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# Thin structures in nonlinear elasticity and in plasticity: a variational approach

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# Thin structures in nonlinear elasticity and in plasticity: a variational approach

Ph.D. Thesis

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*To Matteo,*
*"Because I came here with a load*
*And it feels so much lighter*
*Since I met you*
*And you should know*
*That I could never go on without you".*
*(Coldplay, Green Eyes).*

# Contents

# Introduction

Thin structures are three-dimensional bodies whose thickness in one direction is much smaller than the other dimensions (such as a membrane, a plate, or a shell), or whose cross-section is much smaller than the length (as a string or a rod). The subject of this thesis is the rigorous deduction of lower dimensional models for thin structures in the framework of nonlinear elasticity and of plasticity.

The rigorous derivation of lower dimensional theories for thin structures is a classical question in mechanics. Indeed, both from an analytical and a numerical point of view, one- or two-dimensional models can be handled in an easier way than their three-dimensional counterparts. In the classical approach, lower dimensional models are typically deduced arguing by formal asymptotic expansions of the three-dimensional theories in terms of the thickness parameters or by assuming a priori kinematical restrictions on the structure of the admissible deformations (see e.g. [12, 13, 46] for an overview of the classical results). Hence, the range of validity of these limit theories is often unclear.

The first rigorous results have been obtained in the framework of linearized elasticity ([5, 8]). However, since thin elastic bodies can easily undergo large rotations, even under small loads, this linearized theories have only limited applications.

In the early 90's a rigorous approach to dimension reduction problems has emerged in the stationary framework and in the context of nonlinear elasticity [2, 40, 41]. This approach is based on $\Gamma$-convergence: a variational convergence which guarantees, roughly speaking, convergence of minimizers (and of minima) of the three-dimensional energies to minimizers (and minima) of the reduced models. For the definition and properties of $\Gamma$-convergence we refer to the monograph [14].

The $\Gamma$ convergence method consists in proving two inequalities: a liminf inequality, which provides a lower bound for the limit functional, together with some compactness properties for sequences with equibounded energies, and a limsup inequality, based on the construction of a recovery sequence, which guarantees that the lower bound is indeed optimal. In our framework, to prove compactness of deformations with equibounded energies, two key tools are Korn inequalities and their nonlinear counterpart, i.e. the rigidity estimate proved by Friesecke, James and Müller in [33] (see Section 1.2).

The seminal paper [33] has paved the way for the identification, through the $\Gamma$-convergence method, of hierarchies of limit models for plates [33, 34], rods [53, 54, 57, 58], and shells [32, 42, 43]. The different limit models in the hierarchy correspond to different scaling of the elastic energy, which, in turn, are determined by the scaling of the applied loads in terms of the thickness parameters. In particular, high scalings of the applied forces lead at the limit to linearized models. In this regime it is shown that deformations converge to the identity deformation. To obtain a nontrivial limit model therefore, one is led to introduce some linearized quantities associated to the deformations, to study their asymptotic behaviour, and to express the $\Gamma$-limit energy in terms of their limits.

In the last years, the $\Gamma$-convergence approach has gained attention also in dimension reduction problems arising in the evolutionary framework: in nonlinear elasticity [1], crack propagation [6, 31], linearized elastoplasticity with hardening [44, 45], and delamination problems [51]. In the previous setting, we mention in particular an abstract criterion of evolutionary $\Gamma$-convergence for rate-independent systems by Mielke, Roubíček and Stefanelli [50].

In this thesis we focus on the derivation, by $\Gamma$-convergence, of linearized lower dimensional models for thin structures in the frameworks of nonlinear elasticity, perfect plasticity, and finite plasticity with hardening. The thesis consists of two parts. The first part concerns nonlinearly elastic thin-walled beams in a stationary framework. In the second part we consider thin plastic plates in a quasistatic evolutionary setting.

Thin-walled beams are slender, three-dimensional structural elements, whose length is much larger than the diameter of the cross-section, which, in turn, is much larger than the thickness of the cross-section. This kind of beams are commonly used in mechanical engineering, since they combine good resistance properties with a reasonably low weight. From a mathematical point of view, these thin structures are of interest because their behaviour is determined by the interplay of two different thickness parameters: the diameter and the thickness of the cross-section.

In the framework of nonlinear elasticity, an analysis of lower dimensional models for thin-walled beams has been performed in the recent papers [29, 30], under the assumption of a rectangular cross-section. In Chapter 2, instead, we analyse the case where the cross-section of the beam is a thin tubular neighbourhood of a smooth curve. More precisely, let $\gamma : [0,1] \longrightarrow \mathbb{R}^3$, $\gamma(s) = \gamma_2(s)e_2 + \gamma_3(s)e_3$, be a smooth and simple planar curve, whose curvature is not identically equal to zero, and let $n(s)$ denote the normal vector to the curve at the point $\gamma(s)$. We consider an elastic beam of reference configuration

$$\Omega_h := \Big\{ x_1 e_1 + h\gamma(s) + \delta_h t n(s) : x_1 \in (0,L), s \in (0,1), t \in \Big(-\frac{1}{2}, \frac{1}{2}\Big)\Big\},$$

where $L$ is the length of the beam and $h, \delta_h$ are positive parameters. To model a thin-walled beam, we assume

$$h \to 0 \quad \text{ and } \quad \frac{\delta_h}{h} \to 0 \quad (\text{as } h \to 0).$$

In other words, the diameter of the cross-section is of order $h$ and is assumed to be much larger than the cross-sectional thickness $\delta_h$.

To any deformation $u \in W^{1,2}(\Omega_h;\mathbb{R}^3)$ of the beam, we associate the elastic energy (per unit cross-section) defined as

$$\mathcal{E}^h(u) := \frac{1}{h\delta_h} \int_{\Omega_h} W(\nabla u(x))dx,$$

where the energy density $W$ satisfies the usual assumptions of nonlinear elasticity (see Section 2.2). We are interested in understanding the behaviour, as $h \to 0$, of sequences of deformations $(u^h)$ satisfying

$$\mathcal{E}^h(u^h) \leq C\epsilon_h^2, \tag{0.0.1}$$

where $(\epsilon_h)$ is a given sequence of positive numbers. Estimate (0.0.1) is satisfied, for instance, by global minimizers of the total energy

$$\mathcal{E}^h(u) - \frac{1}{h\delta_h}\int_{\Omega_h} u \cdot f^h dx$$

when the applied body force $f^h : \Omega_h \longrightarrow \mathbb{R}^3$ is of a suitable order of magnitude with respect to $\epsilon_h$ (see [29, 30]). The asymptotic behaviour of $(u^h)$, as $h \to 0$, can be characterized by identifying the $\Gamma$-limit of the sequence of functionals $(\epsilon_h^{-2}\mathcal{E}^h)$. Here we focus on the case where the sequence $(\epsilon_h)$ is infinitesimal and satisfies

$$\lim_{h\to 0} \frac{\epsilon_h}{\delta_h^2} =: \ell \in [0, +\infty). \tag{0.0.2}$$

In analogy with the results of [30], these scalings are expected to correspond at the limit to partially or fully linearized models.

Assuming $\epsilon_h = o(\delta_h)$, as $h \to 0$, we first show (Theorem 2.5.2) that any sequence $(u^h)$ satisfying (0.0.1) converges, up to a rigid motion, to the identity deformation on the mid-fiber of the rod; more precisely, defining

$$\Omega := (0, L) \times (0, 1) \times \Big( -\frac{1}{2}, \frac{1}{2}\Big)$$

and considering a change of variables $\psi^h : \Omega \longrightarrow \Omega_h$, given by

$$\psi^h(x_1, s, t) := x_1 e_1 + h\gamma(s) + \delta_h t n(s)$$

for every $(x_1, s, t) \in \Omega$, we have that, up to rigid motions,

$$y^h := u^h \circ \psi^h \to x_1 e_1$$

strongly in $W^{1,2}(\Omega; \mathbb{R}^3)$.

As we already mentioned, to express the limiting functional, we introduce and study the compactness properties of some linearized quantities associated with the scaled deformations $y^h$. We consider the tangential derivative of the tangential displacement

$$g^h(x_1, s, t) := \frac{1}{\epsilon_h}\partial_1(y_1^h - x_1)$$

for a.e. $(x_1, s, t) \in \Omega$, and the twist function

$$w^h(x_1, s) := \frac{\delta_h}{h\epsilon_h}\int_{-\frac{1}{2}}^{\frac{1}{2}} \partial_s(y^h - \psi^h) \cdot n \, dt$$

for a.e. $(x_1, s) \in (0, L) \times (0, 1)$. In Theorem 2.5.2, under assumption (0.0.2), we prove that

$$\begin{aligned} g^h &\rightharpoonup g \quad \text{weakly in } L^2(\Omega), \\ w^h &\to w \quad \text{strongly in } L^2((0, L) \times (0, 1)), \end{aligned}$$

for some $g \in L^2((0,L)\times(0,1))$ and $w \in W^{1,2}(0,L)$. Moreover, the sequence of bending moments $\left(\frac{1}{h}\partial_s w^h\right)$ converges in the following sense:

$$\frac{1}{h}\partial_s w^h \rightharpoonup b \text{ weakly in } W^{-1,2}((0,L)\times(0,1))$$

for some $b \in L^2((0,L)\times(0,1))$ (see Proposition 2.5.3). In Theorem 2.6.2 we show that the limit quantities $w, g, b$ must satisfy some compatibility conditions that depend on the relative order of magnitude between $\delta_h$ and $h$. More precisely, assuming the existence of the limit

$$\mu := \lim_{h\to 0}\frac{\delta_h}{h^3},$$

three main regimes can be identified:

- $\mu = +\infty$,
- $\mu \in (0,+\infty)$,
- $\mu = 0$.

Heuristically, we expect that in the regime $\mu = 0$ (i.e. when $\delta_h$ is much smaller than $h^3$), the thin-walled beam behaves like a shell whose thickness is converging to zero, whereas for $\delta_h$ much bigger than $h^3$ its asymptotic description is closer to the one of a thin beam.

In the first regime $\mu = +\infty$, one has that $g$ is the tangential derivative of the first component of a Bernoulli-Navier displacement in curvilinear coordinates, that is, there exists $v \in W^{1,2}((0,L)\times(0,1);\mathbb{R}^3)$ such that

$$\partial_1 v\cdot e_1 = g, \quad \partial_s v\cdot\tau = 0, \quad \partial_s v\cdot e_1 + \partial_1 v\cdot\tau = 0 \quad \text{on } (0,L)\times(0,1),$$

where $\tau(s)$ denotes the tangent vector to the curve $\gamma$ at the point $\gamma(s)$. The structure of the cross-sectional components of $v$ depends on the existence and the value of the limit

$$\lambda := \lim_{h\to 0}\frac{\delta_h}{h^2}.$$

Indeed, if $\lambda = +\infty$, there exist $\alpha, \beta \in W^{1,2}(0,L)$ such that

$$v(x_1,s)\cdot e_2 = \alpha(x_1) \quad \text{and} \quad v(x_1,s)\cdot e_3 = \beta(x_1)$$

for every $(x_1,s) \in (0,L)\times(0,1)$. If $\lambda \in (0,+\infty)$, one can show that the twist function $w$ belongs to $W^{2,2}(0,L)$ and the cross-sectional components of $v$ depend on $w$ in the following way:

$$v(x_1,s)\cdot e_2 = \alpha(x_1) - \tfrac{1}{\lambda}\, w(x_1)\gamma_3(s) \quad \text{and} \quad v(x_1,s)\cdot e_3 = \beta(x_1) + \tfrac{1}{\lambda}\, w(x_1)\gamma_2(s)$$

for every $(x_1,s) \in (0,L)\times(0,1)$ and for some $\alpha, \beta \in W^{1,2}(0,L)$. Finally, if $\lambda = 0$, the twist function $w$ is affine, while the cross-sectional components of $v$ satisfy

$$v(x_1,s)\cdot e_2 = \alpha(x_1) - \delta(x_1)\gamma_3(s) \quad \text{and} \quad v(x_1,s)\cdot e_3 = \beta(x_1) + \delta(x_1)\gamma_2(s)$$

for every $(x_1,s) \in (0,L)\times(0,1)$ and for some $\alpha, \beta, \delta \in W^{1,2}(0,L)$. In other words, in the regime $\mu = +\infty$, the structure of $g$ is essentially one-dimensional. As for the bending moment $b$, it simply belongs to $L^2((0,L)\times(0,1))$.

In the regime $\mu = 0$ we still have that $g$ is the tangential derivative of the first component of a Bernoulli-Navier displacement in curvilinear coordinates, but only in an approximate sense (see the definition of the class $\mathcal{G}$ in Section 2.4). Moreover, the bending moment $b$ is associated with an infinitesimal isometry of the cylindrical surface

$$\{x_1 e_1 + \gamma(s) : x_1 \in (0, L), s \in (0, 1)\},$$

in the sense that there exists $\phi \in L^2((0, L) \times (0, 1); \mathbb{R}^3)$, with $\partial_s \phi \in L^2((0, L) \times (0, 1); \mathbb{R}^3)$, such that

$$\partial_1 \phi \cdot e_1 = 0, \quad \partial_s \phi \cdot \tau = 0, \quad \partial_s \phi \cdot e_1 + \partial_1 \phi \cdot \tau = 0 \quad \text{on } (0, L) \times (0, 1)$$

and

$$\partial_s(\partial_s \phi \cdot n) = b \quad \text{on } (0, L) \times (0, 1).$$

The equalities are intended in the sense of distributions; some higher regularity for $\phi$ can be proved (see Remark 2.4.6). In other words, in this regime the limit kinematic description of the thin-walled beam is intrinsically two-dimensional.

In the intermediate regime $\mu \in (0, +\infty)$, the limit quantities $g$ and $b$ are no more mutually independent but they must satisfy the following constraint: there exists $\phi \in L^2((0, L) \times (0, 1); \mathbb{R}^3)$, with $\partial_s \phi \in L^2((0, L) \times (0, 1); \mathbb{R}^3)$, such that

$$\partial_1 \phi \cdot e_1 = \mu g, \quad \partial_s \phi \cdot \tau = 0, \quad \partial_s \phi \cdot e_1 + \partial_1 \phi \cdot \tau = 0 \quad \text{on } (0, L) \times (0, 1)$$

and

$$\partial_s(\partial_s \phi \cdot n) = b \quad \text{on } (0, L) \times (0, 1).$$

Finally, the twist function $w$ is affine for $\mu \in [0, +\infty)$.

The $\Gamma$-limit functional is expressed in terms of the limit quantities $w, g, b$ and, according to the values of $\lambda$ and $\mu$, is finite only on the class $\mathcal{A}_{\lambda,\mu}$ of triples $(w, g, b)$ with the structure described above. In Theorems 2.6.3 and 2.7.1 we prove that for $(w, g, b) \in \mathcal{A}_{\lambda,\mu}$ the $\Gamma$-limit is given by the functional

$$\mathcal{J}_{\lambda,\mu}(g, w, b) = \frac{1}{24} \int_0^L \int_0^1 Q_{tan}(s, \partial_1 w, b)\, ds dx_1 + \frac{1}{2} \int_0^L \int_0^1 \mathbb{E} g^2\, ds dx_1,$$

where $Q_{tan}$ is a positive definite quadratic form and $\mathbb{E}$ is a positive constant, for which explicit formulas are provided (see (2.6.43) and (2.6.44)).

The dependence of the $\Gamma$-limits on the rate of convergence of the thickness parameter $\delta_h$ with respect to the cross-section diameter $h$ is an effect of the nontrivial geometry of the cross-section. Indeed, in the case of a rectangular cross-section this phenomenon is not observed for the scalings (0.0.2) and is conjectured to arise only for scalings $\epsilon_h$ such that $\delta_h^2 \ll \epsilon_h \leq \delta_h$ (see [29, 30]).

Another difference with respect to [30] is that, in general, one can not rely on a three-dimensional Korn inequality on $\Omega$ to guarantee compactness of the sequence of cross-sectional displacements. However, one can use a rescaled two-dimensional Korn inequality in curvilinear coordinates (Theorem 2.3.2) to implicitly determine the cross-sectional displacements in the limit models through the characterization of $g$ (see the proof of Theorem 2.6.2).

The proofs of compactness and of the liminf inequality rely both on the rigidity estimate (Theorem 1.2.1) and on the rescaled two-dimensional Korn inequality. The key ingredients in the construction of the recovery sequences are some approximation results for triples in the classes $\mathcal{A}_{\lambda,\mu}$ in terms of smooth functions (see Section 2.4). In the regime $\mu = 0$ the approximation result is proved under the additional assumption that the set where the curvature of $\gamma$ vanishes is the union of a finite number of intervals and isolated points. Therefore, for $\mu = 0$ the $\Gamma$-convergence result is valid only under this additional restriction.

The second part of the thesis concerns the rigorous justification of quasistatic evolution models for thin elasto-plastic plates. We consider a thin plate of reference configuration

$$\Omega_\varepsilon := \omega \times (-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2}),$$

where $\varepsilon > 0$ is the thickness parameter and $\omega$ is a domain in $\mathbb{R}^2$ with a $C^2$ boundary. We assume that $\partial\omega$ can be partitioned into the union of two disjoint sets $\gamma_d$ and $\gamma_n$ and their common boundary, and we prescribe a time-dependent boundary datum on a subset $\Gamma_\varepsilon := \gamma_d \times (-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2})$ of the lateral surface.

In Chapter 3 we consider the linear framework of perfect plasticity, for which existence of three-dimensional quasistatic evolutions is guaranteed by [15, 59]. In Chapters 4 and 5 we discuss the more difficult case of finite plasticity.

The quasistatic evolution problem in linearized perfect plasticity can be formulated as follows. Assume that the elastic behaviour of the plate is linear and isotropic and its plastic response is governed by the Prandtl-Reuss flow rule without hardening. Let $u^\varepsilon(t)$ denote the displacement field at time $t$ and let $Eu^\varepsilon(t)$ denote the infinitesimal strain tensor at $t$, that is, the symmetric part of $Du^\varepsilon(t)$. Let $\sigma^\varepsilon(t)$ be the stress tensor at $t$ and let $e^\varepsilon(t)$ and $p^\varepsilon(t)$ (a deviatoric symmetric matrix) be the elastic and plastic strain tensors at $t$. Let $w^\varepsilon(t)$ be the time-dependent boundary condition prescribed on $\Gamma_\varepsilon$. Assume also that for simplicity there are no applied loads. The classical formulation of the quasistatic evolution problem on a time interval $[0,T]$ consists in finding $u^\varepsilon(t)$, $e^\varepsilon(t)$, $p^\varepsilon(t)$, and $\sigma^\varepsilon(t)$ such that the following conditions are satisfied for every $t \in [0,T]$:

(cf1) *kinematic admissibility:* $Eu^\varepsilon(t) = e^\varepsilon(t) + p^\varepsilon(t)$ in $\Omega_\varepsilon$ and $u^\varepsilon(t) = w^\varepsilon(t)$ on $\Gamma_\varepsilon$;

(cf2) *constitutive law:* $\sigma^\varepsilon(t) = \mathbb{C}e^\varepsilon(t)$ in $\Omega_\varepsilon$, where $\mathbb{C}$ is the elasticity tensor;

(cf3) *equilibrium:* $\operatorname{div}\sigma^\varepsilon(t) = 0$ in $\Omega_\varepsilon$ and $\sigma^\varepsilon(t)\nu_{\partial\Omega_\varepsilon} = 0$ on $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$, where $\nu_{\partial\Omega_\varepsilon}$ is the outer unit normal to $\partial\Omega_\varepsilon$;

(cf4) *stress constraint:* $\sigma^\varepsilon_D(t) \in K$, where $\sigma^\varepsilon_D$ is the deviatoric part of $\sigma^\varepsilon$ and $K$ is a given convex and compact subset of deviatoric $3\times3$ matrices, representing the set of admissible stresses;

(cf5) *flow rule:* $\dot{p}^\varepsilon(t) = 0$ if $\sigma^\varepsilon_D(t) \in \operatorname{int} K$, while $\dot{p}^\varepsilon(t)$ belongs to the normal cone to $K$ at $\sigma^\varepsilon_D(t)$ if $\sigma^\varepsilon_D(t) \in \partial K$.

The first existence result of a quasistatic evolution in perfect plasticity has been proved in [59] by means of viscoplastic approximations. More recently, in [15] the problem has been reformulated within the framework of the variational theory for rate-independent processes, developed in [47]. This variational formulation reads as follows: to find a triple $(u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t))$ such that for every $t \in [0,T]$ we have

(qs1) *global stability:* $(u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t))$ satisfies $Eu^\varepsilon(t) = e^\varepsilon(t) + p^\varepsilon(t)$ in $\Omega_\varepsilon$, $u^\varepsilon(t) = w^\varepsilon(t)$ on $\Gamma_\varepsilon$, and minimizes

$$\tfrac{1}{2}\int_{\Omega_\varepsilon} \mathbb{C}f : f\,dx + \int_{\Omega_\varepsilon} H(q - p^\varepsilon(t))\,dx$$

among all kinematically admissible triples $(v, f, q)$, where $H$ is the support function of $K$, i.e., $H(p) := \sup\{\sigma : p\ :\ \sigma \in K\}$;

(qs2) *energy balance:*

$$\tfrac{1}{2}\int_{\Omega_\varepsilon} \mathbb{C}e^\varepsilon(t) : e^\varepsilon(t)\,dx + \int_0^t \int_{\Omega_\varepsilon} H(\dot{p}^\varepsilon(s))\,dxds = \tfrac{1}{2}\int_{\Omega_\varepsilon} \mathbb{C}e^\varepsilon(0) : e^\varepsilon(0)\,dx + \int_0^t \int_{\Omega_\varepsilon} \mathbb{C}e^\varepsilon(s) : E\dot{w}^\varepsilon(s)\,dxds.$$

The existence of a quasistatic evolution according to the previous formulation and the extent to which this is equivalent to the original formulation is the main focus of [15].

Our purpose is to characterize the limiting behaviour of a sequence of solutions $(u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t))$, as $\varepsilon \to 0$. We observe that the abstract theory of evolutionary $\Gamma$-convergence for rate-independent systems developed in [50] cannot be directly applied here. Indeed, it consists in studying separately the $\Gamma$-limit of the stored-energy functionals and that of the dissipation distances and in coupling them through the construction of a joint recovery sequence. This technique has been applied, e.g., in [44, 45], where the presence of hardening gives rise to a stored-energy functional that is coercive in the $L^2$ norm both with respect to $e$ and $p$. This approach is not suited to our case, since the elastic energy is coercive only with respect to the elastic strain $e$, while the plastic strain $p$ can be controlled only through the dissipation. For this reason, to identify the correct limiting energy we study the $\Gamma$-convergence of the total energy functional, given by the sum of the stored energy with the dissipation distance.

We first focus on the static case, that is, we consider a boundary displacement $w^\varepsilon$ independent of time, we introduce the functional

$$\mathcal{E}_\varepsilon(u, e, p) := \tfrac{1}{2}\int_{\Omega_\varepsilon} \mathbb{C}e : e\,dx + \int_{\Omega_\varepsilon} H(p)\,dx \tag{0.0.3}$$

defined on the class $\mathcal{A}_\varepsilon(w^\varepsilon, \Omega_\varepsilon)$ of all triples $(u, e, p)$ satisfying $Eu = e + p$ in $\Omega_\varepsilon$ and $u = w^\varepsilon$ on $\Gamma_\varepsilon$, and we study its limit, as $\varepsilon \to 0$, in the sense of $\Gamma$-convergence.

As pointed out in [15], because of the linear growth of $H$, the functional $\mathcal{E}_\varepsilon$ is not coercive in any Sobolev norm. The natural setting for a weak formulation is the space $BD(\Omega_\varepsilon)$ of functions with bounded deformation for the displacement $u$ and the space $M_b(\Omega_\varepsilon \cup \Gamma_\varepsilon; \mathbb{M}^{3\times3}_D)$ of trace-free $\mathbb{M}^{3\times3}$-valued bounded Borel measures on $\Omega_\varepsilon \cup \Gamma_\varepsilon$ for the plastic strain $p$ (see Section 1.4 for an overview on the basic properties of $BD$ functions). This is also natural from a mechanical point of view, because in absence of hardening it is well known that displacements may develop jump discontinuities along so-called slip surfaces, on which plastic strain concentrates.

In particular, the functional

$$\int_{\Omega_\varepsilon} H(p)\, dx$$

has to be interpreted according to the theory of convex functions of measures, developed in [35, 60] (see also Section 3.2), as

$$\int_{\Omega_\varepsilon \cup \Gamma_\varepsilon} H\Big(\frac{dp}{d|p|}\Big)\, d|p|,$$

where $dp/d|p|$ is the Radon-Nicodym derivative of $p$ with respect to its total variation $|p|$. Moreover, the boundary condition is relaxed by requiring that

$$p = (w^\varepsilon - u) \odot \nu_{\partial\Omega_\varepsilon} \mathcal{H}^2 \quad \text{on } \Gamma_\varepsilon, \tag{0.0.4}$$

where $\odot$ denotes the symmetric tensor product. The mechanical interpretation of (0.0.4) is that $u$ may not attain the boundary condition: in this case a plastic slip is developed along $\Gamma_\varepsilon$, whose amount is proportional to the difference between the prescribed boundary value and the actual value.

For simplicity we assume that the prescribed boundary datum $w^\varepsilon$ is a displacement of Kirchhoff-Love type of Sobolev regularity (see (3.2.6)). As observed in Remark 3.4.3, more general boundary conditions can also be considered.

Setting $\Gamma_d := \gamma_d \times (-\frac{1}{2}, \frac{1}{2})$, we show that the $\Gamma$-limit of $\mathcal{E}_\varepsilon$ (rescaled to the domain $\Omega := \omega \times (-\frac{1}{2}, \frac{1}{2})$ independent of $\varepsilon$) is finite only on the class $\mathcal{A}_{KL}(w)$ of triples $(u, e, p)$ such that $u \in BD(\Omega)$, $e \in L^2(\Omega; \mathbb{M}^{3\times3}_{sym})$, $p \in M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_{sym})$, and

$$Eu = e + p \quad \text{in } \Omega, \qquad p = (w - u) \odot \nu_{\partial\Omega} \mathcal{H}^2 \quad \text{on } \Gamma_d, \tag{0.0.5}$$

$$e_{i3} = 0 \quad \text{in } \Omega, \quad p_{i3} = 0 \quad \text{in } \Omega \cup \Gamma_d, \quad i = 1, 2, 3, \tag{0.0.6}$$

where $\nu_{\partial\Omega}$ is the outer unit normal to $\partial\Omega$. On this class the $\Gamma$-limit is given by the functional

$$\mathcal{J}(u, e, p) := \tfrac{1}{2} \int_\Omega \mathbb{C}_r e : e\, dx + \mathcal{H}_r(p) \tag{0.0.7}$$

where

$$\mathcal{H}_r(p) := \int_{\Omega \cup \Gamma_d} H_r\Big(\frac{dp}{d|p|}\Big)\, d|p|$$

and the tensor $\mathbb{C}_r$ and the function $H_r$ are defined through pointwise minimization formulas (see (3.4.1), (3.4.5), and (3.4.7)).

Conditions (0.0.5)–(0.0.6) imply that $u$ is a Kirchhoff-Love displacement in $BD(\Omega)$, that is, $u_3$ belongs to the space $BH(\omega)$ of functions with bounded Hessian (see Section 1.4) and there exists $\bar{u} \in BD(\omega)$ such that

$$u(x) = \big(\bar{u}_1(x') - x_3\partial_1 u_3(x'), \bar{u}_2(x') - x_3\partial_2 u_3(x'), u_3(x')\big) \quad \text{for a.e. } x = (x', x_3) \in \Omega.$$

Moreover,

$$(Eu)_{\alpha\beta} = (E\bar{u})_{\alpha\beta} - x_3\partial^2_{\alpha\beta}u_3 \quad \text{for } \alpha, \beta = 1, 2.$$

We note that the averaged tangential displacement $\bar{u}$ may exhibit jump discontinuities, while, because of the embedding of $BH(\omega)$ into $C(\overline{\omega})$, the normal displacement $u_3$ is continuous, but its gradient may have jump discontinuities. Moreover, the second equality in (0.0.5), together with the second condition in (0.0.6), implies that $u_3$ satisfies the boundary condition $u_3 = w_3$ on $\gamma_d$. In particular, in the limit model slip surfaces are vertical surfaces whose projection on $\omega$ is the union of the jump set of $\bar{u}$ and the jump set of $\nabla u_3$.

We also remark that conditions (0.0.5)–(0.0.6) do not imply that $e$ and $p$ are affine with respect to the $x_3$ variable. Therefore, in contrast with the case of linearized elasticity [5, 8], the limit functional $\mathcal{J}$ cannot be in general expressed in terms of two-dimensional quantities only. A precise characterization of conditions (0.0.5)–(0.0.6) in terms of the moments of $e$ and $p$ is given in Proposition 3.3.5.

We then introduce time and study the convergence of quasistatic evolutions. We prescribe on $\Gamma_\varepsilon$ a boundary datum $w^\varepsilon(t)$ of Kirchhoff-Love type and we consider a sequence of initial data $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon)$, that is compact in a suitable sense. We show (Theorem 3.5.4) that, if for every $\varepsilon > 0$ the triple $(u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t))$ is a quasistatic evolution in the sense of (qs1)–(qs2) for the boundary datum $w^\varepsilon(t)$ and the initial datum $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon)$, then, up to a suitable scaling, $(u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t))$ converges, as $\varepsilon \to 0$, to a limit triple $(u(t), e(t), p(t))$ that satisfies:

(qs1)$_r$ *reduced global stability:* for every $t \in [0,T]$ $(u(t), e(t), p(t)) \in \mathcal{A}_{KL}(w(t))$ and minimizes

$$\tfrac{1}{2}\int_\Omega \mathbb{C}_r f : f\,dx + \mathcal{H}_r(q - p(t))$$

among all triples $(v, f, q)$ in $\mathcal{A}_{KL}(w(t))$;

(qs2)$_r$ *reduced energy balance:* for every $t \in [0,T]$

$$\tfrac{1}{2}\int_\Omega \mathbb{C}_r e(t) : e(t)\,dx + \int_0^t \mathcal{H}_r(\dot{p}(s))\,ds = \tfrac{1}{2}\int_\Omega \mathbb{C}_r e(0) : e(0)\,dx + \int_0^t\int_\Omega \mathbb{C}_r e(s) : E\dot{w}(s)\,dxds.$$

We call a triple satisfying (qs1)$_r$–(qs2)$_r$ a reduced quasistatic evolution.

The proof of Theorem 3.5.4 mainly relies on the $\Gamma$-convergence result in the static case. Even if the abstract theory of [50] cannot be directly applied, we follow the general scheme proposed in that paper. In particular, the role of the so-called joint recovery sequence is played in our case by the recovery sequence constructed at fixed time.

In the last part of the Chapter 3 we discuss some properties of reduced quasistatic evolutions. We show three equivalent formulations in rate form (Theorem 3.6.13). In all of them the global stability condition is replaced by a system of two equilibrium conditions, one for the stretching component of the stress and the other for the bending component. These two components are coupled in the energy balance, which is rephrased in the three different formulations in terms of a maximal dissipation principle, of a flow rule in a weak form, and of a variational inequality for the stress (analogous to the formulation considered in [59] in the case of three-dimensional perfect plasticity), respectively. To prove these results we define a suitable notion of duality between stresses and plastic strains in the footsteps of [37] and [22].

In the last subsection of Chapter 3 we focus on two examples, where a reduced quasistatic evolution can be characterized in terms of two-dimensional quantities only. In particular, (Proposition 3.6.16) we show that, if the set $K$ is symmetric with respect to the origin and the boundary datum and the initial data are properly chosen, our notion of reduced quasistatic evolution coincides with that studied in [9, 24, 25].

In Chapters 4 and 5 we turn our attention to a model in finite plasticity. We consider a plate of reference configuration $\Omega_\epsilon$ and assume that the deformations $\eta \in W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)$ of the plate fulfill the multiplicative decomposition

$$\nabla\eta(x) = F_{el}(x)F_{pl}(x) \quad \text{for a.e. } x \in \Omega_\varepsilon,$$

where $F_{el} \in L^2(\Omega_\varepsilon;\mathbb{M}^{3\times 3})$ represents the elastic strain, $F_{pl} \in L^2(\Omega_\varepsilon;SL(3))$ is the plastic strain and $SL(3) := \{F \in \mathbb{M}^{3\times 3} : \det F = 1\}$. To guarantee coercivity in the plastic strain variable, we suppose to be in a hardening regime. More precisely, the stored energy associated to a deformation $\eta$ and to its elastic and plastic strains is expressed as

$$\begin{aligned}\mathcal{E}(\eta, F_{pl}) &:= \int_{\Omega_\varepsilon} W_{el}(\nabla\eta(x)F_{pl}^{-1}(x))\,dx + \int_{\Omega_\varepsilon} W_{hard}(F_{pl}(x))\,dx \\ &= \int_{\Omega_\varepsilon} W_{el}(F_{el}(x))\,dx + \int_{\Omega_\varepsilon} W_{hard}(F_{pl}(x))\,dx,\end{aligned}$$

where $W_{el}$ is a nonlinear frame-indifferent elastic energy density and $W_{hard}$, which is finite only on a compact subset of $SL(3)$ having the identity as an interior point, describes hardening. The plastic dissipation is expressed by means of a dissipation distance $D : \mathbb{M}^{3\times 3} \times \mathbb{M}^{3\times 3} \to [0, +\infty]$, which is given via a positively 1-homogeneous potential $H_D$, and represents the minimum amount of energy that is dissipated when the system moves from a plastic configuration to another (see Section 4.2).

The existence of a quasistatic evolution in this nonlinear setting is a quite delicate issue, and it has only recently been solved in [48] by adding to the stored-energy functional some further regularizing terms in the plastic component. We shall not add these further terms here, we rather show, in the last section, that our convergence result can be extended to sequences of approximate discrete-time quasistatic evolutions, whose existence is always guaranteed (see Theorem 5.5.2).

In Chapter 4, as in the case of linearized perfect plasticity, we first consider the static problem and we study the asymptotic behaviour of sequences of pairs $(\eta^\varepsilon, F^\varepsilon_{pl})$ whose total energy per unit thickness satisfies

$$\frac{1}{\varepsilon}\Big(\mathcal{E}(\eta^\varepsilon, F^\varepsilon_{pl}) + \varepsilon^{\alpha-1}\int_{\Omega_\varepsilon} D(F^{\varepsilon,0}_{pl}, F^\varepsilon_{pl})\,dx\Big) \leq C\varepsilon^{2\alpha-2}, \tag{0.0.8}$$

where $\alpha \geq 3$ is a positive parameter and $(F^{\varepsilon,0}) \subset L^2(\Omega_\varepsilon; SL(3))$ is a given sequence representing preexistent plastic strains. It was proved in [34] that in the absence of plastic deformation (that is, when $F^{\varepsilon,0} = F_{pl} = Id$) these energy scalings lead to the Von Kármán plate theory for $\alpha = 3$ and to the linear plate theory for $\alpha > 3$. The scaling of the dissipation energy is motivated by its linear growth (see (4.2.20)). In analogy with the results of [34] in the framework of nonlinear elasticity, we expect these scalings to correspond to partially or fully linearized plastic models.

On $\Gamma_\varepsilon$ we prescribe a boundary datum

$$\phi^\varepsilon(x) := \begin{pmatrix} x' \\ x_3 \end{pmatrix} + \begin{pmatrix} \varepsilon^{\alpha-1} u^0(x') \\ \varepsilon^{\alpha-2} v^0(x') \end{pmatrix} - \varepsilon^{\alpha-2} x_3 \nabla v^0(x') \tag{0.0.9}$$

for $x = (x', \varepsilon x_3) \in \Omega_\varepsilon$, where $u^0 \in W^{1,\infty}(\omega; \mathbb{R}^2)$ and $v^0 \in W^{2,\infty}(\omega)$. This structure of the boundary conditions is compatible with that of the minimal energy configurations in the absence of plastic deformations (see Remark 4.2.5).

We first show that, given any sequence of pairs $(\eta^\varepsilon, F^\varepsilon_{pl})$ satisfying (0.0.8) and the boundary conditions

$$\eta^\varepsilon = \phi^\varepsilon \quad \mathcal{H}^2 \text{ - a.e. on } \gamma_d \times \big(-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2}\big), \tag{0.0.10}$$

as $\varepsilon \to 0$, the deformations $\eta^\varepsilon$ converge to the identity deformation on the mid-section of the plate, and the plastic strains $F^\varepsilon_{pl}$ tend to the identity matrix. More precisely, defining $\Omega := \omega \times \big(-\frac{1}{2}, \frac{1}{2}\big)$ and $\psi^\varepsilon(x) := (x', \varepsilon x_3)$ for every $(x', x_3) \in \overline{\Omega}$, and assuming

$$F^{\varepsilon,0}_{pl} \circ \psi^\varepsilon = Id + \varepsilon^{\alpha-1} p^{\varepsilon,0}$$

with

$$p^{\varepsilon,0} \rightharpoonup p^0 \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{3\times 3}), \tag{0.0.11}$$

we show that

$$y^\varepsilon := \eta^\varepsilon \circ \psi^\varepsilon \to \begin{pmatrix} x' \\ 0 \end{pmatrix} \quad \text{strongly in } W^{1,2}(\Omega; \mathbb{R}^3)$$

and

$$P^\varepsilon := F^\varepsilon_{pl} \circ \psi^\varepsilon \to Id \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}).$$

To express the limit functional, we introduce and study the compactness properties of some linearized quantities associated with the scaled deformations and plastic strains: the in-plane displacements

$$u^\varepsilon(x') := \frac{1}{\varepsilon^{\alpha-1}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Big( \begin{pmatrix} y^\varepsilon_1 \\ y^\varepsilon_2 \end{pmatrix} - x' \Big)\, dx_3$$

for a.e. $x' \in \omega$, the out-of-plane displacements

$$v^\varepsilon(x') := \frac{1}{\varepsilon^{\alpha-2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} y^\varepsilon_3(x)\, dx_3,$$

for a.e. $x' \in \omega$, and the linearized plastic strains

$$p^\varepsilon(x) := \frac{P^\varepsilon(x) - Id}{\varepsilon^{\alpha-1}}$$

for a.e. $x \in \Omega$. We prove (Theorem 4.3.3) that, under assumptions (0.0.8), (0.0.10) and (0.0.11) the sequence of triples $(u^\varepsilon, v^\varepsilon, p^\varepsilon)$ converges in a suitable sense to a triple $(u, v, p) \in W^{1,2}(\omega; \mathbb{R}^2) \times W^{2,2}(\omega) \times L^2(\Omega; \mathbb{M}^{3\times 3})$, such that $\operatorname{tr} p = 0$, and

$$u = u^0, \quad v = v^0, \quad \nabla v = \nabla v^0 \quad \mathcal{H}^1\text{-a.e. on } \gamma_d.$$

Moreover, we show that the $\Gamma$-limit functional can be expressed in terms of the limit quantities $u, v$, and $p$, and is given by

$$\begin{aligned} \mathcal{J}_\alpha(u, v, p) \quad &:= \quad \int_\Omega Q_2\big(\mathrm{sym}\nabla' u + \tfrac{L_\alpha}{2}\nabla' v \otimes \nabla' v - x_3(\nabla')^2 v - p'\big)\, dx + \int_\Omega B(p)\, dx \\ &+ \quad \int_\Omega H(p - p^0)\, dx, \end{aligned} \tag{0.0.12}$$

where $L_\alpha = 0$ for $\alpha > 3$ and $L_\alpha = 1$ for $\alpha = 3$ (see Theorems 4.3.3, 4.4.1 and 4.5.1). In the previous formulas, $\nabla'$ denotes the gradient with respect to $x'$, $p'$ is the $2 \times 2$ minor given by the first two rows and columns of the map $p$, and $Q_2$ and $B$ are positive definite quadratic forms on $\mathbb{M}^{2\times 2}$ and $\mathbb{M}^{3\times 3}$, respectively, for which an explicit characterization is provided (see Sections 3.4 and 4.3).

The constant $L_\alpha$ in the limit problem encodes the main differences between the cases $\alpha > 3$ and $\alpha = 3$. Indeed, for $\alpha = 3$, the limit energy contains the nonlinear term $\frac{1}{2}\nabla' v \otimes \nabla' v$, which accounts for the stretching due to the out-of-plane displacement. For $\alpha > 3$ the limit problem is completely linearized and, in the absence of hardening, coincides with the functional (0.0.7) identified starting from three-dimensional linearized elasto-plasticity under the assumption that $D^2 W_{el}(Id) = \mathbb{C}$ (where $\mathbb{C}$ is the tensor in (0.0.3)). However, we point out that the role of the hardening term in the present formulation is fundamental to deduce compactness of the three-dimensional evolutions (see Step 1, Proof of Theorem 5.3.9).

We also remark that in the absence of plastic dissipation ($p^0 = p = 0$) the two $\Gamma$-limits reduce to the functionals deduced in [34] in the context of nonlinear elasticity. As in the case of linearized elasto-plasticity though, also in this context the limit functional $\mathcal{J}_\alpha$ cannot be, in general, expressed in terms of two-dimensional quantities only because the limit plastic strain $p$ depends nontrivially on the $x_3$ variable (see Section 4.5).

The setting of the problem and some proof arguments are very close to those of [52], where it is shown that three-dimensional linearized plasticity can be obtained as $\Gamma$-limit of three-dimensional finite plasticity. The proof of the compactness and the liminf inequality rely on the rigidity estimate (Theorem 1.2.1). This theorem can be applied owing to the presence of the hardening term, which provides one with a uniform bound on the $L^\infty$ norm of the scaled plastic strains $P^\varepsilon$. The construction of the recovery sequence is obtained by combining some results of [34, Sections 6.1 and 6.2] about dimension reduction in nonlinear elasticity and [52, Lemma 3.6].

In Chapter 5 we finally assume that $u^0$ and $v^0$ (and hence $\phi^\varepsilon$) are time-dependent maps, and we study the convergence of quasistatic evolutions associated to $\phi^\varepsilon$, assuming a priori their existence. To deal with the nonlinear structure of the energy, we follow the approach of [28]: we assume $\phi^\varepsilon(t)$ to be a $C^1$ diffeomorphism on $\mathbb{R}^3$ and we write deformations $\eta \in W^{1,2}(\Omega_\varepsilon; \mathbb{R}^3)$ as

$$\eta \circ \psi^\varepsilon = \phi^\varepsilon(t) \circ z,$$

where $z \in W^{1,2}(\Omega;\mathbb{R}^3)$ satisfies the boundary condition

$$z(x) = \psi^\varepsilon(x) = (x', \varepsilon x_3) \quad \mathcal{H}^2 \text{ - a.e. on } \gamma_d \times \big(-\tfrac{1}{2}, \tfrac{1}{2}\big).$$

To any plastic strain $F_{pl} \in L^2(\Omega_\varepsilon; SL(3))$ we associate a scaled plastic strain $P \in L^2(\Omega; SL(3))$ defined as

$$P := F_{pl} \circ \psi^\varepsilon$$

and we rewrite the stored energy as

$$\mathcal{F}_\varepsilon(t,z,P) := \int_\Omega W_{el}(\nabla\phi^\varepsilon(t,z(x))\nabla_\varepsilon z(x))\,dx + \int_\Omega W_{hard}(P(x))\,dx = \frac{1}{\varepsilon}\mathcal{E}(\eta, F_{pl}),$$

where $\nabla_\varepsilon z := (\nabla' z|\frac{1}{\varepsilon}\partial_3 z)$.

In this setting, according to the variational theory for rate-independent processes developed in [47], a quasistatic evolution for the boundary datum $\phi^\varepsilon$ is a function $t \mapsto (z(t), P(t)) \in W^{1,2}(\Omega;\mathbb{R}^3) \times L^2(\Omega; SL(3))$ such that for every $t \in [0,T]$ the following two conditions are satisfied:

(gs) *global stability:* there holds

$$z(t) = \psi^\varepsilon \quad \mathcal{H}^2 \text{ - a.e. on } \gamma_d \times \big(-\tfrac{1}{2}, \tfrac{1}{2}\big)$$

and $(z(t), P(t))$ minimizes

$$\mathcal{F}_\varepsilon(t, \tilde{z}, \tilde{P}) + \varepsilon^{\alpha-1}\int_\Omega D(P(t), \tilde{P})\,dx,$$

among all $(\tilde{z}, \tilde{P}) \in W^{1,2}(\Omega;\mathbb{R}^3) \times L^2(\Omega; SL(3))$ such that $\tilde{z} = \psi^\varepsilon$ $\mathcal{H}^2$ - a.e. on $\gamma_d \times \big(-\tfrac{1}{2}, \tfrac{1}{2}\big)$;

(eb) *energy balance:*

$$\begin{aligned}&\mathcal{F}_\varepsilon(t, z(t), P(t)) + \varepsilon^{\alpha-1}\mathcal{D}(P;0,t)\\ &= \mathcal{F}_\varepsilon(0, z(0), P(0)) + \varepsilon^{\alpha-1}\int_0^t\int_\Omega E^\varepsilon(s) : \Big(\nabla\dot{\phi}^\varepsilon(s,z(s))(\nabla\phi^\varepsilon)^{-1}(s,z(s))\Big)\,dx\,ds.\end{aligned}$$

In the previous formula, $\mathcal{D}(P;0,t)$ is the plastic dissipation in the interval $[0,t]$ (see Section 5.3), $E^\varepsilon(t)$ is the stress tensor, defined as

$$E^\varepsilon(t) := \frac{1}{\varepsilon^{\alpha-1}} DW_{el}\big(\nabla\phi^\varepsilon(t,z(t))\nabla_\varepsilon z(t)(P)^{-1}(t)\big)\big(\nabla\phi^\varepsilon(t,z(t))\nabla_\varepsilon z(t)(P)^{-1}(t)\big)^T,$$

and $\alpha \geq 3$ is the same exponent as in the expression of the boundary datum.

The main result of Chapter 5 is the characterization of the asymptotic behaviour of $(z^\varepsilon(t), P^\varepsilon(t))$, as $\varepsilon \to 0$. More precisely, in Theorem 5.3.9 (and Corollaries 5.4.2 and 5.4.3) we show that, given a sequence of initial data $(z_0^\varepsilon, P_0^\varepsilon)$ which is compact in a suitable sense, if $t \mapsto (z^\varepsilon(t), P^\varepsilon(t))$ is a quasistatic evolution for the boundary datum $\phi^\varepsilon$ (according to (gs)–(eb)), satisfying $z^\varepsilon(0) = z_0^\varepsilon$ and $P^\varepsilon(0) = P_0^\varepsilon$, then defining the in-plane displacement

$$u^\varepsilon(t) := \frac{1}{\varepsilon^{\alpha-1}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\bigg(\begin{pmatrix}\phi_1^\varepsilon(t,z^\varepsilon(t))\\ \phi_2^\varepsilon(t,z^\varepsilon(t))\end{pmatrix} - x'\bigg)\,dx_3,$$

the out-of-plane displacement

$$v^\varepsilon(t) := \frac{1}{\varepsilon^{\alpha-2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi_3^\varepsilon(t, z^\varepsilon(t))\, dx_3$$

and the scaled linearized plastic strain

$$p^\varepsilon(t) := \frac{P^\varepsilon(t) - Id}{\varepsilon^{\alpha-1}},$$

for every $t \in [0,T]$ we have

$$p^\varepsilon(t) \to p(t) \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}),$$

where $p(t) \in L^2(\Omega; \mathbb{M}^{3\times 3})$ with $\mathrm{tr}\, p(t) = 0$ a.e. in $\Omega$. If $\alpha > 3$ there holds

$$u^\varepsilon(t) \to u(t) \quad \text{strongly in } W^{1,2}(\omega; \mathbb{R}^2), \tag{0.0.13}$$

$$v^\varepsilon(t) \to v(t) \quad \text{strongly in } W^{1,2}(\omega), \tag{0.0.14}$$

for every $t \in [0,T]$, where $u(t) \in W^{1,2}(\omega; \mathbb{R}^2)$ and $v(t) \in W^{2,2}(\omega)$. If $\alpha = 3$, the convergence of the in-plane and out-of-plane displacements holds only on a $t$-dependent subsequence. Moreover, $t \mapsto (u(t), v(t), p(t))$ is a solution of the reduced quasistatic evolution problem associated to the functionals $\mathcal{J}_\alpha$ defined in (0.0.12).

The proof of this results follows along the general lines of [50]. A major difficulty in the proof of the reduced energy balance is related to the compactness of the stress tensors $E^\varepsilon(t)$. In fact, due to the physical growth assumptions on $W_{el}$, weak $L^2$ compactness of $E^\varepsilon(t)$ is in general not guaranteed. However, the sequence of stress tensors satisfies the following properties: there exists a sequence of sets $O_\varepsilon(t)$, which converges in measure to $\Omega$, such that on $O_\varepsilon(t)$ the stresses $E^\varepsilon(t)$ are weakly compact in $L^2$, while in the complement of $O_\varepsilon(t)$ their contribution is negligible in the $L^1$ norm. This mixed-type convergence is enough to pass to the limit in the three-dimensional energy balance. This argument of proof is similar to that used in [55] by Mora and Scardia, to prove convergence of critical points for thin plates under physical growth conditions for the energy density.

A further difficulty arises because of the physical growth conditions on $W_{el}$: the global stability (gs) does not secure that $z^\varepsilon(t)$ fulfills the usual Euler-Lagrange equations. This is crucial to identify the limit stress tensor. This issue is overcome by proving that $z^\varepsilon(t)$ satisfies the analogue of an alternative first order condition introduced by Ball in [6, Theorem 2.4] in the context of nonlinear elasticity (see Section 1.3), and by adapting some techniques in [55].

Finally, to obtain the reduced global stability condition, we need an approximation result for triples $(u, v, p) \in W^{1,2}(\omega; \mathbb{R}^2) \times W^{2,2}(\omega) \times L^2(\Omega; \mathbb{M}^{3\times 3})$ such that

$$u = 0, \quad v = 0, \quad \nabla' v = 0 \quad \mathcal{H}^1 \text{ - a.e. on } \gamma_d \tag{0.0.15}$$

in terms of smooth triples. This is achieved arguing as in the linearized elasto-plastic setting (Section 3.3), under additional regularity assumptions on $\partial\omega$ and on $\gamma_d$ (see Lemma 5.2.1).

The results of Chapter 2 will appear in [18]. The results of Chapter 3 have been obtained in collaboration with Maria Giovanna Mora, and will appear in [21]. The content of Chapter 4 corresponds to the article [19] and that of Chapter 5 to the article [20].

# Chapter 1

# Preliminary results

In this chapter we collect some notation and preliminary results that will be useful in the sequel.

The first three sections contain some results related to dimension reduction problems in the framework of elasticity: in Section 1.1 we collect some statements of the Korn inequalities and we recall a lemma due to J.L. Lions that will be crucial in Chapter 2. In Section 1.2 we recall the rigidity estimate proved by Friesecke, James and Müller in [33], whereas Section 1.3 concerns an alternative first order stationarity condition proved by Ball in [7], which is compatible with physical growth conditions for the elastic energy density.

Section 1.4 is a collection of the main properties of functions of bounded deformation and of bounded Hessian, which will play a key role in Chapter 3, whereas the last section concerns to two slightly refined versions of the classical Helly theorem that we will use in Chapter 5.

**Notation**

Throughout the thesis we shall denote the canonical basis of $\mathbb{R}^3$ by $\{e_1, e_2, e_3\}$. The $k$-th component of a vector $v$ will be denoted by $v_k$. For every $v, w \in \mathbb{R}^n$, we shall denote their scalar product by $v \cdot w$. We endow the space $\mathbb{M}^{n\times n}$ of $n \times n$ matrices with the euclidean norm

$$|M| := \sqrt{Tr(M^T M)} = \sqrt{\sum_{i,j=1,\cdots,n} m_{ij}^2}$$

and denote by the colon : the associated scalar product. We shall adopt the classical notation to indicate the following subsets of $\mathbb{M}^{n\times n}$:

$$\begin{aligned}
\mathbb{M}^{n\times n}_{+} &:= \{F \in \mathbb{M}^{n\times n} : \det F > 0\},\\
\mathbb{M}^{n\times n}_{sym} &:= \{F \in \mathbb{M}^{n\times n} : F = F^T\},\\
\mathbb{M}^{n\times n}_{skew} &:= \{F \in \mathbb{M}^{n\times n} : F = -F^T\},\\
\mathbb{M}^{n\times n}_{D} &:= \{F \in \mathbb{M}^{n\times n}_{sym} : \mathrm{tr}\, F = 0\},\\
SO(n) &:= \{F \in \mathbb{M}^{n\times n}_{+} : F^T F = Id\}.
\end{aligned}$$

For every $j \in \mathbb{N}$, we will denote by $C^j_0(A;\mathbb{R}^m)$ and $C^\infty_0(A;\mathbb{R}^m)$ respectively the standard spaces of $C^j$ and $C^\infty$ functions with compact support in $A$.

## 1.1 Korn inequalities

Korn inequalities are an essential tool to establish coerciveness of differential operators in the framework of linear elasticity. Indeed these inequalities allow to bound the $W^{1,p}$ norm of a map with the $L^p$ norms of the symmetric part of its gradient and of the map itself.

The classical statement of Korn inequalities on bounded Lipschitz domains reads as follows.

**Proposition 1.1.1** (Korn inequalities)**.** *Let $U$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and let $1 < p < +\infty$. Consider the space*

$$E^p(U) := \{u \in L^p(U;\mathbb{R}^n) : \operatorname{sym}\nabla u \in L^p(U;\mathbb{M}^{n\times n})\}.$$

*Then $E^p(U) = W^{1,p}(U;\mathbb{R}^n)$,*

$$\|u\|_{W^{1,p}(U;\mathbb{R}^n)} \leq C_p(U)(\|u\|_{L^p(U;\mathbb{R}^n)} + \|\operatorname{sym}\nabla u\|_{L^p(U;\mathbb{M}^{n\times n})})$$

*and*

$$\min\{\|u - Ax - b\|_{W^{1,p}(U;\mathbb{R}^n)} : A \in \mathbb{M}^{n\times n}_{skew},\, b \in \mathbb{R}^n\} \leq C_p(U)\|\operatorname{sym}\nabla u\|_{L^p(U;\mathbb{M}^{n\times n})}.$$

*If $\Gamma \subset \partial U$ has positive $\mathcal{H}^{n-1}$ measure then*

$$\|u\|_{W^{1,p}(U;\mathbb{R}^n)} \leq C_p(U;\Gamma)\|\operatorname{sym}\nabla u\|_{L^p(U;\mathbb{M}^{n\times n})} \quad \text{for all } u \text{ such that } u = 0 \quad \mathcal{H}^{n-1}\text{ - a.e. on } \Gamma.$$

*Proof.* See [34, Proposition 1]. □

For a survey on Korn inequalities on bounded domains we refer to [36]. Some Korn inequalities can be proved also for general surfaces, by introducing a formulation with curvilinear coordinates. A crucial result in this framework is the following lemma, due to J.L. Lions.

**Lemma 1.1.2** (Lemma of J.L. Lions)**.** *Let $U$ be a bounded, connected, open set in $\mathbb{R}^n$ with Lipschitz boundary and let $v$ be a distribution on $U$. If $v \in W^{-1,2}(U)$ and $\partial_i v \in W^{-1,2}(U)$ for $i = 1, \cdots, n$, then $v \in L^2(U)$.*

By combining Lemma 1.1.2 and the closed graph theorem we obtain in particular the following result, that we will use in Chapter 2 to prove a rescaled Korn inequality and to characterize the class of limit displacements and bending moments.

**Corollary 1.1.3.** *Let $U$ be a bounded, connected, open set in $\mathbb{R}^n$ with Lipschitz boundary and let $(v^n)$ be a sequence of distributions in $W^{-1,2}(U)$. If there exists a map $v \in L^2(U)$ such that*

$$v^n \to v \quad \text{strongly in } W^{-1,2}(U),$$
$$\nabla v^n \to \nabla v \quad \text{strongly in } W^{-1,2}(U;\mathbb{R}^n),$$

*then*

$$v^n \to v \quad \text{strongly in } L^2(U).$$

An overview on standard Korn inequalities in curvilinear coordinates as well as a detailed bibliography on Lions Lemma can be found in [13, Sections 2.6 and 2.7] and [13, Section 1.7], respectively.

## 1.2 The rigidity estimate

A tool that will be crucial to establish compactness of deformations with equibounded elastic energies is the following rigidity estimate, due to Friesecke, James, and Müller [33, Theorem 3.1].

**Theorem 1.2.1.** *Let $U$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Then there exists a constant $C(U)$ with the following properties: for every $v \in W^{1,2}(U;\mathbb{R}^n)$ there is an associated rotation $R \in SO(n)$ such that*

$$\|\nabla v - R\|_{L^2(U)} \leq C(U)\|\mathrm{dist}(\nabla v, SO(n))\|_{L^2(U)}.$$

**Remark 1.2.2.** The constant $C(U)$ in Theorem 1.2.1 is invariant by translations and dilations of $U$ and is uniform for families of sets which are uniform bi-Lipschitz images of a cube.

The previous theorem implies, in particular, the following result.

**Corollary 1.2.3** (Liouville Theorem)**.** *Let $U$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Let $v \in W^{1,2}(U,\mathbb{R}^n)$ be such that $\nabla v(x) \in SO(3)$ for a.e. $x \in U$. Then, there exists $R \in SO(3)$ such that $\nabla v = R$, that is $v$ is a rigid motion.*

## 1.3 Ball's first order stationarity condition

In this section we recall a first order stationarity condition proved by Ball in [7, Theorem 2.4] in the framework of nonlinear elasticity. A modified version of (1.3.1) will be essential in Chapter 5 to identify the limit stress tensor.

**Theorem 1.3.1.** *Let $W : \mathbb{M}^{3\times 3} \to [0, +\infty]$ be a map satisfying the following assumptions:*

- *$W(F) = +\infty$ for every $F \in \mathbb{M}^{3\times 3}$, $W(F) \to +\infty$ as $\det F \to 0^+$,*
- *$W$ is $C^1$ on $\mathbb{M}^{3\times 3}_+$,*
- *There exists a constant $k$ such that $|DW(F)F^T| \leq k(W(F)+1)$ for every $F \in \mathbb{M}^{3\times 3}_+$.*

*Let $U \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary $\partial U = \partial U_1 \cup \partial U_2 \cup N$, where $\partial U_1$ and $\partial U_2$ are disjoint and open in the relative topology of $\partial U$, and $N$ has null $\mathcal{H}^2$ measure. Let $\overline{\omega} \in H^{1/2}(\partial U, \mathbb{R}^3)$ and let $f \in L^2(U, \mathbb{R}^3)$. Let $\omega \in W^{1,2}(U, \mathbb{R}^3)$ be a local minimum of the functional*

$$\mathcal{F}(\omega) := \int_U W(\nabla\omega)dz - \int_U f \cdot \omega dz$$

*satisfying the boundary condition*

$$\omega = \overline{\omega} \quad \mathcal{H}^2 \text{ - a.e. on } \partial U_1,$$

*namely, assume there exists $\epsilon > 0$ such that $\mathcal{F}(\omega) \leq \mathcal{F}(v)$ for every $v \in W^{1,2}(U, \mathbb{R}^3)$ satisfying*

$$\begin{cases} \|v - \omega\|_{W^{1,2}(U;\mathbb{R}^3)} \leq \epsilon, \\ v = \overline{\omega} \quad \mathcal{H}^2 \text{ - a.e. on } \partial U_1. \end{cases}$$

*Then:*

$$\int_U DW(\nabla\omega)(\nabla\omega)^T : \nabla\phi(\omega)dz = \int_U f\cdot\phi(\omega)dz \tag{1.3.1}$$

*for every* $\phi \in C^1(\mathbb{R}^3,\mathbb{R}^3)\cap W^{1,\infty}(\mathbb{R}^3,\mathbb{R}^3)$ *such that* $\phi\circ\omega = 0$ $\mathcal{H}^2$ *- a.e. on* $\partial U_1$.

We omit the proof of this result, which can be found in [7, Proof of Theorem 2.4]. We only remark that the main idea of the proof is to perform external variations of the form

$$\omega_\tau(z) := \omega(z) + \tau\phi(\omega(z)),$$

where $\tau\in\mathbb{R}$ is a small parameter, which is supposed to be tending to zero, and $\phi\in C^1(\mathbb{R}^3,\mathbb{R}^3)\cap W^{1,\infty}(\mathbb{R}^3,\mathbb{R}^3)$.

## 1.4 Functions of bounded deformation and bounded Hessian

In this section we recall some notions from measure theory and from the theory of functions with bounded deformation and with bounded Hessian.

**Measures.** Given a Borel set $B\subset\mathbb{R}^N$ and a finite dimensional Hilbert space $X$, $M_b(B;X)$ denotes the space of all bounded Borel measures on $B$ with values in $X$, endowed with the norm $\|\mu\|_{M_b} := |\mu|(B)$, where $|\mu|\in M_b(B;\mathbb{R})$ is the variation of the measure $\mu$. For every $\mu\in M_b(B;X)$ we consider the Lebesgue decomposition $\mu=\mu^a+\mu^s$, where $\mu^a$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^N$ and $\mu^s$ is singular with respect to $\mathcal{L}^N$. If $\mu^s=0$, we always identify $\mu$ with its density with respect to $\mathcal{L}^N$, which is a function in $L^1(B;X)$.

If the relative topology of $B$ is locally compact, by Riesz representation Theorem the space $M_b(B;X)$ can be identified with the dual of $C_0(B;X)$, which is the space of all continuous functions $\varphi : B\to X$ such that the set $\{|\varphi|\geq\delta\}$ is compact for every $\delta>0$. The weak* topology on $M_b(B;X)$ is defined using this duality.

**Convex functions of measures.** For every $\mu\in M_b(B;X)$ let $d\mu/d|\mu|$ be the Radon-Nicodym derivative of $\mu$ with respect to its variation $|\mu|$. Let $H_0 : X\to[0,+\infty)$ be a convex and positively one-homogeneous function such that

$$r_0|\xi|\leq H_0(\xi)\leq R_0|\xi| \quad \text{for every } \xi\in X,$$

where $r_0$ and $R_0$ are two constants, with $0<r_0\leq R_0$. According to the theory of convex functions of measures, developed in [35], we introduce the nonnegative Radon measure $H_0(\mu)\in M_b(B)$ defined by

$$H_0(\mu)(A) := \int_A H_0\Big(\frac{d\mu}{d|\mu|}\Big)\, d|\mu|$$

for every Borel set $A\subset B$. We also consider the functional $\mathcal{H}_0 : M_b(B;X)\to[0,+\infty)$ defined by

$$\mathcal{H}_0(\mu) := H_0(\mu)(B) = \int_B H_0\Big(\frac{d\mu}{d|\mu|}\Big)\, d|\mu|$$

for every $\mu \in M_b(B;X)$. One can prove that $H_0(\mu)$ coincides with the measure studied in [60, Chapter II, Section 4]. Hence,

$$\mathcal{H}_0(\mu) = \sup\Big\{ \int_B \varphi : d\mu \ : \ \varphi \in C_0(B;X),\ \varphi(x) \in K_0 \text{ for every } x \in B \Big\}, \tag{1.4.1}$$

where $K_0 := \partial H_0(0)$ is the subdifferential of $H_0$ at $0$. Moreover, $\mathcal{H}_0$ is lower semicontinuous on $M_b(B;X)$ with respect to weak* convergence.

**Functions with bounded deformation.** Let $U$ be an open set of $\mathbb{R}^N$. The space $BD(U)$ of functions with *bounded deformation* is the space of all functions $u \in L^1(U;\mathbb{R}^N)$ whose symmetric gradient $Eu := \operatorname{sym} Du$ (in the sense of distributions) belongs to $M_b(U;\mathbb{M}^{N\times N}_{sym})$. It is easy to see that $BD(U)$ is a Banach space endowed with the norm

$$\|u\|_{L^1} + \|Eu\|_{M_b}.$$

We say that a sequence $(u^k)$ converges to $u$ weakly* in $BD(U)$ if $u^k \rightharpoonup u$ weakly in $L^1(U;\mathbb{R}^N)$ and $Eu^k \rightharpoonup Eu$ weakly* in $M_b(U;\mathbb{M}^{N\times N}_{sym})$. Every bounded sequence in $BD(U)$ has a weakly* converging subsequence. If $U$ is bounded and has Lipschitz boundary, $BD(U)$ can be embedded into $L^{N/(N-1)}(U;\mathbb{R}^N)$ and every function $u \in BD(U)$ has a trace, still denoted by $u$, which belongs to $L^1(\partial U;\mathbb{R}^N)$. Moreover, if $\Gamma$ is a nonempty open subset of $\partial U$, there exists a constant $C > 0$, depending on $U$ and $\Gamma$, such that

$$\|u\|_{L^1(\Omega)} \leq C\|u\|_{L^1(\Gamma)} + C\|Eu\|_{M_b}. \tag{1.4.2}$$

(see [60, Chapter II, Proposition 2.4 and Remark 2.5]). For the general properties of the space $BD(U)$ we refer to [60].

**Functions with bounded Hessian.** Let $U$ be an open set of $\mathbb{R}^N$. The space $BH(U)$ of functions with *bounded Hessian* is the space of all functions $u \in W^{1,1}(U)$ whose Hessian $D^2u$ belongs to $M_b(U;\mathbb{M}^{N\times N}_{sym})$. It is easy to see that $BH(U)$ is a Banach space endowed with the norm

$$\|u\|_{L^1} + \|\nabla u\|_{L^1} + \|D^2u\|_{M_b}.$$

If $U$ has the cone property, then $BH(U)$ coincides with the space of functions in $L^1(U)$ whose Hessian belongs to $M_b(U;\mathbb{M}^{N\times N}_{sym})$. If $U$ is bounded and has Lipschitz boundary, $BH(U)$ can be embedded into $W^{1,N/(N-1)}(U)$. If, in addition, the boundary of $U$ is $C^2$, then $BH(U)$ is embedded into $C(\overline{U})$, which is the space of all continuous functions on $\overline{U}$. Moreover, if $U$ is bounded and has a $C^2$ boundary, for every function $u \in BH(U)$ one can define the traces of $u$ and of $\nabla u$, still denoted by $u$ and $\nabla u$; they satisfy $u \in W^{1,1}(\partial U)$, $\nabla u \in L^1(\partial U;\mathbb{R}^N)$, and $\frac{\partial u}{\partial \tau} = \nabla u \cdot \tau$ in $L^1(\partial U)$, where $\tau$ is any tangent vector to $\partial U$. For the general properties of the space $BH(U)$ we refer to [23].

## 1.5 Helly Theorem

We conclude this chapter of preliminary results by recalling two generalizations of the classical Helly Theorem for real valued functions with uniformly bounded variation.

Let $X$ be the dual of a separable Banach space. Given $f : [0,T] \to X$ and $a, b \in [0,T]$ with $a \le b$ , denote the total variation of $f$ on $[a,b]$ by

$$\mathcal{V}(f;a,b) := \sup\Big\{ \sum_{i=1}^{N} \|f(t_i) - f(t_{i-1})\|_X \,:\, a = t_0 \le t_1 \le \cdots \le t_N = b,\, N \in \mathbb{N}\Big\}. \quad (1.5.1)$$

The first result of this section is a lemma proved by Dal Maso, DeSimone and Mora in [15, Lemma 7.2], which generalizes the classical Helly Theorem, as well as its extension to reflexive separable Banach spaces (see, e.g., [10, Chapter 1, Theorem 3.5]). We shall use this lemma in Chapter 5 to prove the existence of a quasistatic evolution for our reduced model.

**Lemma 1.5.1.** *Let $f_k : [0,T] \to X$ be a sequence of functions such that $f_k(0)$ and $\mathcal{V}(f_k; 0,T)$ are bounded uniformly with respect to $k$. Then there exist a subsequence, still denoted $f_k$, and a function $f : [0,T] \to X$ with bounded variation on $[0,T]$, such that $f_k(t) \rightharpoonup f(t)$ weakly* for every $t \in [0,T]$.*

In Chapter 5 we shall refer also to a different generalization of Helly Theorem proved by Mielke, Roubíček and Stefanelli in [50, Theorem A.1]. To state this result we first introduce some notations.

Let $\mathcal{Z}$ be a Hausdorff topological space. Assume that $(\mathcal{D}_k)_{k\in\mathbb{N}\cup\{+\infty\}}$ is a sequence of maps $\mathcal{D}_k : \mathcal{Z} \times \mathcal{Z} \to [0,+\infty]$ such that

(A.1) $\mathcal{D}_k(z,z) = 0$ for every $k \in \mathbb{N}$, $z \in \mathcal{Z}$,
and $\mathcal{D}_k(z_1,z_3) \le \mathcal{D}_k(z_1,z_2) + \mathcal{D}_k(z_2,z_3)$ for every $k \in \mathbb{N}$, $z_1,z_2,z_3 \in \mathcal{Z}$;

(A.2) For all sequentially compact $\mathcal{K} \subset \mathcal{Z}$ we have:
if $z_k \in \mathcal{K}$ and $\min\{\mathcal{D}_\infty(z_k,z), \mathcal{D}_\infty(z,z_k)\} \to 0$, then $z_k \to z$;

(A.3) If $z_k \to z$ and $\tilde{z}_k \to \tilde{z}$ then $D_\infty(z,\tilde{z}) \le \liminf_{k\to+\infty} \mathcal{D}_k(z_k,\tilde{z}_k)$.

For every function $z : [0,T] \to \mathcal{Z}$, for every $k \in \mathbb{N}\cup\{+\infty\}$ and $s,t \in [0,T]$ with $s < t$, set

$$Diss_k(z;[s,t]) := \sup\Big\{ \sum_{i=1}^{N} \mathcal{D}_k(z(t_{j-1}), z(t_j)),\, s = t_0 < t_1 < \cdots < t_N \le t,\, N \in \mathbb{N}\Big\}.$$

We are now in a position to state [50, Theorem A.1].

**Theorem 1.5.2.** *Assume that the sequence $(\mathcal{D}_k)$ satisfies conditions (A.1)–(A.3). Let $\mathcal{K}$ be a sequentially compact subset of $\mathcal{Z}$ and $z_k : [0,T] \to \mathcal{Z}$, $k \in \mathbb{N}$ be a sequence satisfying*

*(A.4) (i) $z_k(t) \in \mathcal{K}$ for every $t \in [0,T]$ and $k \in \mathbb{N}$ (ii) $\sup_{k\in\mathbb{N}} Diss_k(z_k;[0,T]) < +\infty$.*

*Then there exist a subsequence $(z_{k_l})_{l\in\mathbb{N}}$ and limit functions $z : [0,T] \to \mathcal{Z}$ and $\delta : [0,T] \to [0,+\infty]$ with the following properties:*

$$\begin{aligned}
&\delta(t) = \lim_{l\to+\infty} Diss_{k_l}(z_{k_l};[0,t]) \quad \text{for every } t \in [0,T],\\
&z_{k_l}(t) \to z(t) \quad \text{for every } t \in [0,T],\\
&Diss_\infty(z;[s,t]) \le \delta(t) - \delta(s) \quad \text{for every } s,t \in [0,T] \text{ with } s < t.
\end{aligned}$$

The previous theorem will be essential in Chapter 5 to prove convergence of time-dependent plastic strains in the framework of finite plasticity.

# Chapter 2

# Thin-walled beams in nonlinear elasticity

## 2.1 Overview of the chapter

A thin-walled beam is a three-dimensional body, whose length is much larger than the diameter of the cross-section, which, in turn, is much larger than the thickness of the cross-section. This kind of beams are commonly used in mechanical engineering, since they combine good resistance properties with a reasonably low weight.

In this chapter we consider a nonlinearly elastic thin-walled beam whose cross-section is a thin tubular neighbourhood of a smooth curve. Denoting by $h$ and $\delta_h$, respectively, the diameter and the thickness of the cross-section, we analyse the case where the scaling factor of the elastic energy is of order $\epsilon_h^2$, with $\epsilon_h/\delta_h^2 \to \ell \in [0, +\infty)$, and we rigorously deduce, by $\Gamma$-convergence techniques, different lower dimensional linearized models, according to the relative order of magnitude between the cross-section diameter and the cross-section thickness.

The chapter is organized as follows. In Section 2.2 we describe the setting of the problem. In Section 2.3 we prove a technical lemma and a rescaled Korn inequality in curvilinear coordinates. In Section 2.4 we discuss some approximation results for displacements and bending moments. Section 2.5 is devoted to the proof of the compactness results, while Section 2.6 to the liminf inequality. Finally, in Section 2.7 we construct the corresponding recovery sequences.

**Notation.** Throughout this chapter if $\alpha : (0, L) \longrightarrow \mathbb{R}^m$ is a function of the $x_1$ variable, we shall denote its derivative, when it exists, by $\alpha'$, while if $\alpha : (0, 1) \longrightarrow \mathbb{R}^m$ is a function of the $s$ variable, we shall denote its derivative by $\dot{\alpha}$.

## 2.2 Setting of the problem

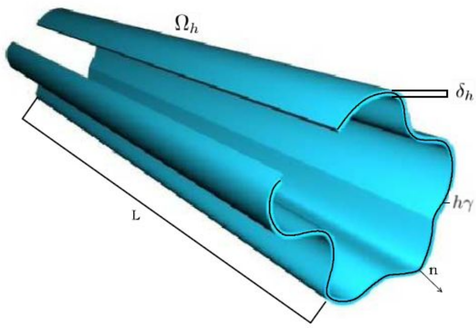
Let $(h), (\delta_h)$ be two sequences of positive numbers such that $h \to 0$ and

$$\lim_{h\to 0} \frac{\delta_h}{h} = 0. \tag{2.2.1}$$

We shall consider a thin-walled elastic beam, whose reference configuration is the set

$$\Omega_h := \big\{x_1 e_1 + h\gamma(s) + \delta_h t n(s) : x_1 \in (0, L),\, s \in (0,1),\, t \in \big(-\tfrac{1}{2}, \tfrac{1}{2}\big)\big\},$$

where $\gamma : [0,1] \longrightarrow \mathbb{R}^3$, $\gamma(s) = (0, \gamma_2(s), \gamma_3(s))$ is a simple, planar curve of class $C^6$ parametrized by arclength and $n(s)$ is the normal vector to the curve $\gamma$ at the point $\gamma(s)$.




We first introduce some notation. We shall denote by $\tau(s) := \dot\gamma(s)$ the tangent vector to $\gamma$ at the point $\gamma(s)$, so that

$$n(s) = \begin{pmatrix} 0 \\ -\tau_3(s) \\ \tau_2(s) \end{pmatrix}$$

for every $s \in [0,1]$. The orthonormal frame associated to the curve $\gamma$ is encoded by the map $R_0 : [0,1] \longrightarrow SO(3)$ given by

$$R_0(s) := \Big(e_1 \Big| \tau(s) \Big| n(s)\Big)$$

for every $s \in [0,1]$. Let $k(s) := \dot\tau(s) \cdot n(s)$ be the curvature of $\gamma$ at the point $\gamma(s)$. We shall assume that $k$ is not identically equal to zero. Finally, let $N, T : [0,1] \longrightarrow \mathbb{R}$ be the functions defined by $N(s) := \gamma(s) \cdot n(s)$ and $T(s) := \gamma(s) \cdot \tau(s)$ for every $s \in [0,1]$.

It will be useful to consider also the following quantities: the two-dimensional vectors

$$\overline{\tau}(s) := \begin{pmatrix} \tau_2(s) \\ \tau_3(s) \end{pmatrix}, \quad \overline{n}(s) := \begin{pmatrix} -\tau_3(s) \\ \tau_2(s) \end{pmatrix}$$

and the $2 \times 2$ rotation

$$\overline{R}_0(s) := (\overline{\tau}(s) \big| \overline{n}(s))$$

for every $s \in [0,1]$.

We define the elastic energy (per unit cross-section) associated with every deformation $u \in W^{1,2}(\Omega_h;\mathbb{R}^3)$ as

$$\mathcal{E}^h(u) := \frac{1}{h\delta_h}\int_{\Omega_h} W(\nabla u(x))dx, \tag{2.2.2}$$

where the stored-energy density $W : \mathbb{M}^{3\times 3} \to [0,+\infty]$ satisfies the usual assumptions in nonlinear elasticity, namely:

(H1) $W$ is continuous;

(H2) $W(RF) = W(F)$ for every $R \in SO(3)$, $F \in \mathbb{M}^{3\times 3}$ (frame indifference);

(H3) $W = 0$ on $SO(3)$;

(H4) $\exists C > 0$ such that $W(F) \geq C\,\mathrm{dist}^2(F, SO(3))$ for every $F \in \mathbb{M}^{3\times 3}$;

(H5) $W$ is of class $C^2$ in a neighbourhood of $SO(3)$.

### 2.2.1 Change of variables and formulation of the problem

As usual in problems of dimension reduction, we scale the deformations and the corresponding energy to a fixed domain. We set $\Omega := (0,L)\times(0,1)\times(-\frac{1}{2},\frac{1}{2})$ and we define the maps $\psi^h : \Omega \longrightarrow \Omega_h$ as

$$\psi^h(x_1,s,t) := x_1e_1 + h\gamma(s) + \delta_h t n(s),$$

for every $(x_1,s,t) \in \Omega$. We notice in particular that there exists $h_0 > 0$ such that $\psi^h$ is a bijection for every $h \in (0,h_0)$. To every deformation $u \in W^{1,2}(\Omega_h;\mathbb{R}^3)$ we associate a scaled deformation $y \in W^{1,2}(\Omega;\mathbb{R}^3)$, given by $y := u \circ \psi^h$. The elastic energy can be rewritten in terms of the scaled deformations as

$$\mathcal{E}^h(u) = \mathcal{J}^h(y) := \int_\Omega \Big(\frac{h-\delta_h tk}{h}\Big) W(\nabla_{h,\delta_h} y R_0^T)\, dx_1dsdt, \tag{2.2.3}$$

where

$$\nabla_{h,\delta_h} y := \Big(\partial_1 y \,\Big|\, \frac{1}{h-\delta_h tk}\,\partial_s y \,\Big|\, \frac{1}{\delta_h}\,\partial_t y\Big).$$

We observe that

$$\nabla_{h,\delta_h}\psi^h = R_0.$$

Moreover, denoting by

$$S := (0,1)\times(-\tfrac{1}{2},\tfrac{1}{2})$$

the scaled cross-section, since $k$ is a bounded function, by (2.2.1) we have

$$\frac{h-\delta_h tk}{h} \to 1 \tag{2.2.4}$$

uniformly in $\overline{S}$. In particular, for $h$ small enough it follows that $h - \delta_h tk > 0$ for every $s \in [0,1]$ and $t \in [-\frac{1}{2}, \frac{1}{2}]$.

Throughout this chapter we shall consider sequences of scaled deformations $(y^h)$ in $W^{1,2}(\Omega;\mathbb{R}^3)$ satisfying

$$\int_\Omega \Big(\frac{h-\delta_h tk}{h}\Big) W(\nabla_{h,\delta_h} y^h R_0^T)\, dx_1 ds dt \le C\epsilon_h^2, \tag{2.2.5}$$

where $(\epsilon_h)$ is a given sequence of positive numbers. We shall mainly focus on the case where $(\epsilon_h)$ is infinitesimal of order larger or equal than $(\delta_h^2)$, that is, we shall assume that

$$\exists \lim_{h\to 0} \frac{\epsilon_h}{\delta_h^2} =: \ell \in [0,+\infty). \tag{2.2.6}$$

## 2.3 Preliminary lemmas

In this section we collect two results which will be useful to prove a liminf inequality for the rescaled energies defined in (2.2.3). A first crucial result in the proof of the liminf inequality is a modified version of the Korn inequality in curvilinear coordinates (see Section 1.1).

We first fix some notation. We recall that $S = (0,1) \times (-\frac{1}{2}, \frac{1}{2})$. For any $\epsilon > 0$ and $v \in W^{1,2}(S;\mathbb{R}^2)$ we set

$$\overline{\nabla}_\epsilon v := \Big(\frac{1}{1-\epsilon tk}\, \partial_s v \Big| \frac{1}{\epsilon}\, \partial_t v\Big) \tag{2.3.1}$$

and we consider the subspace

$$M_\epsilon := \Big\{ v \in W^{1,2}(S;\mathbb{R}^2) : \operatorname{sym}(\overline{\nabla}_\epsilon v \overline{R}_0^T) = 0 \Big\}.$$

We remark that the expression $\operatorname{sym}(\overline{\nabla}_\epsilon v \overline{R}_0^T)$ represents the linearized strain associated with the displacement $v \circ (\overline{\psi}^\epsilon)^{-1}$, where

$$\overline{\psi}^\epsilon(s,t) := \overline{\gamma}(s) + \epsilon t \overline{n}(s) \tag{2.3.2}$$

for every $(s,t) \in S$. Since $M_\epsilon$ is closed in $W^{1,2}(S;\mathbb{R}^2)$, the orthogonal projection

$$\Pi_\epsilon : W^{1,2}(S;\mathbb{R}^2) \longrightarrow M_\epsilon$$

is well defined. We also introduce the set

$$M_0 := \Big\{ v \in W^{1,2}(S;\mathbb{R}^2) : \partial_t v = 0,\ \partial_s v \cdot \overline{\tau} = 0,\ \partial_s(\partial_s v \cdot \overline{n}) = 0 \Big\}, \tag{2.3.3}$$

which will play a key role in the proof of the Korn inequality.

The following characterization of the spaces $M_\epsilon$ and $M_0$ can be given.

**Lemma 2.3.1.** *Let $v \in M_0$. Then there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that*

$$v(s,t) = \begin{pmatrix} \alpha_2 \\ \alpha_3 \end{pmatrix} + \alpha_1 \begin{pmatrix} -\gamma_3(s) \\ \gamma_2(s) \end{pmatrix} \tag{2.3.4}$$

*for every $(s,t) \in S$.*
*Let $v \in M_\epsilon$. Then there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that*

$$v(s,t) = \begin{pmatrix} \alpha_2 \\ \alpha_3 \end{pmatrix} + \alpha_1 \begin{pmatrix} -\gamma_3(s) \\ \gamma_2(s) \end{pmatrix} - \epsilon t \alpha_1 \overline{\tau}(s) \tag{2.3.5}$$

*for every $(s,t) \in S$.*

*Proof.* It is immediate to see that, if $v \in M_0$, then $\partial_s v = \delta \overline{n}$ for some constant $\delta \in \mathbb{R}$, from which (2.3.4) follows.

If $v \in M_\epsilon$, then $v \circ (\overline{\psi}^\epsilon)^{-1}$ is an infinitesimal rigid displacement, that is, there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\Big(v \circ (\overline{\psi}^\epsilon)^{-1}\Big)(x_2, x_3) = \begin{pmatrix} \alpha_2 \\ \alpha_3 \end{pmatrix} + \alpha_1 \begin{pmatrix} -x_3 \\ x_2 \end{pmatrix}$$

for every $(x_2, x_3) \in \overline{\psi}^\epsilon(S)$. This implies (2.3.5). □

We are now in a position to state and prove a rescaled Korn inequality in curvilinear coordinates.

**Theorem 2.3.2** (Korn inequality)**.** *There exist two constants $\epsilon_0 > 0$ and $C > 0$ such that for every $\epsilon \in (0, \epsilon_0)$, $v \in W^{1,2}(S; \mathbb{R}^2)$, there holds*

$$\|v - \Pi_\epsilon(v)\|_{W^{1,2}(S;\mathbb{R}^2)} \le \frac{C}{\epsilon} \|\mathrm{sym}(\overline{\nabla}_\epsilon v \overline{R}_0^T)\|_{L^2(S;\mathbb{M}^{2\times 2})}. \tag{2.3.6}$$

**Remark 2.3.3.** An analogous dependance of Korn constant on the thickness of a thin structure has been proved, e.g. in [38, Proposition 4.1], in the case of a thin plate with rapidly varying thickness.

*Proof of Theorem 2.3.2.* By contradiction, assume there exist a sequence $(\epsilon_j)$ and a sequence of maps $(v^j) \subset W^{1,2}(S; \mathbb{R}^2)$ such that $\epsilon_j \to 0$ and

$$\|v^j - \Pi_{\epsilon_j}(v^j)\|_{W^{1,2}(S;\mathbb{R}^2)} > \frac{j}{\epsilon_j} \|\mathrm{sym}(\overline{\nabla}_{\epsilon_j} v^j \overline{R}_0^T)\|_{L^2(S;\mathbb{M}^{2\times 2})}, \tag{2.3.7}$$

for every $j \in \mathbb{N}$. Up to normalizations, we can assume that

$$\|v^j - \Pi_{\epsilon_j}(v^j)\|_{W^{1,2}(S;\mathbb{R}^2)} = 1. \tag{2.3.8}$$

We set $\phi^j := v^j - \Pi_{\epsilon_j}(v^j)$. By definition $\phi^j \in W^{1,2}(S; \mathbb{R}^2)$, $\phi^j$ is orthogonal to $M_{\epsilon_j}$ in the sense of $W^{1,2}$, and

$$\|\mathrm{sym}(\overline{\nabla}_{\epsilon_j} \phi^j \overline{R}_0^T)\|_{L^2(S;\mathbb{M}^{2\times 2})} < \frac{\epsilon_j}{j} \tag{2.3.9}$$

for every $j$. By the normalization hypothesis (2.3.8), we have $\|\phi^j\|_{W^{1,2}(S;\mathbb{R}^2)} = 1$ for every $j$. Hence, there exists $\phi \in W^{1,2}(S; \mathbb{R}^2)$ such that, up to subsequences, $\phi^j \rightharpoonup \phi$ weakly in $W^{1,2}(S; \mathbb{R}^2)$.

Let now $u \in M_0$. We claim that there exists a sequence $(u^j)$ such that $u^j \in M_{\epsilon_j}$ for every $j \in \mathbb{N}$ and $u^j \to u$ strongly in $W^{1,2}(S; \mathbb{R}^2)$. Indeed, by Lemma 2.3.1, the map $u$ has the following structure:

$$u = \begin{pmatrix} \alpha_2 \\ \alpha_3 \end{pmatrix} + \alpha_1 \begin{pmatrix} -\gamma_3 \\ \gamma_2 \end{pmatrix}$$

for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. Therefore, the maps $u^j$ given by

$$u^j := u - \epsilon_j t \alpha_1 \overline{\tau}$$

have the required properties. Since $\langle \phi^j, u^j \rangle_{W^{1,2}} = 0$ for any $j \in \mathbb{N}$, passing to the limit we deduce

$$\langle \phi, u \rangle_{W^{1,2}} = 0 \quad \text{for every } u \in M_0, \tag{2.3.10}$$

that is $\phi$ is orthogonal to $M_0$ in the sense of $W^{1,2}$.

To deduce a contradiction we shall prove that the convergence of $(\phi^j)$ is actually strong in $W^{1,2}(S;\mathbb{R}^2)$ and $\phi \in M_0$.
To this purpose, we first remark that by (2.3.9) there holds

$$\|\mathrm{sym}(\overline{R}_0{}^T \overline{\nabla}_{\varepsilon_j} \phi^j)\|_{L^2(S;\mathbb{M}^{2\times 2})} = \|\overline{R}_0{}^T (\mathrm{sym}(\overline{\nabla}_{\epsilon_j} \phi^j \overline{R}_0^T))\overline{R}_0\|_{L^2(S;\mathbb{M}^{2\times 2})} \leq \frac{C\varepsilon_j}{j} \tag{2.3.11}$$

for every $j \in \mathbb{N}$. This implies, in particular, that

$$\partial_s \phi^j \cdot \overline{\tau} \to 0, \quad \partial_t \phi^j \cdot \overline{\tau} \to 0, \quad \text{ and } \quad \frac{1}{\epsilon_j} \partial_t \phi^j \cdot \overline{n} \to 0 \tag{2.3.12}$$

strongly in $L^2(S)$. To show the strong convergence of $\phi^j$ in $W^{1,2}(S;\mathbb{R}^2)$, it remains to prove that $\partial_s \phi^j \cdot \overline{n} \to \partial_s \phi \cdot \overline{n}$ strongly in $L^2(S)$. By Lemma 1.1.2 and Corollary 1.1.3, it is enough to prove that

$$\partial_s \phi^j \cdot \overline{n} \to \partial_s \phi \cdot \overline{n} \quad \text{strongly in } W^{-1,2}(S)$$

and

$$\nabla(\partial_s \phi^j \cdot \overline{n}) \to \nabla(\partial_s \phi \cdot \overline{n}) \quad \text{strongly in } W^{-1,2}(S;\mathbb{R}^2).$$

Convergence of $(\partial_s \phi^j \cdot \overline{n})$ in $W^{-1,2}(S)$ is a direct consequence of the strong convergence of $(\phi^j)$ in $L^2(S;\mathbb{R}^2)$, whereas strong convergence of $(\partial_t \partial_s \phi^j \cdot \overline{n})$ in $W^{-1,2}(S;\mathbb{R}^2)$ follows by the identity

$$\partial_t \partial_s \phi^j \cdot \overline{n} = \partial_s(\partial_t \phi^j \cdot \overline{n}) + k \partial_t \phi^j \cdot \overline{\tau}$$

and by property (2.3.12). To prove convergence of $(\partial_s(\partial_s \phi^j \cdot \overline{n}))$ we notice that, by (2.3.11),

$$\frac{1}{\epsilon_j} \|\partial_t (\mathrm{sym}(\overline{R}_0{}^T \overline{\nabla}_{\varepsilon_j} \phi^j)_{11})\|_{W^{-1,2}(S)} \leq \frac{C}{j} \tag{2.3.13}$$

for every $j \in \mathbb{N}$, and

$$\partial_s (\mathrm{sym}(\overline{R}_0{}^T \overline{\nabla}_{\varepsilon_j} \phi^j)_{12}) \to 0 \quad \text{strongly in } W^{-1,2}(S).$$

Furthermore,

$$\begin{aligned}
\frac{1}{\epsilon_j} \partial_t (\mathrm{sym}(\overline{R}_0{}^T \overline{\nabla}_{\varepsilon_j} \phi^j)_{11}) &= \frac{\partial_t \partial_s \phi^j \cdot \overline{\tau}}{\epsilon_j (1 - \epsilon_j tk)} + \frac{k(\partial_s \phi^j \cdot \overline{\tau})}{(1 - \epsilon_j tk)^2} \\
&= \frac{2\partial_s (\mathrm{sym}(\overline{R}_0{}^T \overline{\nabla}_{\varepsilon_j} \phi^j)_{12})}{1 - \epsilon_j tk} - \frac{k}{1 - \epsilon_j tk} \frac{\partial_t \phi^j \cdot \overline{n}}{\epsilon_j} \\
&- \frac{1}{1 - \epsilon_j tk} \partial_s \Big( \frac{\partial_s \phi^j \cdot \overline{n}}{1 - \epsilon_j tk} \Big) + \frac{k(\partial_s \phi^j \cdot \overline{\tau})}{(1 - \epsilon_j tk)^2}.
\end{aligned}$$

By combining (2.3.12) and (2.3.13), we obtain

$$\partial_s (\partial_s \phi^j \cdot \overline{n}) \to 0 \text{ strongly in } W^{-1,2}(S). \tag{2.3.14}$$

By collecting the previous remarks we deduce

$$\phi^j \to \phi \quad \text{strongly in } W^{1,2}(S;\mathbb{R}^2).$$

Now, on the one hand $\|\phi^j\|_{W^{1,2}(S;\mathbb{R}^2)} = 1$ for any $j \in \mathbb{N}$, hence $\|\phi\|_{W^{1,2}(S;\mathbb{R}^2)} = 1$. On the other hand by combining (2.3.10), (2.3.12) and (2.3.14) we deduce that $\phi$ both belong to $M_0$ and is orthogonal to $M_0$. Hence $\phi$ must be identically equal to zero. This leads to a contradiction and completes the proof of the lemma. □

Denote by $\omega$ the set

$$\omega := (0, L) \times (0, 1).$$

We conclude this section by proving a technical lemma.

**Lemma 2.3.4.** *Let* $(\alpha_i^h) \subset W^{-2,2}(0, L)$, $i = 1, 2, 3$, *and let* $f \in W^{-2,2}(\omega)$ *be such that*

$$\alpha_1^h N + \alpha_2^h \tau_2 + \alpha_3^h \tau_3 \rightharpoonup f \quad \textit{weakly in } W^{-2,2}(\omega), \tag{2.3.15}$$

*as* $h \to 0$. *Then, there exist* $\alpha_i \in W^{-2,2}(0, L)$, $i = 1, 2, 3$, *such that for every* $i$

$$\alpha_i^h \rightharpoonup \alpha_i \quad \textit{weakly in } W^{-2,2}(0, L),$$

*as* $h \to 0$, *and*

$$f = \alpha_1 N + \alpha_2 \tau_2 + \alpha_3 \tau_3. \tag{2.3.16}$$

*If, in addition, there exists* $g \in L^2(\omega)$ *such that* $f = \partial_s g$, *then* $\alpha_i \in L^2(0, L)$ *for every* $i = 1, 2, 3$. *If* $f = 0$, *then* $\alpha_i = 0$ *for every* $i = 1, 2, 3$.

*Proof.* To simplify the notation, throughout the proof we shall use the symbol $\langle \cdot \,, \cdot \rangle$ to denote the duality pairing between $W^{-2,2}(\omega)$ and $W_0^{2,2}(\omega)$.

We recall that every $\alpha \in W^{-2,2}(0, L)$ can be identified with an element of the space $W^{-2,2}(\omega)$ by setting

$$\langle \alpha, \delta \rangle := \int_0^1 \langle \alpha, \delta(s, \cdot) \rangle_{W^{-2,2}(0,L), W_0^{2,2}(0,L)} \, ds \tag{2.3.17}$$

for every $\delta \in C_0^\infty(\omega)$, and extending it by density to $W_0^{2,2}(\omega)$. Moreover, for every $\alpha \in W^{-2,2}(0, L)$ and $\beta \in C^2(0, 1)$, we can define the product $\alpha\beta$ as

$$\langle \alpha\beta, \delta \rangle := \langle \alpha, \beta\delta \rangle = \int_0^1 \langle \alpha, \delta(s, \cdot) \rangle_{W^{-2,2}(0,L), W_0^{2,2}(0,L)} \beta(s) \, ds$$

for every $\delta \in C_0^\infty(\omega)$.

Consider now the maps $\varphi \in W_0^{2,2}(0, L)$ and $\psi \in C_0^{j+2}(0, 1)$, with $j \in \mathbb{N}$. We claim that

$$\langle \alpha_i^h, \varphi \partial_s^j \psi \rangle = 0. \tag{2.3.18}$$

Indeed, let $(\varphi^l) \subset C_0^\infty(0, L)$ be such that $\varphi^l \to \varphi$ strongly in $W^{2,2}(0, L)$. Then,

$$\langle \alpha_i^h, \varphi \partial_s^j \psi \rangle = \lim_{l \to +\infty} \langle \alpha_i^h, \varphi^l \partial_s^j \psi \rangle.$$

On the other hand,

$$\langle \alpha_i^h, \varphi^l \partial_s^j \psi \rangle = \int_0^1 \partial_s \langle \alpha_i^h, \varphi^l \partial_s^{j-1} \psi \rangle_{W^{-2,2}(0,L), W_0^{2,2}(0,L)} \, ds = 0$$

for every $l \in \mathbb{N}$. Therefore, we deduce claim (2.3.18).

By (2.3.15), for every $\varphi \in W_0^{2,2}(0,L)$ and $\psi \in C_0^{j+2}(0,1)$, there holds

$$\langle \alpha_1^h N, \varphi \partial_s^j \psi \rangle + \sum_{i=2,3} \langle \alpha_i^h \tau_i, \varphi \partial_s^j \psi \rangle \to \langle f, \varphi \partial_s^j \psi \rangle.$$

Claim (2.3.18) yields

$$\langle \alpha_1^h kT + \alpha_2^h k\tau_3 - \alpha_3^h k\tau_2, \varphi \partial_s^{j-1} \psi \rangle \to \langle f, \varphi \partial_s^j \psi \rangle. \tag{2.3.19}$$

Hence, choosing $j = 1$, we obtain

$$\langle \alpha_1^h kT + \alpha_2^h k\tau_3 - \alpha_3^h k\tau_2, \varphi \psi \rangle \to \langle f, \varphi \partial_s \psi \rangle \tag{2.3.20}$$

for every $\varphi \in W_0^{2,2}(0,L)$ and $\psi \in C_0^3(0,1)$.

Let now $\varphi \in W_0^{2,2}(0,L)$ and $\psi \in C_0^{j+3}(0,1)$. Taking $\varphi \partial_s^j \psi$ as test function in (2.3.20) and applying again (2.3.18), we deduce

$$\langle -\alpha_1^h(\dot{k}T + k + k^2 N) - \alpha_2^h(\dot{k}\tau_3 + k^2\tau_2) + \alpha_3^h(\dot{k}\tau_2 - k^2\tau_3), \varphi \partial_s^{j-1} \psi \rangle \to \langle f, \varphi \partial_s^{j+1} \psi \rangle,$$

which in turn gives

$$\langle -\alpha_1^h(\dot{k}T + k + k^2 N) - \alpha_2^h(\dot{k}\tau_3 + k^2\tau_2) + \alpha_3^h(\dot{k}\tau_2 - k^2\tau_3), \varphi \psi \rangle \to \langle f, \varphi \partial_s^2 \psi \rangle, \tag{2.3.21}$$

for every $\varphi \in W^{2,2}(0,L)$ and $\psi \in C_0^4(0,1)$.

Consider a map $\phi \in C_0^\infty(0,1)$. By regularity of the curve $\gamma$, the map $k\phi$ belongs to $C_0^4(0,1)$. Therefore, for every $\varphi \in W_0^{2,2}(0,L)$ we can choose $\varphi k \phi$ as test function in (2.3.21) and we obtain

$$\langle -\alpha_1^h(k\dot{k}T + k^2 + k^3 N) - \alpha_2^h(k\dot{k}\tau_3 + k^3\tau_2) + \alpha_3^h(k\dot{k}\tau_2 - k^3\tau_3), \varphi \phi \rangle \to \langle f, \varphi \partial_s^2(k\phi) \rangle.$$

On the other hand, by (2.3.15) there holds

$$\langle \alpha_1^h N + \sum_{1=2,3} \alpha_i^h \tau_i, \varphi k^3 \phi \rangle \to \langle f, \varphi k^3 \phi \rangle,$$

whereas (2.3.20) yields

$$\langle \alpha_1^h kT + \alpha_2^h k\tau_3 - \alpha_3^h k\tau_2, \varphi \dot{k} \phi \rangle \to \langle f, \varphi \partial_s(\dot{k}\phi) \rangle.$$

By collecting the previous remark we deduce

$$\langle \alpha_1^h, \varphi k^2 \phi \rangle \to -\langle f, \varphi(\partial_s^2(k\phi) + k^3\phi + \partial_s(\dot{k}\phi)) \rangle \tag{2.3.22}$$

for every $\varphi \in W^{2,2}(0,L)$ and $\phi \in C_0^\infty(0,1)$.

