Corrado Bosca

On the top coefficients of Kazhdan-Lusztig polynomials

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Universitá di Roma "La Sapienza"

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Ph.D. Thesis

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Introduction

In their famous paper [15] Kazhdan and Lusztig defined for any Coxeter group W a family of polynomials indexed by pairs of elements of W, which have become known as Kazhdan-Lusztig polynomials. These polynomials are related to different areas of mathematics such as Schubert varieties, representation theory, Verma module theory, and combinatorics. In order to prove the existence of these polynomials, Kazhdan and Lusztig used another family of polynomials known as R-polynomials of W. Their importance stems mainly from the fact that knowing them is equivalent to knowing the Kazhdan-Lusztig polynomials. These polynomials are intimately related to the Bruhat order of W. This partial order gives to the Coxeter group the structure of a poset and on this structure, we can define a special matching. Our purpose in this work is to investigate a connection between special matchings and the top coefficients of Kazhdan-Lusztig polynomials:

$$\overline{\mu}(u,v) := \begin{cases} [q^{\frac{l(u,v)-1}{2}}]P_{u,v}(q) & \textit{if } l(u,v) \equiv 1 mod 2, \\ 0 & \textit{otherwise.} \end{cases}$$

There are many recent results about this function. Here, we list the most important ones:

- In [18] Lusztig computed this function for the affine Weyl group of type $\widetilde{B}_2$.

- MacLarnan and Warrington in [21] showed that in S_n $\overline{\mu}(u,v) = 0, 1$ is not true for $n \geq 10$.

- Xi showed in [25] that $\overline{\mu}(u,v) \leq 1$ if $u, v \in S_n$ and $a(u) < a(v)$ (where 'a' is the a-function on W defined by Lusztig in [18]).

- Scott and Xi in [22] showed that for elements in the affine Weyl group of type $\widetilde{A}_n$ $\overline{\mu}(u,v) = n + 2$ if $n \geq 4$.

- Jones showed in [14] that $\overline{\mu}(u,v) \leq 1$ when v is a Deodhar element of a finite Weyl group.

- Marietti in [20] find all the Boolean element in a linear Coxeter group such that $\overline{\mu}(u,v) \neq 0$.

In recent years, there have also been many mathematicians who have studied the connection between special matchings and Kazhadan-Lusztig polynomials, but the most important result is due to Brenti, Caselli and Marietti which, using the fact that any lower poset $[e,v]$ (for $v \in W$) has a special matching, show the following.

Theorem 0.0.1 (F. Brenti - F.Caselli - M. Marietti)**.** *Let $v \in W$ and M a special matching of $[e,v]$. Then:*

$$R_{u,v}(q) = q^c R_{M(u),M(v)}(q) + (q^c - 1)R_{u,M(v)}(q),$$

for all $u \leq v$ where $c = 1$ if $M(u) \rhd u$ and $c = 0$ otherwise.

Theorem 0.0.2 (F. Brenti - F.Caselli - M. Marietti)**.** *Let $u, v \in W$, $u < v$ and M a special matching of $[e,v]$. Then:*

$$P_{u,v}(q) = q^{1-c} P_{M(u),M(v)}(q) + q^c P_{u,M(v)}(q) - \sum_{z:M(z) \lhd z} \overline{\mu}(z, M(v)) q^{\frac{l(z,v)}{2}} P_{u,z}(q),$$

where $c = 1$ if $M(u) \rhd u$ and $c = 0$ otherwise.

The previous Theorems open two important questions:

- Can be extended the last results to any interval $[u,v]$ (with $u, v \in W$) that has a special matching?

- Is there a criterion to say if a poset has a special matching or not?

- Can we use special matchings to prove the combinatorial invariance Conjecture?

  Conjecture 0.0.3. *Let $u, v \in S_n$ and $x, y \in S_m$ then:*

  $$[u, v] \cong [x, y] \Rightarrow P_{u,v}(q) = P_{x,y}(q).$$

In my work, beginning with the fact that it's well known that $\overline{\mu}(u, v)$ is often the most important object in the study of Kazhdan-Lusztig polynomials, I studied the previous questions by studying the following Conjecture due to Brenti in [4]:

Conjecture 0.0.4 (F. Brenti)**.** *Let $u, v \in S_n$ with $[u, v]$ irreducible, $l(u, v) > 1$ and $l(u, v)$ odd then:*

$$\overline{\mu}(u, v) = 0 \Leftrightarrow [u, v] \textit{ have a special matching.}$$

The structure of the thesis is as follows. In the first part (Chapter 1) we collect some basic definitions and results about graphs and posets, Coxeter groups, Kazhdan-Lusztig polynomials and special matchings that will be used in the rest of the work. In it we treat the basic examples (using the symmetric group), which will allow the reader to gain some combinatorial intuition in the new object. In the second part (Chapter 2) we disprove one direction of Conjecture 0.0.2 and propose the following new Conjecture which generalizes one direction of Brenti's Conjecture 0.0.4.

Conjecture 0.0.5 (C. Bosca)**.** *Let $u, v \in W$ with $l(u, v) > 1$ then:*

$$[u, v] \textit{ has a special matching} \Rightarrow \overline{\mu}(u, v) = 0.$$

The third part (Chapter 3) is the heart of the thesis. In it we prove the previous Conjecture on the following cases:

- element u, v in S_n such that $u \leq v$ and $D_R(v) \subseteq \{1, n-1\}$.
- Grasmannian elements in S_n^J (quotient set) where $J = S \setminus \{(i, i+1)\}$.
- Boolean element in a linear Coxeter group.

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Finally, it's an honor for me to dedicate this thesis to the memory of Mauro Cibin whose passion for teaching and his friendship will remain indelible in my mind.

In this section, we collect the background used in the rest of this work.

Notation

Here is a list of notation used in the sequel

$\mathbb{Z}$	the set of integers
$\mathbb{N}$	the set of natural numbers
$\mathbb{Q}, \mathbb{R}$	the set of rational and real numbers
$[n]$	the set $\{1, 2, \ldots, n\}$
$[a, b]$	the set $\{c \in \mathbb{Z} : a \leq c \leq b\}$
$[\pm n]$	the set $[-n, n] \setminus \{0\}$
$\|S\|$	the cardinality of the set S
S^*	the set of words with letter from S
$R[q]$	ring of polynomial with coefficients in R
$[q^i]P$	the coefficient of q^i in the polynomial P
$\{s_1, \ldots, s_r\}_<$	the set $\{s_1, \ldots, s_r\}$ where $s_1 < \ldots < s_r$
$odd(a_1, \ldots, a_r)$	is the number of odd integers in the set $\{a_1, \ldots, a_r\}$ where $a_i \in \mathbb{Z}$ for all $i \in [r]$
e	the identity element of the group W

We denote by S_n the symmetric group, that is the set of all bijection $\pi : [n] \to [n]$. If $\sigma \in S_n$ we write $\sigma = \sigma_1 \ldots \sigma_n$ mean that $\sigma(i) = \sigma_i$. In some cases, we will be writing σ in disjoint cycle form. So, for example, the permutation $\sigma = 23451$ can be also written in cycle form as $\sigma = (1, 2, 3, 4, 5)$. Let $u, v \in S_n$, we denote with uv the composition of function $u \circ v$.
Let $v \in S_n$ and $u \in S_m$ $(m \leq n)$. We say that v is u-avoiding if there are no $1 \leq i_i < i_2 < \ldots < i_m \leq n$ if:

$$v(i_{u(1)}) < v(i_{u(2)}) < \ldots < v(i_{u(m)}).$$

The Symmetric group plays a central role throughout the paper and, in some sense, is the archetypal example of a Coxeter group.

