,$$

there exist $\beta^* > 0$, $T \in \mathbb{R}$ and $C > 0$ such that

$$\begin{aligned} &\sup_{t\in[0,T]} \left(\|u(t,\cdot)\|_{H^{1-\theta-\beta^* t}} + \|\partial_t u(t,\cdot)\|_{H^{-\theta-\beta^* t}}\right) \;\le \\ &\qquad\le\; C\left(\|u(0,\cdot)\|_{H^{-\theta+1}} + \|\partial_t u(0,\cdot)\|_{H^{-\theta}} + \int_0^T \|Lu(t,\cdot)\|_{H^{-\theta-\beta^* t}} dt\right) \end{aligned} \tag{2.16}$$

for all $u \in \mathcal{C}^2([0,T]; H^\infty(\mathbb{R}_x))$.

2.3 Proof

Let us tackle the proof of theorem 2.1. Following the main ideas of [19], we will smooth out the coefficient of the second order part both with respect to t and x; in the same time, we will perform a dyadic decomposition of the function u with respect to the space variable. Then we will link the approximation parameter with the dual variable, in order to obtain different approximations in different zones of the phase space.

2.3.1 Approximation of the coefficient $a(t,x)$

Let $\rho \in \mathcal{C}_0^\infty(\mathbb{R})$ be an even function, supported on the interval $[-1,1]$, such that $0 \le \rho \le 1$ and $\int \rho(s)ds = 1$. Moreover, let us suppose also that $|\rho'(s)| \le 2$.

For all $0 < \varepsilon \le 1$, we set $\rho_\varepsilon(s) \;=\; (1/\varepsilon)\,\rho(s/\varepsilon)$, and then we define

$$a_\varepsilon(t,x) \;:=\; \int_{\mathbb{R}_t\times\mathbb{R}_x} \rho_\varepsilon(t-s)\,\rho_\varepsilon(x-y)\,a(s,y)\,ds\,dy\,. \tag{2.17}$$

Let us state some properties of the approximate coefficients.

Lemma 2.2. *The following facts hold true for every $\varepsilon \in]0,1]$.*

1. *For all $(t,x) \in \mathbb{R}^2$, one has*

$$\lambda_0 \;\le\; a_\varepsilon(t,x) \;\le\; \Lambda_0\,. \tag{2.18}$$

Moreover,

$$\sup_{(t,x)} |a_\varepsilon(t,x) - a(t,x)| \;\le\; \frac{3}{2}\,C_0\,\varepsilon\,\log\left(\frac{1}{\varepsilon}+1\right)\,. \tag{2.19}$$

2. *For all $\sigma \in]0,1[$, a constant $C_\sigma > 0$ (independent of ε) exists such that*

$$\sup_{(t,x)} |\partial_t a_\varepsilon(t,x)| \leq C_\sigma \, (\Lambda_0 + C_0)\, \varepsilon^{\sigma-1} . \tag{2.20}$$

3. *Finally, the derivatives of a_ε satisfy:*

$$\sup_{(t,x)} |\partial_x a_\varepsilon(t,x)| \leq C_0 \, \|\rho'\|_{L^1} \log\left(\frac{1}{\varepsilon}+1\right) \tag{2.21}$$

$$\sup_{(t,x)} \left|\partial_t^2 a_\varepsilon(t,x)\right| \leq \frac{C_0}{2} \, \|\rho''\|_{L^1} \, \frac{1}{\varepsilon} \log\left(\frac{1}{\varepsilon}+1\right) \tag{2.22}$$

$$\sup_{(t,x)} |\partial_t \partial_x a_\varepsilon(t,x)| \leq C_0 \, \|\rho'\|_{L^1}^2 \, \frac{1}{\varepsilon} \log\left(\frac{1}{\varepsilon}+1\right) . \tag{2.23}$$

Proof. Inequalities in (2.18) immediately follow from the fact that $|\rho| \leq 1$.

Relation (2.19), instead, follows from (2.11), after one has observed that

$$a_\varepsilon(t,x) - a(t,x) = \frac{1}{2} \int_{\mathbb{R}_t} \rho_\varepsilon(s) \int_{\mathbb{R}_y} \rho_\varepsilon(x-y)(a(t+s,y) + a(t-s,y) - 2a(t,y)) \, dy \, ds \,,$$

where we have used the fact that ρ is an even function.

Moreover, one has

$$\partial_t^2 a_\varepsilon(t,x) = \frac{1}{2} \int \rho_\varepsilon''(s) \int \rho_\varepsilon(x-y)(a(t+s,y) + a(t-s,y) - 2a(t,y)) \, dy \, ds \,,$$

from which one can deduce (2.22).

Inequalities (2.21) and (2.23) derive from (2.12) in a very similar way.

Finally, (2.10) and (2.11) imply that for all $\sigma \in]0,1[$, a constant $c_\sigma > 0$ exists such that, for all $\tau > 0$, one has

$$\sup_{(t,x)} |a(t+\tau,x) - a(t,x)| \leq c_\sigma \, (\Lambda_0 + C_0)\, \tau^\sigma . \tag{2.24}$$

Starting from this relation, it's easy to prove (2.20). □

2.3.2 Approximate and total energy

Let $T_0 > 0$ and $u \in \mathcal{C}^2([0,T_0]; H^\infty(\mathbb{R}_x))$. Let us perform a Littlewood-Paley decomposition of u with respect to the space variable, setting $u_0(t,x) = \chi(D_x)u(t,x)$ and, for all $\nu \geq 1$, $u_\nu(t,x) = \varphi_\nu(D_x)u(t,x)$. So, each u_ν is an entire analytic function belonging to L^2. Moreover, keep in mind that $H^s \equiv B^s_{2,2}$ for all $s \in \mathbb{R}$ and so these spaces enjoy the general properties of Besov spaces stated in chapter 1. In particular, let us recall that for all $s \in \mathbb{R}$ there exists a constant $C_s > 0$ such that

$$\frac{1}{C_s} \sum_{\nu=0}^{+\infty} 2^{2\nu s} \, \|u_\nu\|_{L^2}^2 \leq \|u\|_{H^s}^2 \leq C_s \sum_{\nu=0}^{+\infty} 2^{2\nu s} \, \|u_\nu\|_{L^2}^2 \tag{2.25}$$

and that Bernstein's inequalities hold:

$$\|\partial_x u_\nu\|_{L^2} \leq C \, 2^\nu \, \|u_\nu\|_{L^2} \qquad \text{for all} \quad \nu \geq 0 \tag{2.26}$$

$$\|u_\nu\|_{L^2} \leq C \, 2^{-\nu} \, \|\partial_x u_\nu\|_{L^2} \qquad \text{for all} \quad \nu \geq 1 . \tag{2.27}$$

Moreover, let us quote a result on commutation between localization in phase space operators and multiplication by a log-Lipschitz or a Hölder function (recall also lemma 1.29). Here we denote with $\mathcal{L}(L^2)$ the space of bounded linear operators from L^2 to L^2.

Lemma 2.3. *1. There exist $C > 0$ and $\nu_0 \in \mathbb{N}$ such that, for all $a \in L^\infty(\mathbb{R})$ satisfying, for all $y > 0$,*

$$\sup_{x\in\mathbb{R}} |a(x+y) - a(x)| \;\leq\; C_0\, y\, \log\left(1 + \frac{1}{y}\right)\,,$$

one has, for all $\nu \geq \nu_0$,

$$\|\,[\Delta_\nu, a(x)]\,\|_{\mathcal{L}(L^2)} \;\leq\; C\,(\|a\|_{L^\infty} + C_0)\,2^{-\nu}\,\nu\,. \tag{2.28}$$

2. There exist $C_\omega > 0$ (depending only on the fixed Hölder index ω) and $\nu_0 \in \mathbb{N}$ such that, for all $b \in \mathcal{C}^\omega(\mathbb{R})$ and all $\nu \geq \nu_0$, one has

$$\|\,[\Delta_\nu, b(x)]\,\|_{\mathcal{L}(L^2)} \;\leq\; C_\omega\,\|b\|_{\mathcal{C}^\omega}\,2^{-\omega\nu}\,. \tag{2.29}$$

Proof. Former part of previous lemma is proved in [22]. The latter one, instead, easily follows observing that the kernel of operator $[\Delta_\nu, b]$ is

$$\kappa(x,y) \;=\; \widehat{\varphi}(2^\nu(x-y))\,2^\nu\;\left(b(t,x) - b(t,y)\right)\,,$$

and then applying the Schur's criterion (see also lemma 2.6). □

Let us localize equation (2.15) applying operator Δ_ν: we gather that u_ν satisfies

$$\begin{aligned}(Lu)_\nu \;&=\; \partial_t^2 u_\nu \,-\, \partial_x(a(t,x)\partial_x u_\nu) \,-\, \partial_x([\varphi_\nu(D_x),a]\partial_x u) \,+ \\ &+\; b_0(t,x)\partial_t u_\nu + [\varphi_\nu(D_x),b_0]\partial_t u \,+\, b_1(t,x)\partial_x u_\nu \,+\, [\varphi_\nu(D_x),b_1]\partial_x u \,+ \\ &+\; c(t,x)u_\nu \,+\, [\varphi_\nu(D_x),c]u\,.\end{aligned} \tag{2.30}$$

Now we introduce the approximate energy of u_ν (see [19] and [56]), setting

$$e_{\nu,\varepsilon}(t) := \int_{\mathbb{R}}\left(\frac{1}{\sqrt{a_\varepsilon}}\left|\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right|^2 + \sqrt{a_\varepsilon}\,|\partial_x u_\nu|^2 + |u_\nu|^2\right)\,dx\,. \tag{2.31}$$

This particular quantity will turn out to be suitable for our computations. On the one hand, it is strictly related to the Sobolev norm of u (see also remark 2.4). On the other hand, its time derivative will produce fundamental cancellations which allow us to get rid of the bad behaviour of the coefficient a.

Now, taken θ as in the hypothesis of theorem 2.1, we define the total energy of u:

$$E(t) \;:=\; \sum_{\nu=-1}^{+\infty} e^{-2\beta(\nu+1)t}\,2^{-2\nu\theta}\,e_{\nu,2^{-\nu}}(t)\,, \tag{2.32}$$

where the index $\beta > 0$ will be fixed later on.

Remark 2.4. From (2.25) and Bernstein's inequalities (2.26)-(2.27), it's easy to see that there exist two positive constants C_θ and C'_θ such that

$$\begin{aligned}E(0) \;&\leq\; C_\theta\,(\|\partial_t u(0,\cdot)\|_{H^{-\theta}} \,+\, \|u(0,\cdot)\|_{H^{1-\theta}}) \\ E(t) \;&\geq\; C'_\theta\,(\|\partial_t u(t,\cdot)\|_{H^{-\theta-\beta^* t}} \,+\, \|u(t,\cdot)\|_{H^{1-\theta-\beta^* t}})\end{aligned}$$

where we have set $\beta^* = \beta(\log 2)^{-1}$.

Before going on with computing the time derivative of $e_{\nu,\varepsilon}$, let us note that, as L is a linear operator with real valued coefficients, one has $L(v+iw) = Lv + iLw$. So, without loss of generality, we can assume u to be real valued, too.

Moreover, for notation convenience, we define

$$v_{\nu,\varepsilon} := \partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}\,u_\nu \qquad \text{and} \qquad R_\varepsilon f := \partial_t\left(\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}\right)f - \left(\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}\right)^2 f.$$

Now, let us we differentiate $e_{\nu,\varepsilon}$, defined by (2.31), with respect to t, and we obtain

$$\begin{aligned}
\frac{d}{dt}e_{\nu,\varepsilon}(t) &= \int \frac{2}{\sqrt{a_\varepsilon}}\,\partial_t^2 u_\nu\; v_{\nu,\varepsilon}\,dx + \int \frac{2}{\sqrt{a_\varepsilon}}\,R_\varepsilon u_\nu\; v_{\nu,\varepsilon}\,dx\, + \\
&+ \int \partial_t\sqrt{a_\varepsilon}\;|\partial_x u_\nu|^2\,dx + \int 2\,u_\nu\;\partial_t u_\nu\,dx\, + \\
&+ \int 2\sqrt{a_\varepsilon}\,\partial_x u_\nu\;\partial_x\partial_t u_\nu\,dx\,.
\end{aligned}$$

Now, we can put in the previous relation the value of $\partial_t^2 u_\nu$, given by (2.30). Integrating by parts and taking advantage of the spectral localisation of u_ν, we have

$$\begin{aligned}
\int \frac{2}{\sqrt{a_\varepsilon}}\,\partial_x(a\partial_x u_\nu)\;v_{\nu,\varepsilon}\,dx &= \int 2\frac{\partial_x\sqrt{a_\varepsilon}}{a_\varepsilon}\,a\,\partial_x u_\nu\;v_{\nu,\varepsilon}\,dx - \int \frac{\partial_t\sqrt{a_\varepsilon}}{a_\varepsilon}\,a\,|\partial_x u_\nu|^2\,dx\, - \\
&- \int 2\frac{a}{\sqrt{a_\varepsilon}}\,\partial_x u_\nu\;\partial_x\partial_t u_\nu\,dx\, - \\
&- \int \frac{a}{\sqrt{a_\varepsilon}}\,\partial_x\left(\frac{\partial_t\sqrt{a_\varepsilon}}{\sqrt{a_\varepsilon}}\right)\,\partial_x u_\nu\;u_\nu\,dx\,.
\end{aligned}$$

Finally we obtain the complete expression for the time derivative of the approximate energy:

$$\begin{aligned}
\frac{d}{dt}e_{\nu,\varepsilon}(t) &= \int \frac{2}{\sqrt{a_\varepsilon}}\,(Lu)_\nu\;v_{\nu,\varepsilon}\,dx + \int \frac{2}{\sqrt{a_\varepsilon}}\,R_\varepsilon u_\nu\;v_{\nu,\varepsilon}\,dx\, + \\
&+ \int \partial_t\sqrt{a_\varepsilon}\left(1-\frac{a}{a_\varepsilon}\right)|\partial_x u_\nu|^2 dx + \int 2\left(\sqrt{a_\varepsilon}-\frac{a}{\sqrt{a_\varepsilon}}\right)\partial_x u_\nu\,\partial_x\partial_t u_\nu dx\, + \\
&+ \int 2\frac{\partial_x\sqrt{a_\varepsilon}}{a_\varepsilon}\,a\,\partial_x u_\nu\;v_{\nu,\varepsilon}\,dx - \int \frac{a}{\sqrt{a_\varepsilon}}\;\partial_x\left(\frac{\partial_t\sqrt{a_\varepsilon}}{\sqrt{a_\varepsilon}}\right)\partial_x u_\nu\;u_\nu\,dx\, + \\
&+ \int 2\,u_\nu\;\partial_t u_\nu\,dx + \int \frac{2}{\sqrt{a_\varepsilon}}\,(\partial_x([\varphi_\nu(D_x),a]\partial_x u))\;v_{\nu,\varepsilon}\,dx\, - \\
&- \int \frac{2}{\sqrt{a_\varepsilon}}\,b_0(t,x)\,\partial_t u_\nu\;v_{\nu,\varepsilon}\,dx - \int \frac{2}{\sqrt{a_\varepsilon}}\,([\varphi_\nu(D_x),b_0]\partial_t u)\;v_{\nu,\varepsilon}\,dx\, - \\
&- \int \frac{2}{\sqrt{a_\varepsilon}}\,b_1(t,x)\,\partial_x u_\nu\;v_{\nu,\varepsilon}\,dx - \int \frac{2}{\sqrt{a_\varepsilon}}\,([\varphi_\nu(D_x),b_1]\partial_x u)\;v_{\nu,\varepsilon}\,dx\, - \\
&- \int \frac{2}{\sqrt{a_\varepsilon}}\,c(t,x)\,u_\nu\;v_{\nu,\varepsilon}\,dx - \int \frac{2}{\sqrt{a_\varepsilon}}\,([\varphi_\nu(D_x),c]u)\;v_{\nu,\varepsilon}\,dx\,.
\end{aligned} \tag{2.33}$$

2.3.3 Estimate for the approximate energy

We want to obtain an estimate for the time derivative of the energy (2.32); so, let us start to control each term of (2.33).

Throughout the rest of the proof, we will denote with C, C', C'' and $\widehat{C}$ constants depending only on λ_0, Λ_0, C_0, defined by conditions (2.10)-(2.12), and on the norms of b_0, b_1 and c in their respective functional spaces. These constants are allowed to vary from line to line.

Terms with a and a_ε

Thanks to relations (2.10), (2.20) with $\sigma = 1/2$, (2.22) and Bernstein's inequalities, we deduce that there exists $C > 0$, depending only on λ_0, Λ_0 and C_0, such that, for all $\nu \geq 0$,

$$\left|\int \frac{2}{\sqrt{a_\varepsilon}} R_\varepsilon u_\nu \; v_{\nu,\varepsilon}\, dx\right| \leq C\,\frac{1}{\varepsilon}\,\log\left(\frac{1}{\varepsilon}+1\right)\,2^{-\nu}\,e_{\nu,\varepsilon}(t)\,.$$

In the same way, from (2.10), (2.19) and (2.20), we have

$$\left|\int \partial_t\sqrt{a_\varepsilon}\left(1-\frac{a}{a_\varepsilon}\right)|\partial_x u_\nu|^2\,dx\right| \leq C\,\log\left(\frac{1}{\varepsilon}+1\right)\,e_{\nu,\varepsilon}(t)\,,$$

for a constant C depending again only on λ_0, Λ_0 and C_0.

Moreover, keeping in mind (2.10), (2.19) and Bernstein's inequalities, we obtain

$$\begin{aligned}\left|\int 2\left(\sqrt{a_\varepsilon}-\frac{a}{\sqrt{a_\varepsilon}}\right)\partial_x u_\nu\;\partial_x\partial_t u_\nu\,dx\right| &\leq C\,\varepsilon\,\log\left(\frac{1}{\varepsilon}+1\right)\,\|\partial_x u_\nu\|_{L^2}\,\|\partial_x\partial_t u_\nu\|_{L^2}\\ &\leq C\,\varepsilon\,\log\left(\frac{1}{\varepsilon}+1\right)\,2^{\nu}\,\|\partial_x u_\nu\|_{L^2}\,\|\partial_t u_\nu\|_{L^2}\,.\end{aligned}$$

Now we can estimate $\partial_t u_\nu$ writing

$$\|\partial_t u_\nu\|_{L^2} \leq \left\|\partial_t u_\nu + \frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right\|_{L^2} + \left\|\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right\|_{L^2}\,.$$

The former term is actually the energy (up to multiplication by a constant); the latter one can be brought to it thanks to followig relations:

$$\left\|\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_0\right\|_{L^2} \leq C\,\varepsilon^{-1/2}\|u_0\|_{L^2} \quad \text{and} \quad \forall\,\nu\geq 1\,,\ \left\|\frac{\partial_t\sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}}u_\nu\right\|_{L^2} \leq C\,\varepsilon^{-1/2}\,2^{-\nu}\,\|\partial_x u_\nu\|_{L^2}\,.$$

Hence, we gather the bound

$$\|\partial_t u_\nu\|_{L^2} \leq C\,\left(1+\varepsilon^{-1/2}\,2^{-\nu}\right)\,(e_{\nu,\varepsilon})^{1/2}\,; \tag{2.34}$$

therefore, we finally arrive to

$$\left|\int 2\left(\sqrt{a_\varepsilon}-\frac{a}{\sqrt{a_\varepsilon}}\right)\partial_x u_\nu\;\partial_x\partial_t u_\nu dx\right| \leq C\,(\varepsilon\,2^{\nu}+1)\,\log\left(\frac{1}{\varepsilon}+1\right)\,e_{\nu,\varepsilon}(t)\,.$$

In a very similar way, from (2.21) one has

$$\left|\int 2\frac{\partial_x\sqrt{a_\varepsilon}}{a_\varepsilon}\,a\,\partial_x u_\nu\;v_{\nu,\varepsilon}dx\right| \leq C\log\left(\frac{1}{\varepsilon}+1\right)e_{\nu,\varepsilon}(t)\,,$$

while, from (2.20) with $\sigma = 1/2$, (2.21) and (2.23) we deduce

$$\left|\int \frac{a}{\sqrt{a_\varepsilon}}\,\partial_x\left(\frac{\partial_t\sqrt{a_\varepsilon}}{\sqrt{a_\varepsilon}}\right)\,\partial_x u_\nu\;u_\nu\,dx\right| \leq C\,\frac{1}{\varepsilon}\,\log\left(\frac{1}{\varepsilon}+1\right)\,2^{-\nu}\,e_{\nu,\varepsilon}(t)\,.$$

Finally, arguing as done before, we have

$$\left|\int 2\,u_\nu\;\partial_t u_\nu\,dx\right| \leq C\,\varepsilon^{-1/2}\,2^{-\nu}\,e_{\nu,\varepsilon}(t)\,.$$

Terms with b_0, b_1 and c

Thanks to the hypothesis (2.13)-(2.14), one has that there exist suitable constants, depending only on λ_0, Λ_0, C_0, on the norms of b_0 and b_1 in the space $L^\infty(\mathbb{R}_t;\mathcal{C}^\omega(\mathbb{R}_x))$ and on that of c in $L^\infty(\mathbb{R}_t\times\mathbb{R}_x)$, such that

$$\begin{aligned}
\left|\int \frac{2}{\sqrt{a_\varepsilon}}\, b_0(t,x)\,\partial_t u_\nu\;v_{\nu,\varepsilon}\,dx\right| \;&\leq\; C\,\|\partial_t u_\nu\|_{L^2}\,\|v_{\nu,\varepsilon}\|_{L^2}\\
&\leq\; (C+C'\,2^{-\nu}\varepsilon^{-1/2})\,e_{\nu,\varepsilon}(t)\,;\\
\left|\int \frac{2}{\sqrt{a_\varepsilon}}\, b_1(t,x)\,\partial_x u_\nu\;v_{\nu,\varepsilon}\,dx\right| \;&\leq\; C\int \sqrt[4]{a_\varepsilon}\,|\partial_x u_\nu|\,\frac{1}{\sqrt[4]{a_\varepsilon}}\,|v_{\nu,\varepsilon}|\,dx\\
&\leq\; C\int \sqrt{a_\varepsilon}\,|\partial_x u_\nu|^2\,+\,\frac{1}{\sqrt{a_\varepsilon}}\,|v_{\nu,\varepsilon}|^2\,dx\;\leq\;C\,e_{\nu,\varepsilon}(t)\,;\\
\left|\int \frac{2}{\sqrt{a_\varepsilon}}\, c(t,x)\,u_\nu\;v_{\nu,\varepsilon}\,dx\right| \;&\leq\; C\int |u_\nu|\,\frac{1}{\sqrt[4]{a_\varepsilon}}\,|v_{\nu,\varepsilon}|\,dx\;\leq\;C\,e_{\nu,\varepsilon}(t)\,,
\end{aligned}$$

where we delt with $\|\partial_t u_\nu\|_{L^2}$ as before.

Now, we join the approximation parameter ε with the dual variable ξ, following the original idea of Colombini, De Giorgi and Spagnolo in paper [18]. As $|\xi|\sim 2^\nu$ on the support of $\widehat{u}_\nu$, we set

$$\varepsilon\,=\,2^{-\nu}\,.$$

Therefore, from (2.33) and the previous inequalities, we obtain

$$\begin{aligned}
\frac{d}{dt}e_{\nu,2^{-\nu}}(t)\;\leq\;& \widetilde{C}\,(\nu+1)\,e_{\nu,2^{-\nu}}\,+\,\int\frac{2}{\sqrt{a_{2^{-\nu}}}}\,(Lu)_\nu\;v_{\nu,2^{-\nu}}\,dx\,+\\
&+\,\int\frac{2}{\sqrt{a_{2^{-\nu}}}}\,(\partial_x([\Delta_\nu,a]\partial_x u))\;v_{\nu,2^{-\nu}}\,dx\,-\\
&-\,\int\frac{2}{\sqrt{a_{2^{-\nu}}}}\,([\Delta_\nu,b_0]\partial_t u)\;v_{\nu,2^{-\nu}}\,dx\,-\\
&-\,\int\frac{2}{\sqrt{a_{2^{-\nu}}}}\,([\Delta_\nu,b_1]\partial_x u)\;v_{\nu,2^{-\nu}}\,dx\,-\\
&-\,\int\frac{2}{\sqrt{a_{2^{-\nu}}}}\,([\Delta_\nu,c]u)\;v_{\nu,2^{-\nu}}\,dx\,,
\end{aligned}\tag{2.35}$$

for a suitable constant $\widetilde{C}$, which depends only on λ_0, Λ_0, C_0 and on the norms of the coefficients of the operator L in their respective functional spaces.

2.3.4 Estimates for commutator terms

Now, we have to deal with commutator terms. As we will see, it's useful to consider immediately the sum over $\nu\geq 0$.

First of all, we report an elementary lemma (see also [19]), which we will use very often in next computations.

Lemma 2.5. *There exist two continuous, decreasing functions α_1, $\alpha_2:]0,1[\,\rightarrow\,]0,+\infty[$ such that $\lim_{\delta\to 0^+}\alpha_j(\delta)=+\infty$ for $j=1\,,\,2$ and such that, for all $\delta\in]0,1]$ and all $n\geq 1$, the following inequalities hold:*

$$\sum_{j=1}^{n}e^{\delta j}\,j^{-1/2}\,\leq\,\alpha_1(\delta)\,e^{\delta n}\,n^{-1/2}\,,\qquad\qquad \sum_{j=n}^{+\infty}e^{-\delta j}\,j^{1/2}\,\leq\,\alpha_2(\delta)\,e^{-\delta n}\,n^{1/2}\,.$$

Following what done in [22] and [19], in the present subsection we will often use the next result, sometimes referred in literature as "Schur's Lemma", see e.g. [43]. Here we quote a general version, more suitable for our purposes, whose proof can be found in [57], paragraph 0.5.

Lemma 2.6. *Let (X,μ) be a measure space. Suppose that $k(x,y)$ is a measurable function on $X \times X$ such that*

$$\sup_{y\in X}\int_X |k(x,y)|\,d\mu(x)\;\le\;C_1\qquad\textit{and}\qquad \sup_{x\in X}\int_X |k(x,y)|\,d\mu(y)\;\le\;C_2\,.$$

Then for all $p\in[1,+\infty]$, the operator T, defined by

$$Tu(x)\;=\;\int_X k(x,y)\,u(y)\,d\mu(y)$$

maps continuously $L^p(X,\mu)$ into itself. Moreover we have

$$\|Tu\|_{L^p}\;\le\;C_1^{1/p}\,C_2^{1/p'}\;\|u\|_{L^p}\,,$$

where $p'\in[1,+\infty]$ is such that $(1/p)\,+\,(1/p')\;=\;1$.

Before going on, we fix $\beta>0$ and $T\in]0,T_0]$ such that

$$\beta T\;=\;\frac{\theta}{2}\,\log 2\,. \tag{2.36}$$

Remark 2.7. Notice that, thanks to the hypothesis of theorem 2.1, this condition implies

$$\beta T\;\le\;\frac{\omega-\theta}{2}\,.$$

Moreover, for all $t\in[0,T]$, we have:

$$\begin{gathered} 0\;<\;\tfrac{\theta}{2}\log 2\;\le\;\beta t+\tfrac{\theta}{2}\log 2\;\le\;\theta\log 2\;<\;1\\ 0\;<\;\left(1-\tfrac{3}{2}\theta\right)\log 2\;\le\;(1-\theta)\,\log 2\;-\;\beta t\;\le\;(1-\theta)\log 2\;<\;1\,. \end{gathered}$$

Finally, we set (with the same notations used in chapter 1)

$$\psi_\mu\;=\;\varphi_{\mu-1}+\varphi_\mu+\varphi_{\mu+1}\qquad(\varphi_{-1}\equiv 0)\,.$$

As $\psi_\mu\equiv 1$ on the support of φ_μ, we can write

$$\partial_x u_\mu\;=\;\Delta_\mu\partial_x u\;=\;\Psi_\mu(\Delta_\mu\partial_x u)\;=\;\Psi_\mu\partial_x u_\mu\,,$$

where we have defined $\Psi_\mu\;:=\;\psi_\mu(D_x)$. So, given a generic function $f(t,x)$, one has

$$[\Delta_\nu,f]\partial_x u\;=\;[\Delta_\nu,f]\left(\sum_{\mu\ge0}\partial_x u_\mu\right)\;=\;\sum_{\mu\ge0}\left([\Delta_\nu,f]\Psi_\mu\right)\partial_x u_\mu\,. \tag{2.37}$$

After these preliminary remarks, we can go on with commutators' estimates.

Term with $[\Delta_\nu, a]$

Due to Bernstein's inequalities, we have

$$\left\|\partial_x\left(v_{\nu,2^{-\nu}}\right)\right\|_{L^2} \leq C\,2^\nu \left(e_{\nu,2^{-\nu}}(t)\right)^{1/2} .$$

So, using (2.37) and the fact that a_ε is real-valued, one has

$$\begin{aligned}\left|\int \frac{2}{\sqrt{a_{2^{-\nu}}}}\,\partial_x([\Delta_\nu, a]\partial_x u)\; v_{\nu,2^{-\nu}}\,dx\right| &\leq \\ &\leq C\sum_\mu \|([\Delta_\nu, a]\Psi_\mu)\partial_x u_\mu\|_{L^2}\,2^\nu\left(e_{\nu,2^{-\nu}}(t)\right)^{1/2} \\ &\leq C\sum_\mu \|[\Delta_\nu, a]\Psi_\mu\|_{\mathcal{L}(L^2)}\left(e_{\mu,2^{-\mu}}(t)\right)^{1/2} 2^\nu\left(e_{\nu,2^{-\nu}}(t)\right)^{1/2} ,\end{aligned}$$

with the constant C depending only on λ_0, Λ_0 and C_0. Hence,

$$\begin{aligned}&\left|\sum_{\nu\geq 0} e^{-2\beta(\nu+1)t}2^{-2\nu\theta}\int\frac{2}{\sqrt{a_{2^{-\nu}}}}\,\partial_x([\Delta_\nu, a]\partial_x u)\,v_{\nu,2^{-\nu}}\,dx\right| \leq \\ &\quad\leq C\sum_{\nu,\mu} k_{\nu\mu}\,(\nu+1)^{1/2}e^{-\beta(\nu+1)t}2^{-\nu\theta}\left(e_{\nu,2^{-\nu}}\right)^{1/2}\;(\mu+1)^{1/2}e^{-\beta(\mu+1)t}2^{-\mu\theta}\left(e_{\mu,2^{-\mu}}\right)^{1/2} ,\end{aligned}$$

where we have set

$$k_{\nu\mu} \;=\; e^{-(\nu-\mu)\beta t}\,2^{-(\nu-\mu)\theta}\,2^\nu\,(\nu+1)^{-1/2}\,(\mu+1)^{-1/2}\;\|[\Delta_\nu, a]\Psi_\mu\|_{\mathcal{L}(L^2)} \;. \tag{2.38}$$

Observe that, if $|\nu-\mu|\geq 3$, then $\varphi_\nu\psi_\mu\equiv 0$, so $[\Delta_\nu, a]\Psi_\mu = \Delta_\nu[a,\Psi_\mu]$. Therefore, from lemma 2.3, in particular from (2.28), we deduce that

$$\|[\Delta_\nu, a(t,x)]\Psi_\mu\|_{\mathcal{L}(L^2)} \;\leq\; \begin{cases} C\,2^{-\nu}(\nu+1) & \text{if } |\nu-\mu|\leq 2 \,, \\[1ex] C\,2^{-max\{\nu,\mu\}}(\max\{\nu,\mu\}+1) & \text{if } |\nu-\mu|\geq 3 \,, \end{cases}$$

where the constant C depends only on Λ_0 and C_0.

Now our aim is to apply Schur's lemma 2.6, so to estimate the quantity

$$\sup_\mu\sum_\nu |k_{\nu\mu}| \;+\; \sup_\nu\sum_\mu |k_{\nu\mu}| \;. \tag{2.39}$$

To do this, we will use lemma 2.5 and the inequalities stated in remark 2.7.

1. Fix $\mu\leq 2$. We have

$$\begin{aligned}\sum_{\nu\geq 0}|k_{\nu\mu}| &\leq C\,e^{(\mu+1)\beta t}\,2^{(\mu+1)\theta}\,(\mu+1)^{-\frac{1}{2}}\sum_\nu e^{-(\nu+1)\beta t}\,2^{-(\nu+1)\theta}\,(\nu+1)^{\frac{1}{2}} \\ &= C\,e^{(\mu+1)\beta t}\,2^{(\mu+1)\theta}\,(\mu+1)^{-\frac{1}{2}}\sum_\nu e^{-(\nu+1)(\beta t+\theta\log 2)}\,(\nu+1)^{\frac{1}{2}} \\ &\leq C\,e^{3\beta t}\,2^{3\theta}\,\alpha_2(\beta t+\theta\log 2) \;\leq\; C\,2^{\frac{9}{2}\theta}\,\alpha_2(\theta\log 2)\,.\end{aligned}$$

2. Now, take $\mu \geq 3$ and consider first

$$
\begin{aligned}
\sum_{\nu=0}^{\mu-3} |k_{\nu\mu}| &\leq C\, e^{(\mu+1)\beta t}\, 2^{-(\mu+1)(1-\theta)}\, (\mu+1)^{\frac{1}{2}} \sum_{\nu=0}^{\mu-3} e^{-(\nu+1)\beta t}\, 2^{(\nu+1)(1-\theta)}\, (\nu+1)^{-\frac{1}{2}} \\
&\leq C\, e^{(\mu+1)\beta t}\, 2^{-(\mu+1)(1-\theta)}\, (\mu+1)^{\frac{1}{2}} \sum_{\nu=0}^{\mu-3} e^{(\nu+1)(-\beta t+(1-\theta)\log 2)}\, (\nu+1)^{-\frac{1}{2}} \\
&\leq C\, e^{(\mu+1)\beta t}\, 2^{-(\mu+1)(1-\theta)}\, (\mu+1)^{\frac{1}{2}}\, \alpha_1(-\beta t+(1-\theta)\log 2)\,\cdot \\
&\qquad\qquad \cdot\, e^{(-\beta t+(1-\theta)\log 2)(\mu-2)}\, (\mu-2)^{-\frac{1}{2}} \\
&\leq C\, 2^{\frac{9}{2}\theta}\, \alpha_1\left(\left(1-\frac{3}{2}\theta\right)\log 2\right).
\end{aligned}
$$

For the second part of the sum, one has

$$
\begin{aligned}
\sum_{\nu=\mu-2}^{+\infty} |k_{\nu\mu}| &\leq C\, e^{(\mu+1)\beta t}\, 2^{(\mu+1)\theta}\, (\mu+1)^{-1/2} \sum_{\nu=\mu-2}^{+\infty} e^{-(\nu+1)\beta t}\, 2^{-(\nu+1)\theta}\, (\nu+1)^{1/2} \\
&\leq C\, e^{(\mu+1)\beta t}\, 2^{(\mu+1)\theta}\, (\mu+1)^{-1/2} \alpha_2(\beta t+\theta\log 2)\,\cdot \\
&\qquad\qquad \cdot\, e^{-(\beta t+\theta\log 2)(\mu-1)}\, (\mu-1)^{1/2} \\
&\leq C\, 2^{\frac{7}{2}\theta}\, \alpha_2(\theta\log 2)\,.
\end{aligned}
$$

3. Fix now $\nu \geq 0$; we have

$$
\begin{aligned}
\sum_{\mu=0}^{\nu+2} |k_{\nu\mu}| &\leq C\, e^{-(\nu+1)\beta t}\, 2^{-(\nu+1)\theta}\, (\nu+1)^{1/2} \sum_{\mu=0}^{\nu+2} e^{(\mu+1)\beta t}\, 2^{(\mu+1)\theta}\, (\mu+1)^{-1/2} \\
&\leq C\, e^{-(\nu+1)\beta t}\, 2^{-(\nu+1)\theta}\, (\nu+1)^{1/2}\, \alpha_1(\beta t+\theta\log 2)\,\cdot \\
&\qquad\qquad \cdot\, e^{(\beta t+\theta\log 2)(\nu+3)}\, (\nu+3)^{-1/2} \\
&\leq C\, 2^{\frac{7}{2}\theta}\, \alpha_1(\theta\log 2)\,.
\end{aligned}
$$

For the second part of the series, the following inequality holds:

$$
\begin{aligned}
\sum_{\mu=\nu+3}^{+\infty} |k_{\nu\mu}| &\leq C\, e^{-(\nu+1)\beta t}\, 2^{(\nu+1)(1-\theta)}\, (\nu+1)^{-\frac{1}{2}} \sum_{\mu=\nu+3}^{+\infty} e^{(\mu+1)\beta t}\, 2^{-(\mu+1)(1-\theta)}\, (\mu+1)^{\frac{1}{2}} \\
&\leq C\, e^{-(\nu+1)\beta t}\, 2^{(\nu+1)(1-\theta)}\, (\nu+1)^{-\frac{1}{2}}\, \alpha_2(-\beta t+(1-\theta)\log 2)\,\cdot \\
&\qquad\qquad \cdot\, e^{(-\beta t+(1-\theta)\log 2)(\nu+4)}\, (\nu+4)^{\frac{1}{2}} \\
&\leq C\, 2^{\frac{9}{2}\theta}\, \alpha_2\left(\left(1-\frac{3}{2}\theta\right)\log 2\right).
\end{aligned}
$$

In conclusion, there exists a positive function Π, with $\lim_{\theta\to 0^+}\Pi(\theta)=+\infty$, such that

$$
\sup_\mu \sum_\nu |k_{\nu\mu}| + \sup_\nu \sum_\mu |k_{\nu\mu}| \leq C\, \Pi(\theta)\,;
$$

then, by Schur's lemma we gather

$$
\left|\sum_{\nu\geq 0} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta} \int \frac{2}{\sqrt{a_{2^{-\nu}}}}\, \partial_x([\Delta_\nu, a]\partial_x u)\, v_{\nu,2^{-\nu}}\, dx\right| \leq
$$

$$
\leq C\, \Pi(\theta) \sum_{\nu=0}^{+\infty} (\nu+1)\, e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, e_{\nu,2^{-\nu}}(t)\,.
$$

Terms with $[\Delta_\nu, b_0]$ and $[\Delta_\nu, b_1]$

Now, let us consider commutator terms involving first order coefficients of operator L. The analysis is essentially the same carried out in the previous paragraph, nevertheless we will give here all the details.

Let us start with considering the term

$$\begin{aligned}\left|\int \frac{2}{\sqrt{a_{2^{-\nu}}}}\,[\Delta_\nu, b_0(t,x)]\partial_t u\, v_{\nu,2^{-\nu}}\,dx\right| &\leq 2\,\|[\Delta_\nu, b_0(t,x)]\partial_t u\|_{L^2}\left\|\frac{1}{\sqrt{a_{2^{-\nu}}}}\left|v_{\nu,2^{-\nu}}\right|\right\|_{L^2}\\ &\leq 2\,\|[\Delta_\nu, b_0(t,x)]\partial_t u\|_{L^2}\,(e_{\nu,2^{-\nu}}(t))^{1/2}\,.\end{aligned}$$

Thanks to relation (2.37), we have

$$\begin{aligned}\|[\Delta_\nu, b_0(t,x)]\partial_t u\|_{L^2} &= \left\|[\Delta_\nu, b_0(t,x)]\sum_{\mu\geq 0}\Psi_\mu\partial_t u_\mu\right\|_{L^2}\\ &\leq \sum_{\mu\geq 0}\|[\Delta_\nu, b_0(t,x)]\Psi_\mu\|_{\mathcal{L}(L^2)}\|\partial_t u_\mu\|_{L^2}\,.\end{aligned}$$

On the one hand, estimate (2.34) with $\varepsilon = 2^{-\nu}$ gives us

$$\|\partial_t u_\mu\|_{L^2} \leq C\,\left(e_{\mu,2^{-\mu}}\right)^{1/2}\,.$$

on the other hand, from lemma 2.3 we get

$$\|[\Delta_\nu, b_0(t,x)]\Psi_\mu\|_{\mathcal{L}(L^2)} \leq \begin{cases} C\,2^{-\nu\omega} & \text{if } |\nu-\mu|\leq 2\\ C\,2^{-max\{\mu,\nu\}\omega} & \text{if } |\nu-\mu|\geq 3\end{cases}$$

where C is a constant depending only on $\|b_0\|_{L^\infty(\mathbb{R}_t;\mathcal{C}^\omega(\mathbb{R}_x))}$.

Therefore,

$$\begin{aligned}&\left|\sum_{\nu\geq 0} e^{-2\beta t(\nu+1)}2^{-2\nu\theta}\int\frac{2}{\sqrt{a_{2^{-\nu}}}}\,[\Delta_\nu, b_0]\partial_t u\, v_{\nu,2^{-\nu}}\,dx\right| \leq\\ &\qquad\leq \sum_{\nu,\mu\geq 0} e^{-\beta t(\nu+1)}\,2^{-\nu\theta}\,\left(e_{\nu,2^{-\nu}}\right)^{1/2}\,e^{-\beta t(\mu+1)}\,2^{-\mu\theta}\,\left(e_{\mu,2^{-\mu}}\right)^{1/2}\,l_{\nu\mu}\,,\end{aligned}$$

where we have defined

$$l_{\nu\mu} := e^{-(\nu-\mu)\beta t}\,2^{-(\nu-\mu)\theta}\,\|[\Delta_\nu, b_0(t,x)]\Psi_\mu\|_{\mathcal{L}(L^2)}\,. \tag{2.40}$$

As made before, we are going to estimate $l_{\nu\mu}$ applying Schur's lemma.

1. Let us fix $\mu\leq 2$. Then

$$\begin{aligned}\sum_{\nu\geq 0}|l_{\nu\mu}| &\leq C\,e^{(\mu+1)\beta t}\,2^{(\mu+1)\theta}\sum_{\nu\geq 0}e^{-(\nu+1)\beta t}\,2^{-(\nu+1)\theta}\,2^{-\nu\omega}\\ &\leq C\,e^{3\beta t}\,2^{3\theta}\sum_{\nu\geq 0}e^{-(\nu+1)(\beta t+\theta\log 2)}\,(\nu+1)^{1/2}\\ &\leq C\,e^{3\beta t}\,2^{3\theta}\,\alpha_2(\beta t+\theta\log 2) \leq C\,2^{\frac{9}{2}\theta}\,\alpha_2(\theta\log 2)\,.\end{aligned}$$

2. Now, take $\mu \geq 3$ and consider first

$$\begin{aligned}
\sum_{\nu=0}^{\mu-3} |l_{\nu\mu}| &\leq C\, e^{\mu\beta t}\, 2^{\mu\theta}\, 2^{-\mu\omega} \sum_{\nu=0}^{\mu-3} e^{-\nu\beta t}\, 2^{-\nu\theta} \\
&\leq C\, e^{\mu(\beta t-(\omega-\theta)\log 2)}\, (\mu-2) \\
&\leq C\, e^{-\mu(\omega-\frac{\theta}{2})\log 2}\, (\mu-2) \;\leq\; C\, M(\omega,\theta)\,,
\end{aligned}$$

where $M(\omega,\theta)$ is the maximum of the function $z \mapsto e^{-\gamma z}\,(z-2)$, with $\gamma = \left(\omega - \frac{\theta}{2}\right)\log 2$. For the second part of the sum, we have instead

$$\begin{aligned}
\sum_{\nu=\mu-2}^{+\infty} |l_{\nu\mu}| &\leq C\, e^{(\mu+1)\beta t}\, 2^{(\mu+1)\theta} \sum_{\nu=\mu-2}^{+\infty} e^{-(\nu+1)\beta t}\, 2^{-(\nu+1)\theta}\, 2^{-\nu\omega}\, \frac{(\nu+1)^{1/2}}{(\nu+1)^{1/2}} \\
&\leq C\, e^{(\mu+1)\beta t}\, 2^{(\mu+1)\theta}\, (\mu-1)^{-1/2}\, \alpha_2(\beta t + \theta\log 2)\;\cdot \\
&\qquad\qquad \cdot\, e^{-(\mu-1)\beta t}\, 2^{-(\mu-1)\theta}\, (\mu-1)^{1/2} \\
&\leq C\, e^{2(\beta t+\theta\log 2)}\, \alpha_2(\theta\log 2) \;\leq\; C\, 2^{\frac{9}{2}\theta}\, \alpha_2(\theta\log 2)\,.
\end{aligned}$$

3. Fix now ν. Initially, we have

$$\begin{aligned}
\sum_{\mu=0}^{\nu+2} |l_{\nu\mu}| &\leq C\, e^{-(\nu+1)\beta t}\, 2^{-(\nu+1)\theta}\, 2^{-\nu\omega} \sum_{\mu=0}^{\nu+2} e^{(\mu+1)(\beta t+\theta\log 2)}\, \frac{(\mu+1)^{1/2}}{(\mu+1)^{1/2}} \\
&\leq C\, e^{-(\nu+1)\beta t}\, 2^{-(\nu+1)\theta}\, 2^{-\nu\omega}\, (\nu+3)^{1/2}\, \alpha_1(\beta t+\theta\log 2)\;\cdot \\
&\qquad\qquad \cdot\, e^{(\nu+3)\beta t}\, 2^{(\nu+3)\theta}\, (\nu+3)^{-1/2} \\
&\leq C\, e^{2(\beta t+\theta\log 2)}\, \alpha_1(\theta\log 2) \;\leq\; C\, 2^{3\theta}\, \alpha_1(\theta\log 2)\,.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\sum_{\mu=\nu+3}^{+\infty} |l_{\nu\mu}| &\leq C\, e^{-\nu\beta t}\, 2^{-\nu\theta} \sum_{\mu=\nu+3}^{+\infty} e^{-\mu(-\beta t+(\omega-\theta)\log 2)}\, \frac{\mu^{1/2}}{\mu^{1/2}} \\
&\leq C\, e^{-\nu\beta t}\, 2^{-\nu\theta}\, (\nu+3)^{-1/2}\, \alpha_2((\omega-\theta)\log 2 - \beta t)\;\cdot \\
&\qquad\qquad \cdot\, e^{(\nu+3)\beta t}\, 2^{-(\nu+3)(\omega-\theta)}\, (\nu+3)^{1/2} \\
&\leq C\, e^{3\beta t}\, 2^{3\theta}\, 2^{-(\nu+3)\omega}\, \alpha_2\left(\left(\omega - \frac{3}{2}\theta\right)\log 2\right) \\
&\leq C\, 2^{\frac{9}{2}\theta}\, \alpha_2\left(\left(\omega - \frac{3}{2}\theta\right)\log 2\right)\,.
\end{aligned}$$

From all these inequalities, thanks to Schur's lemma, one has that there exists a constant $\widetilde{M}(\omega,\theta)$, depending only on $\|b_0\|_{L^\infty(\mathbb{R}_t;\mathcal{C}^\omega(\mathbb{R}_x))}$ and on the fixed parameter θ, such that

$$\sup_\mu \sum_\nu |l_{\nu\mu}| + \sup_\nu \sum_\mu |l_{\nu\mu}| \leq C\, \widetilde{M}(\omega,\theta)\,;$$

from this relation, we finally get

$$\begin{aligned}
&\left| \sum_{\nu\geq 0} e^{-2\beta t(\nu+1)} 2^{-2\nu\theta} \int \frac{2}{\sqrt{a_{2^{-\nu}}}}\, [\Delta_\nu, b_0]\partial_t u\, v_{\nu,2^{-\nu}}\, dx \right| \leq \\
&\qquad\qquad \leq C\, \widetilde{M}(\omega,\theta) \sum_{\nu=0}^{+\infty} (\nu+1)\, e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, e_{\nu,2^{-\nu}}(t)\,.
\end{aligned}$$

The term with the commutator $[\Delta_\nu, b_1(t,x)]$ is analogous. Arguing as before, we discover that

$$\begin{aligned}
&\left|\int \frac{2}{\sqrt{a_{2^{-\nu}}}}\,[\Delta_\nu, b_1]\partial_x u\, v_{\nu,2^{-\nu}}\,dx\right| \leq \\
&\qquad\leq C\sum_{\mu\geq 0}\|[\Delta_\nu, b_1]\Psi_\mu\|_{\mathcal{L}(L^2)}\,\|\sqrt[4]{a_{2^{-\nu}}}\,\partial_x u_\mu\|_{L^2}\,\left\|\frac{1}{\sqrt[4]{a_{2^{-\nu}}}}\,\left(v_{\nu,2^{-\nu}}\right)\right\|_{L^2}\\
&\qquad\leq C\sum_{\mu\geq 0}\|[\Delta_\nu, b_1]\Psi_\mu\|_{\mathcal{L}(L^2)}\,\left(e_{\nu,2^{-\nu}}\right)^{1/2}\left(e_{\mu,2^{-\mu}}\right)^{1/2}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\left|\sum_{\nu\geq 0} e^{-2\beta t(\nu+1)}\,2^{-2\nu\theta}\int \frac{2}{\sqrt{a_{2^{-\nu}}}}\,[\Delta_\nu, b_1]\partial_x u\, v_{\nu,2^{-\nu}}\,dx\right| \leq \\
&\qquad\leq C\sum_{\nu,\mu\geq 0} e^{-\beta(\nu+1)t}\,2^{-\nu\theta}\,\left(e_{\nu,2^{-\nu}}\right)^{1/2}\,e^{-\beta(\mu+1)t}\,2^{-\mu\theta}\,\left(e_{\mu,2^{-\mu}}\right)^{1/2}\,l'_{\nu\mu}\,,
\end{aligned}$$

where we have set again

$$l'_{\nu\mu} \;=\; e^{-(\nu-\mu)\beta t}\,2^{-(\nu-\mu)\theta}\,\|[\Delta_\nu, b_1]\Psi_\mu\|_{\mathcal{L}(L^2)}\,.$$

As b_0 and b_1 satisfy to the same hypothesis, the commutator $[\Delta_\nu, b_1]$ verifies the same inequalities as $[\Delta_\nu, b_0]$; so, if we repeat the same computations, we obtain

$$\begin{aligned}
&\left|\sum_{\nu\geq 0} e^{-2\beta t(\nu+1)}\,2^{-2\nu\theta}\int \frac{2}{\sqrt{a_{2^{-\nu}}}}\,[\Delta_\nu, b_1]\partial_x u\, v_{\nu,2^{-\nu}}dx\right| \leq \\
&\qquad\leq C\,\widetilde{M}(\omega,\theta)\sum_{\nu=0}^{+\infty}(\nu+1)\,e^{-2\beta(\nu+1)t}\,2^{-2\nu\theta}\,e_{\nu,2^{-\nu}}(t)\,.
\end{aligned}$$

Term with $[\Delta_\nu, c]$

Finally, we have to deal with the commutator $[\Delta_\nu, c(t,x)]$.