Let now $\overline{\phi} \in C_0^\infty(0,1)$ be such that

$$\int_0^1 k^2 \overline{\phi} \, ds = 1$$

(such $\overline{\phi}$ exists because $k$ is not identically equal to zero in $(0,1)$). Convergence (2.3.22) implies that

$$\alpha_1^h \rightharpoonup \alpha_1 \text{ weakly in } W^{-2,2}(0,L), \tag{2.3.23}$$

where

$$\langle \alpha_1, \varphi \rangle_{W^{-2,2}(0,L),W_0^{2,2}(0,L)} = -\langle f, \varphi(\partial_s^2(k\overline{\phi}) + k^3\overline{\phi} + \partial_s(\dot{k}\overline{\phi}))\rangle \tag{2.3.24}$$

for every $\varphi \in W_0^{2,2}(0,L)$. By definition (2.3.17) it is immediate to see that, identifying $\alpha_1^h, \alpha_1$ with elements of $W^{-2,2}(\omega)$, we also have

$$\alpha_1^h \rightharpoonup \alpha_1 \quad \text{weakly in } W^{-2,2}(\omega). \tag{2.3.25}$$

Let again $\varphi \in W_0^{2,2}(0,L)$ and $\phi \in C_0^\infty(0,1)$. Taking $\varphi k\tau_2\phi$ and $\varphi\tau_3\phi$ as test functions respectively in (2.3.15) and (2.3.20) we deduce

$$\langle \alpha_1^h N + \alpha_2^h \tau_2 + \alpha_3^h \tau_3, \varphi k\tau_2\phi \rangle \to \langle f, \varphi k\tau_2\phi \rangle \tag{2.3.26}$$

and

$$\langle \alpha_1^h kT + \alpha_2^h k\tau_3 - \alpha_3^h k\tau_2, \varphi\tau_3\phi \rangle \to \langle f, \varphi\partial_s(\tau_3\phi)\rangle. \tag{2.3.27}$$

By summing (2.3.26) and (2.3.27) and using (2.3.25), we obtain

$$\langle \alpha_2^h, k\phi\varphi \rangle \to \langle f, \varphi(k\tau_2\phi + \partial_s(\tau_3\phi))\rangle - \langle \alpha_1, \varphi k\gamma_3\phi \rangle$$

for every $\varphi \in W_0^{2,2}(0,L)$ and $\phi \in C_0^\infty(0,1)$.

Choosing $\widehat{\phi}$ such that $\int_0^1 k\widehat{\phi}\, ds = 1$ and arguing as in the proof of (2.3.23), we deduce

$$\alpha_2^h \rightharpoonup \alpha_2 \text{ weakly in } W^{-2,2}(0,L), \tag{2.3.28}$$

where

$$\langle \alpha_2, \varphi \rangle_{W^{-2,2}(0,L),W^{2,2}(0,L)} = \langle f, \varphi(k\tau_2\widehat{\phi} + \partial_s(\tau_3\widehat{\phi}))\rangle - \langle \alpha_1, \varphi k\gamma_3\widehat{\phi} \rangle \tag{2.3.29}$$

for every $\varphi \in W_0^{2,2}(0,L)$.

Similarly, one can prove that

$$\alpha_3^h \rightharpoonup \alpha_3 \text{ weakly in } W^{-2,2}(0,L) \tag{2.3.30}$$

where

$$\langle \alpha_3, \varphi \rangle_{W^{-2,2}(0,L),W_0^{2,2}(0,L)} = \langle f, \varphi(k\tau_3\widehat{\phi} - \partial_s(\tau_2\widehat{\phi}))\rangle + \langle \alpha_1, \varphi k\gamma_2\widehat{\phi} \rangle \tag{2.3.31}$$

for every $\varphi \in W_0^{2,2}(0,L)$.

By combining (2.3.15), (2.3.25), (2.3.28), and (2.3.30), we obtain the representation (2.3.16).

If $f = \partial_s g$, with $g \in L^2(\omega)$, then by (2.3.24) there holds

$$\langle \alpha_1, \varphi \rangle_{W^{-2,2}(0,L),W_0^{2,2}(0,L)} = \int_0^L \int_0^1 g\partial_s(\partial_s^2(k\overline{\phi}) + k^3\overline{\phi} + \partial_s(\dot{k}\overline{\phi}))\varphi\, ds dx_1,$$

for every $\varphi \in W_0^{2,2}(0,L)$. This implies that $\alpha_1 \in L^2(0,L)$. Similarly equalities (2.3.29) and (2.3.31) yield $\alpha_2, \alpha_3 \in L^2(0,L)$.

Finally, if $f = 0$, by properties (2.3.24), (2.3.29) and (2.3.31) it follows immediately that $\alpha_i = 0$ for every $i$. □

## 2.4 Limit classes of displacements and bending moments and approximation results

In this section we introduce some classes of displacements and bending moments, that will play a key role in the characterization of the limit models, and we discuss their properties and their approximation by means of smooth functions.

We begin by introducing the limit class of the tangential derivatives of the tangential displacements

$$\mathcal{G} := \Big\{ g \in L^2(\omega) : \exists (v^\epsilon) \subset C^5(\overline{\omega}; \mathbb{R}^3) \text{ such that } \partial_s v_1^\epsilon + \partial_1 v^\epsilon \cdot \tau = 0, \\ \partial_s v^\epsilon \cdot \tau = 0 \text{ for every } \epsilon > 0 \text{ and } g = \lim_{\epsilon \to 0} \partial_1 v_1^\epsilon \Big\}, \quad (2.4.1)$$

where the limit is intended with respect to the strong convergence in $L^2(\omega)$. In other words, if for every $v \in W^{1,2}(\omega; \mathbb{R}^3)$ we consider the symmetric gradient $e(v) \in L^2(\omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ of $v$, defined by

$$e(v) := \begin{pmatrix} \partial_1 v_1 & \frac{1}{2}(\partial_s v_1 + \partial_1 v \cdot \tau) \\ \frac{1}{2}(\partial_s v_1 + \partial_1 v \cdot \tau) & \partial_s v \cdot \tau \end{pmatrix}, \quad (2.4.2)$$

a function $g \in L^2(\omega)$ belongs to $\mathcal{G}$ if and only if there exists a sequence $(v^\epsilon) \subset C^5(\overline{\omega}; \mathbb{R}^3)$ such that

$$e(v^\epsilon) = \begin{pmatrix} \partial_1 v_1^\epsilon & 0 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}$$

strongly in $L^2(\omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$, as $\epsilon \to 0$.

The following characterization of the class $\mathcal{G}$ can be proved.

**Lemma 2.4.1.** *Let $g \in L^2(\omega)$ and assume there exists a sequence $(v^\epsilon) \subset W^{1,2}(\omega; \mathbb{R}^3)$ such that*

$$e(v^\epsilon) \rightharpoonup \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} \quad (2.4.3)$$

*weakly in $L^2(\omega; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ as $\epsilon \to 0$. Then $g \in \mathcal{G}$.*

*Proof.* Condition (2.4.3) can be rewritten as

$$\partial_1 v_1^\epsilon \rightharpoonup g \text{ weakly in } L^2(\omega), \quad (2.4.4)$$

$$\partial_s v_1^\epsilon + \partial_1 v^\epsilon \cdot \tau \rightharpoonup 0 \text{ weakly in } L^2(\omega), \quad (2.4.5)$$

$$\partial_s v^\epsilon \cdot \tau \rightharpoonup 0 \text{ weakly in } L^2(\omega). \quad (2.4.6)$$

Moreover, by Mazur Lemma, we may assume that the convergence in (2.4.4), (2.4.5) and (2.4.6) is strong in $L^2(\omega)$.

For every $\epsilon$, let $\widetilde{u}^\epsilon \in W^{1,2}(\omega)$, with $\partial_1^2 \widetilde{u}^\epsilon \in L^2(\omega)$, be such that $\partial_1 \widetilde{u}^\epsilon = v_1^\epsilon$. By (2.4.5) and Poincaré inequality

$$\partial_s \widetilde{u}^\epsilon + v^\epsilon \cdot \tau - \fint_0^L \partial_s \widetilde{u}^\epsilon \, dx_1 - \fint_0^L v^\epsilon \cdot \tau \, dx_1 \to 0 \quad (2.4.7)$$

strongly in $L^2(\omega)$. Let now $\nu^\epsilon \in W^{1,2}(\omega)$ be such that $\partial_s \nu^\epsilon = v^\epsilon \cdot \tau$. Setting

$$u^\epsilon := \widetilde{u}^\epsilon - \fint_0^L \widetilde{u}^\epsilon \, dx_1 - \fint_0^L \nu^\epsilon \, dx_1,$$

then $u^\epsilon \in W^{1,2}(\omega)$, $\partial_1^2 u^\epsilon \in L^2(\omega)$ and (2.4.7) yields

$$\partial_s u^\epsilon + v^\epsilon \cdot \tau \to 0 \quad \text{strongly in } L^2(\omega). \tag{2.4.8}$$

Finally, by (2.4.4) there holds

$$\partial_1^2 u^\epsilon \to g \quad \text{strongly in } L^2(\omega). \tag{2.4.9}$$

We want to approximate $u^\epsilon$ and $v^\epsilon$ by smooth functions in such a way that (2.4.9) holds and the quantities in (2.4.6) and (2.4.8) are equal to zero for every $\epsilon > 0$. To this purpose, we first extend $u^\epsilon$ and $v^\epsilon$ to the set

$$\omega_\delta := (-\delta, L+\delta) \times (0,1),$$

with $0 < \delta < \frac{L}{3}$. For every $\epsilon$, we define

$$\mathring{v}^\epsilon(x_1, s) := \begin{cases} v^\epsilon(x_1, s) & \text{in } \omega, \\ 6v^\epsilon(-x_1, s) - 8v^\epsilon(-2x_1, s) + 3v^\epsilon(-3x_1, s) & \text{in } (-\delta, 0) \times (0,1), \\ 6v^\epsilon(2L - x_1, s) - 8v^\epsilon(3L - 2x_1, s) + 3v^\epsilon(4L - 3x_1, s) & \text{in } (L, L+\delta) \times (0,1) \end{cases}$$

and

$$\mathring{u}^\epsilon(x_1, s) := \begin{cases} u^\epsilon(x_1, s) & \text{in } \omega, \\ 6u^\epsilon(-x_1, s) - 8u^\epsilon(-2x_1, s) + 3u^\epsilon(-3x_1, s) & \text{in } (-\delta, 0) \times (0,1), \\ 6u^\epsilon(2L - x_1, s) - 8u^\epsilon(3L - 2x_1, s) + 3u^\epsilon(4L - 3x_1, s) & \text{in } (L, L+\delta) \times (0,1). \end{cases}$$

Clearly, $\mathring{v}^\epsilon$ and $\mathring{u}^\epsilon$ are extensions of $v^\epsilon$ and $u^\epsilon$, respectively, to $\omega_\delta$. Moreover, we have $\mathring{u}^\epsilon \in W^{1,2}(\omega_\delta)$ with $\partial_1^2 \mathring{u}^\epsilon \in L^2(\omega_\delta)$, and $\mathring{v}^\epsilon \in W^{1,2}(\omega_\delta)$. Finally, by (2.4.6) and (2.4.8) we deduce

$$\partial_s \mathring{v}^\varepsilon \cdot \tau \to 0, \quad \text{and} \quad \partial_s \mathring{u}^\varepsilon + \mathring{v}^\varepsilon \cdot \tau \to 0 \quad \text{strongly in } L^2(\omega_\delta). \tag{2.4.10}$$

Furthermore, defining

$$\mathring{g} := \begin{cases} g(x_1, s) & \text{in } \omega, \\ 6g(-x_1, s) - 32g(-2x_1, s) + 27g(-3x_1, s) & \text{in } (-\delta, 0) \times (0,1), \\ 6g(2L - x_1, s) - 32g(3L - 2x_1, s) + 27g(4L - 3x_1, s) & \text{in } (L, L+\delta) \times (0,1) \end{cases}$$

there holds $\mathring{g} \in L^2(\omega_\delta)$, $\mathring{g} = g$ a.e. in $\omega$, and

$$\partial_1^2 \mathring{u}^\epsilon \to \mathring{g} \quad \text{strongly in } L^2(\omega_\delta). \tag{2.4.11}$$

We set $\mathring{v}_t^\epsilon := \mathring{v}^\epsilon \cdot n$ and $\mathring{v}_s^\epsilon := \mathring{v}^\epsilon \cdot \tau$. For every $\epsilon$, let $\overline{v}_t^\epsilon \in C^\infty(\overline{\omega}_\delta)$ be such that

$$\|\overline{v}_t^\epsilon - \mathring{v}_t^\epsilon\|_{W^{1,2}(\omega_\delta)} \le C\epsilon. \tag{2.4.12}$$

Let now $\overline{v}_s^\epsilon \in C^5(\overline{\omega}_\delta)$ be the solution of

$$\partial_s \overline{v}_s^\epsilon = k\overline{v}_t^\epsilon \quad \text{in } \omega_\delta, \tag{2.4.13}$$

satisfying $\int_0^1 \overline{v}_s^\epsilon(x_1,s)ds \in C^\infty(-\delta, L+\delta)$, with

$$\int_0^1 \overline{v}_s^\epsilon(x_1,s)ds - \int_0^1 \mathring{v}_s^\epsilon(x_1,s)ds \to 0 \quad \text{strongly in } L^2(-\delta, L+\delta).$$

By (2.4.13) we deduce

$$\|\partial_s(\overline{v}_s^\epsilon - \mathring{v}_s^\epsilon)\|_{L^2(\omega_\delta)} \le \|k(\overline{v}_t^\epsilon - \mathring{v}_t^\epsilon)\|_{L^2(\omega_\delta)} + \|k\mathring{v}_t^\epsilon - \partial_s\mathring{v}_s^\epsilon\|_{L^2(\omega_\delta)}.$$

Hence, owing to (2.4.10) and (2.4.12),

$$\|\partial_s(\overline{v}_s^\epsilon - \mathring{v}_s^\epsilon)\|_{L^2(\omega_\delta)} \to 0, \tag{2.4.14}$$

and by Poincaré inequality

$$\|\overline{v}_s^\epsilon - \mathring{v}_s^\epsilon\|_{L^2(\omega_\delta)} \to 0. \tag{2.4.15}$$

Finally, let $\overline{u}^\epsilon \in C^6(\overline{\omega}_\delta)$ be such that

$$\partial_s \overline{u}^\epsilon + \overline{v}_s^\epsilon = 0 \quad \text{in } \omega_\delta, \tag{2.4.16}$$

with $\int_0^1 \overline{u}^\epsilon(x_1,s)\,ds \in C^\infty(-\delta, L+\delta)$ and

$$\int_0^1 \overline{u}^\epsilon(x_1,s)\,ds - \int_0^1 \mathring{u}^\epsilon(x_1,s)\,ds \to 0 \quad \text{strongly in } L^2(-\delta, L+\delta).$$

By (2.4.16) there holds

$$\|\partial_s(\overline{u}^\epsilon - \mathring{u}^\epsilon)\|_{L^2(\omega_\delta)} \le \|\partial_s\mathring{u}^\epsilon + \mathring{v}_s^\epsilon\|_{L^2(\omega_\delta)} + \|\mathring{v}_s^\epsilon - \overline{v}_s^\epsilon\|_{L^2(\omega_\delta)}.$$

Therefore, by (2.4.10) and (2.4.15), we deduce

$$\partial_s(\overline{u}^\epsilon - \mathring{u}^\epsilon) \to 0 \quad \text{strongly in } L^2(\omega_\delta),$$

which in turn, by Poincaré inequality, yields

$$\overline{u}^\epsilon - \mathring{u}^\epsilon \to 0 \quad \text{strongly in } L^2(\omega_\delta). \tag{2.4.17}$$

To guarantee convergence of the second derivative in the $x_1$ variable of the sequence $(\overline{u}^\epsilon)$, we regularize both $(\overline{u}^\epsilon)$ and $(\overline{v}^\epsilon)$ by mollification in the $x_1$ variable. To this purpose, we consider a map $\rho \in C_0^\infty(-\lambda, \lambda)$ with $0 < \lambda < \delta$, and we define

$$\begin{cases} \widehat{v}_s^\epsilon(x_1,s) := (\overline{v}_s^\epsilon(\cdot,s) * \rho)(x_1), \\ \widehat{v}_t^\epsilon(x_1,s) := (\overline{v}_t^\epsilon(\cdot,s) * \rho)(x_1), \\ \widehat{u}^\epsilon(x_1,s) := (\overline{u}^\epsilon(\cdot,s) * \rho)(x_1), \end{cases}$$

for a.e. $(x_1,s) \in \omega$ and for every $\epsilon > 0$. The regularized maps satisfy $(\widehat{v}_t^\epsilon) \subset C^\infty(\overline{\omega})$, $(\widehat{v}_s^\epsilon) \subset C^5(\overline{\omega})$, and $(\widehat{u}^\epsilon) \subset C^6(\overline{\omega})$. Moreover, by (2.4.13) and (2.4.16), there holds

$$\partial_s \widehat{v}_s^\epsilon = k\widehat{v}_t^\epsilon \quad \text{and} \quad \partial_s \widehat{u}^\epsilon + \widehat{v}_s^\epsilon = 0 \quad \text{in } \omega.$$

Finally, (2.4.17) yields

$$\partial_1^2(\widehat{u}^\epsilon - (\mathring{u}^\epsilon(\cdot,s) * \rho)) = (\overline{u}^\epsilon(\cdot,s) - \mathring{u}^\epsilon(\cdot,s)) * \rho'' \to 0$$

strongly in $L^2(\omega)$ as $\epsilon \to 0$. On the other hand, by (2.4.11) we have

$$\partial_1^2(\mathring{u}^\epsilon(\cdot,s) * \rho) = \partial_1^2\mathring{u}^\epsilon(\cdot,s) * \rho \to \mathring{g}(\cdot,s) * \rho$$

strongly in $L^2(\omega)$ as $\epsilon \to 0$; hence we deduce

$$\partial_1^2\widehat{u}^\epsilon \to \mathring{g}(\cdot,s) * \rho$$

strongly in $L^2(\omega)$ as $\epsilon \to 0$.

The conclusion of the lemma follows now by considering a sequence of convolution kernels in the $x_1$ variable, and by applying a diagonal argument. □

**Remark 2.4.2.** An equivalent characterization of the class $\mathcal{G}$ is the following:

$$\mathcal{G} = \Big\{ g \in L^2(\omega) : \exists (u^\epsilon) \subset C^5(\overline{\omega}), (z^\epsilon) \subset C^4(\overline{\omega}) \text{ such that } \partial_s^2 u^\epsilon = kz^\epsilon \text{ for every } \epsilon > 0 \text{ and } g = \lim_{\epsilon\to 0} \partial_1 u^\epsilon \Big\}, \quad (2.4.18)$$

where the limit is intended with respect to the strong convergence in $L^2(\omega)$.

Indeed, let $\mathcal{G}'$ be the class defined in the right-hand side of (2.4.18). If $g \in \mathcal{G}$, setting $u^\epsilon = v_1^\epsilon$ and $z^\epsilon = -\partial_1 v^\epsilon \cdot n$ for every $\epsilon > 0$, it is easy to check that $g \in \mathcal{G}'$.
Viceversa, if $g \in \mathcal{G}'$, it is enough to define

$$v^\epsilon(x_1,s) = u^\epsilon(x_1,s)e_1 - \int_0^{x_1} (\partial_s u^\epsilon(\xi,s)\tau(s) + z^\epsilon(\xi,s)n(s))d\xi$$

for every $(x_1,s) \in \omega$ and for every $\epsilon > 0$. The conclusion follows then by Lemma 2.4.1.

By (2.4.18) it follows in particular that if $g \in \mathcal{G}$, then there exist $(\widehat{u}^\varepsilon) \subset C^5(\overline{\omega})$ and $(\widehat{z}^\varepsilon) \subset C^4(\overline{\omega})$ such that $\partial_s\widehat{u}^\varepsilon = k\widehat{z}^\varepsilon$ and $\partial_1^2\widehat{u}^\varepsilon \to g$ strongly in $L^2(\omega)$. Indeed, let $(u^\varepsilon)$ and $(z^\varepsilon)$ be the sequences in (2.4.18), and for every $\varepsilon$ let $\tilde{u}^\varepsilon \in C^5(\overline{\omega})$ be such that $\partial_1\tilde{u}^\varepsilon = u^\varepsilon$. Then

$$\partial_1\Big(\partial_s^2\tilde{u}^\varepsilon + k\int_0^{x_1} z^\varepsilon(\xi,s)\,d\xi\Big) = 0.$$

Hence, setting

$$\widehat{z}^\varepsilon := \int_0^{x_1} z^\varepsilon(\xi,s)\,d\xi,$$

we have $\widehat{z}^\varepsilon \in C^4(\overline{\omega})$ and there exists $\phi \in C^2([0,1])$ such that

$$\partial_s^2\tilde{u}^\varepsilon + k\widehat{z}^\varepsilon = \ddot{\phi}.$$

The thesis follows now by taking $\widehat{u}^\varepsilon := \tilde{u}^\varepsilon - \phi$.

**Remark 2.4.3.** The class $\mathcal{G}$ is always nonempty as it contains all $g \in L^2(\omega)$ which are affine with respect to $s$. Indeed, assume there exist $a_0, a_1 \in L^2(0,L)$ such that

$$g(x_1,s) = a_0(x_1) + sa_1(x_1)$$

for a.e. $(x_1, s) \in \omega$ and let $\widehat{a}_i \in W^{1,2}(0, L)$ be a map satisfying $\widehat{a}_i' = a_i$, $i = 0, 1$. Then, there exists $(\widehat{a}_i^\epsilon) \subset C^\infty([0, L])$ such that $\widehat{a}_i^\epsilon \to \widehat{a}_i$ strongly in $W^{1,2}(0, L)$ as $\epsilon \to 0$, $i = 0, 1$. Hence, setting

$$u^\epsilon(x_1, s) := \widehat{a}_0^\epsilon(x_1) + s\widehat{a}_1^\epsilon(x_1)$$

for every $(x_1, s) \in \omega$ and $z^\epsilon = 0$ for every $\epsilon > 0$, the claim follows by Remark 2.4.2.

We also remark that if $g \in L^2(\omega)$ and there exist $\alpha_i \in L^2(0, L)$, $i = 1, 2, 3$, such that

$$\partial_s g = \alpha_1 N + \alpha_2 \tau_2 + \alpha_3 \tau_3, \tag{2.4.19}$$

then $g \in \mathcal{G}$. Indeed, by (2.4.19) there exists $\alpha_4 \in L^2(0, L)$ such that

$$g = \alpha_1 \int_0^s N(\xi) d\xi + \alpha_2 \gamma_2 + \alpha_3 \gamma_3 + \alpha_4.$$

Let $\widehat{\alpha}_i \in W^{1,2}(0, L)$ be such that $\widehat{\alpha}_i' = \alpha_i$ for $i = 1, 2, 3$. Then, setting

$$u := \widehat{\alpha}_1 \int_0^s N(\xi) d\xi + \widehat{\alpha}_2 \gamma_2 + \widehat{\alpha}_3 \gamma_3 + \widehat{\alpha}_4,$$

and

$$z := -\widehat{\alpha}_1 T - \widehat{\alpha}_2 \tau_3 + \widehat{\alpha}_3 \tau_2,$$

we have $u \in W^{1,2}(\omega)$, $\partial_s^i u \in L^2(\omega)$ for $i = 2, \cdots, 6$, and $z \in W^{1,2}(\omega)$, with $\partial_s^i z \in L^2(\omega)$ for $i = 2, \cdots, 5$ and $\partial_s^2 u = kz$. For every $i = 1, \cdots, 4$ consider a sequence $(\alpha_i^\epsilon) \in C^\infty([0, L])$ such that $\alpha_i^\epsilon \to \widehat{\alpha}_i$ strongly in $W^{1,2}(0, L)$, as $\epsilon \to 0$. By defining

$$u^\epsilon := \alpha_1^\epsilon \int_0^s N(\xi) d\xi + \alpha_2^\epsilon \gamma_2 + \alpha_3^\epsilon \gamma_3 + \alpha_4^\epsilon,$$

and

$$z^\epsilon := -\alpha_1^\epsilon T - \alpha_2^\epsilon \tau_3 + \alpha_3^\epsilon \tau_2,$$

there holds

$$\partial_1 u^\epsilon \to g \quad \text{strongly in } L^2(\omega),$$

$$\partial_s^2 u^\epsilon = kz^\epsilon \quad \text{for every } \epsilon > 0,$$

and both sequences $(u^\epsilon)$ and $(z^\epsilon)$ have the required regularity.

**Remark 2.4.4.** The structure of the class $\mathcal{G}$ depends on the behaviour of the curvature $k$ of the curve $\gamma$.

For instance, if $k$ vanishes only at a finite number of points, then $\mathcal{G} = L^2(\omega)$. Indeed, let

$$0 = p_0 < p_1 < \cdots < p_m = 1$$

be such that $k(s) \neq 0$ for every $s \in (p_i, p_{i+1})$, $i = 0, \cdots, m-1$. For any function $g \in L^2(\omega)$ there exists a sequence $(g^\epsilon) \subset C_0^\infty\big((0, L) \times \bigcup_{i=0}^{m-1} (p_i, p_{i+1})\big)$ such that $g^\epsilon \to g$ strongly in $L^2(\omega)$. By choosing

$$u^\epsilon(x_1, s) = \int_0^{x_1} g^\epsilon(\xi, s) d\xi$$

for every $s \in (0,1)$, then $(u^\epsilon) \subset C^\infty(\overline{\omega})$ and for every $\epsilon > 0$ there exists $\lambda^\epsilon > 0$ such that

$$2\lambda^\epsilon < \min_{i=0,\cdots,m-1}(p_{i+1} - p_i)$$

and $\partial_s^2 u^\epsilon = 0$ in

$$(0,L) \times \bigcup_{i=0,\cdots,m-1} \big((p_i, p_i + \lambda^\epsilon) \cup (p_{i+1} - \lambda^\epsilon, p_{i+1})\big).$$

By setting

$$z^\epsilon = \begin{cases} \frac{\partial_s^2 u^\epsilon}{k} & \text{in } (0,L) \times \bigcup_{i=0}^{m-1} (p_i + \lambda^\epsilon, p_{i+1} - \lambda^\epsilon), \\ 0 & \text{otherwise} \end{cases}$$

we deduce immediately by Remark 2.4.2 that $g \in \mathcal{G}$.

Assume instead that the sign of $k$ has the following behaviour: there exists a finite number of points

$$0 = p_0 < p_1 < \cdots < p_m = 1$$

such that, for every $i = 0, \cdots, m-1$, there holds $k(s) > 0$ for every $s \in (p_i, p_{i+1})$, or $k(s) < 0$ for every $s \in (p_i, p_{i+1})$, or $k(s) = 0$ for every $s \in (p_i, p_{i+1})$. In other words,

$$\{s \in [0,1] : k(s) = 0\} = \bigcup_{i \in I_1} [p_{i-1}, p_i] \cup \bigcup_{i \in I_2} \{p_i\}.$$

with $I_1 \subset \{1, \cdots, m\}$, $I_2 \subset \{0, \cdots, m\}$ disjoint. Then

$$\mathcal{G} := \Big\{g \in L^2(\omega) : g \text{ is affine in the } s \text{ variable in } (0,L) \times \bigcup_{i \in I_1} (p_i, p_{i+1})\Big\}. \qquad (2.4.20)$$

In particular, if $k \equiv 0$ on $[0,1]$, then $\mathcal{G}$ is the set of all functions $g \in L^2(\omega)$ that are affine in the $s$ variable.

To prove (2.4.20), assume for simplicity that $m = 2$ and $\{s \in [0,1] : k(s) = 0\} = [p_1, p_2]$. Denoting by $\mathcal{G}'$ the class in the right hand side of (2.4.20), the inclusion $\mathcal{G} \subset \mathcal{G}'$ follows by Remark 2.4.2. Viceversa, let $g$ be affine in the $s$ variable in $(0,L) \times (p_1, p_2)$. Then, there exist $a, b \in L^2(0,L)$ such that

$$g(x_1, s) = a(x_1) + s b(x_1)$$

for a.e. $(x_1, s) \in (0,L) \times (p_1, p_2)$. Let now $0 < \delta < \frac{L}{3}$ and let $\epsilon > 0$. We define

$$g^\epsilon(x_1, s) = \begin{cases} a(x_1) + s b(x_1) & \text{in } (0,L) \times (p_1 - \epsilon, p_2 + \epsilon), \\ g(x_1, s) & \text{otherwise in } \omega, \end{cases}$$

and arguing as in the proof of Lemma 2.4.1, we extend $g^\varepsilon$ to the set

$$\omega^\delta := (-\delta, L + \delta) \times (-\delta, 1 + \delta).$$

It is easy to see that $g^\epsilon \to g$ strongly in $L^2(\omega^\delta)$ and $\partial_s^2 g^\epsilon = 0$ in the sense of distributions in the set $(-\delta, L + \delta) \times (p_1 - \epsilon, p_2 + \epsilon)$ for every $\epsilon > 0$.

Fix $\epsilon$, let $0 < \lambda < \min\{\frac{\epsilon}{2}, \frac{\delta}{2}\}$ and let $\rho \in C_0^\infty((-\lambda,\lambda)^2)$. By setting $\widehat{g}^\epsilon := g^\epsilon * \rho$, we obtain $\widehat{g}^\epsilon \in C^\infty(\overline{\omega})$ and $\partial_s^2 \widehat{g}^\epsilon = 0$ in $(0,L)\times(p_1-\lambda, p_2+\lambda)$. Define now

$$u^\epsilon(x_1,s) = \int_0^{x_1} \widehat{g}^\epsilon(\xi,s)d\xi.$$

Then, $u^\epsilon \in C^\infty(\overline{\omega})$ and $\partial_s^2 u^\epsilon = 0$ in $(0,L)\times(p_1-\lambda,p_2+\lambda)$. Hence, setting

$$z^\epsilon = \begin{cases} 0 & \text{in } (0,L)\times(p_1-\lambda,p_2+\lambda) \\ \frac{\partial_s^2 u^\epsilon}{k} & \text{otherwise,} \end{cases}$$

the claim follows by Remark 2.4.2, considering a sequence of convolution kernels and applying a diagonal argument.

An easy adaptation of the previous argument leads to the proof of (2.4.20) in the general case.

From here to the end of the section we shall assume that

$$\exists \lim_{h\to 0} \frac{\delta_h}{h^2} := \lambda \quad \text{and} \quad \exists \lim_{h\to 0}\frac{\delta_h}{h^3} := \mu. \tag{2.4.21}$$

For every $0 < \mu < +\infty$, we introduce the class

$$\mathcal{C}_\mu := \Big\{(g,b) \in L^2(\omega)\times L^2(\omega) : \exists v \in L^2(\omega;\mathbb{R}^3) \text{ such that}$$
$$\partial_s v \in L^2(\omega;\mathbb{R}^3),\ \partial_s v\cdot\tau = 0,\ \partial_s(\partial_s v\cdot n) = b \text{ and } \partial_1^2 v\cdot\tau + \mu\partial_s g = 0\Big\}, \tag{2.4.22}$$

where the last two equalities hold in the sense of distributions.

For $\mu = 0$ we define

$$\mathcal{C}_0 := \mathcal{G}\times\mathcal{B}, \tag{2.4.23}$$

where

$$\mathcal{B} := \Big\{b \in L^2(\omega) : \exists v \in L^2(\omega;\mathbb{R}^3) \text{ such that}$$
$$\partial_s v \in L^2(\omega,\mathbb{R}^3),\ \partial_s v\cdot\tau = 0,\ \partial_s(\partial_s v\cdot n) = b \text{ and } \partial_1^2 v\cdot\tau = 0\Big\}, \tag{2.4.24}$$

and again the last two equalities hold in the sense of distributions.

**Remark 2.4.5.** Let $b \in \mathcal{B}$ and let $v$ be as in (2.4.24). Then the tangential component $v\cdot\tau$ belongs to $W^{3,2}(\omega)$. Indeed, since $\partial_s(\partial_s v\cdot n) = b$ and $\partial_s v \in L^2(\omega;\mathbb{R}^3)$, we deduce that $\partial_s^2(v\cdot n) \in L^2(\omega)$. Since $\partial_s v\cdot\tau = 0$, we have $\partial_s(v\cdot\tau) = k(v\cdot n)$ and then $\partial_s^2(v\cdot\tau), \partial_s^3(v\cdot\tau) \in L^2(\omega)$. By the last condition in (2.4.24), there holds

$$\partial_1(v\cdot\tau) \in W^{-1,2}(\omega), \quad \partial_1^2(v\cdot\tau) \in L^2(\omega)$$

and

$$\partial_s\partial_1(v\cdot\tau) = \partial_1\partial_s(v\cdot\tau) \in W^{-1,2}(\omega).$$

Therefore, by Lemma 1.1.2, we obtain $\partial_1(v\cdot\tau) \in L^2(\omega)$. Arguing analogously, by Lemma 1.1.2 we deduce that $\partial_1\partial_s(v\cdot\tau) \in L^2(\omega)$, therefore $v\cdot\tau \in W^{2,2}(\omega)$ and $\partial_s^3(v\cdot\tau) \in L^2(\omega)$. Applying again Lemma 1.1.2, it is straightforward to see that $v\cdot\tau \in W^{3,2}(\omega)$. No regularity conditions can be deduced for the derivatives with respect to $x_1$ of the normal component of $v$.

In the case where $\mu \neq 0$, if $(g,b) \in \mathcal{C}_\mu$ and $v$ is as in (2.4.22), then the regularity of $v \cdot \tau$ and $v \cdot n$ with respect to $s$ is the same as in the previous case. It is still true that $\partial_1(v\cdot\tau) \in L^2(\omega)$ but in general one cannot guarantee that $v \cdot \tau \in W^{2,2}(\omega)$.

**Remark 2.4.6.** A function $b \in L^2(\omega)$ belongs to $\mathcal{B}$ if and only if there exists a map $\phi \in L^2(\omega;\mathbb{R}^3)$, with

$$\phi\cdot\tau \in W^{3,2}(\omega), \quad \phi\cdot e_1 \in W^{1,2}(\omega) \quad \text{and} \quad \partial_s(\phi\cdot n), \partial_s^2(\phi\cdot n) \in L^2(\omega),$$

such that

$$e(\phi) = 0 \tag{2.4.25}$$

and

$$\partial_s(\partial_s\phi\cdot n) = b. \tag{2.4.26}$$

In other words, $\phi$ is an infinitesimal isometry of the cylindrical surface

$$\Sigma := \Big\{x_1e_1 + \gamma(s) : x_1 \in (0,L), s \in (0,1)\Big\}$$

satisfying (2.4.26).

We first observe that the regularity of $\phi$ is sufficient to guarantee that $e(\phi)$, defined as in (2.4.2), belongs to $L^2(\omega;\mathbb{M}^{2\times2}_{\mathrm{sym}})$. Moreover, if $b \in L^2(\omega)$ and $v$ is as in (2.4.24), then there exists $v_1 \in W^{1,2}(\omega)$ such that

$$\begin{cases} \partial_1 v_1 = 0, \\ \partial_s v_1 = -\partial_1 v \cdot \tau. \end{cases}$$

The map $\phi := v_1e_1 + v$ satisfies (2.4.25) and (2.4.26). The converse statement is trivial.

Similarly, a pair $(g,b) \in L^2(\omega)\times L^2(\omega)$ belongs to $\mathcal{C}_\mu$ if and only if there exists a function $\phi \in L^2(\omega;\mathbb{R}^3)$ with $\phi\cdot\tau \in W^{1,2}(\omega)$, $\partial_s^2(\phi\cdot\tau), \partial_s^3(\phi\cdot\tau) \in L^2(\omega)$, $\phi\cdot e_1 \in W^{1,2}(\omega)$ and $\partial_s(\phi\cdot n), \partial_s^2(\phi\cdot n) \in L^2(\omega)$, such that

$$e(\phi) = \begin{pmatrix} \mu g & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\partial_s(\partial_s\phi\cdot n) = b.$$

**Remark 2.4.7.** As in the case of the class $\mathcal{G}$ introduced in (2.4.1), the structure of $\mathcal{B}$ and $\mathcal{C}_\mu$ depends on the behaviour of the curvature $k$ of $\gamma$.

For instance, if $k \equiv 0$ on $[0,1]$, then $\mathcal{B} = L^2(\omega)$. Indeed, condition (2.4.25) implies in this case that there exist some $\alpha,\beta,\delta \in \mathbb{R}$ such that

$$\phi(x_1,s) = (\alpha s + \beta)e_1 + (-\alpha x_1 + \delta)\tau + \phi_t(x_1,s)n$$

for a.e. $(x_1,s) \in \omega$, while condition (2.4.26) reads as $\partial_s^2\phi_t = b$. Hence $\mathcal{B} = L^2(\omega)$. Similarly, it can be deduced that $\mathcal{C}_\mu = \{g \in L^2(\omega) : g \text{ is affine in } s\} \times L^2(\omega)$.

If, instead, $k(s) \neq 0$ for every $s \in [0,1]$, then $\mathcal{B} = \{b \in L^2(\omega) : b \text{ is affine in } x_1\}$.

We conclude this section by proving some approximation results. The first result concerns the class $\mathcal{C}_\mu$ in the case $\mu \neq 0$.

**Lemma 2.4.8.** *Let $(g,b) \in \mathcal{C}_\mu$ with $\mu \neq 0$. Then, there exists a sequence $(\phi^\epsilon) \subset C^5(\overline{\omega};\mathbb{R}^3)$ such that*

$$e(\phi^\epsilon) = \begin{pmatrix} \partial_1\phi_1^\epsilon & 0 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} \mu g & 0 \\ 0 & 0 \end{pmatrix} \tag{2.4.27}$$

*strongly in $L^2(\omega;\mathbb{M}^{2\times 2}_{\text{sym}})$ and*

$$\partial_s(\partial_s\phi^\epsilon \cdot n) \to b \tag{2.4.28}$$

*strongly in $L^2(\omega)$.*

**Remark 2.4.9.** By Lemma 2.4.8 it follows, in particular, that if $(g,b) \in \mathcal{C}_\mu$ and $\mu \neq 0$, then $g \in \mathcal{G}$.

*Proof of Lemma 2.4.8.* Without loss of generality we may assume that $\mu = 1$. By the definition of $\mathcal{C}_\mu$ and by Remark 2.4.6 there exists $\phi \in L^2(\omega;\mathbb{R}^3)$ with $\phi\cdot\tau \in W^{1,2}(\omega)$, $\partial_s^2(\phi\cdot\tau)$, $\partial_s^3(\phi\cdot\tau) \in L^2(\omega)$, $\phi\cdot e_1 \in W^{1,2}(\omega)$ and $\partial_s(\phi\cdot n), \partial_s^2(\phi\cdot n) \in L^2(\omega)$, such that

$$e(\phi) = \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} \tag{2.4.29}$$

and $\partial_s(\partial_s\phi\cdot n) = b$. By (2.4.29) it follows that

$$\partial_1\phi\cdot\tau + \partial_s\phi\cdot e_1 = 0. \tag{2.4.30}$$

Hence, there exists $u \in W^{1,2}(\omega)$, with $\partial_1 u \in W^{1,2}(\omega)$ such that $\partial_1 u = \phi\cdot e_1$ and the equality

$$\phi\cdot\tau + \partial_s u = 0 \tag{2.4.31}$$

holds in the sense of $L^2(\omega)$. Indeed, by (2.4.30), if $\overline{u} \in W^{1,2}(\omega)$ satisfies $\partial_1\overline{u} = \phi\cdot e_1$, there exists $\varphi \in W^{1,2}(0,1)$ such that

$$\phi\cdot\tau + \partial_s\overline{u} = \dot{\varphi}.$$

Defining $u := \overline{u} - \varphi$, then $u$ has the required properties.

We set

$$v = (\phi\cdot\tau)\tau + (\phi\cdot n)n.$$

For the sake of simplicity, we divide the proof into two steps.

**Step 1.**

We claim that we can always reduce to the case where

$$u \in W^{4,2}(\omega), \quad v_s := v\cdot\tau \in W^{3,2}(\omega), \quad \text{and} \quad v_t := v\cdot n \in W^{2,2}(\omega),$$

with $\partial_1^i u, \partial_1^i v_t, \partial_1^i v_s, \partial_1^i g \in L^2(\omega)$ for every $i \in \mathbb{N}$.

Indeed, let $0 < \delta < \frac{L}{3}$. Arguing as in the proof of Lemma 2.4.1 we extend $u$ and $v$ to the set

$$\omega_\delta := (-\delta, L+\delta) \times (0,1)$$

in such a way that, denoting by $\widetilde{v}$ and $\widetilde{u}$ the extended maps and setting

$$\widetilde{g} = \partial_1^2 \widetilde{u} \quad \text{and} \quad \widetilde{b} = \partial_s(\partial_s \widetilde{v} \cdot n)$$

in $\omega_\delta$, then $\widetilde{g}$ and $\widetilde{b}$ are respectively extensions of $g$ and $b$ to $\omega_\delta$. Moreover, there holds $\widetilde{u} \in W^{1,2}(\omega_\delta)$ with $\partial_1 \widetilde{u} \in W^{1,2}(\omega_\delta)$, $\widetilde{v}_s \in W^{1,2}(\omega_\delta)$ with $\partial_s^2 \widetilde{v}_s, \partial_s^3 \widetilde{v}_s \in L^2(\omega_\delta)$ and $\widetilde{v}_t, \partial_s \widetilde{v}_t, \partial_s^2 \widetilde{v}_t \in L^2(\omega_\delta)$. Finally, by (2.4.29) and (2.4.31), the pair $(\widetilde{u}, \widetilde{v})$ solves

$$\partial_s \widetilde{v} \cdot \tau = 0 \quad \text{and} \quad \widetilde{v} \cdot \tau + \partial_s \widetilde{u} = 0 \quad \text{a.e. in } \omega_\delta.$$

We now mollify the functions $\widetilde{u}, \widetilde{v}, \widetilde{g}$, and $\widetilde{b}$ with respect to the $x_1$ variable. Let $0 < \epsilon < \delta$, consider a sequence $(\rho^\epsilon) \subset C_0^\infty(-\epsilon, \epsilon)$ of convolution kernels and set

$$\begin{cases} \widetilde{u}^\epsilon(x_1, s) := (\widetilde{u}(\cdot, s) * \rho^\epsilon)(x_1), \\ \widetilde{v}_s^\epsilon(x_1, s) := (\widetilde{v}_s(\cdot, s) * \rho^\epsilon)(x_1), \\ \widetilde{v}_t^\epsilon(x_1, s) := (\widetilde{v}_s(\cdot, s) * \rho^\epsilon)(x_1), \\ \widetilde{b}^\epsilon(x_1, s) := (\widetilde{b}(\cdot, s) * \rho^\epsilon)(x_1), \\ \widetilde{g}^\epsilon(x_1, s) := (\widetilde{g}(\cdot, s) * \rho^\epsilon)(x_1) \end{cases}$$

for a.e. $(x_1, s) \in \omega$ and for every $\epsilon$. By defining $\tilde{v}^\varepsilon := \tilde{v}_s^\varepsilon \tau + \tilde{v}_t^\varepsilon n$, the pair $(\widetilde{u}^\epsilon, \widetilde{v}^\epsilon)$ solves

$$\partial_1^2 \widetilde{u}^\epsilon = \widetilde{g}^\epsilon, \quad \partial_s \widetilde{v}^\epsilon \cdot \tau = 0, \quad \widetilde{v}^\epsilon \cdot \tau + \partial_s \widetilde{u}^\epsilon = 0 \quad \text{and} \quad \partial_s(\partial_s \widetilde{v}^\epsilon \cdot n) = \widetilde{b}^\epsilon \tag{2.4.32}$$

a.e. in $\omega$ for every $\epsilon$. Moreover $\widetilde{b}^\epsilon \to \widetilde{b}$ in $L^2(\omega)$ and $\widetilde{g}^\epsilon \to \widetilde{g}$ in $L^2(\omega)$. Now,

$$(\widetilde{v}_s^\epsilon) \subset W^{3,2}(\omega) \quad \text{and} \quad (\widetilde{v}_t^\epsilon) \subset W^{2,2}(\omega)$$

with $(\partial_1^i \widetilde{v}_s^\epsilon), (\partial_1^i \widetilde{v}_t^\epsilon) \subset L^2(\omega)$ for every $i \in \mathbb{N}$. Therefore, by (2.4.32) we deduce that $(\partial_s \widetilde{u}^\epsilon) \subset W^{3,2}(\omega)$. As $(\partial_1^i \widetilde{u}^\epsilon) \subset L^2(\omega)$ for every $i \in \mathbb{N}$, it follows that $(\widetilde{u}^\epsilon) \subset W^{4,2}(\omega)$ and the proof of the claim is completed.

**Step 2.**

Assume now that $u \in W^{4,2}(\omega)$,

$$v_s := v \cdot \tau \in W^{3,2}(\omega) \quad \text{and} \quad v_t := v \cdot n \in W^{2,2}(\omega),$$

with $\partial_1^i u, \partial_1^i v_t, \partial_1^i v_s, \partial_1^i g \in L^2(\omega)$ for every $i \in \mathbb{N}$. Since $v_t \in W^{2,2}(\omega)$, there exists a sequence $(v_t^\epsilon) \subset C^\infty(\overline{\omega})$ such that

$$v_t^\epsilon \to v_t \quad \text{strongly in } W^{2,2}(\omega). \tag{2.4.33}$$

Let $v_s^\epsilon \in C^5(\overline{\omega})$ be the solution of

$$\partial_s v_s^\epsilon = k v_t^\epsilon \tag{2.4.34}$$

in $\omega$, with $\int_0^1 v_s^\epsilon(x_1, s)\, ds \in C^\infty([0, L])$ for every $\epsilon > 0$ and

$$\int_0^1 v_s^\epsilon(x_1, s)\, ds \to \int_0^1 v_s(x_1, s)\, ds \quad \text{strongly in } W^{3,2}(0, L). \tag{2.4.35}$$

By Poincaré inequality, (2.4.29) and (2.4.34) we deduce

$$\|v_s^\epsilon - v_s\|_{L^2(\omega)} \le C\Big(\Big\| \int_0^1 (v_s^\epsilon - v_s)\, ds\Big\|_{L^2(\omega)} + \|k(v_t^\epsilon - v_t)\|_{L^2(\omega)}\Big)$$

and hence, by (2.4.33)–(2.4.35)

$$v_s^\epsilon \to v_s \quad \text{and} \quad \partial_s v_s^\epsilon \to \partial_s v_s \quad \text{strongly in } L^2(\omega). \tag{2.4.36}$$

Let $u^\epsilon \in C^6(\overline{\omega})$ be the solution of

$$\partial_s u^\epsilon + v_s^\epsilon = 0 \tag{2.4.37}$$

in $\omega$, with $\int_0^1 u^\epsilon(x_1, s)\, ds \in C^\infty([0, L])$,

$$\int_0^1 u^\epsilon(x_1, s)\, ds \to \int_0^1 u(x_1, s)\, ds \quad \text{strongly in } W^{4,2}(0, L). \tag{2.4.38}$$

By Poincaré inequality, (2.4.29), (2.4.31) and (2.4.37), there holds

$$\begin{aligned}\|\partial_1^2 \partial_s(u^\epsilon - u)\|_{L^2(\omega)} &= \|\partial_1^2(v_s^\epsilon - v_s)\|_{L^2(\omega)} \\ &\le C\Big(\Big\| \int_0^1 \partial_1^2(v_s^\epsilon - v_s)\, ds \Big\|_{L^2(\omega)} + \|k \partial_1^2(v_t^\epsilon - v_t)\|_{L^2(\omega)}\Big).\end{aligned}$$

The right-hand side of the previous inequality converges to zero due to (2.4.33) and (2.4.35). Hence, by (2.4.38) and Poincaré inequality

$$\partial_1^2 u^\epsilon \to \partial_1^2 u = g \quad \text{strongly in } L^2(\omega). \tag{2.4.39}$$

By defining

$$\phi^\epsilon := \partial_1 u^\epsilon e_1 + v^\epsilon,$$

property (2.4.27) follows by (2.4.34), (2.4.37) and (2.4.39). Moreover

$$\partial_s(\partial_s \phi^\epsilon \cdot n) = \partial_s^2 v_t^\epsilon + \dot{k} v_s^\epsilon + k \partial_s v_s^\epsilon.$$

Therefore, (2.4.28) follows by (2.4.33) and (2.4.36), and the proof of the lemma is completed. □

The next lemma, under a suitable additional condition on the sign of the curvature, provides us with an approximation result for the elements of the class $\mathcal{B}$ introduced in (2.4.24).

**Lemma 2.4.10.** *Assume there exists a finite number of points*

$$0 = p_0 < p_1 < \cdots < p_m = 1$$

*such that, for every* $i = 0, \cdots, m-1$, *there holds* $k(s) > 0$ *for every* $s \in (p_i, p_{i+1})$, *or* $k(s) < 0$ *for every* $s \in (p_i, p_{i+1})$ *or* $k(s) = 0$ *for every* $s \in (p_i, p_{i+1})$. *Let* $b \in \mathcal{B}$. *Then, there exists a sequence* $(\phi^\epsilon) \subset C^5(\overline{\omega}; \mathbb{R}^3)$ *such that*

$$e(\phi^\epsilon) = 0 \quad \text{for every } \epsilon > 0 \tag{2.4.40}$$

*and*

$$\partial_s(\partial_s \phi^\epsilon \cdot n) \to b \tag{2.4.41}$$

*strongly in* $L^2(\omega)$ *as* $\epsilon \to 0$.

*Proof.* By definition of $\mathcal{B}$ there exists $v \in L^2(\omega;\mathbb{R}^3)$, with $\partial_s v \in L^2(\omega;\mathbb{R}^3)$, such that

$$\partial_s v \cdot \tau = 0, \tag{2.4.42}$$

$$\partial_s(\partial_s v \cdot n) = b, \tag{2.4.43}$$

$$\partial_1^2 v \cdot \tau = 0. \tag{2.4.44}$$

Arguing as in Step 1 of the proof of Lemma 2.4.8, we may extend both $v$ and $b$ to the set $\omega_\delta := (-\delta, L+\delta) \times (0,1)$ for $0 < \delta < \frac{L}{3}$. By Remark 2.4.6 up to a regularization in the $x_1$ variable, we may assume that

$$v_t := v{\cdot}n \in W^{2,2}(\omega), \quad v_s := v{\cdot}\tau \in W^{3,2}(\omega) \quad \text{and} \quad \partial_1^i v_t, \partial_1^i v_s, \partial_1^i b \in L^2(\omega) \quad \text{for every } i \in \mathbb{N}.$$

Moreover, by (2.4.44) there exist $\alpha_0, \alpha_1 \in W^{3,2}(0,1)$ such that

$$v_s(x_1, s) = \alpha_0(s) + x_1\alpha_1(s), \tag{2.4.45}$$

for a.e. $(x_1, s) \in \omega$.

Let $Z := \{s \in [0,1] : k(s) = 0\}$. By assumption, $Z$ is the union of a finite number of intervals with a finite number of isolated points. For simplicity, we divide the proof into three steps. We first consider the case where $Z$ is a finite union of points. In the second step, we assume $Z$ to be a finite union of closed intervals and in the third step we study the general case.

**Step 1.**
Assume that $Z = \bigcup_{i\in I}\{p_i\}$ for some $I \subset \{0, \cdots, m\}$. By (2.4.42) and (2.4.44), there holds

$$k\partial_1^2 v_t = 0$$

a.e. in $\omega$, which in turn implies

$$\partial_1^2 v_t = 0 \tag{2.4.46}$$

a.e. in $\omega$. Hence, by (2.4.42), (2.4.45), and (2.4.46), there exist $\beta_0, \beta_1 \in W^{2,2}(0,1)$ such that

$$v_t(x_1, s) = \beta_0(s) + x_1\beta_1(s) \quad \text{and} \quad \dot{\alpha}_i(s) = k(s)\beta_i(s), \quad i = 0, 1, \tag{2.4.47}$$

a.e. in $\omega$. In particular, there exist two sequences $(\beta_0^\epsilon), (\beta_1^\epsilon) \subset C^\infty([0,1])$ such that

$$\beta_i^\epsilon \to \beta_i \tag{2.4.48}$$

strongly in $W^{2,2}(0,1)$, for $i = 0, 1$. Let $\alpha_i^\epsilon \in C^5([0,1])$ be the solution of

$$\dot{\alpha}_i^\epsilon = k\beta_i^\epsilon \quad \text{in } (0,1) \tag{2.4.49}$$

such that $\int_0^1 \alpha_i^\epsilon\, ds = \int_0^1 \alpha_i\, ds$ for every $\epsilon$, for $i = 0, 1$. By Poincaré inequality and (2.4.49), we deduce

$$\|\alpha_i^\epsilon - \alpha_i\|_{L^2(0,1)} \le C\|k(\beta_i^\epsilon - \beta_i)\|_{L^2(0,1)},$$

hence (2.4.48) implies

$$\alpha_i^\epsilon \to \alpha_i \quad \text{strongly in } W^{1,2}(0,1), \quad i = 0, 1. \tag{2.4.50}$$

Taking $\phi_1^\epsilon \in C^6([0,1])$ to be a solution of

$$\dot{\phi}_1^\epsilon = -\alpha_1^\epsilon \tag{2.4.51}$$

for every $\epsilon$ and setting

$$\phi^\epsilon := \phi_1^\epsilon e_1 + (\alpha_0^\epsilon + x_1\alpha_1^\epsilon)\tau + (\beta_0^\epsilon + x_1\beta_1^\epsilon)n,$$

we have $\phi^\epsilon \in C^5(\overline{\omega}, \mathbb{R}^3)$, property (2.4.40) holds owing to (2.4.49) and (2.4.51), while convergence (2.4.41) is a straightforward consequence of (2.4.43), (2.4.45), (2.4.47), (2.4.48) and (2.4.50).

**Step 2.**

Assume that $Z = [p_1, 1]$, with $0 < p_1 < 1$. By (2.4.42) and (2.4.44), there holds

$$\partial_1^2 v_t = 0 \quad \text{in } (0,L)\times(0,p_1).$$

Arguing as in the proof of Lemma 2.4.1, we define

$$\omega^\delta := (-\delta, L+\delta)\times(-\delta, 1+\delta)$$

and we extend $v_t$ to the set $\omega^\delta$ for a suitable $\delta > 0$ in such a way that $v_t \in W^{2,2}(\omega^\delta)$ and $\partial_1^2 v_t = 0$ in $(-\delta, L+\delta)\times(-\delta, p_1)$.

We slightly modify the map $v_t$ close to the point $p_1$ so that it remains affine with respect to $x_1$ in a neighbourhood of this point. More precisely, for $\epsilon < \frac{\delta}{2}$, we set

$$v_t^\epsilon(x_1, s) := v_t(x_1, s-\epsilon) \qquad \text{in } \omega^{\frac{\delta}{2}}.$$

It is easy to see that $(v_t^\epsilon) \subset W^{2,2}(\omega^{\frac{\delta}{2}})$, moreover

$$v_t^\epsilon \to v_t, \quad \partial_s v_t^\epsilon \to \partial_s v_t \quad \text{ and } \partial_s^2 v_t^\epsilon \to \partial_s^2 v_t \quad \text{strongly in } L^2(\omega^{\frac{\delta}{2}})$$

and

$$\partial_1^2 v_t^\epsilon = 0 \quad \text{ in } \Big(-\frac{\delta}{2}, L+\frac{\delta}{2}\Big)\times(-\epsilon, p_1+\epsilon).$$

To conclude, we regularize the sequence $(v_t^\epsilon)$ by mollification. Let $0 < \lambda < \epsilon$ and let $\rho \in C_0^\infty((-\lambda,\lambda)^2)$. Defining $\widetilde{v}_t^\epsilon := v_t^\epsilon * \rho$, we have $\widetilde{v}_t^\epsilon \in C^\infty(\overline{\omega})$ and

$$\partial_1^2 \widetilde{v}_t^\epsilon = 0 \quad \text{ in } (0,L)\times(0,p_1). \tag{2.4.52}$$

By considering a sequence of convolution kernels and applying a diagonal argument we may also assume that

$$\widetilde{v}_t^\epsilon \to v_t, \quad \partial_s \widetilde{v}_t^\epsilon \to \partial_s v_t \quad \text{ and } \quad \partial_s^2 \widetilde{v}_t^\epsilon \to \partial_s^2 v_t \quad \text{strongly in } L^2(\omega). \tag{2.4.53}$$

By (2.4.52), for every $\epsilon$ we may choose a map $v_s^\epsilon \in C^5(\overline{\omega})$ such that

$$\partial_s v_s^\epsilon = k\widetilde{v}_t^\epsilon, \quad \partial_1^2 v_s^\epsilon = 0 \quad \text{ and } \quad \int_0^1 v_s^\epsilon\, ds = \int_0^1 v_s\, ds \quad \text{ in } \omega.$$

The conclusion of the lemma follows now arguing as in Step 1.

The same argument applies to the case where $Z = [0, p_1]$, with $0 < p_1 < 1$, by choosing

$$v_t^\epsilon(x_1, s) := v_t(x_1, s + \epsilon) \quad \text{in } \omega^{\frac{\delta}{2}}$$

and by arguing as in the previous case.

Finally, assume that

$$Z = [p_1, p_2] \cup [p_3, 1]$$

with $0 < p_1 < p_2 < p_3 < 1$. Let $\varphi \in C_0^\infty(\mathbb{R})$ be such that $0 \leq \varphi(s) \leq 1$ for every $s \in \mathbb{R}$, $\varphi(s) = 1$ for all $s \in [p_2 - \eta, p_2 + \eta]$ and $\varphi(s) = 0$ for $s \leq p_1 + \eta$ or $s \geq p_3 - \eta$ for some $\eta > 0$ such that

$$\eta < \min\{p_1, \tfrac{p_2 - p_1}{2}, \tfrac{p_3 - p_2}{2}, 1 - p_3\}.$$

The argument shown at the beginning of this step applies now choosing

$$v_t^\epsilon(x_1, s) := (1 - \varphi(s)) v_t(x_1, s - \epsilon) + \varphi(s) v_t(x_1, s + \epsilon) \quad \text{in } \omega^{\frac{\delta}{2}}$$

for $\epsilon$ small enough.

The case where $Z$ is a finite union of disjoint intervals is a simple adaptation of the previous cases.

**Step 3.**
Consider now the general case and assume there exist $I_1 \subset \{1, \cdots, m\}$, $I_2 \subset \{0, \cdots, m\}$ disjoint such that

$$Z = \bigcup_{i \in I_1} [p_{i-1}, p_i] \cup \bigcup_{i \in I_2} \{p_i\}.$$

Then $\partial_1^2 v_t = 0$ a.e. in $(0, L) \setminus \left( \bigcup_{i \in I_1} [p_{i-1}, p_i] \right)$ and the thesis follows arguing as in Step 2. $\square$

## 2.5 Compactness results

In this section we deduce some compactness properties for sequences of deformations $(y^h)$ satisfying the uniform energy estimate (2.2.5).

Assumption (H4) on $W$ provides us with a control on the $L^2$ norm of the distance of the rescaled gradients from $SO(3)$. Applying Theorem 1.2.1 on a scale of order $\delta_h$, we can construct a sequence of approximating rotations $(R^h)$, whose $L^2$ distance from the rescaled gradients is still of order $\epsilon_h$. Because of the different scaling of the cross-section diameter and the cross-section thickness, the approximating rotations turn out to depend both on the mid-fiber coordinate $x_1$ and on the arc-length coordinate s. Moreover, the derivatives of $(R^h)$ in the two variables have a different order of decay, as $h \to 0$.

More precisely, the following result holds true.

**Theorem 2.5.1.** *Assume that* $\frac{\epsilon_h}{\delta_h} \to 0$. *Let* $(y^h)$ *be a sequence of deformations in* $W^{1,2}(\Omega; \mathbb{R}^3)$ *satisfying* (2.2.5). *Then, there exists a sequence of constant rotations* $(P^h)$ *and a sequence* $(R^h) \subset C^\infty(\overline{\omega}; \mathbb{M}^{3\times 3})$ *with the following properties: setting* $Y^h := (P^h)^T y^h - c^h$, *where* $(c^h)$

*is any sequence of constants in* $\mathbb{R}^3$*, for every* $h>0$ *there holds*

$$\|\nabla_{h,\delta_h} Y^h R_0^T - R^h\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} \leq C\epsilon_h, \tag{2.5.1}$$

$$\int_\Omega \Big(\nabla_{h,\delta_h} Y^h R_0^T - (\nabla_{h,\delta_h} Y^h R_0^T)^T\Big)\, dx_1 ds dt = 0, \tag{2.5.2}$$

$$R^h(x_1,s) \in SO(3) \text{ for every } (x_1,s)\in\overline{\omega}, \tag{2.5.3}$$

$$\|R^h - Id\|_{L^2(\omega;\mathbb{M}^{3\times 3})} \leq C\frac{\epsilon_h}{\delta_h}, \tag{2.5.4}$$

$$\|\partial_1 R^h\|_{L^2(\omega;\mathbb{M}^{3\times 3})} \leq C\frac{\epsilon_h}{\delta_h}, \tag{2.5.5}$$

$$\|\partial_s R^h\|_{L^2(\omega;\mathbb{M}^{3\times 3})} \leq C\frac{h\epsilon_h}{\delta_h}. \tag{2.5.6}$$

*Proof.* By (2.2.5) and (H4), the sequence $(y^h\circ(\psi^h)^{-1})$ satisfies

$$\int_{\Omega_h} \operatorname{dist}^2(\nabla(y^h\circ(\psi^h)^{-1}), SO(3))dx \leq Ch\delta_h\epsilon_h^2. \tag{2.5.7}$$

Let us consider the sets

$$A_h^i := \Big\{x_1e_1 + h\gamma(s) + \delta_h t n(s) : x_1 \in \big(\tfrac{i_1 L}{\eta_h}, \tfrac{(i_1+1)L}{\eta_h}\big),$$
$$s\in\big(\tfrac{i_2}{k_h}, \tfrac{(i_2+1)}{k_h}\big),\ t\in\big(-\tfrac12,\tfrac12\big)\Big\},$$

where

$$\eta_h = \Big[\frac{L}{\delta_h}\Big], \quad k_h = \Big[\frac{h}{\delta_h}\Big] \quad \text{and} \quad i=(i_1,i_2),$$

with $i_1 = 0,\cdots,\eta_h-1$, and $i_2=0,\cdots,k_h-1$. By Theorem 1.2.1 and Remark 1.2.2 there exist a sequence of constant rotations $(\overline{Q}_h^i)\subset SO(3)$ and a constant $C$ independent of $h$ and $i$ satisfying

$$\int_{A_h^i} |\nabla(y^h\circ(\psi^h)^{-1}) - \overline{Q}_h^i|^2 dx \leq C\int_{A_h^i} \operatorname{dist}^2(\nabla(y^h\circ(\psi^h)^{-1}), SO(3))dx. \tag{2.5.8}$$

To see that $C$ does not depend on $h$, we first notice that each set $A_h^i$ has the same rigidity constant of the set $\widetilde{A}_h^i$ that is obtained by a uniform dilation of $A_h^i$ of factor $\frac{1}{\delta_h}$. Defining $\phi_h^i : (0,1)^3 \longrightarrow \widetilde{A}_h^i$ as

$$\phi_h^i(x_1,s,t) = \big(\tfrac{(i_1+x_1)L}{\eta_h\delta_h}, \tfrac{h}{\delta_h}\gamma\big(\tfrac{i_2+s}{k_h}\big) + \big(t-\tfrac12\big)n\big(\tfrac{i_2+s}{k_h}\big)\big),$$

we conclude that the sets $\widetilde{A}_h^i$ are images of the unitary cube through a family of uniformly bi-Lipschitz transformations. Therefore by Remark 1.2.2 the constant $C$ is the same for every $i$ and for every $h$.

Let $Q^h : \omega \longrightarrow SO(3)$ be the piecewise constant map given by

$$Q^h(x_1,s) := \overline{Q}_h^i \quad \text{for } (x_1,s)\in\Big(\frac{i_1L}{\eta_h}, \frac{(i_1+1)L}{\eta_h}\Big)\times\Big(\frac{i_2}{k_h},\frac{i_2+1}{k_h}\Big),$$

where $i_1 = 0,\cdots,\eta_h-1$ and $i_2 = 0,\cdots,k_h-1$. By summing (2.5.8) over $i$, changing variables and using (2.2.4) and (2.5.7), we deduce that

$$\int_\Omega |\nabla_{h,\delta_h} y^h R_0^T - Q^h|^2 dx \leq C\int_\Omega \operatorname{dist}^2(\nabla_{h,\delta_h} y^h R_0^T, SO(3))dx. \leq C\epsilon_h^2. \tag{2.5.9}$$

Consider now the sets

$$B_h^i := \Big\{x_1e_1 + h\gamma(s) + \delta_h t n(s) : x_1 \in \big((i_1-1)\tfrac{L}{\eta_h}, (i_1+2)\tfrac{L}{\eta_h}\big),$$
$$s \in \big((i_2-1)\tfrac{1}{k_h}, (i_2+2)\tfrac{1}{k_h}\big),\ t \in \big(-\tfrac{1}{2}, \tfrac{1}{2}\big)\Big\},$$

for $i_1 = 1, \cdots, \eta_h - 2$, and $i_2 = 1, \cdots, k_h - 2$, and for every $h > 0$. Applying the rigidity estimate to the sets $B_h^i$ we obtain that for every $(i_1, i_2)$ there exists a map $\widehat{Q}_h^i \subset SO(3)$ satisfying

$$\int_{B_h^i} |\nabla(y^h \circ (\psi^h)^{-1}) - \widehat{Q}_h^i|^2 dx \le C \int_{B_h^i} \operatorname{dist}^2(\nabla(y^h \circ (\psi^h)^{-1}), SO(3)) dx.$$

Let now $j_k$ be an integer in the set $\{i_k - 1, i_k, i_k + 1\}$, $k = 1,2$ and let $j = (j_1, j_2)$. As $A_h^j \subset B_h^i$, there holds

$$\mathcal{L}^3(A_h^j)\big|Q^h\big(\tfrac{j_1 L}{\eta_h}, \tfrac{j_2}{k_h}\big) - \widehat{Q}_h^i\big|^2 \le 2\int_{A_h^j} \big|Q^h\big(\tfrac{j_1 L}{\eta_h}, \tfrac{j_2}{k_h}\big) - \nabla(y^h \circ (\psi^h)^{-1})\big|^2 dx \quad (2.5.10)$$

$$+2\int_{B_h^i} |\nabla(y^h \circ (\psi^h)^{-1}) - \widehat{Q}_h^i|^2 dx \le C \int_{B_h^i} \operatorname{dist}^2(\nabla(y^h \circ (\psi^h)^{-1}), SO(3)) dx \quad (2.5.11)$$

Hence, by (2.5.7) we deduce

$$\mathcal{L}^3(A_h^i)\big|Q^h\big(\tfrac{(i_1 \pm 1)L}{\eta_h}, \tfrac{i_2 \pm 1}{k_h}\big) - Q_h\big(\tfrac{i_1 L}{\eta_h}, \tfrac{i_2}{k_h}\big)\big|^2 \le C h \delta_h \epsilon_h^2, \quad (2.5.12)$$

for every $i_1 = 1, \cdots \eta_h - 2$, and $i_2 = 1, \cdots k_h - 2$.

We first extend the map $Q^h$ to the strip $\mathbb{R} \times (0,1)$ by setting

$$Q^h(x_1, s) = \begin{cases} Q^h(0, s) & \text{if } (x_1, s) \in (-\infty, 0) \times (0,1), \\ Q^h(L, s) & \text{if } (x_1, s) \in (L, +\infty) \times (0,1), \end{cases}$$

and then to the whole $\mathbb{R}^2$ by

$$Q^h(x_1, s) = \begin{cases} Q^h(x_1, 0) & \text{if } (x_1, s) \in \mathbb{R} \times (-\infty, 0), \\ Q^h(x_1, 1) & \text{if } (x_1, s) \in \mathbb{R} \times (1, +\infty). \end{cases}$$

Since $Q^h$ is constant in each set $A_h^i$, inequality (2.5.12) yields

$$|Q^h(x_1 + \xi, s + \lambda) - Q^h(x_1, s)|^2 \le C \frac{h\epsilon_h^2}{\delta_h^2} \quad (2.5.13)$$

for every $(x_1, s) \in \omega$, for $|\xi| \le \frac{L}{\eta_h}$ and $|\lambda| \le \frac{1}{k_h}$. Moreover, since $Q^h$ is piecewise constant, (2.5.12) implies

$$\int_{\big(\frac{i_1 L}{\eta_h}, \frac{(i_1+1)L}{\eta_h}\big) \times \big(\frac{i_2}{k_h}, \frac{i_2+1}{k_h}\big)} |Q^h(x_1 + \xi, s + \lambda) - Q^h(x_1, s)|^2 dx_1 ds$$
$$\le \frac{C}{h\delta_h} \int_{B_h^i} \operatorname{dist}^2\big(\nabla(y^h \circ (\psi^h)^{-1}), SO(3)\big), \quad (2.5.14)$$

for every $i_1 = 1, \cdots, \eta_h - 2$, and $i_2 = 1, \cdots, k_h - 2$.

Let now $\omega' \subset\subset \omega$. For $h$ small enough, there holds

$$\omega' \subset \Big(\frac{L}{\eta_h}, L - \frac{L}{\eta_h}\Big) \times \Big(\frac{1}{\eta_h}, 1 - \frac{1}{\eta_h}\Big).$$

Hence, by (2.5.7) and (2.5.14), as every $x \in \Omega_h$ belongs to at most nine sets of the form $B_h^i$, summing over the indices $i_k$, we deduce

$$\int_{\omega'} |Q^h(x_1+\xi, s+\lambda) - Q^h(x_1, s)|^2 dx_1 ds \leq C\epsilon_h^2, \tag{2.5.15}$$

for $|\xi| \leq \delta_h$, and $|\lambda| \leq \frac{\delta_h}{h}$.

To obtain a sequence of smooth rotations, we regularize $(Q^h)$ by means of convolution kernels. Let $\eta \in C_0^\infty(0,1)$, $\eta \geq 0$, $\int_0^1 \eta(s)ds = 1$. We define

$$\varphi^h(\xi, \lambda) := \frac{h}{\delta_h^2}\eta\Big(\frac{\xi}{\delta_h}\Big)\eta\Big(\frac{h\lambda}{\delta_h}\Big)$$

for every $\xi \in (0, \delta_h)$ and $\lambda \in \big(0, \frac{\delta_h}{h}\big)$, and we notice that, for $h$ small enough, supp $\varphi^h$ is contained into a ball whose radius is smaller than the distance between $\omega'$ and the boundary of $\omega$.

Setting $\widetilde{Q}^h := Q^h * \varphi^h$, by Holder inequality and (2.5.15) we obtain

$$\int_{\omega'} |\widetilde{Q}^h(x_1, s) - Q^h(x_1, s)|^2 dx_1 ds \leq C\epsilon_h^2,$$

which in turn implies

$$\|\widetilde{Q}^h - Q^h\|_{L^2(\omega;\mathbb{M}^{3\times3})} \leq C\epsilon_h \tag{2.5.16}$$

for the constant $C$ does not depend on the choice of $\omega'$. Analogously we deduce the estimate

$$\|\partial_1 \widetilde{Q}^h\|_{L^2(\omega;\mathbb{M}^{3\times3})} \leq C\frac{\epsilon_h}{\delta_h} \tag{2.5.17}$$

and

$$\|\partial_s \widetilde{Q}^h\|_{L^2(\omega;\mathbb{M}^{3\times3})} \leq C\frac{h\epsilon_h}{\delta_h}. \tag{2.5.18}$$

Finally, let $U$ be a neighbourhood of $SO(3)$ where the projection

$$\Pi : U \longrightarrow SO(3)$$

is well defined and regular. By (2.5.13), there holds

$$|\widetilde{Q}^h(x_1, s) - Q^h(x_1, s)|^2 \leq \|\varphi^h\|^2_{L^2\big((0,\delta_h)\times(0,\frac{\delta_h}{h})\big)} \frac{\delta_h^2}{h}\frac{h\epsilon_h^2}{\delta_h^2} \leq C\frac{h\epsilon_h^2}{\delta_h^2}, \tag{2.5.19}$$

for every $(x_1, s) \in \omega$. Since $\frac{\epsilon_h}{\delta_h} \to 0$, then $\widetilde{Q}^h \in U$ for $h$ small enough and, thus, its projection on $SO(3)$

$$\widetilde{R}^h := \Pi(\widetilde{Q}^h)$$

is well defined. It is immediate to see that, for every $h > 0$, the map $\widetilde{R}^h$ satisfies (2.5.3). Furthermore, by (2.5.17) and (2.5.18) and by the regularity of $\Pi$, properties(2.5.5) and (2.5.6) hold true. By definition of $\widetilde{R}^h$,

$$\|\widetilde{R}^h - \widetilde{Q}^h\|_{L^2(\omega;\mathbb{M}^{3\times3})} \leq \|Q^h - \widetilde{Q}^h\|_{L^2(\omega;\mathbb{M}^{3\times3})} \tag{2.5.20}$$

therefore (2.5.1) follows by (2.5.9) and (2.5.16).

By Poincaré inequality, given

$$\overline{R}^h := \fint_\omega \widetilde{R}^h dx_1 ds,$$

properties (2.5.5) and (2.5.6) yield

$$\|\widetilde{R}^h - \overline{R}^h\|_{L^2(\omega;\mathbb{M}^{3\times3})} \leq C(\|\partial_1 \widetilde{R}^h\|_{L^2(\omega;\mathbb{M}^{3\times3})} + \|\partial_s \widetilde{R}^h\|_{L^2(\omega;\mathbb{M}^{3\times3})}) + \leq C\frac{\epsilon_h}{\delta_h}.$$

This implies that $\mathrm{dist}(\overline{R}^h, SO(3)) \leq C\frac{\epsilon_h}{\delta_h}$. Hence, there exists a sequence of constant rotations $(S^h) \in SO(3)$ such that $|\overline{R}^h - S^h| \leq C\frac{\epsilon_h}{\delta_h}$, which in turn implies

$$\|\widetilde{R}^h - S^h\|_{L^2(\omega;\mathbb{M}^{3\times3})} \leq C\frac{\epsilon_h}{\delta_h}. \tag{2.5.21}$$

We define $\hat{R}^h := (S^h)^T \widetilde{R}^h$ and $\hat{y}^h = (S^h)^T y^h$. By the properties of the sequence $(\widetilde{R}^h)$ and by (2.5.21), $\hat{R}^h$ satisfies (2.5.1) and (2.5.3)–(2.5.6).

To construct a sequence of rotations satisfying also (2.5.2), we argue as in [29, Lemma 3.1] and we introduce the matrices

$$F^h := \fint_\Omega \nabla_{h,\delta_h} \hat{y}^h R_0^T dx_1 ds dt.$$

We notice that

$$|F^h - Id| \leq \fint_\Omega |\nabla_{h,\delta_h} \hat{y}^h R_0^T - Id| dx_1 ds dt \leq C\frac{\epsilon_h}{\delta_h}, \tag{2.5.22}$$

as $\hat{R}^h$ satisfies (2.5.1) and (2.5.4). It turns out that $\det F^h > 0$ for $h$ small enough, therefore by polar decomposition theorem, for every $h$ there exist $P^h \in SO(3)$ and $U^h \in \mathbb{M}^{3\times3}_{sym}$ such that

$$F^h = P^h U^h,$$

and

$$|U^h - Id| = \mathrm{dist}(F^h, SO(3)) \leq |F^h - Id|. \tag{2.5.23}$$

The symmetry of $U^h$, together with (2.5.22) and (2.5.23), yields

$$|P^h - Id| \leq |P^h - U^h| + |U^h - Id| \leq C|F^h - Id| \leq C\frac{\epsilon_h}{\delta_h} \tag{2.5.24}$$

for every $h > 0$. Defining $R^h := (P^h)^T \hat{R}^h$ and $Y^h := (P^h)^T \hat{y}^h$, then (2.5.1), (2.5.3), (2.5.5) and (2.5.6) follow immediately. Moreover,

$$\|R^h - Id\|_{L^2(\omega)} \leq \|R^h - \hat{R}^h\|_{L^2(\omega)} + \|\hat{R}^h - Id\|_{L^2(\omega)} \leq C(\|P^h - Id\|_{L^2(\omega)} + \|\hat{R}^h - Id\|_{L^2(\omega)}).$$

Hence, (2.5.4) holds due to (2.5.24) and from the fact that $\hat{R}^h$ satisfies (2.5.4). Finally, by symmetry of $U^h$, for every $h > 0$ we obtain

$$\begin{aligned} \int_\Omega (\nabla_{h,\delta_h} Y^h R_0^T - (\nabla_{h,\delta_h} Y^h R_0^T)^T)\, dx_1 ds dt &= \mathcal{L}^3(\Omega)((P^h)^T F^h - (F^h)^T P^h) \\ &= \mathcal{L}^3(\Omega)(U^h - (U^h)^T) = 0, \end{aligned}$$

which concludes the proof of (2.5.2) and of the proposition. □

From now on we shall refer to the sequence of deformations $(Y^h)$ introduced in Theorem 2.5.1, where the constants $c^h$ are chosen in such a way to satisfy

$$\int_\Omega (Y^h - \psi^h)\, dx_1 ds dt = 0. \tag{2.5.25}$$

We introduce the tangential derivative of the tangential displacement, associated with $Y^h$, given by

$$g^h(x_1, s, t) := \frac{1}{\epsilon_h}\partial_1(Y_1^h - \psi_1^h), \tag{2.5.26}$$

for a.e. $(x_1, s, t) \in \Omega$, and the (averaged) twist function, associated with $Y^h$, given by

$$w^h(x_1, s) := \frac{\delta_h}{h\epsilon_h}\int_{-\frac{1}{2}}^{\frac{1}{2}} \partial_s(Y^h - \psi^h)\cdot n\, dt, \tag{2.5.27}$$

for a.e. $(x_1, s) \in \omega$.

We are now in a position to prove the first compactness result.

**Theorem 2.5.2.** *Assume that* $\frac{\varepsilon_h}{\delta_h} \to 0$. *Let* $(y^h)$ *be a sequence of deformations in* $W^{1,2}(\Omega;\mathbb{R}^3)$ *satisfying* (2.2.5). *Let* $(R^h)$ *and* $(Y^h)$ *be the sequences introduced in Theorem 2.5.1, with* $(c^h)$ *such that* (2.5.25) *holds. Then,*

$$Y^h \to x_1 e_1 \text{ strongly in } W^{1,2}(\Omega;\mathbb{R}^3). \tag{2.5.28}$$

*Let* $(g^h)$ *and* $(w^h)$ *be the sequences defined in* (2.5.26) *and* (2.5.27). *Then, there exist* $g \in L^2(\Omega)$ *and* $w \in W^{1,2}(0, L)$ *such that, up to subsequences,*

$$w^h \to w \quad \text{strongly in } L^2(\omega), \tag{2.5.29}$$

$$A^h := \frac{\delta_h}{\epsilon_h}(R^h - Id) \rightharpoonup A \quad \text{weakly in } W^{1,2}(\omega;\mathbb{M}^{3\times3}), \tag{2.5.30}$$

$$\frac{\delta_h}{\epsilon_h}(\nabla_{h,\delta_h} Y^h R_0^T - Id) \to A \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}), \tag{2.5.31}$$

$$\frac{\delta_h^2}{\epsilon_h^2}\,\mathrm{sym}(R^h - Id) \to \frac{A^2}{2} \quad \text{strongly in } L^2(\omega;\mathbb{M}^{3\times3}), \tag{2.5.32}$$

*where*

$$A(x_1) = w(x_1)(e_3 \otimes e_2 - e_2 \otimes e_3) \tag{2.5.33}$$

*for a.e.* $x_1 \in (0, L)$, *and if* (2.2.6) *holds*

$$g^h \rightharpoonup g \quad \text{weakly in } L^2(\Omega). \tag{2.5.34}$$

*Moreover,* $(Y^h)$ *satisfies*

$$\|\mathrm{sym}(\nabla_{h,\delta_h} Y^h R_0^T - Id)\|_{L^2(\Omega;\mathbb{M}^{3\times3})} \le C\Big(\epsilon_h + \frac{\epsilon_h^2}{\delta_h^2}\Big). \tag{2.5.35}$$

*Finally, there exists* $b \in L^2(\omega)$ *such that, setting*

$$B(x_1, s) = \begin{pmatrix} 0 & w'(x_1)\tau_3(s) & -w'(x_1)\tau_2(s) \\ -w'(x_1)\tau_3(s) & 0 & -b(x_1, s) \\ w'(x_1)\tau_2(s) & b(x_1, s) & 0 \end{pmatrix} \tag{2.5.36}$$

*for a.e.* $(x_1, s) \in \omega$, *up to subsequences there holds*

$$\frac{\delta_h}{h\epsilon_h}\partial_s R^h \rightharpoonup B \quad \textit{weakly in } L^2(\omega; \mathbb{M}^{3\times 3}). \tag{2.5.37}$$

*Proof.* By properties (2.5.4), (2.5.5) and (2.5.6), the sequence $(A^h)$ is uniformly bounded in $W^{1,2}(\omega; \mathbb{M}^{3\times 3})$. Therefore, there exists $A \in W^{1,2}(\omega; \mathbb{M}^{3\times 3})$ such that, up to subsequences, (2.5.30) holds. Since

$$\|\partial_s A^h\|_{L^2(\omega;\mathbb{M}^{3\times 3})} \leq Ch$$

by (2.5.6), we deduce that $A = A(x_1)$.

By the Sobolev embedding theorems, convergence of $(A^h)$ is actually strong in $L^q(\omega; \mathbb{M}^{3\times 3})$ for every $q \in [1, +\infty)$. Hence, the equality

$$\mathrm{sym}\, A^h = -\frac{\epsilon_h}{\delta_h}\frac{(A^h)^T A^h}{2} \tag{2.5.38}$$

yields (2.5.32) and implies that $A(x_1) \in \mathbb{M}^{3\times 3}_{skew}$ for a.e. $x_1 \in (0, L)$.

By (2.5.1) and by strong convergence of $(A^h)$ in $L^2$, we obtain (2.5.31). In particular,

$$\partial_1 Y^h \to e_1 \quad \text{and} \quad \partial_s Y^h, \partial_t Y^h \to 0 \quad \text{strongly in } L^2(\Omega; \mathbb{R}^3).$$

Now (2.5.28) follows owing to (2.5.25) and Poincaré inequality. Moreover,

$$\|\mathrm{sym}(\nabla_{h,\delta_h} Y^h R_0^T - Id)\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} \leq \|\mathrm{sym}(\nabla_{h,\delta_h} Y^h R_0^T - R^h)\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} + \|\mathrm{sym}(R^h - Id)\|_{L^2(\Omega;\mathbb{M}^{3\times 3})}.$$

Hence, (2.5.35) holds due to (2.5.1) and (2.5.32).

By (2.5.6), there exists a map $B \in L^2(\omega; \mathbb{M}^{3\times 3})$ satisfying (2.5.37). Differentiating the identity

$$(R^h)^T R^h = Id,$$

we obtain

$$(\partial_s R^h)^T (R^h - Id) + (R^h - Id)^T \partial_s R^h = -2\mathrm{sym}\, \partial_s R^h.$$

Thus, by (2.5.30) and (2.5.37), we deduce that $B$ is skew-symmetric.

We claim that

$$Be_1 = A'\tau. \tag{2.5.39}$$

Indeed, let $\varphi \in W^{1,2}_0(\Omega; \mathbb{R}^3)$. Then

$$\langle \frac{\delta_h}{h\epsilon_h}\partial_s\partial_1(Y^h - \psi^h), \varphi\rangle_{W^{-1,2}\times W^{1,2}_0} = \\ -\int_\Omega \frac{\delta_h}{h\epsilon_h}(\nabla_{h,\delta_h} Y^h - R^h R_0)e_1 \cdot \partial_s\varphi\, dx_1 ds dt + \int_\Omega \frac{\delta_h}{h\epsilon_h}\partial_s R^h e_1 \cdot \varphi\, dx_1 ds dt. \tag{2.5.40}$$

The first term in (2.5.40) is infinitesimal due to (2.5.1), whereas (2.5.37) yields

$$\int_\Omega \frac{\delta_h}{h\epsilon_h}\partial_s R^h e_1 \cdot \varphi\, dx_1 ds dt \to \int_\Omega Be_1 \cdot \varphi\, dx_1 ds dt.$$

On the other hand, we have

$$\langle \frac{\delta_h}{h\epsilon_h}\partial_s\partial_1(Y^h - \psi^h), \varphi\rangle_{W^{-1,2}\times W^{1,2}_0} = \\ -\int_\Omega \frac{\delta_h(h - \delta_h tk)}{h\epsilon_h}(\nabla_{h,\delta_h} Y^h - R_0)e_2 \cdot \partial_1\varphi\, dx_1 ds dt,$$

which in turn gives

$$\langle \frac{\delta_h}{h\epsilon_h}\partial_s\partial_1(Y^h-\psi^h),\,\varphi\rangle_{W^{-1,2}\times W_0^{1,2}} \to \int_\Omega A'\tau\cdot\varphi\,dx_1dsdt. \tag{2.5.41}$$

owing to (2.2.4) and (2.5.31). By combining (2.5.40) and (2.5.41), we obtain (2.5.39).

Since $B$ is skew-symmetric, the following equality holds true

$$0 = B_{11}(x_1,s) = A'_{12}(x_1)\tau_2(s) + A'_{13}(x_1)\tau_3(s),$$

for a.e. $x_1 \in (0,L)$ and $s\in(0,1)$. This last condition, together with the assumption that $k$ is not identically zero, implies

$$A'_{12} = A'_{13} \equiv 0. \tag{2.5.42}$$

On the other hand, by (2.5.2) and (2.5.31) we deduce that

$$\int_0^L A(x_1)\,dx_1 = 0.$$

Hence,

$$A_{12} = A_{13} = 0. \tag{2.5.43}$$

To conclude the proof of the Theorem, we consider the sequences $(g^h)$ and $(w^h)$. To prove (2.5.34), we notice that

$$g^h = \frac{1}{\epsilon_h}\Big((\partial_1 Y_1^h - R_{11}^h) + (R_{11}^h - 1)\Big). \tag{2.5.44}$$

Since we are assuming that (2.2.6) holds, it follows from (2.5.1) and (2.5.32) that the sequence $(g^h)$ is uniformly bounded in $L^2(\Omega)$. Therefore, there exists $g\in L^2(\Omega)$ such that (2.5.34) holds up to subsequences.

As for the twist function $w^h$, by (2.2.4) and (2.5.31) we deduce

$$\frac{\delta_h}{h\epsilon_h}\partial_s(Y^h-\psi^h) \to A\tau \text{ strongly in } L^2(\Omega;\mathbb{R}^3),$$

which in turn yields (2.5.29). In particular, by (2.5.43) and by skew-symmetry of $A$ there holds $w = A_{32}$, hence $w\in W^{1,2}(0,L)$ and the proof of (2.5.33) is complete. Finally, by (2.5.39) we deduce the representation (2.5.36). □

In the next proposition, under stronger assumptions on the order of decay of $\epsilon_h$ with respect to the cross-sectional thickness $\delta_h$, we show further compactness properties of the twist functions $w^h$.

**Proposition 2.5.3.** *Under the same assumptions of Theorem 2.5.2, let $w^h$ and $b$ be the functions introduced in* (2.5.27) *and* (2.5.36). *If $\frac{\epsilon_h}{h\delta_h}\to 0$, there holds*

$$\frac{1}{h}\partial_s w^h \rightharpoonup b \text{ weakly in } W^{-1,2}(\omega). \tag{2.5.45}$$

*Proof.* Assume that $\frac{\epsilon_h}{h\delta_h}\to 0$. By definition of the functions $w^h$, we deduce

$$\frac{1}{h}\partial_s w^h = \frac{\delta_h}{h^2\epsilon_h}\partial_s\int_{-\frac12}^{\frac12}(\nabla_{h,\delta_h}Y^h - R^hR_0)e_2\cdot n(h-\delta_h tk)\,dt + \frac{\delta_h}{h\epsilon_h}\partial_s((R^h - Id)\tau\cdot n). \tag{2.5.46}$$

By (2.2.4) and (2.5.1), the first term in the right-hand side of (2.5.46) converges to zero strongly in $W^{-1,2}(\omega)$. The second term can be further decomposed as

$$\frac{\delta_h}{h\epsilon_h}\partial_s((R^h - Id)\tau\cdot n) = \frac{\delta_h}{h\epsilon_h}\partial_s R^h\tau\cdot n + \frac{\delta_h}{h\epsilon_h}(R^h n\cdot n - R^h\tau\cdot\tau)k.$$

Hence, (2.5.45) follows by combining (2.5.32), (2.5.36) and (2.5.37). □

## 2.6 Characterization of the limit strain and liminf inequality

In this section we shall prove a liminf inequality for the rescaled energies $\frac{1}{\epsilon_h^2}\mathcal{J}^h$ defined in (2.2.3). To this purpose we introduce the strains:

$$G^h := \frac{1}{\epsilon_h}\big((R^h)^T\nabla_{h,\delta_h}Y^h R_0^T - Id\big), \tag{2.6.1}$$

where $(R^h)$ and $(Y^h)$ are the sequences introduced in Theorem 2.5.1, and we prove their convergence to a limit strain $G$. In Theorem 2.6.2 we deduce a characterization of $G$, together with some further properties of the limit functions $g$, $w$, and $b$ introduced in (2.5.34), (2.5.29), and (2.5.36).

We first prove a characterization of $g$.

**Proposition 2.6.1.** *Under the same assumptions of Theorem 2.5.2, let* (2.2.6) *be satisfied. Let $g$ be the function introduced in* (2.5.34) *and let $\mathcal{G}$ be the class defined in* (2.4.1). *Then $g\in\mathcal{G}$.*

*Proof.* Let $(Y^h)$ be as in Theorem 2.5.2. For every $h>0$ let

$$v^h := \frac{1}{\epsilon_h}\int_{-\frac12}^{\frac12}(Y_1^h - x_1)e_1\,dt + \frac{h}{\epsilon_h}\int_{-\frac12}^{\frac12}\big((Y_2^h-\psi_2^h)e_2 + (Y_3^h-\psi_3^h)e_3\big)\,dt.$$

By definition, $v^h\in W^{1,2}(\omega;\mathbb{R}^3)$ for every $h>0$; moreover by (2.2.6) and (2.5.35), there holds

$$\Big\|\frac{h}{\epsilon_h}\partial_s(Y^h-\psi^h)\cdot\tau\Big\|_{L^2(\Omega)} = \Big\|\frac{h(h-\delta_h tk)}{\epsilon_h}(\nabla_{h,\delta_h}Y^h R_0^T - Id)\tau\cdot\tau\Big\|_{L^2(\Omega)} \le Ch^2,$$

which implies

$$\partial_s v^h\cdot\tau\to 0 \quad \text{strongly in } L^2(\omega).$$

Similarly, by (2.5.35) we deduce

$$\partial_s v_1^h + \partial_1 v^h\cdot\tau\to 0 \quad \text{strongly in } L^2(\omega).$$

By (2.5.26) and (2.5.34) we also have

$$\partial_1 v_1^h \rightharpoonup g \text{ weakly in } L^2(\omega).$$

The thesis follows now by Lemma 2.4.1. □

We are now in a position to state the first theorem of this section. For every matrix $M \in \mathbb{M}^{3\times 3}$ we use the notation $M_{tan}$ to denote the matrix

$$M_{tan} := (e_1|\tau)^T (Me_1|M\tau).$$

**Theorem 2.6.2.** *Let the assumptions of Theorem 2.5.2 be satisfied. Assume in addition* (2.2.6). *Let* $(Y^h)$ *and* $(R^h)$ *be as in Theorem 2.5.2 and let* $G^h$ *be defined as in* (2.6.1). *Then there exists* $G \in L^2(\Omega;\mathbb{M}^{3\times 3})$ *such that, up to subsequences,*

$$G^h \rightharpoonup G \text{ weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}). \tag{2.6.2}$$

*Let* $g, w, b$ *be the maps introduced in* (2.5.34), (2.5.29), *and* (2.5.36). *Then*

$$G_{tan}(x_1,s,t) = -t\begin{pmatrix} 0 & w'(x_1) \\ w'(x_1) & b(x_1,s) \end{pmatrix} + G_{tan}(x_1,s,0) \tag{2.6.3}$$

*for a.e.* $(x_1,s,t) \in \Omega$ *and*

$$(G_{tan})_{11} = G_{11} = g \tag{2.6.4}$$

*a.e. in* $\Omega$.

*If in addition* (2.4.21) *holds, then:*

*a) if* $\mu = +\infty$, *there exist* $\alpha_1, \alpha_2, \alpha_3 \in L^2(0,L)$ *such that*

$$\partial_s g = \alpha_1 N + \alpha_2 \tau_2 + \alpha_3 \tau_3; \tag{2.6.5}$$

*b) if* $\lambda = +\infty$, *then* (2.6.5) *holds with* $\alpha_1 = 0$;

*c) if* $0 < \lambda < +\infty$, *then* $w \in W^{2,2}(0,L)$ *and* (2.6.5) *holds with* $\alpha_1 = \frac{1}{\lambda} w''$;

*d) if* $\lambda = 0$, *then* $w'' = 0$;

*e) if* $0 \le \mu < +\infty$, *then* $(g,b) \in \mathcal{C}_\mu$, *where* $\mathcal{C}_\mu$ *is the class defined in* (2.4.22)–(2.4.23).

*Proof.* By (2.5.1), the sequence $(G^h)$ is uniformly bounded in $L^2(\Omega;\mathbb{M}^{3\times 3})$; therefore there exists $G \in L^2(\Omega;\mathbb{M}^{3\times 3})$ such that (2.6.2) holds. By (2.6.2), and since $R^h$ converges to the identity boundedly in measure by (2.5.4), we deduce

$$\partial_t(R^h G^h R_0 e_1) \rightharpoonup \partial_t G e_1 \quad \text{weakly in } W^{-1,2}(\Omega;\mathbb{R}^3).$$

On the other hand, by (2.5.31) there holds

$$\begin{aligned} \partial_t(R^h G^h R_0 e_1) &= \frac{1}{\epsilon_h}\partial_t(\nabla_{h,\delta_h} Y^h - R^h R_0)e_1 = \frac{1}{\epsilon_h}\partial_t(\partial_1 Y^h) \\ &= \frac{\delta_h}{\epsilon_h}\partial_1\Big(\frac{\partial_t Y^h}{\delta_h}\Big) = \frac{\delta_h}{\varepsilon_h}\partial_1\Big(\frac{\partial_t Y^h}{\delta_h} - n\Big) \to A'n \end{aligned}$$

strongly in $W^{-1,2}(\Omega)$. Hence,

$$G(x_1,s,t)e_1 = tA'(x_1)n(s) + G(x_1,s,0)e_1 \tag{2.6.6}$$

for a.e. $(x_1,s,t) \in \Omega$.

To characterize $G\tau$ we observe that

$$\begin{aligned}\partial_t(R^hG^hR_0e_2) &= \frac{1}{\epsilon_h}\,\partial_t(\nabla_{h,\delta_h}Y^h - R^hR_0)e_2 = \frac{1}{\epsilon_h}\,\partial_t\big(\frac{\partial_s(Y^h-\psi^h)}{h-\delta_h tk}\big)\\ &= \frac{1}{\epsilon_h}\,\frac{\delta_h}{h-\delta_h tk}\,\partial_s\big(\frac{\partial_t(Y^h-\psi^h)}{\delta_h}\big) + \frac{\delta_h k}{\epsilon_h(h-\delta_h tk)}\,\frac{\partial_s(Y^h-\psi^h)}{h-\delta_h tk}\\ &= \frac{1}{\epsilon_h}\,\frac{\delta_h}{h-\delta_h tk}\,\big(\partial_s(\nabla_{h,\delta_h}Y^h - R^hR_0)e_3 + k(\nabla_{h,\delta_h}Y^h - R^hR_0)e_2\big)\\ &+ \frac{1}{\epsilon_h}\,\frac{\delta_h}{h-\delta_h tk}\,(\partial_s R^h)n.\end{aligned}$$

The first term on the right hand side of the previous equality is converging to zero strongly in $W^{-1,2}(\Omega;\mathbb{R}^3)$ due to (2.5.1), therefore by (2.2.4) and (2.5.37) we deduce

$$\partial_t(R^hG^hR_0e_2) \rightharpoonup Bn \quad \text{weakly in } W^{-1,2}(\Omega;\mathbb{R}^3).$$

On the other hand, (2.6.2) yields

$$\partial_t(R^hG^hR_0e_2) \rightharpoonup \partial_t G\tau \quad \text{weakly in } W^{-1,2}(\Omega;\mathbb{R}^3).$$

Hence

$$G(x_1,s,t)\tau(s) = tB(x_1,s)n(s) + G(x_1,s,0)\tau(s) \tag{2.6.7}$$

for a.e. $(x_1,s,t)\in\Omega$. By combining (2.5.33), (2.5.36), (2.6.6) and (2.6.7), we obtain (2.6.3).

By (2.5.32) and (2.6.1), there holds

$$\frac{1}{\epsilon_h}\partial_1(Y_1^h - \psi_1^h) \rightharpoonup G_{11} = (G_{tan})_{11} \quad \text{weakly in } L^2(\Omega). \tag{2.6.8}$$

Therefore (2.6.4) follows owing to (2.5.26) and (2.5.34).

Assume now that (2.4.21) holds true. To prove properties a)–e), we first claim that

$$\frac{(h-\delta_h tk)}{\epsilon_h}\partial_1^2(Y^h-\psi^h)\cdot\tau \rightharpoonup -\partial_s g \text{ weakly in } W^{-1,2}(\Omega). \tag{2.6.9}$$

Indeed, by (2.5.35) we deduce the following estimate

$$\begin{aligned}&\Big\|\frac{h-\delta_h tk}{\epsilon_h}\Big(\frac{\partial_s(Y_1^h-\psi_1^h)}{h-\delta_h tk} + \partial_1(Y^h-\psi^h)\cdot\tau\Big)\Big\|_{L^2(\Omega)}\\ &\qquad\le 2\Big\|\frac{h-\delta_h tk}{\epsilon_h}\mathrm{sym}(\nabla_{h,\delta_h}Y^hR_0^T - Id)\Big\|_{L^2(\Omega;\mathbb{M}^{3\times3})} \le Ch\Big(1+\frac{\epsilon_h}{\delta_h^2}\Big),\end{aligned}$$

where the quantity in the last inequality converges to zero by (2.2.6). Thus, (2.6.9) follows by (2.6.8) and (2.6.4).