1 Chapter I: Notation and preliminaries

1.1 Graphs and Posets

By a graph $G := (V, E)$ we denote:

- V a set of nodes or vertices.
- $E \subseteq \binom{V}{2}$ the set of edges.

A path in G is a sequence $(x_1, \ldots, x_r) \in V^r$ such that $\{x_i, x_{i+1}\} \in E$ (where $i \in [r-1]$). A graph is connected if for all $x, y \in V$ there exists a path that connects x with y. By a digraph (directed graph) $D := (V, A)$ we denote:

- V a set of nodes or vertices.
- $A \subseteq V^2$ the set of directed edges.

When we have a pair of elements $(x, y) \in A$ we write $x \to y$. A directed path in a directed graph is a sequence $(x_1, \ldots, x_r) \in V^r$ such that $x_1 \to \ldots \to x_r$, $r-1$ is called the length of the path $(x_1, \ldots, x_r)$.
A poset P (partially ordered set) is a set together with a partial order relation $\leq$ and is denoted usually as $(P, \leq)$ (where the order relation is suppressed from the notation when it is clear from the context). If $Q \subseteq P$ then Q has the structure of poset with the order relation induced by P.
Two elements $x, y \in P$ are said to be comparable if $x \leq y$ or $y \leq x$ and incomparable otherwise. Given $x, y \in P$ we define the interval:

$$[x, y] := \{z \in P : x \leq z \leq y\}.$$

An interval $[x, y]$ is called irreducible in P if there are no $a, b, c, d \in P$ such that:

$$[x, y] = [a, b] \times [c, d],$$

where with "$\times$" we denote the cartesian product.
We say that z covers t (or also that t is covered by z)in $[x, y]$, and we write

$z \triangleright t$ (as well as $t \triangleright z$), if $z, t \in [x, y]$ and $[t, z] = \{t, z\}$.
The standard way of depicting a finite poset P is to draw its Hasse diagram. This is the graph with:

- vertex set $V := P$.
- edges set $E := \{\{z, y\} \in P^2 : z \triangleleft t \text{ or } t \triangleleft zt\}$.

Given a poset $[x, y]$ we define the coatom and atom set as follows:

$$c(x, y) := \{z \in [x, y] : z \triangleleft y\},$$

$$a(x, y) := \{z \in [x, y] : x \triangleleft z\}.$$

For these two sets and their connection with Kazhdan-Lusztig polynomials and Bruhat order in the symmetric group see [7], [1] .
A sequence of elements of P $(x_0, \ldots, x_k)$ is called a chain if $x_0 < \ldots < x_k$ and k is its length. The rank of P is the maxinum length of a chain in P. If all maximal chains have the same length, then P is called pure. If P has a bottom element $\widehat{0}$ (an element such that $\widehat{0} \leq x$ for all $x \in P$) and every interval $[\widehat{0}, x]$ is pure, then P is graded. P has a top element $\widehat{1}$ if $x \leq \widehat{1}$ for each $x \in P$. If P has a bottom and top element, then P is bounded.
In the sequel, we use the following rank function:

$$r : P \longrightarrow [0, k],$$

where $r(x)$ is the rank of the subposet $\{y \in P : y \leq x\}$ and decompose P into rank levels:

$$P_i := \{x \in P : r(x) = i\}.$$

If the rank function is defined for a graded poset (or for a pure poset) and if $|P_i| < \infty$ then we can define the rank generating function:

$$\sum_{i \geq 0} |P_i| q^i.$$

A map $f : P \to Q$ of posets is order-preserving (order-reversing) if $x \leq y$ implies $f(x) \leq f(y)$ $(f(x) \geq f(y))$. Two posets P and Q are isomorphic (and we denote this by $P \cong Q$) if there exists an order-preserving function $f : P \to Q$ such that f is a bijection and that f^{-1} is also order-preserving. $\phi : P \to P$ is called an automorphism (anti-automorphism) if ϕ is a bijection and ϕ,ϕ^{-1} an order preserving (order reversing).

Example 1.1.1. *S_n with the order relation:*

$$u \leq v$$

$$\Updownarrow$$

$$u \leq v \Leftrightarrow u = (a_1, b_1) \cdots (a_r, b_r)v,$$

$$l(u) < l(u(a_1, b_1)) < \ldots < l(u(a_1, b_1) \cdots (a_r, b_r)) = l(v),$$

where $a_1, b_1, \ldots, a_r, b_r \in [n]$ has a structure of poset. A permutation u is covered by v if u is obtained from v by a transposition $(a, b) \in S_n$ such that $v(a) > v(b)$ and $a < b$. In figure 1.1.1 we show the Hasse diagram of the poset $[1234, 4321]$. In this example, we have that:

$$c(u, v) = \{4312, 3421, 4231\},$$

$$a(u, v) = \{1243, 2134, 1324\},$$

while the rank generating function is:

$$r(1234, 4321) = 1 + 3q + 5q^2 + 6q^3 + 5q^4 + 3q^5 + q^6.$$

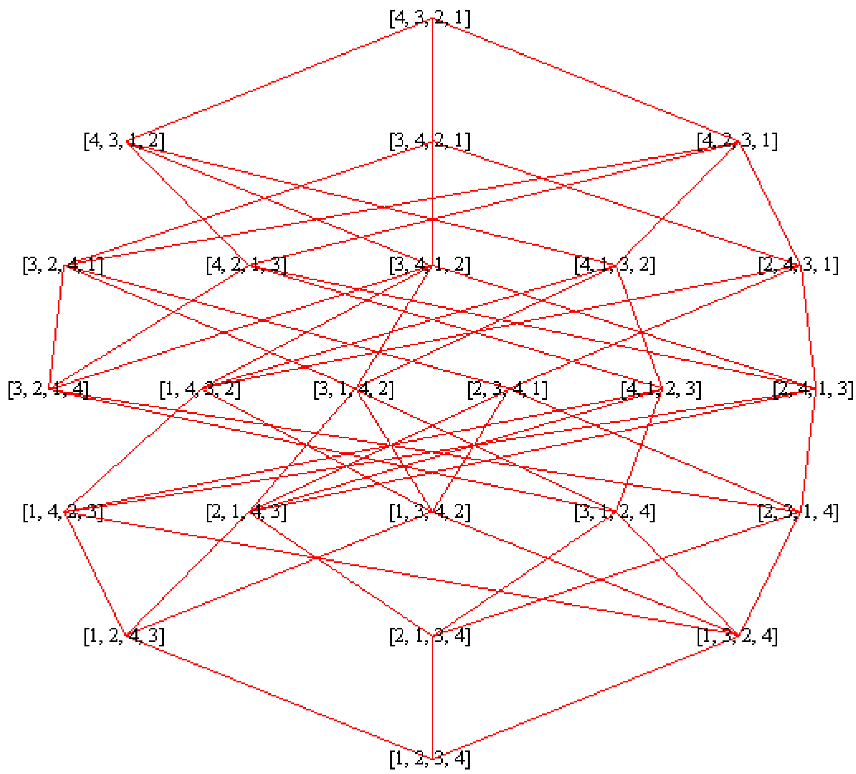

Figure 1.1.1: $[1234, 4321]$.