First of all, observe that there exist constants such that

$$\begin{aligned}
\left|\int \frac{2}{\sqrt{a_{2^{-\nu}}}}\,[\Delta_\nu, c]\partial_x u\, v_{\nu,2^{-\nu}}\,dx\right| \;&\leq\; C\,\|[\Delta_\nu, c]u\|_{L^2}\,\left\|\frac{1}{\sqrt[4]{a_{2^{-\nu}}}}\,\left(v_{\nu,2^{-\nu}}\right)\right\|_{L^2}\\
&\leq\; C\sum_{\mu\geq 0}\|[\Delta_\nu, c]\Psi_\mu\|_{\mathcal{L}(L^2)}\,\|u_\mu\|_{L^2}\,\left(e_{\nu,2^{-\nu}}(t)\right)^{1/2}\\
&\leq\; 2\,C\sum_{\mu\geq 0}\|[\Delta_\nu, c]\Psi_\mu\|_{\mathcal{L}(L^2)}\,2^{-\mu}\,\|\partial_x u_\mu\|_{L^2}\,\left(e_{\nu,2^{-\nu}}(t)\right)^{1/2}\\
&\leq\; 2\,C\sum_{\mu\geq 0}\|[\Delta_\nu, c]\Psi_\mu\|_{\mathcal{L}(L^2)}\,2^{-\mu}\,\left(e_{\mu,2^{-\mu}}(t)\right)^{1/2}\,\left(e_{\nu,2^{-\nu}}(t)\right)^{1/2}.
\end{aligned}$$

Thereby, we get the estimate

$$\begin{aligned}
&\left|\sum_{\nu\geq 0} e^{-2\beta t(\nu+1)}\,2^{-2\nu\theta}\int \frac{2}{\sqrt{a_{2^{-\nu}}}}\,[\Delta_\nu, c]\partial_x u\, v_{\nu,2^{-\nu}}\,dx\right| \leq \\
&\qquad\leq 2\,C\sum_{\nu,\mu\geq 0} e^{-\beta t(\nu+1)}\,2^{-\nu\theta}\,\left(e_{\nu,2^{-\nu}}(t)\right)^{1/2}\,e^{-\beta t(\mu+1)}\,2^{-\mu\theta}\,\left(e_{\mu,2^{-\mu}}(t)\right)^{1/2}\,m_{\nu\mu}\,,
\end{aligned}$$

where we have defined, as usual,

$$m_{\nu\mu} \;=\; e^{-(\nu-\mu)\beta t}\, 2^{-(\nu-\mu)\theta}\, 2^{-\mu}\, \|[\Delta_\nu, c]\Psi_\mu\|_{\mathcal{L}(L^2)}\,.$$

Now, the kernel of the operator $[\Delta_\nu, c]$ is

$$h(x,y) \;=\; \widehat{\psi}(2^\nu(x-y))\, 2^\nu\, (c(t,y)-c(t,x))\,;$$

so, remembering that c is bounded over $\mathbb{R}\times\mathbb{R}$, from Schur's lemma one gets

$$\|[\Delta_\nu, c]\|_{\mathcal{L}(L^2)} \;\leq\; C \qquad\qquad \forall\, \nu \geq 0\,,$$

where the constant C depends only on $\|c\|_{L^\infty(\mathbb{R}_t\times\mathbb{R}_x)}$.

Again, we are going to estimate the kernel $m_{\nu\mu}$ to apply Schur's lemma.

1. First, we take $\mu \leq 2$ and we have

$$\begin{aligned}
\sum_{\nu\geq 0} |m_{\nu\mu}| \;&\leq\; C\, e^{(\mu+1)\beta t}\, 2^{(\mu+1)\theta}\, 2^{-\mu} \sum_{\nu\geq 0} e^{-(\nu+1)\beta t}\, 2^{-(\nu+1)\theta} \\
&\leq\; C\, e^{3\beta t}\, 2^{3\theta} \sum_{\nu\geq 0} e^{-(\nu+1)(\beta t+\theta\log 2)}\, (\nu+1)^{1/2} \\
&\leq\; C\, e^{3\beta t}\, 2^{3\theta}\, \alpha_2(\beta t+\theta\log 2) \;\leq\; C\, 2^{\frac{9}{2}\theta}\, \alpha_2(\theta\log 2)\,.
\end{aligned}$$

2. Now, we fix $\mu \geq 3$ and we consider the first part of the series:

$$\begin{aligned}
\sum_{\nu=0}^{\mu-3} |m_{\nu\mu}| \;&\leq\; C\, e^{(\mu+1)\beta t}\, 2^{-(\mu+1)(1-\theta)}\, 2^{-\mu} \sum_{\nu=0}^{\mu-3} e^{-(\nu+1)\beta t}\, 2^{(\nu+1)(1-\theta)}\, 2^{\mu-\nu}\, \frac{(\nu+1)^{1/2}}{(\nu+1)^{1/2}} \\
&\leq\; C\, e^{(\mu+1)\beta t}\, 2^{-(\mu+1)(1-\theta)}\, (\mu-2)^{1/2} \sum_{\nu=0}^{\mu-3} e^{(\nu+1)((1-\theta)\log 2-\beta t)}\, (\nu+1)^{-1/2} \\
&\leq\; C\, e^{(\mu+1)\beta t}\, 2^{-(\mu+1)(1-\theta)}\, (\mu-2)^{1/2}\, \alpha_1((1-\theta)\log 2-\beta t)\;\cdot \\
&\qquad\qquad \cdot\, e^{-(\mu-2)\beta t}\, 2^{(\mu-2)(1-\theta)}\, (\mu-2)^{-1/2} \\
&\leq\; C\, e^{3\beta t}\, 2^{3\theta}\, \alpha_1\left(\left(1-\frac{3}{2}\theta\right)\log 2\right) \;\leq\; C\, 2^{\frac{9}{2}\theta}\, \alpha_1\left(\left(1-\frac{3}{2}\theta\right)\log 2\right)\,.
\end{aligned}$$

For the second part, one has:

$$\begin{aligned}
\sum_{\nu=\mu-2}^{+\infty} |m_{\nu\mu}| \;&\leq\; C\, e^{(\mu+1)\beta t}\, 2^{(\mu+1)\theta}\, 2^{-\mu} \sum_{\nu=\mu-2}^{+\infty} e^{-(\nu+1)\beta t}\, 2^{-(\nu+1)\theta}\, \frac{(\nu+1)^{1/2}}{(\nu+1)^{1/2}} \\
&\leq\; C\, e^{(\mu+1)\beta t}\, 2^{(\mu+1)\theta}\, 2^{-\mu}\, \alpha_2(\beta t+\theta\log 2)\, e^{-(\mu-1)\beta t}\, 2^{-(\mu-1)\theta} \\
&\leq\; C\, 2^{3\theta}\, \alpha_2(\theta\log 2)\,.
\end{aligned}$$

3. Now, we fix $\nu \geq 0$. Initially, we consider

$$\begin{aligned}
\sum_{\mu=0}^{\nu+2} |m_{\nu\mu}| \;&\leq\; C\, e^{-(\nu+1)\beta t}\, 2^{-(\nu+1)\theta} \sum_{\mu=0}^{\nu+2} e^{(\mu+1)(\beta t+\theta\log 2)}\, 2^{-\mu}\, \frac{(\mu+1)^{1/2}}{(\mu+1)^{1/2}} \\
&\leq\; C\, e^{-(\nu+1)\beta t}\, 2^{-(\nu+1)\theta}\, (\nu+3)^{1/2}\, \alpha_1(\beta t+\theta\log 2)\;\cdot \\
&\qquad\qquad \cdot\, e^{(\nu+3)\beta t}\, 2^{(\nu+3)\theta}\, (\nu+3)^{-1/2} \\
&\leq\; C\, e^{2(\beta t+\theta\log 2)}\, \alpha_1(\theta\log 2) \;\leq\; C\, 2^{3\theta}\, \alpha_1(\theta\log 2)\,.
\end{aligned}$$

The second part of the series, instead, can be treated as follow:

$$\begin{aligned}
\sum_{\mu=\nu+3}^{+\infty} |m_{\nu\mu}| &\leq C\, e^{-(\nu+1)\beta t}\, 2^{(\nu+1)(1-\theta)} \sum_{\mu=\nu+3}^{+\infty} e^{(\mu+1)\beta t}\, 2^{-\mu}\, 2^{-(\mu+1)(1-\theta)}\, 2^{\mu-\nu}\, \frac{(\mu+1)^{\frac{1}{2}}}{(\mu+1)^{\frac{1}{2}}} \\
&\leq C\, e^{-(\nu+1)\beta t}\, 2^{(\nu+1)(1-\theta)}\, (\nu+4)^{-\frac{1}{2}} \sum_{\mu=\nu+3}^{+\infty} e^{-(\mu+1)((1-\theta)\log 2-\beta t)}\, (\mu+1)^{\frac{1}{2}} \\
&\leq C\, e^{-(\nu+1)\beta t}\, 2^{(\nu+1)(1-\theta)}\, \alpha_2((1-\theta)\log 2-\beta t)\, e^{(\nu+4)\beta t}\, 2^{-(\nu+4)(1-\theta)} \\
&\leq C\, 2^{\frac{9}{2}\theta}\, \alpha_2\left(\left(1-\frac{3}{2}\theta\right)\log 2\right).
\end{aligned}$$

Finally, we obtain:

$$\left|\sum_{\nu\geq 0} e^{-2\beta t(\nu+1)}\, 2^{-2\nu\theta} \int \frac{2}{\sqrt{a_{2^{-\nu}}}}\, [\Delta_\nu, c]\partial_x u\, v_{\nu,2^{-\nu}}\, dx\right| \leq$$

$$\leq C\, \varPi(\theta) \sum_{\nu=0}^{+\infty} (\nu+1)\, e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, e_{\nu,2^{-\nu}}(t)\,,$$

where the function $\varPi$ is the same used in the estimate of the term $[\Delta_\nu, a]$.

2.3.5 End of the proof of theorem 2.1

Now we are able to complete the proof of theorem 2.1.

First of all, remembering the definition of the total energy given by (2.32), we gather that there exists a constant $C > 0$, depending only on θ, such that

$$\left|\sum_{\nu=0}^{+\infty} e^{-\beta(\nu+1)t}\, 2^{-2\nu\theta} \int \frac{2}{\sqrt{a_\varepsilon}}\, (Lu)_\nu \cdot v_{\nu,2^{-\nu}}\, dx\right| \leq C\, (E(t))^{1/2}\, \|Lu\|_{H^{-\theta-\beta^* t}}\,.$$

Now, we put estimates just proved in paragraph 2.3.4 into relation (2.35). Therefore, if we set $\widetilde{\varPi}(\omega,\theta) = \max\{\widetilde{M}(\omega,\theta), \varPi(\theta)\}$, we have that, for suitable constants, depending only on λ_0, Λ_0, C_0 and on the norms of the coefficients of operator L in their respective functional spaces, the following inequality holds true for all $t \in [0, T]$:

$$\begin{aligned}
\frac{d}{dt}E(t) &\leq \left(C + C'\, \widetilde{\varPi}(\omega,\theta) - 2\beta\right) \sum_{\nu=0}^{+\infty} (\nu+1)\, e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, e_{\nu,2^{-\nu}}(t)\, + \\
&+ C''\, (E(t))^{1/2}\, \|Lu\|_{H^{-\theta-\beta^* t}}\,.
\end{aligned}$$

Now, let us fix β large enough, such that $C + C'\, \widetilde{\varPi}(\omega,\theta) - 2\beta \leq 0$. We can always do this, on condition that we take T small enough: recall that, by (2.36), only the product βT has been fixed untill now. With this choice, we have

$$\frac{d}{dt}E(t) \leq C''\, (E(t))^{1/2}\, \|Lu\|_{H^{-\theta-\beta^* t}}\,;$$

and the conclusion of the theorem follows from Gronwall's lemma, keeping in mind remark 2.4.

Chapter 3

Non-Lipschitz coefficients: the general N-dimensional case

In this chapter we will keep studying the Cauchy problem for strictly hyperbolic operators with low regularity coefficients. As in previous chapter, we will suppose the coefficients to be log-Zygmund continuous in time and log-Lipschitz continuous in space, but we will tackle the case of any space dimension $N \geq 1$. Again, we will find an energy estimate with a time-dependent loss of derivatives, which allows to get the well-posedness issue in the space H^∞ for the related Cauchy problem (if the coefficients are smooth enough with respect to x).

Paradifferential calculus with parameters will be the main tool to handle the problem and get the improvement with respect to the previous chapter. Let us note that, thanks to it, it will be needed to perform a mollification of the coefficients only in the time variable.

Let us point out that here, for simplicity, we will focus only on a homogeneous second order hyperbolic operator, but the same techniques work also for dealing with lower order terms.

3.1 Introduction

This chapter is devoted to the study of the Cauchy problem for a second order strictly hyperbolic operator defined in a strip $[0,T] \times \mathbb{R}^N$, for some $T > 0$ and $N \geq 1$. Consider a second order operator of the form

$$Lu \;:=\; \partial_t^2 u \,-\, \sum_{j,k=1}^{N} \partial_j \left(a_{jk}(t,x)\, \partial_k u\right) \tag{3.1}$$

and assume that L is strictly hyperbolic with bounded coefficients, i.e. there exist two constants $0 < \lambda_0 \leq \Lambda_0$ such that

$$\lambda_0\, |\xi|^2 \;\leq\; \sum_{j,k=1}^{N} a_{jk}(t,x)\, \xi_j\, \xi_k \;\leq\; \Lambda_0\, |\xi|^2$$

for all $(t,x) \in [0,T] \times \mathbb{R}^N$ and all $\xi \in \mathbb{R}^N$.

It is well-known (see e.g. [42] or [53]) that, if the coefficients a_{jk} are Lipschitz continuous with respect to t and even only measurable in x, then the Cauchy problem for L is well-posed in H^1–L^2. If the a_{jk}'s are Lipschitz continuous with respect to t and C_b^∞ (i.e. $\mathcal{C}^\infty$ and bounded with all their derivatives) with respect to the space variables, one can recover the well-posedness in H^{s+1}–H^s for all $s \in \mathbb{R}$. Moreover, in the latter case, one gets, for all $s \in \mathbb{R}$ and for a constant

C_s depending only on it, the following energy estimate:

$$\sup_{0\leq t\leq T}\left(\|u(t,\cdot)\|_{H^{s+1}}+\|\partial_t u(t,\cdot)\|_{H^s}\right)\leq \tag{3.2}$$
$$\leq C_s\left(\|u(0,\cdot)\|_{H^{s+1}}+\|\partial_t u(0,\cdot)\|_{H^s}+\int_0^T\|Lu(t,\cdot)\|_{H^s}\,dt\right)$$

for all $u\in\mathcal{C}([0,T];H^{s+1}(\mathbb{R}^N))\cap\mathcal{C}^1([0,T];H^s(\mathbb{R}^N))$ such that $Lu\in L^1([0,T];H^s(\mathbb{R}^N))$. Let us explicitly remark that previous inequality involves no loss of regularity for the function u: it holds true for every $u\in\mathcal{C}^2([0,T];H^\infty(\mathbb{R}^N))$ and the Cauchy problem for L is well-posed in H^∞ *with no loss of derivatives.*

If the Lipschitz continuity (in time) hypothesis is not fulfilled, then (3.2) is no more true. Nevertheless, one can still try to recover H^∞-well-posedness, with a *loss of derivatives* in the energy estimate.

The first case to consider is when the coefficients a_{jk} depend only on t:

$$Lu\;=\;\partial_t^2u\;-\;\sum_{j,k=1}^N a_{jk}(t)\,\partial_j\partial_k u\,.$$

In [18], Colombini, De Giorgi and Spagnolo assumed the coefficients to satisfy an integral log-Lipschitz condition:

$$\int_0^{T-\varepsilon}|a_{jk}(t+\varepsilon)\;-\;a_{jk}(t)|\,dt\;\leq\;C\,\varepsilon\,\log\left(1\;+\;\frac{1}{\varepsilon}\right)\,, \tag{3.3}$$

for some constant $C>0$ and all $\varepsilon\in]0,T]$. To get the energy estimate, they first smoothed out coefficients using a mollifier kernel (ρ_ε). Then, by Fourier transform, they defined an approximated energy $E_\varepsilon(\xi,t)$ in phase space, where the problem becomes a family of ordinary differential equations. At that point, the key idea was to perform a different approximation of the coefficients in different zones of the phase space: in particular, they set $\varepsilon=|\xi|^{-1}$. Finally, they obtained an energy estimate *with a fixed loss of derivatives*: there exists a constant $\delta>0$ such that, for all $s\in\mathbb{R}$, the inequality

$$\sup_{0\leq t\leq T}\left(\|u(t,\cdot)\|_{H^{s+1-\delta}}+\|\partial_t u(t,\cdot)\|_{H^{s-\delta}}\right)\leq \tag{3.4}$$
$$\leq C_s\left(\|u(0,\cdot)\|_{H^{s+1}}+\|\partial_t u(0,\cdot)\|_{H^s}+\int_0^T\|Lu(t,\cdot)\|_{H^s}\,dt\right)$$

holds true for all $u\in\mathcal{C}^2([0,T];H^\infty(\mathbb{R}^N))$, for some constant C_s depending only on s. Let us remark that if the coefficients a_{jk} are not Lipschitz continuous, then a loss of regularity cannot be avoided, as shown by Cicognani and Colombini in [16]. Besides, in this paper the authors prove that, if the regularity of the coefficients a_{jk} is measured by a modulus of continuity, any intermediate modulus of continuity between the Lipschitz and the log-Lipschitz ones necessarily entails a loss of regularity, which, however, can be made arbitrarily small.

Recently Tarama (see paper [56]) analysed the problem when coefficients satisfy an integral log-Zygmund condition: there exists a constant $C>0$ such that, for all j, k and all $\varepsilon\in]0,T/2[$, one has

$$\int_\varepsilon^{T-\varepsilon}|a_{jk}(t+\varepsilon)\;+\;a_{jk}(t-\varepsilon)\;-\;2\,a_{jk}(t)|\,dt\;\leq\;C\,\varepsilon\,\log\left(1\;+\;\frac{1}{\varepsilon}\right)\,. \tag{3.5}$$

On the one hand, this assumption is somehow related to the pointwise condition (for a function $a\in\mathcal{C}^2([0,T])$) $|a(t)|+|t\,a'(t)|+|t^2\,a''(t)|\;\leq\;C$, considered by Yamazaki in [62], and hence it is a requirement on the growth of the second derivative of a. On the other hand, it's obvious that, if the a_{jk}'s satisfy (3.3), then they satisfy also (3.5): so, a more general class of functions is considered. Again, Fourier transform, smoothing out the cofficients and linking the approximation parameter with the dual variable were fundamental tools in the analysis of Tarama. The improvement with respect to paper [18], however, was obtained defining a new energy, which involved (by derivation in time) second derivatives of the approximated coefficients. Finally, he got an estimate analogous to (3.4), which implies, in particular, well-posedness in the space H^∞.

In paper [22], Colombini and Lerner considered instead the case in which coefficients a_{jk} depend both on time and on space variables. In particular, they assumed an isotropic pointwise log-Lipschitz condition, i.e. there exists a constant $C > 0$ such that, for all $\zeta = (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^N$, one has

$$\sup_{z=(t,x)\in\mathbb{R}\times\mathbb{R}^N} |a_{jk}(z+\zeta) - a_{jk}(z)| \leq C\,|\zeta|\,\log\left(1 + \frac{1}{|\zeta|}\right).$$

Once again, smoothing out coefficients with respect to the time variable is required; on the contrary, one cannot use the Fourier transform, due to the dependence of a_{jk} on x. The authors bypassed this problem appealing to the Littlewood-Paley decomposition and paradifferential calculus. They defined an energy concerning each localized part $\Delta_\nu u$ of the solution u, and then they performed a weighed summation to put all these pieces together. Also in this case, they had to consider a different approximation of the coefficients in different zones of the phase space, which was obtained setting $\varepsilon = 2^{-\nu}$ (recall that 2^ν is the "size" of the frequencies in the ν-th ring, see chapter 1). In the end, they got the following statement: for all $s \in]0, 1/4]$, there exist positive constants β and C_s and a time $T^* \in]0, T]$ such that

$$\text{(3.6)}\quad \sup_{0\leq t\leq T^*}\left(\|u(t,\cdot)\|_{H^{-s+1-\beta t}} + \|\partial_t u(t,\cdot)\|_{H^{-s-\beta t}}\right) \leq$$
$$\leq C_s\left(\|u(0,\cdot)\|_{H^{-s+1}} + \|\partial_t u(0,\cdot)\|_{H^{-s}} + \int_0^{T^*}\|Lu(t,\cdot)\|_{H^{-s-\beta t}}\,dt\right)$$

for all $u \in \mathcal{C}^2([0,T]; H^\infty(\mathbb{R}^N))$. Let us point out that the bound on s was due to this reason: the product by a log-Lipschitz function is well-defined in H^s if and only if $|s| < 1$. Note also that this fact gives us a bound on the lifespan of the solution: the regularity index $-s + 1 - \beta T^*$ has to be strictly positive, so one can expect only local in time existence of a solution. Moreover in the case the coefficients a_{jk} are $\mathcal{C}_b^\infty$ in space, the authors proved inequality (3.6) for all s: so, they still got well-posedness in H^∞, but *with a loss of derivatives increasing in time.*

The case of a complete strictly hyperbolic second order operator,

$$Lu = \sum_{j,k=0}^{N} \partial_{y_j}\left(a_{jk}\,\partial_{y_k}u\right) + \sum_{j=0}^{N}\left(b_j\,\partial_{y_j}u + \partial_{y_j}(c_j\,u)\right) + d\,u$$

(here we set $y = (t,x) \in \mathbb{R}_t \times \mathbb{R}_x^N$), was tackled by Colombini and Métivier in [23]. They assumed the same isotropic log-Lipschitz condition of [22] on the coefficients of the second order part of L, while b_j and c_j were supposed to be α-Hölder continuous (for some $\alpha \in]1/2, 1[$) and d to be only bounded. The authors considered questions such as local existence and uniqueness, and also finite propagation speed for local solutions.

Recently, Colombini and Del Santo, in [19] (for a first approach to the problem see also [31], where smoothness in space were required), came back to the Cauchy problem for the operator (3.1), mixing up a Tarama-like hypothesis (concerning the dependence on the time variable) with the one of Colombini and Lerner (with respect to x). More precisely, they assumed a pointwise log-Zygmund condition in time and a pointwise log-Lipschitz condition in space, uniformly with respect to the other variable (see conditions (3.9) and (3.10) below). However, they had to restrict themselves to the case of space dimension $N = 1$: as a matter of fact, a Tarama-type energy was somehow necessary to compensate the bad behaviour of the coefficients with respect to t, but it was not clear how to define it in higher space dimensions. Again, localizing energy by use of Littlewood-Paley decomposition and linking approximation parameter and dual variable together lead to an estimate analogous to (3.6).

The aim of the present chapter is to extend the result of Colombini and Del Santo to any dimension $N \geq 1$. As just pointed out, the main difficulty was to define a suitable energy related to the solution. So, the first step is to pass from functions $a(t,x)$ with low regularity modulus of continuity, to more general symbols $\sigma_a(t,x,\xi)$ (obviously related to the initial function a) satisfying the same hypothesis in t and x, and then to consider paradifferential operators associated to these symbols. Nevertheless, positivity hypothesis on a (required for defining a strictly hyperbolic problem) does not translate, in general, to positivity of the corresponding operator, which is fundamental in obtaining energy estimates. At this point, paradifferential calculus depending on a parameter $\gamma \geq 1$, as presented in chapter 1, comes into play and allows us to recover positivity of the (new) paradifferential operator associated to a. Defining a localized energy and an approximation of the coefficients depending on the dual variable are, once again, basic ingredients in closing estimates. Hence, in the end we will get an inequality similar to (3.6), for any $s \in]0,1[$.

The chapter is organized as follows.

First of all, we will present the work hypothesis for our strictly hyperbolic problem, and we will state our main results.

A complete treatement about functions with low regularity modulus of continuity follows. In particular, we will focus on log-Zymgund and log-Lipschitz conditions: by a broad use of paradifferential calculus, we will state properties of functions satisfying such hypothesis and of the relative smoothed-in-time (by a convolution kernel) ones. Hence, we will pass to consider more general symbols and the associated paradifferential operators, for which we will develop also a symbolic calculus and we will state a fundamental positivity estimate. This section is deeply based on the theory developed in chapter 1

This having been done, we will be then ready to tackle the proof of our main result: we will go back to the main ideas of paper [19]. First of all, taking advantage of a convolution kernel, we will smooth out the coefficients, but with respect to the time variable only. As a matter of fact, low regularity in x will be compensated by considering paradifferential operators associated to our coefficients. Then, we will decompose the solution u to the Cauchy problem for (3.1) into dyadic blocks $\Delta_\nu u$, for which we will define an approximate localized energy e_ν: the dependence on the approximation parameter ε will be linked to the phase space localization, setting $\varepsilon = 2^{-\nu}$. The piece of energy e_ν will be of Tarama type, but this time multiplication by functions will be replaced by action of paradifferential operators associated to them. A weighted summation of these pieces will define the total energy $E(t)$ associated to u. The rest of the proof is classical: we will differentiate E with respect to time and, using Gronwall's lemma, we will get a control for it in terms of initial energy $E(0)$ and external force Lu only.

3.2 Basic definitions and main result

This section is devoted to the presentation of our work setting and of our main result.

Let us consider the operator over $[0,T_0] \times \mathbb{R}^N$ (for some $T_0 > 0$ and $N \geq 1$)

$$Lu = \partial_t^2 u - \sum_{i,j=1}^{N} \partial_i \left(a_{ij}(t,x)\, \partial_j u \right) , \tag{3.7}$$

and let us suppose L to be strictly hyperbolic with bounded coefficients, i.e. there exist two positive constants $0 < \lambda_0 \leq \Lambda_0$ such that, for all $(t,x) \in \mathbb{R}_t \times \mathbb{R}^N_x$ and all $\xi \in \mathbb{R}^N$, one has

$$\lambda_0\,|\xi|^2 \;\leq\; \sum_{i,j=1}^{N} a_{ij}(t,x)\,\xi_i\,\xi_j \;\leq\; \Lambda_0\,|\xi|^2\,. \tag{3.8}$$

Moreover, let us suppose the coefficients to be log-Zygmund-continuous in the time variable t, uniformly with respect to x, and log-Lipschitz-continuous in the space variables, uniformly with respect to t. This hypothesis reads as follow: there exists a constant K_0 such that, for all $\tau > 0$ and all $y \in \mathbb{R}^N \setminus \{0\}$, one has

$$\sup_{(t,x)} |a_{ij}(t+\tau,x) + a_{ij}(t-\tau,x) - 2a_{ij}(t,x)| \;\leq\; K_0\,\tau\,\log\left(1+\frac{1}{\tau}\right) \tag{3.9}$$

$$\sup_{(t,x)} |a_{ij}(t,x+y) - a_{ij}(t,x)| \;\leq\; K_0\,|y|\,\log\left(1+\frac{1}{|y|}\right). \tag{3.10}$$

Now, let us state our main result, i.e. an energy estimate for the operator (3.7).

Theorem 3.1. *Let us consider the operator L defined in* (3.7), *and let us suppose L to be strictly hyperbolic with bounded coefficients, i.e. relation* (3.8) *holds true. Moreover, let us suppose that the a_{ij}'s satisfy also conditions* (3.9) *and* (3.10).

Then, for all fixed $\theta \in]0,1[$, there exist some $\beta^* > 0$, some time $T > 0$ and some constant $C > 0$ such that the following estimate,

$$\sup_{0\leq t\leq T}\left(\|u(t,\cdot)\|_{H^{-\theta+1-\beta^* t}} + \|\partial_t u(t,\cdot)\|_{H^{-\theta-\beta^* t}}\right) \leq$$
$$\leq C\left(\|u(0,\cdot)\|_{H^{-\theta+1}} + \|\partial_t u(0,\cdot)\|_{H^{-\theta}} + \int_0^T \|Lu(t,\cdot)\|_{H^{-\theta-\beta^* t}}\,dt\right), \tag{3.11}$$

holds true for all $u \in \mathcal{C}^2([0,T];H^\infty(\mathbb{R}^N))$.

So, it's possible to control the Sobolev norms of solutions to (3.7) in terms of those of initial data and of the external force only: the price to pay is a loss of derivatives, increasing (linearly) in time.

3.3 Tools

The main tools we need to prove our statement all come from Fourier Analysis. We will broadly make use of the methods developed in chapter 1: in particular, Littlewood-Paley decomposition, logarithmic Sobolev spaces and paradifferential calculus depending on parameters.

In this section we use the just mentioned techniques to study functions having low regularity modulus of continuity. In particular, we will focus on log-Zygmund and log-Lipschitz functions: dyadic decomposition allows us to get some of their properties. Moreover, we will analyse also the convolution of a log-Zygmund function by a smoothing kernel.
Finally, taking advantage of paradifferential calculus with parameters, we will consider general symbols having such a low regularity in time and space variables. Under suitable hypothesis on such a symbol, we will also get positivity estimates for the associated paradifferential operator.

3.3.1 On log-Lipschitz and log-Zygmund functions

Let us now give the rigorous definitions of the modulus of continuity of funtions we are dealing with, and state some of their properties.

Definition 3.2. A function $f \in L^\infty(\mathbb{R}^N)$ is said to be log-Lipschitz, and we write $f \in LL(\mathbb{R}^N)$, if the quantity

$$|f|_{LL} := \sup_{x\in\mathbb{R}^N} \sup_{0<|y|<1} \left(\frac{|f(x+y) - f(x)|}{|y| \log\left(1 + \frac{1}{|y|}\right)}\right) < +\infty .$$

We define $\|f\|_{LL} := \|f\|_{L^\infty} + |f|_{LL}$.

Let us define also the space of log-Zygmund functions. We will give the general definition in $\mathbb{R}^N$, even if one dimensional case will be the only relevant one for our purposes.

Definition 3.3. A function $g \in L^\infty(\mathbb{R}^N)$ is said to be log-Zygmund, and we write $g \in LZ(\mathbb{R}^N)$, if the quantity

$$|g|_{LZ} := \sup_{x\in\mathbb{R}^N} \sup_{0<|y|<1} \left(\frac{|g(x+y) + g(x-y) - 2\,g(x)|}{|y| \log\left(1 + \frac{1}{|y|}\right)}\right) < +\infty .$$

We define $\|g\|_{LZ} := \|g\|_{L^\infty} + |g|_{LZ}$.

Remark 3.4. Let us immediately point out that, by monotonicity of logarithmic function, we can replace the factor $\log(1+1/|y|)$ in previous definitions with $\log(1+\gamma+1/|y|)$, for all parameters $\gamma \geq 1$. As paradifferential calculus with parameters will play a fundamental role in our computations, it's convenient to perform such a change, and so does also in hypothesis (3.9) and (3.10) of section 3.2.

Let us give a characterization of the space LZ. Recall that the space of Zygmund functions is actually $B^1_{\infty,\infty}$: following the same proof of this case (see e.g. [13]) one can prove next proposition.

Proposition 3.5. *The space $LZ(\mathbb{R}^N)$ coincides with the logarithmic Besov space $B^{1-\log}_{\infty,\infty}$, i.e. the space of tempered distributions u such that*

$$\sup_{k\geq 0}\left(2^k\,(1+k)^{-1}\,\|\Delta_k u\|_{L^\infty}\right) < +\infty . \tag{3.12}$$

Proof. (i) Let us first consider a $u \in B^{1-\log}_{\infty,\infty}$ and take x and $y \in \mathbb{R}^N$, with $|y| < 1$. For all fixed $n \in \mathbb{N}$ we can write:

$$\begin{aligned} u(x+y) + u(x-y) - 2u(x) &= \sum_{k<n} \left(\Delta_k u(x+y) + \Delta_k u(x-y) - 2\Delta_k u(x)\right) + \\ &\quad + \sum_{k\geq n} \left(\Delta_k u(x+y) + \Delta_k u(x-y) - 2\Delta_k u(x)\right) . \end{aligned}$$

First, we take advantage of the Taylor's formula up to second order to handle the former terms; then, we use property (3.12). Hence we get

$$\begin{aligned} |u(x+y) + u(x-y) - 2u(x)| &\leq C\,|y|^2 \sum_{k<n} \left\|\nabla^2 \Delta_k u\right\|_{L^\infty} + 4 \sum_{k\geq n} \|\Delta_k u\|_{L^\infty} \\ &\leq C\left(|y|^2 \sum_{k<n} 2^k\,(k+1) + \sum_{k\geq n} 2^{-k}(k+1)\right) \\ &\leq C\,(n+1)\left(|y|^2\,2^n + 2^{-n}\right) . \end{aligned}$$

Now, as $|y| < 1$, the choice $n = 1 + [\log_2(1/|y|)]$ (where with $[\varrho]$ we mean the greatest positive integer less than or equal to ϱ) completes the proof of the first part.

(ii) Now, given a log-Zygmund function u, we want to estimate the L^∞ norm of its localized part $\Delta_k u$.

Let us recall that applying the operator Δ_k is the same of the convolution with the inverse Fourier transform of the function $\varphi(2^{-k}\cdot)$, which we call h_k. Now, $h_k(x) = 2^{kN}h(2^k x)$, where we set $h = \mathcal{F}_\xi^{-1}(\varphi)$. As φ is an even function, so does h; moreover we have

$$\int h(z)\,dz \;=\; \int \mathcal{F}_\xi^{-1}(\varphi)(z)\,dz \;=\; \varphi(\xi)_{|\xi=0} \;=\; 0\,.$$

Therefore, we can write:

$$\Delta_k u(x) \;=\; 2^{kN-1}\int h(2^k y)\,(u(x+y)+u(x-y)-2u(x))\;dy\,,$$

and noting that $\varrho \mapsto \varrho\log(1+\gamma+1/\varrho)$ is increasing over $]0,+\infty[$ completes the proof to the second part of the proposition.

□

From definitions 3.2 and 3.3, it's obvious that $LL(\mathbb{R}^N) \hookrightarrow LZ(\mathbb{R}^N)$: proposition 3.3 of [22] explains this property in terms of dyadic decomposition. We quote here its statement.

Proposition 3.6. *There exists a constant C such that, for all $a \in LL(\mathbb{R}^N)$ and all integers $k > 0$, we have*

$$\left\|\Delta_k a\right\|_{L^\infty} \;\leq\; C\,(k+1)\,2^{-k}\,\|a\|_{LL}\,. \tag{3.13}$$

Moreover, for all $k \in \mathbb{N}$ we have

$$\left\|a \;-\; S_k a\right\|_{L^\infty} \;\leq\; C\,(k+1)\,2^{-k}\,\|a\|_{LL} \tag{3.14}$$

$$\left\|S_k a\right\|_{\mathcal{C}^{0,1}} \;\leq\; C\,(k+1)\,\|a\|_{LL}\,. \tag{3.15}$$

Remark 3.7. Note that, again from proposition 3.3 of [22], property (3.15) is a characterization of the space $LL(\mathbb{R}^N)$.

Using dyadic characterization of the space LZ and following the same ideas as those of the proof of proposition 3.5, we can prove the following property. This time we consider a log-Zygmund function a depending only on the time variable t, which is enough for our purposes, but the same reasoning holds true also in higher dimensions.

Lemma 3.8. *For all $a \in LZ(\mathbb{R})$, there exists a constant C, depending only on the LZ norm of a, such that, for all $\gamma \geq 1$ and all $0 < |\tau| < 1$ one has*

$$\sup_{t\in\mathbb{R}}|a(t+\tau)-a(t)| \;\leq\; C\,|\tau|\,\log^2\!\left(1+\gamma+\frac{1}{|\tau|}\right). \tag{3.16}$$

Proof. As done in proving proposition 3.5, for all $n \in \mathbb{N}$ we can write

$$a(t+\tau)-a(t) \;=\; \sum_{k<n}\left(\Delta_k a(t+\tau)-\Delta_k a(t)\right) \;+\; \sum_{k\geq n}\left(\Delta_k a(t+\tau)-\Delta_k a(t)\right)\,,$$

where, obviously, the localization in frequencies is performed in one dimension (with respect to the time variable). For the former terms we use the mean value theorem, while for the latter ones we use characterization (3.13); hence, we get

$$\begin{aligned}
|a(t+\tau)-a(t)| \;&\leq\; \sum_{k<n}\left\|\frac{d}{dt}\Delta_k a\right\|_{L^\infty}|\tau| \;+\; 2\sum_{k\geq n}\left\|\Delta_k a\right\|_{L^\infty} \\
&\leq\; C\left(n^2\,|\tau| \;+\; \sum_{k\geq n}2^{-k}k\right).
\end{aligned}$$

The series in the right-hand side of the previous inequality can be bounded, up to a multiplicative constant, by $2^{-n}n$; therefore

$$|a(t+\tau)-a(t)| \,\leq\, C\,n\left(n\,|\tau|\,+\,2^{-n}\right)\,,$$

and the choice $n = 1 + [\log_2(1/|\tau|)]$ completes the proof. □

Now, given a log-Zygmund function $a(t)$, we can regularize it by convolution. So, take an even function $\rho \in \mathcal{C}_0^\infty(\mathbb{R}_t)$, $0 \leq \rho \leq 1$, whose support is contained in the interval $[-1,1]$ and such that $\int \rho(t)dt = 1$. Define then the mollifier kernel

$$\rho_\varepsilon(t) \,:=\, \frac{1}{\varepsilon}\,\rho\left(\frac{t}{\varepsilon}\right) \qquad\qquad \forall\,\varepsilon \in]0,1]\,.$$

We smooth out the function a setting, for all $\varepsilon \in]0,1]$,

$$a_\varepsilon(t) \,:=\, \left(\rho_\varepsilon\,*\,a\right)(t) \,=\, \int_{\mathbb{R}_s} \rho_\varepsilon(t-s)\,a(s)\,ds\,. \tag{3.17}$$

The following proposition holds true.

Proposition 3.9. *Let a be a log-Zygmund function. For all $\gamma \geq 1$, there exist constants C_γ such that*

$$|a_\varepsilon(t)-a(t)| \;\leq\; C_\gamma\,\|a\|_{LZ}\;\varepsilon\,\log\left(1+\gamma+\frac{1}{\varepsilon}\right) \tag{3.18}$$

$$|\partial_t a_\varepsilon(t)| \;\leq\; C_\gamma\,\|a\|_{LZ}\;\log^2\left(1+\gamma+\frac{1}{\varepsilon}\right) \tag{3.19}$$

$$\left|\partial_t^2 a_\varepsilon(t)\right| \;\leq\; C_\gamma\,\|a\|_{LZ}\;\frac{1}{\varepsilon}\;\log\left(1+\gamma+\frac{1}{\varepsilon}\right)\,. \tag{3.20}$$

Proof. For first and third inequalities, the proof is the same as in [19]. We have to pay attention only to (3.19). As ρ' has null integral, the relation

$$\partial_t a_\varepsilon(t) \,=\, \frac{1}{\varepsilon^2}\int_{|s|\leq\varepsilon} \rho'\left(\frac{s}{\varepsilon}\right)\,\left(a(t-s)-a(t)\right)ds$$

holds, and hence, taking advantage of (3.16), it implies

$$|\partial_t a_\varepsilon(t)| \,\leq\, \frac{C}{\varepsilon^2}\int_{|s|\leq\varepsilon}\left|\rho'\left(\frac{s}{\varepsilon}\right)\right|\,|s|\,\log^2\left(1+\gamma+\frac{1}{|s|}\right)ds\,.$$

Observing that the function $\nu \mapsto \nu\log^2(1+\gamma+1/\nu)$ is increasing in the interval $[0,1]$, and so does in $[0,\varepsilon]$, allows us to complete the proof. □

3.3.2 Low regularity symbols and calculus

For the analysis of our strictly hyperbolic problem, it's important to pass from $LZ_t - LL_x$ functions to more general symbols in variables (t,x,ξ) which have this same regularity in t and x.

We want to investigate properties of such symbols and of the associated operators. For reasons which will appear clear in the sequel, we will have to take advantage not of the classical paradifferential calculus, but of the calculus with parameters. Therefore, we will allow also the symbols to depend on a parameter $\gamma \geq 1$.

So, let us take a symbol $a(t,x,\xi,\gamma)$ of order $m \geq 0$, such that a is log-Zygmund in t and log-Lipschitz in x, uniformly with respect to the other variables. Then we smooth out a with respect to time, as done in (3.17). As a matter of fact, paradifferential calculus already implies a regularization of symbols with respect to x, so that we don't need to apply convolution also in the space variable (as done, for instance, in chapter 2).

Next lemma provides us some estimates on classical symbols associated to a_ε (recall formula (1.15)) and its time derivatives. For notation convenience, in what follows we drop out the dependence of the construction on the admissible cut-off function ψ (recall also remark 1.56).

Lemma 3.10. *The classical symbols associated to a_ε and its time derivatives satisfy:*

$$\begin{aligned}
\left|\partial_\xi^\alpha \sigma_{a_\varepsilon}\right| &\leq C_\alpha\,(\gamma+|\xi|)^{m-|\alpha|} \\
\left|\partial_x^\beta\partial_\xi^\alpha \sigma_{a_\varepsilon}\right| &\leq C_{\beta,\alpha}\,(\gamma+|\xi|)^{m-|\alpha|+|\beta|-1}\,\log\left(1+\gamma+|\xi|\right) \\
\left|\partial_\xi^\alpha \sigma_{\partial_t a_\varepsilon}\right| &\leq C_\alpha\,(\gamma+|\xi|)^{m-|\alpha|}\,\log^2\!\left(1+\gamma+\frac{1}{\varepsilon}\right) \\
\left|\partial_x^\beta\partial_\xi^\alpha \sigma_{\partial_t a_\varepsilon}\right| &\leq C_{\beta,\alpha}\;(\gamma+|\xi|)^{m-|\alpha|+|\beta|-1}\,\log\left(1+\gamma+|\xi|\right)\,\frac{1}{\varepsilon} \\
\left|\partial_\xi^\alpha \sigma_{\partial_t^2 a_\varepsilon}\right| &\leq C_\alpha\,(\gamma+|\xi|)^{m-|\alpha|}\,\log\!\left(1+\gamma+\frac{1}{\varepsilon}\right)\,\frac{1}{\varepsilon} \\
\left|\partial_x^\beta\partial_\xi^\alpha \sigma_{\partial_t^2 a_\varepsilon}\right| &\leq C_{\beta,\alpha}\;(\gamma+|\xi|)^{m-|\alpha|+|\beta|-1}\,\log\left(1+\gamma+|\xi|\right)\,\frac{1}{\varepsilon^2}\,.
\end{aligned}$$

Proof. The first inequality is a quite easy computation.

For the second one, we have to observe that

$$\int \partial_i G(x-y,\xi)dx \;=\; \int \partial_i G(z,\xi)dz \;=\; \int \mathcal{F}_\eta^{-1}\left(\eta_i\,\psi(\eta,\xi)\right)dz \;=\; \left(\eta_i\,\psi(\eta,\xi)\right)_{|\eta=0} \;=\; 0\,.$$

So, we have

$$\partial_i\sigma_{a_\varepsilon} \;=\; \int \partial_i G(y,\xi)\left(a_\varepsilon(t,x-y,\xi,\gamma)-a_\varepsilon(t,x,\xi,\gamma)\right)dy\,,$$

and from this, remembering lemma 1.53, we get the final control.

The third estimate immediately follows from the hypothesis on a and from (3.19).

Moreover, in the case of space derivatives, we can take advantage once again of the fact that $\partial_i G$ has null integral:

$$\begin{aligned}
\partial_i\sigma_{\partial_t a_\varepsilon} &= \int \partial_i G(x-y,\xi)\,\partial_t a_\varepsilon(t,y,\xi,\gamma)\,dy \\
&= \int_{\mathbb{R}_s}\frac{1}{\varepsilon^2}\,\rho'\left(\frac{t-s}{\varepsilon}\right)\left(\int_{\mathbb{R}^N_y}\partial_i G(y,\xi)\left(a(s,x-y,\xi,\gamma)-a(s,x,\xi,\gamma)\right)dy\right)ds\,.
\end{aligned}$$

Hence, the estimate follows from the log-Lipschitz continuity hypothesis and from inequality (1.11) about G.

Now we handle the $\partial_t^2 a_\varepsilon$ term. The first estimate comes from (3.20), while for the second one we argue as before:

$$\begin{aligned}
\partial_i\sigma_{\partial_t^2 a_\varepsilon} &= \int \partial_i G(x-y,\xi)\,\partial_t^2 a_\varepsilon(t,y,\xi,\gamma)\,dy \\
&= \int_{\mathbb{R}^N_y}\partial_i G(x-y,\xi)\,\frac{1}{\varepsilon^3}\left(\int_{\mathbb{R}_s}\rho''\left(\frac{t-s}{\varepsilon}\right)\left(a(s,y,\xi,\gamma)-a(s,x,\xi,\gamma)\right)ds\right)dy \\
&= \frac{1}{\varepsilon^3}\int_{\mathbb{R}_s}\rho''\left(\frac{t-s}{\varepsilon}\right)\left(\int_{\mathbb{R}^N_y}\partial_i G(y,\xi)\left(a(s,x-y,\xi,\gamma)-a(s,x,\xi,\gamma)\right)dy\right)ds\,,
\end{aligned}$$

and the thesis follows again from log-Lipschitz continuity condition and from (1.11). □

Note that first and second inequalities are fulfilled also by the symbol a (not smoothed with respect to the time variable).