We introduce the maps $\overline{v}^h\in W^{1,2}(\Omega;\mathbb{R}^2)$, given by

$$\overline{v}^h := \begin{pmatrix} v_2^h \\ v_3^h \end{pmatrix} = \frac{h}{\epsilon_h}\begin{pmatrix} Y_2^h - \psi_2^h \\ Y_3^h - \psi_3^h \end{pmatrix} \tag{2.6.10}$$

for every $h>0$. By (2.2.4) and (2.6.9), we have

$$\partial_1^2\overline{v}^h\cdot\overline{\tau} \rightharpoonup -\partial_s g \text{ weakly in } W^{-1,2}(\Omega). \tag{2.6.11}$$

Let $\overline{\nabla}_{\frac{\delta_h}{h}}$ be the operator introduced in (2.3.1), with $\epsilon$ replaced by $\frac{\delta_h}{h}$. By straightforward computations and by (2.5.35), we obtain

$$\|\mathrm{sym}(\overline{\nabla}_{\frac{\delta_h}{h}}\overline{v}^h\overline{R}_0^T)\|_{L^2(\Omega;\mathbb{M}^{2\times 2})} \leq \frac{h^2}{\epsilon_h}\Big\|\mathrm{sym}(\nabla_{h,\delta_h}Y^hR_0^T - Id)\Big\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} \leq Ch^2 \tag{2.6.12}$$

for every $h > 0$. Applying the Korn inequality proved in (2.3.6) and using the notation of Theorem 2.3.2, we deduce

$$\|\overline{v}^h - \Pi_{\frac{\delta_h}{h}}(\overline{v}^h)\|_{W^{1,2}(S;\mathbb{R}^2)} \leq C\frac{h}{\delta_h}\|\mathrm{sym}(\overline{\nabla}_{\frac{\delta_h}{h}}v^h\overline{R}_0^T)\|_{L^2(S;\mathbb{M}^{2\times 2})}, \tag{2.6.13}$$

for a.e. $x_1 \in (0, L)$. Integrating (2.6.13) with respect to $x_1$, by (2.6.12) it follows that

$$\|\overline{v}^h - \Pi_{\frac{\delta_h}{h}}(\overline{v}^h)\|_{L^2(\Omega;\mathbb{R}^2)} \leq C\frac{h^3}{\delta_h}, \tag{2.6.14}$$

$$\|\partial_s(\overline{v}^h - \Pi_{\frac{\delta_h}{h}}(\overline{v}^h))\|_{L^2(\Omega;\mathbb{R}^2)} \leq C\frac{h^3}{\delta_h}, \tag{2.6.15}$$

$$\|\partial_t(\overline{v}^h - \Pi_{\frac{\delta_h}{h}}(\overline{v}^h))\|_{L^2(\Omega;\mathbb{R}^2)} \leq C\frac{h^3}{\delta_h}. \tag{2.6.16}$$

By Lemma 2.3.1, for every $h > 0$ there exist $\alpha_1^h, \alpha_2^h, \alpha_3^h \in L^2(0, L)$ such that $\Pi_{\frac{\delta_h}{h}}(\overline{v}^h)$ has the following structure:

$$\Pi_{\frac{\delta_h}{h}}(\overline{v}^h) = \begin{pmatrix} \alpha_2^h \\ \alpha_3^h \end{pmatrix} + \alpha_1^h\begin{pmatrix} -\gamma_3 \\ \gamma_2 \end{pmatrix} - \frac{\delta_h}{h}t\alpha_1^h\overline{\tau}. \tag{2.6.17}$$

Moreover, by (2.5.27) there holds

$$\frac{\delta_h}{h^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \partial_s\overline{v}^h \cdot \overline{n}\, dt = w^h \tag{2.6.18}$$

for every $h > 0$ and for a.e. $(x_1, s) \in \omega$. On the other hand, by (2.6.17)

$$\frac{\delta_h}{h^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \partial_s\Pi_{\frac{\delta_h}{h}}(\overline{v}^h) \cdot \overline{n}\, dt = \frac{\delta_h}{h^2}\alpha_1^h \tag{2.6.19}$$

for every $h > 0$ and for a.e. $(x_1, s) \in \omega$. Therefore, by estimate (2.6.15), we obtain

$$\|\alpha_1^h - \frac{h^2}{\delta_h}w^h\|_{L^2(\omega)} \leq C\frac{h^3}{\delta_h}, \tag{2.6.20}$$

which in turn, by (2.5.29) implies

$$\frac{\delta_h}{h}t\alpha_1^h\overline{\tau} \to 0 \quad \text{strongly in } L^2(\Omega). \tag{2.6.21}$$

We first consider the case where $\mu = +\infty$. By (2.6.11) and (2.6.14), we have

$$\partial_1^2(\Pi_{\frac{\delta_h}{h}}(\overline{v}^h)) \cdot \overline{\tau} \rightharpoonup -\partial_s g \text{ weakly in } W^{-2,2}(\Omega). \tag{2.6.22}$$

Hence, by (2.6.17), (2.6.21) and by Lemma 2.3.4 there exist $\alpha_1, \alpha_2, \alpha_3 \in L^2(0, L)$ such that (2.6.5) holds true and the proof of a) is completed.

The proof of b) follows immediately once we notice that if $\lambda = +\infty$, then by (2.6.20), $\alpha_1 = 0$.

Consider now the case where $0 \leq \lambda < +\infty$. By (2.6.11) and (2.6.14), there holds

$$\frac{\delta_h}{h^2}\partial_1^2(\Pi_{\frac{\delta_h}{h}}(\overline{v}^h)) \cdot \overline{\tau} \rightharpoonup -\lambda \partial_s g \text{ weakly in } W^{-2,2}(\Omega) \tag{2.6.23}$$

for every $0 \leq \lambda < +\infty$. By (2.6.17), (2.6.21) and by Lemma 2.3.4, there exist $\beta_1, \beta_2, \beta_3 \in L^2(0, L)$ such that

$$\frac{\delta_h}{h^2}(\alpha_i^h)'' \rightharpoonup \beta_i \quad \text{weakly in } W^{-2,2}(0, L),\ i = 1, 2, 3 \tag{2.6.24}$$

and

$$\lambda \partial_s g = \beta_1 N + \beta_2 \tau_2 + \beta_3 \tau_3. \tag{2.6.25}$$

Now, if $0 < \lambda < +\infty$, by (2.6.20) and (2.6.24) we obtain $\beta_1 = w''$ and $w \in W^{2,2}(0, L)$. This proves c). To prove d) we observe that if $\lambda = 0$, by (2.6.25) and by Lemma 2.3.4 we have $\beta_1 = \beta_2 = \beta_3 = 0$, hence $w'' = 0$.

Consider now the case where $0 < \mu < +\infty$. Defining

$$\widehat{v}^h := \overline{v}^h - \Pi_{\frac{\delta_h}{h}}(\overline{v}^h),$$

by (2.6.14)–(2.6.16) there exists $\widehat{v} \in L^2(\Omega; \mathbb{R}^2)$ with $\partial_s \widehat{v}, \partial_t \widehat{v} \in L^2(\Omega; \mathbb{R}^2)$ such that, up to subsequences

$$\widehat{v}^h \rightharpoonup \widehat{v}, \quad \partial_s \widehat{v}^h \rightharpoonup \partial_s \widehat{v}, \quad \partial_t \widehat{v}^h \rightharpoonup \partial_t \widehat{v}, \quad \text{weakly in } L^2(\Omega; \mathbb{R}^2). \tag{2.6.26}$$

Since

$$\text{sym}(\overline{\nabla}_{\frac{\delta_h}{h}} \widehat{v}^h \overline{R}_0^T) = \text{sym}(\overline{\nabla}_{\frac{\delta_h}{h}} v^h \overline{R}_0^T), \tag{2.6.27}$$

for every $h > 0$, by combining (2.6.12) with (2.6.26), we deduce

$$\partial_s \widehat{v} \cdot \overline{\tau} = 0, \qquad \text{and} \qquad \partial_t \widehat{v} = 0. \tag{2.6.28}$$

By (2.5.45) and (2.6.18), it follows that

$$\frac{\delta_n}{h^3}\partial_s \int_{-\frac{1}{2}}^{\frac{1}{2}} \partial_s \overline{v}^h \cdot \overline{n}\, dt \rightharpoonup b \quad \text{weakly in } W^{-1,2}(\omega). \tag{2.6.29}$$

On the other hand, by (2.6.19), we have

$$\frac{\delta_h}{h^3}\partial_s\Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} \partial_s \overline{v}^h \cdot \overline{n}\, dt) = \frac{\delta_h}{h^3}\partial_s\Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} (\partial_s \overline{v}^h - \partial_s \Pi_{\frac{\delta_h}{h}}(\overline{v}^h)) \cdot \overline{n}\, dt) = \frac{\delta_h}{h^3}\partial_s\Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} \partial_s \widehat{v}^h \cdot \overline{n}\, dt\Big). \tag{2.6.30}$$

Therefore, (2.6.26) and (2.6.29) yield

$$\mu \partial_s(\partial_s \widehat{v} \cdot \overline{n}) = b, \tag{2.6.31}$$

whenever $0 < \mu < +\infty$. By (2.6.11), (2.6.14) and (2.6.26) we obtain that

$$\partial_1^2(\Pi_{\frac{\delta_h}{h}}(\overline{v}^h)) \cdot \overline{\tau} \rightharpoonup -\partial_s g - \partial_1^2 \widehat{v} \cdot \overline{\tau} \quad \text{weakly in } W^{-2,2}(\Omega).$$

By Lemma 2.3.4, by (2.6.17) and (2.6.21) there exist $\alpha_1, \alpha_2, \alpha_3 \in W^{-2,2}(0,L)$ such that

$$\partial_s g = -\alpha_2 \tau_2 - \alpha_3 \tau_3 + \alpha_1 N - \partial_1^2 \widehat{v} \cdot \overline{\tau}. \tag{2.6.32}$$

For $i = 1,2,3$, let now $\widehat{\alpha}_i \in L^2(0,L)$ be such that $(\widehat{\alpha}_i)'' = \alpha_i$ and let

$$v = \mu \left( \begin{array}{c} 0 \\ \widehat{v} + \begin{pmatrix} \widehat{\alpha}_2 \\ \widehat{\alpha}_3 \end{pmatrix} + \widehat{\alpha}_1 \begin{pmatrix} -\gamma_3 \\ \gamma_2 \end{pmatrix} \end{array} \right).$$

By (2.6.28), (2.6.31), and (2.6.32) we deduce

$$\partial_s v \cdot \tau = 0, \quad \partial_s(\partial_s v \cdot n) = b, \quad \text{ and } \partial_1^2 v \cdot \tau + \mu \partial_s g = 0,$$

where the last two equalities hold in the sense of distributions. Therefore, in particular, $(g,b) \in \mathcal{C}_\mu$.

Finally, we study the case where $\mu = 0$. For every $h > 0$, we define

$$\widetilde{v}^h := \frac{\delta_h}{h^3} \widehat{v}^h. \tag{2.6.33}$$

By (2.6.12), there holds

$$\|\mathrm{sym}(\overline{\nabla}_{\frac{\delta_h}{h}} \widetilde{v}^h \overline{R}_0^T)\|_{L^2} \leq C \frac{\delta_h}{h}. \tag{2.6.34}$$

By (2.6.14)–(2.6.16) there exists $\widetilde{v} \in L^2(\Omega;\mathbb{R}^2)$, with $\partial_s \widetilde{v}, \partial_t \widetilde{v} \in L^2(\Omega;\mathbb{R}^2)$, such that, up to subsequences,

$$\widetilde{v}^h \rightharpoonup \widetilde{v}, \quad \partial_s \widetilde{v}^h \rightharpoonup \partial_s \widetilde{v}, \quad \partial_t \widetilde{v}^h \rightharpoonup \partial_t \widetilde{v}, \quad \text{weakly in } L^2(\Omega;\mathbb{R}^2). \tag{2.6.35}$$

By (2.6.29), (2.6.30) and (2.6.34), $\widetilde{v}$ satisfies

$$\partial_s \widetilde{v} \cdot \overline{\tau} = 0, \quad \partial_t \widetilde{v} = 0 \quad \text{ and } \partial_s(\partial_s \widetilde{v} \cdot \overline{n}) = b. \tag{2.6.36}$$

Moreover, by (2.6.11) and by (2.6.35) we deduce that

$$\partial_1^2 (\Pi_{\frac{\delta_h}{h}}(\widetilde{v}^h)) \cdot \overline{\tau} \rightharpoonup -\partial_1^2 \widetilde{v} \cdot \overline{\tau} \quad \text{weakly in } W^{-2,2}(\Omega). \tag{2.6.37}$$

Hence, by (2.6.21) and Lemma 2.3.4, there exist $\alpha_1, \alpha_2, \alpha_3 \in W^{-2,2}(0,L)$ such that

$$\partial_1^2 \widetilde{v} \cdot \overline{\tau} = -\alpha_2 \tau_2 - \alpha_3 \tau_3 + \alpha_1 N. \tag{2.6.38}$$

Let now $\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\alpha}_3 \in L^2(0,L)$ be such that $\alpha_1 = (\widehat{\alpha}_1)''$, $\alpha_2 = (\widehat{\alpha}_2)''$ and $\alpha_3 = (\widehat{\alpha}_3)''$. By defining

$$v = \left( \begin{array}{c} 0 \\ \widetilde{v} + \begin{pmatrix} \widehat{\alpha}_2 \\ \widehat{\alpha}_3 \end{pmatrix} + \widehat{\alpha}_1 \begin{pmatrix} -\gamma_3 \\ \gamma_2 \end{pmatrix} \end{array} \right),$$

properties (2.6.36) and (2.6.38) yield

$$\partial_s v \cdot \tau = 0, \quad \partial_s(\partial_s v \cdot n) = b \quad \text{ and } \quad \partial_1^2 v \cdot \tau = 0,$$

where the last two equalities hold in the sense of distributions. By Proposition 2.6.1, the proof of the theorem is complete. □

We are now in a position to deduce a lower bound for the rescaled energies $\epsilon_h^{-2}\mathcal{J}^h$. To this purpose, from here to the end of the paper we shall assume that (2.4.21) holds and we introduce the classes $\mathcal{A}_{\lambda,\mu}$ defined as follows. We set

$$\mathcal{A}_{\infty,\infty} := \big\{(w,g,b) \in W^{1,2}(0,L) \times L^2(\omega) \times L^2(\omega) : \partial_s g = \alpha_2\tau_2 + \alpha_3\tau_3, \text{ with } \alpha_i \in L^2(0,L),\ i = 2,3\big\}. \quad (2.6.39)$$

For $\lambda \in (0,+\infty)$ we define

$$\mathcal{A}_{\lambda,\infty} := \big\{(w,g,b) \in W^{2,2}(0,L) \times L^2(\omega) \times L^2(\omega) : \partial_s g = \tfrac{1}{\lambda} w'' N + \alpha_2\tau_2 + \alpha_3\tau_3, \text{ with } \alpha_i \in L^2(0,L),\ i = 2,3\big\}, \quad (2.6.40)$$

and for $\lambda = 0$

$$\mathcal{A}_{0,\infty} := \big\{(w,g,b) \in W^{2,2}(0,L) \times L^2(\omega) \times L^2(\omega) : w'' = 0 \text{ and } \partial_s g = \alpha_1 N + \alpha_2\tau_2 + \alpha_3\tau_3, \text{ with } \alpha_i \in L^2(0,L),\ i = 1,2,3\big\}. \quad (2.6.41)$$

Finally, for $\mu \in [0,+\infty)$, let

$$\mathcal{A}_{0,\mu} := \big\{(w,g,b) \in W^{2,2}(0,L) \times \mathcal{C}_\mu : w'' = 0\big\}. \quad (2.6.42)$$

A key role will be played by the quadratic form of linearized elasticity

$$Q : \mathbb{M}^{3\times 3} \longrightarrow [0,+\infty)$$

defined by

$$Q(F) := D^2W(Id)F : F \qquad \text{for every } F \in \mathbb{M}^{3\times 3}.$$

The limiting functionals will involve the constant

$$\mathbb{E} := \min_{a,b\in\mathbb{R}^3} Q(e_1|a|b) \quad (2.6.43)$$

and the quadratic form $Q_{tan} : [0,1] \times \mathbb{R}^2 \longrightarrow [0,+\infty)$ defined by

$$Q_{tan}(s,a,b) = \min_{\sigma_i\in\mathbb{R}} Q\left( R_0(s) \begin{pmatrix} 0 & a & \sigma_1 \\ a & b & \sigma_2 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} R_0^T(s) \right) \quad (2.6.44)$$

for any $s \in [0,1]$ and for any $(a,b) \in \mathbb{R}^2$. It is well known that, owing to (H2)–(H5), $Q$ is a positive semi-definite quadratic form and is positive definite on symmetric matrices. Hence, $\mathbb{E} > 0$ and $Q_{tan}(s,a,b)$ is strictly positive for every $s \in [0,1]$ and every $(a,b) \neq (0,0)$.

We consider the functionals $\mathcal{J}_{\lambda,\mu} : W^{1,2}(0,L) \times L^2(\omega) \times L^2(\omega) \longrightarrow [0,+\infty]$, defined as

$$\mathcal{J}_{\lambda,\mu}(w,g,b) := \frac{1}{24}\int_0^L\int_0^1 Q_{tan}(s,w',b)\,ds dx_1 + \frac{1}{2}\int_0^L\int_0^1 \mathbb{E} g^2\,ds dx_1 \quad (2.6.45)$$

for $(w,g,b) \in \mathcal{A}_{\lambda,\mu}$, and $\mathcal{J}_{\lambda,\mu}(w,g,b) = +\infty$ otherwise, where $Q_{tan}$ and $\mathbb{E}$ are the quadratic form and the constant given by (2.6.44) and (2.6.43), respectively.

With these definitions at hand the following liminf inequality for the scaled energy functionals can be proved.

**Theorem 2.6.3.** *Assume that* (2.2.6) *and* (2.4.21) *hold. Let* $\mathcal{A}_{\lambda,\mu}$ *be the classes defined in* (2.6.39)–(2.6.42) *and let* $(y^h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ *be a sequence of deformations satisfying* (2.2.5). *Then, there exist rotations* $P^h \in SO(3)$ *and constants* $c^h \in \mathbb{R}^3$ *such that, setting* $Y^h := (P^h)^T y^h - c^h$ *and defining* $g^h$ *and* $w^h$ *as in* (2.5.26) *and* (2.5.27), *up to subsequences there holds*

$$\begin{aligned} &g^h \rightharpoonup g \text{ weakly in } L^2(\Omega), \\ &w^h \to w \text{ in } L^2(\omega), \\ &\tfrac{1}{h}\partial_s w^h \rightharpoonup b \text{ weakly in } W^{-1,2}(\omega), \end{aligned} \tag{2.6.46}$$

*where* $(w,g,b) \in \mathcal{A}_{\lambda,\mu}$. *Moreover,*

$$\liminf_{h\to 0} \frac{1}{\epsilon_h^2}\mathcal{J}^h(Y^h) \geq \mathcal{J}_{\lambda,\mu}(w,g,b), \tag{2.6.47}$$

*where* $\mathcal{J}_{\lambda,\mu}$ *is the functional defined in* (2.6.45).

*Proof.* The convergence properties (2.6.46) follow by Theorem 2.5.2 and Proposition 2.5.3. Moreover, Proposition 2.6.1 and Theorem 2.6.2 guarantee that

$$(w,g,b) \in \mathcal{A}_{\lambda,\mu}.$$

The proof of the lower bound (2.6.47) is an adaptation of [33, Proof of Corollary 2]. We sketch some details for convenience of the reader.

Let $G^h$ be defined as in (2.6.1). We introduce the functions

$$\chi^h(x) := \begin{cases} 1 & \text{if } |G^h| < \frac{1}{\sqrt{\epsilon_h}} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\chi^h \to 1$ in measure and $\chi^h G^h \rightharpoonup G$ weakly in $L^2(\Omega;\mathbb{M}^{3\times 3})$. By frame indifference of $W$,

$$\begin{aligned} \liminf_{h\to 0} \frac{\mathcal{J}^h(Y^h)}{\epsilon_h^2} &= \liminf_{h\to 0} \frac{1}{\epsilon_h^2}\int_\Omega W(\nabla_{h,\delta_h} Y^h R_0^T)\,dx_1 ds dt \\ &= \liminf_{h\to 0} \frac{1}{\epsilon_h^2}\int_\Omega W(Id + \epsilon_h G^h)\,dx_1 ds dt \\ &\geq \liminf_{h\to 0} \frac{1}{\epsilon_h^2}\int_\Omega \chi^h W(Id + \epsilon_h G^h)\,dx_1 ds dt. \end{aligned} \tag{2.6.48}$$

Owing to assumptions (H2), (H3), and (H5), a Taylor expansion of $W$ around the identity yields:

$$W(Id + F) = \frac{1}{2}Q(F) + \eta(F),$$

for every $F \in \mathbb{M}^{3\times 3}$, where $\frac{\eta(F)}{|F|^2} \to 0$ as $|F| \to 0$. Setting

$$\xi(t) := \sup_{|F|\leq t} \frac{\eta(F)}{|F|^2},$$

then $\xi(t) \to 0$ as $t \to 0$ and

$$\chi_h W(Id + \epsilon_h G^h) \geq \chi_h \frac{\epsilon_h^2}{2} Q(G^h) - \chi_h \epsilon_h^2 \xi(\epsilon_h |G^h|)|G^h|^2.$$

Thus, we can continue the chain of inequalities in (2.6.48) as

$$\liminf_{h\to 0} \frac{\mathcal{J}^h(Y^h)}{\epsilon_h^2} \geq \liminf_{h\to 0} \Big\{ \frac{1}{2}\int_\Omega Q(\chi_h G^h)\, dx_1 ds dt - \int_\Omega \chi_h \xi(\epsilon^h |G^h|)|G^h|^2\, dx_1 ds dt \Big\}. \tag{2.6.49}$$

By the assumptions on $W$, $Q$ is a positive semi-definite quadratic form, hence the first term in (2.6.49) is lower semicontinuous with respect to the weak convergence in $L^2$. By definition of the sequence of functions $(\chi_h)$ and by the uniform boundedness of $\|G^h\|_{L^2(\Omega;\mathbb{M}^{3\times 3})}$, the second term in (2.6.49) can be bounded as

$$\frac{1}{2}\int_\Omega \chi_h \xi(\epsilon^h|G^h|)|G^h|^2\, dx_1 ds dt \leq C\xi(\sqrt{\epsilon_h})$$

and therefore converges to zero as $h \to 0$. By collecting the previous remarks, it follows that

$$\liminf_{h\to 0} \frac{\mathcal{J}^h(Y^h)}{\epsilon_h^2} \geq \frac{1}{2}\int_\Omega Q(G)\, dx_1 ds dt.$$

We can decompose $G$ as

$$G = \Big(G - \int_{-\frac{1}{2}}^{\frac{1}{2}} G\, dt\Big) + \int_{-\frac{1}{2}}^{\frac{1}{2}} G\, dt,$$

where by the characterizations (2.6.3) and (2.6.4)

$$\Big(G - \int_{-\frac{1}{2}}^{\frac{1}{2}} G\, dt\Big)_{tan} = -t\begin{pmatrix} 0 & w' \\ w' & b \end{pmatrix} \quad \text{and} \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} G_{11}\, dt = g.$$

Therefore, by developing the quadratic form and using (2.6.43) and (2.6.44), we obtain

$$\begin{aligned}\int_\Omega Q(G)\, dx_1 ds dt &= \int_\Omega Q\Big(G - \int_{-\frac{1}{2}}^{\frac{1}{2}} G\, dt\Big)\, dx_1 ds dt + \int_0^L \int_0^1 Q\Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} G\, dt\Big)\, ds dx_1 \\ &\geq \frac{1}{12}\int_0^L \int_0^1 Q_{tan}(s, w', b)\, ds dx_1 + \int_0^L \int_0^1 \mathbb{E} g^2\, ds dx_1.\end{aligned}$$

This last inequality concludes the proof of the theorem. □

## 2.7 Construction of the recovery sequence

In this section we shall prove that the lower bound obtained in Theorem 2.6.3 is optimal by exhibiting a recovery sequence. The structure of such an optimal sequence varies according to the values of $\lambda$ and $\mu$.

**Theorem 2.7.1.** *Assume* (2.2.6) *and* (2.4.21)*. Let* $\mathcal{A}_{\lambda,\mu}$ *be the classes defined in* (2.6.39)–(2.6.42)*. Then, if* $\mu > 0$*, for every* $(w, g, b) \in \mathcal{A}_{\lambda,\mu}$ *there exists a sequence of deformations* $(y^h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ *such that, defining* $g^h$ *and* $w^h$ *as in* (2.5.26) *and* (2.5.27)*, there holds*

$$y^h \to x_1 e_1 \text{ strongly in } W^{1,2}(\Omega;\mathbb{R}^3), \tag{2.7.1}$$

$$g^h \to g \text{ strongly in } L^2(\Omega), \tag{2.7.2}$$

$$w^h \to w \text{ strongly in } L^2(\omega), \tag{2.7.3}$$

$$\frac{\partial_s w^h}{h} \to b \text{ strongly in } L^2(\omega). \tag{2.7.4}$$

*Moreover,*

$$\limsup_{h\to 0}\frac{1}{\epsilon_h^2}\mathcal{J}^h(y^h)\leq \mathcal{J}_{\lambda,\mu}(w,u,b), \tag{2.7.5}$$

*where $\mathcal{J}_{\lambda,\mu}$ is the functional defined in* (2.6.45).

*The same conclusion holds if $\mu=0$, assuming in addition the hypotheses of Lemma 2.4.10.*

*Proof.* For the sake of simplicity, we divide the proof into five steps. In the first step we consider the case where $\lambda=+\infty$. Then we show how the recovery sequence must be modified for different values of $\lambda$ and $\mu$.

**Step 1:** $\lambda=\mu=+\infty$**.**

Let $(w,g,b)\in\mathcal{A}_{\infty,\infty}$. We can assume that $w\in C^\infty([0,L])$, $b\in C^\infty(\overline{\omega})$, and there exist $\alpha_i\in C^\infty([0,L])$, $i=2,3,4$, such that

$$g=\alpha_2''\gamma_2+\alpha_3''\gamma_3+\alpha_4''.$$

The general case follows by approximation and standard arguments in $\Gamma$-convergence.

Let $\sigma_i\in C^5(\overline{\omega})$, $i=1,2,3$, be such that

$$Q_{tan}(s,w',b)=Q\left(R_0\begin{pmatrix}0 & w' & \sigma_1\\ w' & b & \sigma_2\\ \sigma_1 & \sigma_2 & \sigma_3\end{pmatrix}R_0^T\right) \tag{2.7.6}$$

for every $(x_1,s)\in\overline{\omega}$, and let $H\in C^5(\overline{\omega};\mathbb{M}^{3\times 3}_{\mathrm{sym}})$, $H=(h_{ij})$, be defined as

$$H:=R_0\begin{pmatrix}0 & 0 & \sigma_1\\ 0 & 0 & \sigma_2\\ \sigma_1 & \sigma_2 & \sigma_3\end{pmatrix}R_0^T.$$

For every $h>0$ we introduce the functions $\sigma^h\in C^5(\overline{\Omega};\mathbb{R}^3)$ given by

$$\sigma^h:=\epsilon_h\delta_h\Big(\frac{t^2}{2}-\frac{1}{24}\Big)\begin{pmatrix}2\sigma_1\\ 2\sigma_2\tau_2-\sigma_3\tau_3\\ 2\sigma_2\tau_3+\sigma_3\tau_2\end{pmatrix}.$$

It is easy to see that

$$\mathrm{sym}(\nabla_{h,\delta_h}\sigma^h R_0^T)=\epsilon_h tH+o(\epsilon_h). \tag{2.7.7}$$

Let also $F\in\mathbb{M}^{3\times 3}$ be the matrix defined by

$$\mathbb{E}=Q(e_1\otimes e_1+F), \tag{2.7.8}$$

where $\mathbb{E}$ is the quantity introduced in (2.6.43).

Finally, let $v\in C^6(\overline{\omega};\mathbb{R}^2)$, $v=(v_2,v_3)$ be a solution to

$$\partial_s v\cdot\overline{\tau}=0 \text{ in } \omega, \tag{2.7.9}$$

$$\partial_s\Big(\partial_s v\cdot\overline{n}\Big)=b \text{ in } \omega \tag{2.7.10}$$

and let $\overline{\psi}^{\frac{\delta_h}{h}}$ be the map introduced in (2.3.2), with $\epsilon=\frac{\delta_h}{h}$.

We consider the sequence

$$\begin{aligned}
\widehat{y}^h &= \psi^h + \epsilon_h \begin{pmatrix} \alpha_2' \\ \alpha_3' \end{pmatrix} \cdot \overline{\psi}^{\frac{\delta_h}{h}} e_1 + \epsilon_h \alpha_4' e_1 - \frac{\epsilon_h}{h} \begin{pmatrix} 0 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \\
&+ \epsilon_h F\Big(h\Big(\alpha_4''\gamma + \sum_{i=2,3} \alpha_i'' \int_0^s \gamma_i(\xi)\tau(\xi)d\xi\Big) + \delta_h t\Big(\alpha_4'' + \sum_{i=2,3} \alpha_i''\gamma_i\Big)n\Big) \\
&+ \frac{\epsilon_h}{\delta_h} w\left(h\begin{pmatrix} 0 \\ -\gamma_3 \\ \gamma_2 \end{pmatrix} - \delta_h t\tau\right) - \frac{h\epsilon_h}{\delta_h} w'\Big(\delta_h tT - h\int_0^s N(\xi)d\xi\Big)e_1 \\
&- th\epsilon_h\Big(\partial_s v \cdot \overline{n}\Big)\tau + \frac{h^2\epsilon_h}{\delta_h}\begin{pmatrix} 0 \\ v \end{pmatrix} \\
&- \sigma^h - \frac{\epsilon_h^2}{2\delta_h^2} w^2(h\gamma + \delta_h tn).
\end{aligned}$$

We briefly comment on the structure of $\widehat{y}^h$: the terms in the first line are related to conditions (2.7.1) and (2.7.2), the second line is a corrective term to obtain the optimal constant $\mathbb{E}$, the terms in the third and the fourth line are introduced to satisfy respectively conditions (2.7.3) and (2.7.4), and the last line contains a further corrective term.

We first prove that $\widehat{y}^h$ satisfies (2.7.1)–(2.7.4). By (2.2.6) we have

$$\|\widehat{y}^h - x_1 e_1\|_{W^{1,2}(\Omega;\mathbb{R}^3)} \leq Ch,$$

which in turn implies (2.7.1). Condition (2.7.2) holds since

$$\partial_1(\widehat{y}_1^h - x_1) = \epsilon_h g + \frac{h^2\epsilon_h}{\delta_h} w'' \int_0^s N(\xi)d\xi + o(\epsilon_h) \tag{2.7.11}$$

and $\lambda = +\infty$. By the equality

$$\frac{\delta_h}{h\epsilon_h}\int_{-\frac{1}{2}}^{\frac{1}{2}} \partial_s(\widehat{y}^h - \psi^h)\cdot n\, dt = w + h\partial_s v \cdot \overline{n} + o(h),$$

and by (2.7.10), we deduce (2.7.3) and (2.7.4).

To prove convergence of the energies, we first compute the rescaled gradient of the deformations. By (2.7.9) and (2.7.10), we obtain

$$\begin{aligned}
\nabla_{h,\delta_h}\widehat{y}^h &= R_0 + \epsilon_h g e_1 \otimes e_1 + \epsilon_h g F\big(0\big|\tau\big|n\big) \\
&+ \frac{\epsilon_h}{h}\begin{pmatrix} 0 & \alpha_2'\tau_2 + \alpha_3'\tau_3 & \alpha_3'\tau_2 - \alpha_2'\tau_3 \\ -\alpha_2' & 0 & 0 \\ -\alpha_3' & 0 & 0 \end{pmatrix} \\
&- \epsilon_h t\big(w'\tau\big|w'e_1 + b\tau\big|0\big) + \Big(\frac{\epsilon_h}{\delta_h}w + \frac{h\epsilon_h}{\delta_h}\partial_s v \cdot \overline{n}\Big)\big(0\big|n\big| - \tau\big) \\
&+ \frac{h\epsilon_h}{\delta_h} w' \begin{pmatrix} 0 & N & -T \\ -\gamma_3 & 0 & 0 \\ \gamma_2 & 0 & 0 \end{pmatrix} - \nabla_{h,\delta_h}\sigma^h - \frac{\epsilon_h^2}{2\delta_h^2} w^2\big(0\big|\tau\big|n\big) + o(\epsilon_h).
\end{aligned}$$

We point out that the two terms

$$\Big(\frac{h^2\epsilon_h}{\delta_h}w''\int_0^s N(\xi)d\xi\Big)e_1\otimes e_1 \quad \text{and} \quad \frac{h^2\epsilon_h}{\delta_h}\begin{pmatrix} 0\\ \partial_1 v_2\\ \partial_1 v_3\end{pmatrix}\otimes e_1$$

are infinitesimal of order larger than $\epsilon_h$ since we are assuming $\lambda = +\infty$. Therefore both terms can be included in the error term $o(\epsilon_h)$.

The previous equality in turn gives:

$$\begin{aligned}\nabla_{h,\delta_h}\widehat{y}^h R_0^T &= Id + \epsilon_h g(e_1\otimes e_1 + F) + \frac{\epsilon_h}{h}\begin{pmatrix} 0 & \alpha_2 & \alpha_3\\ -\alpha_2 & 0 & 0\\ -\alpha_3 & 0 & 0\end{pmatrix}\\ &- \epsilon_h t\big(w'\tau\big|w'e_1 + b\tau\big|0\big)R_0^T + \Big(\frac{\epsilon_h}{\delta_h}w + \frac{h\epsilon_h}{\delta_h}\partial_s v\cdot\overline{n}\Big)\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}\\ &+ \frac{h\epsilon_h}{\delta_h}w'\begin{pmatrix} 0 & \gamma_3 & -\gamma_2\\ -\gamma_3 & 0 & 0\\ \gamma_2 & 0 & 0\end{pmatrix} - \nabla_{h,\delta_h}\sigma^h R_0^T - \frac{\epsilon_h^2}{2\delta_h^2}w^2\begin{pmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\\ &+ o(\epsilon_h).\end{aligned}$$

The identity $(Id+F)^T(Id+F) = Id + 2\text{sym}F + F^TF$ yields

$$(\nabla_{h,\delta_h}\widehat{y}^h R_0^T)^T(\nabla_{h,\delta_h}\widehat{y}^h R_0^T) = Id + 2\epsilon_h M + o(\epsilon_h),$$

where $M$ is given by

$$M := g(e_1\otimes e_1 + \text{sym}F) - t\left(R_0\begin{pmatrix} 0 & w' & 0\\ w' & b & 0\\ 0 & 0 & 0\end{pmatrix}R_0^T + H\right),$$

owing to (2.7.7). Hence, by frame-indifference,

$$W(\nabla_{h,\delta_h}\widehat{y}^h R_0^T) = W\Big(\sqrt{(\nabla_{h,\delta_h}\widehat{y}^h R_0^T)^T(\nabla_{h,\delta_h}\widehat{y}^h R_0^T)}\Big) = W(Id + \epsilon_h M + o(\epsilon_h)).$$

Since $M$ is bounded in $L^\infty$, it follows that there exists $\overline{h}$ such that if $h < \overline{h}$, then $Id + \epsilon_h M + o(\epsilon_h)$ belongs to the neighbourhood of SO(3) where $W$ is $C^2$, therefore a Taylor expansion around the identity gives:

$$\frac{1}{\epsilon_h^2}W(\nabla_{h,\delta_h}\widehat{y}^h R_0^T) \to \frac{1}{2}Q(M) \text{ pointwise },$$

and

$$\frac{1}{\epsilon_h^2}W(\nabla_{h,\delta_h}\widehat{y}^h R_0^T) \le C(|M|^2+1),$$

for some constant $C$. By the dominated convergence theorem and by (2.7.6) and (2.7.8) we deduce

$$\begin{aligned}\lim_{h\to 0}\frac{\mathcal{J}^h(\widehat{y}^h)}{\epsilon_h^2} &= \frac{1}{2}\int_\Omega Q(M)\,dx_1 ds dt\\ &= \frac{1}{24}\int_0^L\int_0^1 Q_{tan}(s,w',b)\,ds dx_1 + \frac{1}{2}\int_0^L\int_0^1 \mathbb{E}g^2\,ds dx_1,\end{aligned}$$

which concludes the proof of (2.7.5) in the case where $\lambda = +\infty$.

**Step 2:** $0 < \lambda < +\infty$ **and** $\mu = +\infty$**.**

Let $(w, g, b) \in \mathcal{A}_{\lambda,\infty}$. We can assume that $w \in C^\infty([0, L])$, $b \in C^\infty(\overline{\omega})$, and there exist $\alpha_i \in C^\infty(0, L)$, $i = 2, 3, 4$, such that

$$g = \frac{1}{\lambda} w'' \int_0^s N(\xi) d\xi + \alpha_2'' \gamma_2 + \alpha_3'' \gamma_3 + \alpha_4''.$$

Let $v$ be defined as in (2.7.9)–(2.7.10) and let $u \in C^6(\overline{\omega})$ be such that $\partial_s u + \partial_1 v \cdot \overline{\tau} = 0$ in $\omega$.

We consider the sequence

$$\begin{aligned} y^h &= \widehat{y}^h + \frac{h^2 \epsilon_h}{\delta_h} F\Big(hw'' \int_0^s \Big(\int_0^\xi N(\eta) d\eta\Big) \tau(\xi) d\xi + \delta_h t w'' \Big(\int_0^s N(\xi) d\xi\Big) n\Big) \\ &+ \frac{h^3 \epsilon_h}{\delta_h} \Big(u - \frac{\delta_h}{h} t \partial_1 v \cdot \overline{n}\Big) e_1, \end{aligned}$$

which is obtained adding to the sequence $(\widehat{y}^h)$ introduced in Step 1 two corrective terms. The first corrective term is due to the different structure of $g$, while the second one is needed to cancel the contribution to the energy of the quantity

$$\frac{h^2 \epsilon_h}{\delta_h} \begin{pmatrix} 0 \\ \partial_1 v_2 \\ \partial_1 v_3 \end{pmatrix} \otimes e_1,$$

which is now of order $\epsilon_h$. We observe that the term

$$\Big(\frac{h^2 \epsilon_h}{\delta_h} w'' \int_0^s N(\xi) d\xi\Big) e_1 \otimes e_1$$

is now included in the expression of $g$.

The proof of (2.7.1)–(2.7.4) is analogous to the one in Step 1. To prove convergence of the energies, we argue as in Step 1 and we deduce

$$\lim_{h \to 0} \frac{\mathcal{J}^h(y^h)}{\epsilon_h^2} = \frac{1}{24} \int_0^L \int_0^1 Q_{tan}(s, w', b)\, ds dx_1 + \frac{1}{2} \int_0^L \int_0^1 \mathbb{E} g^2\, ds dx_1.$$

A standard approximation argument leads then to the conclusion.

**Step 3:** $\lambda = 0$ **and** $\mu = +\infty$.

Let $(w, g, b) \in \mathcal{A}_{0,\infty}$. Then $w$ is affine. Moreover, we can assume that $b \in C^\infty(\overline{\omega})$, and there exist $\alpha_i \in C^\infty([0, L])$, $i = 1, \cdots, 4$, such that

$$g = \alpha_1'' \int_0^s N(\xi) d\xi + \alpha_2'' \gamma_2 + \alpha_3'' \gamma_3 + \alpha_4''.$$

Let $v$ and $u$ be defined as in the previous step. We consider the sequence:

$$\begin{aligned} y^h &= \widehat{y}^h + \epsilon_h \alpha_1' \int_0^s N(\xi) d\xi e_1 - \frac{\epsilon_h \delta_h t}{h} \alpha_1' T e_1 + \frac{\epsilon_h}{h} \alpha_1 \begin{pmatrix} 0 \\ -\gamma_3 \\ \gamma_2 \end{pmatrix} - \frac{\epsilon_h \delta_h t}{h^2} \alpha_1 \tau \\ &+ \epsilon_h F\Big(h\Big(\alpha_1'' \int_0^s \Big(\int_0^\xi N(\eta) d\eta\Big) \tau(\xi) d\xi\Big) + \delta_h t \alpha_1'' \Big(\int_0^s N(\xi) d\xi\Big) n\Big) \\ &+ \frac{h^3 \epsilon_h}{\delta_h} \Big(u - \frac{\delta_h}{h} t \partial_1 v \cdot \overline{n}\Big) e_1, \end{aligned}$$

where $(\widehat{y}^h)$ is the sequence introduced in Step 1.

We observe that the previous sequence is obtained by a slight modification of the recovery sequence introduced in Step 2, due to the fact that, since $\lambda = 0$, the contribution of $w''$ to the energy is zero and the role of $w''$ in the structure of $g$ is now played by $\alpha_1''$.

Arguing as in the previous steps, it is straightforward to prove (2.7.1)–(2.7.4). The same computations of Step 1 yield also convergence of the energies and the conclusion follows by approximation.

**Step 4: $\lambda = 0$ and $0 < \mu < +\infty$.**
Let $(w, g, b) \in \mathcal{A}_{0,\mu}$. Then $w$ is affine. Moreover, by Lemma 2.4.8 we can reduce to the case where $g \in C^4(\overline{\omega})$, $b \in C^3(\overline{\omega})$, and there exists $\phi \in C^5(\overline{\omega}; \mathbb{R}^3)$ such that

$$\partial_1 \phi_1 = \mu g, \quad \partial_s \phi \cdot \tau = 0, \quad \partial_s \phi_1 + \partial_1 \phi \cdot \tau = 0, \quad \text{and} \quad \partial_s(\partial_s \phi \cdot n) = b.$$

We define

$$\begin{aligned} y^h \; := \; & \psi^h + \frac{h^3 \epsilon_h}{\delta_h} \phi_1 e_1 + \epsilon_h F\Big(h \int_0^s g(x_1, \xi) \tau(\xi) d\xi + \delta_h t g n\Big) \\ + \; & \epsilon_h \left( - t w \tau + \frac{h}{\delta_h} w \begin{pmatrix} 0 \\ -\gamma_3 \\ \gamma_2 \end{pmatrix} \right) - \epsilon_h \Big( t h w' T - \frac{h^2}{\delta_h} w' \int_0^s N(\xi) d\xi \Big) e_1 \\ - \; & t h \epsilon_h (\partial_s \phi \cdot n) \tau + \frac{h^2 \epsilon_h}{\delta_h} \begin{pmatrix} 0 \\ \phi_2 \\ \phi_3 \end{pmatrix} - h^2 \epsilon_h t \partial_1 \phi \cdot n e_1 \\ - \; & \sigma^h - \frac{\epsilon_h^2}{2\delta_h^2} w^2 (h\gamma + \delta_h t n), \end{aligned}$$

where the terms in the first line are related to conditions (2.7.1) and (2.7.2) and to the optimal constant $\mathbb{E}$, whereas the second and the third lines are related to conditions (2.7.3) and (2.7.4) and to the quadratic form $Q_{tan}$.

Arguing as in the previous steps it is straightforward to prove that conditions (2.7.1)–(2.7.4) are satisfied and that

$$\lim_{h \to 0} \frac{\mathcal{J}^h(y^h)}{\epsilon_h^2} = \frac{1}{24} \int_0^L \int_0^1 Q_{tan}(s, w', b)\, ds dx_1 + \frac{1}{2} \int_0^L \int_0^1 \mathbb{E} g^2 \, ds dx_1.$$

**Step 5: $\lambda = \mu = 0$.**
Assume that there exists a finite number of points

$$0 = p_0 < p_1 < \cdots < p_m = 1$$

such that for every $i = 0, \cdots, m-1$ there holds $k(s) > 0$ for every $s \in (p_i, p_{i+1})$, or $k(s) < 0$ for every $s \in (p_i, p_{i+1})$ or $k(s) = 0$ for every $s \in (p_i, p_{i+1})$.

Let $(w, g, b) \in \mathcal{A}_{0,0}$. Then $w$ is affine. Moreover, by Remark 2.4.2, we can reduce to the case where $g \in C^3(\overline{\omega})$ and there exist two maps $u \in C^5(\overline{\omega})$ and $z \in C^4(\overline{\omega})$ such that $\partial_1^2 u = g$ and $\partial_s^2 u = kz$. By Lemma 2.4.10 we can also assume that $b \in C^3(\overline{\omega})$ and there exists $\phi \in C^5(\overline{\omega}; \mathbb{R}^3)$ such that

$$\partial_1 \phi_1 = 0, \quad \partial_s \phi \cdot \tau = 0, \quad \partial_s \phi_1 + \partial_1 \phi \cdot \tau = 0 \quad \text{and} \quad \partial_s(\partial_s \phi \cdot n) = b.$$

We define:

$$
\begin{aligned}
y^h \quad := \quad & \psi^h + \epsilon_h\Big(\partial_1 u + \frac{\delta_h}{h}t\partial_1 z\Big)e_1 - \frac{\epsilon_h}{h}(\partial_s u\tau + zn) + \frac{\epsilon_h\delta_h}{h^2}t(\partial_s uk + \partial_s z)\tau \\
+ \quad & \epsilon_h F\Big(h\int_0^s g\tau d\xi + \delta_h tgn\Big) \\
+ \quad & \epsilon_h\Bigg(-tw\tau + \frac{h}{\delta_h}w\begin{pmatrix} 0 \\ -\gamma_3 \\ \gamma_2 \end{pmatrix}\Bigg) - \epsilon_h\Big(thw'T - \frac{h^2}{\delta_h}w'\int_0^s Nd\xi\Big)e_1 \\
- \quad & th\epsilon_h(\partial_s\phi\cdot n)\tau + \frac{h^2\epsilon_h}{\delta_h}\begin{pmatrix} 0 \\ \phi_2 \\ \phi_3 \end{pmatrix} - h^2\epsilon_h t\partial_1\phi\cdot ne_1 + \frac{h^3\epsilon_h}{\delta_h}\phi_1 e_1 \\
- \quad & \sigma^h - \frac{\epsilon_h^2}{2\delta_h^2}w^2(h\gamma + \delta_h tn),
\end{aligned}
$$

where the first line contains now some corrective terms to compensate the contribution given by $\partial_s u$, and the terms in the other lines play the same role as in the previous steps.

Arguing as in the previous steps, it is immediate to prove (2.7.1)–(2.7.4). The same computations of Step 1 yield also (2.7.5). Hence, the proof of the theorem is completed. □

# Chapter 3

# A quasistatic evolution model for perfectly plastic thin plates

## 3.1 Overview of the chapter

The subject of this chapter is the rigorous derivation of a quasistatic evolution model for a three-dimensional plate of small thickness, whose elastic behaviour is linear and isotropic and whose plastic response is governed by the Prandtl-Reuss flow rule without hardening. As the thickness of the plate tends to zero, we prove via $\Gamma$-convergence techniques that solutions to the three-dimensional quasistatic evolution problem of Prandtl-Reuss elastoplasticity converge to a quasistatic evolution of a suitable reduced model. In this limiting model the admissible displacements are of Kirchhoff-Love type and the stretching and bending components of the stress are coupled through a plastic flow rule. Some equivalent formulations of the limiting problem in rate form are derived, together with some two-dimensional characterizations for suitable choices of the data.

The chapter is organised as follows: in Section 3.2 we recall some preliminary results and describe the formulation of the problem. In Section 3.3 we discuss the properties of Kirchhoff-Love admissible triples and prove some approximation results. Section 3.4 is devoted to the $\Gamma$-convergence result in the stationary case, while Section 3.5 concerns the convergence of quasistatic evolutions. Finally, in Section 3.6 we show some equivalent formulations of the reduced quasistatic evolution problem and discuss some examples.

## 3.2 Preliminaries and setting of the problem

### 3.2.1 Formulation of the problem

Throughout the chapter $\omega$ is a bounded and connected open set of $\mathbb{R}^2$ with a $C^2$ boundary. We suppose that the boundary $\partial\omega$ is partitioned into two disjoint open subsets $\gamma_d$, $\gamma_n$ and their common boundary $\partial\lfloor_{\partial\omega}\gamma_d = \partial\lfloor_{\partial\omega}\gamma_n$ (topological notions refer here to the relative topology of $\partial\omega$). We assume that $\gamma_d \neq \emptyset$ and that $\partial\lfloor_{\partial\omega}\gamma_d = \{P_1, P_2\}$, where $P_1, P_2$ are two points in $\partial\omega$.

The reference configuration of the plate is given by the set

$$\Omega_\varepsilon := \omega \times (-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2}),$$

where $\varepsilon > 0$. We denote by $\Gamma_\varepsilon$ the Dirichlet part of the boundary, given by $\Gamma_\varepsilon := \gamma_d \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, and by $\nu_{\partial\Omega_\varepsilon}$ the outer unit normal to $\partial\Omega_\varepsilon$.

**The elasticity tensor.** Let $\mathbb{C}$ be the *elasticity tensor*, considered as a symmetric positive definite linear operator $\mathbb{C} : \mathbb{M}^{3\times3}_{sym} \to \mathbb{M}^{3\times3}_{sym}$ and let $Q : \mathbb{M}^{3\times3}_{sym} \to [0, +\infty)$ be the quadratic form associated with $\mathbb{C}$, given by

$$Q(\xi) := \tfrac{1}{2}\mathbb{C}\xi : \xi \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{sym}. \tag{3.2.1}$$

It follows that there exist two constants $r_\mathbb{C}$ and $R_\mathbb{C}$, with $0 < r_\mathbb{C} \leq R_\mathbb{C}$, such that

$$r_\mathbb{C}|\xi|^2 \leq Q(\xi) \leq R_\mathbb{C}|\xi|^2 \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{sym}. \tag{3.2.2}$$

These inequalities imply

$$|\mathbb{C}\xi| \leq 2R_\mathbb{C}|\xi| \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{sym}. \tag{3.2.3}$$

**The dissipation potential.** Let $\mathbb{M}^{3\times3}_D$ be the space of all matrices in $\mathbb{M}^{3\times3}_{sym}$ with zero trace. Let $K$ be a closed convex set of $\mathbb{M}^{3\times3}_D$ such that there exist two constants $r_K$ and $R_K$, with $0 < r_K \leq R_K$, such that

$$\{\xi \in \mathbb{M}^{3\times3}_D : \ |\xi| \leq r_K\} \subset K \subset \{\xi \in \mathbb{M}^{3\times3}_D : \ |\xi| \leq R_K\}.$$

The boundary of $K$ is interpreted as the *yield surface*. The *plastic dissipation potential* is given by the support function $H : \mathbb{M}^{3\times3}_D \to [0, +\infty)$ of $K$, defined as

$$H(\xi) := \sup_{\sigma \in K} \sigma : \xi.$$

It follows that $H$ is a convex and positively one-homogeneous function such that

$$r_K|\xi| \leq H(\xi) \leq R_K|\xi| \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_D. \tag{3.2.4}$$

In particular, $H$ satisfies the triangle inequality

$$H(\xi + \zeta) \leq H(\xi) + H(\zeta) \quad \text{for every } \xi, \zeta \in \mathbb{M}^{3\times3}_D. \tag{3.2.5}$$

**Admissible triples and energy.** On $\Gamma_\varepsilon$ we prescribe a boundary datum $w^\varepsilon \in W^{1,2}(\Omega_\varepsilon; \mathbb{R}^3)$ of the following form:

$$w^\varepsilon(z) := \big(\bar{w}_1(z') - \tfrac{z_3}{\varepsilon}\partial_1 w_3(z'), \bar{w}_2(z') - \tfrac{z_3}{\varepsilon}\partial_2 w_3(z'), \tfrac{1}{\varepsilon}w_3(z')\big) \ \text{ for a.e. } z = (z', z_3) \in \Omega_\varepsilon, \tag{3.2.6}$$

where $\bar{w}_\alpha \in W^{1,2}(\omega)$, $\alpha = 1,2$, and $w_3 \in W^{2,2}(\omega)$. The *set of admissible displacements and strains* for the boundary datum $w^\varepsilon$ is denoted by $\mathcal{A}(\Omega_\varepsilon, w^\varepsilon)$ and is defined as the class of all triples $(v, f, q) \in BD(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon; \mathbb{M}^{3\times3}_{sym}) \times M_b(\Omega_\varepsilon; \mathbb{M}^{3\times3}_D)$ satisfying

$$
\begin{aligned}
Ev &= f + q \quad \text{in } \Omega_\varepsilon,\\
q &= (w^\varepsilon - v) \odot \nu_{\partial\Omega_\varepsilon} \mathcal{H}^2 \quad \text{on } \Gamma_\varepsilon,
\end{aligned}
$$

where $\odot$ stands for the symmetrized tensor product and $\mathcal{H}^2$ is the two-dimensional Hausdorff measure. The function $v$ represents the *displacement* of the plate, while $f$ and $q$ are called the *elastic* and *plastic strain*, respectively.

For every admissible triple $(v, f, q) \in \mathcal{A}(\Omega_\varepsilon, w^\varepsilon)$ we define the associated *energy* as

$$
\mathcal{E}_\varepsilon(v, f, q) := \int_{\Omega_\varepsilon} Q(f(z))\,dz + \int_{\Omega_\varepsilon \cup \Gamma_\varepsilon} H\Big(\frac{dq}{d|q|}\Big) d|q|. \tag{3.2.7}
$$

The first term represents the elastic energy, while the second term accounts for plastic dissipation.

### 3.2.2 The rescaled problem

As usual in dimension reduction problems, it is convenient to perform a change of variable in such a way to rewrite the system on a fixed domain independent of $\varepsilon$. To this purpose, we set

$$
\Omega := \omega \times (-\tfrac{1}{2}, \tfrac{1}{2}), \quad \Gamma_d := \gamma_d \times (-\tfrac{1}{2}, \tfrac{1}{2}), \quad \Gamma_n := \gamma_n \times (-\tfrac{1}{2}, \tfrac{1}{2}),
$$

and we denote by $\nu_{\partial\Omega}$ the outer unit normal to $\partial\Omega$. We consider the change of variable $\psi_\varepsilon : \overline{\Omega} \to \overline{\Omega}_\varepsilon$ given by

$$
\psi_\varepsilon(x) := (x', \varepsilon x_3) \quad \text{for every } x = (x', x_3) \in \overline{\Omega}
$$

and the linear operator $\Lambda_\varepsilon : \mathbb{M}^{3\times3}_{sym} \to \mathbb{M}^{3\times3}_{sym}$ given by

$$
\Lambda_\varepsilon \xi := \begin{pmatrix} \xi_{11} & \xi_{12} & \frac{1}{\varepsilon}\xi_{13} \\ \xi_{21} & \xi_{22} & \frac{1}{\varepsilon}\xi_{23} \\ \frac{1}{\varepsilon}\xi_{31} & \frac{1}{\varepsilon}\xi_{32} & \frac{1}{\varepsilon^2}\xi_{33} \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{3\times3}_{sym}.
$$

To any triple $(v, f, q) \in \mathcal{A}(\Omega_\varepsilon, w^\varepsilon)$ we associate a triple $(u, e, p) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times3}_{sym}) \times M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_{sym})$ defined as follows:

$$
u := (v_1 \circ \psi_\varepsilon, v_2 \circ \psi_\varepsilon, \varepsilon v_3 \circ \psi_\varepsilon), \qquad e := \Lambda_\varepsilon^{-1} f \circ \psi_\varepsilon, \qquad p := \tfrac{1}{\varepsilon} \Lambda_\varepsilon^{-1} \psi_\varepsilon^\#(q).
$$

Here the measure $\psi_\varepsilon^\#(q) \in M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_D)$ is the pull-back measure of $q$, satisfying

$$
\int_{\Omega \cup \Gamma_d} \varphi : d\psi_\varepsilon^\#(q) = \int_{\Omega_\varepsilon \cup \Gamma_\varepsilon} \varphi \circ \psi_\varepsilon^{-1} : dq \quad \text{for every } \varphi \in C_0(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_D).
$$

According to this change of variable we have

$$
\mathcal{E}_\varepsilon(v, f, q) = \varepsilon \mathcal{Q}(\Lambda_\varepsilon e) + \varepsilon \mathcal{H}(\Lambda_\varepsilon p),
$$

where

$$\mathcal{Q}(\Lambda_\varepsilon e) := \int_\Omega Q(\Lambda_\varepsilon e(x))\,dx, \qquad \mathcal{H}(\Lambda_\varepsilon p) := \int_{\Omega\cup\Gamma_d} H\Big(\frac{d\Lambda_\varepsilon p}{d|\Lambda_\varepsilon p|}\Big)\,d|\Lambda_\varepsilon p|.$$

We also introduce the scaled Dirichlet boundary datum $w \in W^{1,2}(\Omega;\mathbb{R}^3)$, given by

$$w(x) := (\bar{w}_1(x') - x_3\partial_1 w_3(x'), \bar{w}_2(x') - x_3\partial_2 w_3(x'), w_3(x')) \quad \text{for every } x \in \Omega.$$

From the definition of the class $\mathcal{A}(\Omega_\varepsilon, w^\varepsilon)$ it immediately follows that the scaled triple $(u, e, p)$ satisfies the equalities

$$Eu = e + p \quad \text{in } \Omega, \tag{3.2.8}$$

$$p = (w - u) \odot \nu_{\partial\Omega}\mathcal{H}^2 \quad \text{on } \Gamma_d, \tag{3.2.9}$$

$$p_{11} + p_{22} + \tfrac{1}{\varepsilon^2} p_{33} = 0 \quad \text{in } \Omega \cup \Gamma_d. \tag{3.2.10}$$

We are thus led to introduce the class $\mathcal{A}_\varepsilon(w)$ of all triples $(u, e, p) \in BD(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times3}_{sym}) \times M_b(\Omega\cup\Gamma_d;\mathbb{M}^{3\times3}_{sym})$ satisfying (3.2.8)–(3.2.10), and to define the functional

$$\mathcal{J}_\varepsilon(u, e, p) := \mathcal{Q}(\Lambda_\varepsilon e) + \mathcal{H}(\Lambda_\varepsilon p) \tag{3.2.11}$$

for every $(u, e, p) \in \mathcal{A}_\varepsilon(w)$. In the following we shall study the asymptotic behaviour of the minimizers of $\mathcal{J}_\varepsilon$ and of the quasistatic evolution associated with $\mathcal{J}_\varepsilon$, as $\varepsilon \to 0$.

## 3.3 The class of Kirchhoff-Love admissible triples

In this section we introduce the class of Kirchhoff-Love admissible triples, which will be the domain of the minimum problem describing the asymptotic behaviour of minimizers of $\mathcal{J}_\varepsilon$, as $\varepsilon \to 0$, and the space where the limiting quasistatic evolution takes place. We prove some approximation results, which will be crucial in the proofs of both convergence results. To this purpose we first define the set of Kirchhoff-Love displacements as

$$KL(\Omega) := \big\{u \in BD(\Omega) : \ (Eu)_{i3} = 0 \quad \text{for } i = 1, 2, 3\big\}.$$

**Remark 3.3.1.** Note that $u \in KL(\Omega)$ if and only if $u_3 \in BH(\omega)$ and there exists $\bar{u} \in BD(\omega)$ such that

$$u_\alpha = \bar{u}_\alpha - x_3\partial_\alpha u_3, \quad \alpha = 1, 2.$$

In particular, if $u \in KL(\Omega)$, then $(Eu)_{\alpha\beta} = (E\bar{u})_{\alpha\beta} - x_3\partial^2_{\alpha\beta}u_3$ for $\alpha, \beta = 1, 2$. If, in addition, $u \in W^{1,p}(\Omega;\mathbb{R}^3)$, then $\bar{u} \in W^{1,p}(\omega;\mathbb{R}^2)$ and $u_3 \in W^{2,p}(\omega)$. We call $\bar{u}, u_3$ the *Kirchhoff-Love components* of $u$.

For every $w \in W^{1,2}(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ we define the class $\mathcal{A}_{KL}(w)$ of Kirchhoff-Love admissible triples for the boundary datum $w$ as the set of all triples $(u, e, p) \in KL(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times3}_{sym}) \times M_b(\Omega\cup\Gamma_d;\mathbb{M}^{3\times3}_{sym})$ satisfying

$$Eu = e + p \quad \text{in } \Omega, \qquad p = (w - u) \odot \nu_{\partial\Omega}\mathcal{H}^2 \quad \text{on } \Gamma_d,$$

$$e_{i3} = 0 \quad \text{in } \Omega, \quad p_{i3} = 0 \quad \text{in } \Omega\cup\Gamma_d, \quad i = 1, 2, 3.$$

**Remark 3.3.2.** The space

$$\{\xi \in \mathbb{M}^{3\times 3}_{sym} : \ \xi_{i3} = 0 \text{ for } i = 1,2,3\}$$

is canonically isomorphic to $\mathbb{M}^{2\times 2}_{sym}$. Therefore, in the following, given a triple $(u,e,p) \in \mathcal{A}_{KL}(w)$ we will usually identify $e$ with a function in $L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$ and $p$ with a measure in $M_b(\Omega \cup \Gamma_d;\mathbb{M}^{2\times 2}_{sym})$.

We notice that the set $\mathcal{A}_{KL}(w)$ is always nonempty as it contains the triple $(w, Ew, 0)$. We also point out that if $(u,e,p) \in \mathcal{A}_{KL}(w)$, then in general one cannot conclude that $e$ and $p$ are affine in the $x_3$ variable. However, some conditions on the structure of $e$ and $p$ can be deduced. To this purpose, we introduce the following definitions.

**Definition 3.3.3.** Let $f \in L^2(\Omega;\mathbb{M}^{3\times 3}_{sym})$. We denote by $\bar{f}, \hat{f} \in L^2(\omega;\mathbb{M}^{3\times 3}_{sym})$ and by $f_\perp \in L^2(\Omega;\mathbb{M}^{3\times 3}_{sym})$ the following orthogonal components (in the sense of $L^2(\Omega;\mathbb{M}^{3\times 3}_{sym})$) of $f$:

$$\bar{f}(x') := \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x', x_3)\, dx_3, \quad \hat{f}(x') := 12 \int_{-\frac{1}{2}}^{\frac{1}{2}} x_3 f(x', x_3)\, dx_3$$

for a.e. $x' \in \omega$, and

$$f_\perp(x) := f(x) - \bar{f}(x') - x_3 \hat{f}(x')$$

for a.e. $x \in \Omega$. The component $\bar{f}$ is called the *zero-th order moment* of $f$, while $\hat{f}$ is called the *first order moment* of $f$.

**Definition 3.3.4.** Let $q \in M_b(\Omega \cup \Gamma_d;\mathbb{M}^{3\times 3}_{sym})$. The *zero-th order moment* of $q$ is the measure $\bar{q} \in M_b(\omega \cup \gamma_d;\mathbb{M}^{3\times 3}_{sym})$ defined by

$$\int_{\omega \cup \gamma_d} \varphi : d\bar{q} := \int_{\Omega \cup \Gamma_d} \varphi : dq$$

for every $\varphi \in C_0(\omega \cup \gamma_d;\mathbb{M}^{3\times 3}_{sym})$, while the *first order moment* of $q$ is the measure $\hat{q} \in M_b(\omega \cup \gamma_d;\mathbb{M}^{3\times 3}_{sym})$ defined by

$$\int_{\omega \cup \gamma_d} \varphi : d\hat{q} := 12 \int_{\Omega \cup \Gamma_d} x_3 \varphi : dq$$

for every $\varphi \in C_0(\omega \cup \gamma_d;\mathbb{M}^{3\times 3}_{sym})$. We also define $q_\perp \in M_b(\Omega \cup \Gamma_d;\mathbb{M}^{3\times 3}_{sym})$ as the measure given by

$$q_\perp := q - \bar{q} \otimes \mathcal{L}^1 - \hat{q} \otimes x_3 \mathcal{L}^1,$$

where the symbol $\otimes$ denotes the usual product of measures.

With these definitions at hand one can easily prove the following characterization of the class $\mathcal{A}_{KL}(\Omega)$.

**Proposition 3.3.5.** *Let $w \in W^{1,2}(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ and $(u,e,p) \in KL(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times 3}_{sym}) \times M_b(\Omega \cup \Gamma_d;\mathbb{M}^{3\times 3}_{sym})$ with $e_{i3} = 0$ in $\Omega$ and $p_{i3} = 0$ in $\Omega \cup \Gamma_d$ for $i = 1,2,3$. Let $\bar{u} \in BD(\omega)$, $u_3 \in BH(\omega)$, and $\bar{w} \in W^{1,2}(\omega;\mathbb{R}^2)$, $w_3 \in W^{2,2}(\omega)$ be the Kirchhoff-Love components of $u$ and $w$, respectively. Finally, let $\bar{e}, \hat{e} \in L^2(\omega;\mathbb{M}^{3\times 3}_{sym})$, $e_\perp \in L^2(\Omega;\mathbb{M}^{3\times 3}_{sym})$, $\bar{p}, \hat{p} \in M_b(\omega \cup \gamma_d;\mathbb{M}^{3\times 3}_{sym})$, and $p_\perp \in M_b(\Omega \cup \Gamma_d;\mathbb{M}^{3\times 3}_{sym})$ be the moments of $e$ and $p$, according to Definitions 3.3.3 and 3.3.4. Then $(u,e,p) \in \mathcal{A}_{KL}(\Omega)$ if and only if the following three conditions are satisfied:*

*(i)* $E\bar{u} = \bar{e} + \bar{p}$ *in* $\omega$ *and* $\bar{p} = (\bar{w} - \bar{u}) \odot \nu_{\partial\omega}\mathcal{H}^1$ *on* $\gamma_d$ *;*

*(ii)* $D^2 u_3 = -(\hat{e} + \hat{p})$ *in* $\omega$ *,* $u_3 = w_3$ *on* $\gamma_d$ *, and* $\hat{p} = (\nabla u_3 - \nabla w_3) \odot \nu_{\partial\omega}\mathcal{H}^1$ *on* $\gamma_d$ *;*

*(iii)* $p_\perp = -e_\perp$ *in* $\Omega$ *and* $p_\perp = 0$ *on* $\Gamma_d$ *,*

*where we have identified* $\bar{e}, \hat{e}$ *with functions in* $L^2(\omega; \mathbb{M}^{2\times2}_{sym})$ *and* $\bar{p}, \hat{p}$ *with measures in* $M_b(\omega \cup \gamma_d; \mathbb{M}^{2\times2}_{sym})$*. Here* $\nu_{\partial\omega}$ *denotes the outer unit normal to* $\partial\omega$ *and* $\mathcal{H}^1$ *is the one-dimensional Hausdorff measure.*

We now prove some approximation results for Kirchhoff-Love admissible triples. We first need a technical lemma.

**Lemma 3.3.6.** *Let* $\mu \in M_b(\overline{\omega} \times (-\frac{1}{2}, \frac{1}{2}); \mathbb{M}^{2\times2}_{sym})$ *be such that*

$$\mu = \bar{\mu} \otimes \mathcal{L}^1 + \hat{\mu} \otimes x_3\mathcal{L}^1 + \mu_\perp,$$

*where* $\bar{\mu}, \hat{\mu} \in M_b(\overline{\omega}; \mathbb{M}^{2\times2}_{sym})$ *with* $|\bar{\mu}|(\partial\omega) = |\hat{\mu}|(\partial\omega) = 0$ *and* $\mu_\perp \in L^2(\Omega; \mathbb{M}^{2\times2}_{sym})$*. Let* $(\rho_\delta) \subset C^\infty_c(\mathbb{R}^2)$ *be a sequence of mollifiers, with* $\operatorname{supp}\rho_\delta \subset B_\delta(0)$*. Then*

$$\lim_{\delta\to0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Big( \int_\omega |\rho_\delta * \mu_{x_3}|\, dx' \Big)\, dx_3 = |\mu|(\Omega),$$

*where we have set* $\mu_{x_3} := \bar{\mu} + x_3\hat{\mu} + \mu_\perp(\cdot, x_3) \in M_b(\overline{\omega}; \mathbb{M}^{2\times2}_{sym})$ *for* $\mathcal{L}^1$*-a.e.* $x_3 \in (-\frac{1}{2}, \frac{1}{2})$*.*

*Proof.* We first observe that, from the assumption $\mu_\perp \in L^2(\Omega; \mathbb{M}^{2\times2}_{sym})$ it follows that

$$\begin{aligned} \mu^a &= \bar{\mu}^a + x_3\hat{\mu}^a + \mu_\perp, \\ \mu^s &= \bar{\mu}^s \otimes \mathcal{L}^1 + \hat{\mu}^s \otimes x_3\mathcal{L}^1. \end{aligned}$$

Since $\bar{\mu}^s + x_3\hat{\mu}^s$ belongs to $L^\infty((-\frac{1}{2}, \frac{1}{2}); M_b(\overline{\omega}; \mathbb{M}^{2\times2}_{sym}))$, by [4, Corollary 2.29] we have

$$|\mu^s| = |\bar{\mu}^s + x_3\hat{\mu}^s| \overset{gen.}{\otimes} \mathcal{L}^1,$$

where $\overset{gen.}{\otimes}$ denotes the generalized product of measures (see, e.g., [4, Definition 2.27]). The equalities above imply that

$$\begin{aligned} |\mu|(\Omega) &= \int_\Omega |\mu^a(x)|\, dx + |\mu^s|(\Omega) \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_\omega |\bar{\mu}^a(x') + x_3\hat{\mu}^a(x') + \mu_\perp(x)|\, dx' dx_3 + \int_{-\frac{1}{2}}^{\frac{1}{2}} |\bar{\mu}^s + x_3\hat{\mu}^s|(\omega)\, dx_3 \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} |\mu_{x_3}|(\omega)\, dx_3. \end{aligned}$$

We now extend $\mu_{x_3}$ to 0 outside $\overline{\omega}$, so that the convolutions $\rho_\delta * \mu_{x_3}$ are well defined on $\mathbb{R}^2$. By Fubini-Tonelli Theorem and the assumption $|\bar{\mu}|(\partial\omega) = |\hat{\mu}|(\partial\omega) = 0$ we obtain

$$\begin{aligned} \int_\omega |\rho_\delta * \mu_{x_3}|\, dx' &= \int_\omega \Big| \int_{\mathbb{R}^2} \rho_\delta(x' - y')\, d\mu_{x_3}(y') \Big|\, dx' \\ &\leq \int_\omega \int_{\mathbb{R}^2} \rho_\delta(x' - y')\, d|\mu_{x_3}|(y')\, dx' \leq |\mu_{x_3}|(\omega) \end{aligned}$$

for $\mathcal{L}^1$-a.e. $x_3 \in (-\frac{1}{2}, \frac{1}{2})$. By integrating with respect to $x_3$ we deduce

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \Big( \int_\omega |\rho_\delta * \mu_{x_3}| \, dx' \Big) \, dx_3 \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} |\mu_{x_3}|(\omega) \, dx_3 = |\mu|(\Omega).$$

On the other hand, we have that $\rho_\delta * \mu_{x_3} \rightharpoonup \mu_{x_3}$ weakly* in $M_b(\omega; \mathbb{M}^{2\times 2}_{sym})$ for $\mathcal{L}^1$-a.e. $x_3 \in (-\frac{1}{2}, \frac{1}{2})$. Hence, by lower semicontinuity

$$|\mu_{x_3}|(\omega) \leq \liminf_{\delta \to 0} \int_\omega |\rho_\delta * \mu_{x_3}| \, dx'$$

for $\mathcal{L}^1$-a.e. $x_3 \in (-\frac{1}{2}, \frac{1}{2})$. Integration with respect to $x_3$ and Fatou's Lemma yield the thesis. $\square$

The next lemma allows one to approximate in energy any Kirchhoff-Love admissible triple by means of triples $(u^\varepsilon, e^\varepsilon, p^\varepsilon) \in \mathcal{A}_{KL}(\Omega)$ with $u^\varepsilon$ smooth. The proof of this result is based on an adaptation of [23, Proposition 1.4].

**Lemma 3.3.7.** *Let $w \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$ and let $(u, e, p) \in \mathcal{A}_{KL}(w)$. Then, there exists a sequence of triples $(u^\varepsilon, e^\varepsilon, p^\varepsilon) \in \mathcal{A}_{KL}(w)$ such that*

$$u^\varepsilon \in C^\infty(\Omega; \mathbb{R}^3) \cap W^{1,1}(\Omega; \mathbb{R}^3)$$

*and the following properties hold:*

$$u^\varepsilon \rightharpoonup u \quad \text{weakly* in } BD(\Omega), \tag{3.3.1}$$

$$e^\varepsilon \to e \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}), \tag{3.3.2}$$

$$p^\varepsilon \rightharpoonup p \quad \text{weakly* in } M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times 3}_{sym}), \tag{3.3.3}$$

$$\|p^\varepsilon\|_{M_b} \to \|p\|_{M_b}. \tag{3.3.4}$$

*Proof.* **Step 1.** We first show that any triple $(u, e, p) \in \mathcal{A}_{KL}(w)$ can be approximated in the sense of (3.3.1)–(3.3.4) by a sequence of triples $(u^\varepsilon, e^\varepsilon, p^\varepsilon) \in \mathcal{A}_{KL}(w)$ with $u^\varepsilon \in C^\infty(\Omega; \mathbb{R}^3) \cap BD(\Omega)$.

Let $w \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$ and let $(u, e, p) \in \mathcal{A}_{KL}(w)$. By Proposition 3.3.5 the Kirchhoff-Love components $\bar{u} \in BD(\omega)$ and $u_3 \in BH(\omega)$ of $u$ satisfy

$$E\bar{u} = \bar{e} + \bar{p} \text{ in } \omega, \quad \bar{p} = (\bar{w} - \bar{u}) \odot \nu_{\partial\omega} \mathcal{H}^1 \text{ on } \gamma_d,$$

$$D^2 u_3 = -(\hat{e} + \hat{p}) \text{ in } \omega, \quad u_3 = w_3 \text{ on } \gamma_d, \quad \hat{p} = (\nabla u_3 - \nabla w_3) \odot \nu_{\partial\omega} \mathcal{H}^1 \text{ on } \gamma_d,$$

where $\bar{e}, \hat{e}$ have been identified with functions in $L^2(\omega; \mathbb{M}^{2\times 2}_{sym})$ and $\bar{p}, \hat{p}$ with measures in $M_b(\omega \cup \gamma_d; \mathbb{M}^{2\times 2}_{sym})$. Moreover,

$$p_\perp = -e_\perp \text{ in } \Omega, \quad p_\perp = 0 \text{ on } \Gamma_d.$$

Fix $\varepsilon > 0$. Let $r > 0$ be such that the set

$$\omega_0 := \big\{ x' \in \omega : \ \operatorname{dist}(x', \partial\omega) > \tfrac{1}{r} \big\}$$

satisfies

$$|\bar{p}|(\omega \setminus \omega_0) + |\hat{p}|(\omega \setminus \omega_0) \leq \varepsilon. \tag{3.3.5}$$

We set

$$\omega_j := \big\{x' \in \omega : \ \mathrm{dist}(x', \partial\omega) > \tfrac{1}{j+r}\big\} \quad \text{for every } j \in \mathbb{N},$$
$$A_j := \omega_{j+1} \setminus \overline{\omega}_{j-1} \quad \text{for } j \geq 2, \quad A_1 := \omega_2.$$

Let $\{\varphi_j\}$ be a $C^\infty$ partition of unity for $\omega$ subordinate to the covering $\{A_j\}$, that is, $\varphi_j \in C_c^\infty(A_j)$, $0 \leq \varphi_j \leq 1$ for every $j \in \mathbb{N}$, and

$$\sum_{j=1}^{\infty} \varphi_j = 1 \quad \text{in } \omega. \tag{3.3.6}$$

Let $(\rho_\delta)$ be a sequence of convolution kernels with $\rho_\delta \in C_0^\infty(B_\delta(0))$ for every $\delta > 0$. For every $j \in \mathbb{N}$ we choose $\delta_j$ such that

$$\{x' \in \omega : \ \mathrm{dist}(x', \mathrm{supp}\, \varphi_j) < \delta_j\} \subset\subset A_j, \tag{3.3.7}$$
$$\|(\varphi_j u_3) * \rho_{\delta_j} - \varphi_j u_3\|_{W^{1,2}} + \|(\varphi_j \bar{u}) * \rho_{\delta_j} - \varphi_j \bar{u}\|_{L^2} \leq \varepsilon 2^{-j}, \tag{3.3.8}$$
$$\|(\varphi_j \bar{e}) * \rho_{\delta_j} - \varphi_j \bar{e}\|_{L^2} + \|(\varphi_j \hat{e}) * \rho_{\delta_j} - \varphi_j \hat{e}\|_{L^2} \leq \varepsilon 2^{-j}, \tag{3.3.9}$$
$$\|(u_3 D^2\varphi_j) * \rho_{\delta_j} - u_3 D^2\varphi_j\|_{L^2} + \|(\nabla u_3 \odot \nabla\varphi_j) * \rho_{\delta_j} - \nabla u_3 \odot \nabla\varphi_j\|_{L^2} \leq \varepsilon 2^{-j}, \tag{3.3.10}$$
$$\|(\bar{u} \odot \nabla\varphi_j) * \rho_{\delta_j} - \bar{u} \odot \nabla\varphi_j\|_{L^2} \leq \varepsilon 2^{-j}. \tag{3.3.11}$$

Moreover, we extend the function $\varphi_j e_\perp$ to 0 outside $A_j \times (-\frac{1}{2}, \frac{1}{2})$ and consider the convolution

$$(\varphi_j e_\perp) * \rho_{\delta_j}(x) := \int_{\mathbb{R}^2} \rho_{\delta_j}(x' - y')\varphi_j(y') e_\perp(y', x_3)\, dy'$$

defined for every $x \in \Omega$. Since $\varphi_j p = \varphi_j \bar{p} \otimes \mathcal{L}^1 + \varphi_j \hat{p} \otimes x_3 \mathcal{L}^1 - \varphi_j e_\perp$, by Lemma 3.3.6 we can assume $\delta_j$ to be so small that

$$\|(\varphi_j e_\perp) * \rho_{\delta_j} - \varphi_j e_\perp\|_{L^2(\Omega)} \leq \varepsilon 2^{-j}, \tag{3.3.12}$$
$$\Big| \int_\Omega \big|(\varphi_j \bar{p}) * \rho_{\delta_j} + x_3(\varphi_j \hat{p}) * \rho_{\delta_j} - (\varphi_j e_\perp) * \rho_{\delta_j}\big|\, dx - |\varphi_j p|(\Omega)\Big| \leq \varepsilon 2^{-j}. \tag{3.3.13}$$

Finally, we define

$$\bar{u}^\varepsilon := \sum_{j=1}^{\infty} (\varphi_j \bar{u}) * \rho_{\delta_j}, \quad u_3^\varepsilon := \sum_{j=1}^{\infty} (\varphi_j u_3) * \rho_{\delta_j}, \quad u_\alpha^\varepsilon := \bar{u}_\alpha^\varepsilon - x_3 \partial_\alpha u_3^\varepsilon \quad (\alpha = 1, 2),$$
$$e^\varepsilon := \bar{e}^\varepsilon + x_3 \hat{e}^\varepsilon + e_\perp^\varepsilon,$$

where

$$\bar{e}^\varepsilon := \sum_{j=1}^{\infty} \big[(\varphi_j \bar{e}) * \rho_{\delta_j} + (\bar{u} \odot \nabla\varphi_j) * \rho_{\delta_j}\big],$$

$$\hat{e}^\varepsilon := \sum_{j=1}^{\infty} \big[(\varphi_j \hat{e}) * \rho_{\delta_j} - (u_3 D^2\varphi_j) * \rho_{\delta_j} - 2(\nabla u_3 \odot \nabla\varphi_j) * \rho_{\delta_j}\big], \quad e_\perp^\varepsilon := \sum_{j=1}^{\infty} (\varphi_j e_\perp) * \rho_{\delta_j},$$

and

$$p^\varepsilon := \begin{cases} \displaystyle\sum_{j=1}^{\infty} \big[(\varphi_j \bar{p}) * \rho_{\delta_j} + x_3(\varphi_j \hat{p}) * \rho_{\delta_j} - (\varphi_j e_\perp) * \rho_{\delta_j}\big] & \text{in } \Omega, \\ (w - u) \odot \nu_{\partial\Omega} \mathcal{H}^2 & \text{on } \Gamma_d. \end{cases}$$

It is easy to see that $\bar{u}^\varepsilon \in C^\infty(\omega;\mathbb{R}^2)\cap BD(\omega)$, $u_3^\varepsilon \in C^\infty(\omega)\cap W^{2,1}(\omega)$, hence $u^\varepsilon \in C^\infty(\Omega;\mathbb{R}^3)\cap BD(\Omega)$. Moreover,

$$E\bar{u}^\varepsilon = \bar{e}^\varepsilon + \bar{p}^\varepsilon \quad \text{and} \quad D^2 u_3^\varepsilon = -(\hat{e}^\varepsilon + \hat{p}^\varepsilon) \quad \text{in } \Omega \tag{3.3.14}$$

for every $\varepsilon$. Arguing as in [23, Proof of Proposition 1.4], one can also show that $u_3^\varepsilon = u_3$, $\nabla u_3^\varepsilon = \nabla u_3$, and $\bar{u}^\varepsilon = \bar{u}$ on $\partial\omega$. By Proposition 3.3.5 this implies that $(u^\varepsilon, e^\varepsilon, p^\varepsilon) \in \mathcal{A}_{KL}(w)$.

By (3.3.6) and (3.3.8) we deduce that

$$u^\varepsilon \to u \quad \text{strongly in } L^2(\Omega;\mathbb{R}^3), \tag{3.3.15}$$

while by (3.3.9)–(3.3.12) we obtain (3.3.2).

To prove (3.3.3) it is enough to show that

$$\bar{p}^\varepsilon \rightharpoonup \bar{p} \quad \text{and} \quad \hat{p}^\varepsilon \rightharpoonup \hat{p} \quad \text{weakly* in } (C_b(\omega;\mathbb{M}^{2\times 2}_{sym}))', \tag{3.3.16}$$

where $C_b(\omega;\mathbb{M}^{2\times 2}_{sym})$ is the space of all bounded functions in $C(\omega;\mathbb{M}^{2\times 2}_{sym})$. Indeed, if (3.3.16) holds, for every $\phi \in C_0(\Omega\cup\Gamma_d;\mathbb{M}^{2\times 2}_{sym})$ we have

$$\int_{\Omega\cup\Gamma_d} \phi : dp^\varepsilon = \int_\omega \bar{\phi} : d\bar{p}^\varepsilon + \tfrac{1}{12}\int_\omega \hat{\phi} : d\hat{p}^\varepsilon - \int_\Omega \phi_\perp : e_\perp^\varepsilon \, dx + \int_{\Gamma_d} \phi : ((w-u)\odot \nu_{\partial\Omega})\, d\mathcal{H}^2,$$

where $\bar{\phi}$, $\hat{\phi}$, $\phi_\perp$ are defined according to Definition 3.3.3. Convergence (3.3.3) follows now by (3.3.2) and (3.3.16).

We prove (3.3.16) for the sequence $(\bar{p}^\varepsilon)$, the same argument applies to $(\hat{p}^\varepsilon)$. By (3.3.2), (3.3.14), and (3.3.15) it is enough to check that

$$\limsup_{\varepsilon\to 0} \|\bar{p}^\varepsilon\|_{M_b(\omega)} \le \|\bar{p}\|_{M_b(\omega)}. \tag{3.3.17}$$

Now, let $\phi \in C_c^\infty(\omega;\mathbb{M}^{2\times 2}_{sym})$ with $\|\phi\|_\infty \le 1$. Denoting by $\check{\rho}_{\delta_j}$ the function $\check{\rho}_{\delta_j}(z') = \rho_{\delta_j}(-z')$ for every $z' \in \mathbb{R}^2$ and for every $j$, we have

$$\begin{aligned}
\Big|\int_\omega \phi : d\bar{p}^\varepsilon\Big| &\le \Big|\sum_{j=1}^\infty \int_\omega \phi(x') : \Big(\int_\omega \varphi_j(y')\rho_{\delta_j}(x'-y')\, d\bar{p}(y')\Big)\, dx'\Big| \\
&= \Big|\sum_{j=1}^\infty \int_\omega \varphi_j(\phi * \check{\rho}_{\delta_j}) : d\bar{p}\Big| \\
&\le \sum_{j=2}^\infty \int_\omega \varphi_j |\phi * \check{\rho}_{\delta_j}|\, d|\bar{p}| + \int_\omega \varphi_1 |\phi * \check{\rho}_{\delta_1}|\, d|\bar{p}| \\
&\le |\bar{p}|(\omega\setminus\omega_0) + |\bar{p}|(\omega).
\end{aligned}$$

Hence, (3.3.17) follows from (3.3.5). Therefore, we deduce (3.3.16), which in turn yields (3.3.3). Combining (3.3.2), (3.3.3), and (3.3.15), we also have (3.3.1).

It remains to prove (3.3.4). We first note that

$$\begin{aligned}
\|p^\varepsilon\|_{M_b} &= |p^\varepsilon|(\Omega) + |p|(\Gamma_d) \\
&\le \sum_{j=1}^\infty \int_\Omega \big|(\varphi_j\bar{p}) * \rho_{\delta_j} + x_3(\varphi_j\hat{p}) * \rho_{\delta_j} - (\varphi_j e_\perp) * \rho_{\delta_j}\big|\, dx + |p|(\Gamma_d) \\
&\le \sum_{j=1}^\infty |\varphi_j p|(\Omega) + |p|(\Gamma_d) + \varepsilon,
\end{aligned}$$

by (3.3.13). Therefore,

$$\begin{aligned}\limsup_{\varepsilon\to 0}\|p^\varepsilon\|_{M_b} &\leq \sum_{j=1}^{\infty}|\varphi_j p|(\Omega)+|p|(\Gamma_d)\\ &= \sum_{j=1}^{\infty}\int_\Omega \varphi_j(x')\,d|p|(x)+|p|(\Gamma_d)=\|p\|_{M_b}.\end{aligned}$$

Since by (3.3.3) and by lower semicontinuity we have

$$\|p\|_{M_b}\leq \liminf_{\varepsilon\to 0}\|p^\varepsilon\|_{M_b},$$

the proof of (3.3.4) and of Step 1 is complete.

**Step 2.** To conclude the proof of the lemma we shall prove that any triple $(u,e,p)\in\mathcal{A}_{KL}(w)$ with $u\in C^\infty(\Omega;\mathbb{R}^3)\cap BD(\Omega)$ can be approximated in the sense of (3.3.1)–(3.3.4) by a sequence of triples $(u^\varepsilon,e^\varepsilon,p^\varepsilon)\in\mathcal{A}_{KL}(w)$ with $u^\varepsilon\in C^\infty(\Omega;\mathbb{R}^3)\cap W^{1,1}(\Omega;\mathbb{R}^3)$.

Let $(u,e,p)\in\mathcal{A}_{KL}(w)$ with $u\in C^\infty(\Omega;\mathbb{R}^3)\cap BD(\Omega)$. The Kirchhoff-Love components of $u$ satisfy $\bar u\in C^\infty(\omega;\mathbb{R}^2)\cap BD(\omega)$ and $u_3\in C^\infty(\omega)\cap W^{2,1}(\omega)$. By [60, Chapter I, Proposition 1.3] and the regularity of $\partial\omega$ we can construct a sequence $(\bar u^\varepsilon)\subset C^\infty(\overline{\omega};\mathbb{R}^2)$ such that

$$\bar u^\varepsilon\to\bar u \quad\text{strongly in } L^1(\omega;\mathbb{R}^2)\quad\text{and}\quad E\bar u^\varepsilon\to E\bar u\quad\text{strongly in } L^1(\omega;\mathbb{M}^{2\times 2}_{sym}). \tag{3.3.18}$$

This implies, in particular, that $\bar u^\varepsilon\to\bar u$ strongly in $L^1(\Gamma_d;\mathbb{R}^2)$. The sequence of triples $(u^\varepsilon,e^\varepsilon,p^\varepsilon)$ defined by

$$u^\varepsilon_\alpha:=\bar u^\varepsilon_\alpha-x_3\partial_\alpha u_3\quad(\alpha=1,2),\qquad u^\varepsilon_3:=u_3,\qquad e^\varepsilon:=e,$$

and

$$p^\varepsilon:=\begin{cases}E\bar u^\varepsilon-e-x_3D^2u_3 & \text{in }\Omega,\\ (w-u^\varepsilon)\odot\nu_{\partial\Omega}\mathcal{H}^2 & \text{on }\Gamma_d,\end{cases}$$

satisfies all the required properties. □

**Remark 3.3.8.** We observe that by (3.3.15) and (3.3.18) and the continuous embedding of $BD(\omega)$ into $L^2(\omega;\mathbb{R}^2)$ the approximating sequence $(u^\varepsilon,e^\varepsilon,p^\varepsilon)$ in Lemma 3.3.7 satisfies also

$$\bar u^\varepsilon\to\bar u\quad\text{strongly in } L^2(\omega;\mathbb{R}^2). \tag{3.3.19}$$

Moreover, the construction of $(u^\varepsilon,e^\varepsilon,p^\varepsilon)$ can be modified in such a way to satisfy also the following convergence properties:

$$\|E\bar u^\varepsilon\|_{L^1}\to\|E\bar u\|_{M_b}, \tag{3.3.20}$$

$$\|D^2u_3^\varepsilon\|_{L^1}\to\|D^2u_3\|_{M_b}, \tag{3.3.21}$$

$$u_3^\varepsilon\to u_3\quad\text{in } C(\overline{\omega}). \tag{3.3.22}$$

Indeed, let us denote by $\bar p^a$, $\hat p^a$ and $\bar p^s$, $\hat p^s$ the absolutely continuous parts and the singular parts of $\bar p$ and $\hat p$, respectively. In Step 1 we can choose $\delta_j$ in such a way to satisfy also the following estimates:

$$\|(\varphi_j \bar{p}^a) * \rho_{\delta_j} - \varphi_j \bar{p}^a\|_{L^1} + \|(\varphi_j \hat{p}^a) * \rho_{\delta_j} - \varphi_j \hat{p}^a\|_{L^1} \le \varepsilon 2^{-j} \tag{3.3.23}$$

$$\big|\|(\varphi_j \bar{p}^s) * \rho_{\delta_j}\|_{L^1} - \|\varphi_j \bar{p}^s\|_{M_b}\big| + \big|\|(\varphi_j \hat{p}^s) * \rho_{\delta_j}\|_{L^1} - \|\varphi_j \hat{p}^s\|_{M_b}\big| \le \varepsilon 2^{-j}, \tag{3.3.24}$$

$$\|(\varphi_j u_3) * \rho_{\delta_j} - \varphi_j u_3\|_{L^\infty} \le \varepsilon 2^{-j}, \tag{3.3.25}$$

where we used the continuous embedding of $BH(\omega)$ into $C(\overline{\omega})$. By (3.3.25) we immediately deduce (3.3.22). By (3.3.23) we have that

$$\sum_{j=1}^{\infty} (\varphi_j \bar{p}^a) * \rho_{\delta_j} \to \bar{p}^a \quad \text{strongly in } L^1(\omega; \mathbb{M}^{2\times 2}_{sym}),$$

while by (3.3.24) we obtain that

$$\Big\|\sum_{j=1}^{\infty} (\varphi_j \bar{p}^s) * \rho_{\delta_j}\Big\|_{L^1} \le \sum_{j=1}^{\infty} \|\varphi_j \bar{p}^s\|_{M_b} + \varepsilon = \sum_{j=1}^{\infty} \int_\omega \varphi_j \, d|\bar{p}^s| + \varepsilon = |\bar{p}^s|(\omega) + \varepsilon.$$

These two facts, together with (3.3.2), yield

$$\limsup_{\varepsilon \to 0} \|E\bar{u}^\varepsilon\|_{L^1} \le \|\bar{e} + \bar{p}^a\|_{L^1} + |\bar{p}^s|(\omega) = \|E\bar{u}\|_{M_b}.$$

The opposite inequality follows from (3.3.1) by lower semicontinuity. A similar argument applies to (3.3.21). Finally, it is easy to see that (3.3.20)–(3.3.22) are preserved in the construction of Step 2, since the approximation result for $\bar{u}$ entails strong convergence of $(E\bar{u}^\varepsilon)$ in $L^1(\omega; \mathbb{M}^{2\times 2}_{sym})$.

We now prove an approximation result for Kirchhoff-Love admissible triples in terms of smooth triples. We denote by $C^\infty_c(\omega \cup \gamma_n; \mathbb{M}^{2\times 2}_{sym})$ the set of smooth maps whose support is a compact subset of $\omega \cup \gamma_n$. Morever, we introduce the set $L^2_{\infty,c}(\Omega; \mathbb{M}^{2\times 2}_{sym})$ of all $p \in L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$ satisfying the following two conditions:

(i) $\partial^k_\alpha \partial^j_\beta p \in L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$ for every $k, j \in \mathbb{N} \cup \{0\}$, $\alpha, \beta = 1, 2$;

(ii) there exists $U \subset\subset \omega \cup \gamma_n$ such that $p = 0$ a.e. on $\omega \setminus \overline{U} \times (-\frac{1}{2}, \frac{1}{2})$.

Note that if $p \in L^2_{\infty,c}(\Omega; \mathbb{M}^{2\times 2}_{sym})$, then $p(\cdot, x_3) \in C^\infty_c(\omega \cup \gamma_n; \mathbb{M}^{2\times 2}_{sym})$ for a.e. $x_3 \in (-\frac{1}{2}, \frac{1}{2})$.

**Theorem 3.3.9.** *Let $w \in W^{1,2}(\Omega, \mathbb{R}^3) \cap KL(\Omega)$ and let $(u, e, p) \in \mathcal{A}_{KL}(w)$. Then, there exists a sequence of triples*

$$(u^\varepsilon, e^\varepsilon, p^\varepsilon) \in \big(W^{1,2}(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}) \times L^2_{\infty,c}(\Omega; \mathbb{M}^{3\times 3}_{sym})\big) \cap \mathcal{A}_{KL}(w)$$

*such that*

$$u^\varepsilon \rightharpoonup u \quad \text{weakly* in } BD(\Omega), \tag{3.3.26}$$

$$e^\varepsilon \to e \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}), \tag{3.3.27}$$

$$p^\varepsilon \rightharpoonup p \quad \text{weakly* in } M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times 3}_{sym}), \tag{3.3.28}$$

$$\|p^\varepsilon\|_{L^1} \to \|p\|_{M_b}. \tag{3.3.29}$$

**Remark 3.3.10.** By Reshetnyak continuity Theorem (see, e.g., [4, Theorem 2.39]), convergences (3.3.28)–(3.3.29) guarantee that $\mathcal{H}_0(p^\varepsilon) \to \mathcal{H}_0(p)$ for every function $H_0 : \mathbb{M}^{2\times 2}_{sym} \to [0,+\infty)$ convex and positively one-homogeneous.

*Proof of Theorem 3.3.9.* Up to translating $u$ by $w$, it is enough to prove the theorem for $w \equiv 0$. Moreover, by Lemma 3.3.7 and by the metrizability of the weak* topology on bounded subsets of $M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times 3}_{sym})$ we can reduce to the case where $u \in W^{1,1}(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ and there exists $q \in L^1(\Omega;\mathbb{M}^{3\times 3}_{sym})$ such that

$$p = q \quad \text{in } \Omega, \quad p = -u \odot \nu_{\partial\Omega}\mathcal{H}^2 \quad \text{on } \Gamma_d. \tag{3.3.30}$$

According to Remark 3.3.2, we identify $e$ and $p$ with a function in $L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$ and a measure in $M_b(\Omega \cup \Gamma_d;\mathbb{M}^{2\times 2}_{sym})$, respectively, and we perform the decomposition of Proposition 3.3.5. By Remark 3.3.1 we have that $\bar{u} \in W^{1,1}(\omega;\mathbb{R}^2)$ and $u_3 \in W^{2,1}(\omega)$, while by (3.3.30) there exist $\bar{q}$, $\hat{q} \in L^1(\omega;\mathbb{M}^{2\times 2}_{sym})$ such that

$$\bar{p} = \bar{q} \quad \text{in } \omega, \quad \bar{p} = -\bar{u} \odot \nu_{\partial\omega}\mathcal{H}^1 \quad \text{on } \gamma_d, \tag{3.3.31}$$

and

$$\hat{p} = \hat{q} \quad \text{in } \omega, \quad \hat{p} = -\nabla u_3 \odot \nu_{\partial\omega}\mathcal{H}^1 \quad \text{on } \gamma_d. \tag{3.3.32}$$

Note also that $u_3 = 0$ on $\gamma_d$.

For the sake of simplicity we split the proof into two steps.

**Step 1.** We claim that we can always reduce to the case where there exists an open set $J \subset \partial\omega$ such that $\gamma_d$ is compactly contained in $J$ and $u_3 = 0$ on $J$ (topological notions refer here to the relative topology of $\partial\omega$).

To prove the claim, it is enough to show that the triple $(u,e,p)$ can be approximated in the sense of (3.3.26)–(3.3.29) by a sequence of triples $(u^\delta, e^\delta, p^\delta)$ in $\mathcal{A}_{KL}(w)$ satisfying the following property: for every $\delta > 0$ there exists an open set $J^\delta \subset \partial\omega$ such that $\gamma_d$ is compactly contained in $J^\delta$ and $u_3^\delta = 0$ on $J^\delta$.

To this purpose, let $\{U_i\}_{i=1,\dots,n}$ be a finite covering of $\partial\omega$ such that for every $i$, up to a $C^2$ change of coordinates, $\partial\omega \cap U_i$ is the graph of a $C^2$ map and $\omega \cap U_i$ is the related subgraph. We also require the covering to be such that for $\alpha = 1,2$ there exists an open neighbourhood $U_{P_\alpha}$ of the point $P_\alpha$ satisfying

$$P_\alpha \in \overline{U}_{P_\alpha} \subset U_\alpha \text{ for } \alpha = 1,2 \quad \text{and} \quad U_{P_\alpha} \cap U_\beta = \varnothing \text{ for } \alpha \neq \beta.$$

We recall that by assumption $\partial\lfloor_{\partial\omega}\gamma_d = \{P_1, P_2\}$. Finally, let $U_0 \subset \mathbb{R}^2$ be an open set, compactly contained in $\omega$, such that $\{U_i\}_{i=0,\dots,n}$ is a finite covering of $\overline{\omega}$, and let $\{\varphi_i\}_{i=0,\dots,n}$ be a subordinate partition of unity, $\varphi_i \in C_c^\infty(U_i)$, $0 \le \varphi_i \le 1$ for $i = 0,\dots,n$, and

$$\sum_{i=0}^{n} \varphi_i = 1 \quad \text{in } \overline{\omega}. \tag{3.3.33}$$

The approximating sequence will be constructed by modifying $u$ in the sets $U_1$ and $U_2$ and keeping it unchanged in the other sets. More precisely, using the $C^2$ regularity, we shall straighten the boundary of $\omega$ in $U_1$ and $U_2$, and shift the function $u$ along the tangential direction in such a way to have the boundary condition satisfied on a set larger than $\gamma_d$.

We first consider the set $U_1$. By our choice of the covering there exist a map $\phi \in C^2(U_1;\mathbb{R}^2)$ and a rectangle $R_1 := (a,b)\times(c,d)$ such that $\phi(U_1) = R_1$ and $\phi^{-1} \in C^2(R_1;U_1)$; moreover, there exists $h \in C^2(a,b)$ such that

$$\phi(U_1 \cap \partial\omega) = \{(s,h(s)) :\ s \in (a,b)\}, \quad \phi(U_1 \cap \omega) := \{(s,t) \in R_1 :\ t < h(s)\}.$$

We can assume that for a suitable $s_1 \in (a,b)$

$$\phi(U_1 \cap \gamma_d) = \{(s,h(s)) :\ s \in (s_1,b)\}.$$

Let $V_1$ be an open set in $\mathbb{R}^2$ such that $\operatorname{supp}\varphi_1 \subset V_1 \subset\subset U_1$. For $\delta$ small enough we define $\psi^\delta : \phi(V_1) \to R_1$ as

$$\psi^\delta(s,t) = (s+\delta, t - h(s) + h(s+\delta))$$

and $\phi^\delta : V_1 \to U_1$ as

$$\phi^\delta := \phi^{-1} \circ \psi_\delta \circ \phi.$$

It is easy to see that for $\delta$ small enough

$$\phi^\delta(V_1 \cap \omega) \subset U_1 \cap \omega, \quad \phi^\delta(V_1 \setminus \overline{\omega}) \subset U_1 \setminus \overline{\omega},$$

and

$$\phi^\delta(V_1 \cap \partial\omega) \subset U_1 \cap \partial\omega.$$

Moreover, setting $K_1 := \operatorname{supp}\varphi_1$, we have that

$$\|\phi^\delta - \mathrm{id}\|_{C^2(K_1)} \to 0, \quad \|(\phi^\delta)^{-1} - \mathrm{id}\|_{C^2(K_1)} \to 0, \tag{3.3.34}$$

as $\delta \to 0$.