1.2 Coxeter systems and Bruhat order

Let S be a finite set and $m : S \times S \to \mathbb{Z}^+ \cup \{\infty\}$ be such that:

- m is symmetric: $m(s,t) = m(t,s) \quad \forall s,t \in S$;
- $m(s,t) \geq 2 \quad \forall s,t \in S, s \neq t$;
- $m(s,s) = 1 \quad \forall s \in S$.

The Coxeter group associated to m is defined as:

$$W := \frac{F_s}{N},$$

where F_s is the free group generated by S and N is the normal subgroup generated by:

$$\{(st)^{m(s,t)} : s,t \in S, m(s,t) < \infty\}.$$

Observation 1.2.1. *m can be represented by a Coxeter graph $G := (V,E)$ where:*

- *the vertex set is the set S,*
- *the edge set is the set of unordered pairs $\{s,t\}$ such that $m(s,t) \geq 3$,*
- *the edges with $m(s,t) \geq 4$ are labelled by $m(s,t)$.*

Example 1.2.2. *Given the matrix:*

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

then its diagram is the following:

Figure 1.2.1.

If W is a group defined as above, then the pair (W, S) is called a Coxeter system, W a Coxeter group and S the set of Coxeter generators. We say that a Coxeter system (W, S) is irreducible if the Coxeter graph is connected. The cardinality of S is called the rank of W.
In a Coxeter group W for each $w \in W$ there exist $s_1, \ldots, s_r \in S$ (not necessarily unique) such that:

$$w = s_1 \cdots s_r.$$

The length function of an element $w \in W$ is defined as the minimum number of generators necessary to express w:

$$l(w) := \min\{r \in \mathbb{N} : \exists s_1, \ldots, s_r \text{ such that } w = s_1 \cdots s_r\},$$

with the advice that $l(e) = 0$.
When an element $v \in W$ is expressed as the product of $l(v)$ generators, then that expression is called a reduced expression for v.

Example 1.2.3. *It can bee seen (see, e.g, [2]) that for S_n the length function of a permutation $v \in S_n$ is the number of inversions of v:*

$$l(v) = |\{(i, j) \in [n]^2 : i < j, v(i) > v(j)\}|.$$

Here is a simply property:

Proposition 1.2.4. *Let $u, v \in W$ then:*

- $l(u) = l(u^{-1})$,
- $l(uv) \leq l(u) + l(v)$,

- *$l(uv) \geq l(u) - l(v)$,*
- *if $s \in S$ then:*

$$l(v) - 1 \leq l(vs) \leq l(v) + 1.$$

Proof. **($l(u) = l(u^{-1})$):** if $u = s_1 \cdots s_r$ is a reduced expression then also $u^{-1} = s_r \cdots s_1$ is a reduced expression.
($l(uv) \leq l(u) + l(v)$): if $u = s_1 \cdots s_r$ and $v = t_1 \cdots t_p$ are reduced expressions then:

$$uv = s_1 \cdots s_r t_1 \cdots t_p \Rightarrow l(uv) \leq r + p.$$

($l(uv) \geq l(u) - l(v)$):

$$l(u) = l(uvv^{-1}) \leq l(uv) + l(v^{-1}) = l(uv) + l(v).$$

($l(v) - 1 \leq l(vs) \leq l(v) + 1$): if $s \in S$ then:

$$l(v) - 1 = l(v) - l(s) \leq l(vs) \leq l(v) + l(s) = l(v) + 1.$$

□

We can now define the following sets:

- the set of reflection $T := \{wsw^{-1} : w \in W, s \in S\}$,
- the right descent set : $D_R(w) := \{s \in S : l(ws) < l(w)\}$,
- the left descent set : $D_L(w) := \{s \in S : l(sw) < l(w)\}$.

Example 1.2.5. *In the case of S_n (see, e.g, [2]), the set of reflections is the set of all transposition in S_n:*

$$T = \{(a, b) : a, b \in [n]\},$$

and given a permutation $u \in S_n$:

$$D_R(u) = \{s_i \in S : u(i) > u(i + 1)\}.$$

Proposition 1.2.6. *Let $v \in W$, with $v = s_1 \cdots s_r$ reduced expression of v where $s_1, \ldots, s_r \in S$. Let $t \in T$ such that:*

$$l(vt) < l(w),$$

then $\exists! 1 \leq i \leq r$ such that:

$$vt = s_1 \cdots \widehat{s_i} \cdots s_r.$$

Proof. The existence of that i is simply and left to the reader. We prove that i is unique. Suppose by contradiction that exixts a $j > i$ such that:

$$wt = s_1 \cdots \widehat{s_i} \cdots s_r = s_1 \cdots \widehat{s_j} \cdots s_r,$$

then:

$$s_{i+1} \cdots s_j = s_i \cdots s_{j-1},$$

$$s_{i+1} \cdots s_{j-1} = s_i \cdots s_j.$$

By this relation we can say that:

$$s_i \cdots s_{i-1} s_{i+1} \cdots s_{j-1} s_{j+1} \cdots s_r = s_1 \cdots s_r = w.$$

Then we have find an expression of v such that $l(v) \leq r - 2$. This is in contradiction with $v = s_1 \cdots s_r$ reduced expression of v. $\square$

Corollary 1.2.7. *For all $s \in S$, the following holds:*

- *$s \in D_L(v)$ if and only if some reduced expression of v begins with the letter s.*
- *$s \in D_R(v)$ if and only if some reduced expression of v ends with the letter s.*

Now we define the Bruhat order which plays a fundamental role in all that follows.

Definition 1.2.8. *Let $u, v \in W$ we say that $u \leq v$ if $\exists t_1, \ldots, t_r \in T$ $(r \geq 0)$ such that:*

$$ut_1 \cdots t_r = v$$

and

$$l(u) < l(ut_1) < \ldots < l(ut_1 \cdots t_r) = l(v).$$

Here are some elementary properties:

Observation 1.2.9. *if $u \leq v$ then $l(u) \leq l(v)$.*

Observation 1.2.10. *$e \leq v$ for all $v \in W$.*

Observation 1.2.11. *if $t \in T$ then $u \leq ut$ if and only if $l(u) \leq l(ut)$.*

The proof of the following fundamental results can be found in [2] Section 2.2.