Now let us quote some basic facts on symbolic calculus, which follow from previous lemma and the general theory developed in section 1.5 (recall in particular theorems 1.61 and 1.62).

Proposition 3.11. *(i) Let a be a symbol of order m which is LL in the x variable. Then T_a maps $H^{s+\alpha\log}_\gamma$ into $H^{s-m+\alpha\log}_\gamma$.*

(ii) Let us take two symbols a, b of order m and m' respectively. Suppose that a, b are LL in the x variable. The composition of the associated operators can be approximated by the symbol associated to the product $a\,b$, up to a remainder term:

$$T_a \circ T_b \,=\, T_{ab} \,+\, R\,.$$

The remainder operator R maps $H^{s+\alpha\log}_\gamma$ into $H^{s-m-m'+1+(\alpha+1)\log}_\gamma$ (recall definition (1.9)).

(iii) Let a be a symbol of order m which is LL in the x variable. The adjoint (over L^2) operator of T_a is, up to a remainder operator, $T_{\overline{a}}$. The remainder operator maps $H^{s+\alpha\log}_\gamma$ into $H^{s-m+1+(\alpha+1)\log}_\gamma$.

Let us end this subsection stating a basic positivity estimate. In this situation, paradifferential calculus with parameters comes into play.

Proposition 3.12. *Let $a(t,x,\xi,\gamma)$ be a symbol of order $2m$, which is log-Lipschitz continuous in the x variable and such that*

$$\mathrm{Re}\left(a(t,x,\xi,\gamma)\right) \,\geq\, \lambda_0\,\left(\gamma+|\xi|\right)^{2m}\,.$$

Then, there exists a constant λ_1, depending only on $|a|_{LL_x}$ and on λ_0 (so, not on γ), such that, for γ large enough, one has

$$\mathrm{Re}\left(T_a u, u\right)_{L^2} \,\geq\, \lambda_1\,\|u\|^2_{H^m_\gamma}\,.$$

Proof. Going along the lines of the proof to theorem 1.63 (see [52]) and keeping in mind lemma 3.10, we arrive to the following estimate, analogous to (1.17):

$$\frac{\lambda_0}{2}\,\|u\|^2_{H^m_\gamma} \,\leq\, \mathrm{Re}\left(T_a u, u\right)_{L^2} \,+\, C\,\|u\|^2_{H^{(m-1)+(\delta+1)\log}_\gamma}\,,$$

where the constant C depends only on $|a|_{LL_x}$. Now, as

$$\lim_{\gamma\to+\infty}\,\frac{\log^{2(\delta+1)}\left(1+\gamma+|\xi|\right)}{\left(\gamma^2+|\xi|^2\right)} \,=\, 0\,,$$

for $\gamma\geq1$ large enough we can absorb the last term of the right-hand side into the left-hand side of the previous relation. □

Remark 3.13. Let us note the following fact, which comes again from theorem 1.63. If the positive symbol a has low regularity in time and we smooth it by convolution with respect to this variable, we obtain a family $\left(a_\varepsilon\right)_\varepsilon$ of positive symbols, with same constant λ_0. Now, all the paradifferential operators associated to these symbols will be positive operators, uniformly in ε: i.e. the constant λ_1 of previous inequality can be choosen independently of ε.

Let us observe that previous proposition generalizes corollary 3.12 of [23] (stated for the paraproduct by a positive LL function) to the more general case of a paradifferential operator with a strictly positive symbol of order m.

Finally, thanks to proposition 3.11 about the remainder operator for the adjoint, we have the following corollary, which turns out to be fundamental in our energy estimates.

Corollary 3.14. *Let a be a positive symbol of order* 1 *and suppose that a is LL in the x variable. Then there exists $\gamma \geq 1$, depending only on the symbol a, such that*

$$\|T_a u\|_{L^2} \sim \|\nabla u\|_{L^2}$$

for all $u \in H^1(\mathbb{R}^N)$.

Proof. Obviously, $\|T_a u\|_{L^2} \leq \|\nabla u\|_{L^2}$, because a is of order 1.

In order to prove the opposite inequality, we use proposition 3.11 and we write

$$\|T_a u\|_{L^2} = ((T_a)^* T_a u, u)_{L^2} = (T_{a^2} u, u) + (Ru, u) ,$$

where R is a remainder operator with symbol equal to $\partial_x \partial_\xi a$, and so it has order $m - 1 + \log$. Hence, applying proposition 3.12 allows us to conclude the proof. □

3.4 Proof of the energy estimate for L

Finally, we are able to tackle the proof of theorem 3.1. We argue in a stadard way: first of all, we define an energy associated to a solution of equation (3.7), and then we prove estimates on its time derivative in terms of the energy itself. In the end, we will close the estimates thanks to Gronwall's lemma.

The key idea to the proof is to split the total energy into localized components e_ν, each one of them associated to the dyadic block $\Delta_\nu u$, and then to put all these pieces together (see also [22] and [19]). Let us see the proof into details.

3.4.1 Approximate and total energy

Let us first regularize coefficients a_{ij} in the time variable by convolution, as done in (3.17): as already pointed out, due to the use of paradifferential calculus, we don't need to perform a regularization of our coefficients also in space.

Then, inspired by Tarama's energy (see [56]), let us define the 0-th order symbol

$$\alpha_\varepsilon(t,x,\xi) := \left(\gamma^2 + |\xi|^2\right)^{-1/2} \left(\gamma^2 + \sum_{i,j} a_{ij,\varepsilon}(t,x)\,\xi_i\,\xi_j\right)^{1/2} .$$

We take $\varepsilon = 2^{-\nu}$ (see also [22] and [19]), and (for notation convenience) we will miss out the ε.

Before going on, let us fix a real number $\gamma \geq 1$, which will depend only on λ_0 and on the $\sup_{i,j} \|a_{ij}\|_{LL_x}$, such that (see corollary 3.14)

$$\|T_{\alpha^{-1/2}}\, w\|_{L^2} \geq \frac{\lambda_0}{2}\, \|w\|_{L^2} \qquad \text{and} \qquad \left\|T_{\alpha^{1/2}(\gamma^2+|\xi|^2)^{1/2}}\, w\right\|_{L^2} \geq \frac{\lambda_0}{2}\, \|\nabla w\|_{L^2} \tag{3.21}$$

for all $w \in H^\infty$. Let us remark that the choice of γ is equivalent to the choice of the parameter μ in (1.8) and from now on, we will consider paraproducts starting from this μ, according to definition (1.8), even if we will omit it in the notations.

Consider in (3.7) a function $u \in \mathcal{C}^2([0,T_0]; H^\infty)$. We want to get energy estimate for u. We rewrite the equation using paraproduct operators by the coefficients a_{ij}:

$$\partial_t^2 u = \sum_{i,j} \partial_i \left(a_{ij}\, \partial_j u\right) + Lu = \sum_{i,j} \partial_i \left(T_{a_{ij}} \partial_j u\right) + \widetilde{L}u,$$

where $\widetilde{L}u = Lu + \sum_{i,j} \partial_i \left((a_{ij} - T_{a_{ij}})\partial_j u\right)$. Let us apply operator Δ_ν: we get

$$\partial_t^2 u_\nu = \sum_{i,j} \partial_i \left(T_{a_{ij}}\, \partial_j u_\nu\right) + \sum_{i,j} \partial_i \left(\left[\Delta_\nu, T_{a_{ij}}\right]\, \partial_j u\right) + (\widetilde{L}u)_\nu, \tag{3.22}$$

where $u_\nu = \Delta_\nu u$, $(\widetilde{L}u)_\nu = \Delta_\nu(\widetilde{L}u)$ and $\left[\Delta_\nu, T_{a_{ij}}\right]$ is the commutator between Δ_ν and the para-multiplication by a_{ij}.

Now, following again the original idea of Tarama in [56], but replacing product with symbols by action of paradifferential operators, we set

$$\begin{aligned} v_\nu(t,x) &:= T_{\alpha^{-1/2}}\,\partial_t u_\nu \,-\, T_{\partial_t(\alpha^{-1/2})}\,u_\nu \\ w_\nu(t,x) &:= T_{\alpha^{1/2}(\gamma^2+|\xi|^2)^{1/2}}\,u_\nu \\ z_\nu(t,x) &:= u_\nu\,. \end{aligned}$$

These functions are relevant in our analysis, because on the one hand they are strictly related to the Sobolev norms of $\partial_t u$ and u (see also inequalities (3.25) and (3.26) below), and on the other hand the presence of the weights (depending on α) will produce fundamental cancellations in our computations.

Now we can define the approximate energy associated to the ν-th component of u (as already done in [19]):

$$e_\nu(t) \,:=\, \|v_\nu(t)\|^2_{L^2} \,+\, \|w_\nu(t)\|^2_{L^2} \,+\, \|z_\nu(t)\|^2_{L^2}\,. \tag{3.23}$$

Remark 3.15. Let us note that, thanks to hypothesis (3.8) and our choice of the frequence μ from which defining the paraproduct, we have that $\|w_\nu(t)\|^2_{L^2} \,\sim\, \|\nabla u_\nu\|^2_{L^2} \,\sim\, 2^{2\nu}\,\|u_\nu\|^2_{L^2}$.

Now, we fix a $\theta \in]0,1[$, as required in hypothesis, and we take a $\beta > 0$ to be chosen later; we can define the total energy associated to the solution u to be the quantity

$$E(t) \,:=\, \sum_{\nu\geq 0} e^{-2\beta(\nu+1)t}\,2^{-2\nu\theta}\,e_\nu(t)\,. \tag{3.24}$$

It's not difficult to prove (see also inequality (3.28) below) that there exist constants C_θ and C'_θ, depending only on the fixed θ, for which one has:

$$(E(0))^{1/2} \,\leq\, C_\theta\left(\|\partial_t u(0)\|_{H^{-\theta}} \,+\, \|u(0)\|_{H^{-\theta+1}}\right) \tag{3.25}$$

$$(E(t))^{1/2} \,\geq\, C'_\theta\left(\|\partial_t u(t)\|_{H^{-\theta-\beta^* t}} \,+\, \|u(t)\|_{H^{-\theta+1-\beta^* t}}\right)\,, \tag{3.26}$$

where we have set $\beta^* = \beta\,(\log 2)^{-1}$.

3.4.2 Time derivative of the approximate energy

We want to find an estimate on time derivative of the energy in order to get a control on it by Gronwall's lemma. Let us start analysing each term of (3.23).

z_ν term

For the third term we have:

$$\frac{d}{dt}\|z_\nu(t)\|^2_{L^2} \,=\, 2\,\mathrm{Re}\,\left(u_\nu\,,\,\partial_t u_\nu\right)_{L^2}\,. \tag{3.27}$$

Now, we have to control the term $\partial_t u_\nu$: using positivity of operator $T_{\alpha^{-1/2}}$, we have

$$\|\partial_t u_\nu\|_{L^2} \,\leq\, C\,\|T_{\alpha^{-1/2}}\partial_t u_\nu\|_{L^2} \,\leq\, C\left(\|v_\nu\|_{L^2} \,+\, \left\|T_{\partial_t\left(\alpha^{-1/2}\right)}u_\nu\right\|_{L^2}\right) \,\leq\, C\,(e_\nu)^{1/2}\,. \tag{3.28}$$

So, we get the estimate:

$$\frac{d}{dt}\|z_\nu(t)\|^2_{L^2} \,\leq\, C\,e_\nu(t)\,. \tag{3.29}$$

v_ν term

Straightforward computations show that

$$\partial_t v_\nu(t,x) \;=\; T_{\alpha^{-1/2}}\partial_t^2 u_\nu \;-\; T_{\partial_t^2(\alpha^{-1/2})}u_\nu\,.$$

Therefore, putting relation (3.22) in the previous one, we easily get:

$$\begin{aligned}\frac{d}{dt}\|v_\nu(t)\|^2_{L^2} \;&=\; -2\,\mathrm{Re}\,\left(v_\nu\,,\,T_{\partial_t^2(\alpha^{-1/2})}u_\nu\right)_{L^2} + \\ &\quad +2\sum_{i,j}\mathrm{Re}\,\left(v_\nu\,,\,T_{\alpha^{-1/2}}\partial_i\left(T_{a_{ij}}\,\partial_j u_\nu\right)\right)_{L^2} + \\ &\quad +2\sum_{i,j}\mathrm{Re}\,\left(v_\nu\,,\,T_{\alpha^{-1/2}}\partial_i\left[\Delta_\nu,T_{a_{ij}}\right]\partial_j u\right)_{L^2} + \\ &\quad +2\,\mathrm{Re}\,\left(v_\nu\,,\,T_{\alpha^{-1/2}}\left(\widetilde{L}u\right)_\nu\right)_{L^2}.\end{aligned} \tag{3.30}$$

Obviously, we have

$$\left|2\,\mathrm{Re}\,\left(v_\nu\,,\,T_{\alpha^{-1/2}}\left(\widetilde{L}u\right)_\nu\right)_{L^2}\right| \;\leq\; C\;(e_\nu)^{1/2}\,\left\|\left(\widetilde{L}u\right)_\nu\right\|_{L^2}\,, \tag{3.31}$$

while from lemma 3.10 we immediately get

$$\begin{aligned}\left|2\,\mathrm{Re}\,\left(v_\nu\,,\,T_{\partial_t^2(\alpha^{-1/2})}u_\nu\right)_{L^2}\right| \;&\leq\; C\,\|v_\nu\|_{L^2}\,\log\!\left(1+\gamma+\frac{1}{\varepsilon}\right)\frac{1}{\varepsilon}\,\|u_\nu\|_{L^2} \\ &\leq\; C\,(\nu+1)\,e_\nu\,,\end{aligned} \tag{3.32}$$

where we have used the fact that $\varepsilon=2^{-\nu}$. The other two terms of (3.30) will be treated later.

w_ν term

We now derive w_ν with respect to the time variable: thanks to a broad use of symbolic calculus, we get the following sequence of equalities:

$$\begin{aligned}\frac{d}{dt}\,\|w_\nu\|^2_{L^2} \;&=\; 2\mathrm{Re}\,\left(T_{\partial_t(\alpha^{1/2})(\gamma^2+|\xi|^2)^{1/2}}u_\nu, w_\nu\right)_{L^2} + 2\mathrm{Re}\,\left(T_{\alpha^{1/2}(\gamma^2+|\xi|^2)^{1/2}}\partial_t u_\nu, w_\nu\right)_{L^2} \\ &=\; 2\,\mathrm{Re}\,\left(T_{\alpha(\gamma^2+|\xi|^2)^{1/2}}T_{-\partial_t(\alpha^{-1/2})}u_\nu\,,\,w_\nu\right)_{L^2} \;+\; 2\,\mathrm{Re}\,\left(R_1u_\nu\,,\,w_\nu\right)_{L^2} \;+ \\ &\quad +2\,\mathrm{Re}\,\left(T_{\alpha(\gamma^2+|\xi|^2)^{1/2}}T_{\alpha^{-1/2}}\partial_t u_\nu\,,\,w_\nu\right)_{L^2} \;+\; 2\,\mathrm{Re}\,\left(R_2\partial_t u_\nu\,,\,w_\nu\right)_{L^2} \\ &=\; 2\,\mathrm{Re}\,\left(v_\nu\,,\,T_{\alpha(\gamma^2+|\xi|^2)^{1/2}}w_\nu\right)_{L^2} \;+\; 2\,\mathrm{Re}\,\left(v_\nu\,,\,R_3w_\nu\right)_{L^2} \;+ \\ &\quad +2\,\mathrm{Re}\,\left(R_1u_\nu\,,\,w_\nu\right)_{L^2} \;+\; 2\,\mathrm{Re}\,\left(R_2\partial_t u_\nu\,,\,w_\nu\right)_{L^2} \\ &=\; 2\,\mathrm{Re}\,\left(v_\nu\,,\,T_{\alpha^{-1/2}}T_{\alpha^{3/2}(\gamma^2+|\xi|^2)^{1/2}}w_\nu\right)_{L^2} \;+\; 2\,\mathrm{Re}\,\left(v_\nu\,,\,R_4w_\nu\right)_{L^2} \;+ \\ &\quad +2\,\mathrm{Re}\,\left(v_\nu\,,\,R_3w_\nu\right)_{L^2} \;+\; 2\,\mathrm{Re}\,\left(R_1u_\nu\,,\,w_\nu\right)_{L^2} \;+\; 2\,\mathrm{Re}\,\left(R_2\partial_t u_\nu\,,\,w_\nu\right)_{L^2} \\ &=\; 2\,\mathrm{Re}\,\left(v_\nu\,,\,T_{\alpha^{-1/2}}T_{\alpha^2(\gamma^2+|\xi|^2)}u_\nu\right)_{L^2} \;+ \\ &\quad +2\,\mathrm{Re}\,\left(v_\nu\,,\,T_{\alpha^{-1/2}}R_5u_\nu\right)_{L^2} \;+\; 2\,\mathrm{Re}\,\left(v_\nu\,,\,R_4w_\nu\right)_{L^2} \;+ \\ &\quad +2\,\mathrm{Re}\,\left(v_\nu\,,\,R_3w_\nu\right)_{L^2} \;+\; 2\,\mathrm{Re}\,\left(R_1u_\nu\,,\,w_\nu\right)_{L^2} \;+\; 2\,\mathrm{Re}\,\left(R_2\partial_t u_\nu\,,\,w_\nu\right)_{L^2}\,.\end{aligned} \tag{3.33}$$

The important fact is that remainder terms are not bad and can be controlled in terms of approximate energy. As a matter of facts, taking advantage of proposition 3.11 and lemma 3.10, we get the following estimates.

- R_1 has principal symbol equal to $\partial_\xi\left(\alpha(\gamma^2+|\xi|^2)^{1/2}\right)\partial_x\partial_t(\alpha^{-1/2})$, so

$$|2\,\mathrm{Re}\,(R_1 u_\nu\,,\,w_\nu)_{L^2}| \;\leq\; C\,(\nu+1)\,e_\nu\,. \tag{3.34}$$

- The principal symbol of R_2 is instead $\partial_\xi\left(\alpha(\gamma^2+|\xi|^2)^{1/2}\right)\partial_x(\alpha^{-1/2})$, so, remembering also the control on $\|\partial_t u_\nu\|_{L^2}$, we have:

$$|2\,\mathrm{Re}\,(R_2\partial_t u_\nu\,,\,w_\nu)_{L^2}| \,\leq\, C\,\nu\,(e_\nu)^{1/2}\,\|w_\nu\|_{L^2}\,\leq\, C\,(\nu+1)\,e_\nu\,. \tag{3.35}$$

- Symbolic calculus tells us that the principal part of R_3 is given by $\partial_\xi\partial_x\left(\alpha(\gamma^2+|\xi|^2)^{1/2}\right)$, therefore

$$|2\,\mathrm{Re}\,(v_\nu\,,\,R_3 w_\nu)_{L^2}| \,\leq\, C\,\|v_\nu\|_{L^2}\,\nu\,\|w_\nu\|_{L^2}\,\leq\, C\,(\nu+1)\,e_\nu\,. \tag{3.36}$$

- Now, R_4 has $\partial_\xi\left(\alpha^{-1/2}\right)\partial_x\left(\alpha^{3/2}(\gamma^2+|\xi|^2)^{1/2}\right)$ as principal symbol, so

$$|2\,\mathrm{Re}\,(v_\nu\,,\,R_4 w_\nu)_{L^2}| \,\leq\, C\,\|v_\nu\|_{L^2}\,\nu\,\|w_\nu\|_{L^2}\,\leq\, C\,(\nu+1)\,e_\nu\,. \tag{3.37}$$

- Finally, R_5 is given, at the higher order, by the product of symbols $\partial_\xi\left(\alpha^{3/2}(\gamma^2+|\xi|^2)^{1/2}\right)$ and $\partial_x\left(\alpha^{1/2}(\gamma^2+|\xi|^2)^{1/2}\right)$, and so we get

$$\left|2\,\mathrm{Re}\,(v_\nu\,,\,T_{\alpha^{-1/2}}R_5 u_\nu)_{L^2}\right| \,\leq\, C\,\|v_\nu\|_{L^2}\,2^\nu\,\nu\,\|u_\nu\|_{L^2}\,\leq\, C\,(\nu+1)\,e_\nu\,. \tag{3.38}$$

Principal part of the operator L

Now, thanks to previous computations, it's natural to pair up the second term of (3.30) with the first one of the last equality of (3.33). As α is a symbol of order 0, we have

$$\left|2\,\mathrm{Re}\left(v_\nu\,,\,T_{\alpha^{-1/2}}\sum_{i,j}\partial_i\left(T_{a_{ij}}\partial_j u_\nu\right)\right)_{L^2} + 2\,\mathrm{Re}\left(v_\nu\,,\,T_{\alpha^{-1/2}}T_{\alpha^2(\gamma^2+|\xi|^2)}u_\nu\right)_{L^2}\right| \,\leq\, C\,\|v_\nu\|_{L^2}\,\|\zeta_\nu\|_{L^2}\,,$$

where we have set

$$\zeta_\nu \,:=\, T_{\alpha^2(\gamma^2+|\xi|^2)}u_\nu \,+\, \sum_{i,j}\partial_i\left(T_{a_{ij}}\,\partial_j u_\nu\right) \,=\, \sum_{ij} T_{a_{ij,\varepsilon}\xi_i\xi_j+\gamma^2}u_\nu \,+\, \partial_i\left(T_{a_{ij}}\,\partial_j u_\nu\right)\,. \tag{3.39}$$

We remark that

$$\partial_i\left(T_{a_{ij}}\,\partial_j u_\nu\right) = T_{\partial_i a_{ij}}\partial_j u_\nu \,-\, T_{a_{ij}\xi_i\xi_j}u_\nu,$$

where, with a little abuse of notations, we have written the derivative $\partial_i a_{ij}$ meaning that we are taking the derivative of the classical symbol associated to a_{ij}.

First of all, we have that

$$\begin{aligned}
\left\|T_{\partial_i a_{ij}}\partial_j u_\nu\right\|_{L^2} \;&\leq\; \|S_\mu\,\partial_i a_{ij}\|_{L^\infty}\,\|S_\mu\partial_j u_\nu\|_{L^2} \,+\, \sum_{k\geq\mu+1}\|\nabla S_{k-1}a_{ij}\|_{L^\infty}\,\|\Delta_k\nabla u_\nu\|_{L^2} \\
&\leq\; C\,(\mu+1)\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right)\|\nabla u_\nu\|_{L^2} \,+ \\
&\qquad + \sum_{k\geq\mu+1\,,\,k\sim\nu}(k+1)\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right)\|\nabla\Delta_k u_\nu\|_{L^2} \\
&\leq\; C_\mu\,(\nu+1)\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right)(e_\nu)^{1/2}\,,
\end{aligned} \tag{3.40}$$

where μ is the parameter fixed in (1.8) and we have also used (3.15). Next, we have to control the term

$$T_{a_{ij,\varepsilon}\xi_i\xi_j+\gamma^2}u_\nu \;-\; T_{a_{ij}\xi_i\xi_j}u_\nu \;=\; T_{(a_{ij,\varepsilon}-a_{ij})\xi_i\xi_j}u_\nu \;+\; T_{\gamma^2}u_\nu\,.$$

It's easy to see that

$$\left\|T_{(a_{ij,\varepsilon}-a_{ij})\xi_i\xi_j}u_\nu\right\|_{L^2} \;\leq\; C\,\varepsilon\,\log\left(1+\frac{1}{\varepsilon}\right)2^\nu\,\|\nabla u_\nu\|_{L^2}\,,$$

and so, keeping in mind that $\varepsilon=2^{-\nu}$,

$$\left\|T_{(a_{ij,\varepsilon}-a_{ij})\xi_i\xi_j+\gamma^2}u_\nu\right\|_{L^2} \;\leq\; C_\gamma\,(\nu+1)\,(e_\nu)^{1/2}\,. \tag{3.41}$$

Therefore, from (3.40) and (3.41) we finally get

$$\left|2\mathrm{Re}\left(v_\nu,T_{\alpha^{-1/2}}\sum_{i,j}\partial_i\left(T_{a_{ij}}\partial_j u_\nu\right)\right)_{L^2} + 2\mathrm{Re}\,\left(v_\nu,T_{\alpha^{-1/2}}T_{\alpha^2(\gamma^2+|\xi|^2)}u_\nu\right)_{L^2}\right| \;\leq\; C(\nu+1)e_\nu, \tag{3.42}$$

where the constant C depends on the log-Lipschitz norm of coefficients a_{ij} of the operator L and on the fixed parameters μ and γ.

To sum up, from inequalities (3.29), (3.31), (3.32) and (3.42) and from estimates of remainder terms (3.34)-(3.38), we can conclude that

$$\begin{aligned}\frac{d}{dt}e_\nu(t) \;\;\leq\;\; & C_1\,(\nu+1)\,e_\nu(t)\;+\;C_2\,(e_\nu(t))^{1/2}\,\left\|\left(\widetilde{L}u\right)_\nu(t)\right\|_{L^2}\;+\\ & +\left|2\sum_{i,j}\mathrm{Re}\,\left(v_\nu\,,\,T_{\alpha^{-1/2}}\partial_i\left[\Delta_\nu,T_{a_{ij}}\right]\partial_j u\right)_{L^2}\right|\,.\end{aligned} \tag{3.43}$$

3.4.3 Commutator term

We want to estimate the quantity

$$\left|\sum_{i,j}\mathrm{Re}\,\left(v_\nu\,,\,T_{\alpha^{-1/2}}\partial_i\left[\Delta_\nu,T_{a_{ij}}\right]\partial_j u\right)_{L^2}\right|\,.$$

We start by remarking that

$$[\Delta_\nu,T_{a_{ij}}]w=[\Delta_\nu,S_\mu a]S_\mu w+\sum_{k=\mu+1}^{+\infty}\left[\Delta_\nu,S_{k-1}a_{ij}\right]\Delta_k w,$$

where μ is fixed, as usual. In fact Δ_ν and Δ_k commute so that

$$\Delta_\nu(S_\mu a_{ij}S_\mu w)-S_\mu a_{ij}(S_\mu\Delta_\nu w)=\Delta_\nu(S_\mu a_{ij}S_\mu w)-S_\mu a_{ij}\Delta_\nu(S_\mu w),$$

and similarly

$$\Delta_\nu(S_{k-1}a_{ij}\Delta_k w)-S_{k-1}a_{ij}\Delta_k(\Delta_\nu w)=\Delta_\nu(S_{k-1}a_{ij}\Delta_k w)-S_{k-1}a_{ij}\Delta_\nu(\Delta_k w).$$

Consequently, taking into account also that S_k and Δ_k commute with ∂_j, we have

$$\partial_i\left(\left[\Delta_\nu,T_{a_{ij}}\right]\partial_j u\right)\;=\;\partial_i\left(\left[\Delta_\nu,S_\mu a_{ij}\right]\partial_j(S_\mu u)\right)\;+\;\partial_i\left(\sum_{k=\mu+1}^{+\infty}\left[\Delta_\nu,S_{k-1}a_{ij}\right]\partial_j(\Delta_k u)\right).$$

Let's consider first the term

$$\partial_i\left([\Delta_\nu, S_\mu a_{ij}]\,\partial_j(S_\mu u)\right).$$

Looking at the support of the Fourier transform of $[\Delta_\nu, S_\mu a_{ij}]\,\partial_j(S_\mu u)$, we have that it is contained in $\{|\xi| \leq 2^{\mu+2}\}$ and moreover $[\Delta_\nu, S_\mu a_{ij}]\,\partial_j(S_\mu u)$ is identically 0 if $\nu \geq \mu+3$. From Bernstein's inequality and theorem 35 of [17] we have that

$$\|\partial_i\left([\Delta_\nu, S_\mu a_{ij}]\,\partial_j(S_\mu u)\right)\|_{L^2} \,\leq\, C_\mu \left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right)\|S_\mu u\|_{L^2}\,,$$

hence, putting all these facts together, we have

$$\begin{aligned}
&\left|\sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\,2^{-2\nu\theta}\sum_{ij} 2\,\mathrm{Re}\Big(v_\nu\,,\,T_{\alpha^{-1/2}}\,\partial_i\left([\Delta_\nu, S_\mu a_{ij}]\partial_j(S_\mu u)\,\right)\Big)_{L^2}\right| \leq \qquad (3.44)\\
&\leq C_\mu\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right)\sum_{\nu=0}^{\mu+2} e^{-2\beta(\nu+1)t}\,2^{-2\nu\theta}\|v_\nu\|_{L^2}\left(\sum_{h=0}^{\mu}\|u_h\|_{L^2}\right)\\
&\leq C_\mu\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right) e^{\beta(\mu+3)T}\,2^{(\mu+2)\theta}\sum_{\nu=0}^{\mu+2} e^{-\beta(\nu+1)t}\,2^{-\nu\theta}\|v_\nu\|_{L^2}\,\cdot\\
&\qquad\qquad\cdot\sum_{h=0}^{\mu+2} e^{-\beta(h+1)t}\,2^{-h\theta}\|u_h\|_{L^2}\\
&\leq C_\mu\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right) e^{\beta(\mu+3)T}\,2^{(\mu+2)\theta}\sum_{\nu=0}^{\mu+2} e^{-2\beta(\nu+1)t}\,2^{-2\nu\theta}e_\nu(t)\,.
\end{aligned}$$

Next, let's consider

$$\partial_i\left(\sum_{k=\mu+1}^{+\infty}[\Delta_\nu, S_{k-1}a_{ij}]\,\partial_j(\Delta_k u)\right).$$

Looking at the support of the Fourier transform, it is possible to see that

$$[\Delta_\nu, S_{k-1}a_{ij}]\,\partial_j(\Delta_k u)$$

is identically 0 if $|k-\nu| \geq 3$. Consequently the sum over k is reduced to at most 5 terms: $\partial_i([\Delta_\nu, S_{\nu-3}a_{ij}]\,\partial_j(\Delta_{\nu-2}u)) + \cdots + \partial_i([\Delta_\nu, S_{\nu+1}a_{ij}]\,\partial_j(\Delta_{\nu+2}u))$, each of them having the support of the Fourier transform contained in $\{|\xi| \leq 2^{\nu+1}\}$. Let's consider one of these terms, e.g. $\partial_i([\Delta_\nu, S_{\nu-1}a_{ij}]\,\partial_j(\Delta_\nu u))$, the computation for the other ones being similar. We have, from Bernstein's inequality,

$$\|\partial_i\left([\Delta_\nu, S_{\nu-1}a_{ij}]\,\partial_j(\Delta_\nu u)\right)\|_{L^2} \,\leq\, C\,2^\nu\,\|[\Delta_\nu, S_{\nu-1}a_{ij}]\,\partial_j(\Delta_\nu u)\|_{L^2}\,.$$

On the other hand, using theorem 35 of [17] again, we have:

$$\|[\Delta_\nu, S_{\nu-1}a_{ij}]\partial_j(\Delta_\nu u)\|_{L^2} \,\leq\, C\,\|\nabla S_{\nu-1}a_{ij}\|_{L^\infty}\,\|\Delta_\nu u\|_{L^2}\,,$$

where C does not depend on ν. Consequently, using also (3.15), we deduce

$$\|\partial_i\left([\Delta_\nu, S_{\nu-1}a_{ij}]\,\partial_j(\Delta_\nu u)\right)\|_{L^2} \,\leq\, C\,2^\nu\,(\nu+1)\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right)\|\Delta_\nu u\|_{L^2}\,.$$

From this last inequality and similar ones for the other terms, it is easy to obtain that

$$\left|\sum_{i,j}\mathrm{Re}\left(v_\nu\,,\,T_{\alpha^{-1/2}}\partial_i\left(\sum_{k=\mu+1}^{+\infty}[\Delta_\nu, S_{k-1}a_{ij}]\,\partial_j(\Delta_k u)\right)\right)_{L^2}\right| \leq C\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right)(\nu+1)\,e_\nu(t)$$

and then

$$
\left|\sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta} \sum_{ij} 2\,\mathrm{Re}\left(v_\nu\,,\, T_{\alpha^{-1/2}}\partial_i\left(\sum_{k=\mu+1}^{+\infty}\left[\Delta_\nu, S_{k-1}a_{ij}\right]\partial_j(\Delta_k u)\right)\right)_{L^2}\right| \leq
$$
$$
\leq C\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right)\sum_{\nu=0}^{+\infty}(\nu+1)\, e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, e_\nu(t)\,. \tag{3.45}
$$

Collecting the informations from (3.44) and (3.45), we obtain

$$
\left|\sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta} \sum_{ij} 2\,\mathrm{Re}\left(v_\nu\,,\, T_{\alpha^{-1/2}}\partial_i\left[\Delta_\nu, T_{a_{ij}}\right]\partial_j u\right)_{L^2}\right| \leq
$$
$$
\leq C_3\sum_{\nu=0}^{+\infty}(\nu+1)\, e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, e_\nu(t)\,, \tag{3.46}
$$

where C_3 depends on μ, $\sup_{i,j}\|a_{ij}\|_{LL_x}$, on θ and on the product $\beta\, T$.

3.4.4 Final estimate

From (3.43) and (3.46) we get

$$
\begin{aligned}
\frac{d}{dt}E(t) \;\leq\;& (C_1 + C_3 - 2\beta)\sum_{\nu=0}^{+\infty}(\nu+1)\, e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, e_\nu(t) \;+\\
&+ C_2\sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\,(e_\nu(t))^{1/2}\,\left\|\left(\widetilde{L}u(t)\right)_\nu\right\|_{L^2}\\
\leq\;& (C_1 + C_3 - 2\beta)\sum_{\nu=0}^{+\infty}(\nu+1)\, e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, e_\nu(t) \;+\\
&+ C_2\sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\,(e_\nu(t))^{1/2}\,\left\|\left(\sum_{i,j}\partial_i\left((a_{ij}-T_{a_{ij}})\partial_j u\right)\right)_\nu\right\|_{L^2} \;+\\
&+ C_2\sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\,(e_\nu(t))^{1/2}\,\left\|(Lu(t))_\nu\right\|_{L^2}\,.
\end{aligned}
$$

Now, applying Hölder inequality for series implies

$$
\sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\,(e_\nu(t))^{1/2}\,\left\|\left(\sum_{i,j}\partial_i\left((a_{ij}-T_{a_{ij}})\partial_j u\right)\right)_\nu\right\|_{L^2} \leq
$$
$$
\leq \left(\sum_{\nu=0}^{+\infty}(\nu+1)\, e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, e_\nu(t)\right)^{1/2}\cdot
$$
$$
\cdot\left(\sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\,(\nu+1)^{-1}\left\|\left(\sum_{i,j}\partial_i\left((a_{ij}-T_{a_{ij}})\partial_j u\right)\right)_\nu\right\|_{L^2}^2\right)^{1/2},
$$

and, by definition, one has

$$\left(\sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\,(\nu+1)^{-1}\left\|\left(\sum_{i,j}\partial_i\left((a_{ij}-T_{a_{ij}})\partial_j u\right)\right)_\nu\right\|_{L^2}^2\right)^{1/2} =$$

$$= \left\|\sum_{i,j}\partial_i\left((a_{ij}-T_{a_{ij}})\partial_j u\right)\right\|_{H^{-\theta-\beta^* t-\frac{1}{2}\log}} .$$

From proposition 3.4 of [23] we have that

$$\left\|\sum_{i,j}\partial_i\left((a_{ij}-T_{a_{ij}})\partial_j u\right)\right\|_{H^{-s-\frac{1}{2}\log}} \leq C\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right)\|u\|_{H^{1-s+\frac{1}{2}\log}}, \tag{3.47}$$

with C uniformly bounded for s in a compact set of $]0,1[$. Consequently,

$$\left(\sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\,(\nu+1)^{-1}\left\|\left(\sum_{i,j}\partial_i\left((a_{ij}-T_{a_{ij}})\partial_j u\right)\right)_\nu\right\|_{L^2}^2\right)^{1/2} \leq$$

$$\leq C\left(\sup_{i,j}\|a_{ij}\|_{LL_x}\right)\|u\|_{H^{1-\theta-\beta^* t+\frac{1}{2}\log}}$$

$$\leq C\left(\sum_{\nu=0}^{+\infty}(\nu+1)\, e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, e_\nu(t)\right)^{1/2},$$

and finally

$$\sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\,(e_\nu(t))^{1/2}\left\|\left(\sum_{i,j}\partial_i\left((a_{ij}-T_{a_{ij}})\partial_j u\right)\right)_\nu\right\|_{L^2} \leq$$

$$\leq C_4\sum_{\nu=0}^{+\infty}(\nu+1)e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, e_\nu(t),$$

with C_4 uniformly bounded for $\beta^* t+\theta$ in a compact set of $]0,1[$. So, if we take $\beta>0$ and $T\in\,]0,T_0]$ such that (recall that $\beta^*=\beta(\log 2)^{-1}$)

$$\beta^*\, T \;=\; \delta \;<\; 1-\theta, \tag{3.48}$$

we have $0<\theta\leq\theta+\beta^* t\leq\theta+\delta<1$. Therefore we obtain

$$\frac{d}{dt}E(t) \;\leq\; (C_1+C_4\,C_2+C_3-2\beta)\sum_{\nu=0}^{+\infty}(\nu+1)\, e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\, e_\nu(t) \;+$$

$$+\, C_2\sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t}\, 2^{-2\nu\theta}\,(e_\nu(t))^{1/2}\,\|(Lu(t))_\nu\|_{L^2} .$$

Now, taking β large enough such that $C_1+C_4\,C_2+C_3-2\beta\leq 0$, which corresponds to take $T>0$ small enough, we finally arrive to the estimate

$$\frac{d}{dt}E(t) \leq C_2\,(E(t))^{1/2}\,\|Lu(t)\|_{H^{-\theta-\beta^* t}} ;$$

applying Gronwall's lemma and keeping in mind (3.25) and (3.26) give us estimate (3.11). □

Remark 3.16. Let us point out that condition (3.48) gives us a condition on the lifespan T of a solution to the Cauchy problem for (3.7). It depends on $\theta \in]0,1[$ and on $\beta^* > 0$, hence on constants $C_1 \dots C_4$. Going after the guideline of the proof, one can see that, in the end, the time T depends only on the index θ, on the parameter μ defined by conditions (3.21), on the constants λ_0 and Λ_0 defined by (3.8) and on the quantities $\sup_{i,j} \|a_{ij}\|_{LZ_t}$ and $\sup_{i,j} \|a_{ij}\|_{LL_x}$.

Part III

Density-dependent Incompressible Euler system

Chapter 4

The well-posedness issue in endpoint Besov spaces

In the recent paper [28], Danchin proved well-posedness for the density-dependent incompressible Euler system in Besov spaces $B^s_{p,r}$ embedded in the set of globally Lipschitz continuous functions, for all $p \in]1,+\infty[$.

In this chapter we will focus on the limit case $B^s_{\infty,r}$ for which condition (1.1) still holds. This functional framework contains also the particular cases of Hölder spaces $\mathcal{C}^{1,\alpha}$ and of the endpoint Besov space $B^1_{\infty,1}$.

In this setting and under non-vacuum assumption, we will establish the local well-posedness and a continuation criterion in the spirit of that of Beale, Kato and Majda (see also [3]). Moreover, in the last part of the chapter we will give lower bounds for the lifespan of a solution, pointing out that, in dimension two, it tends to infinity when the initial density tends to be a constant.

4.1 Introduction and main results

This chapter is, in a certain sense, the continuation of the recent paper [28] by Danchin, devoted to the *density-dependent incompressible Euler equations*:

$$(4.1)\qquad \begin{cases} \partial_t\rho \,+\, u\cdot\nabla\rho \,=\, 0 \\ \rho\,(\partial_t u \,+\, u\cdot\nabla u) \,+\, \nabla\Pi \,=\, \rho\, f \\ \operatorname{div} u \,=\, 0\,. \end{cases}$$

Recall that the above equations describe the evolution of the density $\rho = \rho(t,x) \in \mathbb{R}_+$ and of the velocity field $u = u(t,x) \in \mathbb{R}^N$ of a non-homogeneous inviscid incompressible fluid. The time dependent vector-field f stands for a given body force and the gradient of the pressure $\nabla\Pi$ is the Lagrangian multiplier associated to the divergence free constraint over the velocity. We assume that the space variable x belongs to the whole $\mathbb{R}^N$ with $N \geq 2$.

There is an important literature devoted to the standard incompressible Euler equations, that is to the case where the initial density is a positive constant, an assumption which is preserved during the evolution. In contrast, not so many works have been devoted to the study of (4.1) in the nonconstant density case. In the situation where the equations are considered in a suitably smooth bounded domain of $\mathbb{R}^2$ or $\mathbb{R}^3$, the local well-posedness issue has been investigated by H. Beirão da Veiga and A. Valli in [5], [6], [4] for data with high enough Hölder regularity. In [27] Danchin proved well-posedness in H^s with $s > 1 + N/2$ and studied the inviscid limit in this framework. Data in the limit Besov space $B^{\frac{N}{2}+1}_{2,1}$ were also considered.

As for the standard incompressible Euler equations, any functional space embedded in the set $\mathcal{C}^{0,1}$ of bounded globally Lipschitz functions is a candidate for the study of the well-posedness issue. This stems from the fact that system (4.1) is a coupling between transport equations. Hence preserving the initial regularity requires the velocity field to be at least locally Lipschitz with respect to the space variable. As a matter of fact, the classical Euler equations have been shown to be well posed in any Besov space $B^s_{p,r}$ embedded in $\mathcal{C}^{0,1}$ (see [2], [13], [54], [63] and the references therein), a property which holds if and only if indices $s \in \mathbb{R}$ and $(p,r) \in [1,+\infty]^2$ satisfy condition (1.1), which we recall here:

$$s > 1 + \frac{N}{p} \qquad \text{or} \qquad s = 1 + \frac{N}{p} \quad \text{and} \quad r = 1\,.$$

In [28], Danchin extended the results of the homogeneous case to (4.1) (see also [34] for a similar study in the periodic framework). Under condition (1.1) *with* $1 < p < +\infty$ he established the local well-posedness for any data (ρ_0, u_0) in $B^s_{p,r}$ such that ρ_0 is bounded away from zero. However, he didn't treat the limit case $p = +\infty$ *unless supposing the initial density to be a small perturbation of a constant density state*, a technical artifact due to the method he used to handle the pressure term.

In fact, in contrast to the classical Euler equations, computing the gradient of the pressure involves an elliptic equation *with nonconstant coefficients*, namely

$$\text{div}\,(a\nabla\Pi) = \text{div}\,F\,, \qquad \text{with} \qquad F := f - u\cdot\nabla u \quad \text{and} \quad a := 1/\rho\,. \tag{4.2}$$

Getting appropriate a priori estimates *given that we expect the function* ρ *to have exactly the same regularity as* $\nabla\Pi$ is the main difficulty. In the L^2 framework and, more generally, in the Sobolev framework H^s, this may be achieved by means of a classical energy method. This is also quite straightforward in the $B^s_{p,r}$ framework if a is a small perturbation of some positive constant function $\overline{a}$, as the above equation may be rewritten

$$\overline{a}\,\Delta\Pi = \text{div}\,F + \text{div}\,\big((\overline{a}-a)\nabla\Pi\big)\,.$$

Now, if $a - \overline{a}$ is small enough, then one may take advantage of regularity results for the Laplace operator in order to "absorb" the last term.

If $1 < p < +\infty$ and a is bounded away from zero, then it turns out that combining energy arguments similar to those of the H^s case and a harmonic analysis lemma allows to handle the elliptic equation (4.2). This is the approach Danchin used in [28], but it fails for the limit cases $p = 1$ and $p = +\infty$.

In this chapter, we propose another method for proving a priori estimates for (4.2). In addition to being simpler, this will enable us to treat all the cases $p \in [1,+\infty]$ indistinctly whenever the density is bounded away from zero. Our approach relies on the fact that the pressure Π satisfies (here we take $f \equiv 0$ to simplify)

$$\Delta\Pi = -\rho\,\text{div}\,(u\cdot\nabla u) + \nabla(\log\rho)\cdot\nabla\Pi\,. \tag{4.3}$$

Obviously, the last term is of lower order. In addition, the classical L^2 theory ensures that, if there exists some positive constant m such that $a(t,x) \geq m$ for all $(t,x) \in [0,T]\times\mathbb{R}^N$, then

$$m\,\|\nabla\Pi(t,\cdot)\|_{L^2(\mathbb{R}^N)} \leq \|(u\cdot\nabla u)(t,\cdot)\|_{L^2(\mathbb{R}^N)} \qquad \text{for all} \quad t \in [0,T]\,.$$

Therefore interpolating between the high regularity estimates for the Laplace operator and the L^2 estimate allows to absorb the last term in the right-hand side of (4.3).

In the rest of the chapter, we focus on the case $p = +\infty$ as it is the only definitely new one and as it covers both Hölder spaces with exponent greater than 1 and the limit space $B^1_{\infty,1}$, which is the largest one in which one may expect to get well-posedness.

Before going further into the description of our results, let us introduce a few notation.

- Throughout the chapter, C stands for a harmless "constant" the meaning of which depends on the context.
- If $a = (a^1, a^2)$ and $b = (b^1, b^2)$, then we denote $a \wedge b := a^1 b^2 - a^2 b^1$.
- The vorticity Ω associated to a vector-field u over $\mathbb{R}^N$ is the matrix valued function with entries
$$\Omega_{ij} := \partial_j u^i - \partial_i u^j .$$
If $N = 2$ then the vorticity may be identified with the scalar function $\omega := \partial_1 u^2 - \partial_2 u^1$ and if $N = 3$ with the vector field $\nabla \times u$.
- For all Banach space X and interval I of $\mathbb{R}$, we denote by $\mathcal{C}(I;X)$ (resp. $\mathcal{C}_b(I;X)$) the set of continuous (resp. continuous bounded) functions on I with values in X. If X has predual X^*, then we denote by $\mathcal{C}_w(I;X)$ the set of bounded measurable functions $f : I \to X$ such that for any $\phi \in X^*$, the function $t \mapsto \langle f(t), \phi \rangle_{X \times X^*}$ is continuous over I.
- For $p \in [1, +\infty]$, the notation $L^p(I;X)$ stands for the set of measurable functions on I with values in X such that $t \mapsto \|f(t)\|_X$ belongs to $L^p(I)$. In the case $I = [0,T]$ we alternately use the notation $L^p_T(X)$.
- We denote by $L^p_{loc}(I;X)$ the set of those functions defined on I and valued in X which, restricted to any compact subset J of I, are in $L^p(J;X)$.
- Finally, for any real valued function a over $\mathbb{R}^N$, we denote
$$a_* := \inf_{x \in \mathbb{R}^N} a(x) \qquad \text{and} \qquad a^* := \sup_{x \in \mathbb{R}^N} a(x) .$$

Let us now state our main well-posedness result in the case of a finite energy initial velocity field.

Theorem 4.1. *Let r be in $[1, +\infty]$ and $s \in \mathbb{R}$ satisfy $s > 1$ if $r \neq 1$ and $s \geq 1$ if $r = 1$. Let ρ_0 be a positive function in $B^s_{\infty,r}$ bounded away from 0, and u_0 be a divergence-free vector field with coefficients in $B^s_{\infty,r} \cap L^2$. Finally, suppose that the external force f has coefficients in $L^1([-T_0, T_0]; B^s_{\infty,r}) \cap \mathcal{C}([-T_0, T_0]; L^2)$ for some positive time T_0.*

Then there exists a time $T \in]0, T_0]$ such that system (4.1) with initial data (ρ_0, u_0) has a unique solution $(\rho, u, \nabla\Pi)$ on $[-T,T] \times \mathbb{R}^N$, with:

- *ρ in $\mathcal{C}([-T,T]; B^s_{\infty,r})$ and bounded away from 0,*
- *u in $\mathcal{C}([-T,T]; B^s_{\infty,r}) \cap \mathcal{C}^1([-T,T]; L^2)$ and*
- *$\nabla\Pi$ in $L^1([-T,T]; B^s_{\infty,r}) \cap \mathcal{C}([-T,T]; L^2)$.*

If $r = +\infty$ then one has only weak continuity in time with values in the Besov space $B^s_{\infty,\infty}$.

In the above functional framework, one may state a continuation criterion for the solution to (4.1) similar to that of theorem 2 of [28].