We consider the functions $\bar{u}^{\delta,1} := \varphi_1(\bar{u} \circ \phi^\delta)$ and $u_3^{\delta,1} := \varphi_1(u_3 \circ \phi^\delta)$, which are well defined on $V_1 \cap \omega$ and are extended to zero outside the support of $\varphi_1$. By construction $\bar{u}^{\delta,1} \in W^{1,1}(\omega;\mathbb{R}^2)$, $u_3^{\delta,1} \in W^{2,1}(\omega)$, and

$$u_3^{\delta,1} = 0 \quad \text{on } J^{\delta,1}, \tag{3.3.35}$$

where $J^{\delta,1} := (U_1 \cap \gamma_d) \cup (\phi^\delta)^{-1}(U_1 \cap \gamma_d)$. Moreover, by (3.3.34) we obtain

$$\bar{u}^{\delta,1} \to \varphi_1\bar{u} \quad \text{strongly in } W^{1,1}(\omega;\mathbb{R}^2), \tag{3.3.36}$$

$$u_3^{\delta,1} \to \varphi_1 u_3 \quad \text{strongly in } W^{2,1}(\omega). \tag{3.3.37}$$

Straightforward computations yield the equalities

$$E\bar{u}^{\delta,1} = (\bar{u} \circ \phi^\delta) \odot \nabla\varphi_1 + \varphi_1 \operatorname{sym}\big((D\bar{u} \circ \phi^\delta)D\phi^\delta\big), \tag{3.3.38}$$

$$\begin{aligned} D^2 u_3^{\delta,1} &= (u_3 \circ \phi^\delta)D^2\varphi_1 + 2\nabla\varphi_1 \odot \big((D\phi^\delta)^T(\nabla u_3 \circ \phi^\delta)\big) \\ &\quad + \varphi_1 \sum_{\alpha=1,2} (\partial_\alpha u_3 \circ \phi^\delta) D^2\phi^\delta_\alpha + \varphi_1 (D\phi^\delta)^T (D^2 u_3 \circ \phi^\delta) D\phi^\delta. \end{aligned} \tag{3.3.39}$$

It is therefore natural to introduce the functions $\bar{e}^{\delta,1}, \hat{e}^{\delta,1} \in L^2(\omega;\mathbb{M}^{2\times2}_{sym})$, defined as

$$\begin{aligned} \bar{e}^{\delta,1} &:= (\bar{u} \circ \phi^\delta) \odot \nabla\varphi_1 + \varphi_1 \operatorname{sym}\big((\bar{e} \circ \phi^\delta)D\phi^\delta\big), \\ \hat{e}^{\delta,1} &:= -(u_3 \circ \phi^\delta)D^2\varphi_1 - 2\nabla\varphi_1 \odot ((D\phi^\delta)^T(\nabla u_3 \circ \phi^\delta)) \\ &\quad - \varphi_1 \sum_{\alpha=1,2} (\partial_\alpha u_3 \circ \phi^\delta) D^2\phi^\delta_\alpha + \varphi_1 (D\phi^\delta)^T (\hat{e} \circ \phi^\delta) D\phi^\delta \end{aligned}$$

and the functions $\bar{q}^{\delta,1}, \hat{q}^{\delta,1} \in L^1(\omega;\mathbb{M}^{2\times 2}_{sym})$, defined as

$$\begin{aligned}\bar{q}^{\delta,1} &:= \varphi_1 \operatorname{sym}\big((\bar{q}\circ\phi^\delta)D\phi^\delta\big) + \varphi_1 \operatorname{sym}\big([(D\bar{u} - E\bar{u})\circ\phi^\delta]D\phi^\delta\big),\\ \hat{q}^{\delta,1} &:= \varphi_1 (D\phi^\delta)^T(\hat{q}\circ\phi^\delta)D\phi^\delta.\end{aligned}$$

By (3.3.38) and (3.3.39) there holds

$$E\bar{u}^{\delta,1} = \bar{e}^{\delta,1} + \bar{q}^{\delta,1} \quad \text{in } \omega, \quad D^2 u_3^{\delta,1} = -(\hat{e}^{\delta,1} + \hat{q}^{\delta,1}) \quad \text{in } \omega. \tag{3.3.40}$$

By (3.3.34), (3.3.36), and (3.3.37) we deduce the following convergence properties:

$$\bar{e}^{\delta,1} \to \bar{u}\odot\nabla\varphi_1 + \varphi_1\bar{e} \quad \text{strongly in } L^2(\omega;\mathbb{M}^{2\times 2}_{sym}), \tag{3.3.41}$$

$$\hat{e}^{\delta,1} \to -u_3 D^2\varphi_1 - 2\nabla\varphi_1\odot\nabla u_3 + \varphi_1\hat{e} \quad \text{strongly in } L^2(\omega;\mathbb{M}^{2\times 2}_{sym}), \tag{3.3.42}$$

$$\bar{q}^{\delta,1} \to \varphi_1\bar{q} \quad \text{strongly in } L^1(\omega;\mathbb{M}^{2\times 2}_{sym}), \tag{3.3.43}$$

$$\hat{q}^{\delta,1} \to \varphi_1\hat{q} \quad \text{strongly in } L^1(\omega;\mathbb{M}^{2\times 2}_{sym}). \tag{3.3.44}$$

An analogous construction in the set $U_2$ provides us with two triples

$$\begin{gathered}(\bar{u}^{\delta,2}, \bar{e}^{\delta,2}, \bar{q}^{\delta,2}) \in W^{1,1}(\omega;\mathbb{R}^2)\times L^2(\omega;\mathbb{M}^{2\times 2}_{sym})\times L^1(\omega;\mathbb{M}^{2\times 2}_{sym}),\\ (u_3^{\delta,2}, \hat{e}^{\delta,2}, \hat{q}^{\delta,2}) \in W^{2,1}(\omega)\times L^2(\omega;\mathbb{M}^{2\times 2}_{sym})\times L^1(\omega;\mathbb{M}^{2\times 2}_{sym}),\end{gathered}$$

such that

$$E\bar{u}^{\delta,2} = \bar{e}^{\delta,2} + \bar{q}^{\delta,2} \quad \text{in } \omega, \quad D^2 u_3^{\delta,2} = -(\hat{e}^{\delta,2} + \hat{q}^{\delta,2}) \quad \text{in } \omega, \tag{3.3.45}$$

and the following convergence properties hold:

$$\bar{u}^{\delta,2} \to \varphi_2\bar{u} \quad \text{strongly in } W^{1,1}(\omega;\mathbb{R}^2), \tag{3.3.46}$$

$$u_3^{\delta,2} \to \varphi_2 u_3 \quad \text{strongly in } W^{2,1}(\omega), \tag{3.3.47}$$

$$\bar{e}^{\delta,2} \to \bar{u}\odot\nabla\varphi_2 + \varphi_2\bar{e} \quad \text{strongly in } L^2(\omega;\mathbb{M}^{2\times 2}_{sym}), \tag{3.3.48}$$

$$\hat{e}^{\delta,2} \to -u_3 D^2\varphi_2 - 2\nabla\varphi_2\odot\nabla u_3 + \varphi_2\hat{e} \quad \text{strongly in } L^2(\omega;\mathbb{M}^{2\times 2}_{sym}), \tag{3.3.49}$$

$$\bar{q}^{\delta,2} \to \varphi_2\bar{q} \quad \text{strongly in } L^1(\omega;\mathbb{M}^{2\times 2}_{sym}), \tag{3.3.50}$$

$$\hat{q}^{\delta,2} \to \varphi_2\hat{q} \quad \text{strongly in } L^1(\omega;\mathbb{M}^{2\times 2}_{sym}). \tag{3.3.51}$$

Moreover, the following boundary condition is satisfied:

$$u_3^{\delta,2} = 0 \quad \text{on } J^{\delta,2}, \tag{3.3.52}$$

where $J^{\delta,2}$ is an open subset of $\partial\omega$ strictly containing $U_2\cap\gamma_d$.

To complete the construction of the approximating sequence we set

$$\bar{u}^\delta := \bar{u}^{\delta,1} + \bar{u}^{\delta,2} + \sum_{i\neq 1,2}\varphi_i\bar{u}, \qquad u_3^\delta := u_3^{\delta,1} + u_3^{\delta,2} + \sum_{i\neq 1,2}\varphi_i u_3,$$

and

$$u_\alpha^\delta := \bar{u}_\alpha^\delta - x_3\partial_\alpha u_3^\delta \quad (\alpha = 1,2).$$

It is immediate to see that $u^\delta \in W^{1,1}(\Omega;\mathbb{R}^3)\cap KL(\Omega)$; moreover, by (3.3.35) and (3.3.52) we have

$$u_3^\delta = 0 \quad \text{on } J^\delta,$$

where $J^\delta := J^{\delta,1} \cup J^{\delta,2} \cup \gamma_d$ is an open subset of $\partial\omega$ and satisfies $\gamma_d \subset\subset J^\delta$. By (3.3.33), (3.3.36), (3.3.37), (3.3.46), and (3.3.47) we also have

$$u^\delta \to u \quad \text{strongly in } W^{1,1}(\Omega;\mathbb{R}^3). \tag{3.3.53}$$

By the continuity of the trace operator the previous convergence entails

$$u^\delta \to u \quad \text{strongly in } L^1(\partial\Omega;\mathbb{R}^3). \tag{3.3.54}$$

Finally, we introduce the functions $e^\delta \in L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$ and $q^\delta \in L^1(\Omega;\mathbb{M}^{2\times 2}_{sym})$, defined as

$$\begin{aligned} e^\delta &:= \bar{e}^{\delta,1} + \bar{e}^{\delta,2} + x_3(\hat{e}^{\delta,1} + \hat{e}^{\delta,2}) + (\varphi_1 + \varphi_2)e_\perp \\ &\quad + \sum_{i\neq 1,2} (\varphi_i e + \bar{u} \odot \nabla\varphi_i - x_3 u_3 D^2\varphi_i - 2x_3 \nabla\varphi_i \odot \nabla u_3), \\ q^\delta &:= \bar{q}^{\delta,1} + \bar{q}^{\delta,2} + x_3(\hat{q}^{\delta,1} + \hat{q}^{\delta,2}) - (\varphi_1 + \varphi_2)e_\perp + \sum_{i\neq 1,2} \varphi_i q, \end{aligned}$$

and the measure $p^\delta \in M_b(\Omega \cup \Gamma_d;\mathbb{M}^{2\times 2}_{sym})$, defined as

$$p^\delta := q^\delta \quad \text{in } \Omega, \qquad p^\delta := -u^\delta \odot \nu_{\partial\Omega}\mathcal{H}^2 \quad \text{on } \Gamma_d.$$

Clearly, $(u^\delta, e^\delta, p^\delta) \in \mathcal{A}_{KL}(w)$. Moreover, by (3.3.41)–(3.3.44) and (3.3.48)–(3.3.51) we obtain

$$e^\delta \to e \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}), \tag{3.3.55}$$
$$q^\delta \to q \quad \text{strongly in } L^1(\Omega;\mathbb{M}^{2\times 2}_{sym}). \tag{3.3.56}$$

From (3.3.54) and (3.3.56) it follows immediately that

$$p^\delta \rightharpoonup p \quad \text{weakly* in } M_b(\Omega \cup \Gamma_d;\mathbb{M}^{2\times 2}_{sym})$$

and

$$\|p^\delta\|_{M_b} \to \|p\|_{M_b}.$$

**Step 2.** By Step 1 we can assume that there exists an open set $J \subset \partial\omega$ such that $\gamma_d$ is compactly contained in $J$ and $u_3 = 0$ on $J$.

Let us consider a finite covering $\{Q_i\}_{i=1,\dots,m}$ of $\partial\omega$ made of open squares centered at points on $\partial\omega$, with a face orthogonal to some vector $n_i \in \mathbb{S}^1$ and such that, for every $i = 1,\dots,m$, the set $Q_i \cap \omega$ is a $C^2$ subgraph in the direction $n_i$. We also require that for some $m_0 \in \{1,\dots,m\}$

$$\gamma_d \subset\subset \bigcup_{i=1}^{m_0} Q_i \cap \partial\omega \subset\subset J$$

and

$$\text{dist}(Q_i, \gamma_d) > 0 \quad \text{for every } i = m_0 + 1,\dots,m.$$

Let also $Q_0$ be an open set compactly contained in $\omega$ such that the collection of open sets $\{Q_i\}_{i=0,\dots,m}$ is a finite covering of $\overline{\omega}$. We consider a subordinate partition of unity

$\{\varphi_i\}_{i=0,\dots,m}$, with $0 \le \varphi_i \le 1$, $\varphi_i \in C_c^\infty(Q_i)$ for every $i = 1, \dots, m$, and $\sum_{i=0}^m \varphi_i = 1$ on $\overline{\omega}$.

Denoting by $\tilde{\Omega}$ the set

$$\tilde{\Omega} := \Omega \cup \bigcup_{i=1}^{m_0} \left(Q_i \times (-\tfrac{1}{2}, \tfrac{1}{2})\right),$$

we extend the triple $(u, e, p)$ to $\tilde{\Omega}$ by setting

$$u := 0 \quad \text{in } \tilde{\Omega} \setminus \Omega, \qquad e := 0 \quad \text{in } \tilde{\Omega} \setminus \Omega, \qquad p := \begin{cases} -u \odot \nu_{\partial\Omega} \mathcal{H}^2 & \text{on } \tilde{\Omega} \cap \partial\Omega, \\ 0 & \text{in } \tilde{\Omega} \setminus \overline{\Omega}. \end{cases}$$

The extended maps satisfy

$$u \in BD(\tilde{\Omega}) \cap KL(\tilde{\Omega}), \quad e \in L^2(\tilde{\Omega}; \mathbb{M}^{3\times 3}_{sym}), \quad p \in M_b(\tilde{\Omega}; \mathbb{M}^{3\times 3}_{sym})$$

and

$$Eu = e + p \quad \text{in } \tilde{\Omega}.$$

Note, in particular, that since $u_3 = 0$ and $\nu_{\partial\Omega} = (\nu_{\partial\omega}, 0)$ on $\tilde{\Omega} \cap \partial\Omega$, we have that $p_{i3} = 0$ in $\tilde{\Omega}$ for $i = 1, 2, 3$. Thus, we can as usual identify $e$ with a function in $L^2(\tilde{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ and $p$ with a measure in $M_b(\tilde{\Omega}; \mathbb{M}^{2\times 2}_{sym})$.

For every $i = 1, \dots, m_0$ we introduce the outward translations

$$\tau_{i,\varepsilon}(x') := x' + a_\varepsilon n_i \quad \text{for } x' \in \mathbb{R}^2,$$

while for $i = m_0 + 1, \dots, m$ we consider the inward translations

$$\tau_{i,\varepsilon}(x') := x' - a_\varepsilon n_i \quad \text{for } x' \in \mathbb{R}^2,$$

where $(a_\varepsilon)$ is a sequence converging to $0$, as $\varepsilon \to 0$. We define

$$\bar{u}^\varepsilon := \sum_{i=1}^m (\varphi_i \bar{u}) \circ \tau_{i,\varepsilon} + \varphi_0 \bar{u}, \tag{3.3.57}$$

$$\bar{e}^\varepsilon := \sum_{i=1}^m (\varphi_i \bar{e}) \circ \tau_{i,\varepsilon} + \varphi_0 \bar{e} + \sum_{i=1}^m (\nabla\varphi_i \odot \bar{u}) \circ \tau_{i,\varepsilon} + \nabla\varphi_0 \odot \bar{u}, \tag{3.3.58}$$

$$\bar{p}^\varepsilon := \sum_{i=1}^m \tau^{\#}_{i,\varepsilon}(\varphi_i \bar{p}) + \varphi_0 \bar{p}, \tag{3.3.59}$$

where $\tau^{\#}_{i,\varepsilon}(\varphi_i \bar{p})$ denotes the pull-back measure of $\varphi_i \bar{p}$. Notice that $(\bar{u}^\varepsilon, \bar{e}^\varepsilon, \bar{p}^\varepsilon)$ is well defined in an open neighbourhood $\omega_\varepsilon$ of $\omega$, that is, $\bar{u}^\varepsilon \in BD(\omega_\varepsilon)$, $\bar{e}^\varepsilon \in L^2(\omega_\varepsilon; \mathbb{M}^{2\times 2}_{sym})$, $\bar{p}^\varepsilon \in M_b(\omega_\varepsilon; \mathbb{M}^{2\times 2}_{sym})$, and

$$E\bar{u}^\varepsilon = \bar{e}^\varepsilon + \bar{p}^\varepsilon \quad \text{in } \omega_\varepsilon.$$

Moreover, by construction there exists an open set $U_\varepsilon \subset \mathbb{R}^2$ such that $\gamma_d \subset\subset U_\varepsilon$ and $u^\varepsilon = 0$, $e^\varepsilon = 0$, and $p^\varepsilon = 0$ in $U_\varepsilon$. Finally, we can choose $a_\varepsilon \to 0$ in such a way that

$$\tau^{\#}_{i,\varepsilon}(\varphi_i \bar{p})(\partial\omega \cap Q_i) = 0 \quad \text{for } i = m_0 + 1, \dots, m,$$

so that

$$|\bar{p}^\varepsilon|(\partial\omega) = 0 \quad \text{for every } \varepsilon. \tag{3.3.60}$$

Let now $(\rho_\delta) \subset C_c^\infty(\mathbb{R}^2)$ be a sequence of convolution kernels. For $\delta < a_\varepsilon$ we consider the functions

$$\bar{u}^{\varepsilon,\delta} := \bar{u}^\varepsilon * \rho_\delta, \quad \bar{e}^{\varepsilon,\delta} := \bar{e}^\varepsilon * \rho_\delta, \quad \bar{p}^{\varepsilon,\delta} := \bar{p}^\varepsilon * \rho_\delta.$$

Clearly, we have $\bar{u}^{\varepsilon,\delta} \in C^\infty(\overline{\omega};\mathbb{R}^2)$ and $\bar{e}^{\varepsilon,\delta}, \bar{p}^{\varepsilon,\delta} \in C^\infty(\overline{\omega};\mathbb{M}^{2\times 2}_{sym})$, and

$$E\bar{u}^{\varepsilon,\delta} = \bar{e}^{\varepsilon,\delta} + \bar{p}^{\varepsilon,\delta} \quad \text{in } \omega.$$

Moreover, for $\delta$ small enough there holds

$$\bar{u}^{\varepsilon,\delta} = 0 \quad \text{on } \gamma_d \quad \text{and} \quad \bar{e}^{\varepsilon,\delta}, \bar{p}^{\varepsilon,\delta} \in C_c^\infty(\omega \cup \gamma_n;\mathbb{M}^{2\times 2}_{sym}). \tag{3.3.61}$$

We apply a similar construction to the normal component of $u$ and to the first moments of $e$ and $p$. We first introduce

$$\begin{aligned} u_3^\varepsilon &:= \sum_{i=1}^m (\varphi_i u_3) \circ \tau_{i,\varepsilon} + \varphi_0 u_3, \\ \hat{e}^\varepsilon &:= \sum_{i=1}^m (\varphi_i \hat{e}) \circ \tau_{i,\varepsilon} + \varphi_0 \hat{e} - 2\sum_{i=1}^m (\nabla\varphi_i \odot \nabla u_3) \circ \tau_{i,\varepsilon} - 2\nabla\varphi_0 \odot \nabla u_3 \\ &\quad - \sum_{i=1}^m (D^2\varphi_i u_3) \circ \tau_{i,\varepsilon} - D^2\varphi_0 u_3, \\ \hat{p}^\varepsilon &:= \sum_{i=1}^m \tau_{i,\varepsilon}^{\#}(\varphi_i \hat{p}) + \varphi_0 \hat{p}, \end{aligned}$$

and we then define for $\delta < a_\varepsilon$

$$u_3^{\varepsilon,\delta} := u_3^\varepsilon * \rho_\delta, \quad \hat{e}^{\varepsilon,\delta} := \hat{e}^\varepsilon * \rho_\delta, \quad \hat{p}^{\varepsilon,\delta} := \hat{p}^\varepsilon * \rho_\delta.$$

As before, we can modify the choice of $a_\varepsilon \to 0$ in such a way that

$$|\hat{p}^\varepsilon|(\partial\omega) = 0. \tag{3.3.62}$$

Moreover, for $\delta$ small enough we have that $u_3^{\varepsilon,\delta} \in C^\infty(\overline{\omega})$, $\hat{e}^{\varepsilon,\delta}, \hat{p}^{\varepsilon,\delta} \in C_c^\infty(\omega \cup \gamma_n;\mathbb{M}^{2\times 2}_{sym})$, and $u_3^{\varepsilon,\delta} = 0$ on $\gamma_d$, $\nabla u_3^{\varepsilon,\delta} = 0$ on $\gamma_d$. Finally, there holds

$$D^2 u_3^{\varepsilon,\delta} = -(\hat{e}^{\varepsilon,\delta} + \hat{p}^{\varepsilon,\delta}) \quad \text{in } \omega.$$

Analogously, we define

$$e_\perp^\varepsilon := \sum_{i=1}^m (\varphi_i e_\perp) \circ \tau_{i,\varepsilon} + \varphi_0 e_\perp, \qquad e_\perp^{\varepsilon,\delta} := e_\perp^\varepsilon * \rho_\delta,$$

where, with an abuse of notation, the composition $(\varphi_i e_\perp) \circ \tau_{i,\varepsilon}$ stands for the function

$$(\varphi_i e_\perp) \circ \tau_{i,\varepsilon}(x) = \varphi_i(\tau_{i,\varepsilon}(x'))\, e_\perp(\tau_{i,\varepsilon}(x'), x_3) \quad \text{for a.e. } x \in \Omega,$$

and the convolution is intended with respect to the variable $x' \in \mathbb{R}^2$. It is immediate to see that $e_\perp^{\varepsilon,\delta} \in L^2_{\infty,c}(\Omega;\mathbb{M}^{2\times 2}_{sym})$. We now set

$$\begin{aligned} u_\alpha^{\varepsilon,\delta} &:= \bar{u}_\alpha^{\varepsilon,\delta} - x_3 \partial_\alpha u_3^{\varepsilon,\delta} \quad (\alpha = 1,2), \\ e^{\varepsilon,\delta} &:= \bar{e}^{\varepsilon,\delta} + x_3 \hat{e}^{\varepsilon,\delta} + e_\perp^{\varepsilon,\delta}, \\ p^{\varepsilon,\delta} &:= \bar{p}^{\varepsilon,\delta} + x_3 \hat{p}^{\varepsilon,\delta} - e_\perp^{\varepsilon,\delta}. \end{aligned}$$

By construction we have

$$(u^{\varepsilon,\delta}, e^{\varepsilon,\delta}, p^{\varepsilon,\delta}) \in \big(W^{1,2}(\Omega;\mathbb{R}^3) \times L^2_{\infty,c}(\Omega;\mathbb{M}^{2\times 2}_{sym}) \times L^2_{\infty,c}(\Omega;\mathbb{M}^{2\times 2}_{sym})\big) \cap \mathcal{A}_{KL}(w).$$

It is convenient to introduce also the measure $p^\varepsilon \in M_b(\Omega \cup \Gamma_d;\mathbb{M}^{2\times 2}_{sym})$, defined as

$$p^\varepsilon := \bar{p}^\varepsilon \otimes \mathcal{L}^1 + \hat{p}^\varepsilon \otimes x_3\mathcal{L}^1 - e^\varepsilon_\perp.$$

Lemma 3.3.6, together with equalities (3.3.60) and (3.3.62), guarantees that we can choose $\delta = \delta(\varepsilon)$ small enough, so that

$$\begin{gathered} \|\bar{u}^{\varepsilon,\delta(\varepsilon)} - \bar{u}^\varepsilon\|_{L^2} < \varepsilon, \quad \|u_3^{\varepsilon,\delta(\varepsilon)} - u_3^\varepsilon\|_{W^{1,2}} < \varepsilon, \\ \|\bar{e}^{\varepsilon,\delta(\varepsilon)} - \bar{e}^\varepsilon\|_{L^2} < \varepsilon, \quad \|\hat{e}^{\varepsilon,\delta(\varepsilon)} - \hat{e}^\varepsilon\|_{L^2} < \varepsilon, \quad \|e_\perp^{\varepsilon,\delta(\varepsilon)} - e^\varepsilon_\perp\|_{L^2(\Omega)} < \varepsilon, \\ \big|\|p^{\varepsilon,\delta(\varepsilon)}\|_{L^1(\Omega)} - |p^\varepsilon|(\Omega)\big| < \varepsilon. \end{gathered} \tag{3.3.63}$$

From the convergence properties above we deduce (3.3.26)–(3.3.28). To conclude the proof of the theorem it remains to prove (3.3.29). By (3.3.63) we have

$$\limsup_{\varepsilon\to 0} \|p^{\varepsilon,\delta(\varepsilon)}\|_{L^1(\Omega)} \le \limsup_{\varepsilon\to 0} |p^\varepsilon|(\Omega).$$

On the other hand, since $p$ has been extended to zero on the set $\cup_{i=1}^{m_0}(Q_i \setminus \overline{\omega}) \times (-\frac{1}{2},\frac{1}{2})$, while for $i = m_0+1,\dots,m$ the map $\tau_{i,\varepsilon}$ is an inward translations, we have

$$\begin{aligned} &\limsup_{\varepsilon\to 0} |p^\varepsilon|(\Omega) \\ &\le \ |\varphi_0 p|(\Omega) + \limsup_{\varepsilon\to 0} \sum_{i=1}^m \int_{-\frac{1}{2}}^{\frac{1}{2}} \big|\tau^{\#}_{i,\varepsilon}(\varphi_i\bar{p} + x_3\varphi_i\hat{p} + \varphi_i e_\perp(\cdot,x_3))\big|(\omega\cup\gamma_d)\,dx_3 \\ &\le \ |\varphi_0 p|(\Omega) + \sum_{i=1}^m \int_{-\frac{1}{2}}^{\frac{1}{2}} \big|\varphi_i(\bar{p} + x_3\hat{p} + e_\perp(\cdot,x_3))\big|(\omega\cup\gamma_d)\,dx_3 \\ &= \ \sum_{i=0}^m |\varphi_i p|(\Omega\cup\Gamma_d) = \sum_{i=0}^m \int_{\Omega\cup\Gamma_d} \varphi_i\,d|p| = \|p\|_{M_b}. \end{aligned}$$

This, together with (3.3.28), completes the proof of (3.3.29) and of the theorem. □

**Remark 3.3.11.** Arguing as in Remark 3.3.8, one can modify the construction of the sequence $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$ in Theorem 3.3.9 in such a way that the convergence properties (3.3.19)–(3.3.22) are also satisfied. In particular, (3.3.22) is preserved, since the approximation argument for $u_3$ involves only local translations and convolutions.

**Remark 3.3.12.** We point out that the approximation result provided by Lemma 3.3.7 is crucial in Step 1 of the proof of Theorem 3.3.9. Indeed, it is not in general true that, if $v \in BD(\omega)$ and $\Psi : U \to \omega$ is a smooth bijection with smooth inverse, the composition $v\circ\Psi$ belongs to $BD(U)$. Lemma 3.3.7 allows us to assume $\bar{u} \in W^{1,1}(\omega;\mathbb{R}^2)$ and this regularity guarantees that $\bar{u}\circ\phi^\delta \in W^{1,1}(V_1;\mathbb{R}^2)$, hence, in particular, $\bar{u}\circ\phi^\delta \in BD(V_1)$.

## 3.4 Γ-convergence of the static functionals

In this section we study the Γ-convergence of the rescaled energies $(\mathcal{J}_\varepsilon)$, as $\varepsilon \to 0$. We first introduce the limit functional.

Let $\mathbb{A} : \mathbb{M}^{2\times2}_{sym} \to \mathbb{M}^{3\times3}_{sym}$ be the operator given by

$$\mathbb{A}\xi := \begin{pmatrix} \xi_{11} & \xi_{12} & \lambda_1(\xi) \\ \xi_{12} & \xi_{22} & \lambda_2(\xi) \\ \lambda_1(\xi) & \lambda_2(\xi) & \lambda_3(\xi) \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{2\times2}_{sym}, \tag{3.4.1}$$

where for every $\xi \in \mathbb{M}^{2\times2}_{sym}$ the triple $(\lambda_1(\xi), \lambda_2(\xi), \lambda_3(\xi))$ is the unique solution to the minimum problem

$$\min_{\lambda_i \in \mathbb{R}} Q \begin{pmatrix} \xi_{11} & \xi_{12} & \lambda_1 \\ \xi_{12} & \xi_{22} & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}.$$

We observe that the triple $(\lambda_1(\xi), \lambda_2(\xi), \lambda_3(\xi))$ can be characterized as the unique solution of the linear system

$$\mathbb{C}\mathbb{A}\xi : \begin{pmatrix} 0 & 0 & \zeta_1 \\ 0 & 0 & \zeta_2 \\ \zeta_1 & \zeta_2 & \zeta_3 \end{pmatrix} = 0 \tag{3.4.2}$$

for every $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{R}$. This implies that $\mathbb{A}$ is a linear map.

Let $Q_r : \mathbb{M}^{2\times2}_{sym} \to [0, +\infty)$ be the quadratic form given by

$$Q_r(\xi) := Q(\mathbb{A}\xi) \quad \text{for every } \xi \in \mathbb{M}^{2\times2}_{sym}. \tag{3.4.3}$$

By (3.2.2) it satisfies the estimates

$$r_{\mathbb{C}}|\xi|^2 \le Q_r(\xi) \le R_{\mathbb{C}}|\xi|^2 \quad \text{for every } \xi \in \mathbb{M}^{2\times2}_{sym}. \tag{3.4.4}$$

We also consider the linear operator $\mathbb{C}_r : \mathbb{M}^{2\times2}_{sym} \to \mathbb{M}^{3\times3}_{sym}$ defined as

$$\mathbb{C}_r\xi := \mathbb{C}\mathbb{A}\xi \quad \text{for every } \xi \in \mathbb{M}^{2\times2}_{sym}. \tag{3.4.5}$$

By (3.4.2) we have

$$\mathbb{C}_r\xi : \zeta = \mathbb{C}\mathbb{A}\xi : \zeta = \mathbb{C}\mathbb{A}\xi : \mathbb{A}\zeta'' \quad \text{for every } \xi \in \mathbb{M}^{2\times2}_{sym}, \zeta \in \mathbb{M}^{3\times3}_{sym}, \tag{3.4.6}$$

where $\zeta'' \in \mathbb{M}^{2\times2}_{sym}$ satisfies $\zeta''_{\alpha\beta} = \zeta_{\alpha\beta}$ for $\alpha, \beta = 1, 2$. This implies that

$$Q_r(\xi) = \tfrac{1}{2}\mathbb{C}_r\xi : \begin{pmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{12} & \xi_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{2\times2}_{sym}.$$

We introduce the functional $\mathcal{Q}_r : L^2(\Omega; \mathbb{M}^{2\times2}_{sym}) \to [0, +\infty)$, defined as

$$\mathcal{Q}_r(f) := \int_\Omega Q_r(f(z))\,dz \quad \text{for every } f \in L^2(\Omega; \mathbb{M}^{2\times2}_{sym}).$$

It describes the limiting elastic energy of a configuration of the plate whose elastic strain is given by $f$.

We define $H_r : \mathbb{M}^{2\times 2}_{sym} \to [0, +\infty)$ as

$$H_r(\xi) := \min_{\lambda_1,\lambda_2\in\mathbb{R}} H \begin{pmatrix} \xi_{11} & \xi_{12} & \lambda_1 \\ \xi_{12} & \xi_{22} & \lambda_2 \\ \lambda_1 & \lambda_2 & -(\xi_{11}+\xi_{22}) \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym}. \tag{3.4.7}$$

It turns out that $H_r$ is convex, positively one-homogeneous, and satisfies

$$r_K|\xi| \le H_r(\xi) \le \sqrt{3}R_K|\xi| \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym}. \tag{3.4.8}$$

For every $\mu \in M_b(\Omega \cup \Gamma_d; \mathbb{M}^{2\times 2}_{sym})$ we define

$$\mathcal{H}_r(\mu) := \int_{\Omega\cup\Gamma_d} H_r\Big(\frac{d\mu}{d|\mu|}\Big) d|\mu|. \tag{3.4.9}$$

With the previous notation, we introduce the functional $\mathcal{J} : \mathcal{A}_{KL}(w) \to [0, +\infty]$, defined as

$$\mathcal{J}(u, e, p) := \mathcal{Q}_r(e) + \mathcal{H}_r(p) \tag{3.4.10}$$

for every $(u, e, p) \in \mathcal{A}_{KL}(w)$, where we identify $e$ with a function in $L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$ and $p$ with a measure in $M_b(\Omega; \mathbb{M}^{2\times 2}_{sym})$, according to Remark 3.3.2. We are now in a position to state the main result of the section.

**Theorem 3.4.1.** *Let $\mathcal{J}_\varepsilon$ and $\mathcal{J}$ be the functionals defined in* (3.2.11) *and* (3.4.10)*. Let $w \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$ and for every $\varepsilon > 0$ let $(u_\varepsilon, e_\varepsilon, p_\varepsilon) \in \mathcal{A}_\varepsilon(w)$ be a minimizer of $\mathcal{J}_\varepsilon$. Then there exist a subsequence (not relabelled) and a triple $(u, e, p) \in \mathcal{A}_{KL}(w)$ such that*

$$u^\varepsilon \rightharpoonup u \quad \text{weakly* in } BD(\Omega), \tag{3.4.11}$$

$$e^\varepsilon \to e \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}), \tag{3.4.12}$$

$$\Lambda_\varepsilon e^\varepsilon \to \mathbb{A}e \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}), \tag{3.4.13}$$

$$p^\varepsilon \rightharpoonup p \quad \text{weakly* in } M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times 3}_{sym}), \tag{3.4.14}$$

$$\mathcal{H}(\Lambda_\varepsilon p^\varepsilon) \to \mathcal{H}_r(p). \tag{3.4.15}$$

*Moreover, $(u, e, p)$ is a minimizer of $\mathcal{J}$ and*

$$\lim_{\varepsilon\to 0} \mathcal{J}_\varepsilon(u_\varepsilon, e_\varepsilon, p_\varepsilon) = \mathcal{J}(u, e, p). \tag{3.4.16}$$

**Remark 3.4.2.** The existence of a minimizer for $\mathcal{J}_\varepsilon$ is guaranteed by [15, Theorem 3.3].

**Remark 3.4.3.** More general boundary conditions can be considered in Theorem 3.4.1. For instance, the thesis continues to hold if for every $\varepsilon > 0$ $(u_\varepsilon, e_\varepsilon, p_\varepsilon)$ is a minimizer of $\mathcal{J}_\varepsilon$ in the class $\mathcal{A}_\varepsilon(w^\varepsilon)$ and $w^\varepsilon \in W^{1,2}(\Omega; \mathbb{R}^3)$ is such that $w^\varepsilon \to w$ strongly in $L^2(\Omega; \mathbb{R}^3)$ with $w \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$ and $\Lambda_\varepsilon E w^\varepsilon \to f$ strongly in $L^2(\Omega; \mathbb{M}^{3\times 3}_{sym})$ for some $f \in L^2(\Omega; \mathbb{M}^{3\times 3}_{sym})$.

The proof of Theorem 3.4.1 is in the spirit of Γ-convergence. We first prove a compactness result and a liminf inequality for sequences of triples with equibounded energies.

**Theorem 3.4.4.** *Let $w \in W^{1,2}(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ and let $(u_\varepsilon, e_\varepsilon, p_\varepsilon) \in \mathcal{A}_\varepsilon(w)$ be such that*

$$\mathcal{J}_\varepsilon(u_\varepsilon, e_\varepsilon, p_\varepsilon) \leq C \quad \text{for every } \varepsilon > 0, \tag{3.4.17}$$

*where $C$ is a constant independent of $\varepsilon$. Then, there exist $\tilde{e} \in L^2(\Omega;\mathbb{M}^{3\times3}_{sym})$ and $\tilde{p} \in M_b(\Omega \cup \Gamma_d;\mathbb{M}^{3\times3}_D)$ such that, up to subsequences,*

$$\Lambda_\varepsilon e_\varepsilon \rightharpoonup \tilde{e} \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}_{sym}), \tag{3.4.18}$$

$$\Lambda_\varepsilon p_\varepsilon \rightharpoonup \tilde{p} \quad \text{weakly* in } M_b(\Omega \cup \Gamma_d;\mathbb{M}^{3\times3}_D). \tag{3.4.19}$$

*Moreover, there exists $(u, e, p) \in \mathcal{A}_{KL}(w)$, with $e_{\alpha\beta} = \tilde{e}_{\alpha\beta}$ and $p_{\alpha\beta} = \tilde{p}_{\alpha\beta}$ for $\alpha, \beta = 1, 2$, such that, up to subsequences,*

$$u_\varepsilon \rightharpoonup u \quad \text{weakly* in } BD(\Omega), \tag{3.4.20}$$

$$e_\varepsilon \rightharpoonup e \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}_{sym}), \tag{3.4.21}$$

$$p_\varepsilon \rightharpoonup p \quad \text{weakly* in } M_b(\Omega \cup \Gamma_d;\mathbb{M}^{3\times3}_{sym}), \tag{3.4.22}$$

*and*

$$\mathcal{J}(u, e, p) \leq \liminf_{\varepsilon\to0} \mathcal{J}_\varepsilon(u_\varepsilon, e_\varepsilon, p_\varepsilon). \tag{3.4.23}$$

*Proof.* By the energy estimate (3.4.17) and by (3.2.2) we deduce the bounds

$$\|e_\varepsilon\|_{L^2} \leq \|\Lambda_\varepsilon e_\varepsilon\|_{L^2} \leq C \quad \text{for every } \varepsilon. \tag{3.4.24}$$

Hence, there exist $\tilde{e}, e \in L^2(\Omega;\mathbb{M}^{3\times3}_{sym})$ such that (3.4.18) and (3.4.21) hold up to subsequences, with $e_{\alpha\beta} = \tilde{e}_{\alpha\beta}$ for $\alpha, \beta = 1, 2$ and $e_{i3} = 0$ for $i = 1, 2, 3$. By the lower semicontinuity of $\mathcal{Q}$ with respect to weak convergence in $L^2(\Omega;\mathbb{M}^{3\times3}_{sym})$ and by the definition of $Q_r$ we also deduce

$$\mathcal{Q}_r(e) \leq \mathcal{Q}(\tilde{e}) \leq \liminf_{\varepsilon\to0} \mathcal{Q}(\Lambda_\varepsilon e_\varepsilon). \tag{3.4.25}$$

By (3.4.17) and (3.2.4) we obtain analogously

$$\|p_\varepsilon\|_{M_b} \leq \|\Lambda_\varepsilon p_\varepsilon\|_{M_b} \leq C. \tag{3.4.26}$$

Therefore, there exist $\tilde{p} \in M_b(\Omega \cup \Gamma_d;\mathbb{M}^{3\times3}_D)$ and $p \in M_b(\Omega \cup \Gamma_d;\mathbb{M}^{3\times3}_{sym})$ such that (3.4.19) and (3.4.22) hold up to subsequences, with $p_{\alpha\beta} = \tilde{p}_{\alpha\beta}$ for $\alpha, \beta = 1, 2$ and $p_{i3} = 0$ for $i = 1, 2, 3$. By the lower semicontinuity of $\mathcal{H}$ with respect to weak* convergence in $M_b(\Omega \cup \Gamma_d;\mathbb{M}^{3\times3}_D)$ and by the definition of $H_r$, we have

$$\mathcal{H}_r(p) \leq \mathcal{H}(\tilde{p}) \leq \liminf_{\varepsilon\to0} \mathcal{H}(\Lambda_\varepsilon p_\varepsilon), \tag{3.4.27}$$

which, together with (3.4.25), gives (3.4.23).

Since $(u_\varepsilon, e_\varepsilon, p_\varepsilon) \in \mathcal{A}_\varepsilon(w)$, for every $\varepsilon$ there holds

$$Eu_\varepsilon = e_\varepsilon + p_\varepsilon \quad \text{in } \Omega, \tag{3.4.28}$$

and

$$p_\varepsilon = (w - u_\varepsilon) \odot \nu_{\partial\Omega}\mathcal{H}^2 \quad \text{on } \Gamma_d. \tag{3.4.29}$$

By (3.4.24), (3.4.26), and (3.4.28), the sequence $(Eu_\varepsilon)$ is bounded in $M_b(\Omega;\mathbb{M}^{3\times3}_{sym})$. By (3.4.26) and (3.4.29), the traces of $(u_\varepsilon)$ are uniformly bounded in $L^1(\Gamma_d;\mathbb{R}^3)$. Hence, by (1.4.2) the sequence $(u_\varepsilon)$ is bounded in $BD(\Omega)$ and (3.4.20) holds up to subsequences. Moreover, it is immediate to see that $Eu = e + p$ in $\Omega$, hence $u \in KL(\Omega)$.

To conclude the proof, it remains to check that $p = (w-u)\odot\nu_{\partial\Omega}\mathcal{H}^2$ on $\Gamma_d$. To this purpose we argue as in [15, Lemma 2.1]. Since $\gamma_d$ is an open subset of $\partial\omega$, there exists an open set $A\subset\mathbb{R}^2$ such that $\gamma_d = A\cap\partial\omega$. We set $U := (\omega\cup A)\times(-\frac12,\frac12)$ and we extend the triples $(u_\varepsilon, e_\varepsilon, p_\varepsilon)$ to the set $U$ in the following way:

$$v_\varepsilon := \begin{cases} u_\varepsilon & \text{in } \Omega,\\ w & \text{in } U\setminus\Omega,\end{cases}\qquad f_\varepsilon := \begin{cases} e_\varepsilon & \text{in } \Omega,\\ Ew & \text{in } U\setminus\Omega,\end{cases}\qquad q_\varepsilon := \begin{cases} p_\varepsilon & \text{in } \Omega\cup\Gamma_d,\\ 0 & \text{otherwise.}\end{cases}$$

The symmetric part of the gradient of $v_\varepsilon$ satisfies

$$Ev_\varepsilon = \begin{cases} Eu_\varepsilon & \text{in } \Omega,\\ (w-u_\varepsilon)\odot\nu_{\partial\Omega}\mathcal{H}^2 & \text{on } \Gamma_d,\\ Ew & \text{in } U\setminus\overline{\Omega}.\end{cases}$$

Therefore, by (3.4.20), up to subsequences, $v_\varepsilon \rightharpoonup v$ weakly* in $BD(U)$, where

$$v := \begin{cases} u & \text{in } \Omega,\\ w & \text{in } U\setminus\Omega,\end{cases}\qquad\text{and}\qquad Ev = \begin{cases} Eu & \text{in } \Omega,\\ (w-u)\odot\nu_{\partial\Omega}\mathcal{H}^2 & \text{on } \Gamma_d,\\ Ew & \text{in } U\setminus\overline{\Omega}.\end{cases}\tag{3.4.30}$$

Analogously, up to subsequences, $f_\varepsilon \rightharpoonup f$ weakly in $L^2(U;\mathbb{M}^{3\times3}_{sym})$ and, since the restrictions to $\Omega\cup\Gamma_d$ of functions in $C_0(U;\mathbb{M}^{3\times3}_{sym})$ belong to $C_0(\Omega\cup\Gamma_d;\mathbb{M}^{3\times3}_{sym})$, there holds $q_\varepsilon \rightharpoonup q$ weakly* in $M_b(U;\mathbb{M}^{3\times3}_{sym})$, where

$$f := \begin{cases} e & \text{in } \Omega,\\ Ew & \text{in } U\setminus\Omega,\end{cases}\qquad\text{and}\qquad q := \begin{cases} p & \text{in } \Omega\cup\Gamma_d,\\ 0 & \text{otherwise.}\end{cases}$$

Since $Ev_\varepsilon = f_\varepsilon + q_\varepsilon$ in $U$ for every $\varepsilon$, we deduce that $Ev = f + q$ in $U$. The thesis follows now from (3.4.30). □

In the next theorem we show that the lower bound established in Theorem 3.4.4 is optimal by exhibiting a recovery sequence.

**Theorem 3.4.5.** *Let $w\in W^{1,2}(\Omega;\mathbb{R}^3)\cap KL(\Omega)$ and let $(u,e,p)\in\mathcal{A}_{KL}(w)$. Then, there exists a sequence of triples $(u_\varepsilon, e_\varepsilon, p_\varepsilon)\in\mathcal{A}_\varepsilon(w)$ such that*

$$u^\varepsilon \rightharpoonup u \quad \textit{weakly* in } BD(\Omega),\tag{3.4.31}$$

$$e^\varepsilon \to e \quad \textit{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}_{sym}),\tag{3.4.32}$$

$$p^\varepsilon \rightharpoonup p \quad \textit{weakly* in } M_b(\Omega\cup\Gamma_d;\mathbb{M}^{3\times3}_{sym}),\tag{3.4.33}$$

$$\Lambda_\varepsilon e^\varepsilon \to \mathbb{A}e \quad \textit{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}_{sym}),\tag{3.4.34}$$

$$\mathcal{H}(\Lambda_\varepsilon p^\varepsilon) \to \mathcal{H}_r(p),\tag{3.4.35}$$

*and*

$$\lim_{\varepsilon\to 0}\mathcal{J}_\varepsilon(u^\varepsilon,e^\varepsilon,p^\varepsilon)=\mathcal{J}(u,e,p). \tag{3.4.36}$$

*Proof.* By Theorem 3.3.9, Remark 3.3.10, and the metrizability of the weak* topology on bounded subsets of $M_b(\Omega\cup\Gamma_d;\mathbb{M}^{3\times3}_{sym})$ we can reduce to the case where

$$(u,e,p)\in\big(W^{1,2}(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathbb{M}^{3\times3}_{sym})\times L^2_{\infty,c}(\Omega;\mathbb{M}^{3\times3}_{sym})\big)\cap\mathcal{A}_{KL}(w).$$

In particular, $u=w$ on $\Gamma_d$ and $p=0$ $\mathcal{H}^2$-a.e. on $\Gamma_d$.

Let now $\phi_1,\phi_2,\phi_3\in L^2(\Omega)$ be such that

$$\mathbb{A}e=\begin{pmatrix}e_{11}&e_{12}&\phi_1\\e_{12}&e_{22}&\phi_2\\\phi_1&\phi_2&\phi_3\end{pmatrix}.$$

Since $p\in L^2(\Omega;\mathbb{M}^{3\times3}_{sym})$, by the measurable selection lemma (see, e.g., [26]) and by (3.2.4) and (3.4.8) there exist $\eta_1,\eta_2,\eta_3\in L^2(\Omega)$ such that

$$\mathcal{H}_r(p)=\mathcal{H}\begin{pmatrix}p_{11}&p_{12}&\eta_1\\p_{12}&p_{22}&\eta_2\\\eta_1&\eta_2&-(p_{11}+p_{22})\end{pmatrix}. \tag{3.4.37}$$

We argue as in [44, Proposition 4.1] and we approximate the maps $\phi_i$ and $\eta_i$ by means of elliptic regularizations. For every $\varepsilon$ we define $\phi_i^\varepsilon\in W^{1,2}_0(\Omega)$, $i=1,2,3$, as the solution of the elliptic boundary value problem

$$\begin{cases}-\varepsilon\Delta\phi_i^\varepsilon+\phi_i^\varepsilon=\phi_i & \text{in }\Omega,\\ \phi_i^\varepsilon=0 & \text{on }\partial\Omega,\end{cases}$$

and $\eta_\alpha^\varepsilon\in W^{1,2}_0(\Omega)$, $\alpha=1,2$, as the solution of

$$\begin{cases}-\varepsilon\Delta\eta_\alpha^\varepsilon+\eta_\alpha^\varepsilon=\eta_\alpha & \text{in }\Omega,\\ \eta_\alpha^\varepsilon=0 & \text{on }\partial\Omega.\end{cases}$$

The standard theory of elliptic equations gives

$$\phi_i^\varepsilon\to\phi_i\quad\text{strongly in }L^2(\Omega), \tag{3.4.38}$$

$$\eta_\alpha^\varepsilon\to\eta_\alpha\quad\text{strongly in }L^2(\Omega), \tag{3.4.39}$$

as $\varepsilon\to0$, and

$$\|\nabla\phi_i^\varepsilon\|_{L^2}\le C\varepsilon^{-\frac12},\quad \|\nabla\eta_\alpha^\varepsilon\|_{L^2}\le C\varepsilon^{-\frac12}. \tag{3.4.40}$$

We also introduce the function $f^\varepsilon\in L^2(\omega;\mathbb{M}^{3\times3}_{sym})$, defined componentwise as

$$f^\varepsilon_{\alpha\alpha}(x'):=2\varepsilon\int_0^{x_3}(\partial_\alpha\phi^\varepsilon_\alpha(x',s)+\partial_\alpha\eta^\varepsilon_\alpha(x',s))\,ds\quad(\alpha=1,2),\qquad f^\varepsilon_{33}(x'):=0,$$

$$f^\varepsilon_{12}(x'):=\varepsilon\int_0^{x_3}(\partial_2\phi^\varepsilon_1(x',s)+\partial_2\eta^\varepsilon_1(x',s)+\partial_1\phi^\varepsilon_2(x',s)+\partial_1\eta^\varepsilon_2(x',s))\,ds,$$

$$f^\varepsilon_{\alpha3}(x'):=\frac{\varepsilon^2}{2}\int_0^{x_3}(\partial_\alpha\phi^\varepsilon_3(x',s)-\partial_\alpha p_{11}(x',s)-\partial_\alpha p_{22}(x',s))\,ds\quad(\alpha=1,2)$$

for a.e. $x' \in \omega$.

We are now in a position to define the recovery sequence. Let

$$\begin{aligned} u^\varepsilon_\alpha &:= u_\alpha + 2\varepsilon \int_0^{x_3} (\phi^\varepsilon_\alpha(x',s) + \eta^\varepsilon_\alpha(x',s))\, ds \quad (\alpha = 1,2), \\ u^\varepsilon_3 &:= u_3 + \varepsilon^2 \int_0^{x_3} (\phi^\varepsilon_3(x',s) - p_{11}(x',s) - p_{22}(x',s))\, ds, \end{aligned}$$

and

$$e^\varepsilon := e + \begin{pmatrix} 0 & 0 & \varepsilon\phi^\varepsilon_1 \\ 0 & 0 & \varepsilon\phi^\varepsilon_2 \\ \varepsilon\phi^\varepsilon_1 & \varepsilon\phi^\varepsilon_2 & \varepsilon^2\phi^\varepsilon_3 \end{pmatrix} + f^\varepsilon, \qquad p^\varepsilon := p + \begin{pmatrix} 0 & 0 & \varepsilon\eta^\varepsilon_1 \\ 0 & 0 & \varepsilon\eta^\varepsilon_2 \\ \varepsilon\eta^\varepsilon_1 & \varepsilon\eta^\varepsilon_2 & -\varepsilon^2(p_{11}+p_{22}) \end{pmatrix}.$$

Since $u = w$ on $\Gamma_d$, $p \in L^2_{\infty,c}(\Omega;\mathbb{M}^{3\times3}_{sym})$, and $\phi^\varepsilon_i, \eta^\varepsilon_\alpha \in W^{1,2}_0(\Omega)$, we have that $u_\varepsilon = w$ on $\Gamma_d$. It is also easy to check that $(u_\varepsilon, e_\varepsilon, p_\varepsilon) \in \mathcal{A}_\varepsilon(w)$. From (3.4.38) and (3.4.39) it follows that $u^\varepsilon \to u$ strongly in $L^2(\Omega;\mathbb{R}^3)$. By (3.4.38) and (3.4.40) we deduce (3.4.32) and (3.4.34), while by (3.4.39) we obtain

$$p^\varepsilon \to p \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}_{sym}),$$

hence (3.4.33) and (3.4.31) follow. Finally, by (3.4.37) we have (3.4.35), which, together with (3.4.34), implies the convergence of the energies. □

We are now in a position to prove Theorem 3.4.1.

*Proof of Theorem 3.4.1.* Since $(w, Ew, 0) \in \mathcal{A}_\varepsilon(w)$ for every $\varepsilon > 0$, by minimality we have that

$$\mathcal{J}_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon) \le \mathcal{J}_\varepsilon(w, Ew, 0) \le R_{\mathbb{C}} \|Ew\|^2_{L^2},$$

where the last inequality follows from (3.2.2) and the fact that $w \in KL(\Omega)$. By Theorem 3.4.4 we deduce that there exists $(u,e,p) \in \mathcal{A}_{KL}(\Omega)$ such that, up to subsequences,

$$\begin{aligned} &u^\varepsilon \rightharpoonup u \quad \text{weakly* in } BD(\Omega), \\ &e^\varepsilon \rightharpoonup e \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}_{sym}), \\ &p^\varepsilon \rightharpoonup p \quad \text{weakly* in } M_b(\Omega\cup\Gamma_d;\mathbb{M}^{3\times3}_{sym}), \end{aligned}$$

and

$$\mathcal{J}(u,e,p) \le \liminf_{\varepsilon\to0} \mathcal{J}_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon). \tag{3.4.41}$$

Let now $(v,f,q) \in \mathcal{A}_{KL}(\Omega)$. By Theorem 3.4.5 there exists a sequence of triples $(v^\varepsilon, f^\varepsilon, q^\varepsilon) \in \mathcal{A}_\varepsilon(w)$ such that

$$\mathcal{J}(v,f,q) = \lim_{\varepsilon\to0} \mathcal{J}_\varepsilon(v^\varepsilon, f^\varepsilon, q^\varepsilon) \ge \limsup_{\varepsilon\to0} \mathcal{J}_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon), \tag{3.4.42}$$

where the last inequality follows from the minimality of $(u_\varepsilon, e_\varepsilon, p_\varepsilon)$. Combining (3.4.42) with (3.4.41), we deduce that $(u,e,p)$ is a minimizer of $\mathcal{J}$ and by choosing $(v,f,q) = (u,e,p)$ in (3.4.42) we obtain (3.4.16).

It remains to prove (3.4.12), (3.4.13), and (3.4.15). By the lower semicontinuity of $\mathcal{Q}$ and $\mathcal{H}$ with respect to weak convergence in $L^2(\Omega;\mathbb{M}^{3\times3}_{sym})$ and weak* convergence in $M_b(\Omega\cup\Gamma_d;\mathbb{M}^{3\times3}_{sym})$, respectively, and by the definition of $\mathcal{Q}_r$ and $\mathcal{H}_r$ we have

$$\mathcal{Q}_r(e)\leq\liminf_{\varepsilon\to0}\mathcal{Q}(\Lambda_\varepsilon e^\varepsilon),\qquad \mathcal{H}_r(p)\leq\liminf_{\varepsilon\to0}\mathcal{H}(\Lambda_\varepsilon p^\varepsilon). \tag{3.4.43}$$

Combining (3.4.16) and (3.4.43) yields

$$\lim_{\varepsilon\to0}\mathcal{Q}(\Lambda_\varepsilon e^\varepsilon)=\mathcal{Q}_r(e),\qquad \lim_{\varepsilon\to0}\mathcal{H}(\Lambda_\varepsilon p^\varepsilon)=\mathcal{H}_r(p),$$

so that (3.4.15) is proved. On the other hand, we remark that by (3.4.6)

$$\begin{aligned}\mathcal{Q}(\Lambda_\varepsilon e^\varepsilon-\mathbb{A}e) &= \mathcal{Q}(\Lambda_\varepsilon e^\varepsilon)+\mathcal{Q}_r(e)-\int_\Omega \mathbb{C}\mathbb{A}e:\Lambda_\varepsilon e^\varepsilon\,dx\\ &= \mathcal{Q}(\Lambda_\varepsilon e^\varepsilon)+\mathcal{Q}_r(e)-\int_\Omega \mathbb{C}\mathbb{A}e:e^\varepsilon\,dx \end{aligned}\tag{3.4.44}$$

Therefore, passing to the limit in (3.4.44) and applying again (3.4.6), we obtain

$$\lim_{\varepsilon\to0}\mathcal{Q}(\Lambda_\varepsilon e^\varepsilon-\mathbb{A}e)=0,$$

so that (3.4.13) follows now from (3.2.2). Finally, convergence (3.4.12) is an immediate consequence of (3.4.13). □

## 3.5 Convergence of quasistatic evolutions

In this section we focus on the quasistatic evolution problems associated with the functionals $\mathcal{J}_\varepsilon$ and $\mathcal{J}$, introduced in the previous section. To this purpose, for every $t\in[0,T]$ we prescribe a boundary datum $w(t)\in W^{1,2}(\Omega;\mathbb{R}^3)\cap KL(\Omega)$ and assume the map $t\mapsto w(t)$ to be absolutely continuous from $[0,T]$ into $W^{1,2}(\Omega;\mathbb{R}^3)$.

Let $s_1,s_2\in[0,T]$, $s_1\leq s_2$. For every function $t\mapsto\mu(t)$ of bounded variation from $[0,T]$ into $M_b(\Omega\cup\Gamma_d;\mathbb{M}^{3\times3}_D)$, we define the *dissipation* of $t\mapsto\mu(t)$ in $[s_1,s_2]$ as

$$\mathcal{D}(\mu;s_1,s_2):=\sup\Big\{\sum_{j=1}^n\mathcal{H}(\mu(t_j)-\mu(t_{j-1})):\ s_1=t_0\leq t_1\leq\cdots\leq t_n=s_2,\ n\in\mathbb{N}\Big\}.$$

Analogously, for every function $t\mapsto\mu(t)$ of bounded variation from $[0,T]$ into $M_b(\Omega\cup\Gamma_d;\mathbb{M}^{2\times2}_{sym})$ we define the *reduced dissipation* of $t\to\mu(t)$ in $[s_1,s_2]$ as

$$\mathcal{D}_r(\mu;s_1,s_2):=\sup\Big\{\sum_{j=1}^n\mathcal{H}_r(\mu(t_j)-\mu(t_{j-1})):\ s_1=t_0\leq t_1\leq\cdots\leq t_n=s_2,\ n\in\mathbb{N}\Big\}$$

for every $s_1,s_2\in[0,T]$, $s_1\leq s_2$.

**Definition 3.5.1.** Let $\varepsilon>0$. An $\varepsilon$*-quasistatic evolution* for the boundary datum $w(t)$ is a function $t\mapsto(u^\varepsilon(t),e^\varepsilon(t),p^\varepsilon(t))$ from $[0,T]$ into $BD(\Omega)\times L^2(\Omega;\mathbb{M}^{3\times3}_{sym})\times M_b(\Omega\cup\Gamma_d;\mathbb{M}^{3\times3}_{sym})$ that satisfies the following conditions:

(qs1) for every $t \in [0,T]$ we have $(u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t)) \in \mathcal{A}_\varepsilon(w(t))$ and

$$\mathcal{Q}(\Lambda_\varepsilon e^\varepsilon(t)) \leq \mathcal{Q}(\Lambda_\varepsilon f) + \mathcal{H}(\Lambda_\varepsilon q - \Lambda_\varepsilon p^\varepsilon(t)) \tag{3.5.1}$$

for every $(v,f,q) \in \mathcal{A}_\varepsilon(w(t))$;

(qs2) the function $t \mapsto p^\varepsilon(t)$ from $[0,T]$ into $M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_{sym})$ has bounded variation and for every $t \in [0,T]$

$$\mathcal{Q}(\Lambda_\varepsilon e^\varepsilon(t)) + \mathcal{D}(\Lambda_\varepsilon p^\varepsilon; 0, t) = \mathcal{Q}(\Lambda_\varepsilon e^\varepsilon(0)) + \int_0^t \int_\Omega \mathbb{C}\Lambda_\varepsilon e^\varepsilon(s) : E\dot{w}(s)\, dxds. \tag{3.5.2}$$

**Definition 3.5.2.** A *reduced quasistatic evolution* for the boundary datum $w(t)$ is a function $t \mapsto (u(t), e(t), p(t))$ from $[0,T]$ into $BD(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times3}_{sym}) \times M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_{sym})$ that satisfies the following conditions:

$(\mathrm{qs1})_r$ for every $t \in [0,T]$ we have $(u(t), e(t), p(t)) \in \mathcal{A}_{KL}(w(t))$ and

$$\mathcal{Q}_r(e(t)) \leq \mathcal{Q}_r(f) + \mathcal{H}_r(q - p(t)) \tag{3.5.3}$$

for every $(v,f,q) \in \mathcal{A}_{KL}(w(t))$;

$(\mathrm{qs2})_r$ the function $t \mapsto p(t)$ from $[0,T]$ into $M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_{sym})$ has bounded variation and for every $t \in [0,T]$

$$\mathcal{Q}_r(e(t)) + \mathcal{D}_r(p; 0, t) = \mathcal{Q}_r(e(0)) + \int_0^t \int_\Omega \mathbb{C}_r e(s) : E\dot{w}(s)\, dxds. \tag{3.5.4}$$

**Remark 3.5.3.** Since the functions $t \mapsto p^\varepsilon(t)$ and $t \mapsto p(t)$ from $[0,T]$ into $M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_{sym})$ have bounded variation, they are bounded and the set of their discontinuity points (in the strong topology) is at most countable. By Lemma 3.5.9 below the same properties hold for the functions $t \mapsto e^\varepsilon(t)$ and $t \mapsto e(t)$ from $[0,T]$ into $L^2(\Omega; \mathbb{M}^{3\times3}_{sym})$, and for the functions $t \mapsto u^\varepsilon(t)$ and $t \mapsto u(t)$ from $[0,T]$ into $BD(\Omega)$. Therefore, $t \mapsto e^\varepsilon(t)$ and $t \mapsto e(t)$ belong to $L^\infty([0,T]; L^2(\Omega; \mathbb{M}^{3\times3}_{sym}))$, while $t \mapsto u^\varepsilon(t)$ and $t \mapsto u(t)$ belong to $L^\infty([0,T]; BD(\Omega))$. As $t \mapsto E\dot{w}(t)$ belongs to $L^1([0,T]; L^2(\Omega; \mathbb{M}^{3\times3}_{sym}))$, the integrals on the right-hand side of (3.5.2) and (3.5.4) are well defined.

We are now in a position to state the main result of the chapter.

**Theorem 3.5.4.** *Let $t \mapsto w(t)$ be absolutely continuous from $[0,T]$ into $W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$. Assume there exists a sequence of triples $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon) \in \mathcal{A}_\varepsilon(w(0))$ such that*

$$\mathcal{Q}(\Lambda_\varepsilon e_0^\varepsilon) \leq \mathcal{Q}(\Lambda_\varepsilon f) + \mathcal{H}(\Lambda_\varepsilon q - \Lambda_\varepsilon p_0^\varepsilon) \tag{3.5.5}$$

*for every $(v,f,q) \in \mathcal{A}_\varepsilon(w(0))$ and every $\varepsilon > 0$, and*

$$\Lambda_\varepsilon e_0^\varepsilon \to \tilde{e}_0 \quad \textit{strongly in } L^2(\Omega; \mathbb{M}^{3\times3}_{sym}), \tag{3.5.6}$$

$$\|\Lambda_\varepsilon p_0^\varepsilon\|_{M_b} \leq C \tag{3.5.7}$$

*for some $\tilde{e}_0 \in L^2(\Omega; \mathbb{M}^{3\times3}_{sym})$ and some constant $C > 0$ independent of $\varepsilon$. For every $\varepsilon > 0$ let $t \mapsto (u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t))$ be an $\varepsilon$-quasistatic evolution for the boundary datum $w(t)$ such that*

*$u^\varepsilon(0) = u_0^\varepsilon$, $e^\varepsilon(0) = e_0^\varepsilon$, and $p^\varepsilon(0) = p_0^\varepsilon$. Then, there exists a reduced quasistatic evolution $t \mapsto (u(t), e(t), p(t))$ for the boundary datum $w(t)$ such that, up to subsequences,*

$$u^\varepsilon(t) \rightharpoonup u(t) \quad \text{weakly* in } BD(\Omega), \tag{3.5.8}$$

$$e^\varepsilon(t) \to e(t) \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times3}_{sym}), \tag{3.5.9}$$

$$\Lambda_\varepsilon e^\varepsilon(t) \to \mathbb{A}e(t) \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times3}_{sym}), \tag{3.5.10}$$

$$p^\varepsilon(t) \rightharpoonup p(t) \quad \text{weakly* in } M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_{sym}) \tag{3.5.11}$$

*for every $t \in [0,T]$, where $\mathbb{A}$ is the operator introduced in (3.4.1). Moreover, the functions $t \mapsto u(t)$, $t \mapsto e(t)$, and $t \mapsto p(t)$ are absolutely continuous from $[0,T]$ into $BD(\Omega)$, $L^2(\Omega; \mathbb{M}^{3\times3}_{sym})$, and $M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_{sym})$, respectively.*

**Remark 3.5.5.** From [15, Theorem 4.5] it follows that for every triple $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon) \in \mathcal{A}_\varepsilon(w(0))$ satisfying (3.5.5) there exists an $\varepsilon$-quasistatic evolution $t \mapsto (u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t))$ such that $u^\varepsilon(0) = u_0^\varepsilon$, $e^\varepsilon(0) = e_0^\varepsilon$, and $p^\varepsilon(0) = p_0^\varepsilon$. Moreover, by [15, Theorem 5.2] the functions $t \mapsto u^\varepsilon(t)$, $t \mapsto e^\varepsilon(t)$, and $t \mapsto p^\varepsilon(t)$ are absolutely continuous from $[0,T]$ into $BD(\Omega)$, $L^2(\Omega; \mathbb{M}^{3\times3}_{sym})$, and $M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_{sym})$, respectively, and for a.e. $t \in [0,T]$ we have

$$\|\Lambda_\varepsilon \dot{e}^\varepsilon(t)\|_{L^2} \;\leq\; C_1 \|E\dot{w}(t)\|_{L^2}, \tag{3.5.12}$$

$$\|\Lambda_\varepsilon \dot{p}^\varepsilon(t)\|_{M_b} \;\leq\; C_2 \|E\dot{w}(t)\|_{L^2}, \tag{3.5.13}$$

where $C_1$ and $C_2$ are positive constants depending on $R_K$, $r_{\mathbb{C}}$, $R_{\mathbb{C}}$, $\sup_{t\in[0,T]} \|\Lambda_\varepsilon e^\varepsilon(t)\|_{L^2}$, and $\sup_{t\in[0,T]} \|\Lambda_\varepsilon p^\varepsilon(t)\|_{M_b}$. We notice that these results are proven in [15] under the assumption of a reference configuration of class $C^2$, but, as observed in [27], Lipschitz regularity is enough in the absence of external loads.

**Remark 3.5.6.** The set of admissibile initial data for Theorem 3.5.4 is nonempty. Indeed, for every $\varepsilon > 0$ let $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon) \in \mathcal{A}_\varepsilon(w(0))$ be a minimizer of the functional $\mathcal{J}_\varepsilon$ on $\mathcal{A}_\varepsilon(w(0))$, that is,

$$\mathcal{Q}(\Lambda_\varepsilon e_0^\varepsilon) + \mathcal{H}(\Lambda_\varepsilon p_0^\varepsilon) \leq \mathcal{Q}(\Lambda_\varepsilon f) + \mathcal{H}(\Lambda_\varepsilon q)$$

for every $(v, f, q) \in \mathcal{A}_\varepsilon(w(0))$. Since by (3.2.5)

$$\mathcal{H}(\Lambda_\varepsilon q) \leq \mathcal{H}(\Lambda_\varepsilon q - \Lambda_\varepsilon p_0^\varepsilon) + \mathcal{H}(\Lambda_\varepsilon p_0^\varepsilon),$$

we deduce that $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon)$ satisfies (3.5.5) for every $\varepsilon > 0$. Moreover, by Theorem 3.4.1 we infer the existence of a triple $(u_0, e_0, p_0) \in \mathcal{A}_{KL}(w(0))$ such that (3.5.6) is satisfied with $\tilde{e}_0 = \mathbb{A}e_0$ and

$$\lim_{\varepsilon\to0} \mathcal{H}(\Lambda_\varepsilon p_0^\varepsilon) = \mathcal{H}_r(p_0).$$

This last convergence implies (3.5.7) by (3.2.4).

**Remark 3.5.7.** Theorem 3.5.4 ensures, in particular, the existence of an absolutely continuous reduced quasistatic evolution for every initial datum $(u_0, e_0, p_0) \in \mathcal{A}_{KL}(w(0))$ that is approximable in the sense of (3.5.8)–(3.5.11) by a sequence of triples $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon) \in \mathcal{A}_\varepsilon(w(0))$ satisfying (3.5.5). Note that, again by Theorem 3.5.4, every such datum satisfies

$$\mathcal{Q}_r(e_0) \leq \mathcal{Q}_r(f) + \mathcal{H}_r(q - p_0) \tag{3.5.14}$$

for every $(v, f, q) \in \mathcal{A}_{KL}(w(0))$.

We mention here that existence of a reduced quasistatic evolution can be actually proved for every initial datum $(u_0, e_0, p_0) \in \mathcal{A}_{KL}(w(0))$ satisfying (3.5.14) by applying the abstract method for rate-independent processes developed in [47], namely by discretizing time and by solving suitable incremental minimum problems. Moreover, arguing as in [15, Theorem 5.2], one can show that every reduced quasistatic evolution is absolutely continuous from $[0, T]$ into $BD(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times3}_{sym}) \times M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_{sym})$.

To prove Theorem 3.5.4 we need two technical lemmas concerning some consequences of the minimality condition $(\mathrm{qs1})_r$.

**Lemma 3.5.8.** *Let $w \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$. A triple $(u, e, p) \in \mathcal{A}_{KL}(w)$ is a solution of the minimum problem*

$$\min\big\{\mathcal{Q}_r(f) + \mathcal{H}_r(q-p) : \ (v, f, q) \in \mathcal{A}_{KL}(w)\big\} \tag{3.5.15}$$

*if and only if*

$$-\mathcal{H}_r(q) \le \int_\Omega \mathbb{C}_r e : f \, dx \tag{3.5.16}$$

*for every $(v, f, q) \in \mathcal{A}_{KL}(0)$.*

*Proof.* Let $(u, e, p) \in \mathcal{A}_{KL}(w)$ be a solution to (3.5.15) and let $(v, f, q) \in \mathcal{A}_{KL}(0)$. For every $\eta \in \mathbb{R}$ the triple $(u + \eta v, e + \eta f, p + \eta q)$ belongs to $\mathcal{A}_{KL}(w)$, hence

$$\mathcal{Q}_r(e) \le \mathcal{Q}_r(e + \eta f) + \mathcal{H}_r(\eta q).$$

Using the positive homogeneity of $H_r$, we obtain

$$0 \le \pm\eta \int_\Omega \mathbb{C}_r e : f \, dx + \eta^2 \mathcal{Q}_r(f) + \eta \mathcal{H}_r(\pm q),$$

for every $\eta > 0$. Dividing by $\eta$ and sending $\eta$ to 0 yield (3.5.16).

The converse implication is true by convexity. ☐

**Lemma 3.5.9.** *Let $w_1, w_2 \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$ and for $\alpha = 1, 2$ let $(u_\alpha, e_\alpha, p_\alpha) \in \mathcal{A}_{KL}(w_\alpha)$ be a solution of the minimum problem*

$$\min\big\{\mathcal{Q}_r(f) + \mathcal{H}_r(q - p_\alpha) : \ (v, f, q) \in \mathcal{A}_{KL}(w_\alpha)\big\}. \tag{3.5.17}$$

*Then there exists a positive constant $C$, depending only on $R_K$, $r_{\mathbb{C}}$, $R_{\mathbb{C}}$, $\Omega$, and $\Gamma_d$, such that*

$$\|e_2 - e_1\|_{L^2} \le C\theta_{12}, \tag{3.5.18}$$

$$\|Eu_1 - Eu_2\|_{M_b} \le C\theta_{12}, \tag{3.5.19}$$

$$\|u_1 - u_2\|_{L^1} \le C(\theta_{12} + \|w_1 - w_2\|_{L^2}), \tag{3.5.20}$$

*where $\theta_{12}$ is given by*

$$\theta_{12} := \|p_1 - p_2\|_{M_b} + \|p_1 - p_2\|^{\frac{1}{2}}_{M_b} + \|Ew_1 - Ew_2\|_{L^2}.$$

*Proof.* Since $(u_2 - u_1 - w_2 + w_1, e_2 - e_1 - Ew_2 + Ew_1, p_2 - p_1) \in \mathcal{A}_{KL}(0)$, we can choose $v = u_2 - u_1 - w_2 + w_1$, $f = e_2 - e_1 - Ew_2 + Ew_1$, and $q = p_2 - p_1$ in (3.5.16); thus, by the minimality of $(u_\alpha, e_\alpha, p_\alpha)$, with $\alpha = 1, 2$, and Lemma 3.5.8 we have

$$\begin{aligned} -\mathcal{H}_r(p_2 - p_1) &\leq \int_\Omega \mathbb{C}_r e_1 : (e_2 - e_1 - Ew_2 + Ew_1)\, dx, \\ -\mathcal{H}_r(p_1 - p_2) &\leq \int_\Omega \mathbb{C}_r e_2 : (e_1 - e_2 - Ew_1 + Ew_2)\, dx. \end{aligned}$$

Adding term by term, changing sign, and applying (3.4.8) yield

$$\int_\Omega \mathbb{C}_r(e_2 - e_1) : (e_2 - e_1)\, dx \leq \int_\Omega \mathbb{C}_r(e_2 - e_1) : (Ew_2 - Ew_1)\, dx + 2\sqrt{3} R_K \|p_2 - p_1\|_{M_b}.$$

By (3.4.4) we deduce

$$r_\mathbb{C} \|e_2 - e_1\|_{L^2}^2 \leq R_\mathbb{C} \|e_2 - e_1\|_{L^2} \|Ew_2 - Ew_1\|_{L^2} + 2\sqrt{3} R_K \|p_2 - p_1\|_{M_b},$$

which implies (3.5.18) by the Cauchy inequality. Since $Eu_i = e_i + p_1$ in $\Omega$, Hölder inequality gives

$$\|Eu_2 - Eu_1\|_{M_b} \leq \mathcal{L}^3(\Omega)^{1/2} \|e_2 - e_1\|_{L^2} + \|p_2 - p_1\|_{M_b},$$

so that (3.5.19) follows from (3.5.18). Finally, since $p_2 - p_1 = (w_2 - w_1 - u_2 + u_1) \odot \nu_{\partial\Omega} \mathcal{H}^2$ on $\Gamma_d$, we have

$$\|u_2 - u_1\|_{L^1(\Gamma_d)} \leq \|w_2 - w_1\|_{L^1(\Gamma_d)} + \|p_2 - p_1\|_{M_b} \leq C\|w_2 - w_1\|_{W^{1,2}} + \|p_2 - p_1\|_{M_b}$$

where we used the continuity of the trace operator from $W^{1,2}(\Omega; \mathbb{R}^3)$ into $L^1(\partial\Omega; \mathbb{R}^3)$. Inequality (3.5.20) now follows from (1.4.2) and (3.5.19). □

We are now in a position to prove Theorem 3.5.4.

*Proof of Theorem 3.5.4.* The proof is subdivided into four steps.

**Step 1.** *Compactness estimates.* Let us prove that there exists a constant $C$, depending only on the data, such that

$$\sup_{t \in [0,T]} \|\Lambda_\varepsilon e^\varepsilon(t)\|_{L^2} \leq C, \qquad \sup_{t \in [0,T]} \|\Lambda_\varepsilon p^\varepsilon(t)\|_{M_b} \leq C \tag{3.5.21}$$

for every $\varepsilon$. As $t \mapsto w(t)$ is absolutely continuous with values in $W^{1,2}(\Omega; \mathbb{R}^3)$, the function $t \mapsto \|E\dot{w}(t)\|_2$ is integrable on $[0, T]$. This fact, together with (3.2.2), (3.2.3), and (3.5.2), implies that

$$r_\mathbb{C} \|\Lambda_\varepsilon e^\varepsilon(t)\|_{L^2}^2 \leq R_\mathbb{C} \|\Lambda_\varepsilon e^\varepsilon(0)\|_{L^2}^2 + 2R_\mathbb{C} \sup_{t \in [0,T]} \|\Lambda_\varepsilon e^\varepsilon(t)\|_{L^2} \int_0^T \|E\dot{w}(s)\|_{L^2}\, ds \tag{3.5.22}$$

for every $t \in [0, T]$. The former inequality in (3.5.21) follows now from (3.5.6) and Cauchy inequality. As for the latter, by (3.5.2), (3.5.22), and (3.5.6) we deduce that

$$\mathcal{D}(\Lambda_\varepsilon p^\varepsilon; 0, T) \leq C.$$

By definition of $\mathcal{D}$ and (3.2.4) we infer that

$$r_K \|\Lambda_\varepsilon p^\varepsilon(t) - \Lambda_\varepsilon p_0^\varepsilon\|_{M_b} \le \mathcal{H}(\Lambda_\varepsilon p^\varepsilon(t) - \Lambda_\varepsilon p^\varepsilon(0)) \le \mathcal{D}(\Lambda_\varepsilon p^\varepsilon; 0, t) \le C$$

for every $t \in [0,T]$, which implies the second inequality in (3.5.21) by (3.5.7).

Combining (3.5.12), (3.5.13), and (3.5.21), we obtain

$$\begin{aligned} \|\Lambda_\varepsilon e^\varepsilon(t_1) - \Lambda_\varepsilon e^\varepsilon(t_2)\|_{L^2} &\le C \int_{t_1}^{t_2} \|E\dot{w}(s)\|_{L^2}\, ds \\ \|\Lambda_\varepsilon p^\varepsilon(t_1) - \Lambda_\varepsilon p^\varepsilon(t_2)\|_{M_b} &\le C \int_{t_1}^{t_2} \|E\dot{w}(s)\|_{L^2}\, ds \end{aligned}$$

for every $0 \le t_1 \le t_2 \le T$, where $C$ is a constant depending only on the data. Therefore, by Ascoli-Arzelà Theorem there exist two subsequences, still denoted $\Lambda_\varepsilon e^\varepsilon$ and $\Lambda_\varepsilon p^\varepsilon$, and two absolutely continuous functions $\tilde{e} : [0,T] \to L^2(\Omega; \mathbb{M}^{3\times3}_{sym})$ and $\tilde{p} : [0,T] \to M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_D)$ such that

$$\Lambda_\varepsilon e^\varepsilon(t) \rightharpoonup \tilde{e}(t) \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{3\times3}_{sym}), \tag{3.5.23}$$

$$\Lambda_\varepsilon p^\varepsilon(t) \rightharpoonup \tilde{p}(t) \quad \text{weakly* in } M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_D) \tag{3.5.24}$$

for every $t \in [0,T]$.

Let $e : [0,T] \to L^2(\Omega; \mathbb{M}^{3\times3}_{sym})$ be defined as

$$e_{\alpha\beta}(t) = \tilde{e}_{\alpha\beta}(t) \quad (\alpha, \beta = 1,2) \quad \text{and} \quad e_{i3}(t) = 0 \quad (i = 1,2,3)$$

for every $t \in [0,T]$ and let $p : [0,T] \to M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_{sym})$ be defined as

$$p_{\alpha\beta}(t) = \tilde{p}_{\alpha\beta}(t) \quad (\alpha, \beta = 1,2) \quad \text{and} \quad p_{i3}(t) = 0 \quad (i = 1,2,3) \tag{3.5.25}$$

for every $t \in [0,T]$. Then $t \mapsto e(t)$ is absolutely continuous from $[0,T]$ into $L^2(\Omega; \mathbb{M}^{3\times3}_{sym})$, $t \mapsto p(t)$ is absolutely continuous from $[0,T]$ into $M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_{sym})$, and by (3.5.23) and (3.5.24) we have

$$e^\varepsilon(t) \rightharpoonup e(t) \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{3\times3}_{sym}), \tag{3.5.26}$$

$$p^\varepsilon(t) \rightharpoonup p(t) \quad \text{weakly* in } M_b(\Omega \cup \Gamma_d; \mathbb{M}^{3\times3}_{sym}) \tag{3.5.27}$$

for every $t \in [0,T]$. Using (1.4.2) and the fact that $(u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t)) \in \mathcal{A}_\varepsilon(w(t))$ for every $\varepsilon > 0$, it is easy to see that there exists an absolutely continuous function $u : [0,T] \to BD(\Omega)$ such that

$$u^\varepsilon(t) \rightharpoonup u(t) \quad \text{weakly* in } BD(\Omega)$$

for every $t \in [0,T]$. Moreover, arguing as in the proof of Theorem 3.4.4, one can show that $(u(t), e(t), p(t)) \in \mathcal{A}_{KL}(w(t))$.

**Step 2.** *Reduced stability.* We now show that the triple $(u(t), e(t), p(t))$ is a solution to the minimum problem

$$\min \left\{ \mathcal{Q}_r(f) + \mathcal{H}_r(q - p(t)) : \ (v, f, q) \in \mathcal{A}_{KL}(w(t)) \right\} \tag{3.5.28}$$

for every $t \in [0,T]$.

Let us fix $t \in [0,T]$. By Lemma 3.5.8 it is enough to prove condition (3.5.16). Let $(v,f,q) \in \mathcal{A}_{KL}(0)$. By Theorem 3.4.5 there exists a sequence of triples $(v^\varepsilon, f^\varepsilon, q^\varepsilon) \in \mathcal{A}_\varepsilon(0)$ such that

$$\Lambda_\varepsilon f^\varepsilon \to \mathbb{A} f \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{sym}) \tag{3.5.29}$$

and

$$\mathcal{H}(\Lambda_\varepsilon q^\varepsilon) \to \mathcal{H}_r(q). \tag{3.5.30}$$

By [15, Theorem 3.6] the minimality condition (3.5.1) is equivalent to

$$-\mathcal{H}(\Lambda_\varepsilon \check{q}) \leq \int_\Omega \mathbb{C}\Lambda_\varepsilon e^\varepsilon(t) : \Lambda_\varepsilon \check{f}\, dx \tag{3.5.31}$$

for every $(\check{v}, \check{f}, \check{q}) \in \mathcal{A}_\varepsilon(0)$. Therefore, we have that

$$-\mathcal{H}(\Lambda_\varepsilon q^\varepsilon) \leq \int_\Omega \mathbb{C}\Lambda_\varepsilon e^\varepsilon(t) : \Lambda_\varepsilon f^\varepsilon\, dx$$

for every $\varepsilon > 0$; hence, combining (3.5.23), (3.5.29), and (3.5.30), we obtain

$$-\mathcal{H}_r(q) \leq \int_\Omega \mathbb{C}\tilde{e}(t) : \mathbb{A} f\, dx.$$

Since $\mathbb{C}\tilde{e}(t) : \mathbb{A}f = \mathbb{C}\mathbb{A}e(t) : \mathbb{A}f = \mathbb{C}_r e(t) : f$ a.e. in $\Omega$ by (3.4.6), the inequality above reduces to (3.5.16).

**Step 3.** *Identification of the limiting scaled elastic strain.* We shall prove that the function $\tilde{e}(t)$ in (3.5.23) satisfies

$$\tilde{e}(t) = \mathbb{A}e(t) \tag{3.5.32}$$

for every $t \in [0,T]$.

For every $\psi \in W^{1,2}(\Omega;\mathbb{R}^3)$ with $\psi = 0$ on $\Gamma_d$ we can consider the triples $(\pm\psi, \pm E\psi, 0)$ as test functions in (3.5.31). This leads to the condition

$$\int_\Omega \mathbb{C}\Lambda_\varepsilon e^\varepsilon(t) : \Lambda_\varepsilon E\psi\, dx = 0 \tag{3.5.33}$$

for every $\psi \in W^{1,2}(\Omega;\mathbb{R}^3)$ with $\psi = 0$ on $\Gamma_d$ and for every $\varepsilon$.

Let now $U \subset \omega$, $(a,b) \subset (-\frac{1}{2}, \frac{1}{2})$, and $\lambda_i \in \mathbb{R}$, $i = 1,2,3$. Let us denote the characteristic functions of the sets $U$ and $(a,b)$ by $\chi_U$ and $\chi_{(a,b)}$, respectively. Finally, let $(\varphi_i^k) \subset C^1_c(\omega)$ and $(\xi^k) \subset C^1([-\frac{1}{2}, \frac{1}{2}])$ be such that $\varphi_i^k \to \lambda_i \chi_U$ strongly in $L^4(\omega)$, $i = 1,2,3$, and $(\xi^k)' \to \chi_{(a,b)}$ strongly in $L^4(-\frac{1}{2}, \frac{1}{2})$. For every $\varepsilon$ and $k \in \mathbb{N}$ we consider the function

$$\psi^{\varepsilon,k}(x) := \begin{pmatrix} 2\varepsilon\xi^k(x_3)\varphi_1^k(x') \\ 2\varepsilon\xi^k(x_3)\varphi_2^k(x') \\ \varepsilon^2\xi^k(x_3)\varphi_3^k(x') \end{pmatrix}$$

for every $x \in \Omega$. Since $\psi^{\varepsilon,k} \in W^{1,2}(\Omega;\mathbb{R}^3)$ and $\psi^{\varepsilon,k} = 0$ on $\Gamma_d$, by (3.5.33) we have

$$\int_\Omega \mathbb{C}\Lambda_\varepsilon e^\varepsilon(t) : \Lambda_\varepsilon E\psi^{\varepsilon,k}\, dx = 0$$

for every $\varepsilon$. Passing to the limit with respect to $\varepsilon \to 0$ and then to $k \to \infty$, we deduce

$$\int_{U\times(a,b)} \mathbb{C}\tilde{e}(t) : \begin{pmatrix} 0 & 0 & \lambda_1 \\ 0 & 0 & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} dx = 0.$$

Since $U$ and $(a,b)$ are arbitrary, we conclude that for every $\lambda_i \in \mathbb{R}$.

$$\mathbb{C}\tilde{e}(t) : \begin{pmatrix} 0 & 0 & \lambda_1 \\ 0 & 0 & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} = 0,$$

a.e. in $\Omega$. This implies (3.5.32) by (3.4.2).

**Step 4.** *Reduced energy balance.* By (3.5.2) and lower semicontinuity we have

$$\begin{aligned} \mathcal{Q}_r(e(t)) + \mathcal{D}(p;0,t) &\leq \lim_{\varepsilon\to 0} \Big\{ \mathcal{Q}(\Lambda_\varepsilon e^\varepsilon(0)) + \int_0^t \int_\Omega \mathbb{C}\Lambda_\varepsilon e^\varepsilon(s) : E\dot{w}(s)\, dx\, ds \Big\} \\ &= \mathcal{Q}_r(e_0) + \int_0^t \int_\Omega \mathbb{C}_r e(s) : E\dot{w}(s)\, dx\, ds, \end{aligned}$$

where the last equality follows from (3.5.6), (3.5.21), (3.5.23), (3.5.32), and the dominated convergence theorem. Since by (3.5.25) and the definition of $\mathcal{H}_r$ there holds

$$\mathcal{D}_r(p;0,t) \leq \mathcal{D}(\tilde{p};0,t) \tag{3.5.34}$$

for every $t \in [0,T]$, we conclude that

$$\mathcal{Q}_r(e(t)) + \mathcal{D}_r(p;0,t) \leq \mathcal{Q}_r(e_0) + \int_0^t \int_\Omega \mathbb{C}_r e(s) : E\dot{w}(s)\, dxds. \tag{3.5.35}$$

As it is standard in the variational theory for rate-independent processes, the converse energy inequality follows from the minimality condition $(\text{qs1})_r$. We omit the proof as it follows closely those of [15, Theorem 4.7] and of [47, Theorem 4.4].

Combining (qs2), $(\text{qs2})_r$, and the fact that the right-hand side of (qs2) converges to the right-hand side of $(\text{qs2})_r$, we deduce that

$$\mathcal{Q}(\Lambda_\varepsilon e^\varepsilon(t)) + \mathcal{D}(\Lambda_\varepsilon p^\varepsilon;0,t) \to \mathcal{Q}_r(e(t)) + \mathcal{D}_r(p;0,t) \tag{3.5.36}$$

for every $t \in [0,T]$. On the other hand, by lower semicontinuity of $\mathcal{Q}_r$ and of $\mathcal{D}_r$ we have

$$\mathcal{Q}_r(e(t)) \leq \liminf_{\varepsilon\to 0} \mathcal{Q}(\Lambda_\varepsilon e^\varepsilon(t)) \tag{3.5.37}$$

and

$$\mathcal{D}_r(p;0,t) \leq \liminf_{\varepsilon\to 0} \mathcal{D}(\Lambda_\varepsilon p^\varepsilon;0,t) \tag{3.5.38}$$

for every $t \in [0,T]$. From (3.5.36)–(3.5.38) it follows that

$$\lim_{\varepsilon\to 0} \mathcal{Q}(\Lambda_\varepsilon e^\varepsilon(t)) = \mathcal{Q}_r(e(t)) = \mathcal{Q}(\mathbb{A}e(t))$$

for every $t \in [0,T]$. This, together with (3.5.23) and (3.5.32), implies strong convergence of the scaled strains $\Lambda_\varepsilon e_\varepsilon(t)$, and consequently of the strains $e_\varepsilon(t)$, for every $t \in [0,T]$. This concludes the proof of the theorem. □

## 3.6 Characterization of reduced quasistatic evolutions

In the following we shall consider the space $\Pi_{\Gamma_d}(\Omega)$ of admissible plastic strains, defined as the class of all $p \in M_b(\Omega \cup \Gamma_d; \mathbb{M}^{2\times 2}_{sym})$ for which there exist $u \in BD(\Omega)$, $e \in L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$, and $w \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$ such that $(u, e, p) \in \mathcal{A}_{KL}(w)$.

We shall also use the set

$$\Sigma(\Omega) := \{\sigma \in L^\infty(\Omega; \mathbb{M}^{2\times 2}_{sym}) : \ \mathrm{div}_{x'}\bar{\sigma} \in L^2(\omega; \mathbb{R}^2), \ \mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma} \in M_b(\omega)\},$$

where $\bar{\sigma}, \hat{\sigma} \in L^\infty(\omega; \mathbb{M}^{2\times 2}_{sym})$ are the zero-th and first order moments of $\sigma$, defined according to Definition 3.3.3. In the first subsection we shall introduce a duality pairing between stresses $\sigma \in \Sigma(\Omega)$ and plastic strains $p \in \Pi_{\Gamma_d}(\Omega)$. In the second subsection we shall use this duality pairing to deduce a weak formulation of the classical flow rule for a reduced quasistatic evolution. In the last subsection we discuss some examples, where reduced quasistatic evolutions can be characterized in terms of two-dimensional quantities.

### 3.6.1 Stress-strain duality

We first introduce a notion of duality for the zero-th order moments of the stress and the plastic strain. We essentially follow the theory developed in [37] and [15, Subsection 2.3].

For every $\sigma \in \Sigma(\Omega)$ we can define the trace $[\bar{\sigma}\nu_{\partial\omega}] \in L^\infty(\partial\omega; \mathbb{R}^2)$ of its zero-th order moment $\bar{\sigma}$ through the formula

$$\int_{\partial\omega} [\bar{\sigma}\nu_{\partial\omega}] \cdot \varphi \, d\mathcal{H}^1 := \int_\omega \mathrm{div}_{x'}\bar{\sigma} \cdot \varphi \, dx' + \int_\omega \bar{\sigma} : E\varphi \, dx' \tag{3.6.1}$$

for every $\varphi \in W^{1,1}(\omega; \mathbb{R}^2)$. This is well defined since $W^{1,1}(\omega; \mathbb{R}^2)$ is embedded into $L^2(\omega; \mathbb{R}^2)$.

Let $\sigma \in \Sigma(\Omega)$ and $\xi \in BD(\omega)$. We define the distribution $[\bar{\sigma} : E\xi]$ on $\omega$ by

$$\langle [\bar{\sigma} : E\xi], \varphi \rangle := -\int_\omega \varphi \, \mathrm{div}_{x'}\bar{\sigma} \cdot \xi \, dx' - \int_\omega \bar{\sigma} : (\nabla\varphi \odot \xi) \, dx' \tag{3.6.2}$$

for every $\varphi \in C^\infty_c(\omega)$. From [37, Theorem 3.2] it follows that $[\bar{\sigma} : E\xi]$ is a bounded measure on $\omega$, whose variation satisfies

$$\big|[\bar{\sigma} : E\xi]\big| \le \|\bar{\sigma}\|_{L^\infty} |E\xi| \quad \text{in } \omega. \tag{3.6.3}$$

We can now define a duality between the zero-th order moments of elements in $\Sigma(\Omega)$ and $\Pi_{\Gamma_d}(\Omega)$. Given $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\Gamma_d}(\Omega)$, we fix $(u, e, w) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times (W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega))$ such that $(u, e, p) \in \mathcal{A}_{KL}(w)$. Let $\bar{u} \in BD(\omega)$, $u_3 \in BH(\omega)$ and $\bar{w} \in W^{1,2}(\omega; \mathbb{R}^2)$, $w_3 \in W^{2,2}(\omega)$ be the Kirchhoff-Love components of $u$ and $w$, respectively. We then define the measure $[\bar{\sigma} : \bar{p}] \in M_b(\omega \cup \gamma_d)$ by setting

$$[\bar{\sigma} : \bar{p}] := \begin{cases} [\bar{\sigma} : E\bar{u}] - \bar{\sigma} : \bar{e} & \text{in } \omega, \\ [\bar{\sigma}\nu_{\partial\omega}] \cdot (\bar{w} - \bar{u})\mathcal{H}^1 & \text{on } \gamma_d, \end{cases}$$

so that

$$\int_{\omega \cup \gamma_d} \varphi \, d[\bar{\sigma} : \bar{p}] = \int_\omega \varphi \, d[\bar{\sigma} : E\bar{u}] - \int_\omega \varphi\bar{\sigma} : \bar{e} \, dx' + \int_{\gamma_d} [\bar{\sigma}\nu_{\partial\omega}] \cdot \varphi(\bar{w} - \bar{u}) \, d\mathcal{H}^1 \tag{3.6.4}$$

for every $\varphi \in C(\overline{\omega})$.

**Remark 3.6.1.** Arguing as in [15], one can prove that the definition of $[\bar\sigma : \bar p]$ is independent of the choice of the triple $(u, e, w)$. Moreover, if $\bar\sigma \in C^1(\overline{\omega}; \mathbb{M}^{2\times 2}_{sym})$, then

$$\int_{\omega\cup\gamma_d} \varphi\, d[\bar\sigma : \bar p] = \int_{\omega\cup\gamma_d} \varphi\bar\sigma : d\bar p$$

for every $\varphi \in C^1(\overline{\omega})$. One can prove by approximation that the same equality is true for every $\bar\sigma \in C(\overline{\omega}; \mathbb{M}^{2\times 2}_{sym})$ and $\varphi \in C(\overline{\omega})$.

The following integration by parts formula can be proved.

**Proposition 3.6.2.** *Let $\sigma \in \Sigma(\Omega)$, $w \in W^{1,2}(\omega;\mathbb{R}^3) \cap KL(\Omega)$, and $(u, e, p) \in \mathcal{A}_{KL}(w)$. Let also $\bar u \in BD(\omega)$ and $\bar w \in W^{1,2}(\omega;\mathbb{R}^2)$ be the tangential Kirchhoff-Love components of $u$ and $w$. Then*

$$\int_{\omega\cup\gamma_d} \varphi\, d[\bar\sigma : \bar p] + \int_\omega \varphi\bar\sigma : (\bar e - E\bar w)\, dx' + \int_\omega \bar\sigma : (\nabla\varphi \odot (\bar u - \bar w))\, dx'$$
$$= -\int_\omega \mathrm{div}_{x'}\bar\sigma \cdot \varphi(\bar u - \bar w)\, dx' + \int_{\gamma_n} [\bar\sigma\nu_{\partial\omega}] \cdot \varphi(\bar u - \bar w)\, d\mathcal{H}^1 \quad (3.6.5)$$

*for every $\varphi \in C^1(\overline{\omega})$.*

*Proof.* The result is a corollary of [15, Proposition 2.2]. □

We now introduce a notion of duality for the first order moments of the stress and of the plastic strain. We follow the lines of [22, Subsection 3.2] and [24, Subsection 2.3].

We start with a proposition concerning the traces of the first order moment of a stress in $\Sigma(\Omega)$. To this purpose we introduce the space

$$\hat\Sigma(\omega) := \big\{\vartheta \in L^\infty(\omega; \mathbb{M}^{2\times 2}_{sym}) : \ \mathrm{div}_{x'}\mathrm{div}_{x'}\vartheta \in M_b(\omega)\big\},$$

endowed with the norm $\|\vartheta\|_{L^\infty} + \|\mathrm{div}_{x'}\mathrm{div}_{x'}\vartheta\|_{M_b}$. We also denote by $T_{\partial\omega} : W^{2,1}(\omega) \to W^{1,1}(\partial\omega)$ the trace operator on $W^{2,1}(\omega)$. We recall that $T_{\partial\omega}(W^{2,1}(\omega)) \neq W^{1,1}(\partial\omega)$, see [23, Théorème 2].

**Proposition 3.6.3.** *There exists a surjective continuous linear operator*

$$\begin{array}{rccl} \mathcal{L}: & \hat\Sigma(\omega) & \to & (T_{\partial\omega}(W^{2,1}(\omega)))' \times L^\infty(\partial\omega) \\ & \vartheta & \mapsto & (b_0(\vartheta), b_1(\vartheta)) \end{array}$$

*such that for every $\vartheta \in \hat\Sigma(\omega)$ and $v \in W^{2,1}(\omega)$ there holds*

$$\int_\omega \vartheta : D^2 v\, dx' - \int_\omega v\, d(\mathrm{div}_{x'}\mathrm{div}_{x'}\vartheta) = -\langle b_0(\vartheta), v\rangle + \int_{\partial\omega} b_1(\vartheta)\frac{\partial v}{\partial\nu_{\partial\omega}}\, d\mathcal{H}^1, \quad (3.6.6)$$

*where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(T_{\partial\omega}(W^{2,1}(\omega)))'$ and $T_{\partial\omega}(W^{2,1}(\omega))$. Moreover, if $\vartheta \in C^2(\overline{\omega}; \mathbb{M}^{2\times 2}_{sym})$, then*

$$b_0(\vartheta) = \mathrm{div}_{x'}\vartheta \cdot \nu_{\partial\omega} + \frac{\partial}{\partial\tau_{\partial\omega}}(\vartheta\nu_{\partial\omega} \cdot \tau_{\partial\omega}), \quad (3.6.7)$$
$$b_1(\vartheta) = \vartheta\nu_{\partial\omega} \cdot \nu_{\partial\omega}, \quad (3.6.8)$$

*where $\tau_{\partial\omega}$ is the tangent vector to $\partial\omega$.*

*Proof.* See [22, Théorème 2.3]. □

**Remark 3.6.4.** The second integral on the left-handside of (3.6.6) is well defined because of the embedding of $W^{2,1}(\omega)$ into $C(\overline{\omega})$ (see [3, Theorem 4.12]).