Proposition 1.2.12. *Let $u, v \in W$ with $u \leq v$ and $v = s_1 \cdots s_r$ $(s_1, \ldots, s_r \in S)$. Then $\exists i_1, \ldots, i_q \in [r]$ with $1 \leq i_i < i_2 < \ldots < i_q \leq r$ such that:*

$$u = s_{i_1} \cdots s_{i_q}.$$

Proof. Follow from Proposition 1.2.6. □

Proposition 1.2.13. *Let $w \in W$, $s \in S$ and $t \in T$ be such that $l(wt) < l(v)$. Then $wts \leq w$ or $wts \leq ws$.*

Proof. If $s = t$ there is nothing to prove. Suppose that $s \neq t$, we have to distinguish two cases:
(*Case I:* $l(wts) = l(wt) - 1$): In that case it is easy to see:

$$wts \leq wt \leq w.$$

(*Case II:* $l(wts) = l(wt) + 1$): We note that:

$$wts \leq ws \Leftrightarrow l(wts) \leq l(ws).$$

Suppose by contradiction that $l(wts) > l(ws)$. By Proposition 1.2.6:

$$wt = s_1 \cdots s_r \text{ reduced}$$

$$\Downarrow$$

$$wts = s_1 \cdots s_r s \text{ reduced,}$$

where $s_1, \ldots, s_r \in S$.
By Proposition 1.2.6 $\exists 1 \leq i \leq r+1$ such that:

$$ws = s_1 \cdots \widehat{s_i} \cdots s_{r+1},$$

where $s_{r+1} := s$. We say that $i \neq r+1$ otherwise $ws = wt \Rightarrow s = t$. Then:

$$ws = s_1 \cdots \widehat{s_i} \cdots s_r s,$$

$$w = s_1 \cdots \widehat{s_i} \cdots s_r,$$

this imply that $l(w) \leq r-1$ but this is in contradiction with the relation:

$$r = l(wt) < l(w) \leq r-1.$$

□

Corollary 1.2.14. *Let $u, v \in W$, $u \leq v$ and $s \in S$. Then $us \leq vs$ or $us \leq vs$*

Proof. If $u \leq v$ then $\exists t_1, \ldots, t_r \in T$ such that:

$$l(u) < l(ut_1) < \ldots < l(ut_1 \cdots t_r) = l(v).$$

By the previous Proposition:

$$us \leq ut_1 \text{ or } us \leq ut_1 s,$$

then by induction:

$$ut_1 s \leq v \text{ or } ut_1 s \leq vs.$$

□

Theorem 1.2.15 (Lifting Lemma)**.** *Let $u, v \in W$ with $u < v$ and $s \in D_R(v) \setminus D_R(u)$. Then $us \leq v$ and $u \leq vs$.*

Proof. Let $vs = s_1 \cdots s_q$ a reduced expression. Then:

$$v = s_1 \cdots s_q s,$$

is reduced ($s_{q+1} := s$). By hypotesis $u \leq v$ then by Propositon 1.2.12 $\exists 1 \leq i_i < i_2 < \ldots < i_q \leq q+1$ such that:

$$u = s_{i_1} \cdots s_{i_h}$$

and this expression is reduced. Observe that $i_h \neq q+1$ otherwise $u \not< us$. Then:

$$1 \leq i_i < i_2 < \ldots < i_q \leq q \Rightarrow u = s_{i_1} \cdots s_{i_h} \leq s_1 \cdots s_q = vs$$

and

$$us = s_{i_1} \cdots s_{i_h} s \leq s_1 s_{q+1} \leq s_1 \cdots s_q s_{q+1} = v.$$

□

Given S^* the monoid generated by S and $s, s' \in S$ such that $m(s, s') < \infty$ we define:

$$\alpha_{s,s'} := \underbrace{ss'ss' \ldots ss'}_{m(s,s')-\text{letters}}.$$

Given $\alpha, \beta \in S^*$ we say that α and β are linked by a braid move if $\exists s, s' \in S$ such that α is obtained from β by changing a factor $\alpha_{s,s'}$ to a factor $\alpha_{s',s}$. We say that α is obtained from β by a null move if α is obtained from β by cancelling a factor ss ($s \in S$).

Theorem 1.2.16 (Tits' word Theorem)**.** *Let $v \in W$. Then:*

- *each reduced expression of v is obtained from any other reduced expression of v by a sequence of braid moves.*

- *each expression of v is linked to any reduced expressions of v by braid moves and null moves.*

Example 1.2.17. *Let $S = \{a, b, c\}$ and the Coxeter matrix:*

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Then $\alpha = acbab$ is linked by a braid move to $\beta = cabab$ that is linked by a braid move to $\gamma = caaba$ that is linked by a null move to $\delta = cba$. δ is a reduced expression of α.

In the rest of this work, we work with finite Coxeter groups. In this case $\exists w_0 \in W$ (necessarily unique) such that $x \leq w_0$ for all $x \in W$ (see. e.g., [2]).

Proposition 1.2.18. *Let W a finite Coxeter group. Then:*

- $w_0 w_0 = e$.
- $l(w_0 w) = l(w_0) - l(w)$ *for all* $w \in W$.

Proof. **($\boldsymbol{w_0 w_0 = e}$):** w_0 is the grater element in W so $w_0^{-1} \leq w_0$. We know also that $l(w_0) = l(w_0^{-1})$ and then $w_0 = w_0^{-1}$.
($\boldsymbol{l(w_0 w) = l(w_0) - l(w)}$):

$$l(w_0) = l(w_0 w w^{-1}) \leq l(w_0 w) + l(w^{-1}) = l(w_0 w) + l(w)$$

and then $l(w_0 w) \geq l(w_0) - l(w)$. We prove now that $l(w_0 w) \leq l(w_0) - l(w)$ by induction on $l(w_0) - l(w)$.
If $l(w_0) - l(w) = 0$ then it is clear by the previous point.
If $l(w_0) - l(w) > 0$ observing that $\exists s \in S$ such that $l(ws) > l(w)$ then by induction:

$$l(w_0 w) - 1 \leq l(w_0 w s) \leq l(w_0) - l(ws) = l(w_0) - l(w) - 1.$$

□

For finite Coxeter groups, there is an important theorem useful on the sequel.