Theorem 4.2. *Let $(\rho, u, \nabla\Pi)$ be a solution to system (4.1) on $[0, T^*[\times \mathbb{R}^N$, with the properties described in theorem 4.1 for all $T < T^*$. Suppose also that we have*

$$\int_0^{T^*} \left(\|\nabla u\|_{L^\infty} + \|\nabla\Pi\|_{B^{s-1}_{\infty,r}} \right) dt < +\infty . \tag{4.4}$$

If T^* is finite, then $(\rho, u, \nabla\Pi)$ can be continued beyond T^* into a solution of (4.1) with the same regularity. Moreover, if $s > 1$ then one may replace in (4.4) the term $\|\nabla u\|_{L^\infty}$ with $\|\Omega\|_{L^\infty}$.

A similar result holds true also for negative times.

From this result, as our assumption on (r, s) implies that $B^{s-1}_{\infty,r} \hookrightarrow B^0_{\infty,1} \hookrightarrow L^\infty$, we immediately get the following corollary.

Corollary 4.3. *The lifespan of a solution in $B^s_{\infty,r}$ with $s > 1$ is the same as the lifespan in $B^1_{\infty,1}$. In particular, condition* (4.4) *holds true with $\|\nabla\Pi\|_{B^0_{\infty,1}}$.*

As pointed out in [28], hypothesis $u_0 \in L^2$ is somewhat restrictive in dimension $N = 2$ as if, say, the initial vorticity ω_0 is in L^1, then ω_0 is forced to have average 0 over $\mathbb{R}^2$. In particular, assuming $u_0 \in L^2(\mathbb{R}^2)$ precludes our considering general data with compactly supported nonnegative initial vorticity (e.g. vortex patches as in [13]; see also next chapter).

The following statement aims at considering initial data *with infinite energy*. For simplicity, we suppose the external force to be 0.

Theorem 4.4. *Let (s, r) be as in theorem 4.1. Let $\rho_0 \in B^s_{\infty,r}$ be bounded away from 0 and $u_0 \in B^s_{\infty,r} \cap W^{1,4}$.*

Then there exist a positive time T and a unique solution $(\rho, u, \nabla\Pi)$ on $[-T, T] \times \mathbb{R}^N$ of system (4.1) *with external force $f \equiv 0$, satisfying the following properties:*

- $\rho \in \mathcal{C}([-T, T]; B^s_{\infty,r})$ *bounded away from* 0,
- $u \in \mathcal{C}([-T, T]; B^s_{\infty,r} \cap W^{1,4})$ *and* $\partial_t u \in \mathcal{C}([-T, T]; L^2)$,
- $\nabla\Pi \in L^1([-T, T]; B^s_{\infty,r}) \cap \mathcal{C}([-T, T]; L^2)$.

As above, the continuity in time with values in $B^s_{\infty,r}$ is only weak if $r = +\infty$.

Remark 4.5. Under the above hypothesis, a continuation criterion in the spirit of theorem 4.2 may be proved. The details are left to the reader.

Remark 4.6. Let us also point out that in dimension $N \geq 2$, any velocity field with suitably smooth compactly supported vorticity is in $W^{1,4}$. Furthermore, there is some freedom over the $W^{1,4}$ assumption (see remark 4.14 below).

On the one hand, the existence results we stated so far are *local in time* even in the two-dimensional case. On the other hand, it is well known that the classical two-dimensional incompressible Euler equations are globally well-posed, a result that goes back to the pioneering work by V. Wolibner in [61] (see also [47], [40], [60] for global results in the case of less regular data). In the homogeneous case, the global existence stems from the fact that the vorticity ω is transported by the flow associated to the solution: we have

$$\partial_t \omega \,+\, u \cdot \nabla\omega \,=\, 0\,.$$

In the non-homogeneous context, we have instead the following more complicated identity:

$$\partial_t \omega \,+\, u \cdot \nabla\omega \,+\, \nabla\left(\frac{1}{\rho}\right) \wedge \nabla\Pi \,=\, 0\,. \tag{4.5}$$

If the classical homogeneous case has been deeply studied, to our knowledge there is no literature about the time of existence of solutions for the density-dependent incompressible Euler system. In the last section of this paper, we establish lower bounds for the lifespan of a solution to (4.1).

We first show that in any space dimension, if the initial velocity is of order ε (ε small enough), *without any restriction on the non-homogeneity*, then the lifespan is at least of order ε^{-1} (see the exact statement in theorem 4.15).

Next, taking advantage of equality (4.5) and of an estimate for the transport equation that has been established recently by M. Vishik in [60] (and generalized by T. Hmidi and S. Keraani in

[41]), we show that the lifespan of the solution tends to infinity if $\rho_0 - 1$ goes to 0. More precisely, theorem 4.16 states that if

$$\|\rho_0 - 1\|_{B^1_{\infty,1}} = \varepsilon \qquad \text{and} \qquad \|\omega_0\|_{B^0_{\infty,1}} + \|u_0\|_{L^2} = U_0$$

with ε small enough, then the lifespan is at least of order $U_0^{-1}\log(\log\varepsilon^{-1})$.

The chapter is organized as follows. In the next section, we introduce the tools (in addition to those presented in chapter 1) needed for proving our results: some classical results about transport equations in the $B^s_{p,r}$ framework and about elliptic equations in divergence form with non-constant coefficients. Sections 4.3 and 4.4 are devoted to the proof of our local existence statements first in the finite energy case and next if the initial velocity is in $W^{1,4}$. Finally, in the last section we state and prove results about the lifespan of a solution to our system, focusing on the particular case of space dimension $N = 2$.

4.2 Tools

Our results concerning equations (4.1) rely strongly on a priori estimates in Besov spaces for the transport equation

$$(T) \qquad \begin{cases} \partial_t a + v\cdot\nabla a = f \\ a_{|t=0} = a_0\,. \end{cases}$$

Therefore we shall often use the following result, the proof of which may be found e.g. in chapter 3 of [2].

Proposition 4.7. *Let $1 \leq r \leq +\infty$ and $\sigma > 0$ ($\sigma > -1$ if $\operatorname{div} v = 0$). Let $a_0 \in B^\sigma_{\infty,r}$, $f \in L^1([0,T];B^\sigma_{\infty,r})$ and v be a time dependent vector field in $\mathcal{C}_b([0,T]\times\mathbb{R}^N)$ such that*

$$\begin{aligned} \nabla v &\in L^1([0,T];L^\infty) \quad \text{if} \quad \sigma < 1, \\ \nabla v &\in L^1([0,T];B^{\sigma-1}_{\infty,r}) \quad \text{if} \quad \sigma > 1, \quad \text{or} \quad \sigma = r = 1. \end{aligned}$$

Then equation (T) has a unique solution a in:

- *the space $\mathcal{C}([0,T];B^\sigma_{\infty,r})$ if $r < \infty$,*
- *the space $\left(\bigcap_{\sigma'<\sigma}\mathcal{C}([0,T];B^{\sigma'}_{\infty,\infty})\right)\bigcap\mathcal{C}_w([0,T];B^\sigma_{\infty,\infty})$ if $r = +\infty$.*

Moreover, for all $t \in [0,T]$, we have

$$(4.6) \qquad e^{-CV(t)}\|a(t)\|_{B^\sigma_{\infty,r}} \leq \|a_0\|_{B^\sigma_{\infty,r}} + \int_0^t e^{-CV(\tau)}\|f(\tau)\|_{B^\sigma_{\infty,r}}\,d\tau$$

$$\textit{with} \quad V'(t) := \begin{cases} \|\nabla v(t)\|_{L^\infty} & \textit{if } \sigma < 1, \\ \|\nabla v(t)\|_{B^{\sigma-1}_{\infty,r}} & \textit{if } \sigma > 1, \quad \textit{or} \quad \sigma = r = 1. \end{cases}$$

If $v \equiv a$ then, for all $\sigma > 0$ ($\sigma > -1$ if $\operatorname{div} v = 0$), estimate (4.6) holds with $V'(t) := \|\nabla a(t)\|_{L^\infty}$.

Finally, we shall make an extensive use of energy estimates for the following elliptic equation:

$$(4.7) \qquad -\operatorname{div}(a\nabla\Pi) = \operatorname{div} F \qquad \text{in } \mathbb{R}^N,$$

where $a = a(x)$ is a measurable bounded function satisfying

$$(4.8) \qquad a_* := \inf_{x\in\mathbb{R}^N} a(x) > 0\,.$$

The following result based on Lax-Milgram's theorem (see section 3 of [28] for the proof), will be of great importance for us.

Lemma 4.8. *For all vector field F with coefficients in L^2, there exists a tempered distribution Π, unique up to constant functions, such that $\nabla\Pi \in L^2$ and equation (4.7) is satisfied. In addition, we have*

$$a_* \left\|\nabla\Pi\right\|_{L^2} \leq \left\|F\right\|_{L^2} . \tag{4.9}$$

4.3 Proof of the main well-posedness result

Obviously, one may extend the force term for any time so that it is not restrictive to assume that $T_0 = +\infty$. Owing to time reversibility of system (4.1), we can consider the problem of evolution for positive times only. For convenience we will assume $r < +\infty$; for treating the case $r = +\infty$, it is enough to replace the strong topology by the weak topology, whenever regularity up to index s is involved.

We will not work on system (4.1) directly, but rather on

$$\begin{cases} \partial_t a + u\cdot\nabla a = 0 \\ \partial_t u + u\cdot\nabla u + a\nabla\Pi = f \\ -\operatorname{div}(a\nabla\Pi) = \operatorname{div}\left(u\cdot\nabla\mathcal{P}u - f\right) , \end{cases} \tag{4.10}$$

where we have set $a := 1/\rho$.

The equivalence between (4.1) and (4.10) is given in the following statement (see again [28], section 4 for its proof).

Lemma 4.9. *Let u be a vector field with coefficients in $\mathcal{C}^1([0,T]\times\mathbb{R}^N)$ and such that $\mathcal{Q}u \in \mathcal{C}^1([0,T];L^2)$. Suppose also that $\nabla\Pi \in \mathcal{C}([0,T];L^2)$. Finally, let ρ be a continuous function on $[0,T]\times\mathbb{R}^N$ such that*

$$0 < \rho_* \leq \rho \leq \rho^* , \tag{4.11}$$

and define $a := 1/\rho$.

If $\operatorname{div} u(0,\cdot) \equiv 0$ in $\mathbb{R}^N$, then $(\rho, u, \nabla\Pi)$ is a solution to (4.1) if and only if $(a, u, \nabla\Pi)$ is a solution to (4.10).

This section unfolds as follows. First, we shall prove a priori estimates for suitably smooth solutions of (4.1) or (4.10). They will be most helpful to get the existence. As a matter of fact, the construction of solutions which will be proposed in the next subsection amounts to solving inductively a sequence of *linear* equations. The estimates for those approximate solutions turn out to be the same as those for the true solutions. In the last two subsections, we shall concentrate on the proof of the uniqueness part of theorem 4.1 and of the continuation criterion stated in theorem 4.2 (up to the endpoint case $s = r = 1$ which will be studied in the next section).

4.3.1 A priori estimates

Let $(a, u, \nabla\Pi)$ be a smooth solution of system (4.10) with the properties described in the statement of theorem 4.1. In this subsection, we show that on a suitably small time interval (the length of which depends only on the norms of the data), the norm of $(a, u, \nabla\Pi)$ may be bounded in terms of the data.

Recall that, according to proposition 1.31, the quantities $\|a\|_{B^s_{\infty,r}}$ and $\|\rho\|_{B^s_{\infty,r}}$ are equivalent under hypothesis (4.11). This fact will be used repeatedly in what follows.

Estimates for the density and the velocity field

Let us assume for a while that $\operatorname{div} u = 0$. Then $(\rho, u, \nabla\Pi)$ satisfies system (4.1) and the following energy equality holds true:

$$\left\|\sqrt{\rho(t)}\, u(t)\right\|_{L^2}^2 \;=\; \|\sqrt{\rho_0}\, u_0\|_{L^2}^2 \;+\; 2\int_0^t \left(\int_{\mathbb{R}^N} \rho\, f\cdot u\, dx\right) d\tau\,. \tag{4.12}$$

Moreover, from the equation satisfied by the density, we have that $\rho(t,x) = \rho_0\left(\psi_t^{-1}(x)\right)$, where ψ is the flow associated with u; so, ρ satisfies (4.11). Hence, from relation (4.12), we obtain the control of the L^2 norm of the velocity field: for all $t\in[0,T_0]$, we have, for some constant C depending only on ρ_* and ρ^*,

$$\|u(t)\|_{L^2} \;\leq\; C\left(\|u_0\|_{L^2} \;+\; \int_0^t \|f(\tau)\|_{L^2}\, d\tau\right)\,. \tag{4.13}$$

Next, in the general case where $\operatorname{div} u$ need not be 0, applying proposition 4.7 yields the following estimates:

$$\|a(t)\|_{B^s_{\infty,r}} \;\leq\; \|a_0\|_{B^s_{\infty,r}} \exp\left(C\int_0^t \|u\|_{B^s_{\infty,r}}\, d\tau\right) \tag{4.14}$$

$$\begin{aligned}\|u(t)\|_{B^s_{\infty,r}} \;\leq\; & \exp\left(C\int_0^t \|u\|_{B^s_{\infty,r}}\, d\tau\right)\cdot\left(\|u_0\|_{B^s_{\infty,r}} \;+\right.\\ & \left.+\int_0^t e^{-C\int_0^\tau \|u\|_{B^s_{\infty,r}}\, d\tau'}\left(\|f\|_{B^s_{\infty,r}} + \|a\|_{B^s_{\infty,r}}\|\nabla\Pi\|_{B^s_{\infty,r}}\right) d\tau\right),\end{aligned} \tag{4.15}$$

where, in the last line, we have used the fact that $B^s_{\infty,r}$, under our hypothesis, is an algebra.

Remark 4.10. Of course, as ρ and a verify the same equations, they satisfy the same estimates.

Estimates for the pressure term

Let us use the low frequency localization operator Δ_{-1} to separate $\nabla\Pi$ into low and high frequencies. We get

$$\|\nabla\Pi\|_{B^s_{\infty,r}} \;\leq\; \|\Delta_{-1}\nabla\Pi\|_{B^s_{\infty,r}} \;+\; \|(\mathrm{Id}-\Delta_{-1})\nabla\Pi\|_{B^s_{\infty,r}}\,.$$

Observe that $(\mathrm{Id}-\Delta_{-1})\nabla\Pi$ may be computed from $\Delta\Pi$ by means of a homogeneous multiplier of degree -1 in the sense of proposition 1.14. Hence

$$\|(\mathrm{Id}-\Delta_{-1})\nabla\Pi\|_{B^s_{\infty,r}} \;\leq\; C\,\|\Delta\Pi\|_{B^{s-1}_{\infty,r}}\,. \tag{4.16}$$

For the low frequencies term, however, the above inequality fails. Now, remembering the definition of $\|\cdot\|_{B^s_{\infty,r}}$ and the spectral properties of operator Δ_{-1}, one has that

$$\|\Delta_{-1}\nabla\Pi\|_{B^s_{\infty,r}} \;\leq\; C\,\|\Delta_{-1}\nabla\Pi\|_{L^\infty} \;\leq\; C\,\|\nabla\Pi\|_{L^2}\,,$$

where we used also Bernstein's inequality in writing last relation.

So putting together (4.16) and the above inequality, we finally obtain

$$\|\nabla\Pi\|_{B^s_{\infty,r}} \;\leq\; C\left(\|\nabla\Pi\|_{L^2} \;+\; \|\Delta\Pi\|_{B^{s-1}_{\infty,r}}\right)\,. \tag{4.17}$$

First of all, let us see how to control $\|\Delta\Pi\|_{B^{s-1}_{\infty,r}}$. Recall the third equation of (4.10):

$$\operatorname{div}\,(a\,\nabla\Pi) \;=\; F \quad\text{with}\quad F := \operatorname{div}\,(f \;-\; u\cdot\nabla\mathcal{P}u).$$

Developing the left-hand side of this equation, we obtain

$$\Delta\Pi = -\nabla(\log a)\cdot\nabla\Pi + \frac{F}{a}\cdot \tag{4.18}$$

Let us consider the first term of the right-hand side of the previous equation.

If $s>1$ then one may use that $B^{s-1}_{\infty,r}$ is an algebra and bound $\|\nabla(\log a)\|_{B^{s-1}_{\infty,r}}$ with $\|\nabla a\|_{B^{s-1}_{\infty,r}}$ according to proposition 1.31. So we get

$$\|\nabla(\log a)\cdot\nabla\Pi\|_{B^{s-1}_{\infty,r}} \leq C\,\|\nabla a\|_{B^{s-1}_{\infty,r}}\,\|\nabla\Pi\|_{B^{s-1}_{\infty,r}}\,.$$

Now, as $L^2\hookrightarrow B^{-\frac{N}{2}}_{\infty,\infty}$ (see proposition 1.6) and $B^{s-1}_{\infty,r}$ is an intermediate space between $B^{-\frac{N}{2}}_{\infty,\infty}$ and $B^s_{\infty,r}$, standard interpolation inequalities (see theorem 1.17) ensure that

$$\|\nabla\Pi\|_{B^{s-1}_{\infty,r}} \leq C\,\|\nabla\Pi\|^{\theta}_{L^2}\,\|\nabla\Pi\|^{1-\theta}_{B^s_{\infty,r}} \qquad \text{for some} \quad \theta\in]0,1[\,. \tag{4.19}$$

Plugging this relation in (4.17) and applying Young's inequality, we finally obtain

$$\|\nabla\Pi\|_{B^s_{\infty,r}} \leq C\left(\left(1+\|\nabla a\|^{\gamma}_{B^{s-1}_{\infty,r}}\right)\|\nabla\Pi\|_{L^2} + \left\|\frac{F}{a}\right\|_{B^{s-1}_{\infty,r}}\right), \tag{4.20}$$

where the exponent γ depends only on the space dimension N and on s.

In the limit case $s=r=1$, the space $B^0_{\infty,1}$ is no more an algebra and we have to modify the above argument: we use the Bony decomposition (1.3) to write

$$\nabla(\log a)\cdot\nabla\Pi = T_{\nabla(\log a)}\nabla\Pi + T_{\nabla\Pi}\nabla(\log a) + R(\nabla(\log a),\nabla\Pi)\,.$$

To estimate first and second term, we can apply theorem 1.24 and proposition 1.31: we get

$$\begin{aligned}\|T_{\nabla(\log a)}\nabla\Pi\|_{B^0_{\infty,1}} &\leq C\,\|\nabla(\log a)\|_{L^\infty}\,\|\nabla\Pi\|_{B^0_{\infty,1}}\\ &\leq C\,\|\nabla a\|_{L^\infty}\|\nabla\Pi\|_{B^0_{\infty,1}},\end{aligned} \tag{4.21}$$

$$\begin{aligned}\|T_{\nabla\Pi}\nabla(\log a)\|_{B^0_{\infty,1}} &\leq C\,\|\nabla\Pi\|_{L^\infty}\,\|\nabla(\log a)\|_{B^0_{\infty,1}}\\ &\leq C\,\|\nabla\Pi\|_{L^\infty}\,\|\nabla a\|_{B^0_{\infty,1}}\,.\end{aligned} \tag{4.22}$$

A similar inequality is no more true for the remainder term, though. However, one may use that $\nabla\Pi$ is in fact more regular: it belongs to $B^{1/2}_{\infty,1}$ for instance. Hence, using the embedding $B^{1/2}_{\infty,1}\hookrightarrow B^0_{\infty,1}$ and theorem 1.24, we can write

$$\begin{aligned}\|R(\nabla(\log a),\nabla\Pi)\|_{B^0_{\infty,1}} &\leq C\,\|\nabla(\log a)\|_{L^\infty}\,\|\nabla\Pi\|_{B^{1/2}_{\infty,1}},\\ &\leq C\,\|\nabla a\|_{L^\infty}\,\|\nabla\Pi\|_{B^{1/2}_{\infty,1}}\,.\end{aligned}$$

Putting the above inequality together with (4.21) and (4.22), and using that $B^0_{\infty,1}\hookrightarrow L^\infty$, we conclude that

$$\|\nabla(\log a)\cdot\nabla\Pi\|_{B^0_{\infty,1}} \leq C\|\nabla a\|_{B^0_{\infty,1}}\|\nabla\Pi\|_{B^{1/2}_{\infty,1}}\,.$$

Now, using interpolation between Besov spaces, as done for proving (4.19), we get for some suitable $\theta\in]0,1[$,

$$\|\nabla(\log a)\cdot\nabla\Pi\|_{B^0_{\infty,1}} \leq C\,\|\nabla a\|_{B^0_{\infty,1}}\,\|\nabla\Pi\|^{1-\theta}_{B^1_{\infty,1}}\,\|\nabla\Pi\|^{\theta}_{L^2}\,.$$

Hence $\|\nabla\Pi\|_{B^1_{\infty,1}}$ still satisfies Inequality (4.20) for some convenient $\gamma>0$.

Next, let us bound the last term of (4.18). By virtue of Bony's decomposition (1.3), we have

$$F/a \,=\, \rho\, F \,=\, T_\rho F \,+\, T_F \rho \,+\, R(\rho, F)\,;$$

from theorem 1.24 we infer that:

- $\|T_\rho F\|_{B^{s-1}_{\infty,r}} \,\leq\, C\,\rho^*\,\|F\|_{B^{s-1}_{\infty,r}}$,
- $\|T_F\rho\|_{B^{s-1}_{\infty,r}} \,\leq\, C\,\|F\|_{B^{-1}_{\infty,\infty}}\|\rho\|_{B^s_{\infty,r}} \,\leq\, C\,\|F\|_{B^{s-1}_{\infty,r}}\|\rho\|_{B^s_{\infty,r}}$,
- $\|R(\rho,F)\|_{B^{s-1}_{\infty,r}} \,\leq\, \|R(\rho,F)\|_{B^s_{\infty,r}} \,\leq\, C\,\|\rho\|_{B^1_{\infty,\infty}}\|F\|_{B^{s-1}_{\infty,r}} \,\leq\, C\,\|\rho\|_{B^s_{\infty,r}}\|F\|_{B^{s-1}_{\infty,r}}$.

Now the problem is to bound the Besov norm of $F \,=\, {\rm div}\,(f - u\cdot\nabla u)$. It is clear that $\|{\rm div}\, f\|_{B^{s-1}_{\infty,r}}$ can be bounded by $\|f\|_{B^s_{\infty,r}}$. For the second term of F we have to take advantage, once again, of Bony's decomposition (1.3) as follows:

$${\rm div}\;(u\cdot\nabla\mathcal{P}u) \,=\, \sum_{i,j}\partial_i u^j\,\partial_j\,(\mathcal{P}u)^i \,=\, \sum_{i,j}\Big(T_{\partial_i u^j}\partial_j\mathcal{P}u^i \,+\, T_{\partial_j\mathcal{P}u^i}\partial_i u^j \,+\, \partial_i R(u^j,\partial_j\mathcal{P}u^i)\Big)\,,$$

where we have used also the fact that ${\rm div}\,\mathcal{P}u = 0$. Now, for all i and j we have:

$$\begin{aligned}
\left\|T_{\partial_i u^j}\partial_j\mathcal{P}u^i\right\|_{B^{s-1}_{\infty,r}} &\leq\; C\,\|\nabla u\|_{L^\infty}\,\|\nabla\mathcal{P}u\|_{B^{s-1}_{\infty,r}}\\
\left\|T_{\partial_j\mathcal{P}u^i}\partial_i u^j\right\|_{B^{s-1}_{\infty,r}} &\leq\; C\,\|\nabla\mathcal{P}u\|_{L^\infty}\,\|\nabla u\|_{B^{s-1}_{\infty,r}}\\
\left\|\partial_i R(u^j,\partial_j\mathcal{P}u^i)\right\|_{B^{s-1}_{\infty,r}} &\leq\; \left\|R(u^j,\partial_j\mathcal{P}u^i)\right\|_{B^s_{\infty,r}}\\
&\leq\; C\,\|u\|_{B^s_{\infty,r}}\,\|\nabla\mathcal{P}u\|_{B^{s-1}_{\infty,r}}\,.
\end{aligned}$$

Because, by embedding, $\|\nabla\mathcal{P}u\|_{L^\infty} \,\leq\, C\,\|\nabla\mathcal{P}u\|_{B^{s-1}_{\infty,r}}$, we have

$$\|{\rm div}\,(u\cdot\nabla\mathcal{P}u)\|_{B^{s-1}_{\infty,r}} \,\leq\, C\,\|u\|_{B^s_{\infty,r}}\,\|\nabla\mathcal{P}u\|_{B^{s-1}_{\infty,r}}\,.$$

In order to bound $\mathcal{P}u$, let us decompose it into low and high frequencies as follows:

$$\mathcal{P}u = \Delta_{-1}\mathcal{P}u + ({\rm Id}\,-\,\Delta_{-1})\mathcal{P}u.$$

On the one hand, combining Bernstein's inequality and the fact that $\mathcal{P}$ is an orthogonal projector over L^2 yields

$$\|\Delta_{-1}\nabla\mathcal{P}u\|_{L^\infty} \,\leq\, C\,\|u\|_{L^2}\,.$$

On the other hand, according to remark 1.15, one may write that

$$\|({\rm Id}\,-\,\Delta_{-1})\mathcal{P}u\|_{B^s_{\infty,r}} \leq C\|u\|_{B^s_{\infty,r}}.$$

Therefore we get

$$\|\nabla\mathcal{P}u\|_{B^{s-1}_{\infty,r}} \,\leq\, C\,\|u\|_{B^s_{\infty,r}\cap L^2}\,, \tag{4.23}$$

from which it follows that

$$\left\|\frac{F}{a}\right\|_{B^{s-1}_{\infty,r}} \,\leq\, C\,\|a\|_{B^s_{\infty,r}}\,\Big(\|f\|_{B^s_{\infty,r}} + \|u\|^2_{B^s_{\infty,r}\cap L^2}\Big)\,. \tag{4.24}$$

It remains us to control $\|\nabla\Pi\|_{L^2}$. Keeping in mind lemma 4.8, from the third equation of system (4.10) and inequality (4.23), we immediately get

$$\begin{aligned}
a_*\,\|\nabla\Pi\|_{L^2} &\leq\; \|f\|_{L^2} \,+\, \|u\cdot\nabla\mathcal{P}u\|_{L^2}\\
&\leq\; \|f\|_{L^2} \,+\, \|u\|_{L^2}\,\|\nabla\mathcal{P}u\|_{L^\infty}\\
&\leq\; \|f\|_{L^2} \,+\, C\|u\|^2_{B^s_{\infty,r}\cap L^2}\,.
\end{aligned}$$

Putting all these inequalities together, we finally obtain

$$\|\nabla\Pi\|_{L^1_t(L^2)} \leq C\left(\|f\|_{L^1_t(L^2)} + \int_0^t \|u\|^2_{B^s_{\infty,r}\cap L^2}\,d\tau\right) \tag{4.25}$$

$$\begin{aligned}\|\nabla\Pi\|_{L^1_t(B^s_{\infty,r})} &\leq C\Bigg(\left(1+\|\nabla a\|^\gamma_{L^\infty_t(B^{s-1}_{\infty,r})}\right)\|\nabla\Pi\|_{L^1_t(L^2)} + \\ &\qquad + \|a\|_{L^\infty_t(B^s_{\infty,r})}\left(\|f\|_{L^1_t(B^s_{\infty,r})} + \int_0^t \|u\|^2_{B^s_{\infty,r}\cap L^2} d\tau\right)\Bigg)\,.\end{aligned} \tag{4.26}$$

Final estimate

First of all, let us fix $T>0$ so small as to satisfy

$$\exp\left(C\int_0^T \|u\|_{B^s_{\infty,r}}\,dt\right) \leq 2\,, \tag{4.27}$$

which is always possible because of the boundedness of u with respect to the time variable.

Then, setting

$$\begin{aligned} U(t) &:= \|u(t)\|_{L^2\cap B^s_{\infty,r}} = \|u(t)\|_{L^2} + \|u(t)\|_{B^s_{\infty,r}} \\ U_0(t) &:= \|u_0\|_{L^2\cap B^s_{\infty,r}} + \int_0^t \|f\|_{L^2\cap B^s_{\infty,r}}\,d\tau \end{aligned}$$

and combining estimates (4.13), (4.14), (4.15), (4.25) and (4.26), we get

$$U(t) \leq C\left(U_0(t) + \int_0^t U^2(\tau)\,d\tau\right) \quad \text{for all} \quad t\in[0,T], \tag{4.28}$$

where the constant C depends only on s, N, $\|a_0\|_{B^s_{\infty,r}}$, a_* and a^*.

So, taking T small enough and changing once more the multiplying constant C if needed, a standard bootstrap argument allows to show that

$$U(t) \leq C\,U_0(t) \qquad\qquad \text{for all} \quad t \in [0,T]\,.$$

4.3.2 Existence of a solution to density-dependent Euler system

We proceed in two steps: first we construct inductively a sequence of smooth global approximate solutions, defined as solutions of linear equations, and then we prove the convergence of this sequence to a solution of the nonlinear system (4.10) with the required properties. Recall that to simplify the presentation we have assumed $T_0 = +\infty$ and that we focus on the evolution for positive times only.

Construction of the sequence of approximate solutions

First, we smooth out the data (by convolution for instance) so as to get a sequence $(a_0^n, u_0^n, f^n)_{n\in\mathbb{N}}$ such that $u_0^n \in H^\infty$, $f^n \in \mathcal{C}(\mathbb{R}_+;H^\infty)$, a_0^n and its derivatives at any order are bounded and

$$a_* \leq a_0^n \leq a^*\,, \tag{4.29}$$

with in addition

- $a_0^n \to a_0$ in $B^s_{\infty,r}$,
- $u_0^n \to u_0$ in $L^2\cap B^s_{\infty,r}$,
- $f^n \to f$ in $\mathcal{C}(\mathbb{R}_+;L^2)\cap L^1(\mathbb{R}_+;B^s_{\infty,r})$.

In order to construct a sequence of smooth approximate solutions, we argue by induction. We first set $a^0 = a_0^0$, $u^0 = u_0^0$ and $\nabla\Pi^0 = 0$.

Now, suppose we have already built a smooth approximate solution $(a^n, u^n, \nabla\Pi^n)$ over $\mathbb{R}_+ \times \mathbb{R}^N$ with a^n satisfying (4.11). In order to construct the $(n+1)$-th term of the sequence, we first define a^{n+1} to be the solution of the linear transport equation

$$\partial_t a^{n+1} \,+\, u^n \cdot \nabla a^{n+1} \,=\, 0$$

with initial datum $a^{n+1}|_{t=0} = a_0^{n+1}$.

Given that u^n is Lipschitz continuous in the space varaible (in fact, it belongs to $B^s_{\infty,r}$ by a priori estimates), its flow ψ^n is smooth too. In particular, ψ^n_t is, at every fixed time t, a $\mathcal{C}^1$ diffeomorphism on the whole $\mathbb{R}^N$ (see also proposition 3.10 of [2]). Hence $a^{n+1}(t,x) = a_0^{n+1}\left((\psi^n_t)^{-1}(x)\right)$ is smooth and satisfies (4.11). Furthermore, by virtue of proposition 4.7,

$$\|a^{n+1}(t)\|_{B^s_{\infty,r}} \,\leq\, \|a_0^{n+1}\|_{B^s_{\infty,r}} \,\exp\left(C\int_0^t \|u^n\|_{B^s_{\infty,r}}\,d\tau\right)\,. \tag{4.30}$$

Note that the reciprocal function ρ^{n+1} of a^{n+1} satisfies $\rho^{n+1}(t,x) \,=\, \rho_0^{n+1}\left(\left(\psi^n_t\right)^{-1}(x)\right)$, together with (4.11) and the equation

$$\partial_t \rho^{n+1} \,+\, u^n \cdot \nabla\rho^{n+1} \,=\, 0\,. \tag{4.31}$$

Hence it also fulfills inequality (4.30) up to a change of a_0^{n+1} in ρ_0^{n+1}.

At this point, we define u^{n+1} as the unique smooth solution of the linear transport equation

$$\begin{cases} \partial_t u^{n+1} \,+\, u^n \cdot \nabla u^{n+1} \,=\, f^{n+1} \,-\, a^{n+1}\,\nabla\Pi^n \\ u^{n+1}|_{t=0} \,=\, u_0^{n+1}\,. \end{cases}$$

Since the right-hand side belongs to $L^1_{loc}(\mathbb{R}_+; L^2)$, from classical results for transport equation we get that $u^{n+1} \in \mathcal{C}(\mathbb{R}_+; L^2)$. Besides, as $\rho^n = (a^n)^{-1}$ for all n, if we differentiate $\|\sqrt{\rho^{n+1}}u^{n+1}\|^2_{L^2}$ with respect to time and use the equations for ρ^{n+1} and u^{n+1}, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|\sqrt{\rho^{n+1}}u^{n+1}\right\|^2_{L^2} \,=\, \frac{1}{2}\int \rho^{n+1}|u^{n+1}|^2\,\mathrm{div}\,u^n\,dx \,+\, \int \rho^{n+1}u^{n+1}\cdot f^{n+1}\,dx \,-\, \int \nabla\Pi^n \cdot u^{n+1}\,dx.$$

Observe that u^n and u^{n+1} need not to be divergence free; nevertheless, applying Gronwall's lemma, it is easy to see that

$$\begin{aligned}\left\|\sqrt{\rho^{n+1}(t)}u^{n+1}(t)\right\|_{L^2} \;\leq\;& \left(\left\|\sqrt{\rho_0^{n+1}}u_0^{n+1}\right\|_{L^2} + C\int_0^t \left(\|f^{n+1}\|_{L^2} + \|\nabla\Pi^n\|_{L^2}\right)d\tau\right)\times \\ &\times\exp\left(\frac{1}{2}\int_0^t \|\mathrm{div}\,u^n\|_{L^\infty}\,d\tau\right).\end{aligned} \tag{4.32}$$

Finally, we have to define the approximate pressure Π^{n+1}. We have already proved that a^{n+1} satisfies the ellipticity condition (4.11); so we can consider the elliptic equation

$$\mathrm{div}\,\left(a^{n+1}\,\nabla\Pi^{n+1}\right) \,=\, \mathrm{div}\,\left(f^{n+1} \,-\, u^{n+1}\cdot\nabla\mathcal{P}u^{n+1}\right)\,.$$

As f^{n+1} and u^{n+1} are in $\mathcal{C}(\mathbb{R}_+; H^\infty)$, the classical theory for elliptic equations ensures that the above equation has a unique solution $\nabla\Pi^{n+1}$ in $\mathcal{C}(\mathbb{R}_+; H^\infty)$. In addition, going along the lines of the proof of (4.25), we get

$$\left\|\nabla\Pi^{n+1}\right\|_{L^1_t(L^2)} \,\leq\, C\left(\|f^{n+1}\|_{L^1_t(L^2)} \,+\, \int_0^t \|u^{n+1}\|^2_{B^s_{\infty,r}\cap L^2}\,d\tau\right)\,. \tag{4.33}$$

Of course, by embedding, we have $\nabla\Pi^{n+1} \in \mathcal{C}(\mathbb{R}_+; B^s_{\infty,r})$. Hence, arguing as for proving (4.26), we gather

$$
\begin{aligned}
(4.34)\quad \left\|\nabla\Pi^{n+1}\right\|_{L^1_t(B^s_{\infty,r})} \;\leq\; & C\|a^{n+1}\|_{L^\infty_t(B^s_{\infty,r})}\left(\|f^{n+1}\|_{L^1_t(B^s_{\infty,r})} + \int_0^t \|u^{n+1}\|^2_{B^s_{\infty,r}\cap L^2}\,d\tau\right) + \\
& + C\left(1 + \|\nabla a^{n+1}\|^\gamma_{L^\infty_t(B^{s-1}_{\infty,r})}\right)\|\nabla\Pi^{n+1}\|_{L^1_t(L^2)}.
\end{aligned}
$$

Note also that the norms of the approximate data that we use in (4.30), (4.32), (4.33) and (4.34) may be bounded *independently of* n. Therefore, repeating the arguments leading to (4.28) and to theorem 1 of [28], one may find some positive time T, which may depend on $\|\rho_0\|_{B^s_{\infty,r}}$, $\|u_0\|_{B^s_{\infty,r}\cap L^2}$ and $\|f\|_{L^1([0,T];B^s_{\infty,r}\cap L^2)}$ but is *independent of* n, such that

- $(a^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty([0,T]; B^s_{\infty,r})$,
- $(u^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty([0,T]; B^s_{\infty,r}\cap L^2)$,
- $(\nabla\Pi^n)_{n\in\mathbb{N}}$ is bounded in $L^1([0,T]; B^s_{\infty,r}) \cap L^\infty([0,T]; L^2)$.

Convergence of the sequence

Let us observe that the function $\widetilde{a}^n := a^n - a^n_0$ satisfies

$$
\begin{cases}
\partial_t\widetilde{a}^n &= \; -\,u^{n-1}\cdot\nabla a^n \\
\widetilde{a}^n|_{t=0} &= \; 0\,.
\end{cases}
$$

Because $u^{n-1} \in \mathcal{C}([0,T]; L^2)$ and $\nabla a^n \in \mathcal{C}_b([0,T]\times\mathbb{R}^N)$, it immediately follows that $\widetilde{a}^n \in \mathcal{C}^1([0,T]; L^2)$. Now we want to prove that the sequence $(\widetilde{a}^n, u^n, \nabla\Pi^n)_{n\in\mathbb{N}}$, built in this way, is a Cauchy sequence in $\mathcal{C}([0,T]; L^2)$. So let us define for $n\in\mathbb{N}$ and $p\in\mathbb{N}^*$,

$$
\begin{aligned}
\delta a^n_p &:= a^{n+p} - a^n\,,\\
\delta\widetilde{a}^n_p &:= \widetilde{a}^{n+p} - \widetilde{a}^n \;=\; \delta a^n_p - \delta a^n_p(0)\,,\\
\delta\rho^n_p &:= \rho^{n+p} - \rho^n\,,\\
\delta u^n_p &:= u^{n+p} - u^n\,,\\
\delta\Pi^n_p &:= \Pi^{n+p} - \Pi^n\,,\\
\delta f^n_p &:= f^{n+p} - f^n\,.
\end{aligned}
$$

Let us emphasize that, by assumption and embedding, we have

$$
(4.35)\qquad a^n_0 \to a_0 \;\text{ in }\; \mathcal{C}^{0,1}, \quad u^n_0 \to u_0 \;\text{ in }\; L^2, \quad f^n \to f \;\text{ in }\; \mathcal{C}([0,T]; L^2).
$$

This will be the key to our proof of convergence.

Let us first focus on $\widetilde{a}^n$. By construction, $\delta\widetilde{a}^n_p$ belongs to $\mathcal{C}^1([0,T]; L^2)$ and satisfies the equation

$$
\partial_t\delta\widetilde{a}^n_p \;=\; -u^{n+p-1}\cdot\nabla\delta\widetilde{a}^n_p \;-\; \delta u^{n-1}_p\cdot\nabla a^n - u^{n+p-1}\cdot\nabla\delta a^n_p(0)
$$

from which, taking the scalar product in L^2 with $\delta\widetilde{a}^n$, we obtain

$$
\frac{1}{2}\frac{d}{dt}\|\delta\widetilde{a}^n_p\|^2_{L^2} \;=\; \frac{1}{2}\int\left(\delta\widetilde{a}^n_p\right)^2 \operatorname{div} u^{n+p-1}\,dx \;-\; \int \delta u^{n-1}_p\cdot\nabla a^n\,\delta\widetilde{a}^n_p\,dx - \int u^{n+p-1}\cdot\nabla\delta a^n_p(0)\,\delta\widetilde{a}^n_p\,dx\,.
$$

So, keeping in mind that $\delta\widetilde{a}^n_p(0) = 0$ and integrating with respect to the time variable one has

$$
\begin{aligned}
(4.36)\quad \left\|\delta\widetilde{a}^n_p(t)\right\|_{L^2} \;\leq\; & \int_0^t\left(\frac{1}{2}\left\|\operatorname{div} u^{n+p-1}\right\|_{L^\infty}\left\|\delta\widetilde{a}^n_p\right\|_{L^2} + \right.\\
& \left. + \|\nabla a^n\|_{L^\infty}\|\delta u^{n-1}_p\|_{L^2} + \|u^{n+p-1}\|_{L^2}\|\nabla\delta a^n_p(0)\|_{L^\infty}\right)d\tau\,.
\end{aligned}
$$

Equally easily, one can see that the following equality holds true:

$$\rho^{n+p}\left(\partial_t\delta u_p^n + u^{n+p-1}\cdot\nabla\delta u_p^n\right) + \nabla\delta\Pi_p^{n-1} = \rho^{n+p}\left(\delta f_p^n - \delta u_p^{n-1}\cdot\nabla u^n - \delta a_p^n\,\nabla\Pi^{n-1}\right);$$

taking the scalar product in L^2 with δu_p^n, integrating by parts and remembering equation (4.31) at $(n+p)$-th step, we finally get

$$\begin{aligned}\left\|\sqrt{\rho^{n+p}(t)}\delta u_p^n(t)\right\|_{L^2} &\leq \int_0^t \left\|\operatorname{div} u^{n+p-1}\right\|_{L^\infty}\left\|\sqrt{\rho^{n+p}}\delta u_p^n\right\|_{L^2}\,d\tau\\ &+\int_0^t\Biggl(\|\nabla u^n\|_{L^\infty}\left\|\sqrt{\rho^{n+p}}\delta u_p^{n-1}\right\|_{L^2}+\left\|\sqrt{\rho^{n+p}}\nabla\Pi^{n-1}\right\|_{L^\infty}\|\widetilde{\delta a}_p^n\|_{L^2}\\ &\qquad+\left\|\sqrt{\rho^{n+p}}\nabla\Pi^{n-1}\right\|_{L^2}\left\|\delta a_p^n(0)\right\|_{L^\infty}+\left\|\frac{\nabla\delta\Pi_p^{n-1}}{\sqrt{\rho^{n+p}}}\right\|_{L^2}+\sqrt{\rho^*}\|\delta f_p^n\|_{L^2}\Biggr)d\tau\,.\end{aligned}$$

From (4.36), Gronwall's Lemma and (4.29), we thus get, for some constant C depending only on a_* and a^*,

$$\begin{aligned}\left\|(\widetilde{\delta a}_p^n,\delta u_p^n)(t)\right\|_{L^2} \leq C\Biggl(&e^{A_p^n(t)}\|\delta u_p^n(0)\|_{L^2}+\int_0^t e^{A_p^n(t)-A_p^n(\tau)}\Bigl(\|(\nabla a^n,\nabla u^n)\|_{L^\infty}\left\|\delta u_p^{n-1}\right\|_{L^2}\\ &+\left\|\nabla\delta\Pi_p^{n-1}\right\|_{L^2}+\left\|\nabla\Pi^{n-1}\right\|_{L^2}\left\|\delta a_p^n(0)\right\|_{L^\infty}+\|u^{n+p-1}\|_{L^2}\|\nabla\delta a_p^n(0)\|_{L^\infty}+\|\delta f_p^n\|_{L^2}\Bigr)\,d\tau\Biggr),\end{aligned}$$

where we have set

$$A_p^n(t) := \int_0^t\left(\left\|\operatorname{div} u^{n+p-1}\right\|_{L^\infty}+\left\|\sqrt{\rho^{n+p}}\,\nabla\Pi^{n-1}\right\|_{L^\infty}\right)d\tau\,.$$

Of course, the uniform a priori estimates of the previous step allow us to control the exponential term for all $t\in[0,T]$ by some constant C_T.

Next, we have to deal with the term $\nabla\delta\Pi_p^{n-1}$. We notice that it satisfies the elliptic equation

$$-\operatorname{div}\left(a^{n-1}\nabla\delta\Pi_p^{n-1}\right) = \operatorname{div}\left(-\delta a_p^{n-1}\nabla\Pi^{n-1+p} - u^{n-1}\cdot\nabla\mathcal{P}\delta u_p^{n-1} - \delta u_p^{n-1}\cdot\nabla\mathcal{P}u^{n+p-1}+\delta f_p^{n-1}\right).$$

Now we apply the following algebraic identity,

$$\operatorname{div}(v\cdot\nabla w) = \operatorname{div}(w\cdot\nabla v) + \operatorname{div}(v\operatorname{div} w) - \operatorname{div}(w\operatorname{div} v)\,,$$

to $v=u^{n-1}$ and $w=\mathcal{P}\delta u_p^{n-1}$. Remembering that $\operatorname{div}\mathcal{P}\delta u_p^{n-1}=0$, from the previous relation we infer

$$\begin{aligned}\operatorname{div}\left(a^{n-1}\nabla\delta\Pi_p^{n-1}\right) = \operatorname{div}\Biggl(&\mathcal{P}\delta u_p^{n-1}\operatorname{div} u^{n-1} - \mathcal{P}\delta u_p^{n-1}\cdot\nabla u^{n-1} -\\ &-\delta u_p^{n-1}\cdot\nabla\mathcal{P}u^{n+p-1} - \delta a_p^{n-1}\nabla\Pi^{n-1+p} + \delta f_p^{n-1}\Biggr).\end{aligned}$$

Then, from lemma 4.8 and the fact that $\|\mathcal{P}\|_{\mathcal{L}(L^2;L^2)}=1$, one immediately has the following inequality:

$$\begin{aligned}(4.37)\quad a_*\left\|\nabla\delta\Pi_p^{n-1}\right\|_{L^2} \leq\; &\left\|\widetilde{\delta a}_p^{n-1}\right\|_{L^2}\|\nabla\Pi^n\|_{L^\infty}+\left\|\delta a_p^{n-1}(0)\right\|_{L^\infty}\|\nabla\Pi^n\|_{L^2}+\|\delta f_p^{n-1}\|_{L^2}\\ &+\left\|\delta u_p^{n-1}\right\|_{L^2}\left(\left\|\operatorname{div} u^{n-1}\right\|_{L^\infty}+\left\|\nabla u^{n-1}\right\|_{L^\infty}+\left\|\nabla\mathcal{P}u^{n+p-1}\right\|_{L^\infty}\right).\end{aligned}$$

Due to a priori estimates, we finally obtain, for all $t\in[0,T]$,

$$\begin{aligned}\left\|(\widetilde{\delta a}_p^n,\delta u_p^n)(t)\right\|_{L^2} &\leq C_T\Biggl(\|\delta u_p^n(0)\|_{L^2}+\int_0^t\Bigl(\left\|(\delta a_p^{n-1},\delta u_p^{n-1})\right\|_{L^2}+\\ &\qquad+\left\|\nabla\delta\Pi_p^{n-1}\right\|_{L^2}+\|\delta a_p^n(0)\|_{\mathcal{C}^{0,1}}+\|\delta f_p^n\|_{L^2}\Bigr)\,d\tau\Biggr)\\ \left\|\nabla\delta\Pi_p^{n-1}\right\|_{L^2} &\leq C_T\left(\left\|\widetilde{\delta a}_p^{n-1}\right\|_{L^2}+\left\|\delta u_p^{n-1}\right\|_{L^2}+\|\delta a_p^{n-1}(0)\|_{L^\infty}+\|\delta f_p^{n-1}\|_{L^2}\right).\end{aligned}$$

Therefore, plugging the second inequality in the first one, we find out that, for all $t \in [0,T]$ and all integers $p \geq 1$ and $n \geq 1$,

$$\left\|(\delta\widetilde{a}^n_p, \delta u^n_p)(t)\right\|_{L^2} \leq \varepsilon_n + C_T \int_0^t \left\|(\delta\widetilde{a}^{n-1}_p, \delta u^{n-1}_p)\right\|_{L^2} d\tau \tag{4.38}$$

where we have defined

$$\varepsilon_n := C_T \sup_{p\geq 1}\left(\|\delta u^n_p(0)\|_{L^2} + \int_0^T \left(\|\delta f^{n-1}_p\|_{L^2} + \|\delta f^n_p\|_{L^2} + \|\delta a^{n-1}_p(0)\|_{\mathcal{C}^{0,1}} + \|\delta a^n_p(0)\|_{\mathcal{C}^{0,1}}\right) dt\right).$$

Now, bearing (4.35) in mind, we have

$$\lim_{n\to+\infty} \varepsilon_n = 0\,.$$

Hence, one may conclude that

$$\lim_{n\to+\infty} \sup_{p\geq 1} \sup_{t\in[0,T]} \left(\|\delta\widetilde{a}^n_p(t)\|_{L^2} + \|\delta u^n_p(t)\|_{L^2}\right) = 0\,.$$

In other words, $(\widetilde{a}^n)_{n\in\mathbb{N}}$ and $(u^n)_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathcal{C}([0,T];L^2)$; therefore they converge to some functions $\widetilde{a}$, $u \in \mathcal{C}([0,T];L^2)$. In the same token, it is clear that $(\nabla\Pi^n)_{n\in\mathbb{N}}$ converges to some $\nabla\Pi \in \mathcal{C}([0,T];L^2)$.