Let $\sigma \in \Sigma(\Omega)$ and $v \in BH(\omega)$. We define the distribution $[\hat{\sigma} : D^2 v]$ on $\omega$ by

$$\langle [\hat{\sigma} : D^2 v], \varphi \rangle := \int_\omega \varphi v \, d(\mathrm{div}_{x'} \mathrm{div}_{x'} \hat{\sigma}) - 2 \int_\omega \hat{\sigma} : (\nabla \varphi \odot \nabla v) \, dx' - \int_\omega v \hat{\sigma} : \nabla^2 \varphi \, dx'$$

for every $\varphi \in C_c^\infty(\omega)$. From [24, Proposition 2.1] it follows that $[\hat{\sigma} : D^2 v]$ is a bounded measure on $\omega$, whose variation satisfies

$$\big|[\hat{\sigma} : D^2 v]\big| \leq \|\hat{\sigma}\|_{L^\infty} |D^2 v| \quad \text{in } \omega.$$

We can now define a duality between the first order moments of elements in $\Sigma(\Omega)$ and $\Pi_{\Gamma_d}(\Omega)$. Given $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\Gamma_d}(\Omega)$, we fix $(u, e, w) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times (W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega))$ such that $(u, e, p) \in \mathcal{A}_{KL}(w)$. We then define the measure $[\hat{\sigma} : \hat{p}] \in M_b(\omega \cup \gamma_d)$ by setting

$$[\hat{\sigma} : \hat{p}] := \begin{cases} -[\bar{\sigma} : D^2 u_3] - \hat{\sigma} : \hat{e} & \text{in } \omega, \\ b_1(\hat{\sigma}) \dfrac{\partial (u_3 - w_3)}{\partial \nu_{\partial \omega}} \mathcal{H}^1 & \text{on } \gamma_d, \end{cases}$$

so that

$$\int_{\omega \cup \gamma_d} \varphi \, d[\hat{\sigma} : \hat{p}] = - \int_\omega \varphi \, d[\hat{\sigma} : D^2 u_3] - \int_\omega \varphi \hat{\sigma} : \hat{e} \, dx' + \int_{\gamma_d} \varphi b_1(\hat{\sigma}) \frac{\partial (u_3 - w_3)}{\partial \nu_{\partial \omega}} \, d\mathcal{H}^1$$

for every $\varphi \in C(\overline{\omega})$.

**Remark 3.6.5.** The definition of $[\hat{\sigma} : \hat{p}]$ does not depend on the choice of the triple $(u, e, w)$. Moreover, if $\hat{\sigma} \in C^2(\overline{\omega}; \mathbb{M}^{2\times 2}_{sym})$ and $p \in \Pi_{\Gamma_d}(\Omega)$, then

$$\int_{\omega \cup \gamma_d} \varphi \, d[\hat{\sigma} : \hat{p}] = \int_{\omega \cup \gamma_d} \varphi \hat{\sigma} : d\hat{p} \tag{3.6.9}$$

for every $\varphi \in C^2(\overline{\omega})$. This follows from the equality

$$\int_{\gamma_d} \varphi b_1(\hat{\sigma}) \frac{\partial (u_3 - w_3)}{\partial \nu_{\partial \omega}} \, d\mathcal{H}^1 = \int_{\gamma_d} \varphi \hat{\sigma} : (\nabla (u_3 - w_3) \odot \nu_{\partial \omega}) \, d\mathcal{H}^1,$$

which, in turn, is a consequence of (3.6.8). By an approximation argument one can show that (3.6.9) holds true for every $\hat{\sigma} \in C(\overline{\omega}; \mathbb{M}^{2\times 2}_{sym})$ and $\varphi \in C(\overline{\omega})$.

As a corollary of [24, Proposition 2.1], we have the following integration by parts formula.

**Proposition 3.6.6.** *Let $\sigma \in \Sigma(\Omega)$, $w \in W^{1,2}(\omega; \mathbb{R}^3) \cap KL(\Omega)$, and $(u, e, p) \in \mathcal{A}_{KL}(w)$. Then*

$$\begin{aligned} &\int_{\omega \cup \gamma_d} \varphi \, d[\hat{\sigma} : \hat{p}] + \int_\omega \varphi \hat{\sigma} : (\hat{e} + D^2 w_3) \, dx' \\ &\qquad - 2 \int_\omega \hat{\sigma} : (\nabla \varphi \odot \nabla (u_3 - w_3)) \, dx' - \int_\omega (u_3 - w_3) \hat{\sigma} : \nabla^2 \varphi \, dx' \\ &= - \int_\omega \varphi (u_3 - w_3) \, d(\mathrm{div}_{x'} \mathrm{div}_{x'} \hat{\sigma}) + \langle b_0(\hat{\sigma}), \varphi (u_3 - w_3) \rangle - \int_{\gamma_n} b_1(\hat{\sigma}) \frac{\partial (\varphi (u_3 - w_3))}{\partial \nu_{\partial \omega}} \, d\mathcal{H}^1 \end{aligned} \tag{3.6.10}$$

*for every* $\varphi \in C^2(\overline{\omega})$, *where* $\langle \cdot, \cdot \rangle$ *denotes the duality pairing between* $(T_{\partial\omega}(W^{2,1}(\omega)))'$ *and* $T_{\partial\omega}(W^{2,1}(\omega))$.

**Remark 3.6.7.** The duality product $\langle b_0(\hat{\sigma}), \varphi(u_3 - w_3)\rangle$ in (3.6.10) is well defined, since one can show that $T_{\partial\omega}(BH(\omega)) = T_{\partial\omega}(W^{2,1}(\omega))$ (see, e.g., [23, Section 2]).

We are now in a position to introduce a duality pairing between $\Sigma(\Omega)$ and $\Pi_{\Gamma_d}(\Omega)$. For every $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\Gamma_d}(\Omega)$ we define the measure $[\sigma : p] \in M_b(\Omega \cup \Gamma_d)$ as

$$[\sigma : p] := [\bar{\sigma} : \bar{p}] \otimes \mathcal{L}^1 + \tfrac{1}{12}[\hat{\sigma} : \hat{p}] \otimes \mathcal{L}^1 - \sigma_\perp : e_\perp. \tag{3.6.11}$$

By Remarks 3.6.1 and 3.6.5 we have that

$$\int_{\Omega\cup\Gamma_d} \varphi\, d[\sigma : p] = \int_\omega \varphi\bar{\sigma} : d\bar{p} + \tfrac{1}{12}\int_\omega \varphi\hat{\sigma} : d\hat{p} - \int_\Omega \varphi\sigma_\perp : e_\perp\, dx \tag{3.6.12}$$

for every $\sigma \in \Sigma(\Omega)$ with $\bar{\sigma}, \hat{\sigma} \in C(\overline{\omega}; \mathbb{M}^{2\times 2}_{sym})$ and every $\varphi \in C(\overline{\omega})$. In particular, this implies that

$$\int_{\Omega\cup\Gamma_d} \varphi\, d[\sigma : p] = \int_\Omega \varphi\sigma : dp \tag{3.6.13}$$

for every $\sigma \in \Sigma(\Omega) \cap C(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ and every $\varphi \in C(\overline{\omega})$.

Following [15], for every $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\Gamma_d}(\Omega)$ we consider the duality pairings

$$\langle \bar{\sigma}, \bar{p}\rangle := [\bar{\sigma} : \bar{p}](\omega \cup \gamma_d), \qquad \langle \hat{\sigma}, \hat{p}\rangle := [\hat{\sigma} : \hat{p}](\omega \cup \gamma_d),$$

and

$$\langle \sigma, p\rangle := [\sigma : p](\Omega \cup \Gamma_d) = \langle \bar{\sigma}, \bar{p}\rangle + \tfrac{1}{12}\langle \hat{\sigma}, \hat{p}\rangle - \int_\Omega \sigma_\perp : e_\perp\, dx. \tag{3.6.14}$$

We shall now discuss the connection between the duality (3.6.14) and the functional $\mathcal{H}_r$ introduced in (3.4.9). To this purpose, we consider the set

$$K_r := \{\sigma \in \mathbb{M}^{2\times 2}_{sym} :\ \sigma : \xi \le H_r(\xi) \text{ for every } \xi \in \mathbb{M}^{2\times 2}_{sym}\},$$

which coincides with the subdifferential of $H_r$ at the origin. We also set

$$\mathcal{K}_r(\Omega) := \{\sigma \in L^\infty(\Omega; \mathbb{M}^{2\times 2}_{sym}) :\ \sigma(x) \in K_r \text{ for a.e. } x \in \Omega\}.$$

By (1.4.1) we have that for every $\mu \in M_b(\Omega \cup \Gamma_d; \mathbb{M}^{2\times 2}_{sym})$

$$\mathcal{H}_r(\mu) = \sup\Big\{ \int_{\Omega\cup\Gamma_d} \tau : d\mu :\ \tau \in C_0(\Omega \cup \Gamma_d; \mathbb{M}^{2\times 2}_{sym}) \cap \mathcal{K}_r(\Omega)\Big\}.$$

A variant of this equality can be proved using the duality defined in (3.6.14).

**Proposition 3.6.8.** *Let* $p \in \Pi_{\Gamma_d}(\Omega)$. *Then the following equalities hold:*

$$\begin{aligned} \mathcal{H}_r(p) &= \sup\{\langle \sigma, p\rangle :\ \sigma \in \Sigma(\Omega) \cap \mathcal{K}_r(\Omega)\} && (3.6.15)\\ &= \sup\{\langle \sigma, p\rangle :\ \sigma \in \Theta(\Omega)\}, && (3.6.16)\end{aligned}$$

*where* $\Theta(\Omega)$ *is the set of all* $\sigma \in \Sigma(\Omega) \cap \mathcal{K}_r(\Omega)$ *such that* $[\bar{\sigma}\nu_{\partial\omega}] = 0$ *on* $\gamma_n$, $b_1(\hat{\sigma}) = 0$ *on* $\gamma_n$, *and* $\langle b_0(\hat{\sigma}), v\rangle = 0$ *for every* $v \in W^{2,1}(\omega)$ *with* $v = 0$ *on* $\gamma_d$.

*Proof.* Let us set $\Gamma_0 := \Gamma_n \cup \big(\omega \times \big\{ \pm \frac{1}{2} \big\}\big)$. By [60, Chapter II, Section 4] and (3.6.13) we have that

$$\begin{aligned} \mathcal{H}_r(p) &= \sup\Big\{ \int_{\Omega\cup\Gamma_d} \sigma : dp \,:\, \sigma \in C^\infty(\mathbb{R}^3;\mathbb{M}^{2\times 2}_{sym}) \cap \mathcal{K}_r(\Omega),\ \operatorname{supp}\sigma \cap \Gamma_0 = \varnothing \Big\} \\ &\le \sup\{\langle \sigma, p\rangle :\ \sigma \in \Theta(\Omega)\} \\ &\le \sup\{\langle \sigma, p\rangle :\ \sigma \in \Sigma(\Omega)\cap\mathcal{K}_r(\Omega)\}. \end{aligned} \tag{3.6.17}$$

To prove the converse inequality, let $w \in W^{1,2}(\Omega;\mathbb{R}^3)\cap KL(\Omega)$, $u \in KL(\Omega)$, and $e \in L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$ be such that $(u,e,p) \in \mathcal{A}_{KL}(w)$. By Theorem 3.3.9 and Remark 3.3.10 we can construct a sequence of triples $(u^\varepsilon, e^\varepsilon, p^\varepsilon) \in (W^{1,2}(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})\times L^2_{\infty,c}(\Omega;\mathbb{M}^{2\times 2}_{sym}))\cap \mathcal{A}_{KL}(w)$ such that

$$u^\varepsilon \rightharpoonup u \quad \text{weakly* in } BD(\Omega), \tag{3.6.18}$$

$$e^\varepsilon \to e \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}), \tag{3.6.19}$$

$$\mathcal{H}_r(p^\varepsilon) \to \mathcal{H}_r(p). \tag{3.6.20}$$

By Remark 3.3.11 we can also assume that

$$\bar u^\varepsilon \to \bar u \quad \text{strongly in } L^2(\omega;\mathbb{R}^2), \quad \|E\bar u^\varepsilon\|_{L^1} \to \|E\bar u\|_{M_b}, \tag{3.6.21}$$

$$u_3^\varepsilon \to u_3 \quad \text{in } C(\overline{\omega}), \quad \|D^2 u_3^\varepsilon\|_{L^1} \to \|D^2 u_3\|_{M_b}. \tag{3.6.22}$$

Let now $\sigma \in \mathcal{K}_r(\Omega)\cap\Sigma(\Omega)$. It is clear that

$$\int_\Omega \sigma : p^\varepsilon \, dx \le \mathcal{H}_r(p^\varepsilon). \tag{3.6.23}$$

We now claim that

$$\int_\Omega \sigma : p^\varepsilon \, dx \to \langle \sigma, p\rangle. \tag{3.6.24}$$

If the claim is proved, then passing to the limit in (3.6.23) and applying (3.6.20) yield

$$\langle \sigma, p\rangle \le \mathcal{H}_r(p),$$

which, together with (3.6.17), implies the thesis.

We now prove (3.6.24). Since $\bar u^\varepsilon \in W^{1,2}(\omega;\mathbb{R}^2)$ and $E\bar u^\varepsilon = \bar e^\varepsilon + \bar p^\varepsilon$ in $\omega$, the following equalities hold:

$$\begin{aligned} \int_\omega \bar\sigma : \bar p^\varepsilon \, dx' &= -\int_\omega \bar\sigma : (\bar e^\varepsilon - E\bar w)\, dx' + \int_\omega \bar\sigma : (E\bar u^\varepsilon - E\bar w)\, dx' \\ &= -\int_\omega \bar\sigma : (\bar e^\varepsilon - E\bar w)\, dx' - \int_\omega \operatorname{div}_{x'}\bar\sigma \cdot (\bar u^\varepsilon - \bar w)\, dx' + \int_{\gamma_n} [\bar\sigma \nu_{\partial\omega}]\cdot(\bar u^\varepsilon - \bar w)\, d\mathcal{H}^1, \end{aligned}$$

where we have used (3.6.1) and the fact that $\bar u^\varepsilon = \bar w$ on $\gamma_d$. From (3.6.21) it follows that $\bar u^\varepsilon \to \bar u$ strongly in $L^1(\partial\omega;\mathbb{R}^2)$ (see, e.g., [60, Chapter II, Theorem 3.1]). By (3.6.19) and (3.6.21) we can therefore pass to the limit in the identity above and by (3.6.5) we deduce that

$$\int_\omega \bar\sigma : \bar p^\varepsilon \, dx' \to \langle \bar\sigma, \bar p\rangle. \tag{3.6.25}$$

Similarly, since $u_3^\varepsilon \in W^{2,2}(\omega)$ and $D^2u_3^\varepsilon = -(\hat{e}^\varepsilon + \hat{p}^\varepsilon)$ in $\omega$, we have

$$\begin{aligned}\int_\omega \hat{\sigma} : \hat{p}^\varepsilon \, dx' &= -\int_\omega \hat{\sigma} : (\hat{e}^\varepsilon + D^2 w_3)\, dx' - \int_\omega \hat{\sigma} : (D^2 u_3^\varepsilon - D^2 w_3)\, dx' \\ &= -\int_\omega \hat{\sigma} : (\hat{e}^\varepsilon + D^2 w_3)\, dx' - \int_\omega (u_3^\varepsilon - w_3)\, d(\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma}) \\ &\quad + \langle b_0(\hat{\sigma}),\, u_3^\varepsilon - w_3\rangle - \int_{\gamma_n} b_1(\hat{\sigma}) \frac{\partial(u_3^\varepsilon - w_3)}{\partial \nu_{\partial\omega}}\, d\mathcal{H}^1,\end{aligned}$$

where we have used (3.6.6) and the fact that $\nabla u_3^\varepsilon = \nabla w_3$ on $\gamma_d$. By (3.6.22) and [22, Theorem 3.4] we can pass to the limit in the boundary terms. Therefore, by (3.6.19), (3.6.22), and (3.6.10), we conclude that

$$\int_\omega \hat{\sigma} : \hat{p}^\varepsilon \, dx' \to \langle \hat{\sigma},\, \hat{p}\rangle. \tag{3.6.26}$$

Claim (3.6.24) follows now by combining the identity

$$\int_\Omega \sigma : p^\varepsilon \, dx = \int_\omega \bar{\sigma} : \bar{p}^\varepsilon \, dx' + \tfrac{1}{12}\int_\omega \hat{\sigma} : \hat{p}^\varepsilon \, dx' - \int_\Omega \sigma_\perp : e_\perp^\varepsilon \, dx$$

with (3.6.14) and the convergence properties (3.6.19), (3.6.25), and (3.6.26). □

We are now in a position to show a further equivalent characterization of the minimality condition $(\mathrm{qs1})_r$.

**Proposition 3.6.9.** *Let $\sigma \in L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$. The following conditions are equivalent:*

*(a)* $-\mathcal{H}_r(q) \le \int_\Omega \sigma : f \, dx$ *for every* $(v, f, q) \in \mathcal{A}_{KL}(0)$,

*(b)* $\sigma \in \Theta(\Omega)$, $\mathrm{div}_{x'}\bar{\sigma} = 0$ *in* $\omega$, *and* $\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma} = 0$ *in* $\omega$.

*Proof.* Assume (a). Let $B \subset \Omega$ be a Borel set and let $\chi_B$ denote its characteristic function. Let $\xi \in \mathbb{M}^{2\times 2}_{sym}$ and let $f := \chi_B \xi$. Since $(0, -f, f) \in \mathcal{A}_{KL}(0)$, by (a) we obtain

$$\sigma(x) : \xi \le H_r(\xi) \quad \text{for a.e. } x \in B.$$

Since $B$ is arbitrary, we deduce that $\sigma \in \mathcal{K}_r(\Omega)$.

We observe that $(\pm v, \pm Ev, 0) \in \mathcal{A}_{KL}(0)$ for every $v \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$ such that $v = 0$ on $\Gamma_d$. Hence, by (a) we have that

$$\int_\Omega \sigma : Ev \, dx = 0 \tag{3.6.27}$$

for every $v \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$ with $v = 0$ on $\Gamma_d$. Let now $\bar{v} \in W^{1,2}(\omega; \mathbb{R}^2)$ with $\bar{v} = 0$ on $\gamma_d$. Choosing $v_\alpha = \bar{v}_\alpha$ for $\alpha = 1, 2$ and $v_3 = 0$, we deduce by (3.6.27) that

$$\int_\omega \bar{\sigma} : E\bar{v} \, dx' = 0 \tag{3.6.28}$$

for every $\bar{v} \in W^{1,2}(\omega; \mathbb{R}^2)$ with $\bar{v} = 0$ on $\gamma_d$. Since this is true, in particular, for $\bar{v} \in C_c^\infty(\omega; \mathbb{R}^2)$, we conclude that $\mathrm{div}_{x'}\bar{\sigma} = 0$ in $\omega$. Moreover, by (3.6.1), (3.6.28), and the subsequent Lemma 3.6.10, we obtain that $[\bar{\sigma}\nu_{\partial\omega}] = 0$ on $\gamma_n$.

Let us now consider the function

$$v(x) = \begin{pmatrix} -x_3 \nabla v_3(x') \\ v_3(x') \end{pmatrix} \quad \text{for a.e. } x \in \Omega,$$

where $v_3 \in W^{2,2}(\omega)$ is such that $v_3 = 0$ and $\nabla v_3 = 0$ on $\gamma_d$. Equation (3.6.27) yields

$$\int_\omega \hat{\sigma} : D^2 v_3 \, dx' = 0 \tag{3.6.29}$$

for every $v_3 \in W^{2,2}(\omega)$ with $v_3 = 0$ and $\nabla v_3 = 0$ on $\gamma_d$. Since (3.6.29) is satisfied, in particular, for every $v_3 \in C_c^\infty(\omega)$, we deduce that $\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma} = 0$ in $\omega$. Moreover, by (3.6.6), (3.6.29), and Lemma 3.6.10, we obtain that

$$-\langle b_0(\hat{\sigma}), v_3 \rangle + \int_{\gamma_n} b_1(\hat{\sigma}) \frac{\partial v_3}{\partial \nu_{\partial\omega}} \, d\mathcal{H}^1 = 0$$

for every $v_3 \in W^{2,1}(\omega)$ such that $v_3 = 0$ and $\nabla v_3 = 0$ on $\gamma_d$. By [23, Théorème 1] the trace operator from $W^{2,1}(\omega)$ into $T_{\partial\omega}(W^{2,1}(\omega)) \times L^1(\partial\omega)$ that associates to $u$ the traces of $u$ and of $\frac{\partial u}{\partial \nu_{\partial\omega}}$ on $\partial\omega$ is surjective. We deduce that $b_1(\hat{\sigma}) = 0$ on $\gamma_n$ and $\langle b_0(\hat{\sigma}), v_3 \rangle = 0$ for every $v_3 \in W^{2,1}(\omega)$ with $v_3 = 0$ on $\gamma_d$, hence $\sigma \in \Theta(\Omega)$. This concludes the proof of (b).

Assume now (b). Choosing $\varphi \equiv 1$ in (3.6.5) and (3.6.10) yields

$$\langle \bar{\sigma}, \bar{q} \rangle = -\int_\omega \bar{\sigma} : \bar{f} \, dx', \qquad \langle \hat{\sigma}, \hat{q} \rangle = -\int_\omega \hat{\sigma} : \hat{f} \, dx'$$

for every $(v, f, q) \in \mathcal{A}_{KL}(0)$. Therefore, by (3.6.14)

$$\langle \sigma, q \rangle = -\int_\Omega \sigma : f \, dx.$$

Condition (a) follows now from Proposition 3.6.8. ☐

We conclude this subsection with an approximation lemma, that was needed in the proof of Proposition 3.6.9.

**Lemma 3.6.10.** (i) *Let $\bar{v} \in W^{1,1}(\omega;\mathbb{R}^2)$ with $\bar{v} = 0$ on $\gamma_d$. Then there exists a sequence $(\bar{v}^\varepsilon) \subset W^{1,2}(\omega;\mathbb{R}^2)$ such that $\bar{v}^\varepsilon = 0$ on $\gamma_d$ for every $\varepsilon > 0$ and $\bar{v}^\varepsilon \to \bar{v}$ strongly in $W^{1,1}(\omega;\mathbb{R}^2)$.*
(ii) *Let $v \in W^{2,1}(\omega)$ with $v = 0$ and $\nabla v = 0$ on $\gamma_d$. Then there exists a sequence $(v^\varepsilon) \subset W^{2,2}(\omega)$ such that $v^\varepsilon = 0$ and $\nabla v^\varepsilon = 0$ on $\gamma_d$, and $v^\varepsilon \to v$ strongly in $W^{2,1}(\omega)$.*

*Proof.* We only sketch the proof of (i). Statement (ii) can be proved by similar arguments.

Arguing as in Step 1 of the proof of Theorem 3.3.9, we can reduce, without loss of generality, to the case where there exists an open set $J \subset \partial\omega$ such that $\gamma_d$ is compactly contained in $J$ and $\bar{v} = 0$ on $J$. As in Step 2 of the proof of Theorem 3.3.9 we consider the open covering $\{Q_i\}_{i=0,\dots,m}$ of $\overline{\omega}$, a subordinate partition of unity $\{\varphi_i\}_{i=0,\dots,m}$, and the outward and inward translations $\tau_{i,\varepsilon}$ with $a_\varepsilon = \varepsilon$. We set

$$\tilde{\omega} := \omega \cup \bigcup_{i=1}^{m_0} Q_i$$

and we extend $\bar{v}$ to $\tilde{\omega}$ by setting $\bar{v} = 0$ outside $\overline{\omega}$, so that $\bar{v} \in W^{1,1}(\tilde{\omega};\mathbb{R}^2)$. We define

$$\bar{v}^\varepsilon := \Big( \sum_{i=1}^m (\varphi_i \bar{v}) \circ \tau_{i,\varepsilon} + \varphi_0 \bar{v} \Big) * \rho_{\delta(\varepsilon)},$$

where $\rho_{\delta(\varepsilon)}$ is a mollifier and $\delta(\varepsilon) < \varepsilon$ is chosen small enough in such a way that $\bar{v}^\varepsilon = 0$ on $\gamma_d$. It is now easy to check that the sequence $(\bar{v}^\varepsilon)$ has all the required properties. □

### 3.6.2 Equivalent formulations in rate form

From here to the end of the section we will assume $t \mapsto w(t)$ to be absolutely continuous from $[0,T]$ into $W^{1,2}(\Omega;\mathbb{R}^3) \cap KL(\Omega)$. This implies that the maps $t \mapsto \bar{w}(t)$ and $t \mapsto w_3(t)$ are absolutely continuous from $[0,T]$ into $W^{1,2}(\omega;\mathbb{R}^2)$ and $W^{2,2}(\omega)$, respectively.

We first prove some preliminary results. An easy adaptation of [15, Lemma 5.5] provides us with the following lemma.

**Lemma 3.6.11.** *Let $t \mapsto (u(t), e(t), p(t))$ be an absolutely continuous function from $[0,T]$ into $BD(\Omega) \times L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}) \times M_b(\Omega \cup \Gamma_d;\mathbb{M}^{2\times 2}_{sym})$ with $(u(t), e(t), p(t)) \in \mathcal{A}_{KL}(w(t))$ for every $t \in [0,T]$. Then $(\dot{u}(t), \dot{e}(t), \dot{p}(t)) \in \mathcal{A}_{KL}(\dot{w}(t))$ for a.e. $t \in [0,T]$.*

For absolutely continuous triples the energy balance can be equivalently written as a balance of powers, as shown in the next proposition.

**Proposition 3.6.12.** *Let $t \mapsto (u(t), e(t), p(t))$ be an absolutely continuous function from $[0,T]$ into $BD(\Omega) \times L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}) \times M_b(\Omega \cup \Gamma_d;\mathbb{M}^{2\times 2}_{sym})$ and let $\sigma(t) := \mathbb{C}_r e(t)$. Then, the following conditions are equivalent:*

*(a) for every $t \in [0,T]$*

$$\mathcal{Q}_r(e(t)) + \mathcal{D}_r(p;0,t) = \mathcal{Q}_r(e(0)) + \int_0^t \int_\Omega \sigma(s) : E\dot{w}(s)\,dxds;$$

*(b) for a.e. $t \in [0,T]$*

$$\int_\Omega \sigma(t) : \dot{e}(t)\,dx + \mathcal{H}_r(\dot{p}(t)) = \int_\Omega \sigma(t) : E\dot{w}(t)\,dx.$$

*Proof.* Since $t \mapsto p(t)$ is absolutely continuous, by [15, Theorem 7.1] we have

$$\mathcal{D}_r(p;0,t) = \int_0^t \mathcal{H}_r(\dot{p}(s))\,ds.$$

The equivalence of (a) and (b) follows now by differentiation of (a) and integration of (b). □

We are finally in a position to state the main result of this section.

**Theorem 3.6.13.** *Let $t \mapsto (u(t), e(t), p(t))$ be a function from $[0,T]$ into $BD(\Omega) \times L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}) \times M_b(\Omega \cup \Gamma_d;\mathbb{M}^{2\times 2}_{sym})$ and let $\sigma(t) := \mathbb{C}_r e(t)$. Then the following conditions are equivalent:*

*(a) $t \mapsto (u(t), e(t), p(t))$ is a reduced quasistatic evolution for the boundary datum $w(t)$;*

*(b)* $t \mapsto (u(t), e(t), p(t))$ *is absolutely continuous and*

*(b1) for every* $t \in [0,T]$ *we have* $(u(t), e(t), p(t)) \in \mathcal{A}_{KL}(w(t))$, $\sigma(t) \in \Theta(\Omega)$, $\mathrm{div}_{x'}\bar{\sigma}(t) = 0$ *in* $\omega$, *and* $\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma}(t) = 0$ *in* $\omega$,

*(b2) for a.e.* $t \in [0,T]$ *there holds*

$$\mathcal{H}_r(\dot{p}(t)) = \langle \sigma(t), \dot{p}(t) \rangle = \langle \bar{\sigma}(t), \dot{\bar{p}}(t) \rangle + \tfrac{1}{12} \langle \hat{\sigma}(t), \dot{\hat{p}}(t) \rangle - \int_\Omega \sigma_\perp(t) : \dot{e}_\perp(t);$$

*(c)* $t \mapsto (u(t), e(t), p(t))$ *is absolutely continuous and*

*(c1) for every* $t \in [0,T]$ *we have* $(u(t), e(t), p(t)) \in \mathcal{A}_{KL}(w(t))$, $\sigma(t) \in \Theta(\Omega)$, $\mathrm{div}_{x'}\bar{\sigma}(t) = 0$ *in* $\omega$, *and* $\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma}(t) = 0$ *in* $\omega$,

*(c2) for a.e.* $t \in [0,T]$ *and for every* $\tau \in \Theta(\Omega)$ *there holds*

$$\langle \sigma(t) - \tau, \dot{p}(t) \rangle \geq 0;$$

*(d)* $t \mapsto (u(t), e(t))$ *is absolutely continuous and*

*(d1) for every* $t \in [0,T]$ *we have* $\sigma(t) \in \Theta(\Omega)$, $\mathrm{div}_{x'}\bar{\sigma}(t) = 0$ *in* $\omega$, *and* $\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma}(t) = 0$ *in* $\omega$,

*(d2) for a.e.* $t \in [0,T]$ *and for every* $\tau \in \Theta(\Omega)$ *there holds*

$$\begin{aligned} &\int_\Omega (\tau - \sigma(t)) : \dot{e}(t)\, dx + \int_\omega \mathrm{div}_{x'}\bar{\tau} \cdot \dot{\bar{u}}(t)\, dx' + \tfrac{1}{12} \int_\omega \dot{u}_3(t)\, d(\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\tau}) \\ &\geq \int_{\gamma_d} [(\bar{\tau} - \bar{\sigma})\nu_{\partial\omega}] \cdot \dot{\bar{w}}(t)\, d\mathcal{H}^1 + \tfrac{1}{12} \langle b_0(\hat{\tau} - \hat{\sigma}(t)),\, \dot{w}_3(t) \rangle - \tfrac{1}{12} \int_{\gamma_d} b_1(\hat{\tau} - \hat{\sigma}(t)) \tfrac{\partial \dot{w}_3(t)}{\partial \nu_{\partial\omega}}\, d\mathcal{H}^1, \end{aligned}$$

*(d3) for every* $t \in [0,T]$, $p(t) = Eu(t) - e(t)$ *on* $\Omega$ *and* $p(t) = (w(t) - u(t)) \odot \nu_{\partial\Omega} \mathcal{H}^2$ *on* $\Gamma_d$.

**Remark 3.6.14.** The duality products $\langle \sigma(t), \dot{p}(t) \rangle$ and $\langle \sigma(t) - \tau, \dot{p}(t) \rangle$ in conditions (b) and (c) are well defined since $\dot{p}(t) \in \Pi_{\Gamma_d}(\Omega)$ by Lemma 3.6.11.

*Proof of Theorem 3.6.13.* We first show that (a) is equivalent to (b). By Remark 3.5.7 every reduced quasistatic evolution is absolutely continuous, while Proposition 3.6.9 and Lemma 3.5.8 yield the equivalence of $(\mathrm{qs1})_r$ and (b1). Hence, by Proposition 3.6.12 it is enough to show that for every absolutely continuous function satisfying either (b1) or $(\mathrm{qs1})_r$, (b2) is equivalent to the following condition: for a.e. $t \in [0,T]$

$$\int_\Omega \sigma(t) : \dot{e}(t)\, dx + \mathcal{H}_r(\dot{p}(t)) = \int_\Omega \sigma(t) : E\dot{w}(t)\, dx.$$

This follows from Propositions 3.6.2 and 3.6.6, once we notice that $(\dot{u}(t), \dot{e}(t), \dot{p}(t)) \in \mathcal{A}_{KL}(\dot{w}(t))$ by Lemma 3.6.11.

To show that *(b)* and *(c)* are equivalent, it is enough to prove that, if (b1) holds, then (b2) is equivalent to (c2). Indeed, condition (c2) is equivalent to

$$\langle \sigma(t), \dot{p}(t) \rangle \geq \sup_{\tau \in \Theta(\Omega)} \langle \tau, \dot{p}(t) \rangle.$$

On the other hand, by (b1) there holds

$$\langle \sigma(t),\, \dot p(t)\rangle \le \sup_{\tau\in\Theta(\Omega)} \langle \tau,\, \dot p(t)\rangle.$$

By Proposition 3.6.8 we deduce the thesis.

To conclude the proof of the theorem, we show that (c) is equivalent to (d). We first remark that if $t \mapsto (u(t), e(t))$ is absolutely continuous and (d3) holds, then $t \mapsto p(t)$ is absolutely continuous and $(u(t), e(t), p(t)) \in \mathcal{A}_{KL}(w(t))$ for every $t \in [0,T]$. Hence, it remains only to prove that, if (c1) holds, then (c2) is equivalent to (d2). By Propositions 3.6.2 and 3.6.6 there holds

$$\langle \sigma(t)-\tau,\, \dot p(t)\rangle = \int_\Omega (\tau-\sigma(t)) : (\dot e(t) - E\dot w(t))\, dx \\ + \int_\omega \mathrm{div}_{x'}\bar\tau \cdot (\dot{\bar u} - \dot{\bar w})\, dx' + \tfrac{1}{12}\int_\omega (\dot u_3 - \dot w_3)\, d(\mathrm{div}_{x'}\mathrm{div}_{x'}\hat\tau),$$

therefore (c2) is equivalent to

$$\int_\Omega (\tau-\sigma(t)) : (\dot e(t) - E\dot w(t))\, dx \\ + \int_\omega \mathrm{div}_{x'}\bar\tau \cdot (\dot{\bar u} - \dot{\bar w})\, dx' + \tfrac{1}{12}\int_\omega (\dot u_3 - \dot w_3)\, d(\mathrm{div}_{x'}\mathrm{div}_{x'}\hat\tau) \ge 0 \quad (3.6.30)$$

for a.e. $t\in[0,T]$ and every $\tau\in\Theta(\Omega)$. By (c1), (3.6.1), and (3.6.6) we deduce that

$$\int_\omega (\bar\tau - \bar\sigma(t)) : E\dot{\bar w}(t)\, dx' = \int_{\gamma_d} [(\bar\tau - \bar\sigma(t))\nu_{\partial\omega}] \cdot \dot{\bar w}(t)\, d\mathcal{H}^1 - \int_\omega \mathrm{div}_{x'}\bar\tau \cdot \dot{\bar w}(t)\, dx',$$

and

$$\int_\omega (\hat\tau - \hat\sigma(t)) : D^2\dot w_3(t)\, dx' \\ = -\langle b_0(\hat\tau - \hat\sigma(t)),\, \dot w_3(t)\rangle + \int_{\gamma_d} b_1(\hat\tau - \hat\sigma(t)) \frac{\partial \dot w_3(t)}{\partial\nu_{\partial\omega}}\, d\mathcal{H}^1 + \int_\omega \dot w_3\, d(\mathrm{div}_{x'}\mathrm{div}_{x'}\hat\tau).$$

Therefore, (3.6.30) is in turn equivalent to (d2) and the proof of the theorem is complete. □

### 3.6.3 Two-dimensional characterizations

In this subsection we show that, under some additional hypotheses on the boundary datum and the initial data, a reduced quasistatic evolution can be written in terms of two-dimensional quantities only. The first proposition concerns a quasistatic evolution $(u(t), e(t), p(t))$ with "in-plane" boundary datum and initial data. In this case, the triple given by the tangential component of $u(t)$ and the zero-th order moments of $e(t)$ and $p(t)$ is a two-dimensional quasistatic evolution in $\omega$ in the sense of [15]. It is convenient to introduce the following notation: for every $\bar w \in W^{1,2}(\omega;\mathbb{R}^2)$ we denote by $\bar{\mathcal{A}}_{KL}(\bar w)$ the class of all triples $(v, f, q)$ in $BD(\omega)\times L^2(\omega;\mathbb{M}^{2\times2}_{sym}) \times M_b(\omega\cup\gamma_d;\mathbb{M}^{2\times2}_{sym})$ such that $Ev = f+q$ in $\omega$ and $q = (\bar w - v)\odot\nu_{\partial\omega}\mathcal{H}^1$ on $\gamma_d$. Moreover, we introduce the space

$$\bar\Sigma(\omega) := \left\{\sigma \in L^\infty(\omega;\mathbb{M}^{2\times2}_{sym}) : \mathrm{div}_{x'}\sigma \in L^2(\omega;\mathbb{M}^{2\times2}_{sym})\right\}$$

and the set

$$\mathcal{K}_r(\omega) := \big\{\sigma \in L^\infty(\omega;\mathbb{M}^{2\times 2}_{sym}) : \sigma(x') \in K_r \text{ for a.e. } x' \in \omega\big\}.$$

**Proposition 3.6.15.** *Let $t \mapsto \bar{w}(t)$ be absolutely continuous from $[0,T]$ into $W^{1,2}(\omega;\mathbb{R}^2)$ and let*

$$w(t,x) := \begin{pmatrix} \bar{w}(t,x') \\ 0 \end{pmatrix} \quad \text{for every } t \in [0,T] \text{ and a.e. } x \in \Omega.$$

*Let $(\bar{u}_0, \bar{e}_0, \bar{p}_0) \in \bar{\mathcal{A}}_{KL}(\bar{w}(0))$ and let*

$$u_0(x) := \begin{pmatrix} \bar{u}_0(x') \\ 0 \end{pmatrix}, \quad e_0(x) := \bar{e}_0(x') \quad \text{for a.e. } x \in \Omega, \quad p_0 := \bar{p}_0 \otimes \mathcal{L}^1.$$

*Finally, let $t \mapsto (u(t), e(t), p(t))$ be a reduced quasistatic evolution for the boundary value $w(t)$ such that $u(0) = u_0$, $e(0) = e_0$, and $p(0) = p_0$, and let $\sigma(t) := \mathbb{C}_r e(t)$. Then the map $t \mapsto (\bar{u}(t), \bar{e}(t), \bar{p}(t))$ satisfies the following conditions:*

*(i) $t \mapsto (\bar{u}(t), \bar{e}(t), \bar{p}(t))$ is absolutely continuous from $[0,T]$ into $BD(\omega) \times L^2(\omega;\mathbb{M}^{2\times 2}_{sym}) \times M_b(\omega \cup \gamma_d;\mathbb{M}^{2\times 2}_{sym})$ and $\bar{u}(0) = \bar{u}_0$, $\bar{e}(0) = \bar{e}_0$, and $\bar{p}(0) = \bar{p}_0$;*

*(ii) for every $t \in [0,T]$ we have $(\bar{u}(t), \bar{e}(t), \bar{p}(t)) \in \bar{\mathcal{A}}_{KL}(\bar{w}(t))$, $\bar{\sigma}(t) \in \bar{\Sigma}(\omega) \cap \mathcal{K}_r(\omega)$, $\mathrm{div}_{x'}\bar{\sigma}(t) = 0$ in $\omega$, and $[\bar{\sigma}\nu_{\partial\omega}] = 0$ on $\gamma_n$;*

*(iii) for a.e. $t \in [0,T]$ there holds*

$$\mathcal{H}_r(\dot{\bar{p}}(t)) = \langle \bar{\sigma}(t), \dot{\bar{p}}(t) \rangle. \tag{3.6.31}$$

*Proof.* Condition (i) follows from Remark 3.5.7. By condition (b1) of Theorem 3.6.13 and the convexity of $K_r$ we deduce condition (ii).

By property (b2) of Theorem 3.6.13 and Proposition 3.6.8 we have

$$\begin{aligned} \mathcal{H}_r(\dot{p}(t)) &= \langle \bar{\sigma}(t), \dot{\bar{p}}(t) \rangle + \tfrac{1}{12}\langle \hat{\sigma}(t), \dot{\hat{p}}(t) \rangle - \int_\Omega \sigma_\perp(t) : \dot{e}_\perp(t)\,dx \\ &\le \mathcal{H}_r(\dot{\bar{p}}(t)) + \tfrac{1}{12}\langle \hat{\sigma}(t), \dot{\hat{p}}(t) \rangle - \int_\Omega \sigma_\perp(t) : \dot{e}_\perp(t)\,dx \\ &= \mathcal{H}_r(\dot{\bar{p}}(t)) - \tfrac{1}{12}\int_\omega \hat{\sigma}(t) : \dot{\hat{e}}(t)\,dx - \int_\Omega \sigma_\perp(t) : \dot{e}_\perp(t)\,dx, \end{aligned} \tag{3.6.32}$$

where the last inequality follows from (3.6.10) with $\varphi \equiv 1$ and from the fact that $\sigma(t) \in \Theta(\Omega)$ and $w_3(t) = 0$ for every $t \in [0,T]$. On the other hand, setting

$$\lambda(t) := |\dot{\bar{p}}(t)| + |\dot{\hat{p}}(t)| + \mathcal{L}^2$$

for a.e. $t \in [0,T]$, we have that the measure $\dot{\bar{p}}(t) + x_3\dot{\hat{p}}(t) - \dot{e}_\perp(\cdot, x_3)$ on $\omega \cup \gamma_d$ is absolutely continuous with respect to $\lambda(t)$ for a.e. $x_3 \in (-\frac{1}{2}, \frac{1}{2})$. Therefore, by Jensen inequality we obtain

$$\begin{aligned} \mathcal{H}_r(\dot{p}(t)) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\omega\cup\gamma_d} H_r\Big(\frac{d(\dot{\bar{p}}(t) + x_3\dot{\hat{p}}(t) - \dot{e}_\perp(\cdot, x_3))}{d\lambda(t)}\Big)\,d\lambda(t)dx_3 \\ &\ge \int_{\omega\cup\gamma_d} H_r\Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{d(\dot{\bar{p}}(t) + x_3\dot{\hat{p}}(t) - \dot{e}_\perp(\cdot, x_3))}{d\lambda(t)}dx_3\Big)\,d\lambda(t) \\ &= \int_{\omega\cup\gamma_d} H_r\Big(\frac{d\dot{\bar{p}}(t)}{d\lambda(t)}\Big)\,d\lambda(t) = \mathcal{H}_r(\dot{\bar{p}}(t)) \end{aligned} \tag{3.6.33}$$

for a.e. $t \in [0,T]$. Combining (3.6.32) and (3.6.33), we deduce that

$$-\frac{d}{dt}\Big(\tfrac{1}{12}\mathcal{Q}_r(\hat e(t)) + \mathcal{Q}_r(e_\perp(t))\Big) = -\tfrac{1}{12}\int_\omega \hat\sigma(t) : \dot{\hat e}(t)\,dx - \int_\Omega \sigma_\perp(t) : \dot e_\perp(t)\,dx \geq 0.$$

In particular, this implies that

$$\tfrac{1}{12}\mathcal{Q}_r(\hat e(t)) + \mathcal{Q}_r(e_\perp(t)) \leq \tfrac{1}{12}\mathcal{Q}_r(\hat e(0)) + \mathcal{Q}_r(e_\perp(0)) = 0,$$

hence $\hat e(t) = 0$ and $e_\perp(t) = 0$ for every $t \in [0,T]$. This, together with (3.6.32) and (3.6.33), yields (3.6.31). □

In this last proposition we consider a quasistatic evolution $(u(t), e(t), p(t))$ with "out-of-plane" boundary datum and initial data and we prove that the triple given by the normal component of $u(t)$ and the first order moment of $e(t)$ and $p(t)$ is a two-dimensional quasistatic evolution in $\omega$ in the sense of [24, Definition 4.1]. To this purpose, for every $w_3 \in W^{2,2}(\omega)$ we define the class $\hat{\mathcal{A}}_{KL}(w_3)$ as the set of all triples $(v,f,q) \in BH(\omega) \times L^2(\omega;\mathbb{M}^{2\times2}_{sym}) \times M_b(\omega;\mathbb{M}^{2\times2}_{sym})$ such that $D^2 v = -(f+q)$ in $\omega$, $v = w_3$ on $\gamma_d$, and $q = (\nabla v - \nabla w_3) \odot \nu_{\partial\omega}\mathcal{H}^1$ on $\gamma_d$.

**Proposition 3.6.16.** *Assume the function $H$ to be homogeneous of degree one, i.e.,*

$$H(\lambda\xi) = |\lambda| H(\xi) \quad \text{for every } \lambda \in \mathbb{R},\ \xi \in \mathbb{M}^{3\times3}_{sym}. \tag{3.6.34}$$

*Let $t \mapsto w_3(t)$ be absolutely continuous from $[0,T]$ into $W^{2,2}(\omega)$ and let*

$$w(t,x) := \begin{pmatrix} -x_3 \nabla w_3(t,x') \\ w_3(t,x') \end{pmatrix} \quad \text{for every } t \in [0,T] \text{ and a.e. } x \in \Omega.$$

*Let $(v_0, \hat e_0, \hat p_0) \in \hat{\mathcal{A}}_{KL}(w_3(0))$ and let*

$$u_0(x) := \begin{pmatrix} -x_3 \nabla v_0(x') \\ v_0(x') \end{pmatrix}, \quad e_0(x) := x_3 \hat e_0(x') \quad \text{for a.e. } x \in \Omega, \quad p_0 := x_3 \hat p_0 \otimes \mathcal{L}^1.$$

*Finally, let $t \mapsto (u(t), e(t), p(t))$ be a reduced quasistatic evolution for the boundary value $w(t)$ such that $u(0) = u_0$, $e(0) = e_0$, and $p(0) = p_0$, and let $\sigma(t) := \mathbb{C}_r e(t)$. Then the map $t \mapsto (u_3(t), \hat e(t), \hat p(t))$ satisfies the following conditions:*

*(i)* $t \mapsto (u_3(t), \hat e(t), \hat p(t))$ *is absolutely continuous from $[0,T]$ into $BH(\omega) \times L^2(\omega;\mathbb{M}^{2\times2}_{sym}) \times M_b(\omega \cup \gamma_d;\mathbb{M}^{2\times2}_{sym})$ and $u_3(0) = v_0$, $\hat e(0) = \hat e_0$, and $\hat p(0) = \hat p_0$;*

*(ii)* *for every $t \in [0,T]$ we have $(u_3(t), \hat e(t), \hat p(t)) \in \hat{\mathcal{A}}_{KL}(w_3(t))$, $\hat\sigma(t) \in \hat\Sigma(\omega) \cap \mathcal{K}_r(\omega)$, $\mathrm{div}_{x'}\mathrm{div}_{x'}\hat\sigma(t) = 0$ in $\omega$, $b_1(\hat\sigma(t)) = 0$ on $\gamma_n$, and $\langle b_0(\hat\sigma(t)), v\rangle = 0$ for every $v \in W^{2,1}(\omega)$ with $v = 0$ on $\gamma_d$;*

*(iii)* *for a.e. $t \in [0,T]$ there holds*

$$\mathcal{H}_r(\dot{\hat p}(t)) = \langle \hat\sigma(t), \dot{\hat p}(t)\rangle. \tag{3.6.35}$$

*Proof.* We first remark that (3.6.34) implies that the same property is fullfilled by $H_r$. This latter condition is in turn equivalent to saying that the set $K_r$ is symmetric with respect to the origin.

Condition (i) follows from Remark 3.5.7. By property (b1) of Theorem 3.6.13 we have that $\sigma(t) \in \mathcal{K}_r(\Omega)$ for every $t \in [0,T]$. Since $K_r$ is convex and symmetric with respect with the origin, this implies that $\hat\sigma(t) \in \mathcal{K}_r(\omega)$ for every $t \in [0,T]$. All the other conditions in (ii) follow from Theorem 3.6.13.

By property (b2) of Theorem 3.6.13 and Proposition 3.6.8 we have

$$\begin{aligned}
\mathcal{H}_r(\dot p(t)) &= \langle \bar\sigma(t), \dot{\bar p}(t)\rangle + \tfrac{1}{12}\langle \hat\sigma(t), \dot{\hat p}(t)\rangle - \int_\Omega \sigma_\perp(t) : \dot e_\perp(t)\, dx \\
&\leq \tfrac{1}{12}\mathcal{H}_r(\dot{\hat p}(t)) + \langle \bar\sigma(t), \dot{\bar p}(t)\rangle - \int_\Omega \sigma_\perp(t) : \dot e_\perp(t)\, dx \\
&= \tfrac{1}{12}\mathcal{H}_r(\dot{\hat p}(t)) - \int_\omega \bar\sigma(t) : \dot{\bar e}(t)\, dx - \int_\Omega \sigma_\perp(t) : \dot e_\perp(t)\, dx, \qquad (3.6.36)
\end{aligned}$$

where the last inequality follows from (3.6.5) with $\varphi \equiv 1$ and from the fact that $\sigma(t) \in \Theta(\Omega)$ and $\bar w(t) = 0$ for every $t \in [0,T]$. On the other hand, setting

$$\lambda(t) := |\dot{\bar p}(t)| + |\dot{\hat p}(t)| + \mathcal{L}^2$$

for a.e. $t \in [0,T]$ and applying (3.6.34) and Jensen inequality, we obtain

$$\begin{aligned}
\mathcal{H}_r(\dot p(t)) &\geq \int_{-\frac12}^{\frac12}\int_{\omega\cup\gamma_d} |x_3| H_r\Big(\frac{d(\dot{\bar p}(t) + x_3\dot{\hat p}(t) - \dot e_\perp(\cdot, x_3))}{d\lambda(t)}\Big)\, d\lambda(t) dx_3 \\
&\geq \int_{\omega\cup\gamma_d} H_r\Big(\int_{-\frac12}^{\frac12} x_3 \frac{d(\dot{\bar p}(t) + x_3\dot{\hat p}(t) - \dot e_\perp(\cdot, x_3))}{d\lambda(t)} dx_3\Big)\, d\lambda(t) \\
&= \tfrac{1}{12}\mathcal{H}_r(\dot{\hat p}(t)) \qquad (3.6.37)
\end{aligned}$$

for a.e. $t \in [0,T]$. Combining (3.6.36) and (3.6.37), we deduce that

$$-\frac{d}{dt}\big(\mathcal{Q}_r(\bar e(t)) + \mathcal{Q}_r(e_\perp(t))\big) = -\int_\omega \bar\sigma(t) : \dot{\bar e}(t)\, dx - \int_\Omega \sigma_\perp(t) : \dot e_\perp(t)\, dx \geq 0.$$

In particular, this implies that

$$\mathcal{Q}_r(\bar e(t)) + \mathcal{Q}_r(e_\perp(t)) \leq \mathcal{Q}_r(\bar e(0)) + \mathcal{Q}_r(e_\perp(0)) = 0,$$

hence $\bar e(t) = 0$ and $e_\perp(t) = 0$ for every $t \in [0,T]$. This, together with (3.6.36) and (3.6.37), yields (3.6.35). □

# Chapter 4

# Linearized plastic plate models as $\Gamma$-limits of 3D finite plasticity

## 4.1 Overview of the chapter

The subject of this chapter is the rigorous derivation of reduced models for a three-dimensional plate of small thickness, whose elastic behaviour is nonlinear and whose plastic response is that of finite plasticity with hardening, by means of $\Gamma$-convergence. Denoting by $\varepsilon$ the thickness of the plate, we analyse the case where the scaling factor of the elasto-plastic energy is of order $\varepsilon^{2\alpha-2}$, with $\alpha \geq 3$. According to the value of $\alpha$, partially or fully linearized models are deduced, which correspond, in the absence of plastic deformation, to the Von Kármán plate theory and the linearized plate theory.

The chapter is organized as follows: in Section 4.2 we recall some preliminary results and we discuss the formulation of the problem. Section 4.3 is devoted to prove some compactness results and liminf inequalities, while in Section 4.4 we show that the lower bounds obtained in Section 4.3 are optimal. Finally, in Section 4.5 we deduce convergence of almost minimizers of the three-dimensional energies to minimizers of the limit functionals and we discuss some examples.

## 4.2 Preliminaries and setting of the problem

Let $\omega \subset \mathbb{R}^2$ be a connected, bounded open set with Lipschitz boundary. Let $\varepsilon > 0$. We assume the set $\Omega_\varepsilon := \omega \times \big(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\big)$ to be the reference configuration of a finite-strain elastoplastic plate.

We suppose that the boundary $\partial\omega$ is partitioned into the union of two disjoint sets $\gamma_d$ and $\gamma_n$ and their common boundary, where $\gamma_d$ is such that $\mathcal{H}^1(\gamma_d) > 0$. We denote by $\Gamma_\varepsilon$ the portion of the lateral surface of the plate given by $\Gamma_\varepsilon := \gamma_d \times \big(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\big)$. On $\Gamma_\varepsilon$ we prescribe a boundary datum of the form

$$\phi^\varepsilon(x) := \begin{pmatrix} x' \\ x_3 \end{pmatrix} + \begin{pmatrix} \varepsilon^{\alpha-1}u^0(x') \\ \varepsilon^{\alpha-2}v^0(x') \end{pmatrix} - \varepsilon^{\alpha-2}x_3\nabla v^0(x') \tag{4.2.1}$$

for $x = (x', \varepsilon x_3) \in \Omega_\varepsilon$, where $u^0 \in W^{1,\infty}(\omega;\mathbb{R}^2)$, $v^0 \in W^{2,\infty}(\omega)$ and $\alpha \geq 3$.

We assume that every deformation $\eta \in W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)$ of the plate fulfills the multiplicative decomposition

$$\nabla\eta(x) = F_{el}(x)F_{pl}(x) \quad \text{for a.e. } x \in \Omega_\varepsilon,$$

where $F_{el} \in L^2(\Omega_\varepsilon;\mathbb{M}^{3\times3})$ represents the elastic strain, $F_{pl} \in L^2(\Omega_\varepsilon;SL(3))$ is the plastic strain and $SL(3) := \{F \in \mathbb{M}^{3\times3} : \det F = 1\}$. The stored energy associated to a deformation $\eta$ and to its elastic and plastic strains can be expressed as follows:

$$\begin{aligned}\mathcal{E}(\eta, F_{pl}) &:= \int_{\Omega_\varepsilon} W_{el}(\nabla\eta(x)F_{pl}^{-1}(x))\,dx + \int_{\Omega_\varepsilon} W_{hard}(F_{pl}(x))\,dx \\ &= \int_{\Omega_\varepsilon} W_{el}(F_{el}(x))\,dx + \int_{\Omega_\varepsilon} W_{hard}(F_{pl}(x))\,dx,\end{aligned} \tag{4.2.2}$$

where $W_{el}$ is the elastic energy density and $W_{hard}$ describes hardening.

**Properties of the elastic energy density**

We assume that $W_{el} : \mathbb{M}^{3\times3} \to [0,+\infty]$ satisfies

(H1) $W_{el} \in C^1(\mathbb{M}^{3\times3}_+)$, $\quad W_{el} \equiv +\infty$ on $\mathbb{M}^{3\times3} \setminus \mathbb{M}^{3\times3}_+$,

(H2) $W_{el}(Id) = 0$,

(H3) $W_{el}(RF) = W_{el}(F)$ for every $R \in SO(3)$, $F \in \mathbb{M}^{3\times3}_+$,

(H4) $W_{el}(F) \geq c_1 \mathrm{dist}^2(F; SO(3))$ for every $F \in \mathbb{M}^{3\times3}_+$,

(H5) $|F^T DW_{el}(F)| \leq c_2(W_{el}(F) + 1)$ for every $F \in \mathbb{M}^{3\times3}_+$.

Here $c_1, c_2$ are positive constants, $\mathbb{M}^{3\times3}_+ := \{F \in \mathbb{M}^{3\times3} : \det F > 0\}$ and $SO(3) := \{F \in \mathbb{M}^{3\times3}_+ : F^TF = Id\}$. We also assume that there exists a symmetric, positive semi-definite tensor $\mathbb{C} : \mathbb{M}^{3\times3} \to \mathbb{M}^{3\times3}_{sym}$ such that, setting

$$Q(F) := \frac{1}{2}\mathbb{C}F : F \quad \text{for every } F \in \mathbb{M}^{3\times3}, \tag{4.2.3}$$

the quadratic form $Q$ encodes the local behaviour of $W_{el}$ around the identity, namely

$$\forall\delta > 0\ \exists c_{el}(\delta) > 0 \text{ such that } \forall F \in B_{c_{el}(\delta)}(0) \text{ there holds } |W_{el}(Id + F) - Q(F)| \leq \delta|F|^2. \tag{4.2.4}$$

**Remark 4.2.1.** By [17, Proposition 1.5] and by (H3) and (H5), there holds

$$|DW_{el}(F)F^T| \leq c_3(W_{el}(F) + 1) \quad \text{for every } F \in \mathbb{M}^{3\times3}_+, \tag{4.2.5}$$

where $c_3$ is a positive constant. Moreover, by (H1) and (H5), there exist $c_4$, $c_5$, $\gamma > 0$ such that, for every $G_1$, $G_2 \in B_\gamma(Id)$ and for every $F \in \mathbb{M}^{3\times3}_+$ the following estimate holds true

$$|W_{el}(G_1FG_2) - W_{el}(F)| \leq c_4(W_{el}(F) + c_5)(|G_1 - Id| + |G_2 - Id|) \tag{4.2.6}$$

(see [52, Lemma 4.1]).

**Remark 4.2.2.** As remarked in [52, Section 2], the frame-indifference condition (H3) yields

$$\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{ijlk} \text{ for every } i,j,k,l \in \{1,2,3\}$$

and

$$\mathbb{C}F = \mathbb{C}\,(\operatorname{sym} F) \quad \text{for every } F \in \mathbb{M}^{3\times 3}.$$

Hence, the quadratic form $Q$ satisfies:

$$Q(F) = Q(\operatorname{sym} F) \quad \text{for every } F \in \mathbb{M}^{3\times 3}$$

and by (H4) it is positive definite on symmetric matrices. This, in turn, implies that there exist two constants $r_{\mathbb{C}}$ and $R_{\mathbb{C}}$ such that

$$r_{\mathbb{C}}|F|^2 \leq Q(F) \leq R_{\mathbb{C}}|F|^2 \quad \text{for every } F \in \mathbb{M}^{3\times 3}_{\mathrm{sym}}, \tag{4.2.7}$$

and

$$|\mathbb{C}F| \leq 2R_{\mathbb{C}}|F| \quad \text{for every } F \in \mathbb{M}^{3\times 3}_{\mathrm{sym}}. \tag{4.2.8}$$

**Remark 4.2.3.** We note that (4.2.4) entails, in particular,

$$W_{el}(Id) = 0, \quad DW_{el}(Id) = 0$$

and

$$\mathbb{C} = D^2 W_{el}(Id), \quad \mathbb{C}_{ijkl} = \frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}}(Id) \text{ for every } i,j,k,l \in \{1,2,3\}.$$

By combining (4.2.4) with (4.2.8) we deduce also that there exists a constant $c_{el_2}$ such that

$$|DW_{el}(Id + F)| \leq (2R_{\mathbb{C}} + 1)|F| \tag{4.2.9}$$

for every $F \in \mathbb{M}^{3\times 3}$, $|F| < c_{el_2}$.

**Properties of the hardening functional**

We assume that the hardening map $W_{hard} : \mathbb{M}^{3\times 3} \to [0, +\infty]$ is of the form

$$W_{hard}(F) := \begin{cases} \widetilde{W}_{hard}(F) & \text{for every } F \in K, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.2.10}$$

Here $K$ is a compact set in $SL(3)$ that contains the identity as a relative interior point, and the map $\widetilde{W}_{hard} : \mathbb{M}^{3\times 3} \to [0, +\infty)$ fulfills

$$\begin{aligned} &\widetilde{W}_{hard} \text{ is locally Lipschitz continuous,} \\ &\widetilde{W}_{hard}(Id + F) \geq c_6|F|^2 \quad \text{for every } F \in \mathbb{M}^{3\times 3}, \end{aligned} \tag{4.2.11}$$

where $c_6$ is a positive constant. We also assume that there exists a symmetric, positive definite tensor $\mathbb{B} : \mathbb{M}^{3\times 3} \to \mathbb{M}^{3\times 3}$ such that, setting

$$B(F) := \frac{1}{2}\mathbb{B}F : F \quad \text{for every } F \in \mathbb{M}^{3\times 3},$$

the quadratic form $B$ satisfies

$$\forall\delta>0\,\exists c_h(\delta)>0 \text{ such that } \forall F\in B_{c_h(\delta)}(0) \text{ there holds } |\widetilde{W}_{hard}(Id+F)-B(F)|\le\delta B(F). \tag{4.2.12}$$

In particular, by the hypotheses on $K$ there exists a constant $c_k$ such that

$$|F|+|F^{-1}|\le c_k \quad \text{for every } F\in K, \tag{4.2.13}$$
$$|F-Id|\ge\frac{1}{c_k} \quad \text{for every } F\in SL(3)\setminus K. \tag{4.2.14}$$

Combining (4.2.11) and (4.2.12) we deduce also

$$\frac{c_6}{2}|F|^2\le B(F) \quad \text{for every } F\in\mathbb{M}^{3\times3}. \tag{4.2.15}$$

**Dissipation functional**
Denote by $\mathbb{M}^{3\times3}_D$ the set of trace-free symmetric matrices, namely

$$\mathbb{M}^{3\times3}_D := \{F\in\mathbb{M}^{3\times3}_{\rm sym} : \text{tr } F=0\}.$$

Let $H_D:\mathbb{M}^{3\times3}_D\to[0,+\infty)$ be a convex, positively one-homogeneous function such that

$$r_K|F|\le H_D(F)\le R_K|F| \quad \text{for every } F\in\mathbb{M}^{3\times3}_D. \tag{4.2.16}$$

We define the dissipation potential $H:\mathbb{M}^{3\times3}\to[0,+\infty]$ as

$$H(F):=\begin{cases} H_D(F) & \text{if } F\in\mathbb{M}^{3\times3}_D,\\ +\infty & \text{otherwise.}\end{cases}$$

For every $F\in\mathbb{M}^{3\times3}$, we consider the quantity

$$D(Id,F):=\inf\Big\{\int_0^1 H(\dot c(t)c^{-1}(t))\,dt : c\in C^1([0,1];\mathbb{M}^{3\times3}_+),\ c(0)=Id,\ c(1)=F\Big\}. \tag{4.2.17}$$

Note that if $D(Id,F)<+\infty$, then $F\in SL(3)$.

We define the dissipation distance as the map $D:\mathbb{M}^{3\times3}\times\mathbb{M}^{3\times3}\to[0,+\infty]$, given by

$$D(F_1,F_2):=\begin{cases} D(Id,F_2F_1^{-1}) & \text{if } F_1\in\mathbb{M}^{3\times3}_+, F_2\in\mathbb{M}^{3\times3}\\ +\infty & \text{if } F_1\notin\mathbb{M}^{3\times3}_+, F_2\in\mathbb{M}^{3\times3}.\end{cases}$$

We note that the map $D$ satisfies the triangle inequality

$$D(F_1,F_2)\le D(F_1,F_3)+D(F_3,F_2) \tag{4.2.18}$$

for every $F_1,F_2,F_3\in\mathbb{M}^{3\times3}$.

**Remark 4.2.4.** We remark that there exists a positive constant $c_7$ such that

$$D(F_1,F_2)\le c_7 \quad \text{for every } F_1,F_2\in K, \tag{4.2.19}$$
$$D(Id,F)\le c_7|F-Id| \quad \text{for every } F\in K. \tag{4.2.20}$$

Indeed, by the compactness of $K$ and the continuity of the map $D$ on $SL(3)\times SL(3)$ (see [49]), there exists a constant $\tilde{c}_4$ such that

$$D(F_1,F_2)\le \tilde{c}_4 \quad \text{for every } F_1,F_2\in K. \tag{4.2.21}$$

By the previous estimate, (4.2.20) needs only to be proved in a neighbourhood of the identity. More precisely, let $\delta>0$ be such that $\log F$ is well defined for $F\in K$ and $|F-Id|<\delta$. If $F\in K$ is such that $|F-Id|\ge\delta$, by (4.2.21) we deduce

$$D(Id,F)\le \frac{\tilde{c}_4}{\delta}|F-Id|.$$

If $|F-Id|<\delta$, taking $c(t)=\exp(t\log F)$ in (4.2.17), inequality (4.2.16) yields

$$D(Id,F)\le H_D(\log F)\le R_K|\log F|\le C|F-Id|$$

for every $F\in K$. Collecting the previous estimates we deduce (4.2.19) and (4.2.20).

**Change of variable and formulation of the problem**

As usual in dimension reduction problems we perform a change of variable to formulate the problem on a domain independent of $\varepsilon$. We consider the set $\Omega:=\omega\times\big(-\frac{1}{2},\frac{1}{2}\big)$ and the map $\psi^\varepsilon:\overline{\Omega}\to\overline{\Omega}_\varepsilon$ given by

$$\psi^\varepsilon(x):=(x',\varepsilon x_3)\quad \text{for every } x\in\overline{\Omega}.$$

To every deformation $\eta\in W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)$ satisfying

$$\eta(x)=\phi^\varepsilon(x)\quad \mathcal{H}^2\text{- a.e. on } \Gamma_\varepsilon$$

and to every plastic strain $F_{pl}\in L^2(\Omega_\varepsilon;SL(3))$ we associate the scaled deformation $y:=\eta\circ\psi^\varepsilon$ and the scaled plastic strain $P:=F_{pl}\circ\psi^\varepsilon$. Denoting by $\Gamma_d$ the set $\gamma_d\times\big(-\frac{1}{2},\frac{1}{2}\big)$, the scaled deformation satisfies the boundary condition

$$y(x)=\phi^\varepsilon(x',\varepsilon x_3)\quad \mathcal{H}^2\text{- a.e. on } \Gamma_d. \tag{4.2.22}$$

Applying this change of variable to (4.2.2), the energy functional is now given by

$$\mathcal{I}(y,P):=\frac{1}{\varepsilon}\mathcal{E}(\eta,F_{pl})=\int_\Omega W_{el}(\nabla_\varepsilon y(x)P^{-1}(x))\,dx+\int_\Omega W_{hard}(P(x))\,dx,$$

where $\nabla_\varepsilon y(x):=\big(\partial_1 y(x)\big|\partial_2 y(x)\big|\frac{1}{\varepsilon}\partial_3 y(x)\big)$ for a.e. $x\in\Omega$.

Denote by $\mathcal{A}_\varepsilon(\phi^\varepsilon)$ the class of pairs $(y^\varepsilon,P^\varepsilon)\in W^{1,2}(\Omega;\mathbb{R}^3)\times L^2(\Omega;SL(3))$ such that (4.2.22) is satisfied. We associate to each pair $(y^\varepsilon,P^\varepsilon)\in\mathcal{A}_\varepsilon(\phi^\varepsilon)$ the scaled energy given by

$$\mathcal{J}^\varepsilon_\alpha(y^\varepsilon,P^\varepsilon):=\frac{1}{\varepsilon^{2\alpha-2}}\mathcal{I}(y^\varepsilon,P^\varepsilon)+\frac{1}{\varepsilon^{\alpha-1}}\int_\Omega D(P^{\varepsilon,0},P^\varepsilon)\,dx, \tag{4.2.23}$$

where $\alpha\ge 3$ is the same exponent as in (4.2.1) and $P^{\varepsilon,0}$ is a map in $L^2(\Omega;SL(3))$, which represents a preexistent plastic strain.

**Remark 4.2.5.** We are interested in studying the asymptotic behaviour of sequences of pairs $(y^\varepsilon, P^\varepsilon) \in \mathcal{A}_\varepsilon(\phi^\varepsilon)$ such that the scaled total energies $\mathcal{J}^\varepsilon_\alpha(y^\varepsilon, P^\varepsilon)$ are uniformly bounded. This, in particular, holds for sequences of (almost) minimizers of

$$\mathcal{I}(y, P) - \int_\Omega f^\varepsilon \cdot y \, dx, \tag{4.2.24}$$

whenever the applied forces $f^\varepsilon$ are of order $\varepsilon^\alpha$, with $\alpha \geq 3$. In fact by [34, Theorem 2], in the absence of plastic deformation ($P^\varepsilon \equiv Id$), the elastic energy on (almost) minimizing sequences scales like $\varepsilon^{2\alpha-2}$. In order to have interaction between the elastic and the plastic energy at the limit we are lead to rescale also the hardening functional by $\varepsilon^{2\alpha-2}$. Finally, the scaling of the dissipation functional is motivated by its linear growth and by the estimate (4.2.20).

Our choice of the boundary datum is again motivated by [34, Theorem 2]. Indeed, as remarked in the introduction, the structure of $\phi^\varepsilon$ is compatible with the structure of (almost) minimizers of (4.2.24) in absence of plastic deformation, as $\varepsilon \to 0^+$.

## 4.3 Compactness results and liminf inequality

In this section we study compactness properties of sequences of pairs in $\mathcal{A}_\varepsilon(\phi^\varepsilon)$ satisfying the uniform energy estimate

$$\mathcal{J}^\varepsilon_\alpha(y^\varepsilon, P^\varepsilon) \leq C \quad \text{for every } \varepsilon. \tag{4.3.1}$$

To state the compactness results it is useful to introduce the following notation: given $\varphi : \Omega \to \mathbb{R}^3$, we denote by $\varphi' : \Omega \to \mathbb{R}^2$ the map

$$\varphi' := \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

and for every $\eta \in W^{1,2}(\Omega)$ we denote by $\nabla'\eta$ the vector $\begin{pmatrix} \partial_1 \eta \\ \partial_2 \eta \end{pmatrix}$. Analogously, for every matrix $M \in \mathbb{M}^{3\times 3}$, we use the notation $M'$ to represent the minor

$$M' := \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.$$

Given a sequence of deformations $(y^\varepsilon) \subset W^{1,2}(\Omega; \mathbb{R}^3)$, we consider some associated quantities: the in-plane displacements

$$u^\varepsilon(x') := \frac{1}{\varepsilon^{\alpha-1}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \big((y^\varepsilon)'(x', x_3) - x'\big) \, dx_3 \quad \text{for a.e. } x' \in \omega, \tag{4.3.2}$$

the out-of-plane displacements

$$v^\varepsilon(x') := \frac{1}{\varepsilon^{\alpha-2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} y^\varepsilon_3(x', x_3) \, dx_3 \quad \text{for a.e. } x' \in \omega, \tag{4.3.3}$$

and the first order moments

$$\xi^\varepsilon(x') := \frac{1}{\varepsilon^{\alpha-1}} \int_{-\frac{1}{2}}^{\frac{1}{2}} x_3 \Big( y^\varepsilon(x', x_3) - \begin{pmatrix} x' \\ \varepsilon x_3 \end{pmatrix} \Big) \, dx_3 \quad \text{for a.e. } x' \in \omega. \tag{4.3.4}$$

A key tool to establish compactness of in-plane and out-of-plane displacements is the rigidity estimate due to Friesecke, James and Müller (see Section 1.2). The rigidity estimate provided in Theorem 1.2.1 allows us to approximate sequences of deformations whose distance of the gradient from $SO(3)$ is uniformly bounded, by means of rotations. More precisely, the following theorem holds true.

**Theorem 4.3.1.** *Assume that $\alpha \geq 3$. Let $(y^\varepsilon)$ be a sequence of deformations in $W^{1,2}(\Omega; \mathbb{R}^3)$ satisfying* (4.2.22) *and such that*

$$\|\mathrm{dist}(\nabla_\varepsilon y^\varepsilon, SO(3))\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} \leq C\varepsilon^{\alpha-1}. \tag{4.3.5}$$

*Then, there exists a sequence $(R^\varepsilon) \subset W^{1,\infty}(\omega; \mathbb{M}^{3\times 3})$ such that for every $\varepsilon > 0$*

$$R^\varepsilon(x') \in SO(3) \quad \textit{for every } x' \in \omega, \tag{4.3.6}$$

$$\|\nabla_\varepsilon y^\varepsilon - R^\varepsilon\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} \leq C\varepsilon^{\alpha-1}, \tag{4.3.7}$$

$$\|\partial_i R^\varepsilon\|_{L^2(\omega;\mathbb{M}^{3\times 3})} \leq C\varepsilon^{\alpha-2},\ i = 1, 2 \tag{4.3.8}$$

$$\|R^\varepsilon - Id\|_{L^2(\omega;\mathbb{M}^{3\times 3})} \leq C\varepsilon^{\alpha-2}. \tag{4.3.9}$$

*Proof.* Arguing as in [34, Theorem 6 and Remark 5] we can construct a sequence of maps $R^\varepsilon \in W^{1,\infty}(\omega; \mathbb{M}^{3\times 3})$ satisfying (4.3.6)–(4.3.8). To complete the proof of the theorem it remains only to prove (4.3.9).

To this aim, we preliminarily recall that there exists a neighbourhood $U$ of $SO(3)$ where the projection $\Pi : U \to SO(3)$ onto $SO(3)$ is well defined. By Poincaré inequality, (4.3.8) yields

$$\Big\| R^\varepsilon - \fint_\omega R^\varepsilon \, dx' \Big\|_{L^2(\omega;\mathbb{M}^{3\times 3})} \leq C\varepsilon^{\alpha-2}. \tag{4.3.10}$$

On the other hand, by (4.3.6) we have

$$\mathrm{dist}^2\Big( \fint_\omega R^\varepsilon \, dx', SO(3) \Big) \mathcal{L}^2(\omega) \leq \Big\| R^\varepsilon - \fint_\omega R^\varepsilon \, dx' \Big\|^2_{L^2(\omega;\mathbb{M}^{3\times 3})}.$$

Hence, by (4.3.10) for $\varepsilon$ small enough we can define $\hat{R}^\varepsilon := \Pi(\fint_\omega R^\varepsilon \, dx')$, which fulfills

$$\Big| \hat{R}^\varepsilon - \fint_\omega R^\varepsilon \, dx' \Big| \leq C \Big\| R^\varepsilon - \fint_\omega R^\varepsilon \, dx' \Big\|_{L^2(\omega;\mathbb{M}^{3\times 3})} \leq C\varepsilon^{\alpha-2}.$$

$$\|\hat{R}^\varepsilon - R^\varepsilon\|_{L^2(\omega;\mathbb{M}^{3\times 3})} \leq \Big\| \hat{R}^\varepsilon - \fint_\omega R^\varepsilon \, dx' \Big\|_{L^2(\omega;\mathbb{M}^{3\times 3})} + \Big\| \fint_\omega R^\varepsilon \, dx' - R^\varepsilon \Big\|_{L^2(\omega;\mathbb{M}^{3\times 3})} \leq C\varepsilon^{\alpha-2}.$$

To prove (4.3.9) it is now enough to show that

$$|\hat{R}^\varepsilon - Id| \leq C\varepsilon^{\alpha-2}. \tag{4.3.11}$$

To this purpose, we argue as in [39, Section 4.2, Lemma 13]. We consider the sequences

$$\tilde{R}^\varepsilon := (\hat{R}^\varepsilon)^T R^\varepsilon,$$
$$\tilde{y}^\varepsilon := (\hat{R}^\varepsilon)^T y^\varepsilon - c^\varepsilon,$$
$$\tilde{u}^\varepsilon(x') := \frac{1}{\varepsilon^{\alpha-1}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \big((\tilde{y}^\varepsilon)'(x', x_3) - x'\big)\, dx_3 \quad \text{for a.e. } x' \in \omega,$$
$$\tilde{v}^\varepsilon(x') := \frac{1}{\varepsilon^{\alpha-2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{y}_3^\varepsilon(x', x_3)\, dx_3 \quad \text{for a.e. } x' \in \omega,$$
$$\tilde{\xi}^\varepsilon(x') := \frac{1}{\varepsilon^{\alpha-1}} \int_{-\frac{1}{2}}^{\frac{1}{2}} x_3\Big(\tilde{y}^\varepsilon(x', x_3) - \begin{pmatrix} x' \\ \varepsilon x_3 \end{pmatrix}\Big)\, dx_3 \quad \text{for a.e. } x' \in \omega,$$

where the constants $c^\varepsilon$ are chosen in such a way that

$$\int_\Omega \big(\tilde{y}^\varepsilon(x) - x\big)\, dx = 0.$$

By [34, Lemma 1 and Corollary 1], there exist $\tilde{u} \in W^{1,2}(\omega;\mathbb{R}^2)$, $\tilde{v} \in W^{2,2}(\omega)$ and $\tilde{\xi} \in W^{1,2}(\omega;\mathbb{R}^3)$ such that

$$\tilde{u}^\varepsilon \rightharpoonup \tilde{u} \quad \text{weakly in } W^{1,2}(\omega;\mathbb{R}^2), \tag{4.3.12}$$
$$\tilde{v}^\varepsilon \to \tilde{v} \quad \text{strongly in } W^{1,2}(\omega), \tag{4.3.13}$$
$$\tilde{\xi}^\varepsilon \rightharpoonup \tilde{\xi} \quad \text{weakly in } W^{1,2}(\omega;\mathbb{R}^3). \tag{4.3.14}$$

We now write $u^\varepsilon, v^\varepsilon$ and $\xi^\varepsilon$ in terms of $\tilde{u}^\varepsilon, \tilde{v}^\varepsilon$ and $\tilde{\xi}^\varepsilon$. We have

$$\begin{pmatrix} \varepsilon^{\alpha-1} u^\varepsilon(x') \\ \varepsilon^{\alpha-2} v^\varepsilon(x') \end{pmatrix} = (\hat{R}^\varepsilon - Id)\begin{pmatrix} x' \\ 0 \end{pmatrix} + \hat{R}^\varepsilon \begin{pmatrix} \varepsilon^{\alpha-1}\tilde{u}^\varepsilon(x') \\ \varepsilon^{\alpha-2}\tilde{v}^\varepsilon(x') \end{pmatrix} + \hat{R}^\varepsilon c^\varepsilon, \tag{4.3.15}$$

for a.e. $x' \in \omega$ and

$$\xi^\varepsilon(x') = \frac{1}{12\varepsilon^{\alpha-2}}(\hat{R}^\varepsilon - Id)e_3 + \hat{R}^\varepsilon \tilde{\xi}^\varepsilon(x') \quad \text{for a.e. } x' \in \omega. \tag{4.3.16}$$

By (4.3.14) there exists a constant $C$ such that $\|\tilde{\xi}^\varepsilon\|_{L^2(\gamma_d;\mathbb{R}^3)} \le C$ for every $\varepsilon$. Moreover, by (4.2.1) and (4.2.22) there holds

$$\xi^\varepsilon(x') = \frac{1}{\varepsilon^{\alpha-1}} \int_{-\frac{1}{2}}^{\frac{1}{2}} x_3\Big(\phi^\varepsilon(x', \varepsilon x_3) - \begin{pmatrix} x' \\ \varepsilon x_3 \end{pmatrix}\Big)\, dx_3 = \begin{pmatrix} -\frac{1}{12}\nabla' v^0(x') \\ 0 \end{pmatrix} \quad \mathcal{H}^1\text{- a.e. on } \gamma_d,$$

hence $(\xi^\varepsilon)$ is uniformly bounded in $L^2(\gamma_d;\mathbb{R}^3)$. Therefore, by (4.3.16) we deduce

$$|(\hat{R}^\varepsilon - Id)e_3| \le C\varepsilon^{\alpha-2}\|\xi^\varepsilon - \hat{R}^\varepsilon\tilde{\xi}^\varepsilon\|_{L^2(\Gamma_d;\mathbb{R}^3)} \le C\varepsilon^{\alpha-2}, \tag{4.3.17}$$

for every $\varepsilon$. Since $\hat{R}^\varepsilon \in SO(3)$, (4.3.17) implies that

$$|(\hat{R}^\varepsilon - Id)^T e_3| \le C\varepsilon^{\alpha-2} \tag{4.3.18}$$

for every $\varepsilon$ and there exists a sequence $(\hat{Q}^\varepsilon) \subset SO(2)$ such that

$$|(\hat{R}^\varepsilon)' - \hat{Q}^\varepsilon| \le C\varepsilon^{\alpha-2}. \tag{4.3.19}$$

Now, without loss of generality we can assume that

$$\int_{\gamma_d} x'\, d\mathcal{H}^1(x') = 0 \quad \text{and} \quad \int_{\gamma_d} |x'|^2\, d\mathcal{H}^1(x') = c > 0. \tag{4.3.20}$$

By (4.3.12) and (4.3.13) we have $\|\tilde{u}^\varepsilon\|_{L^2(\gamma_d;\mathbb{R}^2)} + \|\tilde{v}^\varepsilon\|_{L^2(\gamma_d)} \leq C$ for every $\varepsilon$. On the other hand (4.2.1) and (4.2.22) imply that

$$u^\varepsilon(x') = u^0(x') \quad \text{and} \quad v^\varepsilon(x') = v^0(x') \quad \mathcal{H}^1\text{- a.e. on } \gamma_d,$$

hence both $(u^\varepsilon)$ and $(v^\varepsilon)$ are uniformly bounded in $L^2(\gamma_d;\mathbb{R}^2)$ and $L^2(\gamma_d)$, respectively. Therefore, by (4.3.15) and (4.3.19) we deduce

$$|(\hat{Q}^\varepsilon - Id)x' + (\hat{R}^\varepsilon c^\varepsilon)'| \leq C\varepsilon^{\alpha-2}. \tag{4.3.21}$$

The two terms in the left hand side of (4.3.21) are orthogonal in the sense of $L^2(\gamma_d;\mathbb{R}^2)$ by (4.3.20), hence (4.3.21) implies that

$$\|(\hat{Q}^\varepsilon - Id)x'\|^2_{L^2(\gamma_d;\mathbb{R}^2)} \leq C\varepsilon^{2(\alpha-2)}.$$

Since $\hat{Q}^\varepsilon \in SO(2)$, it satisfies

$$2|(\hat{Q}^\varepsilon - Id)x'|^2 = |\hat{Q}^\varepsilon - Id|^2|x'|^2 \quad \text{for every } x' \in \gamma_d.$$

Therefore, applying again (4.3.20) we obtain

$$c|\hat{Q}^\varepsilon - Id|^2 = 2\int_{\gamma_d} |\hat{Q}^\varepsilon - Id|^2|x'|^2\, d\mathcal{H}^1(x') \leq C\varepsilon^{2(\alpha-2)}. \tag{4.3.22}$$

Claim (4.3.11) follows now by collecting (4.3.17)–(4.3.19) and (4.3.22). □

In the remaining of this section we shall establish some compactness results for the displacements defined in (4.3.2) and (4.3.3), and we shall prove a liminf inequality both for the energy functional and the dissipation potential.

We first introduce the limit functional. Let $\mathbb{A} : \mathbb{M}^{2\times 2} \to \mathbb{M}^{3\times 3}_{\mathrm{sym}}$ be the operator given by

$$\mathbb{A}F := \begin{pmatrix} \multicolumn{2}{c}{\mathrm{sym}\, F} & \begin{matrix}\lambda_1(F)\\ \lambda_2(F)\end{matrix} \\ \lambda_1(F) & \lambda_2(F) & \lambda_3(F) \end{pmatrix} \quad \text{for every } F \in \mathbb{M}^{2\times 2},$$

where for every $F \in \mathbb{M}^{2\times 2}$ the triple $(\lambda_1(F), \lambda_2(F), \lambda_3(F))$ is the unique solution to the minimum problem

$$\min_{\lambda_i \in \mathbb{R}} Q\begin{pmatrix} \multicolumn{2}{c}{\mathrm{sym}\, F} & \begin{matrix}\lambda_1\\ \lambda_2\end{matrix} \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}.$$

We remark that for every $F \in \mathbb{M}^{2\times 2}$, $\mathbb{A}(F)$ is given by the unique solution to the linear equation

$$\mathbb{C}\mathbb{A}(F) : \begin{pmatrix} 0 & 0 & \lambda_1 \\ 0 & 0 & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} = 0 \quad \text{for every } \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}. \tag{4.3.23}$$

This implies, in particular, that $\mathbb{A}$ is linear.

We define the quadratic form $Q_2 : \mathbb{M}^{2\times 2} \to [0, +\infty)$ as

$$Q_2(F) = Q(\mathbb{A}(F)) \quad \text{for every } F \in \mathbb{M}^{2\times 2}.$$

By properties of $Q$, we have that $Q_2$ is positive definite on symmetric matrices. We also define the tensor $\mathbb{C}_2 : \mathbb{M}^{2\times 2} \to \mathbb{M}^{3\times 3}_{\mathrm{sym}}$, given by

$$\mathbb{C}_2 F := \mathbb{C}\mathbb{A}(F) \quad \text{for every } F \in \mathbb{M}^{2\times 2}. \tag{4.3.24}$$

We remark that by (4.3.23) there holds

$$\mathbb{C}_2 F : G = \mathbb{C}_2 F : \begin{pmatrix} \operatorname{sym} G & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for every } F \in \mathbb{M}^{2\times 2},\, G \in \mathbb{M}^{3\times 3} \tag{4.3.25}$$

and

$$Q_2(F) = \frac{1}{2}\mathbb{C}_2 F : \begin{pmatrix} \operatorname{sym} F & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for every } F \in \mathbb{M}^{2\times 2}.$$

**Remark 4.3.2.** We note that in the case where the tensors in formulas (3.2.1) and (4.2.3) coincide, then $Q_2(F) = Q_r(F)$ for every $\mathbb{M}^{2\times 2}$, where $Q_r$ is the quadratic form defined in (3.4.3).

Denoting by $\mathcal{A}(u^0, v^0)$ the set of triples $(u, v, p) \in W^{1,2}(\Omega; \mathbb{R}^2) \times W^{2,2}(\Omega) \times L^2(\Omega; \mathbb{M}^{3\times 3}_D)$ such that

$$u(x') = u^0(x'), \quad v(x') = v^0(x'), \quad \text{and } \nabla v(x') = \nabla v^0(x') \quad \mathcal{H}^1 \text{ - a.e. on } \gamma_d,$$

we introduce the functionals $\mathcal{J}_\alpha : \mathcal{A}(u^0, v^0) \to [0, +\infty)$, given by

$$\mathcal{J}_\alpha(u, v, p) := \int_\Omega Q_2(\operatorname{sym}\nabla' u - x_3(\nabla')^2 v - p')\, dx + \int_\Omega B(p)\, dx + \int_\Omega H_D(p - p^0)\, dx \tag{4.3.26}$$

for $\alpha > 3$, and

$$\begin{aligned} \mathcal{J}_3(u, v, p) \;&:=\; \int_\Omega Q_2\big(\operatorname{sym}\nabla' u + \tfrac{1}{2}\nabla' v \otimes \nabla' v - x_3(\nabla')^2 v - p'\big)\, dx + \int_\Omega B(p)\, dx \\ &\quad + \int_\Omega H_D(p - p^0)\, dx, \end{aligned} \tag{4.3.27}$$

for every $(u, v, p) \in \mathcal{A}(u^0, v^0)$. In the expressions of the functionals, $p^0$ is a given map in $L^2(\Omega; \mathbb{M}^{3\times 3}_D)$ that represents the history of the plastic deformations.

Finally, for every sequence $(y^\varepsilon)$ in $W^{1,2}(\Omega; \mathbb{R}^3)$ satisfying both (4.2.22) and (4.3.5), we introduce the strains

$$G^\varepsilon(x) := \frac{(R^\varepsilon(x))^T \nabla_\varepsilon y^\varepsilon(x) - Id}{\varepsilon^{\alpha - 1}} \quad \text{for a.e. } x \in \Omega, \tag{4.3.28}$$

where the maps $R^\varepsilon$ are the pointwise rotations provided by Theorem 4.3.1.

We are now in a position to state the main result of this section.

**Theorem 4.3.3.** *Assume that* $\alpha \geq 3$. *Let* $(y^\varepsilon, P^\varepsilon)$ *be a sequence of pairs in* $\mathcal{A}_\varepsilon(\phi^\varepsilon)$ *satisfying*

$$\mathcal{I}(y^\varepsilon, P^\varepsilon) \leq C\varepsilon^{2\alpha-2} \tag{4.3.29}$$

*for every* $\varepsilon > 0$. *Let* $u^\varepsilon$, $v^\varepsilon$ *and* $G^\varepsilon$ *be defined as in* (4.3.2), (4.3.3) *and* (4.3.28), *respectively. Then, there exists* $(u, v, p) \in \mathcal{A}(u^0, v^0)$ *such that, up to subsequences, there hold*

$$y^\varepsilon \to \begin{pmatrix} x' \\ 0 \end{pmatrix} \quad \textit{strongly in } W^{1,2}(\Omega;\mathbb{R}^3), \tag{4.3.30}$$

$$u^\varepsilon \rightharpoonup u \quad \textit{weakly in } W^{1,2}(\omega;\mathbb{R}^2), \tag{4.3.31}$$

$$v^\varepsilon \to v \quad \textit{strongly in } W^{1,2}(\omega), \tag{4.3.32}$$

$$\frac{\nabla' y_3^\varepsilon}{\varepsilon^{\alpha-2}} \to \nabla' v \quad \textit{strongly in } L^2(\Omega;\mathbb{R}^2), \tag{4.3.33}$$

*and the following estimate holds true*

$$\Big\| \frac{y_3^\varepsilon}{\varepsilon} - x_3 - \varepsilon^{\alpha-3} v^\varepsilon \Big\|_{L^2(\Omega)} \leq C\varepsilon^{\alpha-2}. \tag{4.3.34}$$

*Moreover, there exists* $G \in L^2(\Omega;\mathbb{M}^{3\times 3})$ *such that*

$$G^\varepsilon \rightharpoonup G \quad \textit{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}), \tag{4.3.35}$$

*and the* $2\times 2$ *submatrix* $G'$ *satisfies*

$$G'(x', x_3) = G_0(x') - x_3(\nabla')^2 v(x') \quad \textit{for a.e. } x \in \Omega, \tag{4.3.36}$$

*where*

$$\operatorname{sym} G_0 = \frac{(\nabla' u + (\nabla' u)^T + \nabla' v \otimes \nabla' v)}{2} \quad \textit{if } \alpha = 3, \tag{4.3.37}$$

$$\operatorname{sym} G_0 = \operatorname{sym}\nabla' u \quad \textit{if } \alpha > 3. \tag{4.3.38}$$

*The sequence of plastic strains* $(P^\varepsilon)$ *fulfills*

$$P^\varepsilon(x) \in K \quad \textit{for a.e. } x \in \Omega, \tag{4.3.39}$$

*and*

$$\|P^\varepsilon - Id\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} \leq C\varepsilon^{\alpha-1} \tag{4.3.40}$$

*for every* $\varepsilon$. *Moreover, setting*

$$p^\varepsilon := \frac{P^\varepsilon - Id}{\varepsilon^{\alpha-1}}, \tag{4.3.41}$$

*up to subsequences*

$$p^\varepsilon \rightharpoonup p \quad \textit{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}). \tag{4.3.42}$$

*Finally,*

$$\int_\Omega Q_2(\operatorname{sym} G' - p')\,dx \leq \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^{2\alpha-2}} \int_\Omega W_{el}(\nabla_\varepsilon y^\varepsilon (P^\varepsilon)^{-1})\,dx \tag{4.3.43}$$

*and*

$$\int_\Omega B(p)\,dx \leq \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^{2\alpha-2}} \int_\Omega W_{hard}(P^\varepsilon)\,dx. \tag{4.3.44}$$

*If in addition*

$$\frac{1}{\varepsilon^{\alpha-1}}\int_\Omega D(P^{\varepsilon,0},P^\varepsilon)\,dx \le C \quad \textit{for every } \varepsilon>0 \tag{4.3.45}$$

*and there exist a map* $p^0 \in L^2(\Omega;\mathbb{M}^{3\times3}_D)$ *and a sequence* $(p^{\varepsilon,0}) \subset L^2(\Omega;\mathbb{M}^{3\times3})$ *such that* $P^{\varepsilon,0} = Id + \varepsilon^{\alpha-1}p^{\varepsilon,0}$, *with* $p^{\varepsilon,0} \rightharpoonup p^0$ *weakly in* $L^2(\Omega;\mathbb{M}^{3\times3})$, *then*

$$\int_\Omega H_D(p-p^0)\,dx \le \liminf_{\varepsilon\to0}\frac{1}{\varepsilon^{\alpha-1}}\int_\Omega D(P^{\varepsilon,0},P^\varepsilon)\,dx. \tag{4.3.46}$$

*Proof.* We first remark that by (4.3.29) there holds

$$\int_\Omega W_{hard}(P^\varepsilon)\,dx \le C\varepsilon^{2\alpha-2}, \tag{4.3.47}$$

which, together with (4.2.10), implies (4.3.39). On the other hand, combining (4.2.11) and (4.3.47) we deduce

$$c_3\|P^\varepsilon - Id\|^2_{L^2(\Omega;\mathbb{M}^{3\times3})} \le \int_\Omega \tilde{W}_{hard}(P^\varepsilon)\,dx \le C\varepsilon^{2\alpha-2},$$

which in turn yields (4.3.40) and (4.3.42).

Let $R \in SO(3)$. By (4.2.13), (4.3.39) and (4.3.41) there holds

$$\begin{aligned}|\nabla_\varepsilon y^\varepsilon - R|^2 &= |\nabla_\varepsilon y^\varepsilon - RP^\varepsilon + \varepsilon^{\alpha-1}Rp^\varepsilon|^2 \le 2\big(|\nabla_\varepsilon y^\varepsilon(P^\varepsilon)^{-1} - R|^2|P^\varepsilon|^2 + \varepsilon^{2\alpha-2}|p^\varepsilon|^2\big)\\ &\le 2\,c_K^2|\nabla_\varepsilon y^\varepsilon(P^\varepsilon)^{-1} - R|^2 + 2\varepsilon^{2\alpha-2}|p^\varepsilon|^2.\end{aligned}$$

Hence, the growth condition (H4) implies

$$\|\mathrm{dist}(\nabla_\varepsilon y^\varepsilon, SO(3))\|^2_{L^2(\Omega;\mathbb{M}^{3\times3})} \le C\Big(\int_\Omega W_{el}(\nabla_\varepsilon y^\varepsilon(P^\varepsilon)^{-1})\,dx + \varepsilon^{2\alpha-2}\|p^\varepsilon\|^2_{L^2(\Omega;\mathbb{M}^{3\times3})}\Big),$$

which in turn yields

$$\|\mathrm{dist}(\nabla_\varepsilon y^\varepsilon, SO(3))\|^2_{L^2(\Omega;\mathbb{M}^{3\times3})} \le C\varepsilon^{2\alpha-2}$$

by (4.3.29) and (4.3.42).

Due to (4.2.22), the deformations $(y^\varepsilon)$ fulfill the hypotheses of Theorem 4.3.1. Hence, we can construct a sequence $(R^\varepsilon)$ in $W^{1,\infty}(\omega;\mathbb{M}^{3\times3})$ satisfying (4.3.6)–(4.3.9). Properties (4.3.30)–(4.3.33) and (4.3.35)–(4.3.38) follow arguing as in [34, Lemma 1, Corollary 1 and Lemma 2]. The only difference is due to the fact that compactness is now achieved by using the boundary condition (4.2.22), instead of performing a normalization of the deformations $y^\varepsilon$. Moreover the limit in-plane and out-of-plane displacements satisfy $u = u^0$, $v = v^0$ and $\nabla' v = \nabla' v^0$ $\mathcal{H}^1$- a.e. on $\gamma_d$.

By Poincaré inequality and the definition of $v^\varepsilon$, there holds

$$\Big\|\frac{y_3^\varepsilon}{\varepsilon} - x_3 - \varepsilon^{\alpha-3}v^\varepsilon\Big\|_{L^2(\Omega)} \le C\Big\|\frac{\partial_3 y_3^\varepsilon}{\varepsilon} - 1\Big\|_{L^2(\Omega)},$$

hence (4.3.34) is a consequence of (4.3.7) and (4.3.9).

Inequality (4.3.46) follows by adapting [52, Lemmas 3.4 and 3.5].

The proof of (4.3.43) and (4.3.44) is based on an adaptation of [52, Proof of Lemma 3.3]: we give a sketch for convenience of the reader. Fix $\delta > 0$, let $O_\varepsilon$ be the set

$$O_\varepsilon := \{x : \varepsilon^{\alpha-1}|p^\varepsilon(x)| \le c_h(\delta)\}$$

and let $\chi_\varepsilon$ be its characteristic function. By (4.3.42) and by Chebyshev inequality there holds

$$\mathcal{L}^3(\Omega \setminus O_\varepsilon) \leq C\varepsilon^{2\alpha-2},$$

hence by (4.2.12) and (4.3.29), we deduce

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^{2\alpha-2}} \int_\Omega W_{hard}(P^\varepsilon)\,dx \geq \liminf_{\varepsilon\to 0}(1-\delta)\int_\Omega B(p^\varepsilon)\chi_\varepsilon\,dx \geq (1-\delta)\int_\Omega B(p)\,dx \quad (4.3.48)$$

which yields (4.3.44). To prove the liminf inequality for the elastic energy, we introduce the auxiliary tensors

$$w^\varepsilon := \frac{(P^\varepsilon)^{-1} - Id + \varepsilon^{\alpha-1}p^\varepsilon}{\varepsilon^{\alpha-1}} = \varepsilon^{\alpha-1}(P^\varepsilon)^{-1}(p^\varepsilon)^2. \quad (4.3.49)$$

By (4.2.13) and (4.3.39), there exists a constant $C$ such that

$$\varepsilon^{\alpha-1}\|p^\varepsilon\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})} \leq C \quad (4.3.50)$$

and

$$\varepsilon^{\alpha-1}\|w^\varepsilon\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})} \leq C \quad (4.3.51)$$

for every $\varepsilon$. Furthermore, by (4.3.42),

$$\|w^\varepsilon\|_{L^1(\Omega;\mathbb{M}^{3\times3})} \leq C\varepsilon^{\alpha-1} \quad \text{for every } \varepsilon.$$

By the two previous estimates it follows that $(w^\varepsilon)$ is uniformly bounded in $L^2(\Omega;\mathbb{M}^{3\times3})$ and

$$w^\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}). \quad (4.3.52)$$

For every $\varepsilon$ we consider the map

$$F^\varepsilon := \frac{1}{\varepsilon^{\alpha-1}}\big((Id + \varepsilon^{\alpha-1}G^\varepsilon)(P^\varepsilon)^{-1} - Id\big).$$

By the frame-indifference hypothesis (H3) there holds

$$W_{el}(\nabla_\varepsilon y^\varepsilon (P^\varepsilon)^{-1}) = W_{el}(Id + \varepsilon^{\alpha-1}F^\varepsilon).$$

On the other hand,

$$F^\varepsilon = G^\varepsilon + w^\varepsilon - p^\varepsilon + \varepsilon^{\alpha-1}G^\varepsilon(w^\varepsilon - p^\varepsilon).$$

Combining (4.3.35), (4.3.42) and (4.3.50)–(4.3.52) we deduce

$$F^\varepsilon \rightharpoonup G - p \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}).$$

Therefore, by (4.2.4) and arguing as in the proof of (4.3.48) we conclude that

$$\begin{aligned}\int_\Omega Q_2(\operatorname{sym} G' - p')\,dx &\leq \int_\Omega Q(\operatorname{sym} G - p)\,dx \\ &\leq \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^{2\alpha-2}} \int_\Omega W_{el}(\nabla_\varepsilon y^\varepsilon (P^\varepsilon)^{-1})\,dx, \quad (4.3.53)\end{aligned}$$

which in turn implies (4.3.43). □

## 4.4 Construction of the recovery sequence

In this section, under some additional hypotheses on the sequence $(p^{\varepsilon,0})$ and on $\gamma_d$, we prove that the lower bound obtained in Theorem 4.3.3 is optimal by exhibiting a recovery sequence.