Theorem 1.2.19 (M. Dyer)**.** *Suppose that (W, S) is finite and $[u, v]$ be a Bruhat interval of W with $l(u, v) = r$. Then there exists a reflection subgroup (W', S') of rank $|S'| \leq r$ and a Bruhat interval $[u', v']$ in W' such that $[u, v] \cong [u', v']$.*

So the follow corollary holds:

Corollary 1.2.20. *Let $r \in \mathbb{N}^+$. Then up to isomorphism, there are only finitely many posets that occur as intervals of length r in Bruhat order of finite Coxeter groups.*

Example 1.2.21. *In the symmetric group, for example, we have that:*

- *for $r = 3$ there are 3 intervals of length 3.*
- *for $r = 5$ there are 25 intervals of length 5.*
- *for $r = 7$ there are 217 intervals of length 7.*

In [10] and [13] Hultman and Incitti list all the possibilities for the cases $r = 5, r = 7$. Their result will be used in the sequel.

Here, we briefly recall the classification of the finite irreducible Coxeter group. Namely, a Coxeter system (W, S) is finite and irreducible if and only if its Coxeter graph is one of the following ones.

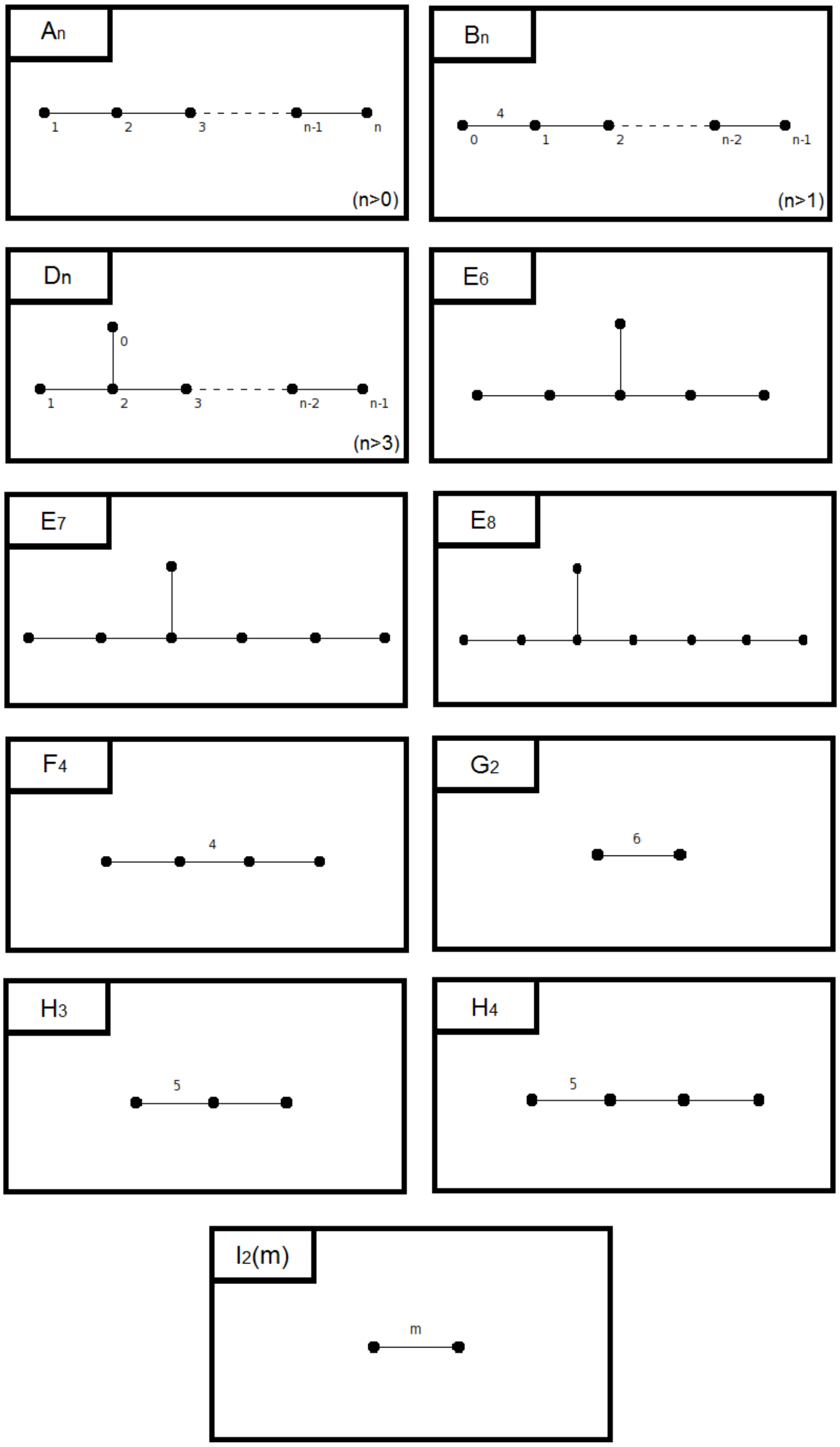
An
1
2
3
n-1
n
(n>0)
Bn
4
0
1
2
n-2
n-1
(n>1)
Dn
0
1
2
3
n-2
n-1
(n>3)
E6
E7
E8
F4
4
G2
6
H3
5
H4
5
I2(m)
m

Figure 1.2.2.

1.3 Parabolic subgroups and quotients

Given $J \subseteq S$ we define the parabolic subgroup W_J as the subgroup of W generated by the set J. We denote the quotient set by:

$$W^J := \{x \in W : D_R(x) \cap J = \emptyset\}.$$

The following holds (see [2] Section 2.4 for proofs):

Observation 1.3.1. *(W_J, J) is a Coxeter system.*

Observation 1.3.2. *$l_J(w) = l(w)$ if $w \in W_J$.*

Observation 1.3.3. *$W_I \cap W_J = W_{I\cap J}$, $< W_I \cup W_J >= W_{I\cup J}$ and $W_I = W_J \Rightarrow I = J$.*

Observation 1.3.4. *$x \in W^J$ if and only if no reduced expression of x ends with an element of J.*

Theorem 1.3.5. *Let $J \subseteq S$ and $v \in W$. Then there exist only one $v^J \in W^J$ and only one $v_J \in W_J$ such that $v = v^J v_J$ and $l(v) = l(v^J) + l(v_J)$.*

Proof. **(Existence):** Let $v \in W$ exist $s_1, \dots, s_k \in S$, with $k \in [l(v)]$ such that:

$$vs_1 \cdots s_j < vs_1 \cdots s_{j-1} \text{ and } vs_1 \cdots s_r s > vs_1 \cdots s_r,$$

for any $s \in S$ and $j \in [r]$.
Choosing $x = s_r \cdots s_1 \in W_J$ and $y = vs_1 \cdots s_r \in W^J$ we have that:

$$v = xy \text{ and } l(v) = l(x) + l(y).$$

(Uniqueness): Suppose by contradiction that:

$$v = xy = zt,$$

where $x, z \in W^J$ and $y, t \in W_J$. Let the following reduced expression of x:

$$x = s_1 \cdots s_k$$

and

$$yt^{-1} = s'_1 \cdots s'_h,$$

where $s_1, \ldots, s_k, s'_1, \ldots s'_h \in S$. Then:

$$z = xyt^{-1} = s_1, \cdots, s_k, s'_1, \cdots s'_h.$$

If we extract from z a reduced subword of z by Observation 1.3.4 we can say that it doesn't end with letters s'_j since $z \in W^J$. Then this reduced subword must be a subword of $s_1 \cdots s_k$ and so $z \le x$. By symmetry $x \le z$ and so $x = z, y = t$. □