Defining $a := \widetilde{a} + a_0$, it remains to show that a, u and $\nabla\Pi$ are indeed solutions of the initial system (4.10). We already know that a, u and $\nabla\Pi \in \mathcal{C}([0,T];L^2)$. In addition,

- thanks to the Fatou's property for Besov spaces, as $(a^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty([0,T];B^s_{\infty,r})$, we obtain that $a \in L^\infty([0,T];B^s_{\infty,r})$ and satisfies (4.29);
- in the same way, $u \in L^\infty([0,T];B^s_{\infty,r})$ because also $(u^n)_{n\in\mathbb{N}}$ is bounded in the same space;
- finally, $\nabla\Pi \in L^1([0,T];B^s_{\infty,r})$ because the sequence $(\nabla\Pi^n)_{n\in\mathbb{N}}$ is bounded in the same functional space.

By interpolation we get that the sequences converge strongly to the solutions in every intermediate space between $\mathcal{C}([0,T];L^2)$ and $\mathcal{C}([0,T];B^s_{\infty,r})$, which is enough to pass to the limit in the equations satisfied by $(a^n, u^n, \nabla\Pi^n)$. So, $(a, u, \nabla\Pi)$ verifies system (4.10).

Finally, continuity properties of the solutions with respect to the time variable can be recovered from the equations fulfilled by them, using proposition 4.7.

4.3.3 Uniqueness of the solution

Uniqueness of the solution to system (4.1) is a straightforward consequence of the following stability result, the proof of which can be found in [28], section 4.

Proposition 4.11. *Let* $(\rho_1, u_1, \nabla\Pi_1)$ *and* $(\rho_2, u_2, \nabla\Pi_2)$ *satisfy System* (4.1) *with external forces* f_1 *and* f_2*, respectively. Suppose that* ρ_1 *and* ρ_2 *both satisfy* (4.11). *Assume also that:*

- $\delta\rho := \rho_2 - \rho_1$ *and* $\delta u := u_2 - u_1$ *both belong to* $\mathcal{C}^1([0,T];L^2)$,
- $\delta f := f_2 - f_1 \in \mathcal{C}([0,T];L^2)$,
- $\nabla\rho_1$, ∇u_1 *and* $\nabla\Pi_1$ *belong to* $L^1([0,T];L^\infty)$.

Then for all $t \in [0,T]$ we have

$$e^{-A(t)} \left(\|\delta\rho(t)\|_{L^2} + \|(\sqrt{\rho_2}\delta u)\,(t)\|_{L^2}\right) \leq \|\delta\rho(0)\|_{L^2} + \|(\sqrt{\rho_2}\delta u)\,(0)\|_{L^2} + \int_0^t e^{-A(\tau)}\,\|(\sqrt{\rho_2}\delta f)\|_{L^2}\,d\tau\,,$$

where we have defined

$$A(t) := \int_0^t \left(\left\|\frac{\nabla\rho_1}{\sqrt{\rho_2}}\right\|_{L^\infty} + \left\|\frac{\nabla\Pi_1}{\rho_1\sqrt{\rho_2}}\right\|_{L^\infty} + \|\nabla u_1\|_{L^\infty}\right) d\tau\,.$$

Proof of uniqueness in theorem 4.1. Let us suppose that there exist two solutions $(\rho_1, u_1, \nabla\Pi_1)$ and $(\rho_2, u_2, \nabla\Pi_2)$ to system (4.1) corresponding to the same data and satisfying the hypotheses of theorem 4.1. Then, as one can easily verify, these solutions satisfy the assumptions of proposition 4.11. For instance, that $\delta\rho \in \mathcal{C}^1([0,T];L^2)$ is an immediate consequence of the fact that, for $i = 1,2$, the velocity field u_i is in $\mathcal{C}([0,T];L^2)$ and $\nabla\rho_i$ is in $\mathcal{C}([0,T];L^\infty)$, so that $\partial_t\rho_i \in \mathcal{C}([0,T];L^2)$.

So, proposition 4.11 implies that $(\rho_1, u_1, \nabla\Pi_1) \equiv (\rho_2, u_2, \nabla\Pi_2)$. $\square$

4.4 The vorticity equation and applications

This section is devoted to the proof of the blow-up criterion and of theorem 4.4. Both results rely on the vorticity equation associated to system (4.1). As done in section 4.3, we shall restrict ourselves to the evolution for positive times and make the usual convention as regards time continuity, if $r < +\infty$.

4.4.1 On the vorticity

As in all this section the vorticity will play a fundamental role, let us spend some words about it. Given a vector-field u, we set ∇u its Jacobian matrix and ${}^t\nabla u$ the transposed matrix of ∇u. We define the vorticity associated to u by

$$\Omega \,:=\, \nabla u \,-\, {}^t\nabla u.$$

Recall that, in dimension $N = 2$, Ω can be identified with the scalar function $\omega \,=\, \partial_1 u^2 - \partial_2 u^1$, while for $N = 3$ with the vector-field $\omega \,=\, \nabla \times u$.

It is obvious that, for all $q \in [1,+\infty]$, if $\nabla u \in L^q$, then also $\Omega \in L^q$. Conversely, if u is divergence-free, then for all $1 \leq i \leq N$ we have $\Delta u^i \,=\, \sum_{j=1}^N \partial_j\Omega_{ij}$, and so, formally,

$$\nabla u^i \;=\; -\nabla\,(-\Delta)^{-1}\sum_{j=1}^N \partial_j\,\Omega_{ij}\,.$$

As the symbol of the operator $-\partial_i\,(-\Delta)^{-1}\,\partial_j$ is $\sigma(\xi) = \xi_i\xi_j/|\xi|^2$, the classical Calderon-Zygmund theorem ensures that[1] for all $q \in]1,+\infty[$ if $\Omega \in L^q$, then $\nabla u \in L^q$ and

$$\|\nabla u\|_{L^q} \leq C\|\Omega\|_{L^q}. \tag{4.39}$$

The above relation also implies that

$$u = \Delta_{-1}u - (\mathrm{Id}\,-\Delta_{-1})(-\Delta)^{-1}\sum_j \partial_j\Omega_{ij}.$$

Hence combining Bernstein's inequality and proposition 1.14, we gather that

$$\|u\|_{B^s_{\infty,r}} \leq C\big(\|u\|_{L^p} + \|\Omega\|_{B^{s-1}_{\infty,r}}\big) \quad \text{for all} \;\; p \in [1,+\infty]. \tag{4.40}$$

[1]this time the extreme values of q are not included.

From now on, let us assume that Ω is the vorticity associated to some solution $(\rho, u, \nabla\Pi)$ of (4.1), defined on $[0,T]\times\mathbb{R}^N$. From the velocity equation, we gather that Ω satisfies the following transport-like equation:

$$\partial_t\Omega\,+\,u\cdot\nabla\Omega\,+\,\Omega\cdot\nabla u\,+\,{}^t\nabla u\cdot\Omega\,+\,\nabla\left(\frac{1}{\rho}\right)\wedge\nabla\Pi\,=\,F \tag{4.41}$$

where $F_{ij}:=\partial_j f^i-\partial_i f^j$ and, for two vector fields v and w, we have set $v\wedge w$ to be the skew-symmetric matrix with components

$$(v\wedge w)_{ij}\,=\,v^jw^i\,-\,v^iw^j\,.$$

Using classical L^q estimates for transport equations and taking advantage of Gronwall's lemma, from (4.41) we immediately get

$$\begin{aligned}\|\Omega(t)\|_{L^q}\;&\leq\;\exp\left(2\int_0^t\|\nabla u\|_{L^\infty}d\tau\right)\\ &\times\left(\|\Omega(0)\|_{L^q}\,+\,\int_0^t e^{-2\int_0^\tau\|\nabla u\|_{L^\infty}d\tau'}\left(\|F\|_{L^q}+\left\|\frac{1}{\rho^2}\nabla\rho\wedge\nabla\Pi\right\|_{L^q}\right)d\tau\right).\end{aligned}\tag{4.42}$$

Let us notice that, in the case of space dimension $N=2$, equation (4.41) becomes

$$\partial_t\omega\,+\,u\cdot\nabla\omega\,+\,\nabla\left(\frac{1}{\rho}\right)\wedge\nabla\Pi\,=\,F\,,$$

so that one obtains the same estimate as before, but without the exponential growth:

$$\|\omega(t)\|_{L^q}\,\leq\,\|\omega(0)\|_{L^q}\,+\,\int_0^t\left(\|F\|_{L^q}+\left\|\frac{1}{\rho^2}\nabla\rho\wedge\nabla\Pi\right\|_{L^q}\right)d\tau\,.$$

Therefore, the two-dimensional case is in a certain sense better. We shall take advantage of that in section 4.5. As concerns the results of this section, the proof will not depend on the dimension. So for the time being we assume that the dimension N is any integer greater than or equal to 2.

4.4.2 Proof of the continuation criterion

Now, we want to prove the continuation criterion for the solution to (4.1). As usual, we will suppose condition (1.1) to be satisfied with $p=+\infty$.

We proceed in two steps. The first one is given by the following lemma.

Lemma 4.12. *Let $(\rho,u,\nabla\Pi)$ be a solution of system (4.1) on $[0,T^*[\times\mathbb{R}^N$ such that*[2]

- $u\in\mathcal{C}([0,T^*[\,;B^s_{\infty,r})\cap\mathcal{C}^1([0,T^*[\,;L^2)$,
- $\rho\in\mathcal{C}([0,T^*[\,;B^s_{\infty,r})$ *and satisfies* (4.11).

Suppose also that condition (4.4) *holds and that T^* is finite. Then*

$$\sup_{t\in[0,T^*[}\left(\|u(t)\|_{B^s_{\infty,r}\cap L^2}\,+\,\|\rho(t)\|_{B^s_{\infty,r}}\right)\,<\,+\infty\,.$$

Proof of lemma 4.12. It is only a matter of repeating a priori estimates of the previous section, but in a more accurate way. Note that $a:=1/\rho$ satisfies the same hypothesis as ρ, so we will

[2] with the usual convention that continuity in time is weak if $r=+\infty$.

work without distinction with these two quantities, according to which is more convenient for us. Hence, set $q=\rho$ or a: recall that it satisfies

$$\partial_t q\,+\,u\cdot\nabla q\,=\,0\,.$$

Hence, applying operator Δ_j yields

$$\partial_t\Delta_j q+u\cdot\nabla\Delta_j q=[u\cdot\nabla,\Delta_j]q\,,$$

whence, for all $t\in[0,T^*[\,$,

$$2^{js}\|\Delta_j q(t)\|_{L^\infty}\leq 2^{js}\|\Delta_j q_0\|_{L^\infty}+\int_0^t 2^{js}\|[u\cdot\nabla,\Delta_j]q\|_{L^\infty}\,d\tau. \tag{4.43}$$

Now, lemma 2.100 in [2] ensures that

$$\left\|\left(2^{js}\|[u\cdot\nabla,\Delta_j]q\|_{L^\infty}\right)_j\right\|_{\ell^r}\leq C\left(\|\nabla u\|_{L^\infty}\|q\|_{B^s_{\infty,r}}+\|\nabla q\|_{L^\infty}\|\nabla u\|_{B^{s-1}_{\infty,r}}\right).$$

Hence, performing an ℓ^r summation in (4.43), we get

$$\|q(t)\|_{B^s_{\infty,r}}\;\leq\;\|q_0\|_{B^s_{\infty,r}}\;+\;C\int_0^t\left(\|\nabla u\|_{L^\infty}\|q\|_{B^s_{\infty,r}}+\|\nabla q\|_{L^\infty}\|u\|_{B^s_{\infty,r}}\right)d\tau. \tag{4.44}$$

As regards to the velocity field, according to (4.13) we have

$$\|u(t)\|_{L^2}\leq C\left(\|u_0\|_{L^2}\;+\;\int_0^t\|f\|_{L^2}\,d\tau\right)\,, \tag{4.45}$$

while, we use (4.40) with $p=2$ to bound its Besov norm. So, the problem is now to control the vorticity in $B^{s-1}_{\infty,r}$. From equation (4.41) and proposition 4.7 (recall that $\operatorname{div}u=0$), we readily get

$$\begin{aligned}\|\Omega(t)\|_{B^{s-1}_{\infty,r}}\;\leq\;&\exp\left(C\int_0^t\|\nabla u\|_{L^\infty}\,d\tau\right)\left(\|\Omega_0\|_{B^{s-1}_{\infty,r}}+\int_0^t\|F\|_{B^{s-1}_{\infty,r}}d\tau+\right.\\&\left.+\int_0^t\left(\|\nabla a\wedge\nabla\Pi\|_{B^{s-1}_{\infty,r}}+\|\Omega\cdot\nabla u+{}^t\nabla u\cdot\Omega\|_{B^{s-1}_{\infty,r}}\right)d\tau\right).\end{aligned} \tag{4.46}$$

The following inequalities hold true:

$$\|\nabla a\wedge\nabla\Pi\|_{B^{s-1}_{\infty,r}}\;\leq\;C\left(\|\nabla a\|_{L^\infty}\|\nabla\Pi\|_{B^{s-1}_{\infty,r}}+\|\nabla\Pi\|_{L^\infty}\|\nabla a\|_{B^{s-1}_{\infty,r}}\right), \tag{4.47}$$

$$\|\Omega\cdot\nabla u+{}^t\nabla u\cdot\Omega\|_{B^{s-1}_{\infty,r}}\;\leq\;C\|\nabla u\|_{L^\infty}\|\nabla u\|_{B^{s-1}_{\infty,r}}\,. \tag{4.48}$$

In the case $s>1$, they immediately come from theorem 1.24. We claim that they are still true in the limit case $s=r=1$: the proof relies on Bony's decomposition (1.3) and algebraic cancellations. Indeed, we observe that

$$\begin{aligned}\partial_i a\,\partial_j\Pi-\partial_j a\,\partial_i\Pi=\,&T_{\partial_i a}\partial_j\Pi-T_{\partial_j a}\partial_i\Pi+T_{\partial_j\Pi}\partial_i a-T_{\partial_i\Pi}\partial_j a+\\&+\partial_i R(a-\Delta_{-1}a,\partial_j\Pi)-\partial_j R(a-\Delta_{-1}a,\partial_i\Pi)+R(\partial_i\Delta_{-1}a,\partial_j\Pi)+R(\partial_j\Delta_{-1}a,\partial_i\Pi).\end{aligned}$$

Applying theorem 1.24 thus yields (4.47).

Next, we notice that, as $\operatorname{div}u=0$, then

$$\begin{aligned}\left(\Omega\cdot\nabla u+{}^t\nabla u\cdot\Omega\right)_{ij}\;&=\;\sum_k\left(\partial_i u^k\partial_k u^j-\partial_j u^k\partial_k u^i\right),\\&=\;\sum_k\Big(\partial_k\big(u^j\partial_i u^k\big)-\partial_k\big(u^i\partial_j u^k\big)\Big).\end{aligned}$$

Therefore,

$$\left(\Omega\cdot\nabla u+{}^t\nabla u\cdot\Omega\right)_{ij}=\sum_k\Big(T_{\partial_i u^k}\partial_k u^j-T_{\partial_j u^k}\partial_k u^i+T_{\partial_k u^j}\partial_i u^k-T_{\partial_k u^i}\partial_j u^k+$$
$$+\partial_k R(u^j-\Delta_{-1}u^j,\partial_i u^k)-\partial_k R(u^i-\Delta_{-1}u^i,\partial_j u^k)+R(\Delta_{-1}\partial_k u^j,\partial_i u^k)+R(\Delta_{-1}\partial_k u^i,\partial_j u^k)\Big).$$

Hence theorem 1.24 again implies (4.48).

Plugging (4.47) and (4.48) in (4.46), using the energy inequality (4.45) and inequality (4.40) with $p=2$, we easily get

$$\begin{aligned}\|u(t)\|_{B^s_{\infty,r}\cap L^2}\ \le\ & C\exp\Big(C\int\|\nabla u\|_{L^\infty}\,d\tau\Big)\Big(\|u_0\|_{B^s_{\infty,r}\cap L^2}+\int_0^t\|f\|_{B^s_{\infty,r}\cap L^2}\,d\tau\ +\\ &+\int_0^t\left(\|\nabla a\|_{L^\infty}\|\nabla\Pi\|_{B^{s-1}_{\infty,r}}+\|\nabla\Pi\|_{L^\infty}\|a\|_{B^s_{\infty,r}}\right)d\tau\Big).\end{aligned}$$

Hence, denoting $X(t):=\|a(t)\|_{B^s_{\infty,r}}+\|u(t)\|_{B^s_{\infty,r}\cap L^2}$, adding up inequality (4.44) and using Gronwall's inequality, we end up with

$$X(t)\le C\exp\Big(C\int_0^t\|(\nabla u,\nabla a,\nabla\Pi)\|_{L^\infty}\,d\tau\Big)\Big(X(0)+\int_0^t\big(\|f\|_{B^s_{\infty,r}\cap L^2}+\|\nabla a\|_{L^\infty}\|\nabla\Pi\|_{B^{s-1}_{\infty,r}}\big)\,d\tau\Big).$$

Now, the equation for ∇a and Gronwall's inequality immediately ensure that

$$\|\nabla a(t)\|_{L^\infty}\ \le\ \|\nabla a_0\|_{L^\infty}\ \exp\left(\int_0^t\|\nabla u\|_{L^\infty}\,d\tau\right),\tag{4.49}$$

which implies, thanks to hypothesis (4.4), that ∇a is bounded in time with values in L^∞. Moreover, by hypothesis $\nabla\Pi\in L^1([0,T^*[\,;B^{s-1}_{\infty,r})$ and $\nabla u\in L^1([0,T^*[\,;L^\infty)$. At this point, keeping in mind the embedding $B^{s-1}_{\infty,r}\hookrightarrow L^\infty$, previous inequality gives us the thesis of the lemma. □

The second result, which will enable us to complete the proof of theorem 4.2, reads in the following way.

Lemma 4.13. *Let $(\rho,u,\nabla\Pi)$ be the solution of system (4.1) such that*[3]

- $\rho\in\mathcal{C}([0,T^*[\,;B^s_{\infty,r})$ *and* (4.11)*;*
- $u\in\mathcal{C}([0,T^*[\,;B^s_{\infty,r})\cap\mathcal{C}^1([0,T^*[\,;L^2)$*;*
- $\nabla\Pi\in\mathcal{C}([0,T^*[\,;L^2)\cap L^1([0,T^*[\,;B^s_{\infty,r})$.

Moreover, suppose that

$$\|u\|_{L^\infty_{T^*}(B^s_{\infty,r}\cap L^2)}+\|\nabla a\|_{L^\infty_{T^*}(B^{s-1}_{\infty,r})}\ <\ +\infty\,.$$

Then $(\rho,u,\nabla\Pi)$ can be continued beyond the time T^* into a solution of (4.1) with the same regularity.

Proof of lemma 4.13. From the proof of theorem 4.1 we know that there exists a time ε, depending only on ρ^*, N, s, $\|u\|_{L^\infty_{T^*}(B^s_{\infty,r}\cap L^2)}$, $\|\nabla a\|_{L^\infty_{T^*}(B^{s-1}_{\infty,r})}$ and on the norm of the data, such that, for all $\widetilde{T}<T^*$, Euler system with data $(\rho(\widetilde{T}),u(\widetilde{T}),f(\widetilde{T}+\cdot))$ has a unique solution until time ε.

Now, taking for example $\widetilde{T}=T^*-\varepsilon/2$, we thus obtain a solution, which is (by uniqueness) the continuation of the initial one, $(\rho,u,\nabla\Pi)$, until time $T+\varepsilon/2$. □

[3] with the usual convention that continuity in time is weak if $r=+\infty$.

Let us complete the proof of theorem 4.2. The first part is a straightforward consequence of these two lemmas. Indeed: lemma 4.12 ensures that $\|u\|_{L^\infty_{T^*}(B^s_{\infty,r}\cap L^2)}$ and $\|\nabla a\|_{L^\infty_{T^*}(B^{s-1}_{\infty,r})}$ are finite. As for the last claim (the Beale-Kato-Majda type continuation criterion), it is a classical consequence of the well-known logarithmic interpolation inequality (see e.g. [2])

$$\|\nabla u\|_{L^\infty} \leq C\left(\|u\|_{L^2} + \|\Omega\|_{L^\infty}\log\left(e + \frac{\|\Omega\|_{B^{s-1}_{\infty,r}}}{\|\Omega\|_{L^\infty}}\right)\right).$$

4.4.3 Proof of theorem 4.4

We first prove a priori estimates, and then we will get from them existence and uniqueness of the solution. In fact, it will turn out to be possible to apply theorem 4.1 after performing a suitable cut-off on the initial velocity field and thus to work directly on system (4.1), without passing through the equivalence with (4.10) or with a sequence of approximate linear systems.

A priori estimates

As in the previous section, remembering also remark 4.10, the following estimates hold true:

$$\text{(4.50)}\qquad \|\nabla\rho(t)\|_{B^{s-1}_{\infty,r}} \leq \|\nabla\rho_0\|_{B^{s-1}_{\infty,r}}\exp\left(C\int_0^t\|u\|_{B^s_{\infty,r}}\,d\tau\right)$$

$$\text{(4.51)}\qquad \begin{aligned}\|u(t)\|_{B^s_{\infty,r}} \leq\ & \exp\left(C\int_0^t\|u\|_{B^s_{\infty,r}}\,d\tau\right)\times\\ & \times\left(\|u_0\|_{B^s_{\infty,r}} + \int_0^t e^{-C\int_0^\tau\|u\|_{B^s_{\infty,r}}\,d\tau'}\|\rho\|_{B^s_{\infty,r}}\|\nabla\Pi\|_{B^s_{\infty,r}}\,d\tau\right).\end{aligned}$$

Moreover, from the transport equation satisfied by the velocity field, we easily gather that

$$\|u(t)\|_{L^4} \leq \|u_0\|_{L^4} + \int_0^t\left\|\frac{\nabla\Pi}{\rho}\right\|_{L^4}\,d\tau.$$

Therefore, using interpolation in Lebesgue spaces and embedding (see proposition 1.6),

$$\text{(4.52)}\qquad \begin{aligned}\|u(t)\|_{L^4} &\leq \|u_0\|_{L^4} + \frac{1}{\rho_*}\int_0^t\|\nabla\Pi\|_{L^\infty}^{\frac12}\|\nabla\Pi\|_{L^2}^{\frac12}\,d\tau\\ &\leq \|u_0\|_{L^4} + \frac{C}{\rho_*}\int_0^t\|\nabla\Pi\|_{B^s_{\infty,r}\cap L^2}\,d\tau.\end{aligned}$$

In order to bound the vorticity in L^4, one may use the fact that

$$\begin{aligned}\left\|\frac{1}{\rho^2}\nabla\rho\wedge\nabla\Pi\right\|_{L^4} &\leq \frac{1}{(\rho_*)^2}\|\nabla\rho\wedge\nabla\Pi\|_{L^4} \leq \frac{1}{(\rho_*)^2}\|\nabla\rho\|_{L^\infty}\|\nabla\Pi\|_{L^4}\\ &\leq \frac{C}{(\rho_*)^2}\|\nabla\rho\|_{B^{s-1}_{\infty,r}}\|\nabla\Pi\|_{B^s_{\infty,r}\cap L^2}.\end{aligned}$$

From this and (4.42), we thus get

$$\text{(4.53)}\qquad \begin{aligned}\|\Omega(t)\|_{L^4} \leq\ & \exp\left(\int_0^t\|\nabla u\|_{B^{s-1}_{\infty,r}}d\tau\right)\left(\|\Omega_0\|_{L^4} +\right.\\ & \left. + \frac{C}{(\rho_*)^2}\int_0^t e^{-\int_0^\tau\|\nabla u\|_{B^{s-1}_{\infty,r}}d\tau'}\|\nabla\rho\|_{B^{s-1}_{\infty,r}}\|\nabla\Pi\|_{B^s_{\infty,r}\cap L^2}\,d\tau\right).\end{aligned}$$

Now, in order to close the estimates, we need to control the pressure term. Its Besov norm can be bounded as in section 4.3, up to a change of $\|u\|_{L^2}$ into $\|u\|_{L^4}$; indeed it is clear that in inequality (4.23) the L^2 norm of u may be replaced by any L^p norm with $p < +\infty$. As a consequence, combining the (modified) inequality (4.24) and (4.20) yields

$$(4.54)\quad \|\nabla\Pi\|_{L^1_t(B^s_{\infty,r})} \leq C\bigg(\Big(1+\|\nabla a\|^{\gamma}_{L^\infty_t(B^{s-1}_{\infty,r})}\Big)\|\nabla\Pi\|_{L^1_t(L^2)} + \|\rho\|_{L^\infty_t(B^s_{\infty,r})}\int_0^t \|u\|^2_{B^s_{\infty,r}\cap L^4} d\tau\bigg).$$

In order to bound the L^2 norm of $\nabla\Pi$, we take the divergence of the second equation of system (4.1). We obtain

$$-\operatorname{div}\left(\frac{\nabla\Pi}{\rho}\right) = \operatorname{div}\,(u\cdot\nabla u)\ ,$$

from which, applying elliptic estimates of lemma 4.8 and

$$(4.55)\qquad \|\nabla u\|_{L^4} \leq C\,\|\Omega\|_{L^4}\ ,$$

we get

$$(4.56)\qquad \frac{1}{\rho^*}\,\|\nabla\Pi\|_{L^2}\ \leq\ \|u\cdot\nabla u\|_{L^2}\ \leq\ \|u\|_{L^4}\,\|\nabla u\|_{L^4}\ \leq\ C\|u\|_{L^4}\,\|\Omega\|_{L^4}\,.$$

Putting together inequalities (4.50), (4.51), (4.52), (4.55), (4.54), (4.53) and (4.56) enables us to close the estimates on some nontrivial time interval $[0,T]$ depending only on the norm of the data. In effect, assuming that T has been chosen so that Inequality (4.27) is satisfied, we get from the above inequalities

$$\begin{aligned}
\|u(t)\|_{B^s_{\infty,r}} &\leq 2\|u_0\|_{B^s_{\infty,r}} + C_0\|\nabla\Pi\|_{L^1_t(B^s_{\infty,r})},\\
\|\nabla\Pi\|_{L^1_t(B^s_{\infty,r})} &\leq C_0\int_0^t \big(\|u\|_{L^4}\|\Omega\|_{L^4} + \|u\|^2_{B^s_{\infty,r}\cap L^4}\big)\,d\tau,\\
\|u(t)\|_{L^4} &\leq \|u_0\|_{L^4} + C_0\|\nabla\Pi\|_{L^1_t(B^s_{\infty,r})} + C_0\int_0^t \|u\|_{L^4}\|\Omega\|_{L^4}\,d\tau,\\
\|\Omega(t)\|_{L^4} &\leq 2\|\Omega_0\|_{L^4} + C_0\|\nabla\Pi\|_{L^1_t(B^s_{\infty,r})},
\end{aligned}$$

where the constant C_0 depends on s, ρ_*, ρ^*, N and $\|\rho_0\|_{B^s_{\infty,r}}$.

Therefore, applying Gronwall Lemma and assuming that T has been chosen so that (in addition to (4.27)) we have

$$\int_0^T \|u\|_{W^{1,4}}\,d\tau \leq c$$

where c is a small enough constant depending only on C_0, it is easy to close the estimates.

Remark 4.14. Exhibiting an L^2 estimate for $\nabla\Pi$ *even though u is not in L^2* is the key to the proof. This has been obtained in (4.56). Note however that we have some freedom there: one may rather assume that $u_0 \in L^p$ and $\nabla u_0 \in L^q$, with p and q in $[2,+\infty]$ such that $1/p + 1/q \geq 1/2$, and get a statement similar to that of theorem 4.4 under these two assumptions. The details are left to the reader.

Existence of a solution

We want to take advantage of the existence theory provided by theorem 4.1. However, under the assumptions of theorem 4.4, the initial velocity does not belong to L^2. To overcome this, we shall introduce a sequence of truncated initial velocities. Then theorem 4.1 will enable us to solve system (4.1) with these modified data and the previous part will provide uniform estimates in the right functional spaces on a small enough (fixed) time interval. Finally, convergence will be proved by an energy method similar to that we used for theorem 4.1.

First step: construction of the sequence of approximate solutions

Take any $\Phi \in C_0^\infty(\mathbb{R}^N_x)$ with $\Phi \equiv 1$ on a neighborhood of the origin, and set $\Phi_n(x) = \Phi(x/n)$. Then let us define $u_0^n := \Phi_n u_0$ for all $n \in \mathbb{N}$. Note that this ensures that $u_0^n \to u_0$ in the distribution meaning.

Given that u_0^n is continuous and compactly supported, it obviously belongs to L^2. Of course, we still have $u_0^n \in B^s_{\infty,r} \cap W^{1,4} \cap L^2$, so we fall back into hypothesis of theorem 4.1. From it, we get the existence of some time T_n and of a solution $(\rho^n, u^n, \nabla\Pi^n)$ to (4.1) with data (ρ_0, u_0^n) such that $\rho^n \in \mathcal{C}([0,T_n]; B^s_{\infty,r})$, $u^n \in \mathcal{C}^1([0,T_n]; L^2) \cap \mathcal{C}([0,T_n]; B^s_{\infty,r})$ and $\nabla\Pi^n \in \mathcal{C}([0,T_n]; L^2) \cap L^1([0,T_n]; B^s_{\infty,r})$. From (4.55), the vorticity equation and the velocity equation, it is easy to see that, in addition, $u^n \in \mathcal{C}([0,T_n]; W^{1,4})$.

Finally, as the norm of u_0^n in $W^{1,4} \cap B^s_{\infty,r}$ may be bounded independently of n, the a priori estimates that have been performed in the previous paragraph ensure that one may find some positive lower bound T for T_n such that $(\rho^n, u^n, \nabla\Pi^n)$ satisfies estimates independent of n on $[0,T]$ in the desired functional spaces.

Second step: convergence of the sequence

As done in the previous section, we define $\widetilde{\rho}^n = \rho^n - \rho_0$, and then

$$\begin{aligned} \delta\rho^n &:= \widetilde{\rho}^{n+1} - \widetilde{\rho}^n \,, \\ \delta u^n &:= u^{n+1} - u^n \,, \\ \delta\Pi^n &:= \Pi^{n+1} - \Pi^n \,. \end{aligned}$$

Resorting to the same type of computations as in the previous section (it is actually easier as, now, $\operatorname{div} u^n = 0$ for all n), we can prove that $(\widetilde{\rho}^n, u^n, \nabla\Pi^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0,T]; L^2)$. Hence it converges to some $(\widetilde{\rho}, u, \nabla\Pi)$ which belongs to the same space.

Now, defining $\rho := \rho_0 + \widetilde{\rho}$, bearing in mind the uniform estimates of the previous step and using the Fatou property, we easily conclude that

- $\rho \in L^\infty([0,T]; B^s_{\infty,r})$ and $\rho_* \le \rho \le \rho^*$;
- $u \in L^\infty([0,T]; B^s_{\infty,r}) \cap L^\infty([0,T]; W^{1,4})$;
- $\nabla\Pi \in L^1([0,T]; B^s_{\infty,r}) \cap L^\infty([0,T]; L^2)$.

Finally, by interpolation we can pass to the limit in the equations at step n, so we get that $(\rho, u, \nabla\Pi)$ satisfies (4.1), while continuity in time follows from proposition 4.7. □

4.5 Remarks on the lifespan of the solution

In this section, we exhibit lower bounds for the lifespan of the solution to the density-dependent incompressible Euler equations. We first establish that, like in the homogeneous case, in any dimension, if the initial velocity is of order ε then the lifespan is at least of order ε^{-1} *even in the fully nonhomogeneous case.* Next we focus on the two-dimensional case: in the second part of this section, we show that for small perturbations of a constant density state, the lifespan tends to be very large. Therefore, for nonhomogeneous incompressible fluids too, the two-dimensional case is somewhat nicer than the general situation.

4.5.1 The general case

Let ρ_0, u_0 and f satisfy the assumptions of theorem 4.1 or 4.4. Denote

$$\widetilde{u}_0(x) := \varepsilon\, u_0(x) \qquad \text{and} \qquad \widetilde{f}(t,x) := \varepsilon^2 f(\varepsilon\, t, x)\,.$$

It is clear that if we set

$$\left(\widetilde{\rho}\,,\,\widetilde{u}\,,\,\nabla\widetilde{\Pi}\right)(t,x)\;:=\;\left(\rho\,,\,\varepsilon\,u\,,\,\varepsilon^2\,\nabla\Pi\right)(\varepsilon\,t\,,\,x)\;,$$

then $(\widetilde{\rho},\widetilde{u},\nabla\widetilde{\Pi})$ is a solution to (4.1) on $[\varepsilon^{-1}\,T_*,\varepsilon^{-1}\,T^*]$ with data $(\rho_0,\widetilde{u}_0,\widetilde{f})$ if and only if $(\rho,u,\nabla\Pi)$ is a solution to (4.1) on $[T_*,T^*]$ with data (ρ_0,u_0,f).

Hence, putting together the results of the previous section, we can conclude to the following statement.

Theorem 4.15. *Let (ρ_0,u_0) satisfy the assumptions of theorem 4.1 or 4.4, and $f\equiv 0$. There exists a positive time T^* depending only on s, N, ρ_*, $\|\rho_0\|_{B^1_{\infty,1}}$ and $\|u_0\|_{B^1_{\infty,1}}$ such that, for any $\varepsilon>0$, the upper bound T^*_ε of the maximal interval of existence for the solution to* (4.1) *with initial data $(\rho_0,\varepsilon\,u_0)$ satisfies*

$$T^*_\varepsilon\;\geq\;\varepsilon^{-1}\,T^*\,.$$

A similar result holds for the lower bound of the maximal interval of existence.

4.5.2 The two-dimensional case

Recall that for the homogeneous equations, any solution corresponding to suitably smooth data is global, a fact which relies on the conservation of the vorticity by the flow. Now, in our case, the vorticity equation reads (if $f\equiv 0$)

$$\partial_t\omega+u\cdot\nabla\omega+\nabla a\wedge\nabla\Pi=0\qquad\text{with}\qquad\nabla a\wedge\nabla\Pi:=\partial_1 a\,\partial_2\Pi-\partial_2 a\,\partial_1\Pi.\tag{4.57}$$

Owing to the new term involving the pressure and the nonhomogeneity, it is not clear at all that global existence still holds. Nevertheless, we are going to prove that the lifespan may be very large if the nonhomogeneity is small.

To simplify the presentation, we focus on the case where $\rho_0\in B^1_{\infty,1}(\mathbb{R}^2)$ and $u_0\in B^1_{\infty,1}(\mathbb{R}^2)$ (note that corollary 4.3 ensures that this is not restrictive) and assume, in addition, that $u_0\in L^2(\mathbb{R}^2)$ (this lower order assumption may be somewhat relaxed too). We aim at proving the following result.

Theorem 4.16. *Under the above assumptions, there exists a constant c, depending only on ρ_* and ρ^*, such that the lifespan of the solution to the two-dimensional density-dependent incompressible Euler equations with initial data (ρ_0,u_0) and no source term is bounded from below by*

$$\frac{c}{\|u_0\|_{L^2\cap B^1_{\infty,1}}}\,\log\!\left(1\,+\,c\,\log\frac{1}{\|\nabla a_0\|_{B^0_{\infty,1}}}\right).$$

Proof. Let $]T_*,T^*[$ denote the maximal interval of existence of the solution $(\rho,u,\nabla\Pi)$ corresponding to (ρ_0,u_0). To simplify the presentation, we focus on the evolution for *positive* times.

The key to the proof relies on the fact that in the two-dimensional case, the vorticity equation satisfies (4.57). Now, it turns out that, as discovered by M. Vishik in [60] and by T. Hmidi and S. Keraani in [41], the norms in Besov spaces *with null regularity index* of solutions to transport equations satisfy better estimates, namely in our case

$$\|\omega(t)\|_{B^0_{\infty,1}}\leq C\biggl(\|\omega_0\|_{B^0_{\infty,1}}+\int_0^t\|\nabla a\wedge\nabla\Pi\|_{B^0_{\infty,1}}\,d\tau\biggr)\biggl(1+\int_0^t\|\nabla u\|_{L^\infty}\,d\tau\biggr)$$

whereas, according to proposition 4.7, the last term has to be replaced with $\exp\Bigl(\int_0^t\|\nabla u\|_{L^\infty}\,d\tau\Bigr)$ for nonzero regularity exponents.

Therefore, using inequality (4.47), we get

$$\|\omega(t)\|_{B^0_{\infty,1}} \leq C\bigg(\|\omega_0\|_{B^0_{\infty,1}} + \int_0^t \|\nabla a\|_{B^0_{\infty,1}} \|\nabla\Pi\|_{B^0_{\infty,1}}\, d\tau\bigg)\bigg(1 + \int_0^t \|\nabla u\|_{L^\infty}\, d\tau\bigg). \tag{4.58}$$

Bearing in mind inequality (4.40) and the energy inequality for u, we thus get

$$X(t) \leq C\bigg(X_0 + \int_0^t A(\tau)\, \|\nabla\Pi(\tau)\|_{B^0_{\infty,1}}\, d\tau\bigg)\bigg(1 + \int_0^t X\, d\tau\bigg), \tag{4.59}$$

where we have defined

$$X(t) := \|u(t)\|_{L^2\cap B^1_{\infty,1}} \qquad \text{and} \qquad A(t) := \|\nabla a(t)\|_{B^0_{\infty,1}}.$$

Bounding A is easy, given that

$$\partial_t\partial_i a + u\cdot\nabla\partial_i a = -\partial_i u\cdot\nabla a \qquad \text{for} \quad i = 1, 2.$$

Indeed, combining inequality (4.6) and paraproduct estimates ensures that

$$\|\nabla a(t)\|_{B^0_{\infty,1}} \leq \|\nabla a_0\|_{B^0_{\infty,1}} \exp\bigg(C\int_0^t \|\nabla u\|_{B^0_{\infty,1}}\, d\tau\bigg).$$

Therefore,

$$A(t) \leq A_0 \exp\bigg(C\int_0^t X\, d\tau\bigg). \tag{4.60}$$

Bounding the pressure term in $B^0_{\infty,1}$ is our next task. In fact, we shall rather bound its $B^1_{\infty,1}$ norm[4]. Recall that, according to inequality (4.20), there exists some exponent $\gamma \geq 1$ so that

$$\|\nabla\Pi\|_{B^1_{\infty,1}} \;\leq\; C\,\bigg(\bigg(1 + \|\nabla a\|^\gamma_{B^0_{\infty,1}}\bigg)\,\|\nabla\Pi\|_{L^2} \;+\; \|\rho\,\mathrm{div}\,(u\cdot\nabla u)\|_{B^0_{\infty,1}}\bigg). \tag{4.61}$$

Combining Bony's decomposition with the fact that $\mathrm{div}\,(u\cdot\nabla u) = \nabla u : \nabla u$, we get

$$\|\mathrm{div}\,(u\cdot\nabla u)\|_{B^0_{\infty,1}} \leq \|u\|^2_{B^1_{\infty,1}}.$$

From the definition of $B^1_{\infty,1}$ and proposition 1.31, it is also clear that

$$\|\rho\|_{B^1_{\infty,1}} \;\leq\; C\,\Big(\rho^* \;+\; \|\nabla a\|_{B^0_{\infty,1}}\Big).$$

Finally, given that

$$-\,\mathrm{div}\;(a\,\nabla\Pi) \;=\; \mathrm{div}\;(u\cdot\nabla u)\;,$$

lemma 4.8 guarantees that

$$a_*\|\nabla\Pi\|_{L^2} \;\leq\; \|u\|_{L^2}\,\|\nabla u\|_{L^\infty}. \tag{4.62}$$

So plugging the above inequalities in (4.61), one may conclude that

$$\|\nabla\Pi\|_{B^1_{\infty,1}} \;\leq\; C\;(1 + A^\gamma)\,X^2, \tag{4.63}$$

for some constant C depending only on a_* and a^*.

[4]We do not know how to take advantage of the fact that only the $B^0_{\infty,1}$ norm is needed.

It is now time to insert inequalities (4.60) and (4.63) in (4.59); setting $\beta = \gamma + 1$, we get

$$X(t) \leq C\left(X_0 + \left(A_0 + A_0^\beta\right)\int_0^t e^{C\int_0^\tau X\,d\tau'}\,X^2\,d\tau\right)\left(1 + \int_0^t X\,d\tau\right). \tag{4.64}$$

Let T_0 denote the supremum of times $t \in [0, T^*[$ so that

$$\left(A_0 + A_0^\beta\right)\int_0^t e^{C\int_0^\tau X\,d\tau'}\,X^2\,d\tau \leq X_0\,. \tag{4.65}$$

From (4.64) and Gronwall's Lemma, we gather that

$$X(t) \leq 2\,C\,X_0\,e^{2\,C\,t\,X_0} \qquad \text{for all} \quad t \in [0, T_0[\,.$$

Note that this inequality implies that for all $t \in [0, T_0[\,$, we have

$$\int_0^t e^{C\int_0^\tau X\,d\tau'}\,X^2\,d\tau \leq C\,X_0\left(e^{4\,C\,t\,X_0} - 1\right)\,\exp\!\left(C\left(e^{2\,C\,t\,X_0} - 1\right)\right).$$

Therefore, using (4.65) and a bootstrap argument (based on the continuation theorems that we proved in the previous sections), it is easy to show that T_0 is greater than any time t such that

$$\left(A_0 + A_0^\beta\right)\left(e^{4\,C\,t\,X_0} - 1\right)\,\exp\!\left(C\left(e^{2\,C\,t\,X_0} - 1\right)\right) \leq 1\,.$$

Taking the logarithm and using that $\log y \leq y - 1$ for $y > 0$, we see that the above inequality is satisfied whenever

$$e^{2\,C\,t\,X_0} \leq 1 + \frac{1}{2C}\,\log\!\left(\frac{1}{A_0 + A_0^\beta}\right).$$

This completes the proof of the lower bound for T^*. $\square$

Remark 4.17. If ω_0 has more regularity (say $\omega_0 \in \mathcal{C}^r$ for some $r \in\,]0,1[\,$), then one may first write an estimate for $\|\omega\|_{L^\infty}$ and next use the classical logarithmic inequality for bounding $\|\nabla u\|_{L^\infty}$ in terms of $\|\omega\|_{L^\infty}$ and $\|\omega\|_{\mathcal{C}^r}$. The proof is longer, requires more regularity and, at the same time, the lower bound for the lifespan does not improve.

Chapter 5

Propagation of geometric structures

In this chapter we obtain a result about propagation of geometric properties for solutions of non-homogeneous incompressible Euler system in any dimension $N \geq 2$. In particular, we investigate conservation of striated and conormal regularity, which is a natural way of generalising the 2-D structure of vortex patches. The results we get are only local in time, even in the dimension $N = 2$: in contrast with the homogeneous case, the global existence issue is still an open problem, because the vorticity is not preserved during the time evolution. Moreover we will be able to give an explicit lower bound for the lifespan of the solution, in terms of the norms of initial data only. In the case of physical dimension $N = 2$ or 3, we will investigate also propagation of Hölder regularity in the interior of a bounded domain.

5.1 Introduction

In this chapter we are interested in studying conservation of geometric properties for solutions of the density-dependent incompressible Euler system

$$\tag{5.1} \begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 \\ \rho \left(\partial_t u + u \cdot \nabla u \right) + \nabla \Pi = 0 \\ \operatorname{div} u = 0 \,, \end{cases}$$

which describes the evolution of a non-homogeneous inviscid fluid with no body force acting on it, an assumption we will make throughout all this chapter to simplify the presentation. Here, $\rho(t,x) \in \mathbb{R}_+$ represents the density of the fluid, $u(t,x) \in \mathbb{R}^N$ its velocity field and $\Pi(t,x) \in \mathbb{R}$ its pressure. The term $\nabla\Pi$ can be also seen as the Lagrangian multiplier associated to the divergence-free constraint over the velocity.

We will always suppose that the variable x belongs to the whole space $\mathbb{R}^N$.

The problem of preserving geometric structures came out already in the homogeneous case, for which $\rho \equiv 1$ and system (5.1) becomes

$$\tag{E} \begin{cases} \partial_t u + u \cdot \nabla u + \nabla \Pi = 0 \\ \operatorname{div} u = 0 \,, \end{cases}$$

in studying 2-dimensional vortex patches, that is to say the initial vorticity Ω_0 is the characteristic function of a bounded domain D_0. As we will explain below, in the case of higher dimension $N \geq 3$ this notion was generalized by the properties of striated and conormal regularity.

We recall here that the vorticity of the fluid is defined as the skew-symmetric matrix

$$\tag{5.2} \Omega := \nabla u - {}^t\nabla u$$

and in the homogenous case it satisfies the equation

$$\partial_t\Omega + u\cdot\nabla\Omega + \Omega\cdot\nabla u + {}^t\nabla u\cdot\Omega = 0\,.$$

In dimension $N=2$ it can be identified with the scalar function $\omega = \partial_1 u^2 - \partial_2 u^1$, while for $N=3$ with the vector-field $\omega = \nabla\times u$. Let us recall also that in the bidimensional case this quantity is transported by the velocity field: it fulfills

$$\partial_t\omega + u\cdot\nabla\omega = 0\,.$$

The notion of vortex patches was introduced in [47] and gained new interest after the survey paper [49] of Majda. In the case $N=2$ Yudovich's theorem ensures the existence of a unique global solution of the homogeneous Euler system, which preserves the geometric structure: the vorticity remains the characteristic function of the evolution (by the flow associated to this solution) of the domain D_0. Vortex patches in bounded domains of $\mathbb{R}^2$ were also studied by Depauw (see [32]), while Dutrifoy in [33] focused on the case of domains in $\mathbb{R}^3$. Moreover, in [11] Chemin proved that, if the initial domain has boundary ∂D_0 of class $\mathcal{C}^{1+\varepsilon}$ for some $\varepsilon>0$, then this regularity is preserved during the evolution for small times; in [12] he also showed a global in time persistence issue. In [24] Danchin considered instead the case in which initial data of the Euler system are vortex patches with singular boundary: he proved that if ∂D_0 is regular apart from a closed subset, then it remains regular for all times, apart from the closed subset transported by the flow associated to the solution.

In the case $N\geq 3$ one can't expect to have global results anymore, nor to preserve the initial vortex patch structure, because of the presence of the stretching term in the vorticity equation. Nevertheless, it's possible to introduce the definition of *striated regularity*, which generalizes in a quite natural way the previous one of vortex patch: it means that the vorticity is more regular along some fixed directions, given by a nondegenerate family of vector-fields (see definition 5.1 below). This notion was introduced first by Bony in [7] in studying hyperbolic equations, and then adapted by Alinhac (see [1]) and Chemin (see [10]) for nonlinear partial differential equations.

In [36], Gamblin and Saint-Raymond proved that striated regularity is preserved during the evolution in any dimension $N\geq 3$, but, as already remarked, only locally in time (see also [55]). They also obtained global results if initial data have other nice properties (e.g., if the initial velocity is axisymmetric).