**Theorem 4.4.1.** *Assume that $\alpha \geq 3$ and $\gamma_d$ is a finite union of disjoint (nontrivial) closed intervals (i.e., maximally connected sets) in $\partial\omega$. Let $p^0 \in L^\infty(\Omega;\mathbb{M}^{3\times 3}_D)$ be such that there exists a sequence $(p^{\varepsilon,0}) \subset L^\infty(\Omega;\mathbb{M}^{3\times 3}_D)$ satisfying*

$$\|p^{\varepsilon,0}\|_{L^\infty(\Omega;\mathbb{M}^{3\times 3}_D)} \leq C \quad \textit{for every } \varepsilon, \tag{4.4.1}$$

$$p^{\varepsilon,0} \to p^0 \quad \textit{strongly in } L^1(\Omega;\mathbb{M}^{3\times 3}_D). \tag{4.4.2}$$

*Assume also that for every $\varepsilon$ the map $P^{\varepsilon,0} := Id + \varepsilon^{\alpha-1}p^{\varepsilon,0}$ satisfies $\det P^{\varepsilon,0} = 1$. Let $(u,v,p) \in \mathcal{A}(u^0,v^0)$. Then, there exists a sequence $(y^\varepsilon, P^\varepsilon) \in \mathcal{A}_\varepsilon(\phi^\varepsilon)$ such that, defining $u^\varepsilon, v^\varepsilon$ and $p^\varepsilon$ as in (4.3.2), (4.3.3) and (4.3.41), we have*

$$y^\varepsilon \to \begin{pmatrix} x' \\ 0 \end{pmatrix} \quad \textit{strongly in } W^{1,2}(\Omega;\mathbb{R}^3), \tag{4.4.3}$$

$$u^\varepsilon \to u \quad \textit{strongly in } W^{1,2}(\omega;\mathbb{R}^2), \tag{4.4.4}$$

$$v^\varepsilon \to v \quad \textit{strongly in } W^{1,2}(\omega), \tag{4.4.5}$$

$$p^\varepsilon \to p \quad \textit{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}). \tag{4.4.6}$$

*Moreover,*

$$\lim_{\varepsilon\to 0} \mathcal{J}^\varepsilon_\alpha(y^\varepsilon, P^\varepsilon) = \mathcal{J}_\alpha(u,v,p), \tag{4.4.7}$$

*where $\mathcal{J}^\varepsilon_\alpha$ and $\mathcal{J}_\alpha$ are the functionals introduced in (4.2.23), (4.3.26) and (4.3.27).*

*Proof.* For the sake of simplicity we divide the proof into two steps.
*Step 1*
Let $(u,v,p) \in \mathcal{A}(u^0,v^0)$. We first remark that by a standard approximation argument we may assume that $p \in C^\infty_c(\Omega;\mathbb{M}^{3\times 3}_D)$. Moreover, we claim that we can always reduce to the case where $u \in W^{1,\infty}(\omega;\mathbb{R}^2)$ and $v \in W^{2,\infty}(\omega)$. That is, we can approximate the pair $(u,v)$ in the sense of (4.4.4)–(4.4.5) by a sequence of pairs $(u^\lambda, v^\lambda)$ in $W^{1,\infty}(\omega;\mathbb{R}^2) \times W^{2,\infty}(\omega)$ satisfying the same boundary conditions as $(u,v)$ on $\gamma_d$, and such that, for $\alpha > 3$,

$$\begin{aligned}&\lim_{\lambda\to+\infty} \int_\Omega Q_2\Big(\mathrm{sym}\nabla' u^\lambda - x_3(\nabla')^2 v^\lambda - p'\Big)\,dx\\ &= \int_\Omega Q_2\Big(\mathrm{sym}\nabla' u - x_3(\nabla')^2 v - p'\Big)\,dx,\end{aligned} \tag{4.4.8}$$

whereas for $\alpha = 3$

$$\begin{aligned}&\lim_{\lambda\to+\infty} \int_\Omega Q_2\Big(\mathrm{sym}\nabla' u^\lambda + \frac{1}{2}\nabla' v^\lambda \otimes \nabla' v^\lambda - x_3(\nabla')^2 v^\lambda - p'\Big)\,dx\\ &= \int_\Omega Q_2\Big(\mathrm{sym}\nabla' u + \frac{1}{2}\nabla' v \otimes \nabla' v - x_3(\nabla')^2 v - p'\Big)\,dx.\end{aligned} \tag{4.4.9}$$

By the hypotheses on $\gamma_d$, we may apply [33, Proposition A.2], and for every $\lambda > 0$ we construct a pair $(u^\lambda, v^\lambda) \in W^{1,\infty}(\omega;\mathbb{R}^2) \times W^{2,\infty}(\omega)$, such that $(u^\lambda, v^\lambda, p) \in \mathcal{A}(u^0, v^0)$,

$$\|u^\lambda\|_{W^{1,\infty}(\omega;\mathbb{R}^2)} + \|v^\lambda\|_{W^{2,\infty}(\omega)} \leq C\lambda, \tag{4.4.10}$$

and setting

$$\omega^\lambda := \{x' \in \omega : u^\lambda(x') \neq u(x') \text{ or } v^\lambda(x') \neq v(x')\},$$

there holds

$$\lim_{\lambda \to +\infty} \lambda^2 \mathcal{L}^2(\omega^\lambda) = 0. \tag{4.4.11}$$

Now, by (4.4.10) we obtain

$$\begin{aligned}\|u^\lambda - u\|_{W^{1,2}(\omega;\mathbb{R}^2)} &\leq C\big(\|u^\lambda - u\|_{L^2(\omega^\lambda;\mathbb{R}^2)} + \|\nabla' u^\lambda - \nabla' u\|_{L^2(\omega^\lambda;\mathbb{M}^{2\times 2})}\big)\\ &\leq C\big(\|u\|_{L^2(\omega^\lambda;\mathbb{R}^2)} + \|\nabla' u\|_{L^2(\omega^\lambda;\mathbb{M}^{2\times 2})} + \lambda\big(\mathcal{L}^2(\omega^\lambda)\big)^{\frac{1}{2}}\big)\end{aligned}$$

and, analogously

$$\|v^\lambda - v\|_{W^{2,2}(\omega;\mathbb{R}^2)} \leq C\big(\|v\|_{L^2(\omega^\lambda)} + \|\nabla' v\|_{L^2(\omega^\lambda;\mathbb{R}^2)} + \|(\nabla')^2 v\|_{L^2(\omega^\lambda;\mathbb{M}^{2\times 2})} + \lambda\big(\mathcal{L}^2(\omega^\lambda)\big)^{\frac{1}{2}}\big).$$

Hence, by (4.4.11) we deduce

$$u^\lambda \to u \quad \text{strongly in } W^{1,2}(\omega;\mathbb{R}^2) \tag{4.4.12}$$

and

$$v^\lambda \to v \quad \text{strongly in } W^{2,2}(\omega), \tag{4.4.13}$$

as $\lambda \to +\infty$. Therefore, in particular

$$\nabla' v^\lambda \to \nabla' v \quad \text{strongly in } L^p(\omega;\mathbb{R}^2) \text{ for every } p \in [2,+\infty). \tag{4.4.14}$$

By (4.4.12)–(4.4.14) we obtain (4.4.8) and (4.4.9).
*Step 2*
To complete the proof of the theorem we shall prove that for every triple $(u, v, p) \in \mathcal{A}(u^0, v^0)$, with $u \in W^{1,\infty}(\omega;\mathbb{R}^2), v \in W^{2,\infty}(\omega)$ and $p \in C^\infty_c(\Omega;\mathbb{M}^{3\times 3}_D)$ we can construct a sequence $(y^\varepsilon, P^\varepsilon) \in \mathcal{A}(\phi^\varepsilon)$ satisfying (4.4.3)–(4.4.7).

To this purpose, consider the functions

$$P^\varepsilon := \exp(\varepsilon^{\alpha-1} p) \quad \text{and} \quad p^\varepsilon := \frac{1}{\varepsilon^{\alpha-1}}(\exp(\varepsilon^{\alpha-1} p) - Id).$$

Since $p \in C^\infty_c(\Omega;\mathbb{M}^{3\times 3}_D)$, it is immediate to see that $\det P^\varepsilon(x) = 1$ for every $\varepsilon$ and for all $x \in \Omega$. Moreover, there exists $\varepsilon_0 > 0$ such that

$$P^\varepsilon(x) \in K \quad \text{for every } x \in \Omega \text{ and for all } 0 \leq \varepsilon < \varepsilon_0,$$

and there holds

$$p^\varepsilon \to p \quad \text{uniformly in } \Omega,$$

which in turn implies (4.4.6). Furthermore,

$$\|P^\varepsilon - Id\|_{L^\infty(\Omega;\mathbb{M}^{3\times 3})} \leq C\varepsilon^{\alpha-1},$$

and by (4.2.12), for every $\delta > 0$ there exists $\varepsilon_\delta$ such that if $0 \leq \varepsilon < \varepsilon_\delta$ there holds

$$\Big|\frac{1}{\varepsilon^{2\alpha-2}}\int_\Omega W_{hard}(P^\varepsilon)\,dx - \int_\Omega B(p^\varepsilon)\,dx\Big| \leq \delta \int_\Omega B(p^\varepsilon)\,dx.$$

By (4.4.6) we deduce that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{2\alpha-2}}\int_\Omega W_{hard}(P^\varepsilon)\,dx = \int_\Omega B(p)\,dx. \tag{4.4.15}$$

To study the dissipation potential, we first remark that by (4.4.1), for $\varepsilon$ small enough, there holds

$$\exp(\varepsilon^{\alpha-1}p^{\varepsilon,0}(x))(P^{\varepsilon,0})^{-1}(x) \in K \quad \text{for every } x \in \Omega. \tag{4.4.16}$$

Hence, by (4.2.18) and (4.2.20) the following estimate holds true:

$$\begin{aligned}\frac{1}{\varepsilon^{\alpha-1}}\int_\Omega D(P^{\varepsilon,0},P^\varepsilon)\,dx &\leq \frac{1}{\varepsilon^{\alpha-1}}\int_\Omega D(P^{\varepsilon,0},\exp(\varepsilon^{\alpha-1}p^{\varepsilon,0}))\,dx\\ &+ \frac{1}{\varepsilon^{\alpha-1}}\int_\Omega D(\exp(\varepsilon^{\alpha-1}p^{\varepsilon,0}),\exp(\varepsilon^{\alpha-1}p))\,dx\\ &\leq \frac{C}{\varepsilon^{\alpha-1}}\int_\Omega |\exp(\varepsilon^{\alpha-1}p^{\varepsilon,0})(P^{\varepsilon,0})^{-1} - Id|\,dx\\ &+ \frac{1}{\varepsilon^{\alpha-1}}\int_\Omega D(Id,\exp(\varepsilon^{\alpha-1}(p-p^{\varepsilon,0})))\,dx.\end{aligned}$$

By the positive homogeneity of $H_D$ and taking $c(t) = \exp(\varepsilon^{\alpha-1}(p-p^{\varepsilon,0})t)$ in (4.2.17), we obtain

$$\frac{1}{\varepsilon^{\alpha-1}}\int_\Omega D(Id,\exp(\varepsilon^{\alpha-1}(p-p^{\varepsilon,0})))\,dx \leq \int_\Omega H_D(p-p^{\varepsilon,0})\,dx.$$

On the other hand, by (4.4.1) there holds

$$\int_\Omega |\exp(\varepsilon^{\alpha-1}p^{\varepsilon,0})(P^{\varepsilon,0})^{-1} - Id|\,dx \leq c_K\int_\Omega |\exp(\varepsilon^{\alpha-1}p^{\varepsilon,0}) - Id - \varepsilon^{\alpha-1}p^{\varepsilon,0}|\,dx \leq C\varepsilon^{2\alpha-2}.$$

Collecting the previous estimates we deduce

$$\frac{1}{\varepsilon^{\alpha-1}}\int_\Omega D(P^{\varepsilon,0},P^\varepsilon)\,dx \leq \int_\Omega H_D(p-p^{\varepsilon,0})\,dx + C\varepsilon^{\alpha-1},$$

which in turn, by (4.4.2), yields

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon^{\alpha-1}}\int_\Omega D(P^{\varepsilon,0},P^\varepsilon)\,dx \leq \int_\Omega H_D(p-p^0)\,dx. \tag{4.4.17}$$

Let $d \in C^\infty_c(\Omega;\mathbb{R}^3)$ and consider the deformations

$$y^\varepsilon(x) := \begin{pmatrix} x' \\ \varepsilon x_3 \end{pmatrix} + \varepsilon^{\alpha-1}\begin{pmatrix} u(x') - x_3\nabla' v(x') \\ 0 \end{pmatrix} + \varepsilon^{\alpha-2}\begin{pmatrix} 0 \\ v(x') \end{pmatrix} + \varepsilon^\alpha \int_{-\frac{1}{2}}^{x_3} d(x',s)\,ds$$

for every $x \in \Omega$. It is immediate to see that the sequence $(y^\varepsilon)$ fulfills both (4.2.22) and (4.4.3). We note that

$$u^\varepsilon(x') = u(x') + \varepsilon\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{x_3} d'(x',s)\,ds\,dx_3$$

and

$$v^\varepsilon(x') = v(x') + \varepsilon^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{x_3} d_3(x', s)\, ds\, dx_3$$

for every $x' \in \omega$, hence both (4.4.4) and (4.4.5) hold true. To complete the proof of the theorem, it remains to show that for $\alpha > 3$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{2\alpha-2}} \int_\Omega W_{el}(\nabla_\varepsilon y^\varepsilon (P^\varepsilon)^{-1})\, dx = \int_\Omega Q\Big(\mathrm{sym}\Big( \begin{matrix} \nabla' u - x_3 (\nabla')^2 v \\ 0 \end{matrix} \Big| d\Big) - p\Big)\, dx, \quad (4.4.18)$$

and for $\alpha = 3$,

$$\begin{aligned} &\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{2\alpha-2}} \int_\Omega W_{el}(\nabla_\varepsilon y^\varepsilon (P^\varepsilon)^{-1})\, dx \\ &= \int_\Omega Q\Big(\mathrm{sym}\Big( \begin{matrix} \nabla' u + \frac{1}{2}\nabla' v \otimes \nabla' v - x_3 (\nabla')^2 v \\ 0 \end{matrix} \Big| \begin{matrix} d' \\ d_3 + |\nabla' v|^2 \end{matrix} \Big) - p\Big)\, dx. \end{aligned} \quad (4.4.19)$$

Indeed, if (4.4.18) holds, then by a standard approximation argument we may assume that

$$Q\Big(\mathrm{sym}\Big( \begin{matrix} \nabla' u - x_3 (\nabla')^2 v \\ 0 \end{matrix} \Big| d\Big) - p\Big) = Q_2\big(\mathrm{sym}\, \nabla' u - x_3 (\nabla')^2 v - p'\big).$$

Analogously, if (4.4.19) holds we may assume that

$$\begin{aligned} &Q\Big(\mathrm{sym}\Big( \begin{matrix} \nabla' u + \frac{1}{2}\nabla' v \otimes \nabla' v - x_3 (\nabla')^2 v \\ 0 \end{matrix} \Big| \begin{matrix} d' \\ d_3 + \frac{|\nabla' v|^2}{2} \end{matrix} \Big) - p\Big) \\ &= Q_2\Big(\mathrm{sym}\, \nabla' u + \frac{1}{2}\nabla' v \otimes \nabla' v - x_3 (\nabla')^2 v - p'\Big). \end{aligned}$$

In both cases by (4.4.15), (4.4.17), and Theorem 4.3.3, we obtain (4.4.7).

To prove (4.4.18) and (4.4.19) we first note that

$$\nabla_\varepsilon y^\varepsilon = Id + \varepsilon^{\alpha-1} \Big( \begin{matrix} \nabla' u - x_3 (\nabla')^2 v \\ 0 \end{matrix} \Big| d\Big) + \varepsilon^{\alpha-2} \begin{pmatrix} 0 & -\nabla' v \\ (\nabla' v)^T & 0 \end{pmatrix} + O(\varepsilon^\alpha).$$

Hence, in particular, $\det(\nabla_\varepsilon y^\varepsilon) > 0$ for $\varepsilon$ small enough. On the other hand, by the frame-indifference hypothesis (H3), there holds

$$W_{el}(\nabla_\varepsilon y^\varepsilon (P^\varepsilon)^{-1}) = W_{el}\Big(\sqrt{(\nabla_\varepsilon y^\varepsilon)^T \nabla_\varepsilon y^\varepsilon}(P^\varepsilon)^{-1}\Big) \quad \text{a.e. in } \Omega.$$

A direct computation yields

$$\begin{aligned} \sqrt{(\nabla_\varepsilon y^\varepsilon)^T \nabla_\varepsilon y^\varepsilon} &= Id + \varepsilon^{\alpha-1} \mathrm{sym}\Big( \begin{matrix} \nabla' u - x_3 (\nabla')^2 v \\ 0 \end{matrix} \Big| d\Big) \\ &+ \frac{\varepsilon^{2\alpha-4}}{2} \begin{pmatrix} \nabla' v \otimes \nabla' v & 0 \\ 0 & |\nabla' v|^2 \end{pmatrix} + o(\varepsilon^{\alpha-1}), \end{aligned}$$

and

$$W_{el}\big(\nabla_\varepsilon y^\varepsilon (P^\varepsilon)^{-1}\big) = W_{el}\big(Id + \varepsilon^{\alpha-1} M_\alpha + o(\varepsilon^{\alpha-1})\big) \quad \text{a.e. in } \Omega,$$

where

$$M_\alpha := \begin{cases} \mathrm{sym}\Big( \begin{array}{c|c} \nabla' u - x_3(\nabla')^2 v & d \\ 0 & \end{array} \Big) - p & \text{if } \alpha > 3, \\ \mathrm{sym}\Big( \begin{array}{c|c} \nabla' u + \frac{1}{2}\nabla' v \otimes \nabla' v - x_3(\nabla')^2 v & d' \\ 0 & d_3 + \frac{|\nabla' v|^2}{2} \end{array} \Big) - p & \text{if } \alpha = 3. \end{cases}$$

Fix $\delta > 0$. For every $\alpha \geq 3$ we have $M_\alpha \in L^\infty(\Omega; \mathbb{M}^{3\times3})$, therefore for $\varepsilon$ small enough

$$\|\varepsilon^{\alpha-1} M_\alpha + o(\varepsilon^{\alpha-1})\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})} \leq c_{el}(\delta).$$

By (4.2.4), we deduce

$$\limsup_{\varepsilon\to0} \Big| \frac{1}{\varepsilon^{2\alpha-2}} \int_\Omega W_{el}(\nabla_\varepsilon y^\varepsilon (P^\varepsilon)^{-1})\, dx - \int_\Omega Q(M_\alpha)\, dx - \frac{o(\varepsilon^{2\alpha-2})}{\varepsilon^{2\alpha-2}} \Big| \leq \delta \int_\Omega Q(M_\alpha)\, dx.$$

Claims (4.4.18) and (4.4.19) follow now by letting $\delta$ tend to zero. □

## 4.5 Convergence of minimizers and characterization of the limit functional

In this section we deduce convergence of almost minimizers of the three-dimensional energies to minimizers of the limit functional and we show some examples where a characterization of the limit functional can be provided in terms of two-dimensional quantities.

The compactness and liminf inequalities proved in Theorem 4.3.3 and the limsup inequality deduced in Theorem 4.4.1 allow us to obtain the main result of the chapter.

**Theorem 4.5.1.** *Assume that $\alpha \geq 3$ and $\gamma_d$ is a finite union of disjoint (nontrivial) closed intervals in the relative topology of $\partial\omega$. Let $p^0 \in L^\infty(\Omega; \mathbb{M}^{3\times3}_D)$ be such that there exists a sequence $(p^{\varepsilon,0}) \subset L^\infty(\Omega; \mathbb{M}^{3\times3}_D)$ satisfying*

$$\|p^{\varepsilon,0}\|_{L^\infty(\Omega;\mathbb{M}^{3\times3}_D)} \leq C,$$
$$p^{\varepsilon,0} \to p^0 \quad \textit{strongly in } L^1(\Omega; \mathbb{M}^{3\times3}_D).$$

*Assume also that for every $\varepsilon$ the map $P^{\varepsilon,0} := Id + \varepsilon^{\alpha-1} p^{\varepsilon,0}$ satisfies $\det P^{\varepsilon,0} = 1$ a.e. in $\Omega$. Let $\phi^\varepsilon$ be defined as in (4.2.1) and let $\mathcal{J}^\varepsilon_\alpha$ and $\mathcal{J}_\alpha$ be the functionals given by (4.2.23), (4.3.26) and (4.3.27). For every $\varepsilon > 0$, let $(y^\varepsilon, P^\varepsilon) \in \mathcal{A}_\varepsilon(\phi^\varepsilon)$ be such that*

$$\mathcal{J}^\varepsilon_\alpha(y^\varepsilon, P^\varepsilon) - \inf_{(y,P)\in\mathcal{A}_\varepsilon(\phi^\varepsilon)} \mathcal{J}^\varepsilon_\alpha(y, P) \leq s_\varepsilon, \tag{4.5.1}$$

*where $s_\varepsilon \to 0^+$ as $\varepsilon \to 0$. Finally, let $u^\varepsilon$, $v^\varepsilon$ and $p^\varepsilon$ be the displacements and scaled plastic strain introduced in (4.3.2), (4.3.3) and (4.3.41). Then, there exists a triple $(u, v, p) \in \mathcal{A}(u^0, v^0)$ such that, up to subsequences, there holds*

$$u^\varepsilon \to u \quad \textit{strongly in } W^{1,2}(\omega; \mathbb{R}^2), \tag{4.5.2}$$
$$v^\varepsilon \to v \quad \textit{strongly in } W^{1,2}(\omega), \tag{4.5.3}$$
$$p^\varepsilon \to p \quad \textit{strongly in } L^2(\Omega; \mathbb{M}^{3\times3}). \tag{4.5.4}$$

*Moreover,* $(u,v,p)$ *is a minimizer of* $\mathcal{J}_\alpha$ *and*

$$\lim_{\varepsilon\to 0}\mathcal{J}^\varepsilon_\alpha(y^\varepsilon,P^\varepsilon)=\mathcal{J}_\alpha(u,v,p). \tag{4.5.5}$$

*Proof.* By Theorems 4.3.3 and 4.4.1 and by standard arguments in Γ-convergence we deduce (4.5.3), we show that

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } W^{1,2}(\omega;\mathbb{R}^2),$$
$$p^\varepsilon \rightharpoonup p \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}),$$

where $(u,v,p)\in\mathcal{A}(u^0,v^0)$ is a minimizer of $\mathcal{J}_\alpha$, and we prove (4.5.5). Strong convergence of $u^\varepsilon$ and $p^\varepsilon$ follows by (4.5.5) and by adaptating Corollaries 5.4.3 and 5.4.2. □

We remark that the limit plastic strain $p$ depends nontrivially on the $x_3$ variable. Therefore, the limit functionals $\mathcal{J}_\alpha$ cannot, in general, be expressed in terms of two-dimensional quantities only. A characterization of the functionals in terms of the zeroth and first order moments of $p$ can be obtained arguing as follows. Denote by $\bar{p},\hat{p}\in L^2(\omega;\mathbb{M}^{3\times 3}_D)$ and $p_\perp\in L^2(\Omega;\mathbb{M}^{3\times 3}_D)$ the following orthogonal components (in the sense of $L^2(\Omega;\mathbb{M}^{3\times 3}_D)$) of the plastic strain $p$:

$$\bar{p}(x'):=\int_{-\frac12}^{\frac12}p(x',x_3)\,dx_3,\quad \hat{p}(x'):=12\int_{-\frac12}^{\frac12}x_3p(x',x_3)\,dx_3\quad\text{for a.e. } x'\in\omega,$$

and

$$p_\perp(x):=p(x)-\bar{p}(x')-x_3\hat{p}(x')\quad\text{for a.e. } x\in\Omega.$$

Then the functionals $\mathcal{J}_\alpha$ can be written in terms of $\bar{p},\hat{p},p_\perp$ as

$$\begin{aligned}\mathcal{J}_\alpha(u,v,p)\quad &=\int_\omega Q_2(\mathrm{sym}\nabla'u-\bar{p}')\,dx'+\frac{1}{12}\int_\omega Q_2((\nabla')^2v+\hat{p}')\,dx'\\ &+\int_\Omega Q_2(p'_\perp)\,dx+\int_\omega B(\bar{p})\,dx'+\frac{1}{12}\int_\omega B(\hat{p})\,dx'\\ &+\int_\Omega B(p_\perp)\,dx+\int_\Omega H_D(p-p^0)\,dx,\end{aligned}$$

for $\alpha>3$, and

$$\begin{aligned}\mathcal{J}_3(u,v,p)\quad &=\int_\omega Q_2\big(\mathrm{sym}\nabla'u+\tfrac12\nabla'v\otimes\nabla'v-\bar{p}'\big)\,dx'\\ &+\frac{1}{12}\int_\omega Q_2((\nabla')^2v+\hat{p}')\,dx'+\int_\Omega Q_2(p'_\perp)\,dx+\int_\omega B(\bar{p})\,dx'\\ &+\frac{1}{12}\int_\omega B(\hat{p})\,dx'+\int_\Omega B(p_\perp)\,dx+\int_\Omega H_D(p-p^0)\,dx,\end{aligned}$$

for every $(u,v,p)\in\mathcal{A}(u^0,v^0)$.

Under additional hypotheses on the boundary data and the preexistent limit plastic strain $p^0$, some two-dimensional characterizations of the limit model can be deduced in the case $\alpha>3$. To this purpose, we introduce the reduced functionals

$$\bar{\mathcal{J}}_\alpha(u,\bar{p}):=\int_\omega Q_2(\mathrm{sym}\nabla'u-\bar{p}')\,dx'+\int_\omega B(\bar{p})\,dx'+\int_\omega H_D(\bar{p}-\bar{p}^0)\,dx' \tag{4.5.6}$$

for every $(u,\bar{p}) \in W^{1,2}(\omega;\mathbb{R}^2) \times L^2(\omega;\mathbb{M}^{3\times3}_D)$ such that $u = u^0$ $\mathcal{H}^1$ - a.e. on $\gamma_d$, and

$$\hat{\mathcal{J}}_\alpha(v,\hat{p}) := \int_\omega Q_2((\nabla')^2 v + \hat{p}')\,dx' \int_\omega B(\hat{p})\,dx' + \int_\omega H_D(\hat{p}-\hat{p}^0)\,dx', \tag{4.5.7}$$

for every $(v,\hat{p}) \in W^{2,2}(\omega) \times L^2(\omega;\mathbb{M}^{3\times3}_D)$ such that $v = v^0$ and $\nabla' v = \nabla' v^0$ $\mathcal{H}^1$ - a.e. on $\gamma_d$.

We first show an example where $\mathcal{J}_\alpha$ reduces to $\bar{\mathcal{J}}_\alpha$, that is the limit model depends just on the in-plane displacement and the zeroth moment of the plastic strain.

**Theorem 4.5.2.** *Under the hypotheses of Theorem 4.5.1, if $\alpha > 3$, $p^0 = \bar{p}^0$, with $\bar{p}^0 \in L^\infty(\omega;\mathbb{M}^{3\times3}_D)$, and $v^0 = 0$ then, denoting by $\bar{p}$ the zeroth moment of the limit plastic strain $p$, the pair $(u,\bar{p})$ is a minimizer of $\bar{\mathcal{J}}_\alpha$ and*

$$\lim_{\varepsilon\to0} \mathcal{J}^\varepsilon_\alpha(y^\varepsilon, P^\varepsilon) = \bar{\mathcal{J}}_\alpha(u,\bar{p}).$$

*Proof.* By Jensen inequality,

$$\int_\Omega H_D(p-p^0)\,dx \geq \int_\omega H_D(\bar{p}-\bar{p}^0)\,dx',$$

hence there holds

$$\mathcal{J}_\alpha(u,v,p) \geq \bar{\mathcal{J}}_\alpha(u,\bar{p}).$$

On the other hand, by setting

$$\widetilde{P}^\varepsilon := \exp(\varepsilon^{\alpha-1}\bar{p})$$

and

$$\widetilde{y}^\varepsilon := \begin{pmatrix} x', \\ \varepsilon x_3 \end{pmatrix} + \varepsilon^{\alpha-1}\begin{pmatrix} u \\ 0 \end{pmatrix} + \varepsilon^\alpha \int_{-\frac{1}{2}}^{x_3} d(x',s)\,ds,$$

with $d \in C^\infty_c(\Omega;\mathbb{R}^3)$, then $(\widetilde{y}^\varepsilon, \widetilde{P}^\varepsilon) \in \mathcal{A}_\varepsilon(\phi^\varepsilon)$ and an adaptation of Theorem 4.4.1 yields

$$\lim_{\varepsilon\to0} \mathcal{J}^\varepsilon_\alpha(\widetilde{y}^\varepsilon, \widetilde{P}^\varepsilon) = \bar{\mathcal{J}}_\alpha(u,\bar{p}).$$

By combining the previous remarks we have

$$\mathcal{J}_\alpha(u,v,p) \geq \bar{\mathcal{J}}_\alpha(u,\bar{p}) = \lim_{\varepsilon\to0} \mathcal{J}^\varepsilon_\alpha(\widetilde{y}^\varepsilon, \widetilde{P}^\varepsilon) \geq \lim_{\varepsilon\to0} \mathcal{J}^\varepsilon_\alpha(y^\varepsilon, P^\varepsilon).$$

The thesis follows now by Theorem 4.5.1 □

We conclude this section by providing an example where, if $H_D$ is homogeneous of degree one, the $\Gamma$-limit $\mathcal{J}_\alpha$ reduces to $\hat{\mathcal{J}}_\alpha$, that is the limit model depends just on the out-of-plane displacement and the first order moment of the plastic strain.

**Theorem 4.5.3.** *Assume the function $H_D$ to be homogeneous of degree one, i.e.,*

$$H_D(\lambda\xi) = |\lambda| H_D(\xi) \quad \text{for every } \lambda \in \mathbb{R},\ \xi \in \mathbb{M}^{3\times3}. \tag{4.5.8}$$

*Under the hypotheses of Theorem 4.5.1, if $\alpha > 3$, $p^0 = x_3\hat{p}^0$, with $\hat{p}^0 \in L^\infty(\omega;\mathbb{M}^{3\times3}_D)$, and $u^0 = 0$ then, denoting by $\hat{p}$ the first order moment of the limit plastic strain $p$, the pair $(v,\hat{p})$ is a minimizer of $\hat{\mathcal{J}}_\alpha$ and*

$$\lim_{\varepsilon\to0} \mathcal{J}^\varepsilon_\alpha(y^\varepsilon, P^\varepsilon) = \frac{1}{12}\hat{\mathcal{J}}_\alpha(v,\hat{p}).$$

*Proof.* By Jensen inequality and (4.5.8) we deduce,

$$\int_\Omega H_D(p - p^0)\,dx \geq \int_\Omega |x_3| H_D(p - p^0)\,dx = \int_\Omega H_D(x_3 p - x_3 p^0)\,dx \geq \tfrac{1}{12} \int_\omega H_D(\hat{p} - \hat{p}^0)\,dx',$$

which in turn implies

$$\mathcal{J}_\alpha(u, v, p) \geq \frac{1}{12} \hat{\mathcal{J}}_\alpha(v, \hat{p}).$$

On the other hand, by setting

$$\widetilde{P}^\varepsilon := \exp\left(\varepsilon^{\alpha-1} x_3 \hat{p}\right)$$

and

$$\widetilde{y}^\varepsilon := \begin{pmatrix} x', \\ \varepsilon x_3 \end{pmatrix} - \varepsilon^{\alpha-1} x_3 \begin{pmatrix} \nabla' v \\ 0 \end{pmatrix} + \varepsilon^{\alpha-2} \begin{pmatrix} v \\ 0 \end{pmatrix} + \varepsilon^\alpha \int_{-\frac{1}{2}}^{x_3} d(x', s)\,ds,$$

with $d \in C_c^\infty(\Omega; \mathbb{R}^3)$, an adaptation of Theorem 4.4.1 yields

$$\lim_{\varepsilon \to 0} \mathcal{J}_\alpha^\varepsilon(\widetilde{y}^\varepsilon, \widetilde{P}^\varepsilon) = \frac{1}{12} \hat{\mathcal{J}}_\alpha(v, \hat{p}).$$

The conclusion follows now arguing as in the proof of Theorem 4.5.2. □

# Chapter 5

# Quasistatic evolution models for thin plates in finite plasticity

## 5.1 Overview of the chapter

In this chapter we deduce by $\Gamma$-convergence some partially and fully linearized quasistatic evolution models for a thin plate, whose elastic behaviour is nonlinear and whose plastic response is governed by finite plasticity with hardening. Denoting by $\varepsilon$ the thickness of the plate, we study the case where the scaling factor of the elasto-plastic energy is of order $\varepsilon^{2\alpha-2}$, with $\alpha \geq 3$. We show that solutions to the three-dimensional quasistatic evolution problems converge, as the thickness of the plate tends to zero, to quasistatic evolutions associated to the reduced models identified in Chapter 4.

The chapter is organized as follows: in Section 5.2 we set the problem and we prove some preliminary results. Section 5.3 concerns the formulation of the quasistatic evolution problems, the statement of the main result of the chapter and the construction of the mutual recovery sequence, whereas Section 5.4 is entirely devoted to the proofs of the convergence of quasistatic evolutions. Finally, in Section 5.5 we discuss convergence of approximate discrete-time quasistatic evolutions. In the appendix (Section 5.6), we show existence of a quasistatic evolution associated to our reduced model for $\alpha = 3$.

## 5.2 Preliminaries and setting of the problem

Let $\omega \subset \mathbb{R}^2$ be a connected, bounded open set with $C^2$ boundary. Let $\varepsilon > 0$. We assume that the set $\Omega_\varepsilon := \omega \times \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$ is the reference configuration of a finite-strain elastoplastic plate, and every deformation $\eta \in W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)$ fulfills the multiplicative decomposition

$$\nabla\eta(x) = F_{el}(x)F_{pl}(x) \quad \text{for a.e. } x \in \Omega_\varepsilon,$$

where $F_{el} \in L^2(\Omega_\varepsilon;\mathbb{M}^{3\times3})$ represents the elastic strain, $F_{pl} \in L^2(\Omega_\varepsilon;SL(3))$ is the plastic strain and $SL(3) := \{F \in \mathbb{M}^{3\times3} : \det F = 1\}$. The stored energy (per unit thickness) associated to a deformation $\eta$ and to its elastic and plastic strains can be expressed as follows:

$$\begin{aligned}\mathcal{E}(\eta, F_{pl}) &:= \int_{\Omega_\varepsilon} W_{el}(\nabla\eta(x)F_{pl}^{-1}(x))\,dx + \int_{\Omega_\varepsilon} W_{hard}(F_{pl}(x))\,dx, \\ &= \int_{\Omega_\varepsilon} W_{el}(F_{el}(x))\,dx + \int_{\Omega_\varepsilon} W_{hard}(F_{pl}(x))\,dx \end{aligned} \tag{5.2.1}$$

where $W_{el}$ is a nonlinear elastic energy density, $W_{hard}$ describes hardening, and both maps satisfy the same assumptions as in Section 4.2.

Given a preexistent plastic strain $F_{pl}^0 \in L^2(\Omega_\varepsilon; SL(3))$, we define the plastic dissipation potential associated to a plastic configuration $F \in L^2(\Omega_\varepsilon; SL(3))$ as

$$\varepsilon^{\alpha-1}\int_{\Omega_\varepsilon} D(F_{pl}^0; F)\,dx, \tag{5.2.2}$$

where $\alpha \geq 3$ is a given parameter and $D$ is the dissipation distance considered in Section 4.2.

### 5.2.1 Change of variables and formulation of the problem

In this chapter we adopt a slightly different formulation of the problem with respect to that of Chapter 4. Indeed we shall add further regularity assumptions both on $\partial\omega$ and $\gamma_d$ and on the boundary datum $\phi^\varepsilon$. We suppose that the boundary $\partial\omega$ is partitioned into two disjoint open subsets $\gamma_d$ and $\gamma_n$, and their common boundary $\partial\lfloor_{\partial\omega}\gamma_d = \partial\lfloor_{\partial\omega}\gamma_n$ (topological notions refer here to the relative topology of $\partial\omega$). We assume that $\gamma_d$ is nonempty and that $\partial\lfloor_{\partial\omega}\gamma_d = \{P_1, P_2\}$, where $P_1, P_2$ are two points in $\partial\omega$. We denote by $\Gamma_\varepsilon$ the portion of the lateral surface of the plate given by $\Gamma_\varepsilon := \gamma_d \times \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$. On $\Gamma_\varepsilon$ we prescribe a boundary datum of the form

$$\phi^\varepsilon(x) := \begin{pmatrix} x' \\ x_3 \end{pmatrix} + \begin{pmatrix} \varepsilon^{\alpha-1}u^0(x') \\ 0 \end{pmatrix} + \varepsilon^{\alpha-2}\begin{pmatrix} -x_3\nabla' v^0(x') \\ v^0(x') \end{pmatrix} \tag{5.2.3}$$

for every $x = (x', \varepsilon x_3) \in \Omega_\varepsilon$, where $u^0 \in C^1(\overline{\omega}; \mathbb{R}^2)$, $v^0 \in C^2(\overline{\omega})$ and $\alpha \geq 3$ is the same parameter as in (5.2.2).

We consider deformations $\eta \in W^{1,2}(\Omega_\varepsilon; \mathbb{R}^3)$ satisfying

$$\eta = \phi^\varepsilon \quad \mathcal{H}^2\text{ - a.e. on } \Gamma_\varepsilon. \tag{5.2.4}$$

Arguing as in Chapter 4, we consider the set $\Omega := \omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ and the map $\psi^\varepsilon : \overline{\Omega} \to \overline{\Omega}_\varepsilon$ given by

$$\psi^\varepsilon(x) := (x', \varepsilon x_3) \quad \text{for every } x \in \overline{\Omega}. \tag{5.2.5}$$

To every deformation $\eta \in W^{1,2}(\Omega_\varepsilon; \mathbb{R}^3)$ satisfying (5.2.4) and to every plastic strain $F_{pl} \in L^2(\Omega_\varepsilon; SL(3))$, we associate the scaled deformation $y := \eta \circ \psi^\varepsilon$ and the scaled plastic strain $P := F_{pl} \circ \psi^\varepsilon$. Denoting by $\Gamma_d$ the set $\gamma_d \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, the scaled deformation satisfies the boundary condition

$$y = \phi^\varepsilon \circ \psi^\varepsilon \quad \mathcal{H}^2\text{ - a.e. on } \Gamma_d. \tag{5.2.6}$$

We still denote by $\mathcal{A}_\varepsilon(\phi^\varepsilon)$ the class of pairs $(y^\varepsilon, P^\varepsilon) \in W^{1,2}(\Omega;\mathbb{R}^3)\times L^2(\Omega; SL(3))$ such that (5.2.6) is satisfied. Applying the change of variable (5.2.5) to (5.2.1) and (5.2.2), the energy functional is now given by

$$\mathcal{I}(y,P) := \frac{1}{\varepsilon}\mathcal{E}(\eta, F_{pl}) = \int_\Omega W_{el}(\nabla_\varepsilon y(x)P^{-1}(x))\,dx + \int_\Omega W_{hard}(P(x))\,dx, \tag{5.2.7}$$

where $\nabla_\varepsilon y(x) := \big(\partial_1 y(x)\big|\partial_2 y(x)\big|\frac{1}{\varepsilon}\partial_3 y(x)\big)$ for a.e. $x\in\Omega$. The plastic dissipation potential is given by

$$\varepsilon^{\alpha-1}\int_\Omega D(P^{\varepsilon,0}, P)\,dx \tag{5.2.8}$$

where $P^{\varepsilon,0} := F^0_{pl}\circ\psi^\varepsilon$ is a scaled preexistent plastic strain.

### 5.2.2 Approximation results

We still denote by $\mathcal{A}(u^0,v^0)$ the set of triples $(u,v,p)\in W^{1,2}(\Omega;\mathbb{R}^2)\times W^{2,2}(\Omega)\times L^2(\Omega;\mathbb{M}^{3\times 3}_D)$ such that

$$u = u^0, \quad v = v^0, \quad \text{and} \quad \nabla' v = \nabla' v^0 \quad \mathcal{H}^1 \text{ - a.e. on } \gamma_d.$$

We conclude this section by stating an approximation result for triples $(u,v,p)\in\mathcal{A}(u^0,v^0)$ by means of smooth triples. Denoting by $C^\infty_c(\omega\cup\gamma_n)$ the sets of smooth maps having compact support in $\omega\cup\gamma_n$, the following lemma holds true.

**Lemma 5.2.1.** *(i) Let $u\in W^{1,2}(\omega)$ with $u=0$ $\mathcal{H}^1$ - a.e. on $\gamma_d$. Then there exists a sequence $u^k\in C^\infty_c(\omega\cup\gamma_n)$ such that $u^k\to u$ strongly in $W^{1,2}(\omega)$. (ii) Let $v\in W^{2,2}(\omega)$ with $v=0$ and $\nabla' v = 0$ $\mathcal{H}^1$ - a.e. on $\gamma_d$. Then there exists a sequence $v^k\in C^\infty_c(\omega\cup\gamma_n)$ such that $v^k\to v$ strongly in $W^{2,2}(\omega)$.*

*Proof.* The proof is an adaptation of the arguments in Theorem 3.3.9 and Lemma 3.6.10. □

In particular, the previous lemma implies the following density result.

**Corollary 5.2.2.** *Let $(u,v,p)\in\mathcal{A}(u^0,v^0)$. Then there exists a sequence of triples $(u^k,v^k,p^k)\in C^\infty_c(\omega\cup\gamma_n;\mathbb{R}^2)\times C^\infty_c(\omega\cup\gamma_n)\times C^\infty_c(\Omega;\mathbb{M}^{3\times3}_D)$ such that*

$$\begin{aligned} &u^k\to u \quad \text{strongly in } W^{1,2}(\omega;\mathbb{R}^2),\\ &v^k\to v \quad \text{strongly in } W^{2,2}(\omega),\\ &p^k\to p \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}_D).\end{aligned}$$

*Proof.* The approximation of the plastic strain $p$ is obtained by standard arguments. The approximation of the in-plane displacements and out-of-plane displacements follows by applying Lemma 5.2.1 to the maps $u-u^0$ and $v-v^0$. □

## 5.3 The quasistatic evolution problems

In this section we set the quasistatic evolution problem for the scaled energy functional defined in (5.2.7).

For every $t \in [0,T]$ we prescribe a boundary datum $\phi^\varepsilon(t) \in W^{1,\infty}(\Omega;\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3;\mathbb{R}^3)$, defined as

$$\phi^\varepsilon(t,x) := \begin{pmatrix} x' \\ x_3 \end{pmatrix} + \varepsilon^{\alpha-1}\begin{pmatrix} u^0(t,x') \\ 0 \end{pmatrix} + \varepsilon^{\alpha-2}\begin{pmatrix} -x_3\nabla' v^0(t,x') \\ v^0(t,x') \end{pmatrix},$$

for every $x \in \mathbb{R}^3$, where the map $t \mapsto u^0(t)$ is assumed to be $C^1([0,T];C^1(\mathbb{R}^2;\mathbb{R}^2))$ and the map $t \mapsto v^0(t)$ is $C^1([0,T];C^2(\mathbb{R}^2))$. We consider deformations $t \mapsto y^\varepsilon(t)$ from $[0,T]$ into $W^{1,2}(\Omega;\mathbb{R}^3)$ that satisfy

$$y^\varepsilon(t,x) = \phi^\varepsilon(t,(x',\varepsilon x_3)) \quad \mathcal{H}^2 \text{ - a.e. on } \Gamma_d,$$

and plastic strains $t \mapsto P^\varepsilon(t)$ from $[0,T]$ into $L^2(\Omega;SL(3))$.

For technical reasons, it is convenient to modify the map $t \mapsto \phi^\varepsilon(t)$ outside the set $\Omega$. We consider a truncation function $\theta^\varepsilon \in W^{1,\infty}(\mathbb{R}) \cap C^1(\mathbb{R})$ satisfying

$$\theta^\varepsilon(s) = s \quad \text{in } (-\ell_\varepsilon,\ell_\varepsilon), \tag{5.3.1}$$

$$|\theta^\varepsilon(s)| \le |s| \text{ for every } s \in \mathbb{R}, \tag{5.3.2}$$

$$\|\theta^\varepsilon\|_{L^\infty(\mathbb{R})} \le 2\ell_\varepsilon, \tag{5.3.3}$$

$$\dot{\theta^\varepsilon}(s) = 0 \quad \text{if } |x_3| \ge \ell_\varepsilon + 1, \tag{5.3.4}$$

$$\|\dot{\theta^\varepsilon}(s)\|_{L^\infty(\mathbb{R})} \le 2, \tag{5.3.5}$$

where $\ell_\varepsilon$ is such that

$$\varepsilon^{\alpha-1-\gamma}\ell_\varepsilon \to 0, \tag{5.3.6}$$

$$\varepsilon\ell_\varepsilon \to +\infty, \tag{5.3.7}$$

$$\varepsilon^{2\alpha-2}\ell_\varepsilon^3 \to 0, \tag{5.3.8}$$

for some $0 < \gamma < \alpha - 2$. For $\alpha > 3$ we also require

$$\varepsilon^{\alpha-1}\ell_\varepsilon^2 \to 0. \tag{5.3.9}$$

**Remark 5.3.1.** A possible choice of $\ell_\varepsilon$ is $\ell_\varepsilon = \frac{1}{\varepsilon^{1+\lambda}}$, with $0 < \lambda < \min\{\frac{\alpha-3}{2}, \alpha-2-\gamma\}$ when $\alpha > 3$, and $0 < \lambda < \min\{\frac{1}{3}, 1-\gamma\}$ in the case $\alpha = 3$.

With a slight abuse of notation, for every $t \in [0,T]$ we still denote by $\phi^\varepsilon(t)$ the map defined as

$$\phi^\varepsilon(t,x) := \begin{pmatrix} x' \\ x_3 \end{pmatrix} + \varepsilon^{\alpha-1}\begin{pmatrix} u^0(t,x') - \theta^\varepsilon\left(\frac{x_3}{\varepsilon}\right)\nabla' v^0(t,x') \\ 0 \end{pmatrix} + \varepsilon^{\alpha-2}\begin{pmatrix} 0 \\ v^0(t,x') \end{pmatrix} \tag{5.3.10}$$

for every $x \in \mathbb{R}^3$.

**Remark 5.3.2.** Conditions (5.3.1) and (5.3.7) guarantee that $\phi^\varepsilon(t)$ is indeed an extension of the originally prescribed boundary datum, for $\varepsilon$ small enough. Conditions (5.3.3) and (5.3.5) provide a uniform bound with respect to $t$ on the $W^{1,\infty}(\mathbb{R}^3;\mathbb{R}^3)$ norm of $\phi^\varepsilon(t) - id$. By (5.3.3), (5.3.5) and (5.3.6), there exists $\varepsilon_0 > 0$ such that, for every $t \in [0,T]$ and $\varepsilon < \varepsilon_0$, the map $\phi^\varepsilon(t) : \mathbb{R}^3 \to \mathbb{R}^3$ is invertible with smooth inverse $\varphi^\varepsilon(t)$. Since

$$\phi^\varepsilon(t,\varphi^\varepsilon(t,x)) = x \quad \text{for every } x \in \mathbb{R}^3,$$

by (5.3.10) there holds

$$(\varphi^\varepsilon)'(t) - x' = -\varepsilon^{\alpha-1}u^0(t,(\varphi^\varepsilon)'(t)) + \varepsilon^{\alpha-1}\theta^\varepsilon\Big(\frac{\varphi_3^\varepsilon(t)}{\varepsilon}\Big)\nabla' v^0(t,(\varphi^\varepsilon)'(t)), \quad (5.3.11)$$

$$\varphi_3^\varepsilon(t) - x_3 = -\varepsilon^{\alpha-2}v^0(t,(\varphi^\varepsilon)'(t)), \quad (5.3.12)$$

for every $t \in [0,T]$. Hence, by the smoothness of $u^0$ and $v^0$ and by (5.3.3), we deduce the estimates

$$\|(\varphi^\varepsilon)'(t) - x'\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^2)} \le C\varepsilon^{\alpha-1}\ell_\varepsilon, \quad (5.3.13)$$

and

$$\|\varphi_3^\varepsilon(t) - x_3\|_{L^\infty(\mathbb{R}^3)} \le C\varepsilon^{\alpha-2}, \quad (5.3.14)$$

where both constants are independent of $t$. In particular, (5.3.11) yields

$$\begin{aligned}\nabla(\varphi^\varepsilon)'(t) - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0\end{pmatrix} \quad &= -\varepsilon^{\alpha-1}\nabla' u^0(t,(\varphi^\varepsilon)'(t))\nabla(\varphi^\varepsilon)'(t) \\ &+\varepsilon^{\alpha-1}\theta^\varepsilon\Big(\frac{\varphi_3^\varepsilon(t)}{\varepsilon}\Big)(\nabla')^2 v^0(t,(\varphi^\varepsilon)'(t))\nabla(\varphi^\varepsilon)'(t) \\ &+\varepsilon^{\alpha-2}\dot{\theta}^\varepsilon\Big(\frac{\varphi_3^\varepsilon(t)}{\varepsilon}\Big)\nabla' v^0(t,(\varphi^\varepsilon)'(t)) \otimes \nabla\varphi_3^\varepsilon(t), \quad (5.3.15)\end{aligned}$$

and (5.3.12) implies

$$\nabla\varphi_3^\varepsilon(t) - e_3 = -\varepsilon^{\alpha-2}(\nabla(\varphi^\varepsilon)'(t))^T\nabla' v^0(t,(\varphi^\varepsilon)'(t)), \quad (5.3.16)$$

for every $t \in [0,T]$.

A direct computation shows that

$$\begin{aligned}\nabla\phi^\varepsilon(t,x) = Id + \varepsilon^{\alpha-1}\begin{pmatrix} \nabla' u^0(t,x') & 0 \\ 0 & 0\end{pmatrix} - \varepsilon^{\alpha-1}\begin{pmatrix} \theta^\varepsilon\big(\frac{x_3}{\varepsilon}\big)(\nabla')^2 v^0(t,x') & 0 \\ 0 & 0\end{pmatrix} \\ +\varepsilon^{\alpha-2}\begin{pmatrix} 0 & -\dot{\theta}^\varepsilon\big(\frac{x_3}{\varepsilon}\big)\nabla' v^0(t,x') \\ (\nabla' v^0(t,x'))^T & 0\end{pmatrix} \quad \text{for every } x \in \mathbb{R}^3. \quad (5.3.17)\end{aligned}$$

Hence by (5.3.3), (5.3.5) and (5.3.6) there holds

$$\|\nabla\varphi^\varepsilon(t)\|_{L^\infty(\mathbb{R}^3;\mathbb{M}^{3\times3})} \le \|(\nabla\phi^\varepsilon(t))^{-1}\|_{L^\infty(\mathbb{R}^3;\mathbb{M}^{3\times3})} \le C, \quad (5.3.18)$$

for every $t \in [0,T]$ and for every $\varepsilon < \varepsilon_0$. Therefore, (5.3.3), (5.3.5), (5.3.7), (5.3.15) and (5.3.16) yield

$$\|\nabla(\varphi^\varepsilon)'(t) - (e_1|e_2|0)\|_{L^\infty(\mathbb{R}^3;\mathbb{M}^{3\times2})} \le C\varepsilon^{\alpha-1}\ell_\varepsilon, \quad (5.3.19)$$

and

$$\|\nabla\varphi_3^\varepsilon(t) - e_3\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \le C\varepsilon^{\alpha-2}. \quad (5.3.20)$$

By Remark 5.3.2 for $\varepsilon$ small enough the function $\phi^\varepsilon(t)$ is a smooth diffeomorphism for every $t \in [0,T]$. This implies that we are allowed to define a map $t \mapsto z^\varepsilon(t)$ from $[0,T]$ into $W^{1,2}(\Omega;\mathbb{R}^3)$ as the pointwise solution of

$$y^\varepsilon(t,x) = \phi^\varepsilon(t,z^\varepsilon(t,x))$$

for every $t \in [0,T]$. We note that

$$z^\varepsilon(t) = (x', \varepsilon x_3) \quad \mathcal{H}^2\text{ - a.e. on } \Gamma_d \tag{5.3.21}$$

for every $t \in [0,T]$. According to this change of variable, the elastic energy at time $t$ associated to the deformation $y^\varepsilon(t)$ can be written in terms of $z^\varepsilon(t)$ as

$$\int_\Omega W_{el}(\nabla_\varepsilon y^\varepsilon(t)(P^\varepsilon)^{-1}(t))\,dx = \int_\Omega W_{el}\big(\nabla\phi^\varepsilon(t, z^\varepsilon(t))\nabla_\varepsilon z^\varepsilon(t)(P^\varepsilon)^{-1}(t)\big)\,dx.$$

For every $t \in [0,T]$ we define the three-dimensional stress as

$$E^\varepsilon(t) := \frac{1}{\varepsilon^{\alpha-1}} DW_{el}\Big(\nabla\phi^\varepsilon(t, z^\varepsilon(t))\nabla_\varepsilon z^\varepsilon(t)(P^\varepsilon)^{-1}(t)\Big)\Big(\nabla\phi^\varepsilon(t, z^\varepsilon(t))\nabla_\varepsilon z^\varepsilon(t)(P^\varepsilon)^{-1}(t)\Big)^T.$$

Let $s_1, s_2 \in [0,T]$, with $s_1 \le s_2$. For every function $t \mapsto P(t)$ from $[0,T]$ into $L^2(\Omega; SL(3))$, we define its dissipation as

$$\mathcal{D}(P; s_1, s_2) := \sup\Big\{\sum_{i=1}^N \int_\Omega D(P(t_{i-1}), P(t_i))\,dx : s_1 = t_0 < t_1 < \cdots < t_N = s_2\Big\}.$$

Analogously, for every function $t \mapsto p(t)$ from $[0,T]$ into $L^2(\Omega; \mathbb{M}^{3\times 3}_D)$, we define its $H_D$-dissipation as

$$\mathcal{D}_{H_D}(p; s_1, s_2) := \sup\Big\{\sum_{i=1}^N \int_\Omega H_D(p(t_i) - p(t_{i-1}))\,dx : s_1 = t_0 < t_1 < \cdots < t_N = s_2\Big\}.$$

Finally, we denote by $\mathcal{F}_\varepsilon(t, z, P)$ the quantity

$$\mathcal{F}_\varepsilon(t, z, P) := \int_\Omega W_{el}\big(\nabla\phi^\varepsilon(t, z)\nabla_\varepsilon z P^{-1}\big)\,dx + \int_\Omega W_{hard}(P)\,dx$$

for every $t \in [0,T]$, $z \in W^{1,2}(\Omega; \mathbb{R}^3)$ and $P \in L^2(\Omega; SL(3))$. We are now in a position to give the definition of quasistatic evolution associated to the boundary datum $t \mapsto \phi^\varepsilon(t)$.

**Definition 5.3.3.** Let $\varepsilon > 0$. An $\varepsilon$*-quasistatic evolution* for the boundary datum $t \mapsto \phi^\varepsilon(t)$ is a function $t \mapsto (z^\varepsilon(t), P^\varepsilon(t))$ from $[0,T]$ into $W^{1,2}(\Omega; \mathbb{R}^3) \times L^2(\Omega; SL(3))$ that satisfies the following conditions:

(gs) *global stability:* for every $t \in [0,T]$ we have $z^\varepsilon(t,x) = (x', \varepsilon x_3)$ $\mathcal{H}^2$ - a.e. on $\Gamma_d$, $P^\varepsilon(t,x) \in K$ for a.e. $x \in \Omega$ and

$$\mathcal{F}_\varepsilon(t, z^\varepsilon(t), P^\varepsilon(t)) \le \mathcal{F}_\varepsilon(t, \tilde{z}, \tilde{P}) + \varepsilon^{\alpha-1}\int_\Omega D(P^\varepsilon(t), \tilde{P})\,dx,$$

for every $(\tilde{z}, \tilde{P}) \in W^{1,2}(\Omega; \mathbb{R}^3) \times L^2(\Omega; SL(3))$ such that $\tilde{z}(x) = (x', \varepsilon x_3)$ $\mathcal{H}^2$ - a.e. on $\Gamma_d$ and $\tilde{P}(x) \in K$ for a.e. $x \in \Omega$;

(eb) *energy balance:* the map

$$s \mapsto \int_\Omega E^\varepsilon(s) : \Big(\nabla\dot{\phi}^\varepsilon(s, z^\varepsilon(s))(\nabla\phi^\varepsilon)^{-1}(s, z^\varepsilon(s))\Big)\,dx$$

is integrable in $[0,T]$ and for every $t \in [0,T]$

$$\begin{aligned}&\mathcal{F}_\varepsilon(t, z^\varepsilon(t), P^\varepsilon(t)) + \varepsilon^{\alpha-1}\mathcal{D}(P^\varepsilon; 0, t)\\ &= \mathcal{F}_\varepsilon(0, z^\varepsilon(0), P^\varepsilon(0)) + \varepsilon^{\alpha-1}\int_0^t\int_\Omega E^\varepsilon(s) : \Big(\nabla\dot{\phi}^\varepsilon(s, z^\varepsilon(s))(\nabla\phi^\varepsilon)^{-1}(s, z^\varepsilon(s))\Big)\,dx\,ds.\end{aligned}$$

**Remark 5.3.4.** We remark that if the function $t \to (z^\varepsilon(t), P^\varepsilon(t))$ satisfies condition (gs), then $E^\varepsilon(t) \in L^1(\Omega; \mathbb{M}^{3\times 3})$ for every $t \in [0,T]$. Indeed, by (gs), taking $\tilde{z}(x) = (x', \varepsilon x_3)$ for every $x \in \Omega$ and $\tilde{P} = P^\varepsilon(t)$, we deduce

$$\int_\Omega W_{el}\big(\nabla\phi^\varepsilon(t, z^\varepsilon(t))\nabla_\varepsilon z^\varepsilon(t)(P^\varepsilon)^{-1}(t)\big)\,dx \le \int_\Omega W_{el}\big(\nabla\phi^\varepsilon(t,(x',\varepsilon x_3))(P^\varepsilon)^{-1}(t)\big)\,dx. \tag{5.3.22}$$

On the other hand, $P^\varepsilon(t,x) \in K$ for a.e. $x \in \Omega$ and for $\varepsilon$ small enough there exists two constants $C_1$ and $C_2$ such that $\det(\nabla\phi^\varepsilon(t,(x',\varepsilon x_3))) \ge C_1$ for every $x \in \Omega$ and $\|\nabla\phi^\varepsilon(t,(x',\varepsilon x_3))\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})} \le C_2$. Therefore, by hypothesis (H1) (see Section 4.2) the quantity in (5.3.22) is finite and

$$\det\big(\nabla\phi^\varepsilon(t, z^\varepsilon(t))\nabla_\varepsilon z^\varepsilon(t)(P^\varepsilon)^{-1}(t)\big) > 0 \quad \text{a.e. in } \Omega \tag{5.3.23}$$

for $\varepsilon$ small enough. Hence, by (4.2.5) we obtain

$$\int_\Omega |E^\varepsilon(t)|\,dx \le \frac{c_3}{\varepsilon^{\alpha-1}}\Big(\int_\Omega W_{el}(\nabla\phi^\varepsilon(t,z^\varepsilon(t))\nabla_\varepsilon z^\varepsilon(t)(P^\varepsilon)^{-1}(t))\,dx + 1\Big) < +\infty.$$

**Remark 5.3.5.** By the frame-indifference (H3) of $W_{el}$ (see Section 4.2), there holds

$$DW_{el}(F)F^T = F(DW_{el}(F))^T \quad \text{for every } F \in \mathbb{M}^{3\times3}_+.$$

Therefore, by (5.3.23), for $\varepsilon$ small enough $E^\varepsilon(t,x) \in \mathbb{M}^{3\times3}_{sym}$ for every $t \in [0,T]$ and for a.e. $x \in \Omega$.

Set

$$L_\alpha := \begin{cases} 0 & \text{if } \alpha > 3 \\ 1 & \text{if } \alpha = 3. \end{cases}$$

For every $\alpha \ge 3$ we define a reduced quasistatic evolution as follows.

**Definition 5.3.6.** For $\alpha \ge 3$, a *reduced quasistatic evolution* for the boundary data $t \mapsto u^0(t)$ and $t \mapsto v^0(t)$ is a map $t \mapsto (u(t), v(t), p(t))$ from $[0,T]$ into $W^{1,2}(\omega;\mathbb{R}^2) \times W^{2,2}(\omega) \times L^2(\Omega;\mathbb{M}^{3\times3}_D)$, that satisfies the following conditions:

$(\mathrm{gs})_{r\alpha}$ for every $t \in [0,T]$ there holds $(u(t), v(t), p(t)) \in \mathcal{A}(u^0(t), v^0(t))$, and setting

$$e_\alpha(t) := \mathrm{sym}\nabla' u(t) + \tfrac{L_\alpha}{2}\nabla' v(t) \otimes \nabla' v(t) - x_3(\nabla')^2 v(t) - p'(t), \tag{5.3.24}$$

we have

$$\int_\Omega Q_2(e_\alpha(t))\,dx + \int_\Omega B(p(t))\,dx \le \int_\Omega Q_2\big(\mathrm{sym}\,\nabla'\hat{u} + \tfrac{L_\alpha}{2}\nabla'\hat{v}\otimes\nabla'\hat{v} - x_3(\nabla')^2\hat{v} - \hat{p}'\big)\,dx$$
$$+ \int_\Omega B(\hat{p})\,dx + \int_\Omega H_D(\hat{p} - p(t))\,dx,$$

for every $(\hat{u}, \hat{v}, \hat{p}) \in \mathcal{A}(u^0(t), v^0(t))$;

$(\mathrm{eb})_{r\alpha}$ the map

$$s \to \int_\Omega \mathbb{C}_2 e_\alpha(s) : \begin{pmatrix} \nabla'\dot{u}^0(s) + L_\alpha\nabla'\dot{v}^0(s)\otimes\nabla' v(s) - x_3(\nabla')^2\dot{v}^0(s) & 0 \\ 0 & 0 \end{pmatrix} dx$$

is integrable in $[0,T]$. Moreover for every $t \in [0,T]$ there holds

$$\int_\Omega Q_2(e_\alpha(t))\,dx + \int_\Omega B(p(t))\,dx + \mathcal{D}_H(p;0,t) = \int_\Omega Q_2(e_\alpha(0))\,dx + \int_\Omega B(p(0))\,dx$$
$$+ \int_0^t \int_\Omega \mathbb{C}_2 e_\alpha(s) : \begin{pmatrix} \nabla'\dot{u}^0(s) + L_\alpha \nabla'\dot{v}^0(s) \otimes \nabla' v(s) - x_3(\nabla')^2 \dot{v}^0(s) & 0 \\ 0 & 0 \end{pmatrix} dx\,ds.$$

**Remark 5.3.7.** An adaptation of [15, Theorem 4.5] guarantees that, if $\alpha > 3$, for every triple $(\overline{u}, \overline{v}, \overline{p}) \in \mathcal{A}(u^0(0), v^0(0))$ satisfying

$$\int_\Omega Q_2\big(\mathrm{sym}\nabla'\overline{u} - x_3(\nabla')^2\overline{v} + \tfrac{L_\alpha}{2}\nabla'\overline{v} \otimes \nabla'\overline{v} - \overline{p}'\big)\,dx + \int_\Omega B(\overline{p})\,dx$$
$$\leq \int_\Omega Q_2\big(\mathrm{sym}\nabla'\hat{u} - x_3(\nabla')^2\hat{v} + \tfrac{L_\alpha}{2}\nabla'\hat{v} \otimes \nabla'\hat{v} - \hat{p}'\big)\,dx + \int_\Omega B(\hat{p})\,dx + \int_\Omega H_D(\hat{p} - \overline{p})\,dx,$$

for every $(\hat{u}, \hat{v}, \hat{p}) \in \mathcal{A}(u^0(0), v^0(0))$, there exists a reduced quasistatic evolution $t \mapsto (u(t), v(t), p(t))$ (according to Definition 5.3.6) such that $u(0) = \overline{u}$, $v(0) = \overline{v}$ and $p(0) = \overline{p}$. Moreover, by adapting [15, Theorem 5.2 and Remark 5.4] one can show that the maps $t \mapsto u(t)$, $t \mapsto v(t)$ and $t \mapsto p(t)$ are Lipschitz continuous from $[0,T]$ into $W^{1,2}(\omega;\mathbb{R}^2)$, $W^{2,2}(\omega)$ and $L^2(\Omega;\mathbb{M}^{3\times 3}_D)$, respectively.

In the case $\alpha = 3$, the existence of a reduced quasistatic evolution $t \mapsto (u(t), v(t), p(t))$ such that $(u(0), v(0), p(0)) = (\overline{u}, \overline{v}, \overline{p})$ can still be proved by adapting [15, Theorem 4.5]. We remark that the proof of this result is more subtle than its counterpart in the case $\alpha > 3$, due to the presence of the nonlinear term $\frac{1}{2}\nabla' v \otimes \nabla' v$. In fact, some further difficulties arise when trying to prove the analogous of [15, Theorem 4.7], that is, to deduce the converse energy inequality by the minimality. To do this, one can apply [16, Lemma 4.12], which guarantees the existence of partitions of $[0,T]$ on which the Bochner integrals of some relevant quantities can be approximated by Riemann sums, and argue as in [6, Lemma 5.7] (see Theorem 5.6.3).

**Remark 5.3.8.** By taking $\hat{p} = p(t)$ in $(\mathrm{gs})_{r\alpha}$, it follows that a reduced quasistatic evolution $t \mapsto (u(t), v(t), p(t))$ satisfies

$$\int_\Omega Q_2(e_\alpha(t))\,dx \leq \int_\Omega Q_2(\mathrm{sym}\,\nabla'\hat{u} + \tfrac{L_\alpha}{2}\nabla'\hat{v} \otimes \nabla'\hat{v} - x_3(\nabla')^2\hat{v} - p'(t))\,dx$$

for every $(\hat{u}, \hat{v}) \in W^{1,2}(\omega;\mathbb{R}^2) \times W^{2,2}(\omega)$ such that

$$\hat{u} = u^0(t),\ \hat{v} = v^0(t) \text{ and } \nabla'\hat{v} = \nabla' v^0(t) \quad \mathcal{H}^1 \text{ - a.e. on } \gamma_d.$$

Hence, in particular, there holds

$$\int_\Omega \mathbb{C}_2 e_\alpha(t) : \nabla'\zeta\,dx = 0$$

for every $\zeta \in W^{1,2}(\omega;\mathbb{R}^2)$ such that $\zeta = 0$ $\mathcal{H}^1$ - a.e. on $\gamma_d$.

With the previous definitions at hand we are in a position to state the main result of the chapter.

**Theorem 5.3.9.** *Let $\alpha \geq 3$. Assume that $t \mapsto u^0(t)$ belongs to $C^1([0,T]; W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2) \cap C^1(\mathbb{R}^2;\mathbb{R}^2))$ and $t \mapsto v^0(t)$ belongs to $C^1([0,T]; W^{2,\infty}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2))$, respectively. For every $t \in [0,T]$, let $\phi^\varepsilon(t)$ be defined as in (5.3.10). Let $(\mathring{u}, \mathring{v}, \mathring{p}) \in \mathcal{A}(u^0(0), v^0(0))$ be such that*

$$\int_\Omega Q_2(\mathrm{sym}\nabla'\mathring{u} - x_3(\nabla')^2\mathring{v} + \tfrac{L_\alpha}{2}\nabla'\mathring{v} \otimes \nabla'\mathring{v} - \mathring{p}')\,dx + \int_\Omega B(\mathring{p})\,dx$$
$$\leq \int_\omega Q_2(\nabla'\hat{u} - x_3(\nabla')^2\hat{v} + \tfrac{L_\alpha}{2}\nabla'\hat{v} \otimes \nabla'\hat{v} - \hat{p}')\,dx' + \int_\Omega B(\hat{p})\,dx + \int_\Omega H(\hat{p} - \mathring{p})\,dx,$$

*for every $(\hat{u}, \hat{v}, \hat{p}) \in \mathcal{A}(u^0(0), v^0(0))$. Assume there exists a sequence of pairs $(y_0^\varepsilon, P_0^\varepsilon) \in \mathcal{A}_\varepsilon(\phi^\varepsilon(0))$ such that*

$$\mathcal{I}(y_0^\varepsilon, P_0^\varepsilon) \leq \mathcal{I}(\hat{y}, \hat{P}) + \varepsilon^{\alpha-1}\int_\Omega D(P_0^\varepsilon, \hat{P})\,dx, \tag{5.3.25}$$

*for every $(\hat{y}, \hat{P}) \in \mathcal{A}_\varepsilon(\phi^\varepsilon(0))$, and*

$$u_0^\varepsilon := \frac{1}{\varepsilon^{\alpha-1}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \big((y_0^\varepsilon)' - x'\big)\,dx_3 \to \mathring{u} \quad \textit{strongly in } W^{1,2}(\omega;\mathbb{R}^2), \tag{5.3.26}$$

$$v_0^\varepsilon := \frac{1}{\varepsilon^{\alpha-2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} (y_0^\varepsilon)_3\,dx_3 \to \mathring{v} \quad \textit{strongly in } W^{1,2}(\omega), \tag{5.3.27}$$

$$p_0^\varepsilon := \frac{P_0^\varepsilon - Id}{\varepsilon^{\alpha-1}} \to \mathring{p} \quad \textit{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}_D), \tag{5.3.28}$$

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon^{2\alpha-2}}\mathcal{I}(y_0^\varepsilon, P_0^\varepsilon) = \int_\Omega Q_2(\mathrm{sym}\nabla'\mathring{u} - x_3(\nabla')^2\mathring{v} + \tfrac{L_\alpha}{2}\nabla'\mathring{v}\otimes\nabla'\mathring{v} - \mathring{p}')\,dx$$
$$+ \int_\Omega B(\mathring{p})\,dx. \tag{5.3.29}$$

*Finally, for every $\varepsilon > 0$, let $t \mapsto (z^\varepsilon(t), P^\varepsilon(t))$ be an $\varepsilon$-quasistatic evolution for the boundary datum $\phi^\varepsilon(t)$ such that*

$$z^\varepsilon(0) = \varphi^\varepsilon(0, y_0^\varepsilon) \quad \textit{a.e. in } \Omega$$

*and*

$$P^\varepsilon(0) = P_0^\varepsilon.$$

*Then, there exists a reduced quasistatic evolution $t \mapsto (u(t), v(t), p(t))$ for the boundary data $(u^0(t), v^0(t))$ (according to Definition 5.3.6), such that $u(0) = \mathring{u}$, $v(0) = \mathring{v}$, $p(0) = \mathring{p}$ and, up to subsequences,*

$$p^\varepsilon(t) := \frac{P^\varepsilon(t) - Id}{\varepsilon^{\alpha-1}} \rightharpoonup p(t) \quad \textit{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}) \tag{5.3.30}$$

*for every $t \in [0,T]$. Moreover, for $\alpha > 3$ up to subsequences there holds*

$$u^\varepsilon(t) := \frac{1}{\varepsilon^{\alpha-1}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \big((\phi^\varepsilon)'(t, z^\varepsilon(t)) - x'\big)\,dx_3 \rightharpoonup u(t) \quad \textit{weakly in } W^{1,2}(\omega;\mathbb{R}^2), \tag{5.3.31}$$

$$v^\varepsilon(t) := \frac{1}{\varepsilon^{\alpha-2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \phi_3^\varepsilon(t, z^\varepsilon(t))\,dx_3 \to v(t) \quad \textit{strongly in } W^{1,2}(\omega), \tag{5.3.32}$$

*for every $t \in [0,T]$. For $\alpha = 3$, for every $t \in [0,T]$ there exists a $t$-dependent subsequence $\varepsilon_j \to 0$ such that*

$$u^{\varepsilon_{jt}}(t) := \frac{1}{\varepsilon_{jt}^{\alpha-1}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \big((\phi^{\varepsilon_{jt}})'(t, z^{\varepsilon_{jt}}(t)) - x'\big)\, dx_3 \rightharpoonup u(t) \text{ weakly in } W^{1,2}(\omega;\mathbb{R}^2), \tag{5.3.33}$$

$$v^{\varepsilon_{jt}}(t) := \frac{1}{\varepsilon_{jt}^{\alpha-2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi_3^{\varepsilon_{jt}}(t, z^{\varepsilon_{jt}}(t))\, dx_3 \to v(t) \quad \text{strongly in } W^{1,2}(\omega). \tag{5.3.34}$$

**Remark 5.3.10.** In the case $\alpha > 3$ the convergence result is stronger than the analogous result for $\alpha = 3$ as the convergence of $u^\varepsilon(t)$ and $v^\varepsilon(t)$ holds on a subsequence independent of $t$. This is related to the fact that, for $\alpha > 3$, once $t \mapsto p(t)$ is identified, both $t \mapsto u(t)$ and $t \mapsto v(t)$ are uniquely determined. In the case $\alpha = 3$ this property is not true anymore because of the presence of the nonlinear term $\frac{1}{2}\nabla' v(t) \otimes \nabla' v(t)$.

We shall prove the previous theorem in the next section. To conclude this section, we prove a technical lemma concerning some properties of the truncation maps $\theta^\varepsilon$ and we provide the construction of the so-called "joint recovery sequence", that will be used in the proof of Theorem 5.3.9.

**Lemma 5.3.11.** *Let $\theta^\varepsilon \in W^{1,\infty}(\mathbb{R}) \cap C^1(\mathbb{R})$ be such that (5.3.1)–(5.3.7) hold and let $(\zeta^\varepsilon)$ be a sequence in $L^2(\Omega)$ such that*

$$\|\zeta^\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon. \tag{5.3.35}$$

*Then,*

$$\Big\|1 - \dot{\theta^\varepsilon}\Big(\frac{\zeta^\varepsilon}{\varepsilon}\Big)\Big\|_{L^2(\Omega)} \leq \frac{3}{\ell_\varepsilon}. \tag{5.3.36}$$

*Moreover, if $\zeta^\varepsilon$ satisfies*

$$\Big\|\frac{\zeta^\varepsilon}{\varepsilon} - x_3 - \varepsilon^{\alpha-3} v\Big\|_{L^2(\Omega)} \to 0, \tag{5.3.37}$$

*for some $v \in L^2(\omega)$, then*

$$\theta^\varepsilon\Big(\frac{\zeta^\varepsilon}{\varepsilon}\Big) \to \begin{cases} x_3 & \text{if } \alpha > 3 \\ x_3 + v & \text{if } \alpha = 3 \end{cases} \quad \text{strongly in } L^2(\Omega). \tag{5.3.38}$$

*Proof.* Denoting by $O_\varepsilon$ the set

$$O_\varepsilon := \Big\{x \in \Omega : |\zeta^\varepsilon(x)| \geq \varepsilon\ell_\varepsilon\Big\},$$

by (5.3.35) and by Chebychev inequality, there holds

$$\mathcal{L}^3(O_\varepsilon) \leq \frac{C}{\ell_\varepsilon^2}.$$

Hence, by (5.3.1) and (5.3.5),

$$\Big\|1 - \dot{\theta^\varepsilon}\Big(\frac{\zeta^\varepsilon}{\varepsilon}\Big)\Big\|_{L^2(\Omega)} = \Big\|1 - \dot{\theta^\varepsilon}\Big(\frac{\zeta^\varepsilon}{\varepsilon}\Big)\Big\|_{L^2(O_\varepsilon)} \leq \frac{3}{\ell_\varepsilon}.$$

To prove (5.3.38), we note that by (5.3.37) there holds

$$\theta^\varepsilon\Big(\frac{\zeta^\varepsilon}{\varepsilon}\Big) \to \begin{cases} x_3 & \text{if } \alpha > 3 \\ x_3 + v & \text{if } \alpha = 3 \end{cases} \quad \text{a.e. in } \Omega.$$

On the other hand, (5.3.2) yields $\big|\theta^\varepsilon\big(\frac{\zeta^\varepsilon}{\varepsilon}\big)\big| \leq \big|\frac{\zeta^\varepsilon}{\varepsilon}\big|$ for every $\varepsilon$ and for a.e. $x \in \Omega$. Therefore (5.3.38) follows by the dominated convergence theorem. □

For the sake of simplicity, in the next theorem we omit the time dependence of $u^0$ and $v^0$. With a slight abuse of notation, we denote by $\phi^\varepsilon$ the map

$$\phi^\varepsilon(x) := \begin{pmatrix} x' \\ x_3 \end{pmatrix} + \varepsilon^{\alpha-1}\begin{pmatrix} u^0(x') - \theta^\varepsilon(\frac{x_3}{\varepsilon})\nabla' v^0(x') \\ 0 \end{pmatrix} + \varepsilon^{\alpha-2}\begin{pmatrix} 0 \\ v^0(x') \end{pmatrix},$$

for a.e. $x \in \Omega$, where $u^0 \in W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2) \cap C^1(\mathbb{R}^2;\mathbb{R}^2)$ and $v^0 \in W^{2,\infty}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)$. We are now in a position to construct the joint recovery sequence.

**Theorem 5.3.12.** *Let $(y^\varepsilon, P^\varepsilon) \in \mathcal{A}_\varepsilon(\phi^\varepsilon)$ satisfy (4.3.29) for every $\varepsilon > 0$. Let $u, v, G, p$ be defined as in Theorem 4.3.3 and let $\hat{u} := u + \tilde{u}$, $\hat{v} := v + \tilde{v}$, and $\hat{p} := p + \tilde{p}$, where $\tilde{u} \in C^\infty_c(\omega \cup \gamma_n;\mathbb{R}^2)$, $\tilde{v} \in C^\infty_c(\omega \cup \gamma_n)$ and $\tilde{p} \in C^\infty_c(\Omega;\mathbb{M}^{3\times 3}_D)$. Then, there exists a sequence of pairs $(\hat{y}^\varepsilon, \hat{P}^\varepsilon) \in \mathcal{A}_\varepsilon(\phi^\varepsilon)$, such that*

$$\hat{y}^\varepsilon \to \begin{pmatrix} x' \\ 0 \end{pmatrix} \quad \textit{strongly in } W^{1,2}(\Omega;\mathbb{R}^3), \tag{5.3.39}$$

$$\hat{u}^\varepsilon := \frac{1}{\varepsilon^{\alpha-1}}\int_{-\frac12}^{\frac12} \big((\hat{y}^\varepsilon)' - x'\big)\,dx_3 \rightharpoonup \hat{u} \quad \textit{weakly in } W^{1,2}(\omega;\mathbb{R}^2) \textit{ for } \alpha > 3, \tag{5.3.40}$$

$$\hat{u}^\varepsilon \rightharpoonup \hat{u} - v\nabla\tilde{v} \quad \textit{weakly in } W^{1,2}(\omega;\mathbb{R}^2) \textit{ for } \alpha = 3, \tag{5.3.41}$$

$$\frac{1}{\varepsilon^{\alpha-2}}\int_{-\frac12}^{\frac12} \hat{y}^\varepsilon_3\,dx_3 \to \hat{v} \quad \textit{strongly in } W^{1,2}(\omega), \tag{5.3.42}$$

$$\hat{P}^\varepsilon(x) \in K \quad \textit{for a.e. } x \in \Omega, \tag{5.3.43}$$

$$\hat{p}^\varepsilon := \frac{\hat{P}^\varepsilon - Id}{\varepsilon^{\alpha-1}} \rightharpoonup \hat{p} \quad \textit{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}). \tag{5.3.44}$$

*Moreover, the following inequalities hold true:*

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^{2\alpha-2}}\Big(\int_\Omega W_{hard}(\hat{P}^\varepsilon)\,dx - \int_\Omega W_{hard}(P^\varepsilon)\,dx\Big) \leq \int_\Omega B(\hat{p})\,dx - \int_\Omega B(p)\,dx, \tag{5.3.45}$$

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^{\alpha-1}}\int_\Omega D(P^\varepsilon, \hat{P}^\varepsilon)\,dx \leq \int_\Omega H_D(\hat{p} - p)\,dx, \tag{5.3.46}$$

*and*

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^{2\alpha-2}}\Big(\int_\Omega W_{el}(\nabla_\varepsilon \hat{y}^\varepsilon(\hat{P}^\varepsilon)^{-1})\,dx - \int_\Omega W_{el}(\nabla_\varepsilon y^\varepsilon(P^\varepsilon)^{-1})\,dx\Big) \\ &\leq \int_\Omega Q_2(\operatorname{sym}\hat{G}' - \hat{p}')\,dx - \int_\Omega Q_2(\operatorname{sym} G' - p')\,dx,\end{aligned} \tag{5.3.47}$$

*where the submatrix $\hat{G}'$ satisfies*

$$\hat{G}'(x', x_3) := \hat{G}_0(x') - x_3(\nabla')^2\hat{v}(x') \quad \textit{for a.e. } x \in \Omega,$$

*and*

$$\operatorname{sym}\hat{G}_0 = \frac{(\nabla'\hat{u} + (\nabla'\hat{u})^T + \nabla'\hat{v} \otimes \nabla'\hat{v})}{2} \quad \textit{for } \alpha = 3,$$
$$\operatorname{sym}\hat{G}_0 = \operatorname{sym}\nabla'\hat{u} \quad \textit{for } \alpha > 3.$$

*Proof.* We divide the proof into four steps. In the first step we exhibit a sequence of deformations $(\hat{y}^\varepsilon)$ satisfying (5.3.39)–(5.3.42). In the second step we construct a sequence $(\hat{P}^\varepsilon)$ of plastic strains and we prove the limsup inequality for the hardening and the dissipation terms. In the third step we rewrite the elastic energy in terms of some auxiliary quantities and in the fourth step we prove the limsup inequality for the elastic energy.

We first remark that by (4.3.29) and the boundary condition

$$y^\varepsilon(x) = \phi^\varepsilon(x', \varepsilon x_3) \quad \mathcal{H}^2 \text{ - a.e. on } \Gamma_d, \tag{5.3.48}$$

the sequence $(y^\varepsilon, P^\varepsilon)$ fulfills the hypotheses of Theorems 4.3.1 and 4.3.3. Hence, there exists a sequence $(R^\varepsilon) \subset W^{1,\infty}(\omega; \mathbb{M}^{3\times3})$ such that (4.3.6)–(4.3.9) hold true, and $(y^\varepsilon)$ satisfies (4.3.30). Moreover, defining $u^\varepsilon, v^\varepsilon$, and $G^\varepsilon$ according to (4.3.2), (4.3.3) and (4.3.28), properties (4.3.31)–(4.3.38) hold true. The sequence of plastic strains $(P^\varepsilon)$ satisfies

$$P^\varepsilon(x) \in K \quad \text{for a.e. } x \in \Omega, \tag{5.3.49}$$

and defining $p^\varepsilon$ as in (4.3.41), there holds

$$p^\varepsilon \rightharpoonup p \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{3\times3}). \tag{5.3.50}$$

Finally, by Theorem 4.3.3, $(u, v, p) \in \mathcal{A}(u^0, v^0)$ and, by (4.3.32) and (4.3.34), the sequence $(y_3^\varepsilon)$ fulfills the hypotheses of Lemma 5.3.11, hence

$$\theta^\varepsilon\Big(\frac{y_3^\varepsilon}{\varepsilon}\Big) \to \begin{cases} x_3 & \text{if } \alpha > 3 \\ x_3 + v & \text{if } \alpha = 3 \end{cases} \quad \text{strongly in } L^2(\Omega), \tag{5.3.51}$$

and by (5.3.7) and (5.3.36) we have

$$\frac{1}{\varepsilon} - \frac{1}{\varepsilon}\dot{\theta}^\varepsilon\Big(\frac{y_3^\varepsilon}{\varepsilon}\Big) \to 0 \quad \text{strongly in } L^2(\Omega). \tag{5.3.52}$$

*Step 1: Construction of the deformations*
Let $d \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$ with $\operatorname{supp} d \subset \Omega$. Consider the map

$$\eta^\varepsilon(x) := \int_{-\frac{1}{2}}^{\frac{x_3}{\varepsilon}} d(x', s)\, ds \quad \text{for every } x \in \mathbb{R}^3.$$

Since $d$ has compact support in $\Omega$, there holds

$$|\eta^\varepsilon(x)| \le \int_{-\frac{1}{2}}^{\left|\frac{x_3}{\varepsilon}\right|} |d(x', s)|\, ds \le \int_{-\frac{1}{2}}^{\frac{1}{2}} |d(x', s)|\, ds \le \|d\|_{L^\infty(\mathbb{R}^3)} \quad \text{for every } x \in \mathbb{R}^3$$

and analogously

$$\|\nabla'\eta^\varepsilon\|_{L^\infty(\mathbb{R}^3;\mathbb{M}^{3\times 2})} \leq \|\nabla' d\|_{L^\infty(\mathbb{R}^3;\mathbb{M}^{3\times 2})}. \tag{5.3.53}$$

A straightforward computation yields

$$\partial_3\eta^\varepsilon(x) = \frac{1}{\varepsilon} d\Big(x', \frac{x_3}{\varepsilon}\Big) \quad \text{for every } x \in \mathbb{R}^3. \tag{5.3.54}$$

Hence,

$$\|\eta^\varepsilon\|_{W^{1,\infty}(\mathbb{R}^3;\mathbb{R}^3)} \leq \frac{C}{\varepsilon}. \tag{5.3.55}$$

In particular, the map $\eta^\varepsilon \circ y^\varepsilon$ satisfies

$$\|\eta^\varepsilon \circ y^\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^3)} \leq C, \tag{5.3.56}$$

$$\|\nabla'(\eta^\varepsilon \circ y^\varepsilon)\|_{L^2(\Omega;\mathbb{M}^{3\times 2})} \leq C\|\nabla'(y^\varepsilon)'\|_{L^2(\Omega;\mathbb{M}^{2\times 2})} + \frac{C}{\varepsilon}\|\nabla' y_3^\varepsilon\|_{L^2(\Omega;\mathbb{R}^2)}. \tag{5.3.57}$$

We extend $\tilde{u}$ and $\tilde{v}$ to zero outside their support, we consider the functions

$$f^\varepsilon(x) := x + \begin{pmatrix} \varepsilon^{\alpha-1}\tilde{u}(x') \\ \varepsilon^{\alpha-2}\tilde{v}(x') \end{pmatrix} - \begin{pmatrix} \varepsilon^{\alpha-1}\theta^\varepsilon\big(\frac{x_3}{\varepsilon}\big)\nabla'\tilde{v}(x') \\ 0 \end{pmatrix} + \varepsilon^\alpha\eta^\varepsilon(x)$$

for every $x \in \mathbb{R}^3$, and we set

$$\hat{y}^\varepsilon := f^\varepsilon \circ y^\varepsilon.$$

It is easy to see that $\hat{y}^\varepsilon \in W^{1,2}(\Omega;\mathbb{R}^3)$, we now check that

$$\hat{y}^\varepsilon = \phi^\varepsilon(x', \varepsilon x_3) \quad \mathcal{H}^2 \text{ - a.e. on } \Gamma_d. \tag{5.3.58}$$

To prove it, we first remark that by (5.3.48)

$$\hat{y}^\varepsilon = f^\varepsilon(\phi^\varepsilon(x', \varepsilon x_3)) \quad \mathcal{H}^2 \text{ - a.e. on } \Gamma_d. \tag{5.3.59}$$

Hence, it remains only to show that

$$f^\varepsilon(\phi^\varepsilon(x', \varepsilon x_3)) = \phi^\varepsilon(x', \varepsilon x_3) \quad \mathcal{H}^2 \text{ - a.e. on } \Gamma_d. \tag{5.3.60}$$

Let $A \subset \mathbb{R}^2$ be an open set such that $\overline{\gamma_d} \subset (A \cap \partial\omega)$ and $\tilde{u}, \tilde{v}, \nabla'\tilde{v} = 0$ in $A$. Since $d$ has compact support in $\Omega$, without loss of generality we may assume that $\eta^\varepsilon(x) = 0$ for all $x \in A \times \mathbb{R}$ and every $\varepsilon$. Therefore, we have $f^\varepsilon(x) = x$ in $A \times \mathbb{R}$. Let now $O \subset \mathbb{R}^2$ be an open set such that $\overline{\gamma_d} \subset (O \cap \partial\omega)$ and $\overline{O} \subset A$, and let $0 < \delta_0 < \mathrm{dist}(O, \partial A)$. By (5.3.2), there holds

$$|(\phi^\varepsilon)'(x', \varepsilon x_3) - x'| \leq \varepsilon^{\alpha-1}\|u^0\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)} + \frac{1}{2}\varepsilon^{\alpha-2}\|\nabla' v^0\|_{L^\infty(\mathbb{R}^2;\mathbb{M}^{2\times 2})} < \frac{\delta_0}{2}$$

for every $x \in O \times \big(-\frac{1}{2}, \frac{1}{2}\big)$, for $\varepsilon$ small enough. Hence, $\phi^\varepsilon(x', \varepsilon x_3) \in A \times \mathbb{R}$ for every $x \in O \times \big(-\frac{1}{2}, \frac{1}{2}\big)$, and $f^\varepsilon(\phi^\varepsilon(x', \varepsilon x_3)) = \phi^\varepsilon(x', \varepsilon x_3)$ for every $x \in O \times \big(-\frac{1}{2}, \frac{1}{2}\big)$. This implies (5.3.60) and (5.3.58).

To prove (5.3.39), we remark that by the smoothness of $\tilde{u}$ and $\tilde{v}$, estimates (5.3.3), (5.3.5), (5.3.7) and (5.3.55) imply

$$\|f^\varepsilon - id\|_{W^{1,\infty}(\mathbb{R}^3;\mathbb{R}^3)} \leq C\varepsilon^{\alpha-1}\ell_\varepsilon. \tag{5.3.61}$$

On the other hand, we have

$$\Big\|\hat{y}^\varepsilon - \begin{pmatrix} x' \\ 0 \end{pmatrix}\Big\|_{W^{1,2}(\Omega;\mathbb{R}^3)} \leq \|\hat{y}^\varepsilon - y^\varepsilon\|_{W^{1,2}(\Omega;\mathbb{R}^3)} + \Big\|y^\varepsilon - \begin{pmatrix} x' \\ 0 \end{pmatrix}\Big\|_{W^{1,2}(\Omega;\mathbb{R}^3)}$$

$$\leq C\|f^\varepsilon - id\|_{W^{1,\infty}(\mathbb{R}^3;\mathbb{R}^3)}\|\nabla y^\varepsilon\|_{L^2(\Omega;\mathbb{M}^{3\times3})} + \Big\|y^\varepsilon - \begin{pmatrix} x' \\ 0 \end{pmatrix}\Big\|_{W^{1,2}(\Omega;\mathbb{R}^3)},$$

so that (5.3.39) follows by (4.3.30), (5.3.6) and (5.3.61).

We now prove convergence of the out-of-plane displacements associated to $(\hat{y}^\varepsilon)$. To show (5.3.42) we note that

$$\hat{v}^\varepsilon = \frac{1}{\varepsilon^{\alpha-2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} f_3^\varepsilon(y^\varepsilon)\,dx_3 = v^\varepsilon + \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{v}((y^\varepsilon)')\,dx_3 + \varepsilon^2\int_{-\frac{1}{2}}^{\frac{1}{2}} \eta_3^\varepsilon(y^\varepsilon)\,dx_3.$$

By (4.3.30), up to subsequences, we can assume

$$(y^\varepsilon)' \to x' \quad \text{and } \nabla'(y^\varepsilon)' \to Id \quad \text{a.e. in } \Omega. \tag{5.3.62}$$

Hence, by the dominated convergence theorem and the smoothness of $\tilde{v}$ we obtain

$$\tilde{v}((y^\varepsilon)') \to \tilde{v} \quad \text{strongly in } L^2(\Omega)$$

and

$$\nabla'\tilde{v}((y^\varepsilon)') \to \nabla'\tilde{v} \quad \text{strongly in } L^2(\Omega;\mathbb{R}^2).$$

By (4.3.30), (4.3.32), (5.3.56) and (5.3.57) we conclude

$$\hat{v}^\varepsilon \to v + \tilde{v} = \hat{v} \quad \text{strongly in } W^{1,2}(\omega).$$

To prove (5.3.40) and (5.3.41) we note that

$$\hat{u}^\varepsilon = u^\varepsilon + \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{u}((y^\varepsilon)')\,dx_3 - \int_{-\frac{1}{2}}^{\frac{1}{2}} \theta^\varepsilon\Big(\frac{y_3^\varepsilon}{\varepsilon}\Big)\nabla'\tilde{v}((y^\varepsilon)')\,dx_3$$
$$+\varepsilon\int_{-\frac{1}{2}}^{\frac{1}{2}} (\eta^\varepsilon)'(y^\varepsilon)\,dx_3. \tag{5.3.63}$$

By (5.3.51), (5.3.62) and the dominated convergence theorem,

$$\tilde{u}((y^\varepsilon)') \to \tilde{u} \quad \text{strongly in } L^2(\Omega;\mathbb{R}^2),$$
$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \theta^\varepsilon\Big(\frac{y_3^\varepsilon}{\varepsilon}\Big)\nabla'\tilde{v}((y^\varepsilon)')\,dx_3 \to 0 \quad \text{strongly in } L^2(\omega;\mathbb{R}^2) \quad \text{for } \alpha > 3,$$
$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \theta^\varepsilon\Big(\frac{y_3^\varepsilon}{\varepsilon}\Big)\nabla'\tilde{v}((y^\varepsilon)')\,dx_3 \to v\nabla\tilde{v} \quad \text{strongly in } L^2(\omega;\mathbb{R}^2) \quad \text{for } \alpha = 3.$$

Hence, by (5.3.56), we have

$$\hat{u}^\varepsilon \to \hat{u} \quad \text{strongly in } L^2(\omega;\mathbb{R}^2) \quad \text{for } \alpha > 3,$$

and

$$\hat{u}^\varepsilon \to \hat{u} - v\nabla\tilde{v} \quad \text{strongly in } L^2(\omega;\mathbb{R}^2) \quad \text{for } \alpha = 3.$$

To complete the proof of (5.3.40) and (5.3.41), it remains to show that

$$\frac{1}{\varepsilon^{\alpha-1}}\nabla'\hat{u}^\varepsilon \text{ is bounded in } L^2(\Omega;\mathbb{M}^{2\times 2}). \tag{5.3.64}$$

By (5.3.63) there holds

$$\frac{1}{\varepsilon^{\alpha-1}}\nabla'\hat{u}^\varepsilon = \nabla' u^\varepsilon + \int_{-\frac12}^{\frac12} \nabla'\tilde{u}((y^\varepsilon)')\nabla'(y^\varepsilon)'\,dx_3$$
$$-\int_{-\frac12}^{\frac12}\theta^\varepsilon\Big(\frac{y_3^\varepsilon}{\varepsilon}\Big)(\nabla')^2\tilde{v}((y^\varepsilon)')\nabla'(y^\varepsilon)'\,dx_3 - \frac{1}{\varepsilon}\int_{-\frac12}^{\frac12}(\theta^\varepsilon)'\Big(\frac{y_3^\varepsilon}{\varepsilon}\Big)\nabla'\tilde{v}((y^\varepsilon)')\otimes\nabla' y_3^\varepsilon\,dx_3$$
$$+\varepsilon\int_{-\frac12}^{\frac12}\nabla'(\eta^\varepsilon\circ y^\varepsilon)\,dx_3.$$

By adding and subtracting the matrix $(R^\varepsilon)'$ we obtain

$$\int_{-\frac12}^{\frac12}\theta^\varepsilon\Big(\frac{y_3^\varepsilon}{\varepsilon}\Big)(\nabla')^2\tilde{v}((y^\varepsilon)')\nabla'(y^\varepsilon)'\,dx_3 = \int_{-\frac12}^{\frac12}\theta^\varepsilon\Big(\frac{y_3^\varepsilon}{\varepsilon}\Big)(\nabla')^2\tilde{v}((y^\varepsilon)')\big(\nabla'(y^\varepsilon)'-(R^\varepsilon)'\big)\,dx_3$$
$$+\int_{-\frac12}^{\frac12}\theta^\varepsilon\Big(\frac{y_3^\varepsilon}{\varepsilon}\Big)(\nabla')^2\tilde{v}((y^\varepsilon)')(R^\varepsilon)'\,dx_3.$$

Combining (4.3.7) and (5.3.3), we deduce

$$\Big\|\theta^\varepsilon\Big(\frac{y_3^\varepsilon}{\varepsilon}\Big)(\nabla')^2\tilde{v}((y^\varepsilon)')\big(\nabla'(y^\varepsilon)'-(R^\varepsilon)'\big)\Big\|_{L^2(\Omega;\mathbb{M}^{2\times 2})}\leq C\varepsilon^{\alpha-1}\ell_\varepsilon.$$

On the other hand, by (4.3.6) and (5.3.51), the maps $\theta^\varepsilon\Big(\frac{y_3^\varepsilon}{\varepsilon}\Big)\nabla^2\tilde{v}((y^\varepsilon)')(R^\varepsilon)'$ are bounded in $L^2(\Omega;\mathbb{M}^{2\times 2})$. The $L^2$-boundedness of the quantity in (5.3.64) follows now by combining (4.3.30), (4.3.31), (4.3.33), (5.3.5) and (5.3.57).