Example 1.3.6. *Let $W = S_n$, $S = \{(1,2), \ldots, (n-1,n)\}$ and $J = S \setminus \{(i, i+1)\}$ ($i \in [n-1]$). Then:*

$$W^J = S_n^{S \setminus \{(i,i+1)\}} = \{x \in S_n : x(1) < x(2) < \ldots < x(i) \text{ and } x(i+1) < \ldots < x(n)\}.$$

About quotients, there are important facts to know:

Theorem 1.3.7 (V. V. Deodhar)**.** *Let $u, v \in W$ then:*

$$u \le v \Leftrightarrow u^J \le v^J \ \forall J \subseteq S : |J| = |S| - 1.$$

In general, it's very difficult to say when two elements are comparable or not.The previous Theorem is very useful for quotient sets. We now introduce a criterion for S_n. First of all we define for $x \in S_n$ the number:

$$x[i,j] := |\{t \in [i] : x(t) \ge j\}|,$$

then the following holds.

Observation 1.3.8. *For any $v \in S_n$:*

$$v[n,i] = n + 1 - i, \tag{1}$$

$$x[i,1] = i, \tag{2}$$

for all $i \in [n]$.

Observation 1.3.9. *Given $v \in S_n$:*

$$v[i,j] - v[k,j] - v[i,l] + v[k,l] = |\{a \in [k+1,i] : j \leq v(a) \leq l\}|, \qquad (3)$$

for all $1 \leq k \leq i \leq n$ and $1 \leq j \leq l \leq n$.

The following criteria (see [2] Section 2.1) play a fundamental role on comparing permutations and for this reason we include its proof.

Theorem 1.3.10. *Given $x, y \in S_n$ then the following are equivalent:*

- $x \leq y$,
- $x[i,j] \leq y[i,j]$ *for all* $i, j \in [n]$.

Proof. Suppose that $x \leq y$.
We can assume that:

$$y = x(a,b),$$

with $x(a) < x(b)$. Then it's easy to see that:

$$y[i,j] = \begin{cases} x[i,j]+1, & \text{if } a \leq i < b, x(a) < j \leq x(b), \\ x[i,j], & \text{otherwise} \end{cases}$$

and so it's true that $x[i,j] \leq y[i,j]$ for all $i,j \in [n]$.
Assume now that $x[i,j] \leq y[i,j]$ for all $i,j \in [n]$.
Let for brevity:

$$M(i,j) := y[i,j] - x[i,j].$$

If $M(i,j) = 0$ for all $i,j \in [n]$ then there is nothing to prove because $x = y$.
Let $(a,b) \in [n]^2$ be such that $M(a,b) > 0$ and:

$$M(i,j) = 0 \text{ for all } (i,j) \in [1,a] \times [b,n] \setminus \{(a,b)\}.$$

Then by the previous equation and the condition $M(a,b) > 0$ we have that:

$$y(a) = b \text{ and } x(a) < b.$$

Let $(c, d) \in [n]^2$ be such that (c, d) is the bottom right corner of a maximal positive connected submatrix of M having (a, b) the upper left corner. By equations 1,2 we have that $c < n$ and $d > 1$ (otherwise $M(c, d) = 0$) and by maximality there exists $f \in [a, c]$, $g \in [b, d]$ such that:

$$M(f, d-1) = 0 \text{ and } M(c+1, g) = 0$$

and then:

$$M(c+1, d-1) - M(f, d-1) - M(c+1, g) + M(f, g) > 0.$$

By equation (3) this implies:

$$|\{e \in [f+1, c+1] : y(e) \in [d-1, g-1]\}| > 0.$$

So let $(\overline{a}, \overline{b}) \in [f+1, c+1] \times [d-1, g-1]$ be such that $y(\overline{a}) = \overline{b}$ then $a < \overline{a}$ and $y(a) = b > \overline{b} = y(\overline{a})$. So we have that:

$$z = y(a, \overline{a}) \text{ and } x[i, j] \leq z[i, j] \text{ for all } i, j \in [n].$$

By induction, we can conclude that $x \leq z$ and so $x \leq y$.

$\square$

Example 1.3.11. $u = 43215 \leq v = 45321$ *because:*

$$u[i, j] = v[i, j] \text{ for all } i, j \in [4].$$

For example:

$$u[2, 4] = 1 < 2 = v[2, 4].$$

Another useful criterion is the following:

Theorem 1.3.12. *Let $x, y \in S_n^{S \setminus \{(i, i+1)\}}$. Then the following are equivalent:*

- $x \leq y$,
- $x(j) \leq y(j)$ *for* $1 \leq j \leq i$,

- $x(j) \geq y(j)$ *for* $i+1 \leq j \leq n$.

Let $v \in S_n$ we call the inversion table the sequence $(v_1, \ldots, v_n)$ where:

$$v_h := |\{k \in [n] : k > h, v^{-1}(k) < v^{-1}(h)\}|.$$

An integer partition is a sequence of non-negative integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 \geq \ldots \geq \lambda_k$ (when we write a partition we omit the zero parts). We identify a partition with its diagram:

$$\{(i,j) \in (\mathbb{N}^+)^2 : 1 \leq i \leq k, 1 \leq j \leq \lambda_i\}$$

and draw the diagram in Russian convention.

Example 1.3.13. *In the following figure we have the diagram of* $(3,2,1,1)$

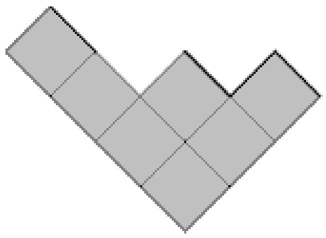

Figure 1.3.1.

We call the elements of λ *cells*. The level of a cell $(i,j) \in (\mathbb{N}^+)^2$ is:

$$lv(i,j) := i + j.$$

Let λ and μ be two partitions such that $\mu \subseteq \lambda$. We consider the skew partition $\lambda \setminus \mu$. A skew partition is called a *Dyck cbs* (Dyck connected border strip) if:

- is connected.
- does not contain a 2×2 square.

- no cell in the diagram has level strictly less than the rightmost and leftmost of its cells.

Observation 1.3.14. *in a* Dyck cbs *the rightmost and leftmost cells have the same level.*

Example 1.3.15. *in the following figure, we can see three different example of no* Dyck cbs *partitions.*

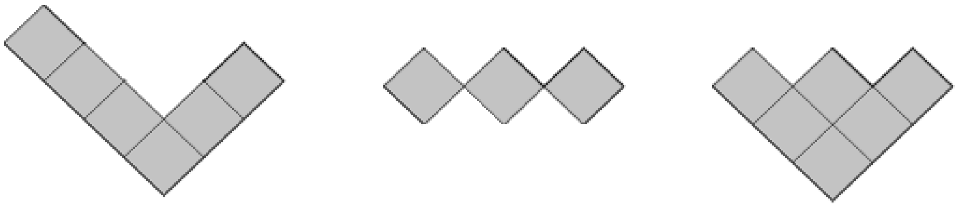

Figure 1.3.2.