As Euler system is, in a certain sense, a limit case of the Navier-Stokes system as the viscosity of the fluid goes to 0, it's interesting to study if there is also "convergence" of the geometric properties of the solutions. Recently Danchin proved results on striated regularity for the solutions of the Navier-Stokes system

$$(NS_\nu)\qquad \begin{cases} \partial_t u + u\cdot\nabla u - \nu\,\Delta u + \nabla\Pi = 0 \\ \operatorname{div} u = 0\,, \end{cases}$$

in [25] for the 2-dimensional case, in [26] for the general one. Already in the former paper, he had to dismiss the vortex patch structure "stricto sensu" due to the presence of the viscous term, which comes out also in the vorticity equation and has a smoothing effect; however, he still got global in time results. Moreover, in both his works he had to handle with spaces of type $B^{1+\varepsilon}_{p,\infty}$ (with $p\in\,]2,+\infty[$ and $\varepsilon\in\,]2/p,1[$) due to technical reasons which come out with a viscous fluid. Let us immediately clarify that these problems have been recently solved by Hmidi in [39] (see also [2]), and this fact allows us to consider again the Hölder spaces framework. In the above mentioned works Danchin proved also a priori estimates for solutions of (NS_ν) independent of the viscosity ν, therefore preservation of the geometric structures in passing from solutions of (NS_ν) to solutions of (E) in the limit $\nu\to 0$.

In this chapter we will come back to the inviscid case and we will study the non-homogeneous incompressible Euler system (5.1). We want to investigate if preservation of geometric properties of initial data, such as striated and conormal regularity, still holds in this setting, as in the classical (homogeneous) one. Let us note that in the 2-dimensional case the equation for the vorticity reads

$$\partial_t\omega \,+\, u\cdot\nabla\omega \,+\, \nabla\left(\frac{1}{\rho}\right)\wedge\nabla\Pi \,=\, 0\,,$$

so it's not better than in higher dimension due to the presence of the density term, which doesn't allow us to get conservation of Lebesgue norms. This is also the reason why it's not clear if Yudovich's theorem still holds true for non-homogeneous fluids: having $\omega_0\in L^q\cap L^\infty$, combined with suitable hypothesis on ρ_0, doesn't give rise to a local solution.
So, we will immediately focus on the general case $N\geq 2$. We will assume the initial velocity u_0 and the initial vorticity Ω_0 to be in some Lebesgue spaces, in order to assure the pressure term to belong to L^2, a requirement we could not bypass. As a matter of fact, $\nabla\Pi$ satisfies an elliptic equation with low regularity coefficient,

$$-\,\mathrm{div}\,\left(a\,\nabla\Pi\right)\,=\,\mathrm{div}\,F\,,$$

and it can be solved independently of a only in the energy space L^2. Moreover, we will suppose Ω_0 to have regularity properties of geometric type. Obviously, we will require some natural but quite general hypothesis also on the initial density ρ_0 of the fluid: we suppose ρ_0 to be strictly positive and bounded with its gradient and that it satisfies geometric assumptions analogous to those for Ω_0. Let us point out that proving the velocity field to be Lipschitz, which was the key part in the homogeneous case, works as in this setting: it relies only on Biot-Savart law and it requires no further hypothesis on the density term. Let us also remark that no smallness conditions over the density are needed. Of course, we will get only local in time results. Moreover, we will see that geometric structures propagate also to the velocity field and to the pressure term.

The present chapter is organized in the following way.

In the first part, we will recall basic facts about Euler system: some properties of the vorticity and how to associate a flow to the velocity field. In this section we will also give the definition of the geometric properties we are studying and we will state the main results we got about striated and conormal regularity.

The mathematical techniques we need to prove our claims are mostly those introduced in chapter 1, even if in the particular case of spaces $\mathcal{C}^s\,\equiv\,B^s_{\infty,\infty}$. Hence, we don't recall them. Nevertheless, we need to introduce the notion of paravector-field, as defined in [26]: it will play a fundamental role in our analysis, because it is, in a certain sense, the principal part of the derivation operator along a fixed vector-field. Moreover, we need also to analyse right composition of a $\mathcal{C}^s$ function with a smooth one. Section 5.3 is devoted to the presentation of these additional tools.

This having been done, we will finally be able to tackle the proof of our result about striated regularity. First of all, we will state a priori estimates for suitable smooth solutions of the Euler system (5.1). Then from them we will get, in a quite classical way, the existence of a solution with the required properties: we will construct a sequence of regular solutions of system (5.1) with approximated data, and, using a compactness argument, we will show the convergence of this sequence to a "real" solution. Proving preservation of the geometric structure requires instead strong convergence in rough spaces of type $\mathcal{C}^{-\alpha}$ (for some $\alpha>0$). Uniqueness of the solution will follow from a stability result for our equations. In the following section, we will also give an estimate from below for the lifespan of the solution.

Finally, we will spend a few words about conormal regularity: proving its propagation from the previous result is standard and can be done as in the homogenous setting. As a consequence, inspired by what done in Huang's paper [44], in the physical case of space dimension $N=2$ or 3 we can improve our result: we will also show that, if the initial data are Hölder continuous in the interior of a suitably smooth bounded domain, the solution conserves this property during the time evolution, i.e. it is still Hölder continuous in the interior of the domain transported by the flow.

5.2 Basic definitions and main results

Let $(\rho, u, \nabla\Pi)$ be a solution of the density-dependent incompressible Euler system (5.1) over $[0,T]\times\mathbb{R}^N$ and let us denote the vorticity of the fluid by Ω. As in the homogeneous case, it will play a fundamental role throughout all this chapter, so let us spend a few words about it.

From the definition (5.2), it is obvious that, for all $q\in[1,+\infty]$, if $\nabla u\in L^q$, then also $\Omega\in L^q$. Conversely, if u is a divergence-free vector-field, then for all $1\leq i\leq N$ we have $\Delta u^i = \sum_{j=1}^N \partial_j\Omega_{ij}$, and so, formally,

$$u^i \,=\, -\,(-\Delta)^{-1}\sum_{j=1}^{N}\partial_j\Omega_{ij}\,. \tag{5.3}$$

This is the Biot-Savart law, and it says that a divergence free vector-field u is completely determined by its vorticity. From (5.3) we immediately get

$$\nabla u^i \,=\, -\nabla\,(-\Delta)^{-1}\sum_{j=1}^{N}\partial_j\Omega_{ij}\,. \tag{5.4}$$

Now, as the symbol of the operator $-\partial_i\,(-\Delta)^{-1}\,\partial_j$ is $\sigma(\xi)\,=\,\xi_i\xi_j/|\xi|^2$, the classical Calderon-Zygmund theorem ensures that[1] for all $q\in]1,+\infty[$, if $\Omega\in L^q$ then $\nabla u\in L^q$ and

$$\|\nabla u\|_{L^q}\leq C\,\frac{q^2}{q-1}\,\|\Omega\|_{L^q}\,. \tag{5.5}$$

In dimension $N=2$ the vorticity equation is simpler than in the general case due to the absence of the stretching term. Nevertheless, as remarked above, the exterior product involving density and pressure terms makes it impossible to get conservation of Lebesgue norms, which was the basic point to get global existence for the classical system (E). So, we immediately focus on the case $N\geq 2$ whatever, in which the vorticity equation reads

$$\partial_t\Omega\,+\,u\cdot\nabla\Omega\,+\,\Omega\cdot\nabla u\,+\,{}^t\nabla u\cdot\Omega\,+\,\nabla\left(\frac{1}{\rho}\right)\wedge\nabla\Pi\,=\,0\,, \tag{5.6}$$

where, for two vector-fields v and w, we have set $v\wedge w$ to be the skew-symmetric matrix with components

$$(v\wedge w)_{ij}\,=\,v^jw^i\,-\,v^iw^j\,.$$

Finally, recall that we can associate a flow ψ to the velocity field u of the fluid: it is defined by the relation

$$\psi(t,x)\,\equiv\,\psi_t(x)\,:=\,x\,+\,\int_0^t u(\tau,\psi_\tau(x))\,d\tau$$

for all $(t,x)\in[0,T]\times\mathbb{R}^N$ and it is, for all fixed $t\in[0,T]$, a diffeomorphism over $\mathbb{R}^N$, if $\nabla u\in L^\infty$. Let us remark that the flow is still well-defined (in a generalized sense) even if u is only log-Lipschitz continuous, but it is no more a diffeomorphism (see e.g. chapter 3 of [2], or [13], for more details).

Let us now introduce the geometric properties we are handling throughout this chapter. The first notion we are interested in is the *striated regularity*, that is to say initial the data are more regular along some given directions.

So, let us take a family $X=(X_\lambda)_{1\leq\lambda\leq m}$ of m vector-fields with components and divergence of class $\mathcal{C}^\varepsilon$ for some fixed $\varepsilon\in]0,1[$. We also suppose this family to be non-degenerate, i.e.

$$I(X)\,:=\,\inf_{x\in\mathbb{R}^N}\;\sup_{\Lambda\in\Lambda^m_{N-1}}\left|\overset{N-1}{\wedge}\,X_\Lambda(x)\right|^{\frac{1}{N-1}}\;>\,0\,.$$

[1] This time the extreme values of q are not included.

Here $\Lambda \in \Lambda^m_{N-1}$ means that $\Lambda = (\lambda_1, \ldots, \lambda_{N-1})$, with each $\lambda_i \in \{1, \ldots, m\}$ and $\lambda_i < \lambda_j$ for $i < j$, while the symbol $\overset{N-1}{\wedge} X_\Lambda$ stands for the element of $\mathbb{R}^N$ such that

$$\forall\, Y \in \mathbb{R}^N\,, \qquad \left(\overset{N-1}{\wedge} X_\Lambda\right) \cdot Y \;=\; \det\left(X_{\lambda_1} \ldots X_{\lambda_{N-1}}, Y\right)\,.$$

For each vector-field of this family we put

$$\widetilde{\|} X_\lambda \|_{\mathcal{C}^\varepsilon} \;:=\; \|X_\lambda\|_{\mathcal{C}^\varepsilon} \,+\, \|\mathrm{div}\, X_\lambda\|_{\mathcal{C}^\varepsilon}\,,$$

while we will use the symbol $|||\cdot|||$ in considering the supremum over all indices $\lambda \in \Lambda^m_1 = \{1 \ldots m\}$.

Definition 5.1. Take a vector-field Y with components and divergence in $\mathcal{C}^\varepsilon$ and fix a $\eta \in [\varepsilon, 1+\varepsilon]$. A function $f \in L^\infty$ is said to be of class $\mathcal{C}^\eta$ along Y, and we write $f \in \mathcal{C}^\eta_Y$, if $\mathrm{div}\,(f\,Y) \in \mathcal{C}^{\eta-1}\left(\mathbb{R}^N\right)$.

If $X = (X_\lambda)_{1\leq\lambda\leq m}$ is a non-degenerate family of vector-fields as above, we define

$$\mathcal{C}^\eta_X \;:=\; \bigcap_{1\leq\lambda\leq m} \mathcal{C}^\eta_{X_\lambda} \qquad \text{and} \qquad \|f\|_{\mathcal{C}^\eta_X} \;:=\; \frac{1}{I(X)}\left(\|f\|_{L^\infty}\, \widetilde{|||} X |||_{\mathcal{C}^\varepsilon} \,+\, |||\,\mathrm{div}\,(f\,X)\,|||_{\mathcal{C}^{\eta-1}}\right)\,.$$

Remark 5.2. Our aim is to investigate Hölder regularity of the derivation of f along the fixed vector-field (say) Y, i.e. the quantity

$$\partial_Y f \;:=\; \sum_{i=1}^N Y^i\,\partial_i f\,.$$

If f is only bounded, however, this expression has no meaning: this is why we decided to focus on $\mathrm{div}\,(f\,Y)$, as done in the literature about this topic (see also [26], section 1). Lemma 5.14 below will clarify the relation between these two quantities.

Now, let us take a vector-field X_0 and define its time evolution $X(t)$:

$$X(t,x) \;\equiv\; X_t(x) \;:=\; \partial_{X_0(x)}\psi_t\left(\psi_t^{-1}(x)\right)\,, \tag{5.7}$$

that is $X(t)$ is the vector-field X_0 transported by the flow associated to u. From this definition, it immediately follows that $[\partial_X\,,\,\partial_t + u\cdot\nabla] = 0$, i.e. $X(t)$ satisfies the following system:

$$\begin{cases} (\partial_t \,+\, u\cdot\nabla)\,X \;=\; \partial_X u \\ X_{|t=0} \;=\; X_0\,. \end{cases} \tag{5.8}$$

We are now ready for stating our first result, on striated regularity.

Theorem 5.3. *Fix $\varepsilon \in]0,1[$ and take a non-degenerate family of vector-fields $X_0 = (X_{0,\lambda})_{1\leq\lambda\leq m}$ over $\mathbb{R}^N$, whose components and divergence are in $\mathcal{C}^\varepsilon$.*
Let the initial velocity field $u_0 \in L^p$, with $p \in]2,+\infty]$, and its vorticity $\Omega_0 \in L^\infty \cap L^q$, with $q \in [2,+\infty[$ such that $1/p + 1/q \geq 1/2$. Let us suppose $\Omega_0 \in \mathcal{C}^\varepsilon_{X_0}$.
Finally, let the initial density $\rho_0 \in W^{1,\infty}$ be such that $0 < \rho_* \leq \rho_0 \leq \rho^*$ and $\nabla\rho_0 \in \mathcal{C}^\varepsilon_{X_0}$.

Then there exist a time $T > 0$ and a unique solution $(\rho, u, \nabla\Pi)$ of system (5.1)*, such that:*

- *$\rho \in L^\infty([0,T];W^{1,\infty}) \cap \mathcal{C}_b([0,T]\times\mathbb{R}^N)$, such that $0 < \rho_* \leq \rho \leq \rho^*$ at every time;*
- *$u \in \mathcal{C}([0,T];L^p) \cap L^\infty([0,T];\mathcal{C}^{0,1})$, with $\partial_t u \in \mathcal{C}([0,T];L^2)$ and vorticity $\Omega \in \mathcal{C}([0,T];L^q)$;*
- *$\nabla\Pi \in \mathcal{C}([0,T];L^2)$, with $\nabla^2\Pi \in L^\infty([0,T];L^\infty)$.*

Moreover, the family of vector-fields transported by the flow still remains, at every time, non-degenerate and with components and divergence in $\mathcal{C}^\varepsilon$, *and striated regularity is preserved: at every time* $t \in [0,T]$, *one has*

- $\nabla\rho(t)$ *and* $\Omega(t) \;\in\; \mathcal{C}^\varepsilon_{X(t)}$,
- $u(t)$ *and* $\nabla\Pi(t) \;\in\; \mathcal{C}^{1+\varepsilon}_{X(t)}$

uniformly on $[0,T]$.

Another interesting notion, strictly related to the previous one, is that of *conormal regularity*. First of all, we have to recall a definition (see again section 1 of [26]).

Definition 5.4. Let $\Sigma \subset \mathbb{R}^N$ be a compact hypersurface of class $\mathcal{C}^{1+\varepsilon}$. Let us denote by $\mathcal{T}^\varepsilon_\Sigma$ the set of all vector-fields X with components and divergence in $\mathcal{C}^\varepsilon$, which are tangent to Σ, i.e. $\partial_X H_{|\Sigma} \equiv 0$ for all local equations H of Σ.

Given a $\eta \in [\varepsilon, 1+\varepsilon]$, we say that a function $f \in L^\infty$ belongs to the space $\mathcal{C}^\eta_\Sigma$ if

$$\forall\, X \,\in\, \mathcal{T}^\varepsilon_\Sigma\;\;, \qquad f\,\in\,\mathcal{C}^\eta_X\,,\;\;\text{i.e. div}\,(f\,X)\,\in\,\mathcal{C}^{\eta-1}\,.$$

Similarly to what happens for striated regularity, also conormal structure propagates during the time evolution.

Theorem 5.5. *Fix* $\varepsilon \in]0,1[$ *and take a compact hypersurface* $\Sigma_0 \subset \mathbb{R}^N$ *of class* $\mathcal{C}^{1+\varepsilon}$.
Let us suppose the initial velocity field $u_0 \in L^p$, *with* $p \in]2,+\infty]$, *and its vorticity* $\Omega_0 \in L^\infty \cap L^q$, *with* $q \in [2,+\infty[$ *such that* $1/p + 1/q \geq 1/2$. *Moreover, let us suppose* $\Omega_0 \in \mathcal{C}^\varepsilon_{\Sigma_0}$.
Finally, let the initial density $\rho_0 \in W^{1,\infty}$ *be such that* $0 < \rho_* \leq \rho_0 \leq \rho^*$ *and* $\nabla\rho_0 \in \mathcal{C}^\varepsilon_{\Sigma_0}$.

Then there exist a time $T > 0$ *and a unique solution* $(\rho, u, \nabla\Pi)$ *of system* (5.1), *which verifies the same properties of theorem 5.3.*
Moreover, if we define

$$\Sigma(t) \,:=\, \psi_t\,(\Sigma_0)\;,$$

$\Sigma(t)$ *is, at every time* $t \in [0,T]$, *a hypersurface of class* $\mathcal{C}^{1+\varepsilon}$ *of* $\mathbb{R}^N$, *and conormal regularity is preserved: at every time* $t \in [0,T]$, *one has*

- $\nabla\rho(t)$ *and* $\Omega(t) \;\in\; \mathcal{C}^\varepsilon_{\Sigma(t)}$,
- $u(t)$ *and* $\nabla\Pi(t) \;\in\; \mathcal{C}^{1+\varepsilon}_{\Sigma(t)}$

uniformly on $[0,T]$.

5.3 More on paradifferential calculus

The proof to our results is essentially based on the Fourier Analysis methods presented in chapter 1. In this section we will introduce some additional tools we need. As we are interested in the class of Hölder spaces, we will focus only on this case.

First of all, let us quote a result (see [25] for the proof) pertaining to the right composition of functions in Besov spaces, which will be of great importance in the sequel. For the sake of completnees, let us also state what proposition 1.31 becomes in the particular case of Hölder continuous functions.

Proposition 5.6. *(i) Let* I *be an open interval of* $\mathbb{R}$ *and* $F : I \to \mathbb{R}$ *a smooth function.*

Then for all compact subset $J \subset I$ *and all* $s > 0$, *there exists a constant* C *such that, for all function* u *valued in* J *and with gradient in* $\mathcal{C}^{s-1}$, *we have* $\nabla(F \circ u) \in \mathcal{C}^{s-1}$ *and*

$$\|\nabla(F\circ u)\|_{\mathcal{C}^{s-1}} \,\leq\, C\,\|\nabla u\|_{\mathcal{C}^{s-1}}\,.$$

(ii) Let $s>0$ and $m\in\mathbb{N}$ be such that $m>s$. Let $u\in\mathcal{C}^s$ and $\psi\in\mathcal{C}^m_b$ such that the Jacobian of ψ^{-1} is bounded.

Then $u\circ\psi\in\mathcal{C}^s$. Moreover, if $s\in]0,1[$ the following estimate holds:

$$\|u\circ\psi\|_{\mathcal{C}^s}\,\leq\,C\,\left(1+\|\nabla\psi\|_{L^\infty}\right)\,\|u\|_{\mathcal{C}^s}\,.$$

Now, let us introduce the notion of paravector-field, which we will broadly use in our computations.

Definition 5.7. Let X be a vector-field with components in $\mathcal{S}'$. We can formally define the paravector-field operator T_X in the following way: for all $u\in\mathcal{S}'$,

$$T_Xu\,:=\,\sum_{i=1}^N T_{X^i}\partial_iu\,.$$

The following result (see section 2 of [26] for the proof) says that the paravector-field operator is, in a certain sense, the principal part of the derivation ∂_X: the derivative along X is more regular if and only if the "paraderivation" along X is.

Lemma 5.8. *For all vector field $X\in\mathcal{C}^s$ and all $u\in\mathcal{C}^t$, we have:*

- *if $t<1$ and $s+t>1$, then*

$$\|\partial_Xu\,-\,T_Xu\|_{\mathcal{C}^{s+t-1}}\,\leq\,\frac{C}{(1-t)\,(s+t-1)}\,\|X\|_{\mathcal{C}^s}\,\|\nabla u\|_{\mathcal{C}^{t-1}}\,;$$

- *if $t<0$, $s<1$ and $s+t>0$, then*

$$\|T_Xu\,-\,\mathrm{div}\,(u\,X)\|_{\mathcal{C}^{s+t-1}}\,\leq\,\frac{C}{t\,(s+t)\,(s-1)}\,\|X\|_{\mathcal{C}^s}\,\|u\|_{\mathcal{C}^t}\,;$$

- *if $t<1$ and $s+t>0$, then*

$$\|\partial_Xu\,-\,T_Xu\|_{\mathcal{C}^{s+t-1}}\,\leq\,\frac{C}{(s+t)\,(1-t)}\,\widetilde{\|}X\|_{\mathcal{C}^s}\,\|\nabla u\|_{\mathcal{C}^{t-1}}\,.$$

Moreover, first and last inequalities are still true even in the case $t=1$, provided that one replaces $\|\nabla u\|_{\mathcal{C}^0_*}$ with $\|\nabla u\|_{L^\infty}$, while the second is still true even if $t=0$, with $\|u\|_{L^\infty}$ instead of $\|u\|_{\mathcal{C}^0_*}$.

We will heavily use also the following statement about composition of paravector-field and paraproduct operators (see the appendix in [26] for its proof).

Lemma 5.9. *Fix $s\in]0,1[$. There exist constants C, depending only on s, such that, for all $t_1<0$ and $t_2\in\mathbb{R}$,*

$$\begin{aligned}\|T_X\,T_u\,v\|_{\mathcal{C}^{s-1+t_1+t_2}}\quad&\leq\quad C\,\big(\|X\|_{\mathcal{C}^s}\,\|u\|_{\mathcal{C}^{t_1}}\,\|v\|_{\mathcal{C}^{t_2}}\,+\\&\qquad+\|v\|_{\mathcal{C}^{t_2}}\,\|T_Xu\|_{\mathcal{C}^{s-1+t_1}}\,+\,\|u\|_{\mathcal{C}^{t_1}}\,\|T_Xv\|_{\mathcal{C}^{s-1+t_2}}\big)\,,\end{aligned}$$

and this is still true in the case $t_1=0$ with $\|u\|_{L^\infty}$ instead of $\|u\|_{\mathcal{C}^0_*}$.

Moreover, if $s-1+t_1+t_2\,>\,0$, then we have also

$$\begin{aligned}\|T_X\,R(u,v)\|_{\mathcal{C}^{s-1+t_1+t_2}}\quad&\leq\quad C\,\big(\|X\|_{\mathcal{C}^s}\,\|u\|_{\mathcal{C}^{t_1}}\,\|v\|_{\mathcal{C}^{t_2}}\,+\\&\qquad+\|v\|_{\mathcal{C}^{t_2}}\,\|T_Xu\|_{\mathcal{C}^{s-1+t_1}}\,+\,\|u\|_{\mathcal{C}^{t_1}}\,\|T_Xv\|_{\mathcal{C}^{s-1+t_2}}\big)\,.\end{aligned}$$

5.4 Propagation of striated regularity

Now we are ready to tackle the proof of theorem 5.3. We will carry out it in a standard way: first of all we will prove a priori estimates for solutions of the non-homogeneous Euler equations. Then, we will construct a sequence of regular approximated solutions. Finally, thanks to upper bounds proved in the first part, we will get convergence of this sequence to a solution of our initial system, with the required properties.

5.4.1 A priori estimates

First of all, we will prove a priori estimates for a smooth solution $(\rho, u, \nabla\Pi)$ of system (5.1).

Estimates for density and velocity field

From first equation of (5.1), it follows that

$$\rho(t,x) \;=\; \rho_0\left(\psi_t^{-1}(x)\right)\,,$$

so, as the flow ψ_t is a diffeomorphism over $\mathbb{R}^N$ at all fixed time, we have that

$$0\;<\;\rho_*\;\leq\;\rho(t)\;\leq\;\rho^*\,. \tag{5.9}$$

Applying the operator ∂_i to the same equation, using classical L^p estimates for the transport equation and Gronwall's lemma, we get

$$\|\nabla\rho(t)\|_{L^\infty}\;\leq\;\|\nabla\rho_0\|_{L^\infty}\,\exp\left(C\int_0^t\|\nabla u\|_{L^\infty}\,d\tau\right)\,. \tag{5.10}$$

From the equation for the velocity, instead, we get, in a classical way,

$$\|u(t)\|_{L^p}\;\leq\;\|u_0\|_{L^p}\,+\,\int_0^t\left\|\frac{\nabla\Pi}{\rho}\right\|_{L^p}\,d\tau\,;$$

so, using (5.9) and Hölder inequalities, for a certain $\theta\in\,]0,1[\,$, the following estimate holds:

$$\|u(t)\|_{L^p}\;\leq\;\|u_0\|_{L^p}\,+\,\frac{C}{\rho_*}\int_0^t\|\nabla\Pi\|_{L^2}^{\theta}\,\|\nabla\Pi\|_{L^\infty}^{1-\theta}\,d\tau\,. \tag{5.11}$$

Remark 5.10. Let us observe that, as regularity of the pressure goes like that of the velocity field, one can try to estimate directly the L^p norm of the pressure term. Unfortunately, we can't solve its (elliptic) equation in this space without assuming a smallness condition on the density or its gradient. So, we will prove that $\nabla\Pi$ is in $L^2\cap L^\infty$, which is actually stronger than previous property and requires no further hypothesis on ρ.

Already from (5.10) it's clear that we need an estimate for the L^∞ norm of the gradient of the velocity. As remarked before, we can't expect to get it from the hypothesis $\Omega\in L^\infty$; the key will be the further assumption, i.e. the hypothesis of more regularity of the vorticity along the fixed directions given by the family X_0.
Here we quote also a fundamental lemma, whose proof can be found in [2] (chapter 7) for the 2-dimensional case, in [26] (section 3) and [36] (again section 3) for the general one. It is the main point to get the velocity field to be Lipschitz and it turns out to be immediately useful in the sequel.

Lemma 5.11. *Fix $\varepsilon\in\,]0,1[$ and an integer $m\geq N-1$, and take a non-degenerate family $Y=(Y_\lambda)_{1\leq\lambda\leq m}$ of $\mathcal{C}^\varepsilon$ vector-fields over $\mathbb{R}^N$ such that also their divergences are in $\mathcal{C}^\varepsilon$.*

Then, for all indices $1 \leq i,j \leq N$, there exist $\mathcal{C}^\varepsilon$ functions a_{ij}, $b_{ij}^{k\lambda}$ (with $1 \leq k \leq N$, $1 \leq \lambda \leq m$) such that, for all $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N$, the following equality holds:

$$\xi_i\,\xi_j \;=\; a_{ij}(x)|\xi|^2 \,+\, \sum_{k,\lambda} b_{ij}^{k\lambda}(x)\,(Y_\lambda(x)\cdot\xi)\,\xi_k\,.$$

Moreover, the functions in the previous relation could be chosen such that

$$\begin{aligned}\|a_{ij}\|_{L^\infty} \;&\leq\; 1\\ \left\|b_{ij}^{k\lambda}\right\|_{\mathcal{C}^\varepsilon} \;&\leq\; C\,\frac{m^{2N-2}}{I(Y)}\,|||Y|||_{\mathcal{C}^\varepsilon}^{9N-10}\,.\end{aligned}$$

Now, we can state the stationary estimate which says that the velocity field u is Lipschitz. This can be done as in the classical case, because it's based only on the Biot-Savart law, or better on it's gradient version (5.4).

Proposition 5.12. *Fix $\varepsilon \in]0,1[$ and $q \in]1,+\infty[$; moreover, take a non-degenerate family $Y = (Y_\lambda)_{1\leq\lambda\leq m}$ of $\mathcal{C}^\varepsilon$ vector-fields over $\mathbb{R}^N$ such that also their divergences are of class $\mathcal{C}^\varepsilon$.*

Then there exists a constant C, depending only on the space dimension N and on the number of vector-fields m, such that, for all skew-symmetric matrices Ω with coefficients in $L^q\cap\mathcal{C}^\varepsilon_Y$, the corresponding divergence-free vector-field u, given by (5.3), satisfies

$$\|\nabla u\|_{L^\infty} \,\leq\, C\left(\frac{q^2}{q-1}\,\|\Omega\|_{L^q} \,+\, \frac{1}{\varepsilon\,(1-\varepsilon)}\,\|\Omega\|_{L^\infty}\log\left(e \,+\, \frac{\|\Omega\|_{\mathcal{C}^\varepsilon_Y}}{\|\Omega\|_{L^\infty}}\right)\right)\,. \tag{5.12}$$

Estimates for the vorticity

Using the well-known L^q estimates for transport equation and taking advantage of Gronwall's lemma and Hölder inequality in Lebesgue spaces, from (5.6) we obtain

$$\begin{aligned}\|\Omega(t)\|_{L^q} \;&\leq\; C\,\exp\left(\int_0^t\|\nabla u\|_{L^\infty}d\tau\right)\times\\ &\quad\times\left(\|\Omega_0\|_{L^q} + \frac{1}{(\rho_*)^2}\int_0^t e^{-\int_0^\tau\|\nabla u\|_{L^\infty}d\tau'}\,\|\nabla\rho\|_{L^\infty}\,\|\nabla\Pi\|_{L^2}^{\gamma}\,\|\nabla\Pi\|_{L^\infty}^{1-\gamma}\,d\tau\right),\end{aligned} \tag{5.13}$$

for a certain $\gamma \in]0,1[$.

Of course an analogue estimate holds true also for the L^∞ norm:

$$\begin{aligned}\|\Omega(t)\|_{L^\infty} \;&\leq\; C\,\exp\left(\int_0^t\|\nabla u\|_{L^\infty}d\tau\right)\times\\ &\quad\times\left(\|\Omega_0\|_{L^\infty} \,+\, \frac{1}{(\rho_*)^2}\int_0^t e^{-\int_0^\tau\|\nabla u\|_{L^\infty}d\tau'}\,\|\nabla\rho\|_{L^\infty}\,\|\nabla\Pi\|_{L^\infty}\,d\tau\right)\,.\end{aligned} \tag{5.14}$$

Remark 5.13. Let us fix the index p pertaining to u and let us call $\overline{q}$ the real number in $[2,+\infty[$ such that $1/p + 1/\overline{q} = 1/2$. From our hypothesis, it's clear that $q \leq \overline{q}$; therefore, thanks to Hölder and Young inequalities, we have

$$\|\Omega\|_{L^{\overline{q}}} \,\leq\, \|\Omega\|_{L^q}^{\eta}\,\|\Omega\|_{L^\infty}^{1-\eta} \,\leq\, \|\Omega\|_{L^q\cap L^\infty}\,.$$

Estimates for the pressure term

Now, let us focus on the pressure term: taking the divergence of the second equation of system (5.1), we discover that it solves the elliptic equation

$$-\operatorname{div}\left(\frac{\nabla\Pi}{\rho}\right) = \operatorname{div}\,(u\cdot\nabla u)\,. \tag{5.15}$$

From this, remembering our hypothesis and remark 5.13, estimate (5.5) and lemma 4.8, the control of the L^2 norm immediately follows:

$$\frac{1}{\rho^*}\,\|\nabla\Pi\|_{L^2} \,\leq\, C\,\|u\|_{L^p}\,\|\Omega\|_{L^q\cap L^\infty}\,. \tag{5.16}$$

Moreover, we have that $\nabla\Pi$ belongs also to L^∞, and so, by interpolation, $\nabla\Pi\in L^b$ for all $b\in[2,+\infty]$. As a matter of fact, now we are going to show a stronger claim, that is to say $\nabla\Pi\in\mathcal{C}^1_*$. Cutting in low and high frequencies, we have that

$$\|\nabla\Pi\|_{\mathcal{C}^1_*} \,\leq\, \|\Delta_{-1}\nabla\Pi\|_{\mathcal{C}^1_*} \,+\, \|(\mathrm{Id}\,-\,\Delta_{-1})\nabla\Pi\|_{\mathcal{C}^1_*} \,\leq\, C\left(\|\nabla\Pi\|_{L^2}\,+\,\|\Delta\Pi\|_{\mathcal{C}^0_*}\right)\,.$$

Now, from (5.15) we get

$$-\,\Delta\Pi \,=\, \nabla\left(\log\rho\right)\cdot\nabla\Pi \,+\, \rho\,\operatorname{div}\,(u\cdot\nabla u)\,. \tag{5.17}$$

From this last relation, from the fact that $\operatorname{div}(u\cdot\nabla u)=\nabla u:\nabla u$ and the immersion $L^\infty\hookrightarrow\mathcal{C}^0_*$, we obtain

$$\begin{aligned}\|\Delta\Pi\|_{\mathcal{C}^0_*} \;\leq\; \|\Delta\Pi\|_{L^\infty} \;&\leq\; \|\nabla\left(\log\rho\right)\cdot\nabla\Pi\|_{L^\infty} \,+\, \|\rho\,\operatorname{div}\,(u\cdot\nabla u)\|_{L^\infty}\\ &\leq\; C\left(\|\nabla\rho\|_{L^\infty}\,\|\nabla\Pi\|_{L^\infty} \,+\, \rho^*\,\|\nabla u\|^2_{L^\infty}\right)\,.\end{aligned}$$

Now, $\mathcal{C}^1_*\hookrightarrow\mathcal{C}^\eta\hookrightarrow L^\infty$ for all $\eta\in]0,1[$; taking for instance $\eta=1/2$ and using interpolation inequalities between Besov spaces, we thus have, for a certain $\beta\in]0,1[$,

$$\|\nabla\Pi\|_{L^\infty} \,\leq\, \|\nabla\Pi\|_{\mathcal{C}^{1/2}} \,\leq\, C\,\|\nabla\Pi\|^\beta_{\mathcal{C}^{-N/2}}\,\|\nabla\Pi\|^{1-\beta}_{\mathcal{C}^1_*} \,\leq\, C\,\|\nabla\Pi\|^\beta_{L^2}\,\|\nabla\Pi\|^{1-\beta}_{\mathcal{C}^1_*}\,.$$

Thanks to Young's inequality, from this relation and (5.16) one finally gets

$$\|\nabla\Pi\|_{\mathcal{C}^1_*} \,\leq\, C\,\left(\left(1+\|\nabla\rho\|^\delta_{L^\infty}\right)\|u\|_{L^p}\,\|\Omega\|_{L^q\cap L^\infty} \,+\, \rho^*\,\|\nabla u\|^2_{L^\infty}\right)\,, \tag{5.18}$$

for some δ depending only on the space dimension N. So we have proved our claim, i.e. $\nabla\Pi\in\mathcal{C}^1_*$, and so it belongs also to L^∞.

Finally, we want to prove boundedness of second derivatives of the pressure term. This property is a consequence of striated regularity for $\nabla\Pi$ we will show in next section: for the time being, let us admit this fact. So, passing in Fourier variables and using lemma 5.11, for all $1\leq i\,,j\leq N$ we can write

$$\xi_i\,\xi_j\,\widehat{\Pi}(\xi) \,=\, a_{ij}(x)|\xi|^2\,\widehat{\Pi}(\xi) \,+\, \sum_{k,\lambda} b^{k\lambda}_{ij}(x)\,(X_\lambda(x)\cdot\xi)\,\xi_k\,\widehat{\Pi}(\xi)\,.$$

Applying the inverse Fourier transform $\mathcal{F}^{-1}_\xi$ and passing to L^∞ norms, we get

$$\left\|\nabla^2\Pi\right\|_{L^\infty} \,\leq\, C\left(\|\Delta\Pi\|_{L^\infty} \,+\, \|\partial_X\nabla\Pi\|_{L^\infty}\right)\,.$$

Proposition 2.104 of [2] tells us that

$$\|\partial_X\nabla\Pi\|_{L^\infty} \,\leq\, \frac{C}{\varepsilon}\,\|\partial_X\nabla\Pi\|_{\mathcal{C}^0_*}\,\log\left(e\,+\,\frac{\|\partial_X\nabla\Pi\|_{\mathcal{C}^\varepsilon}}{\|\partial_X\nabla\Pi\|_{\mathcal{C}^0_*}}\right)\,.$$

Using Bony's paraproduct decomposition to handle the norm in $\mathcal{C}^0_*$ and noticing that the function $\zeta \mapsto \zeta \log(e + k/\zeta)$ is nondecreasing, we finally get

$$\left\|\nabla^2\Pi\right\|_{L^\infty} \leq C\Bigg(\|\nabla\rho\|_{L^\infty}\,\|\nabla\Pi\|_{\mathcal{C}^1_*} + \rho^*\,\|\nabla u\|^2_{L^\infty} + \\ + \widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\,\log\left(e + \frac{\|\partial_X\nabla\Pi\|_{\mathcal{C}^\varepsilon}}{\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\|\nabla\Pi\|_{\mathcal{C}^1_*}}\right)\Bigg). \tag{5.19}$$

5.4.2 A priori estimates for striated regularity

After having established the "classical" estimates, let us now focus on the conservation of striated regularity. The most important step lies in finding a priori estimates for the derivations along the vector-field X. So, let us now state a lemma which explains the relation between the operators ∂_X and $\mathrm{div}\,(\,\cdot\,X)$ (see also remark 5.2).

Lemma 5.14. *For every vector-field X with components and divergence in $\mathcal{C}^\varepsilon$, and every function $f \in \mathcal{C}^\eta$ for some $\eta \in]0,1]$, we have*

$$\|\mathrm{div}\,(f\,X) - \partial_X f\|_{\mathcal{C}^{\min\{\varepsilon,\eta\}}} \leq C\,\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\|f\|_{\mathcal{C}^\eta}\,.$$

Moreover, the previous inequality is still true in the limit case $\eta = 0$, with $\|f\|_{L^\infty}$ instead of $\|f\|_{\mathcal{C}^0_*}$.

Proof. The thesis immediately follows from the identity $\mathrm{div}\,(f\,X) - \partial_X f = f\,\mathrm{div}X$ and from Bony's paraproduct decomposition. □

So, it's enough for us to focus on the operator ∂_X.

The evolution of the family of vector-fields

First of all, we want to prove that the family of vector-fields $X(t) = (X_\lambda(t))_{1\leq\lambda\leq m}$, where each $X_\lambda(t)$ is defined by (5.7), still remains non-degenerate for all t, and that each $X_\lambda(t)$ still has components and divergence in $\mathcal{C}^\varepsilon$. Throughout this paragraph we will denote by $Y(t)$ a generic element of the family $X(t)$.

Applying the divergence operator to (5.8), an easy computation shows us that $\mathrm{div}\,Y$ satisfies

$$(\partial_t + u\cdot\nabla)\,\mathrm{div}\,Y = 0\,,$$

which immediately implies $\mathrm{div}\,Y(t) \in \mathcal{C}^\varepsilon$ for all t and

$$\|\mathrm{div}\,Y(t)\|_{\mathcal{C}^\varepsilon} \leq C\,\|\mathrm{div}\,Y_0\|_{\mathcal{C}^\varepsilon}\exp\left(c\int_0^t\|\nabla u\|_{L^\infty}\,d\tau\right)\,. \tag{5.20}$$

Moreover, starting again from (5.8), we get (for the details, see proposition 4.1 of [26])

$$(\partial_t + u\cdot\nabla)\left(\overset{N-1}{\wedge}X_\lambda\right) = {}^t\nabla u\cdot\left(\overset{N-1}{\wedge}X_\lambda\right)\,,$$

from which it follows

$$\left(\overset{N-1}{\wedge}X_\lambda\right)(t,x) = \left(\overset{N-1}{\wedge}X_\lambda\right)(0,\psi_t^{-1}(x)) - \int_0^t {}^t\nabla u\cdot\left(\overset{N-1}{\wedge}X_\lambda\right)(\tau,\psi_t^{-1}(\psi_\tau(x)))\,d\tau\,.$$

This relation gives us

$$\left|\left(\overset{N-1}{\wedge}X_\lambda\right)(0,\psi_t^{-1}(x))\right| \leq \left|\left(\overset{N-1}{\wedge}X_\lambda\right)(t,x)\right| + \\ + \int_0^t\|\nabla u(t-\tau)\|_{L^\infty}\left|\left(\overset{N-1}{\wedge}X_\lambda\right)(t-\tau,\psi_\tau^{-1}(x))\right|\,d\tau\,,$$

and by Gronwall's lemma one gets

$$\left|\left(\overset{N-1}{\wedge}X_\lambda\right)(t,x)\right| \geq \left|\left(\overset{N-1}{\wedge}X_{0,\lambda}\right)\left(\psi_t^{-1}(x)\right)\right| e^{-c\int_0^t\|\nabla u\|_{L^\infty}d\tau}.$$

From this inequality we immediately gather that the family $X(t)$ still remains non-degenerate at every time t:

$$I(X(t)) \geq I(X_0)\exp\left(-c\int_0^t\|\nabla u\|_{L^\infty}d\tau\right). \tag{5.21}$$

Finally, again from the evolution equation (5.8), it's clear that, to prove that $Y(t)$ is of class $\mathcal{C}^\varepsilon$, we need a control on the norm in this space of the term $\partial_Y u$. To get this, we use, as very often in the sequel, the following decomposition:

$$\partial_Y u = T_Y u + (\partial_Y - T_Y)\,u,$$

with (by lemma 5.8)

$$\|(\partial_Y - T_Y)\,u\|_{\mathcal{C}^\varepsilon} \leq C\,\widetilde{\|}Y\|_{\mathcal{C}^\varepsilon}\|\nabla u\|_{L^\infty}.$$

Moreover, for all $1\leq i\leq N$ thanks to (5.3) we can write

$$T_Y u^i = -\sum_{k,j}\left(\partial_k(-\Delta)^{-1}T_{Y^j}\partial_j\Omega_{ik} - \left[\partial_k(-\Delta)^{-1},T_{Y^j}\partial_j\right]\Omega_{ik}\right).$$

Obviously, from lemma 5.8 we have

$$\left\|\partial_k(-\Delta)^{-1}\sum_j T_{Y^j}\partial_j\Omega_{ik}\right\|_{\mathcal{C}^\varepsilon} \leq \|T_Y\Omega\|_{\mathcal{C}^{\varepsilon-1}} \leq \|\partial_Y\Omega\|_{\mathcal{C}^{\varepsilon-1}} + C\,\widetilde{\|}Y\|_{\mathcal{C}^\varepsilon}\|\Omega\|_{L^\infty},$$

while for the commutator term we use lemma 1.29, which gives us the following control:

$$\left\|\left[\partial_k(-\Delta)^{-1},T_{Y^j}\partial_j\right]\Omega_{ik}\right\|_{\mathcal{C}^\varepsilon} \leq C\,\|Y\|_{\mathcal{C}^\varepsilon}\|\Omega\|_{L^\infty}.$$

So, in the end, from the hypothesis of striated regularity for the vorticity we get that also the velocity field u is more regular along the fixed directions and

$$\|\partial_Y u\|_{\mathcal{C}^\varepsilon} \leq C\left(\|\partial_Y\Omega\|_{\mathcal{C}^{\varepsilon-1}} + \widetilde{\|}Y\|_{\mathcal{C}^\varepsilon}\|\nabla u\|_{L^\infty}\right). \tag{5.22}$$

Moreover, applying proposition 4.7 to (5.8) and using (5.22), (5.20) and Gronwall's inequality finally give us

$$\widetilde{\|}Y(t)\|_{\mathcal{C}^\varepsilon} \leq C\exp\left(c\int_0^t\|\nabla u\|_{L^\infty}d\tau\right)\left(\widetilde{\|}Y_0\|_{\mathcal{C}^\varepsilon} + \int_0^t e^{-c\int_0^\tau\|\nabla u\|_{L^\infty}d\tau'}\|\partial_Y\Omega\|_{\mathcal{C}^{\varepsilon-1}}\,d\tau\right). \tag{5.23}$$

These estimates having being established, from now on for simplicity we will consider the case of a single vector-field $X(t)$: the generalization to the case of a finite family is quite obvious, and where the difference is substantial, we will suggest references for the details.

Striated regularity for the density

Now, we want to investigate propagation of striated regularity for the density. First of all, let us state a stationary lemma.

Lemma 5.15. *Let f be a function in $\mathcal{C}^1_*$.*

(i) If $\partial_X f \in \mathcal{C}^\varepsilon$ *and* $\nabla f \in L^\infty$*, then one has* $\partial_X \nabla f \in \mathcal{C}^{\varepsilon-1}$ *and the following inequality holds:*

$$\|\partial_X \nabla f\|_{\mathcal{C}^{\varepsilon-1}} \,\leq\, C\left(\|\partial_X f\|_{\mathcal{C}^\varepsilon} \,+\, \widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\left(\|f\|_{\mathcal{C}^1_*} \,+\, \|\nabla f\|_{L^\infty}\right)\right). \tag{5.24}$$

(ii) Conversely, if $\partial_X \nabla f \in \mathcal{C}^{\varepsilon-1}$*, then* $\partial_X f \in \mathcal{C}^\varepsilon$ *and one has*

$$\|\partial_X f\|_{\mathcal{C}^\varepsilon} \,\leq\, C\left(\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\left(\|f\|_{\mathcal{C}^1_*} \,+\, \|\nabla f\|_{L^\infty}\right) \,+\, \|\partial_X \nabla f\|_{\mathcal{C}^{\varepsilon-1}}\right). \tag{5.25}$$

Proof. (i) Using the paravector-field operator (remember definition 5.7), we can write:

$$\partial_X \nabla f \,=\, (\partial_X \,-\, T_X)\,\nabla f \,+\, T_X\,\nabla f\,.$$

From lemma 5.8, we have that the first term of the previous equality is in $\mathcal{C}^{\varepsilon-1}$ and

$$\|(\partial_X - T_X)\,\nabla f\|_{\mathcal{C}^{\varepsilon-1}} \,\leq\, C\,\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\|\nabla f\|_{L^\infty}\,. \tag{5.26}$$

Now, we have to estimate the paravector-field term: note that

$$T_X\,\nabla f \,=\, \nabla\,(T_X f) \,+\, [T_X, \nabla]\,f\,.$$

From hypothesis of the lemma, it's obvious that $\nabla\,(T_X f) \,\in\, \mathcal{C}^{\varepsilon-1}$. For the last term, remembering that ∇ and T_X are operators of order 1, we can use lemma 1.29 and get

$$\|[T_X, \nabla]\,f\|_{\mathcal{C}^{\varepsilon-1}} \,\leq\, C\,\|X\|_{\mathcal{C}^\varepsilon}\,\|f\|_{\mathcal{C}^1_*}\,. \tag{5.27}$$

Putting together (5.26), (5.27) and the control for $\|\nabla\,(T_X f)\|_{\mathcal{C}^{\varepsilon-1}}$ gives us the first part of the lemma.

(ii) For the second part, we write again

$$\partial_X f \,=\, T_X f \,+\, (\partial_X - T_X)\,f\,.$$

By definition of the space $\mathcal{C}^\varepsilon_X$, we know that ∇f is bounded: so, second term can be easily controlled in $\mathcal{C}^\varepsilon$ thanks to lemma 5.8. Now let us define the operator Ψ such that, in Fourier variables, for all vector-fields v we have

$$\mathcal{F}_x\,(\Psi v)\,(\xi) \,=\, -\,i\,\frac{1}{|\xi|^2}\,\xi\cdot\widehat{v}(\xi)\,.$$

So, noting that the paravector term involves only high frequencies of f, we can write

$$T_X f \,=\, T_X\,(\Psi\,\nabla f) \,=\, \Psi\,T_X \nabla f \,+\, [T_X, \Psi]\,\nabla f\,.$$

Now, applying lemmas 5.8 and 1.29 completes the proof. □

Remark 5.16. Let us note that, if $f \in L^b$ (for some $b \in [1, +\infty]$) is such that $\nabla f \in L^\infty$, then $f \in \mathcal{C}^1_*$ (indeed $f \in \mathcal{C}^{0,1}$) and (separating low and high frequencies)

$$\|f\|_{\mathcal{C}^1_*} \,\leq\, C\,\left(\|f\|_{L^b} \,+\, \|\nabla f\|_{L^\infty}\right)\,.$$

Both u and ρ satisfy such an estimate, respectively with $b = p$ and $b = +\infty$.