*Step 2: Construction of the plastic strains*

Arguing as in [52, Proof of Lemma 3.6], we introduce the sets

$$S_\varepsilon := \{x\in\Omega : \exp(\varepsilon^{\alpha-1}\tilde{p}(x))P^\varepsilon(x)\in K\},$$

we define

$$\hat{p}^\varepsilon := \begin{cases}\frac{1}{\varepsilon^{\alpha-1}}\big(\exp(\varepsilon^{\alpha-1}\tilde{p})P^\varepsilon - Id\big) & \text{in } S_\varepsilon,\\ p^\varepsilon & \text{in } \Omega\setminus S_\varepsilon,\end{cases}$$

and

$$\hat{P}^\varepsilon := Id + \varepsilon^{\alpha-1}\hat{p}^\varepsilon,$$

so that, by (5.3.49), the sequence $(\hat{P}^\varepsilon)$ satisfies (5.3.43). Since tr $\tilde{p}=0$,

$$\det(\exp(\varepsilon^{\alpha-1}\tilde{p})) = \exp(\varepsilon^{\alpha-1}\text{tr }\tilde{p}) = 1,$$

therefore

$$\exp(\varepsilon^{\alpha-1}\tilde{p}(x))P^\varepsilon(x)\in SL(3) \quad \text{for a.e. } x\in\Omega. \tag{5.3.65}$$

By (5.3.65) we can estimate $\mathcal{L}^3(\Omega \setminus S_\varepsilon)$. Indeed by (4.2.14) and (5.3.50) there holds

$$\begin{aligned}
&\mathcal{L}^3(\Omega \setminus S_\varepsilon) \leq c_k^2 \int_\Omega |\big(\exp(\varepsilon^{\alpha-1}\tilde{p}(x))P^\varepsilon(x) - Id|^2\, dx \\
&= c_k^2 \int_\Omega |\big(\exp(\varepsilon^{\alpha-1}\tilde{p}(x)) + \varepsilon^{\alpha-1}\exp(\varepsilon^{\alpha-1}\tilde{p}(x))p^\varepsilon(x) - Id|^2\, dx \\
&\leq C\varepsilon^{2(\alpha-1)} \int_\Omega (1 + |p^\varepsilon(x)|^2)\, dx \leq C\varepsilon^{2(\alpha-1)}. \qquad (5.3.66)
\end{aligned}$$

Now,

$$\hat{p}^\varepsilon - p^\varepsilon = \begin{cases} \frac{1}{\varepsilon^{\alpha-1}}\big(\exp(\varepsilon^{\alpha-1}\tilde{p}) - Id\big)P^\varepsilon & \text{in } S_\varepsilon, \\ 0 & \text{in } \Omega \setminus S_\varepsilon. \end{cases} \qquad (5.3.67)$$

By (5.3.49), (5.3.50) and (5.3.66) we deduce the following convergence properties:

$$\begin{cases} \|\hat{p}^\varepsilon - p^\varepsilon\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})} \leq C, \\ \hat{p}^\varepsilon - p^\varepsilon \to \tilde{p} \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}), \\ \hat{p}^\varepsilon + p^\varepsilon \rightharpoonup \hat{p} + p \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}), \end{cases} \qquad (5.3.68)$$

hence in particular (5.3.44) holds true. Arguing exactly as in [52, Proof of Lemma 3.6, Step 2 and Step 4], we obtain (5.3.45) and (5.3.46).

*Step 3: Convergence properties of the elastic energy*

To complete the proof of the theorem it remains to prove (5.3.47). To this purpose, let $w^\varepsilon$ be the map defined as

$$w^\varepsilon := \frac{(P^\varepsilon)^{-1} - Id + \varepsilon^{\alpha-1}p^\varepsilon}{\varepsilon^{\alpha-1}} = \varepsilon^{\alpha-1}(P^\varepsilon)^{-1}(p^\varepsilon)^2. \qquad (5.3.69)$$

By (4.2.13) and (5.3.49), there exists a constant $C$ such that

$$\varepsilon^{\alpha-1}\|p^\varepsilon\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})} \leq C$$

and

$$\varepsilon^{\alpha-1}\|w^\varepsilon\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})} \leq C \qquad (5.3.70)$$

for every $\varepsilon$. Furthermore, by (5.3.50),

$$\|w^\varepsilon\|_{L^1(\Omega;\mathbb{M}^{3\times3})} \leq C\varepsilon^{\alpha-1} \quad \text{for every } \varepsilon.$$

By the two previous estimates it follows that $(w^\varepsilon)$ is uniformly bounded in $L^2(\Omega;\mathbb{M}^{3\times3})$ and

$$w^\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}). \qquad (5.3.71)$$

Now, by (4.3.28) and the frame-indifference property (H3) of $W_{el}$ (see Section 4.2) there holds

$$\begin{aligned}
W_{el}(\nabla_\varepsilon y^\varepsilon (P^\varepsilon)^{-1}) &= W_{el}\big((Id + \varepsilon^{\alpha-1}G^\varepsilon)(Id + \varepsilon^{\alpha-1}(w^\varepsilon - p^\varepsilon))\big) \\
&= W_{el}(Id + \varepsilon^{\alpha-1}F^\varepsilon), \qquad (5.3.72)
\end{aligned}$$

for a.e. $x \in \Omega$, where

$$F^\varepsilon := G^\varepsilon + w^\varepsilon - p^\varepsilon + \varepsilon^{\alpha-1}G^\varepsilon(w^\varepsilon - p^\varepsilon). \qquad (5.3.73)$$

We note that

$$\|G^\varepsilon(w^\varepsilon - p^\varepsilon)\|_{L^1(\Omega;\mathbb{M}^{3\times3})} \leq C$$

by (4.3.35), (5.3.50) and (5.3.71). Moreover, by (4.3.35), (5.3.49) and (5.3.70),

$$\varepsilon^{\alpha-1}\|G^\varepsilon(w^\varepsilon - p^\varepsilon)\|_{L^2(\Omega;\mathbb{M}^{3\times3})} \leq \varepsilon^{\alpha-1}\|G^\varepsilon\|_{L^2(\Omega;\mathbb{M}^{3\times3})}\|(w^\varepsilon - p^\varepsilon)\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})} \leq C$$

for every $\varepsilon$. Hence

$$\varepsilon^{\alpha-1}G^\varepsilon(w^\varepsilon - p^\varepsilon) \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}),$$

which in turn, by (4.3.35), (5.3.50) and (5.3.71), yields

$$F^\varepsilon \rightharpoonup G - p \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}). \tag{5.3.74}$$

Analogously, we define

$$\hat{w}^\varepsilon := \frac{(\hat{P}^\varepsilon)^{-1} - Id + \varepsilon^{\alpha-1}\hat{p}^\varepsilon}{\varepsilon^{\alpha-1}} = \varepsilon^{\alpha-1}(\hat{P}^\varepsilon)^{-1}(\hat{p}^\varepsilon)^2. \tag{5.3.75}$$

Then,

$$(\hat{P}^\varepsilon)^{-1} = Id + \varepsilon^{\alpha-1}(\hat{w}^\varepsilon - \hat{p}^\varepsilon),$$

by (4.2.13) and (5.3.43) we deduce

$$\varepsilon^{\alpha-1}\|\hat{w}^\varepsilon\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})} \leq C,$$

and by (5.3.44),

$$\hat{w}^\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}).$$

We define

$$\hat{G}^\varepsilon := G^\varepsilon + \hat{w}^\varepsilon - \hat{p}^\varepsilon + \varepsilon^{\alpha-1}G^\varepsilon(\hat{w}^\varepsilon - \hat{p}^\varepsilon). \tag{5.3.76}$$

Arguing as before, we can prove that

$$\hat{G}^\varepsilon \rightharpoonup G - \hat{p} \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}). \tag{5.3.77}$$

We shall prove that there exists a sequence $(\hat{F}^\varepsilon) \subset L^2(\Omega;\mathbb{M}^{3\times3})$ satisfying

$$W_{el}(\nabla_\varepsilon \hat{y}^\varepsilon(\hat{P}^\varepsilon)^{-1}) = W_{el}(Id + \varepsilon^{\alpha-1}\hat{F}^\varepsilon)$$

and such that

$$\hat{F}^\varepsilon - \hat{G}^\varepsilon \to N_\alpha \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}), \tag{5.3.78}$$

where

$$N_\alpha := \operatorname{sym}\begin{pmatrix} \nabla'\tilde{u} - x_3(\nabla')^2\tilde{v} & \Big| d \\ 0 & \end{pmatrix} \quad \text{for } \alpha > 3, \tag{5.3.79}$$

and

$$N_3 := \operatorname{sym}\begin{pmatrix} \nabla\tilde{u} - (x_3+v)(\nabla')^2\tilde{v} + \frac{\nabla'\tilde{v}\otimes\nabla'\tilde{v}}{2} & \Big| & d'(x', x_3+v) \\ 0 & & d_3(x', x_3+v) + \frac{1}{2}|\nabla'\tilde{v}|^2 \end{pmatrix} \tag{5.3.80}$$

a.e. in $\Omega$. To this purpose, we first observe that by (4.3.28), (5.3.76) and the frame-indifference hypothesis (H3) (see Section 4.2) there holds

$$\begin{aligned} & W_{el}(\nabla_\varepsilon \hat{y}^\varepsilon(\hat{P}^\varepsilon)^{-1}) = W_{el}\big(\nabla f^\varepsilon(y^\varepsilon)\nabla_\varepsilon y^\varepsilon(\hat{P}^\varepsilon)^{-1}\big) \\ & = W_{el}\Big((R^\varepsilon)^T\sqrt{(\nabla f^\varepsilon(y^\varepsilon))^T\nabla f^\varepsilon(y^\varepsilon)}R^\varepsilon(Id+\varepsilon^{\alpha-1}\hat{G}^\varepsilon)\Big). \end{aligned} \tag{5.3.81}$$

We set

$$M^\varepsilon(x) := \frac{\nabla f^\varepsilon(x) - Id}{\varepsilon^{\alpha-1}\ell_\varepsilon}.$$

By (5.3.61) there holds

$$\|M^\varepsilon(y^\varepsilon)\|_{L^\infty(\Omega;\mathbb{M}^{3\times 3})} \leq C \tag{5.3.82}$$

for every $\varepsilon$.

We claim that, to prove (5.3.78) it is enough to show that

$$\ell_\varepsilon(R^\varepsilon)^T\mathrm{sym}\big(M^\varepsilon(y^\varepsilon)\big)R^\varepsilon \to \begin{cases} \mathrm{sym}\begin{pmatrix} \nabla'\tilde{u} - x_3(\nabla')^2\tilde{v} & \Big| d \\ 0 & \end{pmatrix} & \text{if } \alpha > 3 \\ \mathrm{sym}\begin{pmatrix} \nabla'\tilde{u} - (x_3+v)(\nabla')^2\tilde{v} & \Big| d(x', x_3+v) \\ 0 & \end{pmatrix} & \text{if } \alpha = 3 \end{cases} \tag{5.3.83}$$

strongly in $L^2(\Omega;\mathbb{M}^{3\times 3})$, and

$$\varepsilon^2\ell_\varepsilon^2(R^\varepsilon)^T(M^\varepsilon(y^\varepsilon))^T M^\varepsilon(y^\varepsilon)R^\varepsilon \to \begin{pmatrix} \nabla'\tilde{v}\otimes\nabla'\tilde{v} & 0 \\ 0 & |\nabla'\tilde{v}|^2 \end{pmatrix} \quad \text{if } \alpha = 3 \tag{5.3.84}$$

strongly in $L^2(\Omega;\mathbb{M}^{3\times 3})$. Indeed, a Taylor expansion around the identity yields

$$\sqrt{(Id+F)^T(Id+F)} = Id + \mathrm{sym}\,F + \frac{F^TF}{2} - \frac{(\mathrm{sym}\,F)^2}{2} + O(|F|^3)$$

for every $F \in \mathbb{M}^{3\times 3}$. Hence,

$$\begin{aligned} & \sqrt{(\nabla f^\varepsilon(y^\varepsilon))^T\nabla f^\varepsilon(y^\varepsilon)} = Id + \varepsilon^{\alpha-1}\ell_\varepsilon\,\mathrm{sym}\,M^\varepsilon(y^\varepsilon) + \frac{\varepsilon^{2\alpha-2}\ell_\varepsilon^2}{2}(M^\varepsilon(y^\varepsilon))^TM^\varepsilon(y^\varepsilon) \\ & -\frac{\varepsilon^{2\alpha-2}\ell_\varepsilon^2}{2}\big(\mathrm{sym}\,M^\varepsilon(y^\varepsilon)\big)^2 + O(\varepsilon^{3\alpha-3}\ell_\varepsilon^3). \end{aligned}$$

Substituting the previous expression into (5.3.81) we obtain

$$W_{el}(\nabla_\varepsilon\hat{y}^\varepsilon(\hat{P}^\varepsilon)^{-1}) = W_{el}(Id+\varepsilon^{\alpha-1}\hat{F}^\varepsilon) \tag{5.3.85}$$

where

$$\begin{aligned} \hat{F}^\varepsilon \quad = \quad & \hat{G}^\varepsilon + \ell_\varepsilon(R^\varepsilon)^T\mathrm{sym}\,M^\varepsilon(y^\varepsilon)R^\varepsilon + \frac{\varepsilon^{\alpha-1}\ell_\varepsilon^2}{2}(R^\varepsilon)^T(M^\varepsilon(y^\varepsilon))^TM^\varepsilon(y^\varepsilon)R^\varepsilon \\ & -\frac{\varepsilon^{\alpha-1}\ell_\varepsilon^2}{2}(R^\varepsilon)^T\big(\mathrm{sym}\,M^\varepsilon(y^\varepsilon)\big)^2R^\varepsilon + \varepsilon^{\alpha-1}\ell_\varepsilon(R^\varepsilon)^T\mathrm{sym}\,M^\varepsilon(y^\varepsilon)R^\varepsilon\hat{G}^\varepsilon + O(\varepsilon^{2\alpha-2}\ell_\varepsilon^3) \\ & +O(\varepsilon^{2\alpha-2}\ell_\varepsilon^2)\hat{G}^\varepsilon \end{aligned}$$

Now, if $\alpha > 3$, by (4.3.6) and (5.3.82) there holds

$$\begin{aligned} \|\hat{F}^\varepsilon - \hat{G}^\varepsilon - \ell_\varepsilon(R^\varepsilon)^T\mathrm{sym}(M^\varepsilon(y^\varepsilon))R^\varepsilon\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} \quad \leq \quad & C\varepsilon^{\alpha-1}\ell_\varepsilon^2 + C\varepsilon^{\alpha-1}\ell_\varepsilon\|\hat{G}^\varepsilon\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} \\ + \quad & C\varepsilon^{2\alpha-2}\ell_\varepsilon^3 + C\varepsilon^{2\alpha-2}\ell_\varepsilon^2\|\hat{G}^\varepsilon|_{L^2(\Omega;\mathbb{M}^{3\times 3})}. \end{aligned}$$

Hence, by combining (5.3.6), (5.3.9), (5.3.77) and (5.3.83) we deduce (5.3.78).

In the case $\alpha = 3$, by (5.3.82) and (5.3.83) there holds

$$\varepsilon^4 \ell_\varepsilon^4 \int_\Omega |\mathrm{sym}(M^\varepsilon(y^\varepsilon))|^4\,dx \le C\varepsilon^4 \ell_\varepsilon^4 \int_\Omega |\mathrm{sym}(M^\varepsilon(y^\varepsilon))|^2\,dx \le C\varepsilon^4 \ell_\varepsilon^2.$$

Therefore, by (4.3.6) and (5.3.82) we have

$$\begin{aligned}\|\hat{F}^\varepsilon - \hat{G}^\varepsilon - \ell_\varepsilon (R^\varepsilon)^T \mathrm{sym}(M^\varepsilon(y^\varepsilon))R^\varepsilon - \frac{\varepsilon^2\ell_\varepsilon^2}{2}(R^\varepsilon)^T(M^\varepsilon(y^\varepsilon))^T M^\varepsilon(y^\varepsilon)R^\varepsilon\|_{L^2(\Omega;\mathbb{M}^{3\times3})}&\\ \le C\varepsilon^2\ell_\varepsilon + C\varepsilon^2\ell_\varepsilon\|\hat{G}^\varepsilon\|_{L^2(\Omega;\mathbb{M}^{3\times3})} + C\varepsilon^4\ell_\varepsilon^3&\\ + C\varepsilon^4\ell_\varepsilon^2\|\hat{G}^\varepsilon\|_{L^2(\Omega;\mathbb{M}^{3\times3})}.&\end{aligned}$$

Therefore, once (5.3.83) and (5.3.84) are proved, (5.3.78) follows by (5.3.6), (5.3.8) and (5.3.77).

We now prove (5.3.83) and (5.3.84). By straightforward computations we have

$$\begin{aligned}\ell_\varepsilon \mathrm{sym}\,(M^\varepsilon(y^\varepsilon)) &= \mathrm{sym}\begin{pmatrix} \nabla'\tilde{u}((y^\varepsilon)') - \theta^\varepsilon\big(\frac{y_3^\varepsilon}{\varepsilon}\big)(\nabla')^2\tilde{v}((y^\varepsilon)') & 0 \\ 0 & 0\end{pmatrix}\\ &+\frac{1}{\varepsilon}\mathrm{sym}\begin{pmatrix} 0 & \big(1-(\theta^\varepsilon)'\big(\frac{y_3^\varepsilon}{\varepsilon}\big)\big)\nabla'\tilde{v}((y^\varepsilon)') \\ 0 & 0\end{pmatrix} + \varepsilon\,\mathrm{sym}(\nabla'\eta^\varepsilon(y^\varepsilon)|\partial_3\eta^\varepsilon(y^\varepsilon)).\end{aligned}$$

Now, $\varepsilon\nabla'\eta^\varepsilon(y^\varepsilon) \to 0$ strongly in $L^2(\Omega;\mathbb{M}^{3\times2})$ by (5.3.53). Moreover, (5.3.54) yields

$$\varepsilon\partial_3\eta^\varepsilon(y^\varepsilon(x)) = d\Big((y^\varepsilon)'(x), \frac{y_3^\varepsilon(x)}{\varepsilon}\Big) \quad \text{for a.e. } x\in\Omega.$$

Hence, by (4.3.34) and (5.3.62), there holds

$$\varepsilon\,(\nabla'\eta^\varepsilon(y^\varepsilon)|\partial_3\eta^\varepsilon(y^\varepsilon)) \to \begin{cases}(0|d) & \text{if } \alpha>3\\ (0|d(x',x_3+v)) & \text{if } \alpha=3\end{cases} \tag{5.3.86}$$

strongly in $L^2(\Omega;\mathbb{M}^{3\times3})$. On the other hand, by (5.3.51), (5.3.62), and the dominated convergence theorem

$$\nabla'\tilde{u}((y^\varepsilon)') - \theta^\varepsilon\Big(\frac{y_3^\varepsilon}{\varepsilon}\Big)(\nabla')^2\tilde{v}((y^\varepsilon)') \to \begin{cases}\nabla'\tilde{u} - x_3(\nabla')^2\tilde{v} & \text{if } \alpha>3\\ \nabla'\tilde{u} - (x_3+v)(\nabla')^2\tilde{v} & \text{if } \alpha=3\end{cases} \tag{5.3.87}$$

strongly in $L^2(\Omega;\mathbb{M}^{2\times2})$. Claim (5.3.83) follows now by combining (4.3.6), (4.3.9), (5.3.52), (5.3.86) and (5.3.87).

To prove (5.3.84), we observe that by (5.3.52), (5.3.86) and (5.3.87), if $\alpha = 3$ there exists a constant $C$ such that

$$\Big\|\ell_\varepsilon M^\varepsilon(y^\varepsilon) - \frac{1}{\varepsilon}\begin{pmatrix} 0 & -\nabla'\tilde{v}((y^\varepsilon)') \\ (\nabla'\tilde{v}((y^\varepsilon)'))^T & 0\end{pmatrix}\Big\|_{L^2(\Omega;\mathbb{M}^{3\times3})} \le C$$

for every $\varepsilon$. Hence, by (4.3.6) there holds

$$\begin{aligned}&\Big\|\varepsilon^2\ell_\varepsilon^2(R^\varepsilon)^T(M^\varepsilon(y^\varepsilon))^TM^\varepsilon(y^\varepsilon)R^\varepsilon\\&-(R^\varepsilon)^T\begin{pmatrix}0 & -\nabla'\tilde{v}((y^\varepsilon)')\\(\nabla'\tilde{v}((y^\varepsilon)'))^T & 0\end{pmatrix}^T\begin{pmatrix}0 & -\nabla'\tilde{v}((y^\varepsilon)')\\(\nabla'\tilde{v}((y^\varepsilon)'))^T & 0\end{pmatrix}R^\varepsilon\Big\|_{L^2(\Omega;\mathbb{M}^{3\times3})}\\&\leq C\varepsilon^2\ell_\varepsilon\|M^\varepsilon(y^\varepsilon)\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})}+C\varepsilon\Big\|\begin{pmatrix}0 & -\nabla'\tilde{v}((y^\varepsilon)')\\(\nabla'\tilde{v}((y^\varepsilon)'))^T & 0\end{pmatrix}\Big\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})},\end{aligned}\tag{5.3.88}$$

which converges to zero due to (5.3.6) and (5.3.82). On the other hand,

$$\begin{aligned}&\begin{pmatrix}0 & -\nabla'\tilde{v}((y^\varepsilon)')\\(\nabla'\tilde{v}((y^\varepsilon)'))^T & 0\end{pmatrix}^T\begin{pmatrix}0 & -\nabla'\tilde{v}((y^\varepsilon)')\\(\nabla'\tilde{v}((y^\varepsilon)'))^T & 0\end{pmatrix}\\&=\begin{pmatrix}\nabla'\tilde{v}((y^\varepsilon)')\otimes\nabla'\tilde{v}((y^\varepsilon)') & 0\\0 & |\nabla'\tilde{v}((y^\varepsilon)')|^2\end{pmatrix}.\end{aligned}$$

Moreover, by (5.3.62) and by the dominated convergence theorem there holds

$$\begin{pmatrix}\nabla'\tilde{v}((y^\varepsilon)')\otimes\nabla'\tilde{v}((y^\varepsilon)') & 0\\0 & |\nabla'\tilde{v}((y^\varepsilon)')|^2\end{pmatrix}\to\begin{pmatrix}\nabla'\tilde{v}\otimes\nabla'\tilde{v} & 0\\0 & |\nabla'\tilde{v}|^2\end{pmatrix}\tag{5.3.89}$$

strongly in $L^2(\Omega;\mathbb{M}^{3\times3})$. By combining (5.3.88) and (5.3.89), we deduce (5.3.84) and therefore (5.3.78).

*Step 4: Limsup inequality for the elastic energy*

We are now in a position to prove (5.3.47). We argue as in [52, Lemma 3.6]. We fix $\delta>0$ and we introduce the sets

$$U_\varepsilon:=\{x\in\Omega:\varepsilon^{\alpha-1}(|F^\varepsilon|+|\hat{F}^\varepsilon|)\leq c_{el}(\delta)\},$$

where $c_{el}(\delta)$ is the constant in (4.2.4). By (5.3.77) and (5.3.78) it follows that

$$\hat{F}^\varepsilon\rightharpoonup\hat{F}_\alpha:=N_\alpha+G-\hat{p}\quad\text{weakly in }L^2(\Omega;\mathbb{M}^{3\times3}),\ \text{for }\alpha\geq3.\tag{5.3.90}$$

By (5.3.74) and by Chebychev inequality we deduce

$$\mathcal{L}^3(\Omega\setminus U_\varepsilon)\leq C\varepsilon^{2\alpha-2}.\tag{5.3.91}$$

Since

$$\nabla_\varepsilon\hat{y}^\varepsilon(\hat{P}^\varepsilon)^{-1}=\nabla f^\varepsilon(y^\varepsilon)\big(\nabla_\varepsilon y^\varepsilon(P^\varepsilon)^{-1}\big)P^\varepsilon(\hat{P}^\varepsilon)^{-1},$$

property (4.2.6) yields

$$\begin{aligned}&|W_{el}(\nabla_\varepsilon\hat{y}^\varepsilon(\hat{P}^\varepsilon)^{-1})-W_{el}(\nabla_\varepsilon y^\varepsilon(P^\varepsilon)^{-1})|\\&\leq C(1+W_{el}(\nabla_\varepsilon y^\varepsilon(P^\varepsilon)^{-1}))(|\nabla f^\varepsilon(y^\varepsilon)-Id|+|P^\varepsilon(\hat{P}^\varepsilon)^{-1}-Id|)\end{aligned}\tag{5.3.92}$$

a.e. in $\Omega$. By (4.2.13) and (5.3.43) there holds

$$\begin{aligned}\|P^\varepsilon(\hat{P}^\varepsilon)^{-1}-Id\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})}&\leq c_k\|P^\varepsilon-\hat{P}^\varepsilon\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})}\\&\leq c_k\|(Id-\exp(\varepsilon^{\alpha-1}\tilde{p}))P^\varepsilon\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})}\leq C\varepsilon^{\alpha-1},\end{aligned}$$

hence, by combining (5.3.61), (5.3.91) and (5.3.92) we deduce

$$\begin{aligned}&\frac{1}{\varepsilon^{2\alpha-2}}\Big|\int_{\Omega\setminus U_\varepsilon} W_{el}(\nabla_\varepsilon \hat{y}^\varepsilon(\hat{P}^\varepsilon)^{-1}) - \int_{\Omega\setminus U_\varepsilon} W_{el}(\nabla_\varepsilon y^\varepsilon(P^\varepsilon)^{-1})\Big|\\ &\leq C\varepsilon^{\alpha-1}(1+\ell_\varepsilon)\Big(1+\frac{1}{\varepsilon^{2\alpha-2}}\int_\Omega W_{el}(\nabla_\varepsilon y^\varepsilon(P^\varepsilon)^{-1})\,dx\Big),\end{aligned} \tag{5.3.93}$$

which tends to zero owing to (4.3.29) and (5.3.6).

On the other hand, on the sets $U_\varepsilon$ we can use the estimate (4.2.4). Hence, by (5.3.72), (5.3.85) and the quadratic structure of $Q$ we obtain

$$\begin{aligned}&\frac{1}{\varepsilon^{2\alpha-2}}\int_{U_\varepsilon} W_{el}(\nabla_\varepsilon \hat{y}^\varepsilon(\hat{P}^\varepsilon)^{-1})\,dx - \frac{1}{\varepsilon^{2\alpha-2}}\int_{U^\varepsilon} W_{el}(\nabla_\varepsilon y^\varepsilon(P^\varepsilon)^{-1})\,dx\\ &\leq \delta\int_\Omega (|F^\varepsilon|^2+|\hat{F}^\varepsilon|^2)\,dx + \int_\Omega Q(\hat{F}^\varepsilon)-Q(F^\varepsilon)\,dx\\ &= \delta\int_\Omega (|F^\varepsilon|^2+|\hat{F}^\varepsilon|^2)\,dx + \frac{1}{2}\int_\Omega \mathbb{C}(\hat{F}^\varepsilon - F^\varepsilon):(\hat{F}^\varepsilon+F^\varepsilon)\,dx.\end{aligned} \tag{5.3.94}$$

Now, by (5.3.74) and (5.3.90) there holds

$$\hat{F}^\varepsilon + F^\varepsilon \rightharpoonup \hat{F}_\alpha + G - p \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}). \tag{5.3.95}$$

Moreover,

$$\hat{F}^\varepsilon - F^\varepsilon \to \hat{F}_\alpha - G + p \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}). \tag{5.3.96}$$

Indeed, by (5.3.78) and (5.3.90) it is enough to show that

$$\hat{G}^\varepsilon - F^\varepsilon \to p - \hat{p} \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}).$$

By (5.3.73) and (5.3.76) we have

$$\hat{G}^\varepsilon - F^\varepsilon = (Id + \varepsilon^{\alpha-1}G^\varepsilon)(\hat{w}^\varepsilon - \hat{p}^\varepsilon - w^\varepsilon + p^\varepsilon).$$

Now, by (5.3.67), (5.3.69) and (5.3.75), $\hat{w}^\varepsilon - w^\varepsilon = 0$ in $\Omega\setminus S^\varepsilon$, whereas in the sets $S^\varepsilon$ we have

$$\begin{aligned}\hat{w}^\varepsilon - w^\varepsilon &= \varepsilon^{\alpha-1}(\hat{P}^\varepsilon)^{-1}(\hat{p}^\varepsilon)^2 - \varepsilon^{\alpha-1}(P^\varepsilon)^{-1}(p^\varepsilon)^2\\ &= \varepsilon^{\alpha-1}(P^\varepsilon)^{-1}\big(\exp(-\varepsilon^{\alpha-1}\tilde{p})(\hat{p}^\varepsilon)^2 - (p^\varepsilon)^2\big)\\ &= \varepsilon^{\alpha-1}(P^\varepsilon)^{-1}(\exp(-\varepsilon^{\alpha-1}\tilde{p}) - Id)(\hat{p}^\varepsilon)^2 + \varepsilon^{\alpha-1}(P^\varepsilon)^{-1}((\hat{p}^\varepsilon)^2 - (p^\varepsilon)^2).\end{aligned}$$

Therefore, by (5.3.43), (5.3.44), (5.3.49) and (5.3.50), we deduce

$$\begin{aligned}&\|\hat{w}^\varepsilon - w^\varepsilon\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} \leq C(\varepsilon^{\alpha-1} + \|\hat{p}^\varepsilon - p^\varepsilon\|_{L^2(\Omega;\mathbb{M}^{3\times 3})}) \leq C,\\ &\|\hat{w}^\varepsilon - w^\varepsilon\|_{L^1(\Omega;\mathbb{M}^{3\times 3})} \leq C\varepsilon^{\alpha-1},\\ &\|\hat{w}^\varepsilon - w^\varepsilon\|_{L^\infty(\Omega;\mathbb{M}^{3\times 3})} \leq C(1+\|\hat{p}^\varepsilon - p^\varepsilon\|_{L^\infty(\Omega;\mathbb{M}^{3\times 3})}).\end{aligned}$$

Combining these estimates with (4.3.35) and (5.3.68) we obtain (5.3.96).

Consider now the case $\alpha > 3$. By (5.3.93)–(5.3.96) we have

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\Big\{\frac{1}{\varepsilon^{2\alpha-2}}\int_\Omega W_{el}(\nabla_\varepsilon \hat{y}^\varepsilon(\hat{P}^\varepsilon)^{-1})\,dx - \frac{1}{\varepsilon^{2\alpha-2}}\int_\Omega W_{el}(\nabla_\varepsilon y^\varepsilon(P^\varepsilon)^{-1})\,dx\Big\}\\ &\leq \frac{1}{2}\int_\Omega \mathbb{C}(\hat{F}_\alpha - G + p):(\hat{F}_\alpha + G - p)\,dx + C\delta.\end{aligned}$$

Since $\delta$ is arbitrary, we deduce

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\Big\{\frac{1}{\varepsilon^{2\alpha-2}}\int_\Omega W_{el}(\nabla_\varepsilon \hat{y}^\varepsilon(\hat{P}^\varepsilon)^{-1})\,dx - \frac{1}{\varepsilon^{2\alpha-2}}\int_\Omega W_{el}(\nabla_\varepsilon y^\varepsilon(P^\varepsilon)^{-1})\,dx\Big\}\\ &\leq \frac{1}{2}\int_\Omega \mathbb{C}(\hat{F}_\alpha - G + p):(\hat{F}_\alpha + G - p)\,dx\\ &= \int_\Omega Q(\hat{F}_\alpha)\,dx - \int_\Omega Q(G-p)\,dx \leq \int_\Omega Q(\hat{F}_\alpha)\,dx - \int_\Omega Q_2(G'-p')\,dx. \qquad (5.3.97)\end{aligned}$$

By (5.3.79) and (5.3.90), up to an approximation argument, we may assume that $d$ is such that

$$Q(\hat{F}_\alpha) = Q_2(\operatorname{sym}\nabla'\hat{u} - x_3(\nabla')^2\hat{v} - \hat{p}').$$

This, together with (5.3.97), implies (5.3.47).

In the case $\alpha = 3$ a preliminary approximation argument is needed. Let $(\tilde{u}^k)$ be a sequence in $C_c^\infty(\omega\cup\gamma_n;\mathbb{R}^2)$, such that

$$\tilde{u}^k \to \tilde{u} + v\nabla'\tilde{v} \quad \text{strongly in } W^{1,2}(\omega;\mathbb{R}^2)$$

(such a sequence exists by Lemma 5.2.1 because $\tilde{u}\in C_c^\infty(\omega\cup\gamma_n;\mathbb{R}^2)$ and $\tilde{v}\in C_c^\infty(\omega\cup\gamma_n)$). Let also $v^k\in C_c^\infty(\omega)$ be such that

$$v^k \to v \quad \text{strongly in } L^2(\omega)$$

and set

$$d^k(x) := d(x', x_3 - v^k(x')) \quad \text{for a.e. } x\in\Omega.$$

Since $d\in C_c^\infty(\Omega;\mathbb{R}^3)$, there exists an open set $O\subset\mathbb{R}^2$ such that $\overline{O}\subset\omega$ and $d^k(x',x_3)=0$ for every $x\in(\omega\setminus\overline{O})\times\mathbb{R}$. Moreover, $d^k(x',x_3)=0$ for every $x\in\mathbb{R}^3$ such that $|x_3| > \frac{1}{2} + \|v^k\|_{L^\infty(\mathbb{R}^2)}$. Hence, $d^k\in C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ and

$$\operatorname{supp} d^k \subset \overline{O}\times\big(-\tfrac{1}{2} - \|v^k\|_{L^\infty(\mathbb{R}^2)}, \tfrac{1}{2} + \|v^k\|_{L^\infty(\mathbb{R}^2)}\big).$$

It is easy to see that (5.3.80), (5.3.90) and (5.3.93)–(5.3.96) can still be deduced, and for every $k$ we can construct a sequence $(\hat{y}^\varepsilon_k, \hat{P}^\varepsilon_k)$ that satisfies (5.3.39)–(5.3.44) with $\hat{u}$ replaced by $u + \tilde{u}^k$, and

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\Big\{\frac{1}{\varepsilon^{2\alpha-2}}\int_\Omega W_{el}(\nabla_\varepsilon \hat{y}^\varepsilon_k(\hat{P}^\varepsilon_k)^{-1})\,dx - \frac{1}{\varepsilon^{2\alpha-2}}\int_\Omega W_{el}(\nabla_\varepsilon y^\varepsilon(P^\varepsilon)^{-1})\,dx\Big\}\\ &\leq \frac{1}{2}\int_\Omega \mathbb{C}(\hat{F}^k - G + p):(\hat{F}^k + G - p)\,dx,\end{aligned}$$

where

$$\hat{F}^k := \operatorname{sym}\begin{pmatrix} \nabla'\tilde{u}^k - (x_3+v)(\nabla')^2\tilde{v} + \frac{\nabla'\tilde{v}\otimes\nabla'\tilde{v}}{2} & \Big| & d'(x', x_3 + v - v^k) \\ 0 & \Big| & d_3(x', x_3+v-v^k) + \frac{1}{2}|\nabla'\tilde{v}|^2\end{pmatrix} + G - \hat{p}.$$

On the other hand,

$$\hat{F}^k \to \operatorname{sym}\begin{pmatrix} \nabla'\tilde{u} - x_3(\nabla')^2\tilde{v} + \nabla' v\otimes\nabla'\tilde{v} + \frac{\nabla'\tilde{v}\otimes\nabla'\tilde{v}}{2} & \Big| & d' \\ 0 & \Big| & d_3 + \frac{1}{2}|\nabla'\tilde{v}|^2\end{pmatrix} + G - \hat{p} =: \hat{F}$$

strongly in $L^2(\Omega;\mathbb{M}^{3\times3})$, as $k\to+\infty$. A diagonal argument leads then to the estimate

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon^{2\alpha-2}}\Big(\int_\Omega W_{el}(\nabla_\varepsilon\hat{y}^\varepsilon(\hat{P}^\varepsilon)^{-1})\,dx-\int_\Omega W_{el}(\nabla_\varepsilon y^\varepsilon(P^\varepsilon)^{-1})\,dx\Big)$$
$$\leq\frac{1}{2}\int_\Omega\mathbb{C}(\hat{F}-G+p):(\hat{F}+G-p)\,dx. \tag{5.3.98}$$

Up to a further approximation, we may assume that $d$ is such that

$$Q(\hat{F})=Q_2\Big(\mathrm{sym}\,\nabla'\hat{u}-x_3(\nabla')^2\hat{v}+\frac{1}{2}\nabla'\hat{v}\otimes\nabla'\hat{v}-\hat{p}'\Big),$$

hence (5.3.47) follows by (5.3.98). □

## 5.4 Convergence of quasistatic evolutions

The first part of this section is devoted to the proof of Theorem 5.3.9. We first prove the theorem for $\alpha>3$ and then we show how the proof must be modified for $\alpha=3$.

*Proof of Theorem 5.3.9 in the case* $\alpha>3$. The proof is divided into five steps.
*Step 0: A priori estimates on the elasto-plastic energy*
Set $y^\varepsilon(t):=\phi^\varepsilon(t,z^\varepsilon(t))$ for every $t\in[0,T]$. It is immediate to see that

$$y^\varepsilon(t,x)=\phi^\varepsilon(t,(x',\varepsilon x_3))\quad\mathcal{H}^2\text{ - a.e. on }\Gamma_d. \tag{5.4.1}$$

In this step we shall show that there exists a constant $C$ such that for every $t\in[0,T]$ and every $\varepsilon$ there holds

$$\frac{1}{\varepsilon^{\alpha-1}}\|\mathrm{dist}(\nabla_\varepsilon y^\varepsilon(t)(P^\varepsilon)^{-1}(t),SO(3))\|_{L^2(\Omega)}+\|p^\varepsilon(t)\|_{L^2(\Omega;\mathbb{M}^{3\times3})}+\|\varepsilon^{\alpha-1}p^\varepsilon(t)\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})}\leq C. \tag{5.4.2}$$

To this purpose, we first remark that since $t\mapsto(z^\varepsilon(t),P^\varepsilon(t))$ is an $\varepsilon$-quasistatic evolution, then

$$P^\varepsilon(t,x)\in K\quad\text{for a.e. }x\in\Omega,\text{ for every }\varepsilon\text{ and }t, \tag{5.4.3}$$

hence $\varepsilon^{\alpha-1}p^\varepsilon(t)\in K-Id$ for every $\varepsilon$ and $t$ and by (4.2.13) there exists a constant $C$ such that

$$\|\varepsilon^{\alpha-1}p^\varepsilon(t)\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})}\leq C\quad\text{for every }\varepsilon\text{ and }t. \tag{5.4.4}$$

By the minimality condition (gs), taking $\tilde{z}(x)=(x',\varepsilon x_3)$ and $\tilde{P}(x)=Id$ for every $x\in\Omega$, and observing that $W_{hard}(Id)=0$ a.e. in $\Omega$, by (4.2.12) we deduce

$$\frac{1}{\varepsilon^{2\alpha-2}}\mathcal{F}_\varepsilon(t,z^\varepsilon(t),P^\varepsilon(t))\leq\frac{1}{\varepsilon^{2\alpha-2}}\int_\Omega W_{el}\big(\nabla\phi^\varepsilon(t,(x',\varepsilon x_3))\big)\,dx+\frac{1}{\varepsilon^{\alpha-1}}\int_\Omega D(P^\varepsilon(t),Id)\,dx \tag{5.4.5}$$

for every $t\in[0,T]$ and for all $\varepsilon$. By (4.2.20) and (5.4.3), there holds

$$D(P^\varepsilon(t),Id)=D(Id,(P^\varepsilon)^{-1}(t))\leq c_7|(P^\varepsilon)^{-1}(t)-Id|\leq c_7c_K|Id-P^\varepsilon(t)|,$$

where the last inequality follows by (4.2.13). Hence, Holder inequality yields

$$\frac{1}{\varepsilon^{\alpha-1}}\int_\Omega D(P^\varepsilon(t),Id)\,dx\leq\frac{C}{\varepsilon^{\alpha-1}}\|Id-P^\varepsilon(t)\|_{L^2(\Omega;\mathbb{M}^{3\times3})}. \tag{5.4.6}$$

On the other hand, by frame indifference (H3) of $W_{el}$ (see Section 4.2) we obtain

$$W_{el}\big(\nabla\phi^{\varepsilon}(t,(x',\varepsilon x_3))\big) = W_{el}\Big(\sqrt{(\nabla\phi^{\varepsilon})^T(t,(x',\varepsilon x_3))\nabla\phi^{\varepsilon}(t,(x',\varepsilon x_3))}\Big)$$

for every $x \in \Omega$ and for all $t \in [0,T]$. By (5.3.1), (5.3.7) and (5.3.17) there holds

$$\nabla\phi^{\varepsilon}(t,(x',\varepsilon x_3)) = Id + \varepsilon^{\alpha-1}\begin{pmatrix} \nabla' u^0(t,x') - x_3(\nabla')^2 v^0(t,x') & 0 \\ 0 & 0 \end{pmatrix}$$
$$+\varepsilon^{\alpha-2}\begin{pmatrix} 0 & -\nabla' v^0(t,x') \\ (\nabla' v^0(t,x'))^T & 0 \end{pmatrix},$$

for every $x \in \Omega$. Since $\alpha > 3$, we deduce

$$\begin{aligned}&(\nabla\phi^{\varepsilon})^T(t,(x',\varepsilon x_3))\nabla\phi^{\varepsilon}(t,(x',\varepsilon x_3)) \\ &= Id + 2\varepsilon^{\alpha-1}\mathrm{sym}\begin{pmatrix} \nabla' u^0(t,x') - x_3(\nabla')^2 v^0(t,x') & 0 \\ 0 & 0 \end{pmatrix} + o(\varepsilon^{\alpha-1}),\end{aligned}$$

and

$$\sqrt{(\nabla\phi^{\varepsilon})^T(t,(x',\varepsilon x_3))\nabla\phi^{\varepsilon}(t,(x',\varepsilon x_3))} = Id + \varepsilon^{\alpha-1}M(t,x) + o(\varepsilon^{\alpha-1}), \tag{5.4.7}$$

where

$$M(t,x) = \mathrm{sym}\begin{pmatrix} \nabla' u^0(t,x') - x_3(\nabla')^2 v^0(t,x') & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for every } x \in \Omega.$$

Therefore,

$$\frac{1}{\varepsilon^{2\alpha-2}}W_{el}\big(\nabla\phi^{\varepsilon}(t,(x',\varepsilon x_3))\big) = \frac{1}{\varepsilon^{2\alpha-2}}W_{el}\big(Id + \varepsilon^{\alpha-1}M(t,x) + o(\varepsilon^{\alpha-1})\big)$$

for every $x \in \Omega$. Now, by the smoothness of $u^0$ and $v^0$, there exists a constant $C$ such that

$$\sup_{t\in[0,T]} \|M(t)\|_{L^{\infty}(\Omega;\mathbb{M}^{3\times 3})} \le C \tag{5.4.8}$$

and there exist $\overline{\varepsilon}$ such that, if $\varepsilon < \overline{\varepsilon}$, for every $t \in [0,T]$

$$|\varepsilon^{\alpha-1}M(t) + o(\varepsilon^{\alpha-1})| \le c_{el}(1),$$

where $c_{el}$ is the constant in (4.2.4). Therefore, by (4.2.4), (4.2.7), and (5.4.8) we have

$$\frac{1}{\varepsilon^{2\alpha-2}}\int_{\Omega} W_{el}\big(\nabla\phi^{\varepsilon}(t,(x',\varepsilon x_3))\big)\,dx \le C\Big(\int_{\Omega}|M(t)|^2\,dx + 1\Big) \le C \tag{5.4.9}$$

for every $\varepsilon$ and for all $t \in [0,T]$.

By combining (5.4.5), (5.4.6) and (5.4.9) we obtain

$$\begin{aligned}&\frac{1}{\varepsilon^{2\alpha-2}}\int_{\Omega} W_{el}\big(\nabla_{\varepsilon} y^{\varepsilon}(t)(P^{\varepsilon})^{-1}(t)\big)\,dx + \frac{1}{\varepsilon^{2\alpha-2}}\int_{\Omega} W_{hard}(P^{\varepsilon}(t))\,dx \\ &\le C\Big(1 + \frac{1}{\varepsilon^{\alpha-1}}\|Id - P^{\varepsilon}(t)\|_{L^2(\Omega;\mathbb{M}^{3\times 3})}\Big).\end{aligned} \tag{5.4.10}$$

Now, by (4.2.11) there holds

$$\frac{c_6}{\varepsilon^{2\alpha-2}}\int_\Omega |Id - P^\varepsilon(t)|^2\,dx \le C\Big(1 + \frac{1}{\varepsilon^{\alpha-1}}\|Id - P^\varepsilon(t)\|_{L^2(\Omega;\mathbb{M}^{3\times 3})}\Big),$$

which in turn, by Cauchy inequality implies

$$\|p^\varepsilon(t)\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} = \frac{1}{\varepsilon^{\alpha-1}}\|Id - P^\varepsilon(t)\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} \le C \tag{5.4.11}$$

for every $\varepsilon$ and for all $t \in [0,T]$. On the other hand, by (5.4.10) and (5.4.11), we deduce

$$\frac{1}{\varepsilon^{2\alpha-2}}\int_\Omega W_{el}\big(\nabla_\varepsilon y^\varepsilon(t)(P^\varepsilon)^{-1}(t)\big)\,dx \le C, \tag{5.4.12}$$

for every $\varepsilon$ and for all $t \in [0,T]$. Estimate (5.4.2) follows now by (5.4.4), (5.4.11), (5.4.12) and the growth condition (H4) (see Section 4.2).

*Step 1: A priori estimate on the dissipation functional.*

In this step we shall show that there exists a constant $C$, such that

$$\frac{1}{\varepsilon^{\alpha-1}}\mathcal{D}(P^\varepsilon;0,t) \le C \quad \text{for every } \varepsilon \text{ and for all } t \in [0,T]. \tag{5.4.13}$$

By (eb), (5.3.29) and (5.4.10)–(5.4.12) it is enough to show that there exists a constant $C$ such that

$$\Big|\frac{1}{\varepsilon^{\alpha-1}}\int_\Omega E^\varepsilon(t) : \nabla\dot{\phi^\varepsilon}(t, z^\varepsilon(t))(\nabla\phi^\varepsilon)^{-1}(t, z^\varepsilon(t))\,dx\Big| \le C \tag{5.4.14}$$

for every $\varepsilon$ and $t \in [0,T]$. To prove (5.4.14), we first deduce some properties of the map $t \mapsto E^\varepsilon(t)$.

Let $R \in SO(3)$. By (4.2.13) and (5.4.3) there holds

$$\begin{aligned}
&|\nabla_\varepsilon y^\varepsilon(t) - R|^2 = |\nabla_\varepsilon y^\varepsilon(t) - RP^\varepsilon(t) + \varepsilon^{\alpha-1}Rp^\varepsilon(t)|^2\\
&\le 2|\nabla_\varepsilon y^\varepsilon(t)(P^\varepsilon)^{-1}(t) - R|^2|P^\varepsilon(t)|^2 + 2\varepsilon^{2\alpha-2}|p^\varepsilon(t)|^2\\
&\le 2\,c_K^2|\nabla_\varepsilon y^\varepsilon(t)(P^\varepsilon)^{-1}(t) - R|^2 + 2\varepsilon^{2\alpha-2}|p^\varepsilon(t)|^2.
\end{aligned}$$

Hence, the growth condition (H4) (see Section 4.2) implies

$$\|\mathrm{dist}(\nabla_\varepsilon y^\varepsilon(t), SO(3))\|^2_{L^2(\Omega;\mathbb{M}^{3\times 3})} \le C\Big(\int_\Omega W_{el}(\nabla_\varepsilon y^\varepsilon(t)(P^\varepsilon)^{-1}(t))\,dx + \varepsilon^{2\alpha-2}\|p^\varepsilon(t)\|^2_{L^2(\Omega;\mathbb{M}^{3\times 3})}\Big),$$

which in turn yields

$$\|\mathrm{dist}(\nabla_\varepsilon y^\varepsilon(t), SO(3))\|^2_{L^2(\Omega;\mathbb{M}^{3\times 3})} \le C\varepsilon^{2\alpha-2} \tag{5.4.15}$$

by (5.4.2) and (5.4.12). By (5.4.1) and (5.4.15), the sequence $y^\varepsilon(t)$ fulfills the hypotheses of Theorem 4.3.1. Hence, for every $t \in [0,T]$ there exists a sequence of maps $(R^\varepsilon(t)) \subset W^{1,\infty}(\omega;\mathbb{M}^{3\times 3})$ such that

$$R^\varepsilon(t,x') \in SO(3) \quad \text{for every } x' \in \omega, \tag{5.4.16}$$

$$\|\nabla_\varepsilon y^\varepsilon(t) - R^\varepsilon(t)\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} \le C\varepsilon^{\alpha-1}, \tag{5.4.17}$$

$$\|\partial_i R^\varepsilon(t)\|_{L^2(\omega;\mathbb{M}^{3\times 3})} \le C\varepsilon^{\alpha-2}, \quad i = 1,2, \tag{5.4.18}$$

$$\|R^\varepsilon(t) - Id\|_{L^2(\omega;\mathbb{M}^{3\times 3})} \le C\varepsilon^{\alpha-2}, \tag{5.4.19}$$

where the constant $C$ is independent of $\varepsilon$ and $t$.

We consider the auxiliary maps

$$w^\varepsilon(t) := \frac{(Id+\varepsilon^{\alpha-1}p^\varepsilon(t))^{-1} - Id + \varepsilon^{\alpha-1}p^\varepsilon(t)}{\varepsilon^{\alpha-1}},$$

the elastic strains

$$G^\varepsilon(t) := \frac{(R^\varepsilon(t))^T \nabla_\varepsilon y^\varepsilon(t) - Id}{\varepsilon^{\alpha-1}},$$

and the matrices

$$F^\varepsilon(t) := G^\varepsilon(t) + w^\varepsilon(t) - p^\varepsilon(t) + \varepsilon^{\alpha-1}G^\varepsilon(t)(w^\varepsilon(t) - p^\varepsilon(t)), \tag{5.4.20}$$

for all $t \in [0,T]$. Clearly we have

$$(P^\varepsilon)^{-1}(t) = Id + \varepsilon^{\alpha-1}(w^\varepsilon(t) - p^\varepsilon(t)) \quad \text{and} \quad \nabla_\varepsilon y^\varepsilon(t) = R^\varepsilon(t)(Id + \varepsilon^{\alpha-1}G^\varepsilon(t)). \tag{5.4.21}$$

Since

$$w^\varepsilon(t) = \varepsilon^{\alpha-1}(Id + \varepsilon^{\alpha-1}p^\varepsilon(t))^{-1}(p^\varepsilon(t))^2 \tag{5.4.22}$$

for every $t \in [0,T]$, by (5.4.2) and (5.4.3) there holds

$$\|\varepsilon^{\alpha-1}w^\varepsilon(t)\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})} \leq C \quad \text{for every } t \in [0,T], \tag{5.4.23}$$

$$\|w^\varepsilon(t)\|_{L^1(\Omega;\mathbb{M}^{3\times3})} \leq C\varepsilon^{\alpha-1} \quad \text{for every } t \in [0,T], \tag{5.4.24}$$

and

$$\|w^\varepsilon(t)\|_{L^2(\Omega;\mathbb{M}^{3\times3})} \leq C \quad \text{for every } t \in [0,T]. \tag{5.4.25}$$

Combining (5.4.24) and (5.4.25) we deduce

$$w^\varepsilon(t) \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}) \text{ for every } t \in [0,T]. \tag{5.4.26}$$

On the other hand, (5.4.16) and (5.4.17) yield

$$\|G^\varepsilon(t)\|_{L^2(\Omega;\mathbb{M}^{3\times3})} \leq C \tag{5.4.27}$$

for every $\varepsilon$ and for all $t \in [0,T]$. Collecting (5.4.2), (5.4.23), (5.4.25) and (5.4.27), we obtain

$$\begin{aligned}\|F^\varepsilon(t)\|_{L^2(\Omega;\mathbb{M}^{3\times3})} &\leq \|G^\varepsilon(t)\|_{L^2(\Omega;\mathbb{M}^{3\times3})} + \|w^\varepsilon(t)\|_{L^2(\Omega;\mathbb{M}^{3\times3})} + \|p^\varepsilon(t)\|_{L^2(\Omega;\mathbb{M}^{3\times3})}\\ &+\|G^\varepsilon(t)\|_{L^2(\Omega;\mathbb{M}^{3\times3})}\|\varepsilon^{\alpha-1}(w^\varepsilon(t) - p^\varepsilon(t))\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})} \leq C\end{aligned} \tag{5.4.28}$$

for every $\varepsilon$ and for all $t \in [0,T]$.

Now, by (5.4.20), (5.4.21) and the frame-indifference (H3) of $W_{el}$ (see Section 4.2) we deduce the decomposition

$$E^\varepsilon(t) = R^\varepsilon(t)\tilde{E}^\varepsilon(t)(R^\varepsilon(t))^T \tag{5.4.29}$$

for every $t \in [0,T]$, where

$$\tilde{E}^\varepsilon(t) := \frac{1}{\varepsilon^{\alpha-1}} DW_{el}(Id + \varepsilon^{\alpha-1}F^\varepsilon(t))(Id + \varepsilon^{\alpha-1}F^\varepsilon(t))^T.$$

We argue as in [55, Proof of Theorem 3.1, Steps 2–3] and we first show that there exist two positive constants $k_1, k_2$, independent of $\varepsilon$, such that

$$|\tilde{E}^\varepsilon(t)| \leq k_1\Big(\frac{W_{el}(Id+\varepsilon^{\alpha-1}F^\varepsilon(t))}{\varepsilon^{\alpha-1}}+k_2|F^\varepsilon(t)|\Big) \tag{5.4.30}$$

for every $t\in[0,T]$ and for a.e. $x\in\Omega$.

Indeed, let $c_{el_2}$ be the constant in (4.2.9). Suppose that $\varepsilon^{\alpha-1}|F^\varepsilon(t)|\geq c_{el_2}$. We remark that (H1) (see Section 4.2), (5.4.3) and (5.4.12) imply in particular that

$$\det(\nabla_\varepsilon y^\varepsilon(t))>0 \quad \text{a.e. in } \Omega.$$

Therefore, by (4.2.5) there holds

$$|\tilde{E}^\varepsilon(t)| \leq \frac{c_3}{\varepsilon^{\alpha-1}}\Big(W_{el}(Id+\varepsilon^{\alpha-1}F^\varepsilon(t))+1\Big) \leq c_3\Big(\frac{W_{el}(Id+\varepsilon^{\alpha-1}F^\varepsilon(t))}{\varepsilon^{\alpha-1}}+\frac{1}{c_{el_2}}|F^\varepsilon(t)|\Big). \tag{5.4.31}$$

Consider now the case where $\varepsilon^{\alpha-1}|F^\varepsilon(t)|<c_{el_2}$. Then, by (4.2.9) there holds

$$DW_{el}(Id+\varepsilon^{\alpha-1}F^\varepsilon(t)) \leq \varepsilon^{\alpha-1}(2R_{\mathbb{C}}+1)|F^\varepsilon(t)|,$$

which in turn implies

$$|\tilde{E}^\varepsilon(t)| \leq C|F^\varepsilon(t)|(|Id|+|\varepsilon^{\alpha-1}F^\varepsilon(t)|) \leq C|F^\varepsilon(t)|. \tag{5.4.32}$$

Collecting (5.4.31) and (5.4.32), we obtain (5.4.30).

By (5.4.12), (5.4.28) and (5.4.30), for every measurable $\Lambda\subset\Omega$, the following estimate holds true:

$$\int_\Lambda |\tilde{E}^\varepsilon(t)|\,dx \leq k_1\int_\Lambda \Big(\frac{W_{el}(Id+\varepsilon^{\alpha-1}F^\varepsilon(t))}{\varepsilon^{\alpha-1}}+k_2|F^\varepsilon(t)|\Big) \leq C(|\Lambda|^{\frac12}+\varepsilon^{\alpha-1}), \tag{5.4.33}$$

for every $\varepsilon$ and for all $t\in[0,T]$. By (5.4.16) there holds also

$$\int_\Lambda |E^\varepsilon(t)|\,dx \leq C(|\Lambda|^{\frac12}+\varepsilon^{\alpha-1}), \tag{5.4.34}$$

for every $\varepsilon$ and for all $t\in[0,T]$.

Let now $\gamma\in(0,\alpha-2)$ be the positive constant in the definition of the maps $\theta^\varepsilon$. Let $O_\varepsilon(t)$ be the set given by

$$O_\varepsilon(t) := \{x\in\Omega:\ \varepsilon^{\alpha-1-\gamma}|F^\varepsilon(t,x)|\leq c_{el_2}\},$$

and let $\chi_\varepsilon(t):\Omega\to\{0,1\}$ be the map

$$\chi_\varepsilon(t,x)=\begin{cases}1 & \text{if } x\in O_\varepsilon(t),\\ 0 & \text{otherwise.}\end{cases}$$

By Chebychev inequality and (5.4.28) we deduce

$$\mathcal{L}^3(\Omega\setminus O_\varepsilon(t)) \leq C\varepsilon^{2(\alpha-1-\gamma)}, \tag{5.4.35}$$

for every $\varepsilon$ and for all $t \in [0,T]$. By combining (5.4.33) and (5.4.35) we conclude that

$$\|(1-\chi_\varepsilon(t))\tilde{E}^\varepsilon(t)\|_{L^1(\Omega;\mathbb{M}^{3\times 3})} \le C\varepsilon^{\alpha-1-\gamma} \quad \text{for every } t\in[0,T]. \tag{5.4.36}$$

By (5.4.34) the previous estimate implies also

$$\|(1-\chi_\varepsilon(t))E^\varepsilon(t)\|_{L^1(\Omega;\mathbb{M}^{3\times 3})} \le C\varepsilon^{\alpha-1-\gamma} \quad \text{for every } t\in[0,T]. \tag{5.4.37}$$

On the other hand (4.2.9) yields the following estimate on the sets $O_\varepsilon(t)$:

$$|\chi_\varepsilon(t)\tilde{E}^\varepsilon(t)| \le (2R_{\mathbb{C}}+1)|F^\varepsilon(t)||Id+\varepsilon^{\alpha-1}F^\varepsilon(t)| \le C(1+c_{el_2}\varepsilon^\gamma)|F^\varepsilon(t)|,$$

which in turn, by (5.4.28), implies

$$\|\chi_\varepsilon(t)\tilde{E}^\varepsilon(t)\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} \le C \tag{5.4.38}$$

for every $\varepsilon$ and for all $t\in[0,T]$.

By (5.3.6), (5.4.16), (5.4.37) and (5.4.38), and since $E^\varepsilon(t)$ is symmetric by Remark 5.3.5, to prove (5.4.14) it is enough to show that there exists a constant $C$ such that

$$\Big\|\frac{1}{\varepsilon^{\alpha-1}}\nabla\dot{\phi}^\varepsilon(t,z^\varepsilon(t))(\nabla\phi^\varepsilon)^{-1}(t,z^\varepsilon(t))\Big\|_{L^\infty(\Omega;\mathbb{M}^{3\times 3})} \le C\ell_\varepsilon \tag{5.4.39}$$

and

$$\Big\|\frac{1}{\varepsilon^{\alpha-1}}\mathrm{sym}\Big(\nabla\dot{\phi}^\varepsilon(t,z^\varepsilon(t))(\nabla\phi^\varepsilon)^{-1}(t,z^\varepsilon(t))\Big)\Big\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} \le C \tag{5.4.40}$$

for every $\varepsilon$ and for all $t\in[0,T]$. By (5.3.17), there holds

$$\begin{aligned}\frac{1}{\varepsilon^{\alpha-1}}\nabla\dot{\phi}^\varepsilon(t,z^\varepsilon(t)) &= \begin{pmatrix} \nabla'\dot{u}^0(t,(z^\varepsilon)'(t)) - \theta^\varepsilon\big(\frac{z_3^\varepsilon(t)}{\varepsilon}\big)(\nabla')^2\dot{v}^0(t,(z^\varepsilon)'(t)) & 0 \\ 0 & 0\end{pmatrix}\\ &+\frac{1}{\varepsilon}\begin{pmatrix} 0 & -\dot{\theta}^\varepsilon\big(\frac{z_3^\varepsilon(t)}{\varepsilon}\big)\nabla'\dot{v}^0(t,(z^\varepsilon)'(t)) \\ (\nabla'\dot{v}^0(t,(z^\varepsilon)'(t)))^T & 0\end{pmatrix}.\end{aligned} \tag{5.4.41}$$

Estimate (5.4.39) follows directly by (5.3.3), (5.3.5), (5.3.7), and (5.3.18). To prove (5.4.40), we first provide an estimate for the $L^2$ norm of the maps $\frac{1}{\varepsilon}z_3^\varepsilon(t)$. To this purpose, let $v^\varepsilon(t)$ be defined as in (5.3.32). It is easy to see that

$$v^\varepsilon(t) = \frac{1}{\varepsilon^{\alpha-2}}\int_{-\frac12}^{\frac12} y_3^\varepsilon(t)\,dx_3 \quad\text{and}\quad \nabla'v^\varepsilon(t) = \frac{1}{\varepsilon^{\alpha-2}}\int_{-\frac12}^{\frac12}\nabla'y_3^\varepsilon(t)\,dx_3$$

for every $\varepsilon$ and for all $t\in[0,T]$. By (5.4.1), arguing as in the proof of Theorem 4.3.1,

$$v^\varepsilon(t) = v^0(t) \quad \mathcal{H}^1\text{ - a.e. on } \gamma_d.$$

By (5.4.17) and (5.4.19), we have

$$\|\nabla'v^\varepsilon(t)\|_{L^2(\omega;\mathbb{R}^2)} \le C$$

for every $\varepsilon$ and $t\in[0,T]$. Hence, by Poincaré inequality we deduce

$$\|v^\varepsilon(t)-v^0(t)\|_{L^2(\omega)} \le C\|\nabla'v^\varepsilon(t)-\nabla'v^0(t)\|_{L^2(\omega;\mathbb{R}^2)} \le C,$$

which in turn, by the smoothness of $v^0$, yields

$$\|v^\varepsilon(t)\|_{L^2(\omega)} \le C \quad \text{for every } \varepsilon \text{ and for all } t \in [0,T].$$

By (5.4.17), (5.4.19) and Poincaré-Wirtinger inequality, we deduce

$$\Big\| \frac{y_3^\varepsilon(t)}{\varepsilon} - x_3 - \varepsilon^{\alpha-3} v^\varepsilon(t) \Big\|_{L^2(\Omega)} \le C \Big\| \frac{\partial_3 y_3^\varepsilon(t)}{\varepsilon} - 1 \Big\|_{L^2(\Omega)} \le C\varepsilon^{\alpha-2} \tag{5.4.42}$$

for every $t \in [0,T]$, which implies

$$\Big\| \frac{y_3^\varepsilon(t)}{\varepsilon} \Big\|_{L^2(\Omega)} \le C \quad \text{for every } \varepsilon \text{ and } t \in [0,T]. \tag{5.4.43}$$

On the other hand,

$$z^\varepsilon(t) = \varphi^\varepsilon(t, y^\varepsilon(t)) \quad \text{a.e. in } \Omega, \tag{5.4.44}$$

hence by (5.3.12),

$$\frac{z_3^\varepsilon(t)}{\varepsilon} = \frac{y_3^\varepsilon(t)}{\varepsilon} - \varepsilon^{\alpha-3} v^0(t, (\varphi^\varepsilon)'(t, y^\varepsilon(t))). \tag{5.4.45}$$

Therefore (5.3.2) and (5.4.43) yield

$$\Big\| \theta^\varepsilon\Big(\frac{z_3^\varepsilon(t)}{\varepsilon}\Big) \Big\|_{L^2(\Omega)} \le \Big\| \frac{z_3^\varepsilon(t)}{\varepsilon} \Big\|_{L^2(\Omega)} \le C \quad \text{for every } \varepsilon \text{ and } t \in [0,T]. \tag{5.4.46}$$

By Lemma 5.3.11, we deduce

$$\Big\| 1 - \dot\theta^\varepsilon\Big(\frac{z_3^\varepsilon(t)}{\varepsilon}\Big) \Big\|_{L^2(\Omega)} \le \frac{3}{\ell_\varepsilon} \quad \text{for every } \varepsilon \text{ and } t \in [0,T]. \tag{5.4.47}$$

Collecting (5.3.7), (5.4.41), (5.4.46) and (5.4.47), we obtain that there exists a constant $C$ such that

$$\Big\| \frac{1}{\varepsilon^{\alpha-1}} \mathrm{sym} \nabla \dot\phi^\varepsilon(t, z^\varepsilon(t)) \Big\|_{L^2(\Omega;\mathbb{M}^{3\times3})} \le C$$

for every $\varepsilon$ and for all $t \in [0,T]$. Therefore, to prove (5.4.40), it remains only to study the quantity

$$\frac{1}{\varepsilon^{\alpha-1}} \mathrm{sym}\Big( \nabla \dot\phi^\varepsilon(t, z^\varepsilon(t)) \big( (\nabla\phi^\varepsilon)^{-1}(t, z^\varepsilon(t)) - Id \big) \Big).$$

By (5.3.18),

$$\|(\nabla\phi^\varepsilon(t))^{-1} - Id\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})} \le C \quad \text{for every } \varepsilon \text{ and } t \in [0,T].$$

By (5.4.46), the first term in the right hand side of (5.4.41) is uniformly bounded in $L^2(\Omega;\mathbb{M}^{3\times3})$. Therefore, it remains to show that

$$\frac{1}{\varepsilon} \begin{pmatrix} 0 & -\dot\theta^\varepsilon\big(\frac{z_3^\varepsilon(t)}{\varepsilon}\big) \nabla' \dot v^0(t, (z^\varepsilon)'(t)) \\ (\nabla' \dot v^0(t, (z^\varepsilon)'(t)))^T & 0 \end{pmatrix} \Big( (\nabla\phi^\varepsilon)^{-1}(t, z^\varepsilon(t)) - Id \Big) \tag{5.4.48}$$

is uniformly bounded in $L^2(\Omega;\mathbb{M}^{3\times3})$.

By (5.3.5) and by the smoothness of $v^0$, there holds

$$\Big\| \frac{1}{\varepsilon} \begin{pmatrix} 0 & -\dot\theta^\varepsilon\big(\frac{z_3^\varepsilon(t)}{\varepsilon}\big) \nabla' \dot v^0(t, (z^\varepsilon)'(t)) \\ (\nabla' \dot v^0(t, (z^\varepsilon)'(t)))^T & 0 \end{pmatrix} \Big\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})} \le \frac{C}{\varepsilon} \tag{5.4.49}$$

for every $t \in [0,T]$. On the other hand,

$$(\nabla\phi^\varepsilon)^{-1}(t, z^\varepsilon(t)) = \nabla\varphi^\varepsilon(t, y^\varepsilon(t)) \quad \text{a.e. in } \Omega. \tag{5.4.50}$$

Property (5.3.20) yields the estimate

$$\|\nabla\varphi_3^\varepsilon(t, y^\varepsilon(t)) - e_3\|_{L^\infty(\Omega;\mathbb{R}^3)} \le C\varepsilon^{\alpha-2} \tag{5.4.51}$$

for every $t \in [0,T]$, whereas by (5.3.5), (5.3.15) and (5.3.18)

$$\|\nabla\varphi_i^\varepsilon(t, y^\varepsilon(t)) - e_i\|_{L^2(\Omega;\mathbb{R}^3)} \le C\varepsilon^{\alpha-1}\Big\|\theta^\varepsilon\Big(\frac{\varphi_3^\varepsilon(t, y^\varepsilon(t))}{\varepsilon}\Big)\Big\|_{L^2(\omega)} + C\varepsilon^{\alpha-2},$$

hence by (5.4.44) and (5.4.46) we obtain

$$\|\nabla\varphi_i^\varepsilon(t, y^\varepsilon(t)) - e_i\|_{L^2(\Omega;\mathbb{R}^3)} \le C\varepsilon^{\alpha-2}. \tag{5.4.52}$$

By combining (5.4.49)–(5.4.52), we deduce

$$\Big\|\frac{1}{\varepsilon}\begin{pmatrix} 0 & -\dot\theta^\varepsilon\big(\frac{z_3^\varepsilon(t)}{\varepsilon}\big)\nabla'\dot v^0(t,(z^\varepsilon)'(t)) \\ (\nabla'\dot v^0(t,(z^\varepsilon)'(t)))^T & 0 \end{pmatrix}\Big((\nabla\phi^\varepsilon)^{-1}(t, z^\varepsilon(t)) - Id\Big)\Big\|_{L^2(\Omega;\mathbb{M}^{3\times3})} \le C\varepsilon^{\alpha-3} \tag{5.4.53}$$

for every $\varepsilon$ and $t \in [0,T]$, therefore the quantity in (5.4.48) is uniformly bounded in $L^2(\Omega;\mathbb{M}^{3\times3})$, and the proof of (5.4.40) is complete. By (5.4.36)–(5.4.40), since all estimates are uniform both in $\varepsilon$ and $t$, we deduce (5.4.14), which in turn yields (5.4.13).

*Step 2: Reduced Stability*

Owing to the a priori bounds (5.4.2) and (5.4.13), we can apply the generalized version of Helly's Selection Principle in Theorem 1.5.2. To show it, take $\mathcal{Z} := L^2(\Omega;\mathbb{M}^{3\times3})$ endowed with the weak topology of $L^2$, and set

$$\mathcal{D}_\varepsilon(z_1, z_2) := \frac{1}{\varepsilon^{\alpha-1}}\int_\Omega D(Id + \varepsilon^{\alpha-1}z_1, Id + \varepsilon^{\alpha-1}z_2)\,dx$$

and

$$\mathcal{D}_\infty(z_1, z_2) := \int_\Omega H(z_2 - z_1)\,dx$$

for every $z_1, z_2 \in L^2(\Omega;\mathbb{M}^{3\times3})$. Hypotheses (A.1) and (A.2) of Theorem 1.5.2 are satisfied by (4.2.16)–(4.2.18). Hypothesis (A.3) follows by adapting [52, Lemmas 3.4 and 3.5], whereas condition (A.4) follows directly by (5.4.2) and (5.4.13). Hence, by Theorem 1.5.2 there holds

$$\begin{aligned} &p^\varepsilon(t) \rightharpoonup p(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}) \quad \text{for every } t \in [0,T], \\ &\mathcal{D}_{H_D}(p; 0, t) \le \liminf_{\varepsilon\to0}\frac{1}{\varepsilon^{\alpha-1}}\mathcal{D}(P^\varepsilon; 0, t) \quad \text{for every } t \in [0,T]. \end{aligned} \tag{5.4.54}$$

Moreover, by (5.3.28), $p(0) = \mathring{p}$.

Let now $t \in [0,T]$ be fixed. By (5.4.1), (5.4.17), (5.4.19) and Poincaré inequality, up to subsequences there holds

$$y^\varepsilon(t) \to \begin{pmatrix} x' \\ 0 \end{pmatrix} \quad \text{strongly in } W^{1,2}(\Omega;\mathbb{R}^3). \tag{5.4.55}$$

Arguing as in the proof of Theorem 4.3.3 and owing to (5.4.2), we deduce the existence of a pair $(u^*(t), v^*(t)) \in W^{1,2}(\omega;\mathbb{R}^2) \times W^{2,2}(\omega)$ such that $(u^*(t), v^*(t), p(t)) \in \mathcal{A}(u^0(t), v^0(t))$ and a sequence $\varepsilon_j \to 0$ such that

$$u^{\varepsilon_j}(t) \rightharpoonup u^*(t) \quad \text{weakly in } W^{1,2}(\omega;\mathbb{R}^2), \tag{5.4.56}$$

$$v^{\varepsilon_j}(t) \to v^*(t) \quad \text{strongly in } W^{1,2}(\omega). \tag{5.4.57}$$

In particular, by (5.3.26) and (5.3.27) we have $u^*(0) = \mathring{u}$ and $v^*(0) = \mathring{v}$. By (5.4.27) up to extracting a further subsequence, there exists a map $G^*(t) \in L^2(\Omega;\mathbb{M}^{3\times3})$ such that

$$G^{\varepsilon_j}(t) \rightharpoonup G^*(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}) \tag{5.4.58}$$

and the $2\times2$ submatrix $(G^*)'(t)$ satisfies

$$(G^*)'(t,x) = G_0^*(t,x') - x_3(\nabla')^2 v^*(t,x') \quad \text{for a.e. } x \in \Omega, \tag{5.4.59}$$

where

$$\operatorname{sym} G_0^*(t) = \operatorname{sym} \nabla' u^*(t). \tag{5.4.60}$$

We shall show that the triple $(u^*(t), v^*(t), p(t))$ satisfies the reduced stability condition $(\mathrm{gs})_{r\alpha}$. By Corollary 5.2.2, it is enough to prove the inequality for triples $(\hat{u}, \hat{v}, \hat{p}) \in \mathcal{A}(u^0(t), v^0(t))$ such that

$$\tilde{u} := \hat{u} - u^*(t) \in C_c^\infty(\omega \cup \gamma_n;\mathbb{R}^2),$$
$$\tilde{v} := \hat{v} - v^*(t) \in C_c^\infty(\omega \cup \gamma_n),$$
$$\tilde{p} := \hat{p} - p^*(t) \in C_c^\infty(\Omega;\mathbb{M}^{3\times3}_D).$$

By Theorem 5.3.12 there exists a sequence $(\hat{y}^{\varepsilon_j}, \hat{P}^{\varepsilon_j}) \in \mathcal{A}_{\varepsilon_j}(\phi^{\varepsilon_j t}(t))$ satisfying

$$\begin{aligned}
&\int_\Omega Q_2(\operatorname{sym}\hat{G}' - \hat{p}')\,dx + \int_\Omega B(\hat{p})\,dx \\
&- \int_\Omega Q_2(\operatorname{sym}(G^*)'(t) - p'(t))\,dx - \int_\Omega B(p(t))\,dx + \int_\Omega H_D(\hat{p} - p(t))\,dx \\
&\geq \limsup_{\varepsilon_j \to 0} \Big\{ \frac{1}{{\varepsilon_j}^{2\alpha-2}} \int_\Omega W_{el}(\nabla_{\varepsilon_j}\hat{y}^{\varepsilon_j}(\hat{P}^{\varepsilon_j})^{-1})\,dx + \frac{1}{{\varepsilon_j}^{2\alpha-2}} \int_\Omega W_{hard}(\hat{P}^{\varepsilon_j})\,dx \\
&- \frac{1}{{\varepsilon_j}^{2\alpha-2}} \int_\Omega W_{el}(\nabla_{\varepsilon_j} y^{\varepsilon_j}(t)(P^{\varepsilon_j})^{-1}(t))\,dx - \frac{1}{{\varepsilon_j}^{2\alpha-2}} \int_\Omega W_{hard}(P^{\varepsilon_j}(t))\,dx \\
&+ \frac{1}{{\varepsilon_j}^{\alpha-1}} \int_\Omega D(P^{\varepsilon_j}(t), \hat{P}^{\varepsilon_j})\,dx \Big\}
\end{aligned}$$

where

$$\hat{G}'(x', x_3) := \hat{G}_0(x') - x_3(\nabla')^2\hat{v}(x') \quad \text{a.e. in } \Omega,$$

and

$$\operatorname{sym}\hat{G}_0 = \operatorname{sym}\nabla'\hat{u}.$$

Inequality $(\mathrm{gs})_{r\alpha}$ follows now by the $\varepsilon$-stability (gs) of $(y^\varepsilon(t), P^\varepsilon(t))$.

By strict convexity of the quadratic form $Q_2$, an adaptation of [15, Theorem 3.8] yields that, once $p(t)$ is identified, there exist unique $u(t) \in W^{1,2}(\omega;\mathbb{R}^2)$ and $v(t) \in W^{2,2}(\omega)$ such that $(\mathrm{gs})_{r\alpha}$ holds at time $t$. This implies that $u^*(t) = u(t)$, $v^*(t) = v(t)$ for every $t \in [0,T]$ and both (5.4.56) and (5.4.57) hold for the whole sequences $u^\varepsilon(t)$ and $v^\varepsilon(t)$ and for every $t \in [0,T]$. Moreover, by (5.4.58)–(5.4.60) we have

$$\mathrm{sym}\,(G^*)'(t) = \mathrm{sym}\,\nabla' u(t) - x_3(\nabla')^2 v(t)$$

and

$$\mathrm{sym}\,(G^\varepsilon)'(t) \rightharpoonup \mathrm{sym}\,\nabla' u(t) - x_3(\nabla')^2 v(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}) \quad \text{for every } t \in [0,T].$$

*Step 3: Convergence of the scaled stress*

In this step we shall show that for every $t \in [0,T]$ there exists a subsequence $\varepsilon_j$, possibly depending on $t$, such that

$$\chi_{\varepsilon_j}(t) E^{\varepsilon_j}(t) \rightharpoonup E^*(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}), \tag{5.4.61}$$

where

$$E^*(t) = \mathbb{C}(G^*(t) - p(t)). \tag{5.4.62}$$

To this purpose, for $t \in [0,T]$ fixed, let $\varepsilon_j \to 0$ be such that (5.4.58) holds and let $F^{\varepsilon_j}(t)$ be the map defined in (5.4.20). By (5.4.2), (5.4.23) and (5.4.58) we deduce

$$\|\varepsilon^{\alpha-1} G^{\varepsilon_j}(t)(w^{\varepsilon_j}(t) - p^{\varepsilon_j}(t))\|_{L^2(\Omega;\mathbb{M}^{3\times3})} \le C \quad \text{for every } \varepsilon_j.$$

On the other hand, by (5.4.2), (5.4.25), and (5.4.58), there holds

$$\varepsilon^{\alpha-1} G^{\varepsilon_j}(t)(w^{\varepsilon_j}(t) - p^{\varepsilon_j}(t)) \to 0 \quad \text{strongly in } L^1(\Omega;\mathbb{M}^{3\times3}). \tag{5.4.63}$$

Hence, we conclude that

$$\varepsilon^{\alpha-1} G^{\varepsilon_j}(t)(w^{\varepsilon_j}(t) - p^{\varepsilon_j}(t)) \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}). \tag{5.4.64}$$

Collecting (5.4.20), (5.4.26), (5.4.54), (5.4.58) and (5.4.64) we obtain

$$F^{\varepsilon_j}(t) \rightharpoonup G^*(t) - p(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}). \tag{5.4.65}$$

By (5.4.35) we deduce that $\chi_{\varepsilon_j}(t) \to 1$ boundedly in measure, therefore by (5.4.65) there holds

$$\chi_{\varepsilon_j}(t) F^{\varepsilon_j}(t) \rightharpoonup G^*(t) - p(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}).$$

Now, estimate (5.4.33) implies that the sequence $(\tilde{E}^{\varepsilon_j}(t))$ is uniformly bounded in $L^1(\Omega;\mathbb{M}^{3\times3})$ and is equiintegrable, hence by the Dunford-Pettis Theorem, up to extracting a further subsequence, there exists $E^*(t) \in L^1(\Omega;\mathbb{M}^{3\times3}_{sym})$ such that

$$\tilde{E}^{\varepsilon_j}(t) \rightharpoonup E^*(t) \quad \text{weakly in } L^1(\Omega;\mathbb{M}^{3\times3}).$$

Using a Taylor expansion argument in $O_\varepsilon(t)$, and arguing as in [55, Proof of Theorem 3.1, Step 3] we deduce

$$\chi_{\varepsilon_j}(t) \tilde{E}^{\varepsilon_j}(t) \rightharpoonup \mathbb{C}(G^*(t) - p(t)) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}_{sym}).$$

By (5.4.16) and (5.4.19), the sequence $(R^\varepsilon(t))$ converges boundedly in measure to the identity, hence the previous convergence implies in particular (5.4.61) and (5.4.62).

*Step 4: Characterization of the limit stress*

In this step we shall show that

$$E^*(t) = \mathbb{C}_2(\mathrm{sym}\,\nabla' u(t) - x_3(\nabla')^2 v(t) - p'(t)) := E(t) \quad \text{for every } t \in [0,T]. \tag{5.4.66}$$

This, in turn, will imply that all convergence properties established in the previous step hold for the entire sequences and for every $t \in [0,T]$.

We first remark that, choosing $\tilde{P} = P^\varepsilon(t)$ in (gs) there holds

$$\int_\Omega W_{el}(\nabla_\varepsilon y^\varepsilon(t)(P^\varepsilon)^{-1}(t))\,dx \le \int_\Omega W_{el}(\nabla_\varepsilon \tilde{y}(P^\varepsilon)^{-1}(t))\,dx, \tag{5.4.67}$$

for every $\tilde{y} \in W^{1,2}(\Omega;\mathbb{R}^3)$ such that $\tilde{y} = \phi^\varepsilon(t,(x',\varepsilon x_3))$ $\mathcal{H}^2$ - a.e. on $\Gamma_d$.

Let $\eta \in W^{1,\infty}(\mathbb{R}^3;\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ be such that $\eta \circ \phi^\varepsilon(t,(x',\varepsilon x_3)) = 0$ $\mathcal{H}^2$ - a.e. on $\Gamma_d$. Then, in particular, we can consider in (5.4.67) inner variations of the form $y^\varepsilon + \lambda\eta \circ y^\varepsilon$, where $\lambda \in \mathbb{R}$. By the growth hypothesis (4.2.5) and by the minimality condition (5.4.67), an adaptation Theorem 1.3.1 shows that $y^\varepsilon(t)$ satisfies the following Euler-Lagrange equation:

$$\int_\Omega DW_{el}(\nabla_\varepsilon y^\varepsilon(t)(P^\varepsilon)^{-1}(t))(\nabla_\varepsilon y^\varepsilon(t)(P^\varepsilon)^{-1}(t))^T : \nabla\eta(y^\varepsilon(t))\,dx = 0 \tag{5.4.68}$$

for every $t \in [0,T]$ and for every $\eta \in W^{1,\infty}(\mathbb{R}^3;\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ such that $\eta \circ \phi^\varepsilon(t,(x',\varepsilon x_3)) = 0$ $\mathcal{H}^2$ - a.e. on $\Gamma_d$. Hence,

$$\int_\Omega E^\varepsilon(t) : \nabla\eta(y^\varepsilon(t))\,dx = 0 \tag{5.4.69}$$

for every $t \in [0,T]$ and for every $\eta \in W^{1,\infty}(\mathbb{R}^3;\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ such that $\eta \circ \phi^\varepsilon(t,(x',\varepsilon x_3)) = 0$ $\mathcal{H}^2$ - a.e. on $\Gamma_d$.

Now, fix $t \in [0,T]$ and let $\varepsilon_j$ be the sequence selected in the previous step. Let $\eta \in W^{1,\infty}(\mathbb{R}^3;\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ be such that $\eta = 0$ $\mathcal{H}^2$ - a.e. on $\Gamma_d$. We consider the maps $\eta^{\varepsilon_j}(t) \in W^{1,\infty}(\mathbb{R}^3,\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ defined as

$$\eta^{\varepsilon_j}(t) := \varepsilon_j \eta\big(\varphi_1^{\varepsilon_j}(t), \varphi_2^{\varepsilon_j}(t), \tfrac{1}{\varepsilon_j}\varphi_3^{\varepsilon_j}(t)\big).$$

It is clear that $\eta^{\varepsilon_j}(t) \circ \phi^{\varepsilon_j}(t,(x',\varepsilon_j x_3)) = 0$ $\mathcal{H}^2$ - a.e. on $\Gamma_d$, hence we can use $\eta^{\varepsilon_j}(t)$ as a test function in (5.4.69) and we obtain

$$\int_\Omega E^{\varepsilon_j t}(t) : \nabla\eta^{\varepsilon_j}(y^{\varepsilon_j}(t))\,dx = 0 \tag{5.4.70}$$

for every $j$.

Now, set $\xi^{\varepsilon_j}(x) = \big(\varphi_1^{\varepsilon_j}(t,x), \varphi_2^{\varepsilon_j}(t,x), \frac{1}{\varepsilon_j}\varphi_3^{\varepsilon_j}(t,x)\big)$ for every $x \in \mathbb{R}^3$. We can rewrite (5.4.70) as

$$\begin{aligned}
&\sum_{i=1,2,3} \varepsilon_j \int_\Omega E^{\varepsilon_j t}(t)e_i \cdot \sum_{k=1,2} \partial_k\eta(\xi^{\varepsilon_j}(y^{\varepsilon_j}(t)))\partial_i\xi_k^{\varepsilon_j}(y^{\varepsilon_j}(t))\,dx \\
&+\varepsilon_j \sum_{i=1,2} \int_\Omega E^{\varepsilon_j t}(t)e_i \cdot \partial_3\eta(\xi^{\varepsilon_j}(y^{\varepsilon_j}(t)))\partial_i\xi_3^{\varepsilon_j}(y^{\varepsilon_j}(t))\,dx \\
&+\varepsilon_j \int_\Omega E^{\varepsilon_j t}(t)e_3 \cdot \partial_3\eta(\xi^{\varepsilon_j}(y^{\varepsilon_j}(t)))\partial_3\xi_3^{\varepsilon_j}(y^{\varepsilon_j}(t))\,dx = 0.
\end{aligned} \tag{5.4.71}$$

Since $\eta \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$ and $E^{\varepsilon_j t}(t)$ is uniformly bounded in $L^1(\Omega; \mathbb{M}^{3\times 3})$ by (5.4.34), estimate (5.3.18) yields that the term in the first row of (5.4.71) converges to zero. By (5.3.20), the term in the second row of (5.4.71) can be bounded as follows:

$$\Big|\varepsilon_j \sum_{i=1,2} \int_\Omega E^{\varepsilon_j t}(t)e_i \cdot \partial_3 \eta(\xi^{\varepsilon_j}(y^{\varepsilon_j}(t)))\partial_i \xi_3^{\varepsilon_j}(y^{\varepsilon_j}(t))\, dx\Big| \le C\varepsilon_j^{\alpha-2}\|E^{\varepsilon_j t}(t)e_i\|_{L^1(\Omega;\mathbb{R}^3)}$$

and hence converges to zero due to (5.4.34). By (5.3.20), there holds

$$\Big|\varepsilon_j \int_\Omega E^{\varepsilon_j t}(t)e_3 \cdot \partial_3 \eta(\xi^{\varepsilon_j}(y^{\varepsilon_j}(t)))\partial_3 \xi_3^{\varepsilon_j}(y^{\varepsilon_j}(t))\, dx - \int_\Omega E^{\varepsilon_j t}(t)e_3 \cdot \partial_3 \eta(\xi^{\varepsilon_j}(y^{\varepsilon_j}(t)))\, dx\Big|$$
$$\le C\varepsilon_j{}^{\alpha-2}\|E^{\varepsilon_j t}(t)e_3\|_{L^1(\Omega;\mathbb{R}^3)}.$$

which converges to zero, owing to (5.4.34). Therefore, (5.4.71) yields

$$\lim_{\varepsilon_j\to 0} \int_\Omega E^{\varepsilon_j t}(t)e_3 \cdot \partial_3 \eta(\xi^{\varepsilon_j}(y^{\varepsilon_j}(t)))\, dx = 0. \tag{5.4.72}$$

By (5.3.6), (5.3.13) and (5.4.55) we deduce

$$\xi_k^{\varepsilon_j}(y^{\varepsilon_j}(t)) \to x_k \quad \text{strongly in } L^2(\Omega) \quad \text{for } k = 1, 2.$$

Since $\alpha > 3$, by (5.3.14) and (5.4.42) we have $\xi_3^{\varepsilon_j}(y^{\varepsilon_j}(t)) \to x_3$ strongly in $L^2(\Omega)$. Hence, by the regularity of $\eta$,

$$\partial_3 \eta(\xi^{\varepsilon_j}(y^{\varepsilon_j}(t))) \to \partial_3 \eta(t, x) \quad \text{a.e. in } \Omega \text{ as } \varepsilon_j \to 0.$$

By the dominated convergence theorem and by combining (5.3.6), (5.4.37), (5.4.61) and (5.4.72), we conclude that

$$\int_\Omega E^*(t)e_3 \cdot \partial_3 \eta(t)\, dx = 0,$$

for every $\eta \in W^{1,\infty}(\mathbb{R}^3; \mathbb{R}^3) \cap C^\infty(\mathbb{R}^3; \mathbb{R}^3)$ such that $\eta = 0$ $\mathcal{H}^2$ - a.e. on $\Gamma_d$. Hence,

$$E^*(t)e_3 = 0 \quad \text{a.e. in } \Omega. \tag{5.4.73}$$

By combining (4.3.23), (4.3.24), (5.4.62) and (5.4.73) we deduce (5.4.66). Moreover, by (4.3.23) there holds

$$\operatorname{sym} G^*(t) - p(t) = \mathbb{A}(\operatorname{sym} \nabla' u(t) - x_3(\nabla')^2 v(t) - p'(t)), \text{ for every } t \in [0, T]. \tag{5.4.74}$$

*Step 5: Reduced energy balance*

To complete the proof of the theorem it remains to show that the triple $(u(t), v(t), p(t))$ satisfies

$$\int_\Omega Q_2\big(\operatorname{sym}\nabla' u(t) - x_3(\nabla')^2 v(t) - p'(t)\big)\, dx + \int_\Omega B(p(t))\, dx + \mathcal{D}_H(p; 0, t)$$
$$\le \int_\Omega Q_2\big(\operatorname{sym}\nabla' u(0) - x_3(\nabla')^2 v(0) - p'(0)\big)\, dx + \int_\Omega B(p(0))\, dx$$
$$+ \int_0^t \int_\Omega \mathbb{C}_2(\operatorname{sym} \nabla' u(s) - x_3(\nabla')^2 v(s) - p'(s)) : \begin{pmatrix} \nabla \dot{u}^0(s) - x_3(\nabla')^2 \dot{v}^0(s) & 0 \\ 0 & 0 \end{pmatrix} dx\, ds. \tag{5.4.75}$$

Once (5.4.75) is proved, the opposite inequality in $(\mathrm{eb})_{r\alpha}$ follows by adapting [15, Theorem 4.7].

We claim that, to prove (5.4.75) it is enough to show that

$$\frac{1}{\varepsilon^{\alpha-1}}\mathrm{sym}\Big(\nabla\dot{\phi}^\varepsilon(t,z^\varepsilon(t))(\nabla\phi^\varepsilon)^{-1}(t,z^\varepsilon(t))\Big)\to\mathrm{sym}\Big(\begin{array}{cc}\nabla'\dot{u}^0(t)-x_3(\nabla')^2\dot{v}^0(t) & 0\\ 0 & 0\end{array}\Big)\quad(5.4.76)$$

strongly in $L^2(\Omega;\mathbb{M}^{3\times 3})$, for all $t\in[0,T]$. Indeed, if (5.4.76) holds, by (5.3.6), (5.4.37), (5.4.39), (5.4.61) and (5.4.66), one has

$$\frac{1}{\varepsilon^{\alpha-1}}\int_\Omega E^\varepsilon(s):\nabla\dot{\phi}^\varepsilon(s,z^\varepsilon(s))(\nabla\phi^\varepsilon)^{-1}(s,z^\varepsilon(s))\,dx\to\int_\Omega E(s):\mathrm{sym}\Big(\begin{array}{cc}\nabla'\dot{u}^0(s)-x_3(\nabla')^2\dot{v}^0(s) & 0\\ 0 & 0\end{array}\Big)\,dx,$$

for every $s\in[0,t]$. Hence, by (5.4.14) and the dominated convergence theorem we deduce

$$\begin{aligned}&\frac{1}{\varepsilon^{\alpha-1}}\int_0^t\int_\Omega E^\varepsilon(s):\nabla\dot{\phi}^\varepsilon(s,z^\varepsilon(s))(\nabla\phi^\varepsilon)^{-1}(s,z^\varepsilon(s))\,dx\,ds\\ &\to\int_0^t\int_\Omega E(s):\mathrm{sym}\Big(\begin{array}{cc}\nabla\dot{u}^0(s)-x_3\nabla^2\dot{v}^0(s) & 0\\ 0 & 0\end{array}\Big)\,dx\,ds.\end{aligned}\quad(5.4.77)$$

On the other hand, by Theorem 4.3.3 there holds

$$\begin{aligned}&\int_\Omega Q_2\big(\mathrm{sym}\nabla'u(t)-x_3(\nabla')^2v(t)-p'(t)\big)\,dx+\int_\Omega B(p(t))\,dx\\ &\le\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^{2\alpha-2}}\mathcal{F}_\varepsilon(t,z^\varepsilon(t),P^\varepsilon(t)).\end{aligned}$$

Therefore, once (5.4.76) is proved, by (5.4.54) and (5.4.77), passing to the liminf in the $\varepsilon$ energy balance (eb), inequality (5.4.75) follows by (5.3.29).

To prove (5.4.76), we first study some properties of the maps $z^\varepsilon(t)$. By (5.3.11) and (5.4.44) there holds

$$z_i^\varepsilon(t)=y_i^\varepsilon(t)-\varepsilon^{\alpha-1}u_i^0(t,(\varphi^\varepsilon)'(t,y^\varepsilon(t)))+\varepsilon^{\alpha-1}\theta^\varepsilon\Big(\frac{\varphi_3^\varepsilon(t,y^\varepsilon(t))}{\varepsilon}\Big)\partial_i v^0(t,(\varphi^\varepsilon)'(t,y^\varepsilon(t)))$$

for every $t\in[0,T]$, $i=1,2$. Hence, by (5.3.3), (5.3.6) and (5.4.55) we deduce

$$z_i^\varepsilon(t)\to x_i\quad\text{strongly in }L^2(\Omega)\quad\text{for every }t\in[0,T],\ i=1,2.\quad(5.4.78)$$

Moreover, by (5.4.45) we have

$$\begin{aligned}&\Big\|\frac{z_3^\varepsilon(t)}{\varepsilon}-x_3-\varepsilon^{\alpha-3}v(t)+\varepsilon^{\alpha-3}v^0(t)\Big\|_{L^2(\Omega)}\le\Big\|\frac{y_3^\varepsilon(t)}{\varepsilon}-x_3-\varepsilon^{\alpha-3}v^\varepsilon(t)\Big\|_{L^2(\Omega)}\\ &+\varepsilon^{\alpha-3}\|v^\varepsilon(t)-v(t)\|_{L^2(\Omega)}+\varepsilon^{\alpha-3}\|v^0(t)-v^0(t,(\varphi^\varepsilon)'(t,y^\varepsilon(t)))\|_{L^2(\Omega)}.\end{aligned}$$

Hence, by (5.3.13), (5.3.32), (5.4.42) and (5.4.55),

$$\Big\|\frac{z_3^\varepsilon(t)}{\varepsilon}-x_3-\varepsilon^{\alpha-3}v(t)+\varepsilon^{\alpha-3}v^0(t)\Big\|_{L^2(\Omega)}\to 0\quad(5.4.79)$$

for every $t\in[0,T]$. In particular, by Lemma 5.3.11,

$$\theta^\varepsilon\Big(\frac{z_3^\varepsilon(t)}{\varepsilon}\Big)\to x_3\quad\text{strongly in }L^2(\Omega).\quad(5.4.80)$$

Arguing as in the proof of (5.4.40), we perform the decomposition

$$\frac{1}{\varepsilon^{\alpha-1}}\mathrm{sym}\Big(\nabla\dot{\phi}^{\varepsilon}(t,z^{\varepsilon}(t))(\nabla\phi^{\varepsilon})^{-1}(t,z^{\varepsilon}(t))\Big) = \frac{1}{\varepsilon^{\alpha-1}}\mathrm{sym}\big(\nabla\dot{\phi}^{\varepsilon}(t,z^{\varepsilon}(t))\big)$$
$$+\frac{1}{\varepsilon^{\alpha-1}}\mathrm{sym}\Big(\nabla\dot{\phi}^{\varepsilon}(t,z^{\varepsilon}(t))\Big((\nabla\phi^{\varepsilon})^{-1}(t,z^{\varepsilon}(t)) - Id\Big)\Big). \tag{5.4.81}$$

By (5.3.7), (5.4.41), (5.4.47), (5.4.78) and (5.4.80), we obtain

$$\frac{1}{\varepsilon^{\alpha-1}}\mathrm{sym}\big(\nabla\dot{\phi}^{\varepsilon}(t,z^{\varepsilon}(t))\big) \to \mathrm{sym}\begin{pmatrix} \nabla\dot{u}^0(t) - x_3\nabla^2\dot{v}^0(t) & 0 \\ 0 & 0 \end{pmatrix} \tag{5.4.82}$$

strongly in $L^2(\Omega;\mathbb{M}^{3\times 3})$. To study the second term in the right-hand side of (5.4.81), we remark that by (5.4.41) and (5.4.53), there holds

$$\Big\|\frac{1}{\varepsilon^{\alpha-1}}\mathrm{sym}\Big(\nabla\dot{\phi}^{\varepsilon}(t,z^{\varepsilon}(t))(\nabla\phi^{\varepsilon})^{-1}(t,z^{\varepsilon}(t)) - Id)\Big)\Big\|_{L^2(\Omega;\mathbb{M}^{3\times 3})}$$
$$\leq C\Big(1 + \Big\|\theta^{\varepsilon}\Big(\frac{z_3^{\varepsilon}(t)}{\varepsilon}\Big)\Big\|_{L^2(\Omega)}\Big)\|(\nabla\phi^{\varepsilon})^{-1}(t,z^{\varepsilon}(t)) - Id\|_{L^\infty(\Omega;\mathbb{M}^{3\times 3})} + C\varepsilon^{\alpha-3}.$$

On the other hand, (5.3.7), (5.3.19), (5.3.20) and (5.4.50) yield

$$\|(\nabla\phi^{\varepsilon})^{-1}(t,z^{\varepsilon}(t)) - Id\|_{L^\infty(\Omega;\mathbb{M}^{3\times 3})} \leq C\varepsilon^{\alpha-1}\ell_{\varepsilon}.$$

Hence, by (5.3.6) and (5.4.80) we have

$$\frac{1}{\varepsilon^{\alpha-1}}\mathrm{sym}\Big(\nabla\dot{\phi}^{\varepsilon}(t,z^{\varepsilon}(t))(\nabla\phi^{\varepsilon})^{-1}(t,z^{\varepsilon}(t)) - Id)\Big) \to 0 \tag{5.4.83}$$

strongly in $L^2(\Omega;\mathbb{M}^{3\times 3})$. By combining (5.4.82) and (5.4.83) we obtain (5.4.76). This completes the proof of the theorem. □

We give only a sketch of the proof of Theorem 5.3.9 in the case $\alpha = 3$, as it follows closely that of Theorem 5.3.9 for $\alpha > 3$.