Example 1.3.16. *in the following figure, we can see three different examples of skew partitions which are not* Dyck cbs.

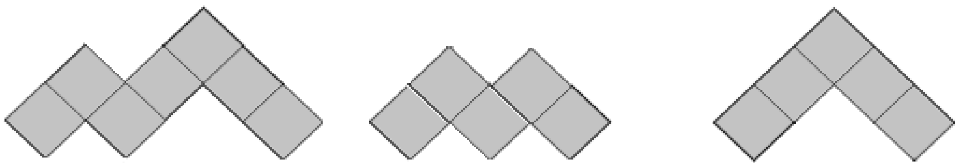

Figure 1.3.3.

In the sequel, we consider elements in the following quotient set:

$$S_n^{S\setminus(i,i+1)} = \{x \in S_n : x(1) < x(2) < \ldots < x(i) \text{ and } x(i+1) < \ldots < x(n)\},$$

these permutations are called Grasmannian permutations by Lascoux in [17] i.e. permutations $v \in S_n$ with a unique descent $v(i) > v(i+1)$.
Given $v \in S_n^{S \setminus \{(i,i+1)\}}$ we associate to v the partition:

$$\Lambda(v) = (v(i) - i, \ldots, v(1) - 1).$$

We recall that:

Proposition 1.3.17. *Given $u, v \in S_n^{S \setminus \{(i,i+1)\}}$, $u \leq v$ then:*

$$\Lambda(u) \subseteq \Lambda(v).$$

For a proof see, e.g., [19].

1.4 Permutation groups

The most important Coxeter group is certainly the symmetric group S_n, but as the reader can see in Section 3.2 , there are other Coxeter groups important in my research. We list in this Section the most important ones. We define S_n^B (the group of signed permutations) as the set of all bijections $\pi : [\pm n] \to [\pm n]$ such that $\pi(-x) = -\pi(x)$ for all $x \in [n]$. If $v \in S_n^B$ we write:

$$v = [v_1, \dots, v_n],$$

to mean that $v(i) = v_i$ (for $i \in [n]$) and call this notation the window notation. We also write v in-line notation:

$$v = v_{-n}, \dots, v_{-1}, v_1, \dots, v_n$$

to mean that $v(i) = v_i$ (for $i \in [\pm n]$). Finally, as for permutations we can write an element of S_n^B in disjoint cycle notation.

Example 1.4.1. *Let $v = [-2, 1, 4, 3]$. Then the line notation and disjoint cycle notation are respectively:*

$$v = -3, -4, -1, 2, -2, 1, 4, 3$$

$$v = (-2, -1, 2, 1)(3, 4)(-3, -4).$$

Let $u, v \in S_n^B$. The group operation uv is defined as a composition of functions.
The set of generators of S_n^B is $S_B = \{s_1^B, \dots, s_{n-1}^B, s_0^B\}$ where:

$$s_i^B = [1, \dots, i-1, i+1, i, i+2, \dots, n],$$

for $i \in [n-1]$ and

$$s_0^B = [-1, 2, \dots, n].$$

Observation 1.4.2. *If we multiply an element $v \in S_n^B$ by s_i^B on the right the result in window notation is an exchange of the values in position i and $i+1$. If we multiply an element $v \in S_n^B$ by s_0^B on the right the result in window notation is to change the sign of the value in position 1.*

(S_n^B, S_B) has the structure of a Coxeter system (see, e.g., [2]) and one can show that the length function is:

$$l_B(v) = |\{(i,j) \in [n]^2 : i < j, v(i) > v(j))\}| + |\{(i,j) \in [n]^2 : i \leq j, v(-i) > v(j)\}|,$$

while the descent and reflection sets are:

$$D_R(v) := \{s_i \in S : v(i) > v(i+1)\},$$

$$T_B := \{(i,j)(-i,-j) : 1 \leq i < |j| \leq n\} \cup \{(i,-i) : i \in [n]\},$$

where $v(0) = 0$. We refer the reader to [2], Section 8.1 for a proof of these results.

There are some interesting subgroups of S_n^B that are Coxeter groups too. In particular:

$$S_n^D = \{v \in S_n^B : neg(v(1), \ldots, v(n)) \equiv 0 mod 2)\}.$$

The set of Coxeter generators of this subgroup is:

$$S_D = S_B \cup \{[-2, -1, 3, \ldots, n]\}.$$

For this Coxeter group the length function, descent and reflection sets are given by:

$$l_D(v) := l_B(v) - odd(v(1), \ldots, v(n)),$$

$$D_R(v) = \{s_i \in S : v(i) > v(i+1)\},$$

where $v(0) := -v(2)$ and

$$T_D = \{(i,j)(-i,-j) : 1 \leq i < |j| \leq n\}.$$

We refer the reader to, e.g., [2] Section 8.2, for a proof of these results.

Another extension of S_n is $\widetilde{S_n}$ $(n \geq 2)$ that is the group of all bijection $\pi : \mathbb{Z} \to \mathbb{Z}$ such that:

$$\pi(x+n) = \pi(x) + n,$$

for all $x \in \mathbb{Z}$ and

$$\sum_{x=1}^{n} \pi(x) = \binom{n+1}{2}.$$

Observation 1.4.3. *Let a $v \in \widetilde{S_n}$. By the previous definition v is uniquely determined by its values on $[n]$.*

By the previous observation, we can write each element $v \in \widetilde{S_n}$ in the window form $v = [v_1, \ldots, v_n]$ to mean that $v(i) = v_i$ for $i \in [n]$. The set of Coxeter generators of this Coxetr group is $\widetilde{S_A} = \{\widetilde{s_1}^A, \ldots, \widetilde{s_n}^A\}$ where:

$$\widetilde{s_i}^A := [1, 2, \ldots, i-1, i+1, i+2, \ldots, n],$$

for $i \in [n-1]$ and

$$\widetilde{s_n}^A := [0, 2, 3, \ldots, n-1, n+1].$$

As before we now list in order the length function and descent sets of $\widetilde{S_n}^A$:

$$l_{\widetilde{A}}(v) = |\{(i,j) \in [n] \times \mathbb{N} : i < j, v(i) > v(j))\}|,$$

$$D_R(v) = \{s_i \in S : v(i) > v(i+1)\}.$$

We refer the reader, e.g., [2] Section 8.3, for proofs of these results.