Thanks to lemma 5.15, we can equally deal with ρ or $\nabla\rho$: as the equation for ρ is very simple, we choose to work with it. Keeping in mind that $[X(t)\,,\,\partial_t+u\cdot\nabla]=0$, we have

$$\partial_t\left(\partial_X\rho\right)\,+\,u\cdot\nabla\left(\partial_X\rho\right)\,=\,0\,,$$

from which (remember also (5.25)) it immediately follows that

$$\left\|\partial_{X(t)}\rho(t)\right\|_{\mathcal{C}^\varepsilon}\,\leq\,C\left(\widetilde{\|}X_0\|_{\mathcal{C}^\varepsilon}\left(\rho^*+\|\nabla\rho_0\|_{L^\infty}\right)+\|\partial_{X_0}\nabla\rho_0\|_{\mathcal{C}^{\varepsilon-1}}\right)\exp\left(c\int_0^t\|\nabla u\|_{L^\infty}\,d\tau\right). \tag{5.28}$$

Therefore, one gets also

$$\begin{aligned}\left\|\partial_{X(t)}\nabla\rho(t)\right\|_{\mathcal{C}^{\varepsilon-1}}\quad\leq\quad & C\,\exp\left(c\int_0^t\|\nabla u\|_{L^\infty}\,d\tau\right)\,\times\\ &\times\left(\left(\rho^*+\|\nabla\rho_0\|_{L^\infty}\right)\widetilde{\|}X_0\|_{\mathcal{C}^\varepsilon}\,+\,\|\partial_{X_0}\nabla\rho_0\|_{\mathcal{C}^{\varepsilon-1}}\,+\right.\\ &\left.\qquad\qquad+\,\int_0^t e^{-\int_0^\tau\|\nabla u\|_{L^\infty}d\tau'}\,\|\partial_X\Omega\|_{\mathcal{C}^{\varepsilon-1}}\,d\tau\right).\end{aligned} \tag{5.29}$$

Striated regularity for the pressure term

In this paragraph we want to show that geometric properties propagates also to the pressure term, i.e. we want to prove $\partial_X\nabla\Pi\in\mathcal{C}^\varepsilon$.

Again, we use the decomposition $\partial_X\nabla\Pi\,=\,T_X\left(\nabla\Pi\right)\,+\,\left(\partial_X-T_X\right)\nabla\Pi$.

As usual, lemma 5.8 gives us

$$\|(\partial_X\,-\,T_X)\,\nabla\Pi\|_{\mathcal{C}^\varepsilon}\,\leq\,C\,\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\left\|\nabla^2\Pi\right\|_{L^\infty}\,.$$

Now we use estimate (5.19), the fact that $\log(e+\zeta)\leq e+\zeta^{1/2}$ and Young's inequality to isolate the term $\|\partial_X\nabla\Pi\|_{\mathcal{C}^\varepsilon}$. As $2z\leq 1+z^2$, we have

$$\widetilde{\|}X\|^2_{\mathcal{C}^\varepsilon}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\,\leq\,C\left(\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\,+\,\widetilde{\|}X\|^3_{\mathcal{C}^\varepsilon}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\right)\,,$$

and finally we can control $\|(\partial_X\,-\,T_X)\,\nabla\Pi\|_{\mathcal{C}^\varepsilon}$ by the quantity

$$C\left(\|\rho\|_{W^{1,\infty}}\,\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\,+\,\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\|\nabla u\|^2_{L^\infty}\,+\,\widetilde{\|}X\|^3_{\mathcal{C}^\varepsilon}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\right)+\,\frac{1}{2}\,\|\partial_X\nabla\Pi\|_{\mathcal{C}^\varepsilon}\,. \tag{5.30}$$

To deal with the paravector term, we keep in mind that $\nabla\Pi\,=\,\nabla\left(-\Delta\right)^{-1}\left(g_1\,+\,g_2\right)$, where we have set

$$g_1\,=\,-\,\nabla\left(\log\rho\right)\cdot\nabla\Pi\qquad\text{and}\qquad g_2\,=\,\rho\,\mathrm{div}\left(u\cdot\nabla u\right).$$

So it's enough to prove that both $T_X\nabla\left(-\Delta\right)^{-1}g_1$ and $T_X\nabla\left(-\Delta\right)^{-1}g_2$ belong to $\mathcal{C}^\varepsilon$.

Let us consider first the term

$$T_X\nabla\left(-\Delta\right)^{-1}g_2\,=\,\nabla\left(-\Delta\right)^{-1}T_Xg_2\,+\,\left[T_X,\nabla\left(-\Delta\right)^{-1}\right]g_2\,. \tag{5.31}$$

From lemma 1.29 one immediately gets that

$$\left\|\left[T_X,\nabla\left(-\Delta\right)^{-1}\right]g_2\right\|_{\mathcal{C}^\varepsilon}\,\leq\,C\,\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\|g_2\|_{\mathcal{C}^0_*}\,\leq\,C\,\rho^*\,\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\|\nabla u\|^2_{L^\infty}\,, \tag{5.32}$$

while it's obvious that

$$\left\|\nabla\left(-\Delta\right)^{-1}T_Xg_2\right\|_{\mathcal{C}^\varepsilon}\,\leq\,C\,\|T_Xg_2\|_{\mathcal{C}^{\varepsilon-1}}\,.$$

Now we use Bony's paraproduct decomposition and write

$$T_Xg_2\,=\,T_XT_\rho\mathrm{div}\left(u\cdot\nabla u\right)\,+\,T_XT_{\mathrm{div}\,(u\cdot\nabla u)}\rho\,+\,T_XR\left(\rho,\mathrm{div}\left(u\cdot\nabla u\right)\right)\,.$$

From theorem 1.24 and the equality $\operatorname{div}(u\cdot\nabla u)=\nabla u:\nabla u$, it follows that

$$\left\|T_X T_{\operatorname{div}(u\cdot\nabla u)}\rho\right\|_{\mathcal{C}^{\varepsilon-1}} \leq C\,\|X\|_{L^\infty}\,\left\|T_{\operatorname{div}(u\cdot\nabla u)}\rho\right\|_{\mathcal{C}^\varepsilon} \leq C\,\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\|\rho\|_{\mathcal{C}^1_*}\,\|\nabla u\|^2_{L^\infty}\,, \tag{5.33}$$

and the same estimate holds true for the remainder term $T_X R(\rho,\operatorname{div}(u\cdot\nabla u))$. Lemma 5.9, instead, provides a control for $\|T_X T_\rho \operatorname{div}(u\cdot\nabla u)\|_{\mathcal{C}^{\varepsilon-1}}$ by (up to multiplication by a constant)

$$\|X\|_{\mathcal{C}^\varepsilon}\,\|\rho\|_{\mathcal{C}^1_*}\,\|\nabla u\|^2_{L^\infty} + \|\nabla u\|^2_{L^\infty}\,\|T_X\rho\|_{\mathcal{C}^{\varepsilon-1}} + \|\rho\|_{\mathcal{C}^1_*}\,\|T_X\operatorname{div}(u\cdot\nabla u)\|_{\mathcal{C}^{\varepsilon-1}}\,,$$

where $\|T_X\rho\|_{\mathcal{C}^{\varepsilon-1}}\leq C\|X\|_{\mathcal{C}^\varepsilon}\,\|\rho\|_{\mathcal{C}^1_*}$ by theorem 1.24. Now the problem is the control of the $\mathcal{C}^{\varepsilon-1}$ norm of $T_X\operatorname{div}(u\cdot\nabla u)$. Writing

$$\begin{aligned}
T_X\operatorname{div}(u\cdot\nabla u) &= \sum_{i,j} 2\,T_X T_{\partial_i u^j}\partial_j u^i + T_X\partial_i R(u^j,\partial_j u^i)\\
&= \sum_{i,j,k} 2\,T_{X^k}\partial_k T_{\partial_i u^j}\partial_j u^i + \partial_i T_{X^k}\partial_k R(u^j,\partial_j u^i) - T_{\partial_i X^k}\partial_k R(u^j,\partial_j u^i)\,,
\end{aligned}$$

by use of lemma 5.9 we can easily see that it's bounded by

$$\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\|u\|_{\mathcal{C}^1_*}\,\|\nabla u\|_{L^\infty} + \|u\|_{\mathcal{C}^1_*}\,\|T_X\nabla u\|_{\mathcal{C}^{\varepsilon-1}} + \|\nabla u\|_{L^\infty}\,\|T_X u\|_{\mathcal{C}^\varepsilon}\,.$$

Hence, keeping in mind lemmas 5.8 and 5.15, we discover

$$\|T_X\operatorname{div}(u\cdot\nabla u)\|_{\mathcal{C}^{\varepsilon-1}} \leq C\left(\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\|u\|^2_{\mathcal{C}^1_*} + \|\partial_X u\|_{\mathcal{C}^\varepsilon}\,\|u\|_{\mathcal{C}^1_*}\right),$$

and therefore

$$\|T_X T_\rho\operatorname{div}(u\cdot\nabla u)\|_{\mathcal{C}^{\varepsilon-1}} \leq C\left(\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\|\rho\|_{\mathcal{C}^1_*}\,\|u\|^2_{\mathcal{C}^1_*} + \|\rho\|_{\mathcal{C}^1_*}\,\|u\|_{\mathcal{C}^1_*}\,\|\partial_X u\|_{\mathcal{C}^\varepsilon}\right). \tag{5.34}$$

Putting inequalities (5.32), (5.33) and (5.34) all together, we finally get

$$\left\|T_X\nabla\left(-\Delta\right)^{-1} g_2\right\|_{\mathcal{C}^\varepsilon} \leq C\left(\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\|\rho\|_{\mathcal{C}^1_*}\,\|u\|^2_{\mathcal{C}^1_*} + \|\rho\|_{\mathcal{C}^1_*}\,\|u\|_{\mathcal{C}^1_*}\,\|\partial_X u\|_{\mathcal{C}^\varepsilon}\right), \tag{5.35}$$

for some constant C which depends also on ρ^* and ρ_*.

Before going on, let us state a simple lemma.

Lemma 5.17. *Fix a $\varepsilon\in]0,1[$ and an open interval $I\subset\mathbb{R}$.*
Let X be a $\mathcal{C}^\varepsilon$ vector-field with divergence in $\mathcal{C}^\varepsilon$, and $F:I\to\mathbb{R}$ be a smooth function.

Then for all compact set $J\subset I$ and all $\rho\in W^{1,\infty}$ valued in J and such that $\partial_X\rho\in\mathcal{C}^\varepsilon$, one has that $\partial_X(F\circ\rho)\in\mathcal{C}^\varepsilon$ and $\partial_X\nabla(F\circ\rho)\in\mathcal{C}^{\varepsilon-1}$. Moreover, the following estimates hold true:

$$\begin{aligned}
\|\partial_X(F\circ\rho)\|_{\mathcal{C}^\varepsilon} &\leq C\,\|\rho\|_{W^{1,\infty}}\,\|\partial_X\rho\|_{\mathcal{C}^\varepsilon}\\
\|\partial_X\nabla(F\circ\rho)\|_{\mathcal{C}^{\varepsilon-1}} &\leq C\,\|\rho\|_{W^{1,\infty}}\left(\|\partial_X\rho\|_{\mathcal{C}^\varepsilon} + \widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\|\rho\|_{W^{1,\infty}}\right),
\end{aligned}$$

for a constant C depending only on F and on the fixed subset J.

Proof. The first inequality is immediate, keeping in mind identity $\partial_X(F\circ\rho) = F'(\rho)\,\partial_X\rho$ and estimate

$$\left\|F'(\rho)\right\|_{\mathcal{C}^\varepsilon} \leq C\,\left\|F''\right\|_{L^\infty(J)}\,\|\rho\|_{\mathcal{C}^\varepsilon} \leq C\,\left\|F''\right\|_{L^\infty(J)}\,\|\rho\|_{W^{1,\infty}}\,.$$

For the second one, we write:

$$\partial_X\nabla(F\circ\rho) = \partial_X\left(F'(\rho)\,\nabla\rho\right) = F'(\rho)\,\partial_X\nabla\rho + F''(\rho)\,\partial_X\rho\,\nabla\rho\,.$$

Let us observe that the first term is well-defined in $\mathcal{C}^{\varepsilon-1}$, and using decomposition in paraproducts and remainder operators, we have

$$\left\|F'(\rho)\,\partial_X\nabla\rho\right\|_{\mathcal{C}^{\varepsilon-1}} \leq C\,\left\|F'(\rho)\right\|_{W^{1,\infty}}\,\|\partial_X\nabla\rho\|_{\mathcal{C}^{\varepsilon-1}}\,.$$

Now, the conclusion immediately follows from lemma 5.15. □

Let us come back to g_1: using the same trick as in (5.31), it's enough to estimate

$$\left\|T_X g_1\right\|_{\mathcal{C}^{\varepsilon-1}} \qquad \text{and} \qquad \left\|\left[T_X, \nabla\left(-\Delta\right)^{-1}\right] g_1\right\|_{\mathcal{C}^{\varepsilon}}.$$

Again, the control of the commutator term follows from lemma 1.29:

$$\left\|\left[T_X, \nabla\left(-\Delta\right)^{-1}\right] g_1\right\|_{\mathcal{C}^{\varepsilon}} \leq C\, \widetilde{\|}X\|_{\mathcal{C}^{\varepsilon}} \left\|g_1\right\|_{\mathcal{C}^0_*} \leq \frac{C}{\rho_*}\, \widetilde{\|}X\|_{\mathcal{C}^{\varepsilon}} \left\|\nabla\rho\right\|_{L^\infty} \left\|\nabla\Pi\right\|_{\mathcal{C}^1_*}. \tag{5.36}$$

For the other term, we use again Bony's paraproduct decomposition:

$$T_X g_1 \,=\, T_X T_{\nabla(\log\rho)}\nabla\Pi \,+\, T_X T_{\nabla\Pi}\nabla(\log\rho) \,+\, T_X R(\nabla(\log\rho), \nabla\Pi).$$

Thanks to theorem 1.24 we immediately find

$$\left\|T_X T_{\nabla(\log\rho)}\nabla\Pi\right\|_{\mathcal{C}^{\varepsilon-1}} \leq C\, \widetilde{\|}X\|_{\mathcal{C}^{\varepsilon}} \left\|\nabla\rho\right\|_{L^\infty} \left\|\nabla\Pi\right\|_{\mathcal{C}^1_*}, \tag{5.37}$$

and the same control holds true also for the remainder. Moreover, a direct application of lemma 5.9 implies

$$\left\|T_X T_{\nabla(\log\rho)}\nabla\Pi\right\|_{\mathcal{C}^{\varepsilon-1}} \leq C\left(\widetilde{\|}X\|_{\mathcal{C}^{\varepsilon}} \left\|\nabla\rho\right\|_{L^\infty} \left\|\nabla\Pi\right\|_{\mathcal{C}^1_*} + \left\|\nabla\Pi\right\|_{\mathcal{C}^1_*} \left\|T_X\nabla(\log\rho)\right\|_{\mathcal{C}^{\varepsilon-1}}\right), \tag{5.38}$$

Now, from lemmas 5.8 and 5.17 we easily get

$$\left\|T_X\nabla(\log\rho)\right\|_{\mathcal{C}^{\varepsilon-1}} \leq C\left(\left\|\partial_X\rho\right\|_{\mathcal{C}^{\varepsilon}} \left\|\rho\right\|_{W^{1,\infty}} + \widetilde{\|}X\|_{\mathcal{C}^{\varepsilon}} \left\|\rho\right\|^2_{W^{1,\infty}}\right).$$

Putting this last relation into (5.38) and keeping in mind inequalities (5.36) and (5.37), we find

$$\left\|T_X\nabla\left(-\Delta\right)^{-1} g_1\right\|_{\mathcal{C}^{\varepsilon}} \leq C\left(\left\|\partial_X\rho\right\|_{\mathcal{C}^{\varepsilon}} \left\|\rho\right\|_{W^{1,\infty}} \left\|\nabla\Pi\right\|_{\mathcal{C}^1_*} + \widetilde{\|}X\|_{\mathcal{C}^{\varepsilon}} \left\|\rho\right\|^2_{W^{1,\infty}} \left\|\nabla\Pi\right\|_{\mathcal{C}^1_*}\right), \tag{5.39}$$

where, as before, C may depend also on ρ^* and ρ_*.

Therefore, putting (5.30), (5.35) and (5.39) together, we finally get

$$\begin{aligned}\left\|\partial_X\nabla\Pi\right\|_{\mathcal{C}^{\varepsilon}} \,\leq\, C\Big(&\left\|\rho\right\|_{W^{1,\infty}} \left\|\partial_X\rho\right\|_{\mathcal{C}^{\varepsilon}} \left\|\nabla\Pi\right\|_{\mathcal{C}^1_*} + \left\|\nabla\Pi\right\|_{\mathcal{C}^1_*} \widetilde{\|}X\|_{\mathcal{C}^{\varepsilon}} \left\|\rho\right\|^2_{W^{1,\infty}} + \\ &+ \widetilde{\|}X\|^3_{\mathcal{C}^{\varepsilon}} \left\|\nabla\Pi\right\|_{\mathcal{C}^1_*} + \left\|\rho\right\|_{W^{1,\infty}} \widetilde{\|}X\|_{\mathcal{C}^{\varepsilon}} \left\|u\right\|^2_{\mathcal{C}^1_*} + \left\|\rho\right\|_{\mathcal{C}^1_*} \left\|u\right\|_{\mathcal{C}^1_*} \left\|\partial_X u\right\|_{\mathcal{C}^{\varepsilon}}\Big).\end{aligned} \tag{5.40}$$

Striated regularity for the vorticity

Let us now establish a control on the regularity of Ω along the vector-fields $(X_\lambda)_{1\leq\lambda\leq m}$. Applying the operator ∂_X to (5.6), we obtain the evolution equation for $\partial_X\Omega$:

$$\partial_t\left(\partial_X\Omega\right) + u\cdot\nabla\left(\partial_X\Omega\right) \,=\, \partial_X\left(\frac{1}{\rho^2}\nabla\rho\wedge\nabla\Pi\right) - \partial_X\left(\Omega\cdot\nabla u\right) - \partial_X\left({}^t\nabla u\cdot\Omega\right). \tag{5.41}$$

Second and third terms of the right-hand side of (5.41) can be treated taking advantage once again of the following decomposition:

$$\partial_X\left(\Omega\cdot\nabla u + {}^t\nabla u\cdot\Omega\right) \,=\, \left(\partial_X - T_X\right)\left(\Omega\cdot\nabla u + {}^t\nabla u\cdot\Omega\right) + T_X\left(\Omega\cdot\nabla u + {}^t\nabla u\cdot\Omega\right).$$

Lemma 5.8 says that the operator $\partial_X - T_X$ maps $\mathcal{C}^0_*$ in $\mathcal{C}^{\varepsilon-1}$ continuously: as $L^\infty \hookrightarrow \mathcal{C}^0_*$, one has

$$\left\|\left(\partial_X - T_X\right)\left(\Omega\cdot\nabla u + {}^t\nabla u\cdot\Omega\right)\right\|_{\mathcal{C}^{\varepsilon-1}} \leq C\, \widetilde{\|}X\|_{\mathcal{C}^{\varepsilon}} \left\|\Omega\right\|_{L^\infty} \left\|\nabla u\right\|_{L^\infty} \leq C\, \widetilde{\|}X\|_{\mathcal{C}^{\varepsilon}} \left\|\nabla u\right\|^2_{L^\infty}.$$

To handle the paravector term, we proceed in the following way. First of all, we note that, as $\operatorname{div} u = 0$, we can write

$$\left(\Omega\cdot\nabla u\,+\,{}^t\nabla u\cdot\Omega\right)_{ij}\;=\;\sum_k\left(\partial_i u^k\,\partial_k u^j\,-\,\partial_j u^k\,\partial_k u^i\right)\;=\;\sum_k\left(\partial_k\left(u^j\,\partial_i u^k\right)\,-\,\partial_k\left(u^i\,\partial_j u^k\right)\right).$$

So, we have to estimate the $\mathcal{C}^{\varepsilon-1}$ norm of terms of the type $T_X T_{\nabla u}\nabla u$ and $T_X\nabla R(u,\nabla u)$. Using the same trick as in (5.31) for the remainder terms and applying lemmas 5.9 and 1.29 give us the control of $T_X\left(\Omega\cdot\nabla u\,+\,{}^t\nabla u\cdot\Omega\right)$ in $\mathcal{C}^{\varepsilon-1}$ by the quantity

$$\|X\|_{\mathcal{C}^\varepsilon}\left(\|u\|_{L^p}+\|\nabla u\|_{L^\infty}\right)^2\,+\,\left(\|u\|_{L^p}+\|\nabla u\|_{L^\infty}\right)\|T_X\nabla u\|_{\mathcal{C}^{\varepsilon-1}}\,+\,\|\nabla u\|_{L^\infty}\,\|T_X u\|_{\mathcal{C}^\varepsilon}\,.$$

So, from lemmas 5.8 and 5.15 it easily follows

$$\left\|\partial_X\left(\Omega\cdot\nabla u\,+\,{}^t\nabla u\cdot\Omega\right)\right\|_{\mathcal{C}^{\varepsilon-1}}\;\leq\;C\bigg(\,\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\left(\|u\|_{L^p}+\|\nabla u\|_{L^\infty}\right)^2\,+\,\|\partial_X u\|_{\mathcal{C}^\varepsilon}\left(\|u\|_{L^p}+\|\nabla u\|_{L^\infty}\right)\bigg).\tag{5.42}$$

Now, let us analyse the first term of (5.41). It can be written as the sum of three items:

$$\partial_X\left(\frac{1}{\rho^2}\,\nabla\rho\wedge\nabla\Pi\right)\;=\;-\,\frac{2}{\rho^3}\,(\partial_X\rho)\,(\nabla\rho\wedge\nabla\Pi)\,+\,\frac{1}{\rho^2}\,(\partial_X\nabla\rho)\wedge\nabla\Pi\,+\,\frac{1}{\rho^2}\,\nabla\rho\wedge(\partial_X\nabla\Pi)\,.$$

So, let us consider each one separately and prove that it belongs to the space $\mathcal{C}^{\varepsilon-1}$.

Obviously, from previous estimates we have that first and third terms are actually in L^∞, which is embedded in $\mathcal{C}^{\varepsilon-1}$, and satisfy

$$\begin{aligned}\left\|\frac{1}{\rho^3}\,(\partial_X\rho)\,(\nabla\rho\wedge\nabla\Pi)\right\|_{\mathcal{C}^{\varepsilon-1}}\;&\leq\;\frac{C}{(\rho_*)^3}\,\|\partial_X\rho\|_{\mathcal{C}^\varepsilon}\,\|\nabla\rho\|_{L^\infty}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\\ \left\|\frac{1}{\rho^2}\,\nabla\rho\wedge(\partial_X\nabla\Pi)\right\|_{\mathcal{C}^{\varepsilon-1}}\;&\leq\;\frac{C}{(\rho_*)^2}\,\|\nabla\rho\|_{L^\infty}\,\|\partial_X\nabla\Pi\|_{\mathcal{C}^\varepsilon}\,.\end{aligned}$$

Now, let us find a $\mathcal{C}^{\varepsilon-1}$ control for the second term. Note that it is well-defined, due to the fact that both ρ and $\nabla\Pi$ are in $\mathcal{C}^1_*$ (the product of a $\mathcal{C}^\sigma$ function, $\sigma<0$, with a L^∞ one is not even well-defined). With a little abuse of notation (in the end, we have to deal with the sum of products of components of the two vector-fields), we write

$$(\partial_X\nabla\rho)\,\nabla\Pi\;=\;T_{(\partial_X\nabla\rho)}\nabla\Pi\,+\,T_{\nabla\Pi}\,(\partial_X\nabla\rho)\,+\,R\,(\partial_X\nabla\rho,\nabla\Pi)\;;$$

remembering theorem 1.24 and the embeddings $\mathcal{C}^1_*\hookrightarrow L^\infty\hookrightarrow\mathcal{C}^0_*$, we get

$$\|(\partial_X\nabla\rho)\wedge\nabla\Pi\|_{\mathcal{C}^{\varepsilon-1}}\;\leq\;C\;\|\partial_X\nabla\rho\|_{\mathcal{C}^{\varepsilon-1}}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\,.$$

In the same way, as $\left\|1/\rho^2\right\|_{\mathcal{C}^1_*}\leq\left\|1/\rho^2\right\|_{W^{1,\infty}}$, we get

$$\left\|\frac{1}{\rho^2}\,(\partial_X\nabla\rho)\wedge\nabla\Pi\right\|_{\mathcal{C}^{\varepsilon-1}}\;\leq\;\frac{C}{(\rho_*)^2}\left(1+\frac{\|\nabla\rho\|_{L^\infty}}{\rho_*}\right)\|\partial_X\nabla\rho\|_{\mathcal{C}^{\varepsilon-1}}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\,.$$

So, using also lemma 5.15, we finally obtain, for a constant C depending also on ρ_* and ρ^*,

$$\left\|\partial_X\left(\frac{1}{\rho^2}\,\nabla\rho\wedge\nabla\Pi\right)\right\|_{\mathcal{C}^{\varepsilon-1}}\;\leq\;C\left(\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\,\|\rho\|^2_{W^{1,\infty}}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\,+\,\|\nabla\rho\|_{L^\infty}\,\|\partial_X\nabla\Pi\|_{\mathcal{C}^\varepsilon}\,+\,\|\rho\|_{W^{1,\infty}}\,\|\partial_X\nabla\rho\|_{\mathcal{C}^{\varepsilon-1}}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\right).\tag{5.43}$$

Therefore, from equation (5.41), classical estimates for transport equation in Hölder spaces and inequalities (5.42) and (5.43), we obtain

$$\begin{aligned} \|\partial_X\Omega(t)\|_{\mathcal{C}^{\varepsilon-1}} \;\leq\; & C\,\exp\left(c\int_0^t\|\nabla u\|_{L^\infty}\,d\tau\right)\times \\ & \times\left(\|\partial_{X_0}\Omega_0\|_{\mathcal{C}^{\varepsilon-1}}+\int_0^t e^{-\int_0^t\|\nabla u\|_{L^\infty}d\tau'}\,\Upsilon(\tau)\,d\tau\right), \end{aligned} \tag{5.44}$$

where we have defined

$$\begin{aligned} \Upsilon(t) \;:=\; & \widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\|u\|^2_{\mathcal{L}^{p,\infty}}+\|\partial_X u\|_{\mathcal{C}^\varepsilon}\|u\|_{\mathcal{L}^{p,\infty}}+\|\nabla\rho\|_{L^\infty}\|\partial_X\nabla\Pi\|_{\mathcal{C}^\varepsilon}+ \\ & +\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\|\rho\|^2_{W^{1,\infty}}\|\nabla\Pi\|_{\mathcal{C}^1_*}+\|\rho\|_{W^{1,\infty}}\|\nabla\Pi\|_{\mathcal{C}^1_*}\|\partial_X\nabla\rho\|_{\mathcal{C}^{\varepsilon-1}}\,. \end{aligned} \tag{5.45}$$

5.4.3 Final estimates

First of all, thanks to Young's inequality and estimates (4.56) and (5.18), for all $\eta\in[0,1]$ we have

$$\|\nabla\Pi\|^\eta_{L^2}\|\nabla\Pi\|^{1-\eta}_{L^\infty}\leq\|\nabla\Pi\|_{L^2\cap\mathcal{C}^1_*}\leq C\left(\left(1+\|\nabla\rho\|^\delta_{L^\infty}\right)\|u\|_{L^p}\|\Omega\|_{L^q\cap L^\infty}+\rho^*\|\nabla u\|^2_{L^\infty}\right). \tag{5.46}$$

So, setting

$$L(t)\;:=\;\|u(t)\|_{L^p}\;+\;\|\Omega(t)\|_{L^q\cap L^\infty}\,,$$

putting (5.10) and (5.46) into (5.11), (5.13) and (5.14), for all fixed $T>0$ we obtain, in the time interval $[0,T]$, an inequality of the form

$$L(t)\;\leq\;C\,\exp\left(c\int_0^t\|\nabla u\|_{L^\infty}d\tau\right)\left(L(0)\;+\;\int_0^t\|\nabla u\|^2_{L^\infty}\,d\tau\;+\;\int_0^t L^2(\tau)\,d\tau\right)\,,$$

with constants C, c depending only on N, ε, ρ_* and ρ^*. Now, if we define

$$T\;:=\;\sup\left\{t>0\,\middle|\,\int_0^t\left(e^{-\int_0^\tau L(\tau')d\tau'}L(\tau)+\|\nabla u(\tau)\|^2_{L^\infty}\right)d\tau\;\leq\;2L(0)\right\}\,, \tag{5.47}$$

from previous inequality and Gronwall's lemma and applying a standard bootstrap procedure, we manage to estimate the norms of the solution on $[0,T]$ in terms of initial data only:

$$L(t)\;\leq\;C\,L(0)\qquad\text{and}\qquad\|\rho(t)\|_{W^{1,\infty}}\;\leq\;C\,\|\rho_0\|_{W^{1,\infty}}\,.$$

From this, keeping in mind (5.46) and (5.47), we also have

$$\|\nabla\Pi\|_{L^\infty_t(L^2)\cap L^1_t(\mathcal{C}^1_*)}\;\leq\;C\left(1\;+\;\|\nabla\rho_0\|^\delta_{L^\infty}\right)L^2(0)\,.$$

Now, let us focus on estimates about striated regularity. First of all, from (5.21) we get that the family $X(t)$ remains non-degenerate: $I(X(t))\geq C\,I(X_0)$.

Now, for notation convenience, let us come back to the case of only one vector-field, which we keep to call X, and set $S(t):=\left\|\partial_{X(t)}\Omega(t)\right\|_{\mathcal{C}^{\varepsilon-1}}$. Let us note that the constants C which will occur in our estimates depend on the functional norms of the initial data, but also on the time T.

From (5.23) and (5.22) we find

$$\begin{aligned} \widetilde{\|}X(t)\|_{\mathcal{C}^\varepsilon} \;&\leq\; C\left(\widetilde{\|}X_0\|_{\mathcal{C}^\varepsilon}\;+\;\int_0^t S(\tau)\,d\tau\right) \\ \left\|\partial_{X(t)}u(t)\right\|_{\mathcal{C}^\varepsilon}\;&\leq\;C\left(S(t)\;+\;\widetilde{\|}X(t)\|_{\mathcal{C}^\varepsilon}\,\|\nabla u(t)\|_{L^\infty}\right)\,, \end{aligned}$$

while (5.28) and (5.29) give us

$$\|\partial_X\rho\|_{\mathcal{C}^\varepsilon}\,\leq\,C\qquad\text{and}\qquad\left\|\partial_{X(t)}\nabla\rho(t)\right\|_{\mathcal{C}^{\varepsilon-1}}\,\leq\,C\left(1\,+\,\int_0^t S(\tau)\,d\tau\right).$$

Before going on, let us notice the following fact, which is a direct consequence of the integral condition in (5.47): for $m=1,\,2$ we have

$$\int_0^t\left(\int_0^\tau S(\tau')d\tau'\right)\|\nabla u(\tau)\|_{L^\infty}^m\,d\tau\,\leq\,C\int_0^t S(\tau)\,d\tau\,. \tag{5.48}$$

We will repeatedly use it in what follows.

Now, let us focus on $\partial_X\nabla\Pi$: for convenience, we want to estimate its $L^1_t(\mathcal{C}^\varepsilon)$ norm, starting from the bound (5.40) and the ones we have just found.
First of all, we have

$$\begin{aligned}\int_0^t\|\rho\|_{W^{1,\infty}}\,\|\partial_X\rho\|_{\mathcal{C}^\varepsilon}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\,d\tau\quad&\leq\quad C\\ \int_0^t\|\nabla\Pi\|_{\mathcal{C}^1_*}\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\|\rho\|^2_{W^{1,\infty}}\,d\tau\quad&\leq\quad\|\nabla\Pi\|_{L^1_t(\mathcal{C}^1_*)}\widetilde{\|}X\|_{L^\infty_t(\mathcal{C}^\varepsilon)}\,\leq\,C\left(1+\int_0^t S(\tau)d\tau\right).\end{aligned}$$

Exactly in the same way, using also Jensen's inequality, we get

$$\int_0^t\widetilde{\|}X\|^3_{\mathcal{C}^\varepsilon}\|\nabla\Pi\|_{\mathcal{C}^1_*}\,d\tau\,\leq\,C\|\nabla\Pi\|_{L^1_t(\mathcal{C}^1_*)}\widetilde{\|}X\|^3_{L^\infty_t(\mathcal{C}^\varepsilon)}\,\leq\,C\left(1+\int_0^t S^3(\tau)\,d\tau\right)\,,$$

while, keeping in mind the definition of the $\mathcal{L}^{p,\infty}$ norm (see remark 5.16) and inequality (5.48), we easily find

$$\begin{aligned}\int_0^t\|\rho\|_{W^{1,\infty}}\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\|u\|^2_{\mathcal{L}^{p,\infty}}\,d\tau\quad&\leq\quad C\left(1+\int_0^t S(\tau)\,d\tau\right)\\ \int_0^t\|\rho\|_{W^{1,\infty}}\|u\|_{\mathcal{L}^{p,\infty}}\,\|\partial_X u\|_{\mathcal{C}^\varepsilon}\,d\tau\quad&\leq\quad C\left(1+\int_0^t S(\tau)\,d\tau+\int_0^t\|\nabla u\|_{L^\infty}\,S(\tau)\,d\tau\right).\end{aligned}$$

Therefore, in the end we get

$$\|\partial_X\nabla\Pi\|_{L^1_t(\mathcal{C}^\varepsilon)}\,\leq\,C\left(1\,+\,\int_0^t(1+\|\nabla u\|_{L^\infty})S(\tau)\,d\tau\,+\,\int_0^t S^3(\tau)\,d\tau\right).$$

Finally, let us handle the term $S(t)$: from (5.44), we see that we have to control the L^1_t norm of ι, defined by (5.45). First of all, we have

$$\int_0^t\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\|u\|^2_{\mathcal{L}^{p,\infty}}\,d\tau\,,\qquad\int_0^t\|u\|_{\mathcal{L}^{p,\infty}}\,\|\partial_X u\|_{\mathcal{C}^\varepsilon}\,d\tau\,\leq\,C\left(1+\int_0^t S(\tau)\,d\tau\right)\,:$$

we have just analysed the same items multiplied by $\|\rho\|_{W^{1,\infty}}$, which we controlled by a constant. Moreover, one immediately find

$$\int_0^t\|\nabla\rho\|_{L^\infty}\,\|\partial_X\nabla\Pi\|_{\mathcal{C}^\varepsilon}\,d\tau\,\leq\,C\,\|\partial_X\nabla\Pi\|_{L^1_t(\mathcal{C}^\varepsilon)}\,,$$

while the term $\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\|\rho\|^2_{W^{1,\infty}}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}$ already occurred in considering $\partial_X\nabla\Pi$, and so it can be absorbed in the previous inequality. Finally, we have

$$\int_0^t\|\rho\|_{W^{1,\infty}}\,\|\nabla\Pi\|_{\mathcal{C}^1_*}\,\|\partial_X\nabla\rho\|_{\mathcal{C}^{\varepsilon-1}}\,d\tau\,\leq\,C\left(1+\int_0^t S(\tau)\,d\tau\right).$$

Putting all these inequalitites together, in the end we find the control for $S(t)$ on $[0,T]$:

$$S(t) \,\leq\, C\left(S(0) \,+\, \int_0^t (1+\|\nabla u\|_{L^\infty})S(\tau)d\tau \,+\, \int_0^t S^3(\tau)d\tau\right).$$

Now, suppose that T was chosen so small that, in addition to (5.47), for all $t\in[0,T]$ one has also

$$\int_0^t S^3(\tau)\,d\tau \,\leq\, 2\,S(0)\,. \tag{5.49}$$

Then Gronwall's lemma allows us to get the bound

$$\left\|\partial_{X(t)}\Omega(t)\right\|_{\mathcal{C}^{\varepsilon-1}} \,\leq\, C\,S(0) \qquad \forall\, t\in[0,T]\,,$$

for a constant C depending only on T, N, p, q, ε, ρ_* and ρ^* and on the norms of initial data in the relative functional spaces.

Let us note that this inequality allows us to recover a uniform bound, on $[0,T]$, for $\|\nabla u\|_{L^\infty}$ and $\|\nabla\Pi\|_{\mathcal{C}^1_*}$, which we previously controlled only in L^1_t.

Remark 5.18. The lifespan T of the solution is essentially determined by conditions (5.47) and (5.49). In section 4.5 we will establish an explicit lower bound for T in terms of the norms of initial data only and we will compare it with the classical result in the case of constant density.

5.4.4 Proof of the existence of a solution

After establishing a priori estimates, we want to give the proof of the existence of a solution for system (5.1) under our assumptions.

We will get it in a classical way: first of all, we will construct a sequence of approximate solutions to our problem, for which a priori estimates of the previous section hold uniformly, and then we will show the convergence of such a sequence to a solution of (5.1).

Now, we will work only for positive times, but it goes without saying that the same argument holds true also for negative times evolution.

Construction of a sequence of approximate solutions

For each $n\in\mathbb{N}$, let us define $u_0^n\,:=\,S_n u_0$; obviously $u_0^n\in L^p$, and an easy computation shows that it belongs also to the space $B^\sigma_{p,r}$ for all $\sigma\in\mathbb{R}$ and all $r\in[1,+\infty]$. Let us notice that $\bigcap_\sigma B^\sigma_{p,r}\subset\mathcal{C}^\infty_b$, so in particular we have that $u_0^n\in L^p\cap B^s_{\infty,r}$, for some fixed $s>1$ and $r\in[1,+\infty]$ such that $B^s_{\infty,r}\hookrightarrow\mathcal{C}^{0,1}$.

Keeping in mind that $[S_n,\nabla]=0$, we have that $\Omega_0^n\,=\,S_n\Omega_0\in L^q\cap B^{s-1}_{\infty,r}$; in particular, from (5.5) we get $\nabla u_0^n\in L^q$.

Now let us take an even radial function $\theta\in\mathcal{C}^\infty_0(\mathbb{R}^N)$, supported in the unitary ball, such that $0\leq\theta\leq1$ and $\int_{\mathbb{R}^N}\theta(x)\,dx=1$, and set $\theta_n(x)=n^N\,\theta(nx)$ for all $n\in\mathbb{N}$. We define $\rho_0^n\,:=\,\theta_n*\rho_0$: it belongs to $B^s_{\infty,r}$ and it satisfies the bounds $0<\rho_*\leq\rho_0^n\leq\rho^*$.

Moreover, by properties of localisation operators S_n and of θ_n, we also have:

- $\rho_0^n\rightharpoonup\rho_0$ in $W^{1,\infty}$ and $\|\nabla\rho_0^n\|_{L^\infty}\,\leq\,c\,\|\nabla\rho_0\|_{L^\infty}$;
- $u_0^n\to u_0$ in the space L^p and $\|u_0^n\|_{L^p}\,\leq\,c\,\|u_0\|_{L^p}$;
- $\Omega_0^n\to\Omega_0$ in L^q and $\|\Omega_0^n\|_{L^q}\,\leq\,c\,\|\Omega_0\|_{L^q}$, $\|\Omega_0^n\|_{L^\infty}\,\leq\,c\,\|\Omega_0\|_{L^\infty}$.

So, for each n, theorem 3 and remark 4 of [29] give us a unique solution of (5.1) such that:

(i) $\rho^n\in\mathcal{C}([0,T^n];B^s_{\infty,r})$, with $0<\rho_*\leq\rho^n\leq\rho^*$;

(ii) $u^n \in \mathcal{C}([0,T^n];L^p \cap B^s_{\infty,r})$, with $\Omega^n \in \mathcal{C}([0,T^n];L^q \cap B^{s-1}_{\infty,r})$;

(iii) $\nabla\Pi^n \in \mathcal{C}([0,T^n];L^2) \cap L^1([0,T^n];B^s_{\infty,r})$.

For such a solution, a priori estimates of the previous section hold at every step n. Moreover, remembering previous properties about approximated initial data and that the function $y \mapsto y\log\left(e+\frac{c}{y}\right)$ is nondecreasing, we can find a control independent of $n \in \mathbb{N}$. So, we can find a positive time $T \leq T^n$ for all $n \in \mathbb{N}$, such that in $[0,T]$ approximate solutions are all defined for every n and satisfy uniform bounds.

Convergence of the sequence of approximate solutions

To prove convergence of the obtained sequence, we appeal to a compactness argument. Actually, we weren't able to apply the classical method used for the homogeneous case, i.e. proving estimates in rough spaces as $\mathcal{C}^{-\alpha}$ ($\alpha > 0$): we couldn't solve the elliptic equation for the pressure term in this framework.

We know that $(\rho^n)_{n\in\mathbb{N}} \subset L^\infty([0,T];W^{1,\infty})$, $(u^n)_{n\in\mathbb{N}} \subset L^\infty([0,T];L^p)$ and $(\nabla\Pi^n)_{n\in\mathbb{N}} \subset L^\infty([0,T];L^2)$ and, thanks to a priori estimates, all these sequences are bounded in the respective functional spaces.

Due to the reflexivity of L^2 and L^p and seeing L^∞ as the dual of L^1, we obtain the existence of functions ρ, u and $\nabla\Pi$ such that (up to a subsequence)

- $\rho^n \overset{*}{\rightharpoonup} \rho$ in the space $L^\infty([0,T];W^{1,\infty})$,
- $u^n \rightharpoonup u$ in $L^\infty([0,T];L^p)$ and
- $\nabla\Pi^n \rightharpoonup \nabla\Pi$ in $L^\infty([0,T];L^2)$.

Nevertheless, we are not able to prove that $(\rho,u,\nabla\Pi)$ is indeed a solution of system (5.1): passing to the limit in nonlinear terms requires strong convergence in (even rough) suitable functional spaces. So let us argue in a different way and establish strong convergence properties, which will be useful also to prove preservation of striated regularity.

First of all, let us recall that, by construction, $u_0^n \to u_0$ in L^p and $\Omega_0^n \to \Omega_0$ in L^q, and $(\rho_0^n)_n$ is bounded in $W^{1,\infty}$. So, for $\alpha > 0$ big enough (for instance, take $\alpha = \max\{N/p\,,N/q\}$), we have that $(\rho_0^n)_n$, $(u_0^n)_n$, $(\Omega_0^n)_n$ are all bounded in the space $\mathcal{C}^{-\alpha}$.

Remark 5.19. It goes without saying that the sequences $(u_0^n)_n$ and $(\Omega_0^n)_n$ still converge in $\mathcal{C}^{-\alpha}$; moreover, also $\rho_0^n \to \rho_0$ in this space. Remember that ρ_0 belongs to the space $\mathcal{C}^1_*$, which coincides (see [13] for the proof) with the Zygmund class, i.e. the set of bounded functions f for which there exists a constant Z_f such that

$$|f(x+y) + f(x-y) - 2f(x)| \leq Z_f\,|y|$$

for all x, $y \in \mathbb{R}^N$. So, using the symmetry of θ, we can write

$$\rho_0^n(x) - \rho_0(x) = \frac{1}{2}n^N\int_{\mathbb{R}^N}\theta(ny)\,(\rho_0(x+y) + \rho_0(x-y) - 2\rho_0(x))\,dy\,;$$

from this identity we get that $\rho_0^n \to \rho_0$ in L^∞, and hence also in $\mathcal{C}^{-\alpha}$.

Now, let us consider the equation for ρ^n:

$$\partial_t\rho^n = -u^n\cdot\nabla\rho^n\,.$$

From a priori estimates we get that $(u^n)_n$ is bounded in $L^\infty([0,T];\mathcal{C}^1_*)$ and $(\nabla\rho^n)_n$ is bounded in the space $L^\infty([0,T];L^\infty)$; so, from the properties of paraproduct and remainder operators, one has that the sequence $(\partial_t\rho^n)_n$ is bounded in $L^\infty([0,T];\mathcal{C}^{-\alpha})$. Therefore $(\rho^n)_n$ is bounded in $\mathcal{C}^{0,1}([0,T];\mathcal{C}^{-\alpha})$, and in particular uniformly equicontinuous in the time variable.

Now, up to multiply by a $\varphi\in\mathcal{D}(\mathbb{R}^N)$ (recall theorem 2.94 of [2]) and extract a subsequence, Ascoli-Arzelà theorem and Cantor diagonal process ensure us that $\rho^n\to\rho$ in the space $\mathcal{C}([0,T];\mathcal{C}^{-\alpha}_{loc})$.

Exactly in the same way, one can show that $(\rho^n)_n$ is bounded in $\mathcal{C}_b([0,T]\times\mathbb{R}^N)$ and it converges to ρ in this space.

Finally, remember that $\rho\in L^\infty([0,T];W^{1,\infty})$, by compacteness of bounded sets of this space for the weak-$*$ topology. Therefore, by interpolation one can recover convergence also in $L^\infty([0,T];\mathcal{C}^{1-\eta}_{loc})$ for all $\eta>0$.

We repeat the same argument for the velocity field. For all n, we have

$$\partial_t u^n \,=\, -\,u^n\cdot\nabla u^n\,-\,a^n\,\nabla\Pi^n\,,$$

where we have set $a^n:=(\rho^n)^{-1}$. Let us notice that, as ρ_0, $a_0:=(\rho_0)^{-1}$ satisfy the same hypothesis and a^n, ρ^n satisfy the same equations, they enjoy also the same properties.

Keeping this fact in mind, let us consider each term separately.

- Thanks to what we have just said, $(a^n)_n$ is a bounded sequence in $\mathcal{C}_b([0,T]\times\mathbb{R}^N)\cap L^\infty([0,T];\mathcal{C}^1_*)$. Moreover, from a priori estimates, we see that also $(\nabla\Pi^n)_n$ is bounded in the space $L^\infty([0,T];\mathcal{C}^1_*)$. Therefore, it follows that the sequence $(a^n\,\nabla\Pi^n)_n$ is bounded in $L^\infty([0,T];\mathcal{C}^{-\alpha})$.

- In the same way, as $(u^n)_n\subset L^\infty([0,T];\mathcal{C}^1_*)$ and $(\nabla u^n)_n\subset L^\infty([0,T];L^\infty)$ are both bounded sequences, one has that the sequence $(u^n\cdot\nabla u^n)_n$ is bounded in $L^\infty([0,T];\mathcal{C}^{-\alpha})$.

Therefore, exactly as done for the density, we get that $(u^n)_n$ is bounded in $\mathcal{C}^{0,1}([0,T];\mathcal{C}^{-\alpha})$, so uniformly equicontinuous in the time variable. This fact implies that $u^n\to u$ in $\mathcal{C}([0,T];\mathcal{C}^{-\alpha}_{loc})$.

Finally, thanks to uniform bounds and Fatou's property of Besov spaces, we have that $u\in L^\infty([0,T];\mathcal{C}^1_*)$ and, by interpolation, that $u^n\to u$ in $\mathcal{C}([0,T];\mathcal{C}^{1-\eta}_{loc})$ for all $\eta>0$.

So, thanks to strong convergence properties, if we test the equations on a $\varphi\in\mathcal{C}^1([0,T];\mathcal{S}(\mathbb{R}^N))$ (here we have set $\mathcal{S}$ to be the Schwartz class), we can pass to the limit and get that $(\rho,u,\nabla\Pi)$ is indeed a solution to the Euler system (5.1).

Before going on with the striated regularity, let us establish continuity properties of the solutions with respect to the time variable.

First of all, from

$$\partial_t\rho\,=\,-\,u\cdot\nabla\rho\,,$$

as $u\in\mathcal{C}([0,T];L^\infty)$ (from the properties of convergence stated before) and $\nabla\rho\in L^\infty([0,T];L^\infty)$, we obtain that $\rho\in\mathcal{C}^{0,1}([0,T];L^\infty)$, and the same holds for $a:=\rho^{-1}$.

Remember that $u\in L^\infty([0,T];L^p)$, ∇u and $a\in L^\infty([0,T];L^\infty)$. Moreover, as $\nabla\Pi\in L^\infty([0,T];L^2)\cap L^\infty([0,T];L^\infty)$, it belongs also to $L^\infty([0,T];L^p)$. So, from the equation

$$\partial_t u\,=\,-\,u\cdot\nabla u\,-\,a\,\nabla\Pi\,,$$

we get that $\partial_t u\in L^\infty([0,T];L^p)$, which implies $u\in\mathcal{C}^{0,1}([0,T];L^p)$.