*Proof of Theorem 5.3.9 in the case $\alpha = 3$. Steps 0–3*
Steps 0–3 follow as a straightforward adaptation of the corresponding steps in the case $\alpha > 3$, where now (5.4.7) holds with

$$M(t,x) := \mathrm{sym}\begin{pmatrix} \nabla' u^0(t,x') - x_3(\nabla')^2 v^0(t,x') & 0 \\ 0 & 0 \end{pmatrix}$$
$$+ \frac{1}{2}\begin{pmatrix} \nabla' v^0(t,x')\otimes\nabla' v^0(t,x') & 0 \\ 0 & |\nabla' v^0(t,x')|^2 \end{pmatrix}$$

for every $x \in \Omega$ and for all $t \in [0,T]$. The only relevant difference is that we can not conclude that $u(t)$ and $v(t)$ are uniquely determined once $p(t)$ is identified. Hence, now all convergence properties hold on $t$-dependent subsequences. In particular the counterparts of (5.4.1)–(5.4.65) still hold for $\alpha = 3$.
*Step 4: Characterization of the limit stress*
Arguing exactly as in Step 4 of the proof of Theorem 5.3.9 for $\alpha > 3$, we obtain

$$\int_\Omega E(t,x)e_3\cdot\partial_3\eta(t,(x',x_3 + v(t,x') - v^0(t,x')))\,dx = 0 \tag{5.4.84}$$

for every $\eta \in W^{1,\infty}(\mathbb{R}^3;\mathbb{R}^3)\cap C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ such that $\eta = 0$ $\mathcal{H}^2$ - a.e. on $\Gamma_d$. Consider now a sequence $(w_k) \subset C^\infty_c(\omega)$ that converges to $v(t) - v^0(t)$ strongly in $L^2(\omega)$. Taking as test functions in (5.4.84) the maps $\eta_k(x) := \eta(x', x_3 - w_k(x'))$, where $\eta \in W^{1,\infty}(\mathbb{R}^3,\mathbb{R}^3)\cap C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ and $\eta = 0$ $\mathcal{H}^2$ - a.e. on $\Gamma_d$, we have

$$\int_\Omega E(t,x)e_3 \cdot \partial_3\eta(t,(x', x_3 + v(t,x') - v^0(t,x') - w_k(x')))\,dx = 0 \quad \text{for every } k.$$

Passing to the limit as $k \to +\infty$ in the previous equation, by the dominated convergence theorem we deduce

$$\int_\Omega E(t)e_3 \cdot \partial_3\eta\,dx = 0$$

for every $\eta \in W^{1,\infty}(\mathbb{R}^3,\mathbb{R}^3)\cap C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ such that $\eta = 0$ $\mathcal{H}^2$ - a.e. on $\Gamma_d$, which implies $E(t)e_3 = 0$ a.e. in $\Omega$. Hence, (4.3.23) and (4.3.24) yield

$$E(t) = \mathbb{C}_2(e_3(t)),$$

and

$$\operatorname{sym} G(t) - p(t) = \mathbb{A}(\operatorname{sym}\nabla' u(t) + \tfrac{1}{2}\nabla' v(t)\otimes\nabla' v(t) - x_3(\nabla')^2 v(t) - p'(t)). \tag{5.4.85}$$

*Step 5: Reduced energy balance*

Arguing as in Step 5 of the case $\alpha > 3$, to prove $(\mathrm{eb})_{r3}$ it is enough to show that

$$\begin{aligned}
&\int_\Omega Q_2(e_3(t))\,dx + \int_\Omega B(p(t))\,dx + \mathcal{D}_{H_D}(p;0,t)\\
&\le \int_\Omega Q_2(e_3(0))\,dx + \int_\Omega B(p(0))\,dx\\
&+ \int_0^t \int_\Omega \mathbb{C}_2(e_3(s)) : \begin{pmatrix} \nabla\dot{u}^0(s) + \nabla' v(s)\otimes\nabla'\dot{v}^0(s) - x_3(\nabla')^2\dot{v}^0(s) & 0 \\ 0 & 0\end{pmatrix} dx\,ds,
\end{aligned} \tag{5.4.86}$$

where $t \mapsto e_3(t)$ is the map defined in (5.3.24). Indeed, once (5.4.86) is proved, $(\mathrm{eb})_{r3}$ follows by adapting [15, Theorem 4.7] according to Remark 5.3.7 (see Lemma 5.6.4). To prove (5.4.86), we argue as in [6, Lemma 5.1] and we set

$$\Theta^\varepsilon(t) := \frac{1}{\varepsilon^2}\int_\Omega E^\varepsilon(t) : \nabla\dot{\phi}^\varepsilon(t, z^\varepsilon(t))(\nabla\phi^\varepsilon)^{-1}(t, z^\varepsilon(t))\,dx,$$

$$\Theta(t) := \limsup_{\varepsilon\to 0}\Theta^\varepsilon(t)$$

for every $t \in [0,T]$. By (5.4.14) (which is still true for $\alpha = 3$), $\Theta(t) \in L^1([0,T])$ and by Fatou Lemma there holds

$$\limsup_{\varepsilon\to 0}\int_0^t \Theta^\varepsilon(s)\,ds \le \int_0^t \Theta(s)\,ds. \tag{5.4.87}$$

Now, by Theorem 4.3.3 we know that

$$\int_\Omega Q_2(e_3(t))\,dx + \int_\Omega B(p(t))\,dx \le \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^{2\alpha-2}}\mathcal{F}^\varepsilon(t, z^\varepsilon(t), P^\varepsilon(t)).$$

By (eb), (5.3.29), (5.4.54) and (5.4.87) we deduce

$$\int_\Omega Q_2(e_3(t))\,dx + \int_\Omega B(p(t))\,dx + \mathcal{D}_{H_D}(p;0,t) \le \int_\Omega Q_2(e_3(0))\,dx + \int_\Omega B(p(0))\,dx + \int_0^t \Theta(s)\,ds.$$

Hence, to prove (5.4.86) it is enough to show that

$$\Theta(t) = \int_\Omega E(t) : \begin{pmatrix} \nabla \dot{u}^0(t) + \nabla' v(t) \otimes \nabla' \dot{v}^0(t) - x_3(\nabla')^2 \dot{v}^0(t) & 0 \\ 0 & 0 \end{pmatrix} dx \tag{5.4.88}$$

for a.e. $t \in [0,T]$.

To this purpose, fix $t \in [0,T]$ and let $\varepsilon_{jt} \to 0$ be such that

$$\Theta(t) = \lim_{\varepsilon_{jt}\to 0} \Theta^{\varepsilon_{jt}}(t).$$

Up to extracting a further subsequence, we may assume that $\varepsilon_{jt}$ is the same subsequence we selected in the previous steps. We claim that

$$\begin{aligned} &\frac{1}{\varepsilon_{jt}^2}\mathrm{sym}\Big(\nabla \dot{\phi^{\varepsilon_{jt}}}(t, z^{\varepsilon_{jt}}(t))(\nabla\phi^{\varepsilon_{jt}})^{-1}(t, z^{\varepsilon_{jt}}(t))\Big) \\ &\to \mathrm{sym}\begin{pmatrix} \nabla' \dot{u}^0(t) + \nabla' \dot{v}^0(t) \otimes \nabla' v^0(t) - (x_3 + v(t) - v^0(t))(\nabla')^2 \dot{v}^0(t) & 0 \\ 0 & \frac{d}{dt}\frac{|\nabla' v^0(t)|^2}{2} \end{pmatrix} \end{aligned} \tag{5.4.89}$$

strongly in $L^2(\Omega;\mathbb{M}^{3\times 3})$. To prove the claim, we perform the decomposition (5.4.81). Now, arguing as in the proof of (5.4.82), and using (5.4.79) (which still holds for $\alpha = 3$) and Lemma 5.3.11 we obtain

$$\frac{1}{\varepsilon_{jt}^2}\mathrm{sym}(\nabla \dot{\phi^{\varepsilon_{jt}}}(t, z^{\varepsilon_{jt}}(t))) \to \mathrm{sym}\begin{pmatrix} \nabla' \dot{u}^0(t) - (x_3 + v(t) - v^0(t))(\nabla')^2 \dot{v}^0(t) & 0 \\ 0 & 0 \end{pmatrix} \tag{5.4.90}$$

strongly in $L^2(\Omega;\mathbb{M}^{3\times 3})$. To study the second term in the right-hand side of (5.4.81), we remark that by (5.3.7), (5.3.19), (5.3.20) and (5.4.50), one has

$$\Big\|(\nabla\phi^{\varepsilon_{jt}})^{-1}(t, z^{\varepsilon_{jt}}(t)) - Id\Big\|_{L^\infty(\Omega;\mathbb{M}^{3\times 3})} \le C\varepsilon_{jt}^2 \ell_{\varepsilon_{jt}}.$$

By (5.4.46), there holds

$$\begin{aligned} &\Big\|\begin{pmatrix} \nabla' \dot{u}^0(t, (z^{\varepsilon_{jt}})'(t)) - \theta^{\varepsilon_{jt}}\big(\frac{z_3^{\varepsilon_{jt}}(t)}{\varepsilon_{jt}}\big)(\nabla')^2 \dot{v}^0(t, (z^{\varepsilon_{jt}})'(t)) & 0 \\ 0 & 0 \end{pmatrix}\Big((\nabla\phi^{\varepsilon_{jt}})^{-1}(t, z^{\varepsilon_{jt}}(t)) - Id\Big)\Big\|_{L^2(\Omega;\mathbb{M}^{3\times 3})} \\ &\le C\varepsilon_{jt}^2 \ell_{\varepsilon_{jt}}, \end{aligned} \tag{5.4.91}$$

which tends to zero due to (5.3.6).

By (5.4.41), it remains only to study the asymptotic behaviour of

$$\frac{1}{\varepsilon_{jt}}\begin{pmatrix} 0 & -\dot{\theta}^{\varepsilon_{jt}}\big(\frac{z_3^{\varepsilon_{jt}}(t)}{\varepsilon_{jt}}\big)\nabla' \dot{v}^0(t, (z^{\varepsilon_{jt}})'(t)) \\ (\nabla' \dot{v}^0(t, (z^{\varepsilon_{jt}})'(t)))^T & 0 \end{pmatrix}\Big((\nabla\phi^{\varepsilon_{jt}})^{-1}(t, z^{\varepsilon_{jt}}(t)) - Id\Big).$$

By (5.4.50), this is the same as studying the quantity

$$\frac{1}{\varepsilon_{jt}}\begin{pmatrix} 0 & -\dot{\theta}^{\varepsilon_{jt}}\big(\frac{z_3^{\varepsilon_{jt}}(t)}{\varepsilon_{jt}}\big)\nabla' \dot{v}^0(t, (z^{\varepsilon_{jt}})'(t)) \\ (\nabla' \dot{v}^0(t, (z^{\varepsilon_{jt}})'(t)))^T & 0 \end{pmatrix}\Big(\nabla\varphi^{\varepsilon_{jt}}(t, y^{\varepsilon_{jt}}(t)) - Id\Big).$$

We claim that

$$\frac{1}{\varepsilon_{jt}}\Big(\nabla\varphi^{\varepsilon_{jt}}(t,y^{\varepsilon_{jt}}(t))-Id\Big)\to\begin{pmatrix}0 & \nabla'v^0(t)\\ -(\nabla'v^0(t))^T & 0\end{pmatrix} \tag{5.4.92}$$

strongly in $L^2(\Omega;\mathbb{M}^{3\times3})$. Indeed, by (5.3.15) and (5.3.18) and the smoothness of $u^0$ and $v^0$,

$$\begin{aligned}&\Big\|\frac{1}{\varepsilon_{jt}}\Big(\nabla(\varphi^{\varepsilon_{jt}})'(t,y^{\varepsilon_{jt}}(t))-\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}\Big)-(0|\nabla'v^0(t))\Big\|_{L^2(\Omega;\mathbb{M}^{2\times3})}\leq C\varepsilon_{jt}\Big\|\theta^{\varepsilon_{jt}}\Big(\frac{\varphi_3^{\varepsilon_{jt}}(t,y^{\varepsilon_{jt}}(t))}{\varepsilon_{jt}}\Big)\Big\|_{L^2(\Omega)}\\&+\Big\|\dot{\theta}^{\varepsilon_{jt}}\Big(\frac{\varphi_3^{\varepsilon_{jt}}(t,y^{\varepsilon_{jt}}(t))}{\varepsilon_{jt}}\Big)\nabla'v^0(t,(\varphi^{\varepsilon})'(t,y^{\varepsilon_{jt}}(t)))\otimes(\nabla\varphi_3^{\varepsilon_{jt}}(t,y^{\varepsilon_{jt}}(t))-e_3)\Big\|_{L^2(\Omega;\mathbb{M}^{2\times3})}\\&+\Big\|\dot{\theta}^{\varepsilon_{jt}}\Big(\frac{\varphi_3^{\varepsilon_{jt}}(t,y^{\varepsilon_{jt}}(t))}{\varepsilon_{jt}}\Big)\nabla'v^0(t,(\varphi^{\varepsilon})'(t,y^{\varepsilon_{jt}}(t)))-\nabla'v^0(t)\Big\|_{L^2(\Omega;\mathbb{R}^2)}+C\varepsilon_j.\end{aligned}$$

By (5.3.2), (5.3.12), and (5.4.43)(which can be proved arguing exactly as in Step 1 of the case $\alpha>3$), we deduce

$$\Big\|\theta^{\varepsilon_{jt}}\Big(\frac{\varphi_3^{\varepsilon_{jt}}(t,y^{\varepsilon_{jt}}(t))}{\varepsilon_{jt}}\Big)\Big\|_{L^2(\Omega)}\leq\Big\|\frac{\varphi_3^{\varepsilon_{jt}}(t,y^{\varepsilon_{jt}}(t))}{\varepsilon_{jt}}\Big\|_{L^2(\Omega)}\leq C\Big(\Big\|\frac{y_3^{\varepsilon_{jt}}(t)}{\varepsilon_{jt}}\Big\|_{L^2(\Omega)}+\|v^0\|_{L^\infty(\omega;\mathbb{R}^2)}\Big)\leq C. \tag{5.4.93}$$

On the other hand, by (5.3.5) and (5.3.20)

$$\begin{aligned}&\Big\|\dot{\theta}^{\varepsilon_{jt}}\Big(\frac{\varphi_3^{\varepsilon_{jt}}(t,y^{\varepsilon_{jt}}(t))}{\varepsilon_{jt}}\Big)\nabla'v^0(t,(\varphi^{\varepsilon})'(t,y^{\varepsilon_{jt}}(t)))\otimes(\nabla\varphi_3^{\varepsilon_{jt}}(t,y^{\varepsilon_{jt}}(t))-e_3)\Big\|_{L^2(\Omega;\mathbb{M}^{2\times3})}\\&\leq C\|\nabla\varphi_3^{\varepsilon_{jt}}(t,y^{\varepsilon_{jt}}(t))-e_3\|_{L^\infty(\Omega;\mathbb{R}^3)}\leq C\varepsilon_{jt}.\end{aligned}$$

Finally, by (5.4.93) and Lemma 5.3.11

$$\begin{aligned}&\Big\|\dot{\theta}^{\varepsilon_{jt}}\Big(\frac{\varphi_3^{\varepsilon_{jt}}(t,y^{\varepsilon_{jt}}(t))}{\varepsilon_{jt}}\Big)\nabla'v^0(t,(\varphi^{\varepsilon})'(t,y^{\varepsilon_{jt}}(t)))-\nabla'v^0(t)\Big\|_{L^2(\Omega;\mathbb{R}^2)}\\&\leq C\Big\|\dot{\theta}^{\varepsilon_{jt}}\Big(\frac{\varphi_3^{\varepsilon_{jt}}(t,y^{\varepsilon_{jt}}(t))}{\varepsilon_{jt}}\Big)-1\Big\|_{L^2(\Omega)}+\|\nabla'v^0(t,(\varphi^{\varepsilon})'(t,y^{\varepsilon_{jt}}(t)))-\nabla'v^0(t)\|_{L^2(\omega;\mathbb{R}^2)}\\&\leq\frac{C}{\ell_{\varepsilon_{jt}}}+\|\nabla'v^0(t,(\varphi^{\varepsilon})'(t,y^{\varepsilon_{jt}}(t)))-\nabla'v^0(t)\|_{L^2(\omega;\mathbb{R}^2)}\end{aligned}$$

which converges to zero owing to (5.3.6), (5.3.13), (5.3.14), (5.4.55) (which can be proved arguing exactly as in Step 2 of the case $\alpha>3$) and the dominated convergence theorem. By collecting the previous remarks, we obtain

$$\Big\|\frac{1}{\varepsilon_{jt}}\Big(\nabla(\varphi^{\varepsilon_{jt}})'(t,y^{\varepsilon_{jt}}(t))-\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}\Big)-(0|\nabla'v^0(t))\Big\|_{L^2(\Omega;\mathbb{M}^{3\times3})}\to0.$$

On the other hand, by (5.3.16) there holds

$$\begin{aligned}&\Big\|\frac{\nabla\varphi_3^{\varepsilon_{jt}}(t,y^{\varepsilon_{jt}}(t))-e_3}{\varepsilon_{jt}}+\begin{pmatrix}\nabla'v^0\\0\end{pmatrix}\Big\|_{L^2(\Omega;\mathbb{R}^3)}\leq C\Big\|\nabla(\varphi^{\varepsilon_{jt}})'(t)-\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}\Big\|_{L^\infty(\Omega;\mathbb{M}^{2\times3})}\\&+\|\nabla'v^0(t,(\varphi^{\varepsilon_{jt}})'(t,y^{\varepsilon_{jt}}(t)))-\nabla'v^0(t)\|_{L^2(\Omega;\mathbb{R}^2)}\end{aligned}$$

which tends to zero owing to (5.3.6), (5.3.13), (5.3.14), (5.3.19), (5.4.55) and the dominated convergence theorem. Therefore, the proof of claim (5.4.92) is completed.

Now, by (5.4.55), (5.4.92) and the dominated convergence theorem we conclude that

$$\frac{1}{\varepsilon_{jt}}\begin{pmatrix} 0 & -\dot\theta^{\varepsilon_{jt}}\big(\frac{z_3^{\varepsilon_{jt}}(t)}{\varepsilon_{jt}}\big)\nabla'\dot v^0(t,(z^{\varepsilon_{jt}})'(t)) \\ (\nabla'\dot v^0(t,(z^{\varepsilon_{jt}})'(t)))^T & 0 \end{pmatrix}\Big(\nabla\varphi^{\varepsilon_{jt}}(t,y^{\varepsilon_{jt}}(t)) - Id\Big)$$
$$\to \begin{pmatrix} \nabla'\dot v^0(t)\otimes\nabla' v^0(t) & 0 \\ 0 & \frac{d}{dt}\frac{|\nabla' v^0(t)|^2}{2} \end{pmatrix} \tag{5.4.94}$$

strongly in $L^2(\Omega;\mathbb{M}^{3\times3})$. By combining (5.4.90), (5.4.91) and (5.4.94) we deduce (5.4.89). Now, by (4.3.25), (5.3.6), (5.4.37), (5.4.39), (5.4.61) (which still hold true for $\alpha=3$), (5.4.85) and (5.4.89) we obtain

$$\Theta(t) = \int_\Omega E(t) : \begin{pmatrix} \nabla'\dot u^0(t) + \nabla'\dot v^0(t)\otimes\nabla' v^0(t) - (x_3 + v(t) - v^0(t))(\nabla')^2\dot v^0(t) & 0 \\ 0 & 0 \end{pmatrix} dx. \tag{5.4.95}$$

On the other hand,

$$\begin{aligned}&\mathrm{sym}(\nabla'\dot v^0(t)\otimes\nabla' v^0(t) - (v(t)-v^0(t))(\nabla')^2\dot v^0(t)) \\ &= -\mathrm{sym}\,\nabla'\big((v(t)-v^0(t))\nabla'\dot v^0(t)\big) + \mathrm{sym}\big(\nabla' v(t)\otimes\nabla'\dot v^0(t)\big)\end{aligned}$$

and

$$\int_\Omega \mathbb{C}_2 E(t) : \nabla'\big((v(t)-v^0(t))\nabla'\dot v^0(t)\big)\,dx = 0 \tag{5.4.96}$$

by Remark 5.3.8. By combining (5.4.95) and (5.4.96), the proof of (5.4.88) and of the theorem is complete. □

To conclude this section we show some corollaries of Theorem 5.3.9. We first prove that under the hypotheses of the theorem we can deduce convergence of the elastic energies and of the hardening functionals. More precisely, the following result holds true.

**Corollary 5.4.1.** *Under the assumptions of Theorem 5.3.9, for $\alpha > 3$ for every $t\in[0,T]$, setting $y^\varepsilon(t) := \phi^\varepsilon(t, z^\varepsilon(t))$ there holds*

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon^{2\alpha-2}}\int_\Omega W_{el}(\nabla_\varepsilon y^\varepsilon(t)(P^\varepsilon)^{-1}(t))\,dx = \int_\Omega Q_2(\mathrm{sym}\,\nabla' u(t) - x_3(\nabla')^2 v(t) - p'(t))\,dx,$$

*and*

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon^{2\alpha-2}}\int_\Omega W_{hard}(P^\varepsilon(t))\,dx = \int_\Omega B(p(t))\,dx. \tag{5.4.97}$$

*The analogous result holds true for $\alpha=3$ on the $t$-dependent subsequence $\varepsilon_{jt}\to0$ selected in Theorem 5.3.9.*

*Proof.* The result follows by combining the liminf inequalities (4.3.43) and (4.3.44) in Theorem 4.3.3, the $\varepsilon$-energy balance (eb) and the reduced energy balance $(\mathrm{eb})_{r\alpha}$. □

In particular, we can deduce strong convergence of the sequence of scaled plastic strains by the convergence of the energies.

**Corollary 5.4.2.** *Under the hypotheses of Theorem 5.3.9, for $\alpha>3$ there holds*

$$p^\varepsilon(t)\to p(t) \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}) \tag{5.4.98}$$

*for every $t\in[0,T]$. The analogous result holds true for $\alpha=3$ on the $t$-dependent subsequence $\varepsilon_{jt}\to0$ selected in Theorem 5.3.9.*

*Proof.* We prove the corollary for $\alpha > 3$. The case $\alpha = 3$ follows by simple adaptations. Fix $\delta > 0$ and let $c_h(\delta)$ be the constant in (4.2.12). By (4.2.12) there holds

$$W_{hard}(Id + F) \geq B(F) - C\delta|F|^2 \quad \text{for every } F \in \mathbb{M}^{3\times3},\ |F| < c_h(\delta). \tag{5.4.99}$$

Fix $t \in [0, T]$ and for every $\varepsilon$ consider the set

$$S_\varepsilon(t) := \Big\{x \in \Omega : |p^\varepsilon(t, x)| < \frac{c_h(\delta)}{\varepsilon}\Big\}.$$

Denoting by $\mu_\varepsilon(t)$ the characteristic function of the set $S_\varepsilon(t)$, by (5.3.30) and Chebychev inequality,

$$\mu_\varepsilon(t) \to 1 \quad \text{boundedly in measure as } \varepsilon \to 0. \tag{5.4.100}$$

and thus

$$\mu_\varepsilon(t)p^\varepsilon(t) \rightharpoonup p(t) \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{3\times3}). \tag{5.4.101}$$

We remark that in the set $S_\varepsilon(t)$ we have $\varepsilon^{\alpha-1}|p^\varepsilon(t)| < \varepsilon^{\alpha-2}c_h(\delta)$. Hence, by (5.4.99) for $\varepsilon$ small enough there holds

$$\frac{1}{\varepsilon^{2\alpha-2}} W_{hard}(P^\varepsilon(t)) \geq \frac{1}{\varepsilon^{2\alpha-2}} \mu_\varepsilon(t) W_{hard}(P^\varepsilon(t)) \geq \mu_\varepsilon(t)\big(B(p^\varepsilon(t)) - C\delta|p^\varepsilon(t)|^2\big).$$

In particular, by (5.3.30), (5.4.97) and the lower semicontinuity of $B$ with respect to weak $L^2$ convergence, we have

$$\begin{aligned}\int_\Omega B(p(t))\,dx &= \lim_{\varepsilon\to0} \frac{1}{\varepsilon^{2\alpha-2}} \int_\Omega W_{hard}(P^\varepsilon(t))\,dx \geq \limsup_{\varepsilon\to0} \frac{1}{\varepsilon^{2\alpha-2}} \int_\Omega \mu_\varepsilon(t) W_{hard}(P^\varepsilon(t))\,dx \\ &\geq \limsup_{\varepsilon\to0} \int_\Omega \mu_\varepsilon(t) B(p^\varepsilon(t))\,dx - C\delta \geq \liminf_{\varepsilon\to0} \int_\Omega \mu_\varepsilon(t) B(p^\varepsilon(t))\,dx - C\delta \geq \int_\Omega B(p(t))\,dx - C\delta.\end{aligned}$$

Since $\delta$ is arbitrary, we obtain

$$\lim_{\varepsilon\to0} \int_\Omega \mu_\varepsilon(t) B(p^\varepsilon(t))\,dx = \int_\Omega B(p(t))\,dx \tag{5.4.102}$$

and by (5.4.97)

$$\lim_{\varepsilon\to0} \frac{1}{\varepsilon^{2\alpha-2}} \int_\Omega (1 - \mu_\varepsilon(t)) W_{hard}(P^\varepsilon(t))\,dx = 0. \tag{5.4.103}$$

By (4.2.11) and (5.4.103) we deduce

$$\lim_{\varepsilon\to0} \int_\Omega (1 - \mu_\varepsilon(t))|p^\varepsilon(t)|^2\,dx \leq \frac{2}{c_6} \lim_{\varepsilon\to0} \frac{1}{\varepsilon^{2\alpha-2}} \int_\Omega (1 - \mu_\varepsilon(t)) W_{hard}(P^\varepsilon(t))\,dx = 0. \tag{5.4.104}$$

Hence, by (4.2.15) there holds

$$\begin{aligned}\int_\Omega |p^\varepsilon(t) - p(t)|^2\,dx &= \int_\Omega \mu_\varepsilon(t)|p^\varepsilon(t) - p(t)|^2\,dx + \int_\Omega (1 - \mu_\varepsilon(t))|p^\varepsilon(t) - p(t)|^2\,dx \\ &\leq \frac{2}{c_6} \int_\Omega \mu_\varepsilon(t) B(p^\varepsilon(t) - p(t))\,dx + 2\int_\Omega (1 - \mu_\varepsilon(t))(|p^\varepsilon(t)|^2 + |p(t)|^2)\,dx.\end{aligned} \tag{5.4.105}$$

Recalling the quadratic structure of $B$, the first term in the second row of (5.4.105) can be decomposed as

$$\begin{aligned}\frac{2}{c_6}\int_\Omega \mu_\varepsilon(t)B(p^\varepsilon(t)-p(t))\,dx &= \frac{2}{c_6}\int_\Omega \mu_\varepsilon(t)B(p^\varepsilon(t))\,dx+\frac{2}{c_6}\int_\Omega \mu_\varepsilon(t)B(p(t))\,dx\\ &- \frac{4}{c_6}\int_\Omega \mu_\varepsilon(t)\mathbb{B}p^\varepsilon(t):p(t)\,dx\end{aligned}$$

and tends to zero due to (5.4.100)–(5.4.102). On the other hand, by (5.4.100) and (5.4.104)

$$\int_\Omega (1-\mu_\varepsilon(t))(|p^\varepsilon(t)|^2+|p(t)|^2)\,dx\to 0.$$

By combining the previous results, we deduce (5.4.98). □

Convergence of the energy implies also strong convergence of the in-plane displacements. More precisely, the following result holds true.

**Corollary 5.4.3.** *Under the assumptions of Theorem 5.3.9, for $\alpha>3$, for every $t\in[0,T]$ there holds*

$$u^\varepsilon(t)\to u(t)\quad \textit{strongly in } W^{1,2}(\omega;\mathbb{R}^2). \tag{5.4.106}$$

*The same result holds true for $\alpha=3$, on the $t$-dependent subsequence $\varepsilon_{jt}\to 0$ selected in Theorem 5.3.9.*

*Proof.* We prove the corollary for $\alpha>3$. The case where $\alpha=3$ follows by simple adaptations. Fix $t\in[0,T]$ and let $F^\varepsilon(t)$ be the map defined in (5.4.20). Fix $\delta>0$ and consider the set

$$U_\varepsilon(t):=\Big\{x\in\Omega:|F^\varepsilon(t,x)|<\frac{c_{el}(\delta)}{\varepsilon}\Big\},$$

where $c_{el}(\delta)$ is the constant in (4.2.4). In particular, in the set $U_\varepsilon(t)$ there holds $\varepsilon^{\alpha-1}|F^\varepsilon(t)|\le\varepsilon^{\alpha-2}c_{el}(\delta)$. Hence, denoting by $\mu_\varepsilon(t)$ the characteristic function of $U_\varepsilon(t)$, by (H3) (see Section 4.2), (4.2.4) and (5.4.21), we have

$$\frac{1}{\varepsilon^{2\alpha-2}}W_{el}(\nabla_\varepsilon y^\varepsilon(t)(P^\varepsilon)^{-1}(t))=\frac{1}{\varepsilon^{2\alpha-2}}W_{el}(Id+\varepsilon^{\alpha-1}F^\varepsilon(t))\ge\mu_\varepsilon(t)Q(F^\varepsilon(t))-\mu_\varepsilon(t)C\delta|F^\varepsilon(t)|^2.$$

By Chebychev inequality and (5.4.28),

$$\mu_\varepsilon(t)\to 1\quad\text{boundedly in measure,} \tag{5.4.107}$$

whereas by (5.4.65) and (5.4.74),

$$\mu_\varepsilon(t)\mathrm{sym}\,F^\varepsilon(t)\rightharpoonup\mathbb{A}(\mathrm{sym}\,\nabla'u(t)-x_3(\nabla')^2v(t)-p'(t))\quad\text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}). \tag{5.4.108}$$

Arguing as in the proof of (5.4.102) we obtain

$$\lim_{\varepsilon\to0}\int_\Omega\mu_\varepsilon(t)Q(F^\varepsilon(t))\,dx=\int_\Omega Q_2(\mathrm{sym}\,\nabla'u(t)-x_3(\nabla')^2v(t)-p'(t))\,dx \tag{5.4.109}$$

and

$$\lim_{\varepsilon\to0}\int_\Omega(1-\mu_\varepsilon(t))W_{el}(Id+\varepsilon^{\alpha-1}F^\varepsilon(t))\,dx=0.$$

By (H4) (see Section 4.2), this implies that

$$\lim_{\varepsilon\to 0}\int_\Omega (1-\mu_\varepsilon(t))\mathrm{dist}^2(Id+\varepsilon^{\alpha-1}F^\varepsilon(t),SO(3))\,dx\to 0. \tag{5.4.110}$$

On the other hand, (4.2.7) and (4.3.25) yield

$$\begin{aligned}
&\int_\Omega \big|\mu_\varepsilon(t)\mathrm{sym}\,F^\varepsilon(t)-\mathbb{A}\big(\nabla' u(t)-x_3(\nabla')^2 v(t)-p'(t)\big)\big|^2\,dx\\
&\leq \frac{1}{r_{\mathbb{C}}}\int_\Omega Q\big(\mu_\varepsilon(t)\mathrm{sym}\,F^\varepsilon(t)-\mathbb{A}\big(\nabla' u(t)-x_3(\nabla')^2 v(t)-p'(t)\big)\big)\,dx\\
&=\frac{1}{r_{\mathbb{C}}}\int_\Omega Q(\mu_\varepsilon(t)F^\varepsilon(t))\,dx+\frac{1}{r_{\mathbb{C}}}\int_\Omega Q_2(\nabla' u(t)-x_3(\nabla')^2 v(t)-p'(t))\,dx\\
&-\frac{2}{r_{\mathbb{C}}}\int_\Omega \mu_\varepsilon(t)\mathbb{C}_2F^\varepsilon(t):(\nabla' u(t)-x_3(\nabla')^2 v(t)-p'(t))\,dx.
\end{aligned}$$

Hence, by (5.4.108) and (5.4.109)

$$\mu_\varepsilon(t)\mathrm{sym}\,F^\varepsilon(t)\to\mathbb{A}(\nabla' u(t)-x_3(\nabla')^2 v(t)-p'(t))\quad\text{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}). \tag{5.4.111}$$

Moreover,

$$\begin{aligned}
&\frac{1}{\varepsilon^{\alpha-1}}\mu_\varepsilon(t)\mathrm{dist}(Id+\varepsilon^{\alpha-1}F^\varepsilon(t),SO(3))\\
&=\mu_\varepsilon(t)|\mathrm{sym}\,F^\varepsilon(t)|+\mu_\varepsilon(t)O(\varepsilon^{\alpha-1}|F^\varepsilon(t)|^2)\to|\mathbb{A}(\nabla' u(t)-x_3(\nabla')^2 v(t)-p'(t))|
\end{aligned} \tag{5.4.112}$$

strongly in $L^2(\Omega)$. By combining (5.4.110) and (5.4.112) we deduce

$$\frac{1}{\varepsilon^{\alpha-1}}\mathrm{dist}(Id+\varepsilon^{\alpha-1}F^\varepsilon(t),SO(3))\to|\mathbb{A}(\nabla' u(t)-x_3(\nabla')^2 v(t)-p'(t))|$$

strongly in $L^2(\Omega)$. In particular, the sequence $\frac{1}{\varepsilon^{2\alpha-2}}\mathrm{dist}^2(Id+\varepsilon^{\alpha-1}F^\varepsilon(t),SO(3))$ is equi-integrable.

Now, recalling that by (5.4.20) there holds

$$Id+\varepsilon^{\alpha-1}F^\varepsilon(t)=(Id+\varepsilon^{\alpha-1}G^\varepsilon(t))(Id+\varepsilon^{\alpha-1}p^\varepsilon(t))^{-1},$$

by (4.2.13) and (5.4.3) for every $R\in SO(3)$ we deduce

$$\begin{aligned}
&\frac{1}{\varepsilon^{2\alpha-2}}|Id+\varepsilon^{\alpha-1}G^\varepsilon(t)-R|^2=\frac{1}{\varepsilon^{2\alpha-2}}|(Id+\varepsilon^{\alpha-1}F^\varepsilon(t))(Id+\varepsilon^{\alpha-1}p^\varepsilon(t))-R|^2\\
&\leq\frac{2}{\varepsilon^{2\alpha-2}}|Id+\varepsilon^{\alpha-1}F^\varepsilon(t)-R|^2+2|p^\varepsilon(t)|^2,
\end{aligned}$$

which in turn implies

$$\frac{1}{\varepsilon^{2\alpha-2}}\mathrm{dist}^2(Id+\varepsilon^{\alpha-1}G^\varepsilon(t),SO(3))\leq\frac{2}{\varepsilon^{2\alpha-2}}\mathrm{dist}^2(Id+\varepsilon^{\alpha-1}F^\varepsilon(t),SO(3))+|p^\varepsilon(t)|^2.$$

Hence, by (5.4.98) $\frac{1}{\varepsilon^{2\alpha-2}}\mathrm{dist}^2(Id+\varepsilon^{\alpha-1}G^\varepsilon(t),SO(3))$ is equi-integrable. Arguing as in [34, Section 7.2, Proof of Theorem 2] we obtain the equi-integrability of $|G^\varepsilon(t)|^2$.

We claim that also $|F^\varepsilon(t)|^2$ is equi-integrable. Indeed, by (5.4.20), there holds

$$|F^\varepsilon(t)|^2\leq C(|G^\varepsilon(t)|^2+|w^\varepsilon(t)|^2+|p^\varepsilon(t)|^2+\varepsilon^{2\alpha-2}|G^\varepsilon(t)w^\varepsilon(t)|^2+\varepsilon^{2\alpha-2}|G^\varepsilon(t)p^\varepsilon(t)|^2).$$

Now, by (5.4.2), (5.4.3) and (5.4.22), we have

$$|w^\varepsilon(t)|^2 \le c_K^2 \varepsilon^{2\alpha-2}|p^\varepsilon(t)|^4 \le C|p^\varepsilon(t)|^2.$$

Hence, by (5.4.98) the maps $|w^\varepsilon(t)|^2$ are equi-integrable. Moreover, by (5.4.2) there holds

$$\varepsilon^{2\alpha-2}|G^\varepsilon(t)p^\varepsilon(t)|^2 \le C|G^\varepsilon(t)|^2$$

and by (5.4.23)

$$\varepsilon^{2\alpha-2}|G^\varepsilon(t)w^\varepsilon(t)|^2 \le C|G^\varepsilon(t)|^2.$$

Therefore, the equi-integrability of $|F^\varepsilon(t)|^2$ follows from the equi-integrability of $|G^\varepsilon(t)|^2$.

By (5.4.111), this implies that

$$\operatorname{sym} F^\varepsilon(t) \to \mathbb{A}(\nabla' u(t) - x_3(\nabla')^2 v(t) - p'(t))$$

strongly in $L^2(\Omega;\mathbb{M}^{3\times 3})$. On the other hand, by (5.4.24) and (5.4.63),

$$w^\varepsilon(t) - \varepsilon^{\alpha-1}G^\varepsilon(t)(p^\varepsilon(t) - w^\varepsilon(t)) \to 0$$

strongly in $L^1(\Omega;\mathbb{M}^{3\times 3})$. Therefore, by (5.4.20) and (5.4.98) we obtain

$$\operatorname{sym} G^\varepsilon(t) \to \mathbb{A}(\nabla' u(t) - x_3(\nabla')^2 v(t) - p'(t)) + p(t) \quad \text{strongly in } L^1(\Omega;\mathbb{M}^{3\times 3}).$$

By the equi-integrability of $|G^\varepsilon(t)|^2$, it follows that

$$\operatorname{sym} G^\varepsilon(t) \to \mathbb{A}(\nabla' u(t) - x_3(\nabla')^2 v(t) - p'(t)) + p(t) \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times 3}).$$

The conclusion follows then arguing as in [34, Section 7.2, Proof of Theorem 2]. □

## 5.5 Convergence of approximate minimizers

Theorems 5.3.9 is actually only a convergence result. Indeed, under our assumptions the existence of an $\varepsilon$-quasistatic evolution according to Definition 5.3.3 is not guaranteed. However, following the same approach as in [52, Theorem 2.3], we can extend our convergence result to sequences of approximate discrete-time $\varepsilon$-quasistatic evolutions. More precisely, setting

$$\begin{aligned} \mathcal{A}_\varepsilon \quad := \quad & \{(z,P) \in W^{1,2}(\Omega;\mathbb{R}^3) \times L^2(\Omega;SL(3)) : \\ & z = (x', \varepsilon x_3) \quad \mathcal{H}^2 \text{ - a.e. on } \Gamma_d \quad \text{and } P(x) \in K \quad \text{a.e. in } \Omega\}, \end{aligned}$$

we give the following definition.

**Definition 5.5.1.** Given a sequence of time-partitions

$$\{0 = t_\varepsilon^0 < t_\varepsilon^1 < \cdots t_\varepsilon^{N^\varepsilon} = T\},$$

with time-steps

$$\tau_\varepsilon := \max_{i=1,\cdots N^\varepsilon}(t_\varepsilon^i - t_\varepsilon^{i-1}) \to 0 \quad \text{as } \varepsilon \to 0, \tag{5.5.1}$$

and a sequence of positive parameters $\delta_\varepsilon \to 0$, we call $\{(z^i_\varepsilon, P^i_\varepsilon)\}$ a sequence of *approximate minimizers* if, for every $\varepsilon > 0$, $(z^0_\varepsilon, P^0_\varepsilon) \in \mathcal{A}_\varepsilon$, and $(z^i_\varepsilon, P^i_\varepsilon) \in \mathcal{A}_\varepsilon$ satisfies

$$\begin{aligned}
&\mathcal{F}_\varepsilon(t^i_\varepsilon, z^i_\varepsilon, P^i_\varepsilon) + \varepsilon^{\alpha-1}\int_\Omega D(P^{i-1}_\varepsilon, P^i_\varepsilon)\,dx \\
&\le \varepsilon^{2\alpha-2}\delta_\varepsilon(t^i_\varepsilon - t^{i-1}_\varepsilon) + \inf_{(z,P)\in\mathcal{A}_\varepsilon}\Big\{\mathcal{F}_\varepsilon(t^i_\varepsilon, z, P) + \varepsilon^{\alpha-1}\int_\Omega D(P^{i-1}_\varepsilon, P)\,dx\Big\} \quad (5.5.2)
\end{aligned}$$

for every $i = 1, \cdots, N^\varepsilon$.

Our final result is to show that every sequence of approximate minimizers converges, as $\varepsilon \to 0$, to a reduced quasistatic evolution.

**Theorem 5.5.2.** *Let $\alpha \ge 3$. Assume that $t \mapsto u^0(t)$ belongs to $C^1([0,T]; W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2) \cap C^1(\mathbb{R}^2;\mathbb{R}^2))$ and $t \mapsto v^0(t)$ belongs to $C^1([0,T]; W^{2,\infty}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2))$, respectively. For every $t \in [0,T]$, let $\phi^\varepsilon(t)$ be defined as in (5.3.10) and let $(\mathring{u}, \mathring{v}, \mathring{p}) \in \mathcal{A}(u^0(0), v^0(0))$ be such that*

$$\begin{aligned}
&\int_\Omega Q_2(\mathrm{sym}\nabla'\mathring{u} - x_3(\nabla')^2\mathring{v} + \tfrac{L_\alpha}{2}\nabla'\mathring{v}\otimes\nabla'\mathring{v} - \mathring{p}')\,dx + \int_\Omega B(\mathring{p})\,dx \\
&\le \int_\omega Q_2(\mathrm{sym}\nabla'\hat{u} - x_3(\nabla')^2\hat{v} + \tfrac{L_\alpha}{2}\nabla'\hat{v}\otimes\nabla'\hat{v} - \hat{p}')\,dx' + \int_\Omega B(\hat{p})\,dx + \int_\Omega H_D(\hat{p} - \mathring{p})\,dx,
\end{aligned} \quad (5.5.3)$$

*for every $(\hat{u}, \hat{v}, \hat{p}) \in \mathcal{A}(u^0(0), v^0(0))$. Given a sequence of time-partitions*

$$\{0 = t^0_\varepsilon < t^1_\varepsilon < \cdots t^{N^\varepsilon}_\varepsilon = T\},$$

*with time-steps*

$$\tau_\varepsilon := \max_{i=1,\cdots N^\varepsilon}(t^i_\varepsilon - t^{i-1}_\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0,$$

*and a sequence of positive parameters $\delta_\varepsilon \to 0$, assume there exists a sequence of pairs $(y^\varepsilon_0, P^\varepsilon_0) \in \mathcal{A}_\varepsilon(\phi^\varepsilon(0))$ such that*

$$\mathcal{I}(y^\varepsilon_0, P^\varepsilon_0) \le \mathcal{I}(\hat{y}, \hat{P}) + \varepsilon^{\alpha-1}\int_\Omega D(P^\varepsilon_0, \hat{P})\,dx + \delta_\varepsilon\tau_\varepsilon, \quad (5.5.4)$$

*for every $(\hat{y}, \hat{P}) \in \mathcal{A}_\varepsilon(\phi^\varepsilon(0))$, and*

$$u^\varepsilon_0 := \frac{1}{\varepsilon^{\alpha-1}}\int_{-\frac12}^{\frac12}\big((y^\varepsilon_0)' - x'\big)\,dx_3 \to \mathring{u} \quad \text{strongly in } W^{1,2}(\omega;\mathbb{R}^2), \quad (5.5.5)$$

$$v^\varepsilon_0 := \frac{1}{\varepsilon^{\alpha-2}}\int_{-\frac12}^{\frac12}(y^\varepsilon_0)_3\,dx_3 \to \mathring{v} \quad \text{strongly in } W^{1,2}(\omega), \quad (5.5.6)$$

$$p^\varepsilon_0 := \frac{P^\varepsilon_0 - Id}{\varepsilon^{\alpha-1}} \to \mathring{p} \quad \text{strongly in } L^2(\Omega;\mathbb{M}^{3\times3}_D), \quad (5.5.7)$$

$$\begin{aligned}
\lim_{\varepsilon\to0}\frac{1}{\varepsilon^{2\alpha-2}}\mathcal{I}(y^\varepsilon_0, P^\varepsilon_0) &= \int_\Omega Q_2(\mathrm{sym}\nabla'\mathring{u} - x_3(\nabla')^2\mathring{v} + \tfrac{L_\alpha}{2}\nabla'\mathring{v}\otimes\nabla'\mathring{v} - \mathring{p}')\,dx \\
&+ \int_\Omega B(\mathring{p})\,dx.
\end{aligned} \quad (5.5.8)$$

*Let $(z^i_\varepsilon, P^i_\varepsilon)$ be a sequence of approximate minimizers and let $(\overline{z}^\varepsilon(t), \overline{P}^\varepsilon(t))$ be the corresponding right-continuous, piecewise constant interpolants on the time partitions. Let $\overline{\phi}^\varepsilon(t)$ be the associated interpolant of $t \mapsto \phi^\varepsilon(t)$. Then, for every $t \in [0,T]$*

$$\overline{p}^\varepsilon(t) := \frac{\overline{P}^\varepsilon(t) - Id}{\varepsilon^{\alpha-1}} \rightharpoonup p(t) \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{3\times 3}).$$

*Moreover, for $\alpha > 3$, for every $t \in [0,T]$ the following convergence properties hold true:*

$$\overline{u}^\varepsilon(t) := \frac{1}{\varepsilon^{\alpha-1}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \big((\overline{\phi}^\varepsilon)'(t, \overline{z}^\varepsilon(t)) - x'\big)\, dx_3 \rightharpoonup u(t) \quad \text{weakly in } W^{1,2}(\omega; \mathbb{R}^2),$$

$$\overline{v}^\varepsilon(t) := \frac{1}{\varepsilon^{\alpha-2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{\phi}^\varepsilon_3(t, \overline{z}^\varepsilon(t))\, dx_3 \to v(t) \quad \text{strongly in } W^{1,2}(\omega),$$

*where $t \mapsto (u(t), v(t), p(t))$ is a reduced quasistatic evolution.*

*For $\alpha = 3$, up to extracting a $t$-dependent subsequence $\varepsilon_{jt} \to 0$, there holds*

$$\overline{u}^{\varepsilon_{jt}}(t) := \frac{1}{\varepsilon_{jt}^{\alpha-1}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \big((\overline{\phi}^{\varepsilon_{jt}})'(t, \overline{z}^{\varepsilon_{jt}}(t)) - x'\big)\, dx_3 \rightharpoonup u(t) \quad \text{weakly in } W^{1,2}(\omega; \mathbb{R}^2),$$

$$\overline{v}^{\varepsilon_{jt}}(t) := \frac{1}{\varepsilon_{jt}^{\alpha-2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{\phi}^{\varepsilon_{jt}}_3(t, \overline{z}^{\varepsilon_{jt}}(t))\, dx_3 \to v(t) \quad \text{strongly in } W^{1,2}(\omega),$$

*where $t \mapsto (u(t), v(t), p(t))$ is a reduced quasistatic evolution.*

**Remark 5.5.3.** The set of admissible data $(\mathring{u}, \mathring{v}, \mathring{p})$ for Theorem 5.5.2 is nonempty.

Indeed, for every $\varepsilon > 0$ let $(y^\varepsilon_0, P^\varepsilon_0) \in \mathcal{A}_\varepsilon(\phi^\varepsilon(0))$ be such that

$$\mathcal{I}(y^\varepsilon_0, P^\varepsilon_0) + \varepsilon^{\alpha-1} \int_\Omega D(Id, P^\varepsilon_0)\, dx \le \inf_{(\hat{y}, \hat{P}) \in \mathcal{A}_\varepsilon(\phi^\varepsilon(0))} \Big\{ \mathcal{I}(\hat{y}, \hat{P}) + \varepsilon^{\alpha-1} \int_\Omega D(Id, \hat{P})\, dx \Big\} + \delta_\varepsilon \tau_\varepsilon.$$

By (4.2.18) there holds

$$D(Id, \hat{P}) \le D(Id, P^\varepsilon_0) + D(P^\varepsilon_0, \hat{P}),$$

hence $(y^\varepsilon_0, P^\varepsilon_0)$ fulfills (5.5.4). By the regularity of $\partial\omega$, the set $\gamma_d$ coincides $\mathcal{H}^1$ - a.e. with its closure in the relative topology of $\partial\omega$, which in turn is a closed (nontrivial) interval in $\partial\omega$. Hence, by Theorem 4.5.1, choosing $p^{\varepsilon,0} = p^0 = 0$ for every $\varepsilon > 0$, and $s_\varepsilon = \delta_\varepsilon \tau_\varepsilon$, we infer the existence of a triple $(\mathring{u}, \mathring{v}, \mathring{p}) \in \mathcal{A}(u^0(0), v^0(0))$ such that (5.5.3) is satisfied and (5.5.5)–(5.5.8) hold true.

*Proof of Theorem 5.5.2.* The proof follows along the general lines of the proof of Theorem 5.3.9. We sketch the main steps in the case $\alpha > 3$. The case $\alpha = 3$ follows by straightforward adaptations.

*Quasi-stability condition*

By (4.2.18) the piecewise constant interpolants fullfill

$$\mathcal{F}_\varepsilon(t, \overline{z}^\varepsilon(t), \overline{P}^\varepsilon(t)) \le \mathcal{F}_\varepsilon(t, \hat{z}, \hat{P}) + \varepsilon^{\alpha-1} \int_\Omega D(\overline{P}^\varepsilon(t), \hat{P})\, dx + \delta_\varepsilon \tau_\varepsilon \varepsilon^{2\alpha-2} \tag{5.5.9}$$

for every $(\hat{z}, \hat{P}) \in \mathcal{A}_\varepsilon$. The previous inequality will play the role of the $\varepsilon$-stability condition (gs).

*Discrete energy inequality*

To adapt the proof of Theorem 5.3.9 we shall need an analogue of condition (eb). To this purpose, we notice that, by (5.5.2) the following discrete energy inequality holds true

$$
\begin{aligned}
&\mathcal{F}_\varepsilon(t^i_\varepsilon, z^i_\varepsilon, P^i_\varepsilon) + \varepsilon^{\alpha-1}\int_\Omega D(P^{i-1}_\varepsilon, P^i_\varepsilon)\,dx \le \varepsilon^{2\alpha-2}\delta_\varepsilon(t^i_\varepsilon - t^{i-1}_\varepsilon) + \mathcal{F}_\varepsilon(t^i_\varepsilon, z^{i-1}_\varepsilon, P^{i-1}_\varepsilon)\\
&= \varepsilon^{2\alpha-2}\delta_\varepsilon(t^i_\varepsilon - t^{i-1}_\varepsilon) + \mathcal{F}_\varepsilon(t^{i-1}_\varepsilon, z^{i-1}_\varepsilon, P^{i-1}_\varepsilon) + \int_{t^{i-1}_\varepsilon}^{t^i_\varepsilon} \partial_s\mathcal{F}_\varepsilon(s, z^{i-1}_\varepsilon, P^{i-1}_\varepsilon)\,ds\\
&= \varepsilon^{2\alpha-2}\delta_\varepsilon(t^i_\varepsilon - t^{i-1}_\varepsilon) + \mathcal{F}_\varepsilon(t^{i-1}_\varepsilon, z^{i-1}_\varepsilon, P^{i-1}_\varepsilon)\\
&+\varepsilon^{2\alpha-2}\int_{t^{i-1}_\varepsilon}^{t^i_\varepsilon}\int_\Omega DW_{el}\big(\nabla\phi^\varepsilon(s, z^{i-1}_\varepsilon)\nabla_\varepsilon z^{i-1}_\varepsilon(P^{i-1}_\varepsilon)^{-1}\big) : \nabla\dot{\phi}^\varepsilon(s, z^{i-1}_\varepsilon)\nabla_\varepsilon z^{i-1}_\varepsilon(P^{i-1}_\varepsilon)^{-1}\,dx\,ds\\
&= \varepsilon^{2\alpha-2}\delta_\varepsilon(t^i_\varepsilon - t^{i-1}_\varepsilon) + \mathcal{F}_\varepsilon(t^{i-1}_\varepsilon, z^{i-1}_\varepsilon, P^{i-1}_\varepsilon)\\
&+\varepsilon^{\alpha-1}\int_{t^{i-1}_\varepsilon}^{t^i_\varepsilon}\int_\Omega E^{i-1}_\varepsilon(s) : \nabla\dot{\phi}^\varepsilon(s, z^{i-1}_\varepsilon)(\nabla\phi^\varepsilon)^{-1}(s, z^{i-1}_\varepsilon)\,dx\,ds,
\end{aligned}
$$

where

$$E^{i-1}_\varepsilon(s) := \frac{1}{\varepsilon^{\alpha-1}} DW_{el}\big(\nabla\phi^\varepsilon(s, z^{i-1}_\varepsilon)\nabla_\varepsilon z^{i-1}_\varepsilon(P^{i-1}_\varepsilon)^{-1}\big)\big(\nabla\phi^\varepsilon(s, z^{i-1}_\varepsilon)\nabla_\varepsilon z^{i-1}_\varepsilon(P^{i-1}_\varepsilon)^{-1}\big)^T$$

for every $s \in [t^{i-1}_\varepsilon, t^i_\varepsilon]$.

By iterating the discrete energy inequality, recalling that $\overline{P}^\varepsilon(t)$ is locally constant, we obtain

$$
\begin{aligned}
&\mathcal{F}_\varepsilon(t, \overline{z}^\varepsilon(t), \overline{P}^\varepsilon(t)) + \varepsilon^{\alpha-1}\mathcal{D}(\overline{P}^\varepsilon; 0, t)\\
&\le \varepsilon^{2\alpha-2}\delta_\varepsilon T + \mathcal{F}_\varepsilon(0, z^\varepsilon_0, P^\varepsilon_0) + \varepsilon^{\alpha-1}\int_0^t\int_\Omega \overline{E}^\varepsilon(s) : \nabla\dot{\phi}^\varepsilon(s, \overline{z}^\varepsilon(s))(\nabla\phi^\varepsilon)^{-1}(s, \overline{z}^\varepsilon(s))\,dx\,ds,
\end{aligned}
\tag{5.5.10}
$$

where $z^\varepsilon_0 := \varphi^\varepsilon(0, y^\varepsilon_0)$ and

$$\overline{E}^\varepsilon(s) := \frac{1}{\varepsilon^{\alpha-1}} DW_{el}\big(\nabla\phi^\varepsilon(s, \overline{z}^\varepsilon(s))\nabla_\varepsilon\overline{z}^\varepsilon(s)(\overline{P}^\varepsilon)^{-1}(s)\big)\big(\nabla\phi^\varepsilon(s, \overline{z}^\varepsilon(s))\nabla_\varepsilon\overline{z}^\varepsilon(s)(\overline{P}^\varepsilon)^{-1}(s)\big)^T$$

for every $s \in [0, t]$.

*Proof of the reduced stability condition and energy balance*

The reduced stability condition can be deduced as in Step 2 of the proof of Theorem 5.3.9. Moreover, arguing as in the proof of Theorem 5.3.9 one can show that $\overline{E}^\varepsilon(t)$ converges in the sense of (5.4.37) and (5.4.61) to a limit stress $E(t)$ such that

$$E(t) = \mathbb{C}(G(t) - p(t)).$$

The crucial step to deduce the reduced energy balance is to show that $E(t)e_3 = 0$ a.e. in $\Omega$, that is,

$$E(t) = \mathbb{C}_2(G'(t) - p'(t)). \tag{5.5.11}$$

The main difference with respect to Theorem 5.3.9 is that in this case we can not deduce this condition starting from the three-dimensional Euler-Lagrange equations because (5.5.10) does not imply (5.4.68).

To cope with this problem, set $\overline{y}^\varepsilon(t) = \overline{\phi}^\varepsilon(t, \overline{z}^\varepsilon(t))$ for every $t \in [0,T]$. Let $\eta \in W^{1,\infty}(\mathbb{R}^3;\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ be such that $\eta = 0$ $\mathcal{H}^2$ - a.e. on $\Gamma_d$. We argue as in the proof of Theorem 1.3.1 and we consider variations of the form

$$\hat{y} = \overline{y}^\varepsilon(t) + \tau_\varepsilon \varepsilon^{\alpha-1} \eta^\varepsilon \circ \overline{y^\varepsilon},$$

where $\eta^\varepsilon$ is the test function considered in Step 4 of the proof of Theorem 5.3.9. By (5.5.9), taking $\hat{P} = \overline{P}^\varepsilon(t)$, we deduce

$$\begin{aligned}
-\delta_\varepsilon &\leq \frac{1}{\varepsilon^{\alpha-1}} \int_\Omega \frac{W_{el}\Big(\Big(Id + \tau_\varepsilon \varepsilon^{\alpha-1} \nabla \eta^\varepsilon(\overline{y}^\varepsilon(t))\Big) \nabla_\varepsilon \overline{y}^\varepsilon(t) (\overline{P}^\varepsilon)^{-1}(t)\Big) - W_{el}(\nabla_\varepsilon \overline{y}^\varepsilon(t)(\overline{P}^\varepsilon)^{-1}(t))}{\tau_\varepsilon \varepsilon^{\alpha-1}} \, dx \\
&= \frac{1}{\varepsilon^{\alpha-1}} \int_\Omega \int_0^1 \frac{d}{ds} \frac{W_{el}\Big(\Big(Id + s\tau_\varepsilon \varepsilon^{\alpha-1} \nabla \eta^\varepsilon(\overline{y}^\varepsilon(t))\Big) \nabla_\varepsilon \overline{y}^\varepsilon(t) (\overline{P}^\varepsilon)^{-1}(t)\Big)}{\tau_\varepsilon \varepsilon^{\alpha-1}} \, ds \, dx \\
&= \int_\Omega \Phi^\varepsilon(t) : \nabla \eta^\varepsilon(\overline{y}^\varepsilon(t)) \, dx,
\end{aligned}$$

where

$$\Phi^\varepsilon(t) := \frac{1}{\varepsilon^{\alpha-1}} \int_0^1 DW_{el}\Big(\Big(Id + s\tau_\varepsilon \varepsilon^{\alpha-1} \nabla \eta^\varepsilon(\overline{y}^\varepsilon(t))\Big) \nabla_\varepsilon \overline{y}^\varepsilon(t)(\overline{P}^\varepsilon)^{-1}(t)\Big)(\nabla_\varepsilon \overline{y}^\varepsilon(t)(\overline{P}^\varepsilon)^{-1}(t))^T \, ds.$$

Since $\overline{P}^\varepsilon(t) \in L^2(\Omega; SL(3))$, $\det \overline{P}^\varepsilon(t) = 1$ a.e. in $\Omega$. Moreover, by (H1) (see Section 4.2) and (5.5.9) we deduce that $\det \nabla_\varepsilon \overline{y}^\varepsilon(t) > 0$ a.e. in $\Omega$. On the other hand, since $\|\nabla \eta^\varepsilon\|_{L^\infty(\Omega;\mathbb{M}^{3\times3})} \leq C$ for every $\varepsilon$ (see Step 4 of the proof of Theorem 5.3.9 and (5.3.18)), by (5.5.1),

$$\det (Id + s\tau_\varepsilon \varepsilon^{\alpha-1} \nabla \eta^\varepsilon(\overline{y}^\varepsilon(t))) > 0 \quad \text{for every } s \in [0,1],$$

for $\varepsilon$ small enough. Hence, by combining (4.2.5) and (4.2.6) we deduce that $\Phi^\varepsilon(t)$ is well defined for $\varepsilon$ small enough. Moreover, there holds

$$\liminf_{\varepsilon \to 0} \Big\{ \int_\Omega \Phi^\varepsilon(t) : \nabla \eta^\varepsilon(\overline{y}^\varepsilon(t)) \, dx \Big\} \geq 0. \tag{5.5.12}$$

We claim that

$$\lim_{\varepsilon \to 0} \int_\Omega \Phi^\varepsilon(t) : \nabla \eta^\varepsilon(\overline{y}^\varepsilon(t)) \, dx = \int_\Omega E(t)e_3 : \partial_3 \eta \, dx. \tag{5.5.13}$$

We note that, once (5.5.13) is proved, from (5.5.12) it follows that

$$\int_\Omega E(t)e_3 : \partial_3 \eta \, dx \geq 0$$

for every $\eta \in W^{1,\infty}(\mathbb{R}^3;\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ such that $\eta = 0$ $\mathcal{H}^2$ - a.e. on $\Gamma_d$, hence the proof of (5.5.11) is complete.

To prove (5.5.13), it is enough to consider the sets

$$O_\varepsilon(t) := \{x : \varepsilon^{\alpha-1-\gamma} |\overline{F}^\varepsilon(t)| < 1\},$$

where the maps $\overline{F}^\varepsilon(t)$ are the piecewise constant interpolants of the maps $F^\varepsilon(t)$ defined in (5.4.20). Arguing as in the proof of (5.4.36) and (5.4.61), one can show that, denoting by $\chi_\varepsilon(t)$ the characteristic function of the set $O_\varepsilon(t)$, there holds

$$||(1 - \chi_\varepsilon(t))\Phi^\varepsilon(t)||_{L^1(\Omega;\mathbb{M}^{3\times3})} \leq C\varepsilon^{\alpha-1-\gamma}$$

and

$$\chi_\varepsilon(t)\Phi^\varepsilon(t) \rightharpoonup E(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}).$$

Claim (5.5.13) follows now arguing as in Step 4 of the proof of Theorem 5.3.9. □

## 5.6 Appendix

This section is devoted to the proof of the existence of a reduced quasistatic evolution for the boundary data $t \mapsto u^0(t)$ and $t \mapsto v^0(t)$ (according to Definition 5.3.6) in the case $\alpha = 3$. We first prove two lemmas that will be useful in the proof of the existence result.

**Lemma 5.6.1.** *Let $p^0 \in L^2(\Omega;\mathbb{M}^{3\times 3}_D)$, $u^0 \in C^1(\overline{\omega};\mathbb{R}^2)$ and $v^0 \in C^2(\overline{\omega})$. Then, there exists a triple $(u,v,p) \in \mathcal{A}(u^0,v^0)$ that solves*

$$\min_{(\tilde u,\tilde v,\tilde p)\in\mathcal{A}(u^0,v^0)} \Big\{ \int_\Omega Q_2\big(\mathrm{sym}\,\nabla'\tilde u + \tfrac12 \nabla'\tilde v\otimes\nabla'\tilde v - x_3(\nabla')^2\tilde v - \tilde p'\big)\,dx + \int_\Omega B(\tilde p)\,dx + \int_\Omega H_D(\tilde p - p^0)\,dx\Big\}. \quad (5.6.1)$$

*Proof.* Let $(u^n,v^n,p^n) \subset \mathcal{A}(u^0,v^0)$ be a minimizing sequence for (5.6.1). Then, there exists a constant $C$ such that

$$\|B(p^n)\|_{L^1(\Omega)} \le C \quad \text{for every } n\in\mathbb{N}.$$

Since $B$ is strictly positive definite, we deduce

$$\|p^n\|_{L^2(\Omega;\mathbb{M}^{3\times 3}_D)} \le C \quad \text{for every } n\in\mathbb{N}. \quad (5.6.2)$$

Hence, there exists a map $p \in L^2(\Omega;\mathbb{M}^{3\times 3}_D)$ such that, up to subsequences

$$p^n \rightharpoonup p \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}_D). \quad (5.6.3)$$

By (4.2.7) and (5.6.2) there holds

$$\|\mathrm{sym}\,\nabla' u^n + \tfrac12\nabla' v^n\otimes\nabla' v^n\|_{L^2(\Omega;\mathbb{M}^{2\times 2})} + \|(\nabla')^2 v^n\|_{L^2(\Omega;\mathbb{M}^{2\times 2})} \le C \quad \text{for every } n\in\mathbb{N}.$$

Therefore, Poincarè inequality yields

$$\|v^n - v^0\|_{L^2(\Omega)} \le C\|\nabla' v^n - \nabla' v^0\|_{L^2(\Omega;\mathbb{R}^2)} \le C\|(\nabla')^2 v^n - (\nabla')^2 v^0\|_{L^2(\Omega;\mathbb{M}^{2\times 2})},$$

which in turn implies that the sequence $(v^n)$ is uniformly bounded in $W^{2,2}(\Omega)$. Thus, there exists $v \in W^{2,2}(\Omega)$ such that, up to subsequences

$$v^n \rightharpoonup v \quad \text{weakly in } W^{2,2}(\Omega). \quad (5.6.4)$$

On the other hand, Proposition 1.1.1 implies

$$\begin{aligned} \|u^n - u^0\|_{W^{1,2}(\Omega;\mathbb{R}^2)} &\le C\|\mathrm{sym}\,(\nabla' u^n - \nabla' u^0)\|_{L^2(\Omega;\mathbb{M}^{2\times 2})} \\ &\le C\|\mathrm{sym}\,\nabla' u^n + \tfrac12\nabla' v^n\otimes\nabla' v^n\|_{L^2(\Omega;\mathbb{M}^{2\times 2})} + C\|u^0\|_{W^{1,2}(\Omega;\mathbb{R}^2)} \\ &+ C\|v^n\|_{W^{2,2}(\Omega)} \le C. \end{aligned}$$

Hence, $(u^n)$ is uniformly bounded in $W^{1,2}(\Omega;\mathbb{R}^2)$ and there exists $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ such that, up to subsequences

$$u^n \rightharpoonup u \quad \text{weakly in } W^{1,2}(\Omega;\mathbb{R}^2). \tag{5.6.5}$$

It is easy to see that $(u,v,p) \in \mathcal{A}(u^0,v^0)$. Moreover, by combining (5.6.3)–(5.6.5) and by the lower semicontinuity of $Q_2$, $B$ and $H_D$ with respect to weak $L^2$ convergence, it follows that the triple $(u,v,p)$ is a solution to (5.6.1). □

**Lemma 5.6.2.** *Let $p^0 \in L^2(\Omega;\mathbb{M}^{3\times 3}_D)$, $u^0 \in C^1(\overline{\omega};\mathbb{R}^2)$ and $v^0 \in C^2(\overline{\omega})$. Let $(u,v,p) \in \mathcal{A}(u^0,v^0)$ be a solution to the minimum problem* (5.6.1)*. Then*

$$\int_\Omega Q_2\big(\mathrm{sym}\,\nabla' u + \tfrac{1}{2}\nabla' v \otimes \nabla' v - x_3(\nabla')^2 v - p'\big)\,dx + \int_\Omega B(p)\,dx$$
$$\leq \int_\Omega Q_2\big(\mathrm{sym}\,\nabla' \tilde{u} + \tfrac{1}{2}\nabla' \tilde{v} \otimes \nabla' \tilde{v} - x_3(\nabla')^2 \tilde{v} - \tilde{p}'\big)\,dx + \int_\Omega B(\tilde{p})\,dx + \int_\Omega H_D(\tilde{p} - p)\,dx$$

*for every $(\tilde{u},\tilde{v},\tilde{p}) \in \mathcal{A}(u^0,v^0)$.*

*Proof.* The thesis follows by (5.6.1), once we notice that by (4.2.18) there holds

$$H_D(\tilde{p} - p^0) \leq H_D(\tilde{p} - p) + H_D(p - p^0)$$

for every $\tilde{p} \in L^2(\Omega;\mathbb{M}^{3\times 3}_D)$. □

We are now in a position to prove the main result of the section.

**Theorem 5.6.3.** *Let $\alpha = 3$. Assume that $t \mapsto u^0(t)$ belongs to $C^1([0,T];W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2) \cap C^1(\mathbb{R}^2;\mathbb{R}^2))$ and $t \mapsto v^0(t)$ belongs to $C^1([0,T];W^{2,\infty}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2))$. Let $(\bar{u},\bar{v},\bar{p}) \in \mathcal{A}(u^0(0),v^0(0))$ be such that*

$$\int_\Omega Q_2\big(\mathrm{sym}\,\nabla' \bar{u} + \tfrac{1}{2}\nabla' \bar{v} \otimes \nabla' \bar{v} - x_3(\nabla')^2 \bar{v} - \bar{p}'\big)\,dx + \int_\Omega B(\bar{p})\,dx$$
$$\leq \int_\Omega Q_2\big(\mathrm{sym}\,\nabla' \tilde{u} + \tfrac{1}{2}\nabla' \tilde{v} \otimes \nabla' \tilde{v} - x_3(\nabla')^2 \tilde{v} - \tilde{p}'\big)\,dx + \int_\Omega B(\tilde{p})\,dx + \int_\Omega H_D(\tilde{p} - \bar{p})\,dx$$

*for every $(\tilde{u},\tilde{v},\tilde{p}) \in \mathcal{A}(u^0(0),v^0(0))$. Then, there exists a reduced quasistatic evolution $t \mapsto (u(t),v(t),p(t))$ for the boundary data $(u^0(t),v^0(t))$ (according to Definition 5.3.6) such that*

$$u(0) = \bar{u}, \quad v(0) = \bar{v} \quad \text{and} \quad p(0) = \bar{p}.$$

*Proof.* Let us consider a sequence of subdivisions $(t^i_k)_{0\leq i\leq k}$ of the interval $[0,T]$, with

$$0 = t^0_k < t^1_k < \cdots < t^{k-1}_k < t^k_k = T$$

and such that

$$\lim_{k\to+\infty} \max_{1\leq i\leq k} (t^i_k - t^{i-1}_k) = 0.$$

Set $(u^0)^i_k := u^0(t^i_k)$ and $(v^0)^i_k := v^0(t^i_k)$, for $0 \leq i \leq k$ and for every $k$ and let $(u^i_k, v^i_k, p^i_k)$, $i = 1,\cdots k$, be defined inductively as solutions to the minimum problem

$$\min_{(u,v,p)\in\mathcal{A}((u^0)^i_k,(v^0)^i_k)} \Big\{ \int_\Omega Q_2\big(\mathrm{sym}\,\nabla' u + \tfrac{1}{2}\nabla' v \otimes \nabla' v - x_3(\nabla')^2 v - p'\big)\,dx + \int_\Omega B(p)\,dx$$
$$+ \int_\Omega H_D(p - p^{i-1}_k)\,dx \Big\}, \tag{5.6.6}$$

with $(u_k^0, v_k^0, p_k^0) = (\bar{u}, \bar{v}, \bar{p})$. For $i = 1, \cdots k$ we set

$$e_k^i := \operatorname{sym} \nabla' u_k^i + \tfrac{1}{2} \nabla' v_k^i \otimes \nabla' v_k^i - x_3 (\nabla')^2 v_k^i - (p_k^i)'$$

and for every $t \in [0, T]$ we consider the piecewise constant interpolants

$$u_k(t) = u_k^i, \quad v_k(t) = v_k^i, \quad p_k(t) = p_k^i, \quad u_k^0(t) = (u^0)_k^i$$

$$v_k^0(t) = (v^0)_k^i \quad \text{and} \quad e_k(t) = e_k^i,$$

where $i$ is the larger integer such that $t_k^i \leq t$. By definition, $(u_k(t), v_k(t), p_k(t)) \in \mathcal{A}(u_k^0(t), v_k^0(t))$ for every $t \in [0, T]$. Moreover, by Lemma 5.6.2 for every $t \in [0, T]$ there holds

$$\begin{aligned}
&\int_\Omega Q_2(e_k(t))\, dx + \int_\Omega B(p_k(t))\, dx \\
&\leq \int_\Omega Q_2\big(\operatorname{sym} \nabla' \tilde{u} + \tfrac{1}{2} \nabla' \tilde{v} \otimes \nabla' \tilde{v} - x_3 (\nabla')^2 \tilde{v} - \tilde{p}'\big)\, dx + \int_\Omega B(\tilde{p})\, dx + \int_\Omega H_D(\tilde{p} - p_k(t))\, dx
\end{aligned} \tag{5.6.7}$$

for every $(\tilde{u}, \tilde{v}, \tilde{p}) \in \mathcal{A}(u_k^0(t), v_k^0(t))$.

We split the construction of the reduced quasistatic evolution into three steps.

*Step 1: A priori estimates*

In this step we shall prove that there exists a constant $C$ such that

$$\|u_k(t)\|_{W^{1,2}(\Omega;\mathbb{R}^2)} + \|v_k(t)\|_{W^{2,2}(\Omega)} + \|p_k(t)\|_{L^2(\Omega;\mathbb{M}_D^{3\times 3})} \leq C \quad \text{for every } k \text{ and for all } t \in [0, T]. \tag{5.6.8}$$

Indeed, by the minimality condition (5.6.6), there holds

$$\begin{aligned}
&\int_\Omega Q_2(e_k(t))\, dx + \int_\Omega B(p_k(t))\, dx \\
&\leq \int_\Omega Q_2\big(\operatorname{sym} \nabla' u_k^0(t) + \tfrac{1}{2} \nabla' v_k^0(t) \otimes \nabla' v_k^0(t) - x_3 (\nabla')^2 v_k^0(t) - \bar{p}'\big)\, dx + \int_\Omega B(\bar{p})\, dx \\
&+ \int_\Omega H_D(\bar{p} - p_k(t))\, dx
\end{aligned}$$

for every $t \in [0, T]$. Since $B$ is strictly positive definite, by (4.2.16) we deduce

$$\|p_k(t)\|^2_{L^2(\Omega;\mathbb{M}_D^{3\times 3})} \leq C(1 + \|p_k(t)\|_{L^1(\Omega;\mathbb{M}_D^{3\times 3})}).$$

Hence, by Holder and Cauchy inequalities,

$$\|p_k(t)\|_{L^2(\Omega;\mathbb{M}_D^{3\times 3})} \leq C \quad \text{for every } k \text{ and } t \in [0, T].$$

Estimate (5.6.8) follows now by (4.2.7), and arguing as in the proof of Lemma 5.6.1.

*Step 2: Discrete energy inequality*

In this step we shall show that there exists a sequence $\delta_k \to 0^+$ such that

$$\begin{aligned}
&\int_\Omega Q_2(e_k(t))\, dx + \int_\Omega B(p_k(t))\, dx + \sum_{r=1}^{i} \int_\Omega H_D(p_k^r - p_k^{r-1})\, dx \leq \int_\Omega Q_2(e_k(0))\, dx + \int_\Omega B(p_k(0))\, dx \\
&+ \int_0^{t_k^i} \int_\Omega \mathbb{C}_2 e_k(s) : \begin{pmatrix} \nabla' \dot{u}^0(s) - x_3 (\nabla')^2 \dot{v}^0(s) + \nabla' v_k(s) \otimes \nabla' \dot{v}^0(s) & 0 \\ 0 & 0 \end{pmatrix} dx + C\delta_k.
\end{aligned} \tag{5.6.9}$$

To prove inequality (5.6.9), we fix $r$ such that $1 \leq r \leq i$, and we consider the maps $\tilde{u} = u_k^{r-1} - (u^0)_k^{r-1} + (u^0)_k^r$, and $\tilde{v} = v_k^{r-1} - (v^0)_k^{r-1} + (v^0)_k^r$. It is immediate to see that $(\tilde{u}, \tilde{v}, p_k^{r-1}) \in \mathcal{A}((u^0)_k^r, (v^0)_k^r)$. Hence, the minimality condition (5.6.7) yields

$$
\begin{aligned}
&\int_\Omega Q_2(e_k^r)\,dx + \int_\Omega B(p_k^r)\,dx + \int_\Omega H(p_k^r - p_k^{r-1})\,dx \\
&\leq \int_\Omega Q_2\big(\mathrm{sym}\,\nabla'\tilde{u} + \tfrac{1}{2}\nabla'\tilde{v}\otimes\nabla'\tilde{v} - x_3(\nabla')^2\tilde{v} - (p_k^{r-1})'\big)\,dx + \int_\Omega B(p_k^{r-1})\,dx \\
&\leq \int_\Omega Q_2(e_k^{r-1})\,dx + \int_\Omega B(p_k^{r-1})\,dx \\
&+2\int_\Omega Q_2\Big(\mathrm{sym}\,\nabla'((u^0)_k^r - (u^0)_k^{r-1}) - x_3(\nabla')^2((v^0)_k^r - (v^0)_k^{r-1})\Big)\,dx \\
&+2\int_\Omega Q_2\Big(\nabla'((v^0)_k^r - (v^0)_k^{r-1}) \otimes \Big(\nabla' v_k^{r-1} + \tfrac{1}{2}\nabla'((v^0)_k^r - (v^0)_k^{r-1})\Big)\Big)\,dx \\
&+\int_\Omega \mathbb{C}_2 e_k^{r-1} : \Big(\mathrm{sym}\,\nabla'((u^0)_k^r - (u^0)_k^{r-1}) - x_3(\nabla')^2((v^0)_k^r - (v^0)_k^{r-1})\Big)\,dx \\
&+\int_\Omega \mathbb{C}_2 e_k^{r-1} : \nabla'((v^0)_k^r - (v^0)_k^{r-1}) \otimes \Big(\nabla' v_k^{r-1} + \tfrac{1}{2}\nabla'((v^0)_k^r - (v^0)_k^{r-1})\Big)\,dx.
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\int_\Omega \mathbb{C}_2 e_k^{r-1} : \Big(\mathrm{sym}\,\nabla'((u^0)_k^r - (u^0)_k^{r-1}) - x_3(\nabla')^2((v^0)_k^r - (v^0)_k^{r-1})\Big)\,dx \\
&= \int_{t_k^{r-1}}^{t_k^r}\int_\Omega \mathbb{C}_2 e_k^{r-1} : (\mathrm{sym}\,\nabla'\dot{u}^0(s) - x_3(\nabla')^2\dot{v}^0(s))\,dx\,ds
\end{aligned}
$$

and by (5.6.8)

$$
\begin{aligned}
&\int_\Omega \mathbb{C}_2 e_k^{r-1} : \nabla'((v^0)_k^r - (v^0)_k^{r-1}) \otimes \big(\nabla' v_k^{r-1} + \tfrac{1}{2}\nabla'((v^0)_k^r - (v^0)_k^{r-1})\big)\,dx \\
&= \int_{t_k^{r-1}}^{t_k^r}\int_\Omega \mathbb{C}_2 e_k^{r-1} : \nabla'\dot{v}^0(s) \otimes (\nabla' v_k^{r-1} + \tfrac{1}{2}\nabla'((v^0)_k^r - (v^0)_k^{r-1}))\,dx\,ds \\
&\leq \int_{t_k^{r-1}}^{t_k^r}\int_\Omega \mathbb{C}_2 e_k^{r-1} : \nabla'\dot{v}^0(s) \otimes \nabla' v_k^{r-1}\,dx\,ds \\
&+C\|\nabla'\dot{v}^0\|_{L^\infty(\omega;\mathbb{R}^2)}\Big(\sup_{t\in[0,T]}\|e_k(t)\|_{L^2(\Omega;\mathbb{M}^{3\times3})}\Big)\int_{t_k^{r-1}}^{t_k^r}\|\nabla'((v^0)_k^r - (v^0)_k^{r-1}))\|_{L^2(\omega;\mathbb{R}^2)}\,ds \\
&\leq \int_{t_k^{r-1}}^{t_k^r}\int_\Omega \mathbb{C}_2 e_k^{r-1} : \nabla'\dot{v}^0(s) \otimes \nabla' v_k^{r-1}\,dx\,ds + C(t_k^r - t_k^{r-1})^2.
\end{aligned}
$$

On the other hand, by (4.2.7) and Holder inequality

$$
\begin{aligned}
&\int_\Omega Q_2\Big(\mathrm{sym}\,\nabla'((u^0)_k^r - (u^0)_k^{r-1}) - x_3(\nabla')^2((v^0)_k^r - (v^0)_k^{r-1})\Big)\,dx \\
&\leq C(t_k^r - t_k^{r-1})\int_{t_k^{r-1}}^{t_k^r}\|\mathrm{sym}\nabla'\dot{u}^0(s) - x_3(\nabla')^2\dot{v}^0(s)\|^2_{L^2(\Omega;\mathbb{M}^{2\times2})}\,ds \\
&\qquad\qquad \leq C(t_k^r - t_k^{r-1})^2
\end{aligned}
$$

and

$$\begin{aligned}
&\int_\Omega Q_2\Big(\nabla'((v^0)^r_k-(v^0)^{r-1}_k)\otimes\Big(\nabla'(v^{r-1}_k)+\tfrac{1}{2}\nabla'((v^0)^r_k-(v^0)^{r-1}_k)\Big)\Big)\,dx\\
&\le C(t^r_k-t^{r-1}_k)\int_{t^{r-1}_k}^{t^r_k}\int_\Omega|\nabla'\dot v^0(s)\otimes\tfrac{1}{2}\nabla'((v^0)^r_k-(v^0)^{r-1}_k)|^2\,dx\,ds\\
&\le C(t^r_k-t^{r-1}_k)^2,
\end{aligned}$$

where the last inequality follows by (5.6.8). By combining the previous estimates and by setting

$$\delta_k:=\max_{1\le i\le k}(t^r_k-t^{r-1}_k),$$

we deduce

$$\begin{aligned}
&\int_\Omega Q_2(e^r_k)\,dx+\int_\Omega B(p^r_k)\,dx+\int_\Omega H(p^r_k-p^{r-1}_k)\,dx\\
&\le\int_\Omega Q_2(e^{r-1}_k)\,dx+\int_\Omega B(p^{r-1}_k)\,dx\\
&+\int_{t^{r-1}_k}^{t^r_k}\int_\Omega \mathbb{C}_2e^{r-1}_k:\Big(\operatorname{sym}\nabla'\dot u^0(s)-x_3(\nabla')^2\dot v^0(s)+\nabla'\dot v^0(s)\otimes\nabla' v^{r-1}_k\Big)\,dx\,ds+C\delta_k(t^r_k-t^{r-1}_k).
\end{aligned}$$

By iterating the previous inequality we obtain (5.6.9).

*Step 3: Reduced global stability*

The discrete energy inequality proved in Step 2 and the a priori estimates deduced in Step 1 imply, in particular, by (4.2.16) that

$$\sum_{0\le t^r_k\le t}\|p^r_k-p^{r-1}_k\|_{L^1(\Omega;\mathbb{M}^{3\times3}_D)}\le C\quad\text{for every } t\in[0,T],$$

which in turn, since $p_k(t)$ is piecewise constant, is equivalent to

$$\mathcal{V}(p_k;0,t)\le C$$

for every $k$ and $t\in[0,T]$ (where $\mathcal{V}$ is the map defined in (1.5.1)). Therefore, by Theorem 1.5.1 there exists a map $t\mapsto p(t)$ which has bounded variation from $[0,T]$ into $L^2(\Omega;\mathbb{M}^{3\times3}_D)$, such that

$$p_k(t)\rightharpoonup p(t)\quad\text{weakly in } L^2(\Omega;\mathbb{M}^{3\times3}_D)$$

for every $t\in[0,T]$. By (5.6.8) for every $t\in[0,T]$ there exists a $t$-dependent subsequence $k_j\to0$ such that

$$u_{k_j}(t)\rightharpoonup u(t)\quad\text{weakly in } W^{1,2}(\Omega;\mathbb{R}^2)\quad\text{and}\quad v_{k_j}(t)\rightharpoonup v(t)\quad\text{weakly in } W^{2,2}(\Omega).\tag{5.6.10}$$

By the continuity of the trace operator, $(u(t),v(t),p(t))\in\mathcal{A}(u^0(t),v^0(t))$.

In this step we shall prove that $(u(t),v(t),p(t))$ fulfills $(\text{gs})_{r3}$ for every $t\in[0,T]$. Indeed, fix $t\in[0,T]$ and $(\tilde u,\tilde v,\tilde p)\in\mathcal{A}(u^0(t),v^0(t))$. We claim that

$$\int_\Omega Q_2(e_3(t))\,dx+\int_\Omega B(p(t))\,dx\le\int_\Omega Q_2(\widetilde e)\,dx+\int_\Omega B(\widetilde p)\,dx+\int_\Omega H_D(\widetilde p-p(t))\,dx,\tag{5.6.11}$$

where $e_3(t)$ is the map defined in (5.3.24) and

$$\tilde{e} := \operatorname{sym} \nabla' \tilde{u} + \tfrac{1}{2} \nabla' \tilde{v} \otimes \nabla' \tilde{v} - x_3 (\nabla')^2 \tilde{v} - \tilde{p}'.$$

Define the maps

$$\hat{u}_k(t) := u_k(t) + \tilde{u} - u(t), \quad \hat{v}_k(t) := v_k(t) + \tilde{v} - v(t), \quad \text{and} \quad \hat{p}_k(t) := p_k(t) + \tilde{p} - p(t).$$

With these definitions $(\hat{u}_k(t), \hat{v}_k(t), \hat{p}_k(t)) \in \mathcal{A}(u_k^0(t), v_k^0(t))$. Moreover,

$$\hat{u}_k(t) \rightharpoonup \tilde{u}, \quad \hat{v}_k(t) \rightharpoonup \tilde{v} \quad \text{and} \quad \hat{p}_k(t) \rightharpoonup p(t)$$

weakly in $W^{1,2}(\omega;\mathbb{R}^2)$, $W^{2,2}(\omega)$ and $L^2(\Omega;\mathbb{M}_D^{3\times 3})$, respectively. By (5.6.7), there holds

$$\int_\Omega Q(e_k(t))\,dx + \int_\Omega B(p_k(t))\,dx \le \int_\Omega Q(\hat{e}_k(t))\,dx + \int_\Omega B(\hat{p}_k(t))\,dx + \int_\Omega H_D(\hat{p}_k(t) - p_k(t))\,dx,$$

where

$$\hat{e}_k(t) := \operatorname{sym} \nabla' \hat{u}_k(t) + \tfrac{1}{2} \nabla' \hat{v}_k(t) \otimes \nabla' \hat{v}_k(t) - x_3 (\nabla')^2 \hat{v}_k(t),$$

which in turn implies

$$\int_\Omega Q(e_k(t))\,dx - \int_\Omega Q(\hat{e}_k(t))\,dx + \int_\Omega B(p_k(t))\,dx - \int_\Omega B(\hat{p}_k(t))\,dx \le \int_\Omega H_D(\tilde{p} - p(t))\,dx. \tag{5.6.12}$$

On the other hand,

$$\int_\Omega B(p_k(t))\,dx - \int_\Omega B(\hat{p}_k(t))\,dx = \tfrac{1}{2} \int_\Omega \mathbb{B}(p_k(t) + \hat{p}_k(t)) : (p(t) - \tilde{p})\,dx$$

and

$$\begin{aligned}
&\int_\Omega Q_2(e_k(t))\,dx - \int_\Omega Q_2(\hat{e}_k(t))\,dx \\
&= \tfrac{1}{2} \int_\Omega \mathbb{C}_2(e_k(t)) + \hat{e}_k(t))) : \Big(\operatorname{sym} \nabla'(u(t) - \tilde{u}) - x_3 (\nabla')^2 (v(t) - \tilde{v}) - (p'(t) - \tilde{p}')\Big)\,dx \\
&+ \tfrac{1}{2} \int_\Omega \mathbb{C}_2(e_k(t)) + \hat{e}_k(t))) : (\nabla')(v(t) - \tilde{v}) \otimes \Big(\nabla' v_k(t) + \tfrac{1}{2} \nabla'(\tilde{v} - v(t))\Big)\,dx.
\end{aligned}$$

Therefore, there holds

$$\lim_{k\to+\infty} \Big( \int_\Omega B(p_k(t))\,dx - \int_\Omega B(\hat{p}_k(t))\,dx \Big) = \int_\Omega B(p(t))\,dx - \int_\Omega B(\tilde{p})\,dx$$

and

$$\lim_{k\to+\infty} \Big( \int_\Omega Q_2(e_k(t))\,dx - \int_\Omega Q_2(\hat{e}_k(t))\,dx \Big) = \int_\Omega Q_2(e_3(t))\,dx - \int_\Omega Q_2(\tilde{e})\,dx,$$

By (5.6.12) we obtain (5.6.11) and hence $(\mathrm{gs})_{r3}$.

*Step 4: Reduced energy balance*

To complete the proof of the lemma it remains to prove that $(u(t), v(t), p(t))$ satisfies $(\mathrm{eb})_{r3}$.

Fix $t \in [0,T]$. Since $p_k(t)$ is piecewise constant in $[t_k^{r-1}, t_k^r[$ there holds

$$\mathcal{D}_{H_D}(p_k; 0, t) \leq \sum_{0 \leq t_k^r \leq t} H_D(p_k^r - p_k^{r-1}).$$

Hence, by lower semicontinuity, we deduce

$$\mathcal{D}_{H_D}(p; 0, t) \leq \liminf_{k \to +\infty} \mathcal{D}_{H_D}(p_k; 0, t) \leq \liminf_{k \to +\infty} \sum_{0 \leq t_k^r \leq t} H_D(p_k^r - p_k^{r-1}).$$

By (5.6.8), (5.6.10) and the dominated convergence theorem, passing to the limit in the discrete energy inequality, we obtain

$$\begin{aligned}\int_\Omega Q_2(e_3(t))\,dx + \int_\Omega B(p(t))\,dx &\leq \int_\Omega Q_2(e_3(0))\,dx + \int_\Omega B(p(0))\,dx \\ &+ \int_0^t \int_\Omega \mathbb{C}_2 e_3(s) : \Big(\mathrm{sym}\,\nabla' \dot{u}^0(s) + \nabla' v(s) \otimes \nabla' \dot{v}^0(s) - x_3 (\nabla')^2 \dot{v}^0(s)\Big)\,dx\,ds.\end{aligned} \tag{5.6.13}$$

The converse inequality in $(\mathrm{eb})_{r3}$ follows by Lemma 5.6.4 below. □

As in [47, Theorem 4.4] and [15, Theorem 4.7], the reduced global stability $(\mathrm{gs})_{r\alpha}$ and the energy inequality (5.6.13) imply the reduced energy balance $(\mathrm{eb})_{r\alpha}$.

**Lemma 5.6.4.** *Let $\alpha = 3$. Assume that $t \mapsto u^0(t)$ belongs to $C^1([0,T]; W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2) \cap C^1(\mathbb{R}^2;\mathbb{R}^2))$, $t \mapsto v^0(t)$ belongs to $C^1([0,T]; W^{2,\infty}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2))$, and $t \mapsto (u(t), v(t), p(t))$ satisfies $(gs)_{r3}$. Then, for every $t \in [0,T]$ there holds*

$$\begin{aligned}\int_\Omega Q_2(e_3(t))\,dx + \int_\Omega B(p(t))\,dx + \mathcal{D}_{H_D}(p;0,t) &\geq \int_\Omega Q_2(e_3(0))\,dx + \int_\Omega B(p(0))\,dx \\ &+ \int_0^t \int_\Omega \mathbb{C}_2 e_3(s) : \begin{pmatrix} \nabla' \dot{u}^0(s) + \nabla' v(s) \otimes \nabla' v^0(s) - x_3 (\nabla')^2 \dot{v}^0(s) & 0 \\ 0 & 0 \end{pmatrix} dx\,ds,\end{aligned}$$

*where $e_3(t)$ is the map defined in* (5.3.24).

*Proof.* Fix $t \in [0,T]$ and let $(s_k^i)_{0 \leq i \leq k}$ be a sequence of subdivisions of $[0,T]$ such that

$$0 = s_k^0 < s_k^1 \cdots < s_k^k = T$$

and

$$\lim_{k \to +\infty} \max_{1 \leq i \leq k} (s_k^i - s_k^{i-1}) = 0.$$

Set $u^i := u(s_k^i) - u^0(s_k^i) + u^0(s_k^{i-1})$ and $v^i := v(s_k^i) - v^0(s_k^i) + v^0(s_k^{i-1})$, and let $e_3(t)$ be the map defined in (5.3.24). As $(u^i, v^i, p(s_k^i)) \in \mathcal{A}(u^0(s_k^i), v^0(s_k^i))$, the reduced global stability condition $(\mathrm{gs})_{r3}$ yields

$$\begin{aligned}\int_\Omega Q_2(e_3(s_k^{i-1}))\,dx + \int_\Omega B(p(s_k^{i-1}))\,dx &\leq \int_\Omega Q_2(\mathrm{sym}\,\nabla' u^i + \tfrac{1}{2}\nabla' v^i \otimes \nabla' v^i - x_3 (\nabla')^2 v^i - p'(s_k^i))\,dx \\ &+ \int_\Omega B(p(s_k^i))\,dx + \int_\Omega H_D(p(s_k^i) - p(s_k^{i-1}))\,dx.\end{aligned}$$

By substituting the definition of the maps $u^i$ and $v^i$ in the previous expression we deduce

$$\begin{aligned}
&\int_\Omega Q_2(e_3(s_k^{i-1}))\,dx + \int_\Omega B(p(s_k^{i-1}))\,dx \\
&\leq \int_\Omega Q_2(e_3(s_k^i))\,dx + \int_\Omega B(p(s_k^i))\,dx + \int_\Omega H_D(p(s_k^i) - p(s_k^{i-1}))\,dx \\
&+2\int_\Omega Q_2\big(\operatorname{sym}\nabla'(u^0(s_k^{i-1}) - u^0(s_k^i)) - x_3(\nabla')^2(v^0(s_k^{i-1}) - v^0(s_k^i))\big)\,dx \\
&+2\int_\Omega Q_2\Big(\nabla'(v^0(s_k^{i-1}) - v^0(s_k^i)) \otimes \Big(\nabla' v(s_k^i) + \tfrac{1}{2}\nabla'(v^0(s_k^{i-1}) - v^0(s_k^i))\Big)\Big)\,dx \\
&+\int_\Omega \mathbb{C}_2 e_3(s_k^i) : (\operatorname{sym}\nabla'(u^0(s_k^{i-1}) - u^0(s_k^i)) - x_3(\nabla')^2(v^0(s_k^{i-1}) - v^0(s_k^i))\,dx \\
&+\int_\Omega \mathbb{C}_2 e_3(s_k^i) : \nabla'(v^0(s_k^{i-1}) - v^0(s_k^i)) \otimes \big(\nabla' v(s_k^i) + \tfrac{1}{2}\nabla'(v^0(s_k^{i-1}) - v^0(s_k^i))\big)\,dx.
\end{aligned} \tag{5.6.14}$$

Consider now the piecewise constant interpolants

$$\bar{u}_k(t) = u(s_k^i), \quad \bar{v}_k(t) = v(s_k^i), \quad \bar{p}_k(t) = p(s_k^i), \quad \text{and} \quad \bar{e}_k(t) = e_3(s_k^i),$$

where $i$ is the smaller integer such that $t \leq s_k^i$. Arguing as in Step 2 of the proof of Theorem 5.6.3 one can show that there exists a sequence $\delta_k \to 0^+$ such that

$$\begin{aligned}
&2\int_\Omega Q_2\big(\operatorname{sym}\nabla'(u^0(s_k^{i-1}) - u^0(s_k^i)) - x_3(\nabla')^2(v^0(s_k^{i-1}) - v^0(s_k^i))\big)\,dx \\
&+2\int_\Omega Q_2\Big(\nabla'(v^0(s_k^{i-1}) - v^0(s_k^i)) \otimes \Big(\nabla' v(s_k^i) + \tfrac{1}{2}\nabla'(v^0(s_k^{i-1}) - v^0(s_k^i))\Big)\Big)\,dx \\
&+\tfrac{1}{2}\int_\Omega \mathbb{C}_2 e_3(s_k^i) : \nabla'(v^0(s_k^{i-1}) - v^0(s_k^i)) \otimes \nabla'(v^0(s_k^{i-1}) - v^0(s_k^i)))\,dx \\
&\leq C\delta_k(s_k^i - s_k^{i-1}).
\end{aligned}$$

Hence, by iterating (5.6.14) we obtain

$$\begin{aligned}
&\int_\Omega Q_2(e_3(0))\,dx + \int_\Omega B(p(0))\,dx \\
&\leq \int_\Omega Q_2(e_3(t))\,dx + \int_\Omega B(p(t))\,dx + \mathcal{D}_{H_D}(p;0,t) + C\delta_k \\
&-\int_0^t\int_\Omega \mathbb{C}_2\bar{e}_k(s) : \begin{pmatrix} \nabla'\dot{u}^0(s) - x_3(\nabla')^2\dot{v}^0(s) + \nabla'\bar{v}_k(s)\otimes\nabla'\dot{v}^0(s) & 0 \\ 0 & 0 \end{pmatrix} dx\,ds,
\end{aligned}$$

which in turn implies

$$\begin{aligned}
&\int_\Omega Q_2(e_3(t))\,dx + \int_\Omega B(p(t))\,dx + \mathcal{D}_{H_D}(p;0,t) - \int_\Omega Q_2(e_3(0))\,dx - \int_\Omega B(p(0))\,dx \\
&\geq \limsup_{k\to+\infty}\int_0^t\int_\Omega \mathbb{C}_2\bar{e}_k(s) : \begin{pmatrix} \nabla'\dot{u}^0(s) - x_3(\nabla')^2\dot{v}^0(s) + \nabla'\bar{v}_k(s)\otimes\nabla'\dot{v}^0(s) & 0 \\ 0 & 0 \end{pmatrix} dx\,ds.
\end{aligned}$$

To conclude the proof of the lemma it remains to check that

$$\begin{aligned}
&\limsup_{k\to+\infty}\int_0^t\int_\Omega \mathbb{C}_2\bar{e}_k(s) : \begin{pmatrix} \nabla'\dot{u}^0(s) - x_3(\nabla')^2\dot{v}^0(s) + \nabla'\bar{v}_k(s)\otimes\nabla'\dot{v}^0(s) & 0 \\ 0 & 0 \end{pmatrix} dx\,ds \\
&\geq \int_0^t\int_\Omega \mathbb{C}_2 e_3(s) : \begin{pmatrix} \nabla'\dot{u}^0(s) - x_3(\nabla')^2\dot{v}^0(s) + \nabla' v(s)\otimes\nabla'\dot{v}^0(s) & 0 \\ 0 & 0 \end{pmatrix} dx\,ds.
\end{aligned} \tag{5.6.15}$$

To this purpose we argue as in [6, Lemma 5.7]. For every $s \in [0,T]$ we define

$$\Theta(s) := \int_\Omega \mathbb{C}_2 e_3(s) : \begin{pmatrix} \nabla'\dot{u}^0(s) - x_3(\nabla')^2\dot{v}^0(s) + \nabla' v(s) \otimes \nabla'\dot{v}^0(s) & 0 \\ 0 & 0 \end{pmatrix} dx.$$

By [16, Lemma 4.12], setting

$$a_k^i := (s_k^i - s_k^{i-1})\Big(\mathrm{sym}\,\nabla'\dot{u}^0(s_k^i) - x_3(\nabla')^2\dot{v}^0(s_k^i)\Big) - \int_{s_k^{i-1}}^{s_k^i} \Big(\mathrm{sym}\nabla'\dot{u}^0(s) - x_3(\nabla')^2\dot{v}^0(s)\Big)\,ds,$$

$$b_k^i := (s_k^i - s_k^{i-1})\nabla'\dot{v}^0(s_k^i) - \int_{s_k^{i-1}}^{s_k^i} \nabla'\dot{v}^0(s)\,ds,$$

$$c_k^i := (s_k^i - s_k^{i-1})\Theta(s_k^i) - \int_{s_k^{i-1}}^{s_k^i} \Theta(s)\,ds,$$

we may assume that our sequence of partitions $(s_k^i)_{0\le i\le k}$ satisfies

$$\lim_{k\to+\infty} \sum_{i=1}^k (\|a_k^i\|_{L^2(\omega;\mathbb{M}^{2\times2})} + \|b_k^i\|_{L^4(\omega;\mathbb{R}^2)} + |c_k^i|) = 0. \tag{5.6.16}$$

By $(\mathrm{gs})_{r3}$, arguing as in Step 1 of the proof of Theorem 5.6.3 we deduce that there exists a constant $C$ such that

$$\sup_{s\in[0,T]} \|e_3(s)\|_{L^2(\Omega;\mathbb{M}^{3\times3})} + \sup_{s\in[0,T]} \|\nabla' v(s)\|_{L^4(\omega;\mathbb{R}^2)} \le C.$$

Hence, by (5.6.16) there holds

$$\lim_{k\to+\infty} \sum_{i=1}^k \Big| \int_\Omega \mathbb{C}_2 e_3(s_k^i) : (a_k^i + \nabla' v(s_k^i) \otimes b_k^i)\,dx \Big|$$

$$\le \lim_{k\to+\infty} C\Big(\sup_{s\in[0,T]} \|e_3(s)\|_{L^2(\Omega;\mathbb{M}^{3\times3})}\Big) \sum_{i=1}^k \Big( \|a_k^i\|_{L^2(\omega;\mathbb{M}^{2\times2})} + \sup_{s\in[0,T]} \|\nabla' v(s)\|_{L^4(\omega;\mathbb{R}^2)} \|b_k^i\|_{L^4(\omega;\mathbb{R}^2)} \Big) = 0.$$

Therefore,

$$\limsup_{k\to+\infty} \int_0^t \int_\Omega \mathbb{C}_2 \bar{e}_k(s) : \begin{pmatrix} \nabla'\dot{u}^0(s) - x_3(\nabla')^2\dot{v}^0(s) + \nabla'\bar{v}_k(s) \otimes \nabla'\dot{v}^0(s) & 0 \\ 0 & 0 \end{pmatrix} dx\,ds$$

$$= \limsup_{k\to+\infty} \sum_{i=1}^k (s_k^i - s_k^{i-1})\Theta(s_k^i) = \limsup_{k\to+\infty} \sum_{i=1}^k \int_{s_k^{i-1}}^{s_k^i} \Theta(s)\,ds = \int_0^t \Theta(s)\,ds.$$

This concludes the proof of (5.6.15) and of the lemma. □

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