In the same way, from (5.6) we get that $\Omega\in\mathcal{C}([0,T];L^q)$, and therefore the same holds true also for ∇u.

Now, using elliptic equation (5.15) and keeping in mind the properties just proved for ρ and a, one can see that $\nabla\Pi\in\mathcal{C}([0,T];L^2)$. So, coming back to the previous equation, we discover that also $\partial_t u$ belongs to the same space.

Final checking about striated regularity

It remains us to prove that also properties of striated regularity are preserved in passing to the limit. For doing this, we will follow the outline of the proof in [25].

1. *Convergence of the flow*

   Let ψ^n and ψ be the flows associated respectively to u^n and u; for all fixed $\varphi \in \mathcal{D}(\mathbb{R}^N)$, by definition we have:

$$
\begin{aligned}
|\varphi(x)\,(\psi^n(t,x)-\psi(t,x))| &\leq \int_0^t |\varphi(x)\,(u^n(\tau,\psi^n(\tau,x))-u(\tau,\psi(\tau,x)))|\,d\tau \\
&\leq \int_0^t |\varphi(x)\,(u^n-u)\,(\tau,\psi^n(\tau,x))| + \\
&\qquad + |\varphi(x)u^n(\tau,\psi^n(\tau,x))-\varphi(x)u^n(\tau,\psi(\tau,x))|\,d\tau \\
&\leq \int_0^t \|\nabla u^n\|_{L^\infty}\,|\varphi(x)\,(\psi^n-\psi)\,(\tau,x)|\,d\tau\; + \\
&\qquad + \int_0^t \|\varphi u^n - \varphi u\|_{L^\infty}\;d\tau\,.
\end{aligned}
$$

   So, from convergence properties stated in previous part, we have that $\psi^n \to \psi$ in the space $L^\infty([0,T];Id+L^\infty_{loc})$. Moreover, it's easy to see that

$$
\|\nabla\psi^n(t)\|_{L^\infty} \leq C\,\exp\left(\int_0^t \|\nabla u^n\|_{L^\infty}\;d\tau\right)\,,
$$

   which tells us that the sequence $(\psi^n)_n$ is bounded in $L^\infty([0,T];Id+\mathcal{C}^{0,1})$. Hence, finally we discover that $\psi^n \to \psi$ also in the spaces $L^\infty([0,T];Id+\mathcal{C}^{1-\eta}_{loc})$ for all $\eta>0$.

2. *Regularity of $\partial_{X_0}\psi$*

   First of all, let us notice that, by definition,

$$
\partial_{X_0(x)}\psi^n(t,x) \;=\; X^n_t(\psi^n(t,x))\,;
$$

   applying proposition 5.6, we get

$$
\|\partial_{X_0}\psi^n_t\|_{\mathcal{C}^\varepsilon} \;=\; \|X^n_t\circ\psi^n_t\|_{\mathcal{C}^\varepsilon} \;\leq\; c\;\|\nabla\psi^n_t\|_{L^\infty}\;\|X^n_t\|_{\mathcal{C}^\varepsilon}\,, \tag{5.50}
$$

   which implies that $(\partial_{X_0}\psi^n)_n$ is bounded in the space $L^\infty([0,T];\mathcal{C}^\varepsilon)$. Now we note that, for every fixed $\varphi\in\mathcal{D}(\mathbb{R}^N)$, we have

$$
\varphi\,\partial_{X_0}\psi^n \;-\; \varphi\,\partial_{X_0}\psi \;=\; \partial_{X_0}(\varphi\psi^n-\varphi\psi) \;-\; (\partial_{X_0}\varphi)\,(\psi^n-\psi)\;;
$$

   the second term is compactly supported, hence it converges in L^∞ because of what we have already proved. So let us focus on the first one and consider the difference

$$
\partial_{X_0}(\varphi\,\psi^n)-\partial_{X_0}(\varphi\,\psi) \;=\; \mathrm{div}\;(X_0\otimes\varphi(\psi^n-\psi)) \;-\; \varphi(\psi^n-\psi)\;\mathrm{div}\,X_0\,;
$$

   decomposing both terms in paraproduct and remainder and remembering hypothesis over X_0, it's easy to see that

$$
\|\partial_{X_0}(\varphi\,\psi^n)-\partial_{X_0}(\varphi\,\psi)\|_{\mathcal{C}^{\varepsilon-1}} \;\leq\; c\;\|\varphi\,\psi^n-\varphi\,\psi\|_{\mathcal{C}^\varepsilon}\;\widetilde{\|}X_0\|_{\mathcal{C}^\varepsilon}\,.
$$

   Therefore, from what we have just proved, $\partial_{X_0}\psi^n \to \partial_{X_0}\psi$ in $L^\infty([0,T];\mathcal{C}^{\varepsilon-1}_{loc})$; moreover, by Fatou's property, one gets that $\partial_{X_0}\psi\in L^\infty([0,T];\mathcal{C}^\varepsilon)$ and it verifies estimate (5.50). So, by interpolation, convergence occurs also in $L^\infty([0,T];\mathcal{C}^{\varepsilon-\eta}_{loc})$ for all $\eta>0$.

3. *Regularity of X_t*

   Remembering the definitions

$$\begin{aligned} X_t(x) &:= \left(\partial_{X_0(x)}\psi\right)(t, \psi_t^{-1}(x)) \\ \operatorname{div} X_t &= \operatorname{div} X_0 \circ \psi_t^{-1}\,, \end{aligned}$$

   from proposition 5.6 it immediately follows that X_t and $\operatorname{div} X_t$ both belong to $\mathcal{C}^\varepsilon$. Moreover, the same proposition implies that $X^n \to X$ in the space $L^\infty([0,T];\mathcal{C}^{\varepsilon-\eta}_{loc})$ for all $\eta > 0$, and the same holds for the divergence. In particular, we have convergence also in $L^\infty([0,T];L^\infty_{loc})$, which finally tells us that X_t remains non-degenerate for all $t \in [0,T]$, i.e. $I(X_t) \geq c\, I(X_0)$.

4. *Striated regularity for the density and the vorticity*

   Let us first prove that regularity of the density with respect to the vector field X_t is preserved during the time evolution. To simplify the presentation, we will omit the localisation by $\varphi \in \mathcal{D}(\mathbb{R}^N)$: formally, we should repeat the same reasoning applied to prove regularity of $\partial_{X_0}\psi$. So, let us consider

$$\partial_{X^n}\rho^n - \partial_X\rho \,=\, \operatorname{div}\left(\rho^n\,(X^n - X)\right) - \rho^n \operatorname{div}(X^n - X) + \operatorname{div}\left((\rho^n - \rho)\,X\right) - (\rho^n - \rho)\operatorname{div} X$$

   and prove the convergence in $L^\infty([0,T];\mathcal{C}^{-1}_{loc})$. Using Bony's paraproduct decomposition, it's not difficult to see that first and third terms can be bounded by $\|\rho^n\|_{L^\infty}\,\|X^n - X\|_{L^\infty} + \|\rho^n - \rho\|_{L^\infty}\,\|X\|_{L^\infty}$, while second and last terms can be controlled by $\|\rho^n\|_{L^\infty}\,\|\operatorname{div}(X^n - X)\|_{\mathcal{C}^{\varepsilon/2}} + \|\rho^n - \rho\|_{L^\infty}\,\|\operatorname{div} X\|_{\mathcal{C}^{\varepsilon/2}}$, for instance. So, from the convergence properties stated for $(\rho^n)_n$ and $(X^n)_n$, we get that $\partial_{X^n}\rho^n \to \partial_X\rho$ in the space $L^\infty([0,T];\mathcal{C}^{-1}_{loc})$, as claimed. Moreover, from a priori bounds and Fatou's property of Besov spaces, we have that $\partial_X\rho \in L^\infty([0,T];\mathcal{C}^\varepsilon)$ and so, by interpolation, convergence occurs also in $L^\infty([0,T];\mathcal{C}^{\varepsilon-\eta}_{loc})$ for all $\eta > 0$.

   Now we consider the vorticity term (again, we omit the multiplication by a $\mathcal{D}(\mathbb{R}^N)$ function):

$$\begin{aligned} \partial_{X^n}\Omega^n \,-\, \partial_X\Omega \;&=\; \operatorname{div}\left((X^n - X)\otimes\Omega^n\right) \,-\, \Omega^n \operatorname{div}(X^n - X) \,+ \\ &\qquad + \operatorname{div}\left(X\otimes(\Omega^n - \Omega)\right) \,-\, (\Omega^n - \Omega)\operatorname{div} X\,. \end{aligned}$$

   From the convergence properties of $(u^n)_n$, we get that $\Omega^n \to \Omega$ in $L^\infty([0,T];\mathcal{C}^{-\eta}_{loc})$ for all $\eta > 0$, so for instance also for $\eta = \varepsilon/2$. From this, using again paraproduct decomposition as done before, one can prove that $\partial_{X^n}\Omega^n \to \partial_X\Omega$ in $L^\infty([0,T];\mathcal{C}^{-1-\varepsilon/2}_{loc})$. Therefore, as usual from a priori estimates and Fatou's property of Besov spaces, we have that $\partial_X\Omega \in L^\infty([0,T];\mathcal{C}^{\varepsilon-1})$, and moreover convergence remains true (by interpolation) in spaces $L^\infty([0,T];\mathcal{C}^{\varepsilon-1-\eta}_{loc})$ for all $\eta > 0$.

So, all the properties linked to striated regularity are now verified, and this concludes the proof of the existence part of theorem 5.3.

5.4.5 Uniqueness

Let us spend a few words on proof of uniqueness: it is an immediate consequence of the following stability result.

Proposition 5.20. *Let $\left(\rho^1, u^1, \nabla\Pi^1\right)$ and $\left(\rho^2, u^2, \nabla\Pi^2\right)$ be solutions of system* (5.1) *with*

$$0 \,<\, \rho_* \,\leq\, \rho^1\,,\,\rho^2 \,\leq\, \rho^*\,.$$

Let us suppose that $\delta\rho := \rho^1 - \rho^2 \in \mathcal{C}([0,T];L^2)$ and that $\delta u := u^1 - u^2 \in \mathcal{C}^1([0,T];L^2)$. Finally, assume that $\nabla\rho^2$, ∇u^1, ∇u^2 and $\nabla\Pi^2$ all belong to $L^1([0,T];L^\infty)$.

Then, for all $t \in [0,T]$, we have the following estimate:

$$\|\delta\rho(t)\|_{L^2} + \|\delta u(t)\|_{L^2} \leq C\, e^{c\,I(t)} \left(\|\delta\rho(0)\|_{L^2} + \|\delta u(0)\|_{L^2}\right),$$

where we have defined

$$I(t) := \int_0^t \left(\left\|\nabla\rho^2\right\|_{L^\infty} + \left\|\nabla u^1\right\|_{L^\infty} + \left\|\nabla u^2\right\|_{L^\infty} + \left\|\nabla\Pi^2\right\|_{L^\infty}\right) d\tau\,.$$

Proof. From $\partial_t\delta\rho + u^1\cdot\nabla\delta\rho = -\,\delta u\cdot\nabla\rho^2$, we immediately get

$$\|\delta\rho(t)\|_{L^2} \leq \|\delta\rho(0)\|_{L^2} + \int_0^t \|\delta u\|_{L^2} \left\|\nabla\rho^2\right\|_{L^\infty} d\tau\,.$$

Moreover, the equation for δu reads as follows:

$$\partial_t\delta u + u^1\cdot\nabla\delta u = -\,\delta u\cdot\nabla u^2 - \frac{\nabla\delta\Pi}{\rho^1} + \frac{\nabla\Pi^2}{\rho^1\,\rho^2}\,\delta\rho\,,$$

where we have set $\delta\Pi = \Pi^1 - \Pi^2$. So, from standard L^p estimates for transport equations, one infers that

$$\|\delta u(t)\|_{L^2} \leq \|\delta u(0)\|_{L^2} + C\int_0^t \left(\|\delta u\|_{L^2}\left\|\nabla u^2\right\|_{L^\infty} + \|\nabla\delta\Pi\|_{L^2} + \left\|\nabla\Pi^2\right\|_{L^\infty}\|\delta\rho\|_{L^2}\right) d\tau\,.$$

Now, in order to get bounds for $\nabla\delta\Pi$, we analyse its equation:

$$\begin{aligned}
-\operatorname{div}\left(\frac{\nabla\delta\Pi}{\rho^1}\right) &= \operatorname{div}\left(-\frac{\nabla\Pi^2}{\rho^1\,\rho^2}\,\delta\rho + u^1\cdot\nabla\delta u + \delta u\cdot\nabla u^2\right) \\
&= \operatorname{div}\left(-\frac{\nabla\Pi^2}{\rho^1\,\rho^2}\,\delta\rho + \delta u\cdot\left(\nabla u^1 + \nabla u^2\right)\right),
\end{aligned}$$

where, to get the second equality, we have used the algebraic identity

$$\operatorname{div}(v\cdot\nabla w) = \operatorname{div}(w\cdot\nabla v) + \operatorname{div}(v\,\operatorname{div} w) - \operatorname{div}(w\,\operatorname{div} v)\,.$$

So, from lemma 4.8 we obtain

$$\|\nabla\delta\Pi\|_{L^2} \leq C\left(\left\|\nabla\Pi^2\right\|_{L^\infty}\|\delta\rho\|_{L^2} + \|\delta u\|_{L^2}\left(\left\|\nabla u^1\right\|_{L^\infty} + \left\|\nabla u^2\right\|_{L^\infty}\right)\right),$$

and Gronwall's inequality completes the proof of the proposition. □

Now, let us prove uniqueness: let $\left(\rho^1, u^1, \nabla\Pi^1\right)$ and $\left(\rho^2, u^2, \nabla\Pi^2\right)$ satisfy system (5.1) with same initial data (ρ_0, u_0), under hypothesis of theorem 5.3.

As $\delta u(0) = 0$ and $u \in \mathcal{C}([0,T];L^p)$, $\nabla u \in \mathcal{C}([0,T];L^q)$, one easily gets that $\delta u \in \mathcal{C}^1([0,T];L^2)$. Moreover, from this fact, observing that also $\delta\rho(0) = 0$, the equation for $\delta\rho$ tells us that $\delta\rho \in \mathcal{C}([0,T];L^2)$. Hence proposition 5.20 can be applied and uniqueness immediately follows.

5.5 On the lifespan of the solution

The aim of this section is to establish, in the most accurate way, an explicit lower bound for the lifespan of the solution of system (4.1) in terms of initial data only.

For notation convenience, let us define

$$L_0 := \|u_0\|_{L^p} + \|\Omega_0\|_{L^q\cap L^\infty} \qquad \text{and} \qquad A_0 := \|\nabla\rho_0\|_{L^\infty}\,.$$

Theorem 5.21. *Under the hypothesis of theorem 5.3, the lifespan T of a solution to system (4.1) with initial data (ρ_0, u_0) is bounded from below, up to multiplication by a constant (depending only on the space dimension N, ε, p, q, ρ_* and ρ^*), by the quantity*

$$\frac{\min\left\{L_0, \|\Omega_0\|_{\mathcal{C}^\varepsilon_{X_0}}\right\} \times \left(L_0 \log\left(e + \frac{\|\Omega_0\|_{\mathcal{C}^\varepsilon_{X_0}}}{L_0}\right)\right)^{-1}}{\left(1 + L_0 + \|\Omega_0\|_{\mathcal{C}^\varepsilon_{X_0}}\right)^2 \left(1 + A_0^{\delta+3}\right)\left(1 + \widetilde{|||} X_0 |||^3_{\mathcal{C}^\varepsilon} + \|\partial_X \nabla \rho_0\|_{\mathcal{C}^\varepsilon_{X_0}}\right)}\,, \tag{5.51}$$

where $\delta > 1$ is the exponent which occurs in (5.18).

Proof. Our starting point is subsection 5.4.3. With the same notations, let us also define the following quantities:

$$\Theta(t) := L(t)\, \log\left(e + \frac{S(t)}{L(t)}\right)\,,\quad U(t) := \int_0^t \|\nabla u(\tau)\|_{L^\infty}\, d\tau\,,$$
$$A(t) := \|\nabla\rho(t)\|_{L^\infty}\,,\quad \Gamma(t) := \widetilde{\|} X(t)\|_{\mathcal{C}^\varepsilon}\,,\quad R(t) := \left\|\partial_{X(t)}\nabla\rho(t)\right\|_{\mathcal{C}^{\varepsilon-1}}\,.$$

It's only matter of repeating previous computations in a more accurate way.

Let us notice that, from inequality (5.12), for all time t one has

$$L(t)\,,\, U'(t) \;=\; \|\nabla u(t)\|_{L^\infty} \;\leq\; C\,\Theta(t)\;: \tag{5.52}$$

we will make a broad use of these facts.

Now, let us define the time $T_1 := \sup\{t > 0 \,|\, U(t) \leq \log 2\}$. Then, on $[0, T_1]$ we have (from (5.9) and (5.10))

$$A(t) \;\leq\; C\, A_0 \qquad \text{and} \qquad \|\rho(t)\|_{W^{1,\infty}} \;\leq\; C\,\|\rho_0\|_{W^{1,\infty}}\,,$$

and so, keeping in mind (5.11), (5.13), (5.14) and (5.46), we get also

$$\begin{aligned} L(t) \;&\leq\; C\left(L(0) + \left(1 + A_0^{\delta+1}\right)\int_0^t \Theta^2(\tau)\, d\tau\right) \\ \|\nabla\Pi\|_{L^2\cap\mathcal{C}^1_*} \;&\leq\; C\left(1 + A_0^{\delta+1}\right)\,\Theta^2(t)\,. \end{aligned} \tag{5.53}$$

In addition, (5.21) implies $I(X(t)) \geq C I(X_0)$, while from (5.23) and (5.22) it follows

$$\Gamma(t) \;\leq\; C\left(\Gamma_0 + \int_0^t S(\tau) d\tau\right) \qquad \text{and} \qquad \|\partial_X u(t)\|_{\mathcal{C}^\varepsilon} \;\leq\; C\Big(S(t) \;+\; \Gamma(t)\,\Theta(t)\Big).$$

Finally, (5.28) and (5.29) together entail

$$\begin{aligned} \|\partial_X\rho\|_{\mathcal{C}^\varepsilon} \;&\leq\; C\,\left((1 + A_0)\Gamma_0 + R_0\right) \\ \|\partial_X\nabla\rho\|_{\mathcal{C}^{\varepsilon-1}} \;&\leq\; C\left((1 + A_0)\Gamma_0 + R_0 + \int_0^t S(\tau)d\tau\right)\,. \end{aligned}$$

From the inequalities we've just established, the control of the striated norm of $\nabla\Pi$ immediately follows.

Let us proceed carefully, as done in subsection 5.4.3. After some simply (even if rough) manipulations, we get (up to multiplication by constant terms)

$$\begin{aligned} \|\rho\|_{W^{1,\infty}}\, \|\partial_X\rho\|_{\mathcal{C}^\varepsilon}\, \|\nabla\Pi\|_{\mathcal{C}^1_*} \;&\leq\; \left(1 + A_0^{\delta+2}\right)(\Gamma_0 + R_0)\,\Theta^2(t) \\ \|\nabla\Pi\|_{\mathcal{C}^1_*}\widetilde{\|} X\|_{\mathcal{C}^\varepsilon}\|\rho\|^2_{W^{1,\infty}} \;&\leq\; \left(1 + A_0^{\delta+2}\right)\left(\Gamma_0 + \int_0^t S(\tau)d\tau\right)\Theta^2(t)\,. \end{aligned}$$

Now, thanks to $(a+b)^3 \leq C(a^3+b^3)$ and Jensen's inequality we infer

$$\widetilde{\|}X\|^3_{\mathcal{C}^\varepsilon}\|\nabla\Pi\|_{\mathcal{C}^1_*} \leq \left(\Gamma_0 + \int_0^t S^3(\tau)d\tau\right)\left(1+A_0^\delta\right)\Theta^2(t)\,.$$

Finally, the fact that $\|u(t)\|_{\mathcal{L}^{p,\infty}} \leq \Theta(t)$ implies

$$\|\rho\|_{W^{1,\infty}}\widetilde{\|}X\|_{\mathcal{C}^\varepsilon}\|u\|^2_{\mathcal{L}^{p,\infty}} \leq (1+A_0)\left(\Gamma_0+\int_0^t S(\tau)d\tau\right)\Theta^2(t)$$

$$\begin{aligned}\|\rho\|_{W^{1,\infty}}\|u\|_{\mathcal{L}^{p,\infty}}\|\partial_X u\|_{\mathcal{C}^\varepsilon} &\leq (1+A_0)\,\Theta(t)\left(S(t)+\left(\Gamma_0+\int_0^t S(\tau)\,d\tau\right)\Theta(t)\right)\\ &\leq (1+A_0)(1+\Gamma_0)\Big(S(t)+\Theta(t)\Big)\Theta(t)+(1+A_0)\Theta^2(t)\int_0^t S(\tau)d\tau\,.\end{aligned}$$

Let us define

$$M_0 := \left(1 + A_0^{\delta+2}\right)\left(1+\Gamma_0^3+R_0\right);$$

as $\int S \leq 1+\int S^3$, in the end we get

$$\left\|\partial_{X(t)}\nabla\Pi(t)\right\|_{\mathcal{C}^\varepsilon} \leq C\,M_0\left(\Theta^2(t)\left(1+\int_0^t S^3(\tau)d\tau\right)+\Theta(t)S(t)\right). \tag{5.54}$$

Now let us focus on the striated norm of the vorticity, estimated in (5.44). Analysing each term which occurs in the definition (5.45) of Υ, we see that first, second and fourth items can be bounded by $\left\|\partial_{X(t)}\nabla\Pi(t)\right\|_{\mathcal{C}^\varepsilon}$, and the third one is controlled as in (5.54), up to replace M_0 by $\widetilde{M}_0 := (1+A_0)M_0$. Finally,

$$\begin{aligned}&\|\rho\|_{W^{1,\infty}}\|\nabla\Pi\|_{\mathcal{C}^1_*}\|\partial_X\nabla\rho\|_{\mathcal{C}^{\varepsilon-1}} \leq\\ &\qquad\leq (1+A_0)\left(1+A_0^\delta\right)\Theta^2(t)\left((1+A_0)\Gamma_0+R_0+\int_0^t S(\tau)d\tau\right)\\ &\qquad\leq \left(1+A_0^{\delta+2}\right)(1+\Gamma_0+R_0)\,\Theta^2(t)\left(1+\int_0^t S(\tau)d\tau\right).\end{aligned}$$

So, putting all these inequalitites together, we discover that, in $[0,T_1]$,

$$S(t) \leq C\left(S_0+\widetilde{M}_0\int_0^t\left(\Theta^2(\tau)\left(1+\int_0^\tau S^3(\tau')d\tau'\right)+\Theta(\tau)S(\tau)\right)d\tau\right),$$

and, in the end, by use of Gronwall's lemma this implies

$$S(t) \leq C\,e^{c\int_0^t\Theta(\tau)d\tau}\left(S_0+\widetilde{M}_0\int_0^t\Theta^2(\tau)\left(1+\int_0^\tau S^3(\tau')d\tau'\right)d\tau\right).$$

Define T_2 as the supremum of the times $t>0$ for which both the relations

$$\widetilde{M}_0\int_0^t\Theta^2(\tau)d\tau \leq 2L_0 \quad\text{and}\quad \widetilde{M}_0\int_0^t\Theta^2(\tau)\left(1+\int_0^\tau S^3(\tau')d\tau'\right)d\tau \leq 2S_0$$

are fulfilled. Note that, by Cauchy-Schwarz inequality, this implies in particular $\int_0^t\Theta(\tau)d\tau \leq C$: so, thanks to (5.52), we can suppose $T_2 \leq T_1$. Hence, keeping in mind (5.53), in $[0,T_2]$ one has

$$S(t) \leq C_1\,S_0\,,\qquad L(t) \leq C_2\,L_0 \qquad\text{and}\qquad \Theta(t) \leq C_3\,\Theta_0\,,$$

because the function $(\lambda,\sigma)\mapsto\lambda\log(e+\sigma/\lambda)$ is increasing both in λ and σ.

Let us put these bounds in the integral conditions defining T_2: we discover that T_2 is greater than or equal to every time t for which

$$\widetilde{M}_0\,\Theta_0^2\,t\;\leq\;\frac{2\,L_0}{C_3^2}\qquad\text{and}\qquad\widetilde{M}_0\,\Theta_0^2\,t\;+\;\widetilde{M}_0\,\Theta_0^2\,S_0^3\,\frac{t^2}{2}\;\leq\;\frac{2\,S_0}{C_3^2(1+C_1^3)}\,.\tag{5.55}$$

Therefore, if we define

$$T\;:=\;K\,\frac{\min\{L_0,S_0\}}{\widetilde{M}_0\,(1+L_0+S_0)^2}\,\Theta_0^{-1}\,,\tag{5.56}$$

where $K>0$ is a constant, then both the inequalities in (5.55) are verified, for some suitable value of K. Hence, $T\leq T_2$, and the theorem is now proved. $\square$

Remark 5.22. Let us notice that, in the classical case (constant density), the lifespan of a solution was controlled from below by

$$T_{cl}\;:=\;C\left(\|\Omega_0\|_{L^q\cap L^\infty}\,\log\left(e+\frac{\|\Omega_0\|_{\mathcal{C}^\varepsilon_{X_0}}}{\|\Omega_0\|_{L^q\cap L^\infty}}\right)\right)^{-1}$$

(see also [26]). We have just proved that in our case the lifespan is given by (5.51), instead. The two lower bounds are quite similar, even if in our case also the initial density comes into play, and there are some additional items, basically due to the more complicate analysis of the pressure term.

Remark 5.23. Note also that, in the two dimensional case, the stretching term in the vorticity equation disappears. This fact translates, at the level of a priori estimates, into the absence of the first two items in the right-hand side of (5.45). Nevertheless, as we have seen, the analysis of $\nabla\Pi$ produces terms of this kind: for this reason, in dimension $N=2$ we weren't able to improve the lower bound (5.51).

5.6 Hölder continuous vortex patches

First of all, let us prove conservation of conormal regularity.

Given a compact hypersurface $\Sigma\subset\mathbb{R}^N$ of class $\mathcal{C}^{1+\varepsilon}$, we can always find, in a canonical way, a family X of $m=N(N+1)/2$ vector-fields such that the inclusion $\mathcal{C}^\eta_\Sigma\subset\mathcal{C}^\eta_X$ holds for all $\eta\in[\varepsilon,1+\varepsilon]$. For completeness, let us recall the result (see proposition 5.1 of [26]), which turns out to be important in the sequel.

Proposition 5.24. *Let Σ be a compact hypersurface of class $\mathcal{C}^{1+\varepsilon}$.*

Then there exists a non-degenerate family of $m=N(N+1)/2$ vector-fields $X\subset\mathcal{T}^\varepsilon_\Sigma$ such that $\mathcal{C}^\eta_\Sigma\subset\mathcal{C}^\eta_X$ for all $\eta\in[\varepsilon,1+\varepsilon]$.

Hence, thanks to theorem 5.3 we propagate striated regularity with respect to this family. Finally, in a classical way, from this fact one can recover conormal properties of the solution, and so get the thesis of theorem 5.5 (see e.g. [36], sections 5 and 6, and [26], section 5, for the details).

Actually, in the case of space dimension $N=2\,,\,3$ (finally, the only relevant ones from the physical point of view) one can improve the statement of theorem 5.5. To avoid traps coming from differential geometry, let us clarify our work setting.

In considering a submanifold $\Sigma\subset\mathbb{R}^N$ of dimension k and of class $\mathcal{C}^{1+\varepsilon}$ (for some $\varepsilon>0$), we mean that Σ is a manifold of dimension k endowed with the differential structure inherited from its inclusion in $\mathbb{R}^N$, and the transition maps are of class $\mathcal{C}^{1+\varepsilon}$.
In particular, for all $x\in\Sigma$ there is an open ball $B\subset\mathbb{R}^N$ containing x, and a $\mathcal{C}^{1+\varepsilon}$ local

parametrization $\varphi : \mathbb{R}^k \to B \cap \Sigma$ with inverse of class $\mathcal{C}^{1+\varepsilon}$. This is equivalent to require local equations $H : B \to \mathbb{R}^k$ of class $\mathcal{C}^{1+\varepsilon}$ such that $H_{|B\cap\Sigma} \equiv 0$.

Let us explicitly point out that, when we speak about generic submanifolds, we always mean submanifolds without boundary, while, in the other case, we have clearly to specify the property "with boundary".

Given a local parametrization φ on $U := \Sigma \cap B$, its differential $\varphi_* : T\mathbb{R}^k \to TU \cong T\Sigma$ induces, in each point $x \in \mathbb{R}^k$, a linear isomorphism between the tangent spaces, $\varphi_{*,x} : T_x\mathbb{R}^k \to T_{\varphi(x)}\Sigma$. Moreover, the dependence of this map on the point $x \in \mathbb{R}^k$ is of class $\mathcal{C}^\varepsilon$: in coordinates, φ_* is given by the Jacobian matrix $\nabla\varphi$.

Finally, we say that a function f defined on Σ is (locally) of class $\mathcal{C}^\alpha$ (for $\alpha > 0$) if the composition $f \circ \varphi : \mathbb{R}^k \to \mathbb{R}$ is α-Hölder continuous for any local parametrization φ.

Before stating our claim, some preliminary results are in order. Let us start with a very simple lemma.

Lemma 5.25. *Let $f \in L^\infty(\mathbb{R}^N)$ such that its gradient is α-Hölder continuous for some $\alpha > 0$. Then $f \in \mathcal{C}^{1+\alpha}(\mathbb{R}^N)$.*

Proof. It's obvious using dyadic characterization of Hölder spaces and Bernstein's inequalities. □

Now, by analogy, one may ask if this property still holds true for a function defined on a submanifold, with Hölder continuous tangential derivatives. In fact, with some additional hypothesis on the submanifold, one can prove that also in this case there is a gain of regularity.

Proposition 5.26. *Let $\Sigma \subset \mathbb{R}^N$ be a submanifold of dimension k and of class $\mathcal{C}^{1+\varepsilon}$, for some $\varepsilon > 0$. Moreover, let us suppose Σ to be compact.*
Let us consider a function $f : \Sigma \to \mathbb{R}$, bounded on Σ and such that $\partial_X f \in \mathcal{C}^\varepsilon(\Sigma)$ for all vector-fields X of class $\mathcal{C}^\varepsilon$ tangent to Σ.

Then $f \in \mathcal{C}^{1+\varepsilon}(\Sigma)$.

Proof. Let us fix a coordinate set $U := B \cap \Sigma$ (for some open ball $B \subset \mathbb{R}^N$) with its $\mathcal{C}^{1+\varepsilon}$ local parametrization $\varphi : \mathbb{R}^k \to U$, and let us define $g := f \circ \varphi : \mathbb{R}^k \to \mathbb{R}$.

Obviously, $g \in L^\infty(\mathbb{R}^k)$, because $f \in L^\infty(\mathbb{R}^N)$.

Moreover, for all $1 \le i \le k$ let us set $\varphi_*(\partial_i) = X_i$: then, X_i is obviously of class $\mathcal{C}^\varepsilon$. Hence we have $\partial_i g(x) = X_i(f)(\varphi(x))$, i.e. $\partial_i g$ in a point x is the derivation X_i applied to the function f, and evaluated in the point $\varphi(x)$. In our notations, we get $\partial_i g = (\partial_{X_i} f) \circ \varphi$.

Therefore, from our hypothesis it follows that $\nabla g \in \mathcal{C}^\varepsilon$, and so, by lemma 5.25, $g \in \mathcal{C}^{1+\varepsilon}(\mathbb{R}^k)$.

In conclusion, we have proved that f composed with any local parametrization φ is of class $\mathcal{C}^{1+\varepsilon}$ on $\mathbb{R}^k$. Therefore $f \in \mathcal{C}^{1,\varepsilon}(\Sigma)$, and, as Σ is compact, we can bound its Hölder norm globally. □

Remark 5.27. Let us note that the operator ∂_X depends linearly on the vector-field X. Hence, in the hypothesis of previous lemma it's enough to assume that one can find, locally on Σ, a family $\{X_1, \ldots, X_k\}$ of linearly independent vector-fields of class $\mathcal{C}^\varepsilon$ such that $\partial_{X_i} f \in \mathcal{C}^\varepsilon(\Sigma)$ for all $1 \le i \le k$.

Corollary 5.28. *Let $\Sigma \subset \mathbb{R}^N$ be a compact hypersurface of class $\mathcal{C}^{1+\varepsilon}$, and let $f \in \mathcal{C}^1_*(\mathbb{R}^N)$. If $f \in \mathcal{C}^{1+\varepsilon}_\Sigma$, then $f_{|\Sigma} \in \mathcal{C}^{1+\varepsilon}(\Sigma)$.*

Proof. By proposition 5.24 and non-degeneracy condition, we can find, locally on Σ, $N-1$ linearly independent vector-fileds $X_1 \ldots X_{N-1}$, defined on the whole $\mathbb{R}^N$ and of class $\mathcal{C}^\varepsilon$, which are tangent to Σ and such that $\mathrm{div}\,(f\,X_i) \in \mathcal{C}^\varepsilon(\mathbb{R}^N)$ for all $1 \le i \le N-1$.

Moreover, also the divergence of these vector-fields is ε-Hölder continuous; therefore, using once again Bony's paraproduct decomposition, we gather that

$$\partial_{X_i} f \;=\; \mathrm{div}\,(f\,X_i) \;-\; f\;\mathrm{div}\,X_i \;\in\; \mathcal{C}^\varepsilon(\mathbb{R}^N) \qquad \forall\, 1 \le i \le N-1\,,$$

and hence this regularity is preserved if we restrict $\partial_{X_i} f$ only to Σ.

So, proposition 5.26 and remark 5.27 both imply that $f_{|\Sigma} \in \mathcal{C}^{1+\varepsilon}(\Sigma)$. $\square$

Now, let us come back to the situation of theorem 5.5. Moreover, let us suppose that the hypersurface Σ_0 is also connected: then it separates the whole space $\mathbb{R}^N$ into two connected components, the first one bounded and the other one unbounded, and whose boundary is exactly Σ_0. In dimension 2, this is nothing but the Jordan curve theorem, while in the general case $N \geq 3$ it's a consequence of the Alexander duality theorem (see e.g. [38], theorem 3.44). For the sake of completeness, we will quote the exact statement and its proof, actually due to A. Lerario, in section 5.7.

So, let us set D_0 to be the bounded domain of $\mathbb{R}^N$ whose boundary is $\partial D_0 = \Sigma_0$ and let us define $D(t) = \psi_t(D_0)$. As the flow ψ_t is a diffeomorphism for every fixed time t, we have that $\partial D(t) = \Sigma(t)$ and also the complementary region is transported by ψ: $D(t)^c = \psi_t(D_0^c)$.

Let us denote by χ_A the characteristic function of a set A.

Theorem 5.29. *Under hypothesis of theorem 5.5, suppose also that the initial data can be decomposed in the following way:*

$$\rho_0(x) \,=\, \rho_0^i(x)\,\chi_{D_0}(x) \,+\, \rho_0^e(x)\,\chi_{D_0^c}(x) \qquad \textit{and} \qquad \Omega_0(x) \,=\, \Omega_0^i(x)\,\chi_{D_0}(x) \,+\, \Omega_0^e(x)\,\chi_{D_0^c}(x)\,,$$

with $\rho_0^i \in \mathcal{C}^{1+\varepsilon}(D_0)$ and $\Omega_0^i \in \mathcal{C}^{\varepsilon}(D_0)$.

Then, the previous decomposition still holds for the solution at every time $t \in [0,T]$:

$$\rho(t,x) \;=\; \rho^i(t,x)\,\chi_{D(t)}(x) \,+\, \rho^e(t,x)\,\chi_{D(t)^c}(x) \tag{5.57}$$

$$\Omega(t,x) \;=\; \Omega^i(t,x)\,\chi_{D(t)}(x) \,+\, \Omega^e(t,x)\,\chi_{D(t)^c}(x)\,. \tag{5.58}$$

Moreover, Hölder continuity in the interior of the domain $D(t)$ is preserved, uniformly on $[0,T]$: at every time t, we have

$$\rho^i(t) \,\in\, \mathcal{C}^{1+\varepsilon}(D(t)) \qquad \textit{and} \qquad \Omega^i(t) \,\in\, \mathcal{C}^{\varepsilon}(D(t))\,.$$

In addition, regularity on $D(t)$ propagates also for the velocity field and the pressure term: $u(t)$ and $\nabla\Pi(t)$ both belong to $\mathcal{C}^{1+\varepsilon}(D(t))$.

Proof. First of all, let us recall that, by theorem 5.5, on $[0,T]$ we have

$$\int_0^T \|\nabla u(t)\|_{L^\infty}\,dt \,\leq\, C\,. \tag{5.59}$$

Thanks to the first equation of (5.1), relation (5.57) obviously holds, with

$$\rho^{i,e}(t,x) \,=\, \rho_0^{i,e}\left(\psi_t^{-1}(x)\right)\,.$$

So, we immediately get that $\rho^i(t)$ belongs to the space $\mathcal{C}^{1+\varepsilon}(D(t))$. Let us observe also that a decomposition analogous to (5.57) holds also for $a = 1/\rho$, and its components $a^{i,e}$ have the same properties of the corresponding ones of ρ.

Now let us handle the vorticity term. We can always decompose the solution in a component localized on $D(t)$ and the other one supported on the complementary set, defining

$$\Omega^i(t,x) \,:=\, \Omega(t,x)\,\chi_{D(t)}(x)\,, \qquad \Omega^e(t,x) \,:=\, \Omega(t,x)\,\chi_{D(t)^c}(x)\,,$$

and therefore obtain relation (5.58). By virtue of this fact, equation (5.6) restricted on the domain $D(t)$ reads as follows:

$$\partial_t\Omega^i \,+\, u\cdot\nabla\Omega^i \,=\, -\left(\Omega^i\cdot\nabla u \,+\, {}^t\nabla u\cdot\Omega^i \,+\, \nabla a^i\wedge\nabla\Pi\right),$$

which gives us the estimate (keep in mind also (5.59))

$$\left\|\Omega^i(t)\right\|_{\mathcal{C}^\varepsilon} \leq C\left(\left\|\Omega_0^i\right\|_{\mathcal{C}^\varepsilon} + \int_0^t \left(\left\|\Omega^i\cdot\nabla u + {}^t\nabla u\cdot\Omega^i\right\|_{\mathcal{C}^\varepsilon} + \left\|\nabla a^i\wedge\nabla\Pi\right\|_{\mathcal{C}^\varepsilon}\right) d\tau\right).$$

We claim that the first term under the integral can be controlled in $\mathcal{C}^\varepsilon$. As a matter of facts, by (5.3) we know that the velocity field satisfies the elliptic equation

$$-\Delta u^k = \sum_{j=1}^N \partial_j\Omega^i_{kj}$$

in $D(t)$, with the boundary condition (by theorem 5.5 and corollary 5.28) $u_{|\partial D(t)}\in\mathcal{C}^{1+\varepsilon}(\partial D(t))$. So (see theorem 8.33 of [37]) we have that $u\in\mathcal{C}^{1+\varepsilon}(D(t))$ and the following inequality holds:

$$\|u\|_{\mathcal{C}^{1+\varepsilon}(D(t))} \leq C\left(\|u(t)\|_{L^\infty(D(t))} + \left\|u_{|\partial D(t)}\right\|_{\mathcal{C}^{1+\varepsilon}(\partial D(t))} + \left\|\Omega^i\right\|_{\mathcal{C}^\varepsilon(D(t))}\right).$$

Let us note that, as pointed out in [37], a priori the constant C depends on $\partial D(t)$ through the $\mathcal{C}^{1+\varepsilon}$ norms of its local parametrizations, so finally on $\exp\left(\int_0^t\|\nabla u\|_{L^\infty}d\tau\right)$. However relation (5.59) allows us to control it uniformy on $[0,T]$. Therefore, in $D(t)$ one gets the following inequality:

$$\left\|\Omega^i\cdot\nabla u + {}^t\nabla u\cdot\Omega^i\right\|_{\mathcal{C}^\varepsilon(D(t))} \leq C\left(\|\nabla u\|_{L^\infty}\left\|\Omega^i\right\|_{\mathcal{C}^\varepsilon(D(t))} + \|\Omega\|_{L^\infty}\|u\|_{\mathcal{C}^{1+\varepsilon}(D(t))}\right),$$

which proves our claim.

Finally, let us handle the pressure term. From what we have proved, ∇a^i is in $\mathcal{C}^\varepsilon$; so

$$\left\|\nabla a^i\wedge\nabla\Pi\right\|_{\mathcal{C}^\varepsilon} \leq C\left\|\nabla a^i\right\|_{\mathcal{C}^\varepsilon}\|\nabla\Pi\|_{\mathcal{C}^1_*(\mathbb{R}^N)}.$$

However, we want to prove that an improvement of regularity in the interior of $D(t)$ occurs also for $\nabla\Pi$. In fact, keeping in mind (5.17), Π satisfies the elliptic equation

$$-\Delta\Pi = \nabla(\log\rho^i)\cdot\nabla\Pi + \rho^i\,\nabla u:\nabla u$$

in the bounded domain $D(t)$. Now, from what we have proved, the right-hand side obviously belongs to $\mathcal{C}^\varepsilon(D(t))$. Moreover, by theorem 5.5 and corollary 5.28, we have $\nabla\Pi_{|\partial D(t)}\in\mathcal{C}^{1+\varepsilon}(\partial D(t))$: in particular, as $\Sigma(t)$ is compact, $\Pi_{|\partial D(t)}$ is continuous and bounded. Finally, as $D(t)$ is of class $\mathcal{C}^{1+\varepsilon}$, it satisfies the exterior cone condition (see [30], page 340). So, theorem 6.13 of [37] applies: from it, we gather $\Pi(t)\in\mathcal{C}^{2+\varepsilon}(D(t))$. Therefore, $\nabla\Pi(t)_{|D(t)}\in\mathcal{C}^{1+\varepsilon}(D(t))$ and its norm is bounded by

$$\left\|\nabla\Pi_{|\partial D(t)}\right\|_{\mathcal{C}^{1+\varepsilon}(\partial D(t))} + \left\|\nabla a^i\right\|_{\mathcal{C}^\varepsilon(D(t))}\|\nabla\Pi\|_{\mathcal{C}^1_*(\mathbb{R}^N)} + \left\|\rho^i\right\|_{\mathcal{C}^{1+\varepsilon}(D(t))}\|\nabla u\|^2_{\mathcal{C}^\varepsilon(D(t))}.$$

Putting all these inequalities together and applying Gronwall's lemma, we finally get a control for the $\mathcal{C}^\varepsilon$ norm of Ω^i in the interior of $D(t)$, and this completes the proof of the theorem. □

5.7 Complements from Algebraic Topology

Here we want to prove the following theorem, which we used in last section of the present chapter. For the technical definitions, notions and results, we refer to [38].

Theorem 5.30. *For any dimension $N\geq 2$, let $\Sigma\subset\mathbb{R}^N$ be a compact, connected hypersurface without boundary.*

Then $\mathbb{R}^N\setminus\Sigma$ has two connected components (say) B and U, one bounded and the other one unbounded, whose boundary is just Σ.

The previous result implies in particular that Σ is orientable (see theorem 5.31).
Let us note that, if we already assumed this (redundant) hypothesis in theorem 5.30, then the proof would be easier (see e.g. [48]).

The proof we quote here is actually due to A. Lerario.

Proof. With standard notations, for a submanifold $\mathcal{M} \subset \mathbb{R}^N$ and an abelian group G, we denote with

$$\widetilde{H}_k(\mathcal{M}; G) \qquad \text{and} \qquad \widetilde{H}^k(\mathcal{M}; G)$$

the k-th reduced homology and cohomology groups of $\mathcal{M}$ with coefficients in G.

Let us compactify $\mathbb{R}^N$ by adding the point at infinity: in this way, we reconduct ourselves to work with the N-dimensional sphere $S^N = \mathbb{R}^N \cup \{\infty\}$.
Obviously, $\mathbb{R}^N \backslash \Sigma$ and $S^N \backslash \Sigma$ have the same number of connected components.

By Alexander duality theorem (see theorem 3.44 of [38]) with coefficients in $\mathbb{Z}_2$, we have

$$\widetilde{H}_k(S^N \backslash \Sigma; \mathbb{Z}_2) \simeq \widetilde{H}^{N-k-1}(\Sigma; \mathbb{Z}_2) \qquad \forall\, k \geq 0\,.$$

In particular, this is true for $k = 0$:

$$\widetilde{H}_0(S^N \backslash \Sigma; \mathbb{Z}_2) \simeq \widetilde{H}^{N-1}(\Sigma; \mathbb{Z}_2)\,.$$

Now, as Σ is compact, connected and without boundary, theorem 3.26 of [38] applies, and gives us

$$\widetilde{H}^{N-1}(\Sigma; \mathbb{Z}_2) \simeq \mathbb{Z}_2$$

(independentely whether Σ is orientable or not). In particular, also $\widetilde{H}_0(S^N \backslash \Sigma; \mathbb{Z}_2)$ is isomorphic to the same group, and this implies that the homology group (not reduced!)

$$H_0(S^N \backslash \Sigma; \mathbb{Z}_2) \simeq \widetilde{H}_0(S^N \backslash \Sigma; \mathbb{Z}_2) \oplus \mathbb{Z}_2 \tag{5.60}$$

has rank equal to 2. But the rank of $H_0(\mathcal{M}; G)$ is always the number of the connected components of $\mathcal{M}$. Hence, $S^N \backslash \Sigma$ has two connected components, A and B.

Let us suppose that $\infty \in A$; then

$$S^N \backslash \Sigma = A \cup B \qquad \Longrightarrow \qquad \mathbb{R}^N \backslash \Sigma = (A \backslash \{\infty\}) \cup B\,.$$

Now, as $N \geq 2$, $U := A \backslash \{\infty\}$ is still connected.

Hence, U and B are the two connected components of $\mathbb{R}^N \backslash \Sigma$.
Moreover, it's easy to see (for instance, by stereographic projection with respect to the point ∞) that U is unbounded, while B is bounded.
Finally, obviously $\partial B \equiv \partial U \equiv \Sigma$. □

As already pointed out, theorem 5.30 entails the following fundamental result. Even if it lies outside of the topics of the present manuscript, we decided to quote it to give a more complete and detailed picture of the framework we adopted in section 5.6.

Again, the proof is due to A. Lerario.

Theorem 5.31. *Let $\Sigma \subset \mathbb{R}^N$ (for some $N \geq 2$) be a compact, connected hypersurface without boundary.*

Then Σ is orientable.

Proof. The starting point is relation (5.60) in the previous proof. Actually, it holds true for any submanifold $\mathcal{M}$ and any abelian group G:

$$H_0(\mathcal{M}; G) \simeq \widetilde{H}_0(\mathcal{M}; G) \oplus G\,. \tag{5.61}$$

Moreover, it is always true that $H_0(\mathcal{M};G)$ is isomorphic to the direct product of n copies of G, where n is the number of connected components of $\mathcal{M}$:

$$H_0(\mathcal{M};G) \simeq G^{\oplus n} \tag{5.62}$$

(see [38] for the proof of these facts).

In the previous proof, we established that the rank of $H_0(S^N \backslash \Sigma; \mathbb{Z}_2)$ is 2. Then, by (5.62) we have that it is still 2 if we consider the homology with coefficients in $\mathbb{Z}$:

$$rk\big(H_0(S^N \backslash \Sigma; \mathbb{Z})\big) = 2\,.$$

Therefore, keeping in mind (5.61) and the Alexander duality theorem, we gather

$$\widetilde{H}_0(S^N \backslash \Sigma; \mathbb{Z}) \simeq \mathbb{Z} \qquad \Longrightarrow \qquad \widetilde{H}^{N-1}(\Sigma; \mathbb{Z}) \simeq \mathbb{Z}\,.$$

Now, by theorem 3.26 of [38], this last condition is equivalent to the fact that Σ is orientable. $\square$

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