1.5 Kazhdan-Lusztig polynomials

In order to have a unified construction of representations of finite Coxeter groups, Kazhdan and Lusztig in [15], proposed an approach of representations based on a set of polynomials now called Kazhdan-Lusztig polynomials. These polynomials are indexed by pairs of elements in the Coxeter groups W and are related to the Brhuat order of W and the descent set of its elements. There are several ways to define Kazhdan-Lusztig polynomials,for example by Hecke algebra (see [11]), here we give a list of Definition-Theorem. In this section, we define also other useful families of polynomials such as R-polynomials and $\widetilde{R}$-polynomials and, in some sense, we will see that these other two families of polynomials are equivalent to the Kazhdan-Lusztig polynomials.

Theorem 1.5.1 (Definition-Theorem)**.** *There is a unique family of polynomials* $\{R_{u,v}(q)\}_{u,v\in W} \subseteq \mathbb{Z}[q]$ *such that:*

- $R_{u,v}(q) = 0$ *if* $u \not\leq v$,
- $R_{u,v}(q) = 1$ *if* $u = v$,
- *if* $s \in D_R(v)$ *then:*

$$R_{u,v}(q) = \begin{cases} R_{us,vs}(q), & \text{if } s \in D_R(u), \\ qR_{us,vs}(q) + (q-1)R_{u,vs}(q), & \text{if } s \notin D_R(u). \end{cases}$$

It is easy to see that the last property of the previous Definition-Theorem can be used in a math software to compute R-polynomials by induction on $l(v)$.

Example 1.5.2. *Let* $(W, S) = (S_3, \{(1, 2), (2, 3)\})$ *and* $u = 123$, $v = 321$ *then by Theorem 1.5.1 we can easily compute:*

$$R_{123,321}(q) = qR_{213,231}(q) + (q-1)R_{123,231}(q).$$

Compute now $R_{213,231}(q)$*:*

$$R_{213,231}(q) = qR_{231,213}(q) + (q-1)R_{213,213}(q) = q-1$$

and $R_{123,231}(q)$*:*

$R_{123,231}(q) = qR_{132,213}+(q-1)R_{123,213} = 0+(q-1)(qR_{213,123}+(q-1)R_{123,123}) = (q-1)^2.$

Then finally:

$$R_{123,321}(q) = q(q-1) + (q-1)^3 = q^3 - 2q^2 + 2q - 1.$$

Here we list some classical results (see [2] Section 5 for the proofs):

Proposition 1.5.3. *Let* $u, v \in W$ *with* $u \leq v$*. Then* $R_{u,v}(0) = (-1)^{l(u,v)}$ *and* $R_{u,v}$ *is a monic polynomial of degree* $l(u,v)$*.*

Proposition 1.5.4. *Let* $u, v \in W$ *with* $u \leq v$*. Then:*

$$q^{l(u,v)} R_{u,v}(\frac{1}{q}) = (-1)^{l(u,v)} R_{u,v}(q).$$

Proposition 1.5.5. *Let* W *be a finite Coxeter group. Then:*

- $R_{u,v}(q) = R_{u^{-1},v^{-1}}(q)$,
- $R_{u,v}(q) = R_{w_0 v, w_0 u}(q)$,

for all $u, v \in W$

Using the R-polynomials, we can define Kazhdan-Lusztig polynomials.

Theorem 1.5.6 (Definition-Theorem)**.** *There is a unique family of polynomials* $\{P_{u,v}(q)\}_{u,v \in W} \subseteq \mathbb{Z}[q]$ *(usually called Kazhdan-Lusztig polynomials) such that:*

- $P_{u,v}(q) = 0$ *if* $u \leq v$,
- $P_{u,v}(q) = 1$ *if* $u = v$,
- $deg(P_{u,v}(q)) \leq \frac{l(u,v)-1}{2}$ *if* $u < v$,

- *if $u \leq v$ then:*

$$q^{l(v)-l(u)}P_{u,v}(\frac{1}{q}) = \sum_{a\in[u,v]} R_{u,a}(q)P_{a,v}(q).$$

As for the R-polynomials the last condition in the previous Definition-Theorem can be used in a math software to compute Kazhdan-Lusztig polynomials.

Example 1.5.7. *Let $(W,S) = (S_3, \{(1,2),(2,3)\})$ and $u = 123$, $v = 321$ then by the previous Theorem we can easily compute:*

$$q^{l(321)-l(123)}P_{123,321}(\frac{1}{q}) - P_{123,321}(q) = \sum_{123<a\leq 321} R_{123,a}(q)P_{a,321}(q) =$$

$$= R_{123,213}(q)P_{213,321}(q) + R_{123,132}(q)P_{132,321}(q) + R_{123,231}(q)P_{231,321}(q)+$$

$$+R_{123,312}(q)P_{312,321}(q) + R_{123,321}(q)P_{321,321}(q)$$

and then we can deduce from this that:

$$q^{l(321)-l(123)}P_{123,321}(\frac{1}{q}) - P_{123,321}(q) = q^3 - 1$$

and using the third point of the previous Theorem that $P_{123,321}(q) = 1$.

As before, here are some results useful to recall.

Proposition 1.5.8. *Let $u, v \in W$, $u \leq v$ then $P_{u,v}(0) = 1$.*

Proposition 1.5.9. *Let W be a Coxeter group. Then:*

- $P_{u,v}(q) = P_{u^{-1},v^{-1}}(q)$ *for all $u, v \in W$,*

- *if W is finite then $P_{u,v}(q) = P_{w_0uw_0,w_0vw_0}(q)$ for all $u, v \in W$.*

Proof. $\boldsymbol{(P_{u,v}(q) = P_{u^{-1},v^{-1}}(q))}$**:** We prove this by induction on $l(u,v)$.
Suppose that $l(u,v) = 0$ then $u = v$ and the relation is clear.
If $l(u,v) > 0$ then:

$$q^{l(u,v)} P_{u,v}(\frac{1}{q}) - P_{u,v}(q) = \sum_{u<a<v} R_{u,a}(q) P_{a,v}(q) + R_{u,v}(q).$$

We know by Proposition 1.5.5 and Proposition 1.2.4 that:

$$l(u,v) = l(u^{-1}, v^{-1}) \text{ and } R_{u,v}(q) = R_{u^{-1},v^{-1}}(q)$$

and so with the first:

$$deg(P_{u^{-1},v^{-1}}(q)) < \frac{l(u^{-1}, v^{-1})}{2} = \frac{l(u,v)}{2}, \tag{4}$$

with the second:

$$q^{l(u,v)} P_{u^{-1},v^{-1}}(\frac{1}{q}) - P_{u^{-1},v^{-1}}(q) = \sum_{u^{-1}<a\leq v^{-1}} R_{u^{-1},a}(q) P_{a,v^{-1}}(q) =$$

$$= \sum_{u^{-1}<a\leq v^{-1}} R_{u,a^{-1}}(q) P_{a^{-1},v}(q) = \sum_{u<a\leq v} R_{u,a}(q) P_{a,v}(q) = q^{l(u,v)} P_{u,v}(\frac{1}{q}) - P_{u,v}(q)$$

and by this relation, (4) and induction, we can conclude that:

$$P_{u,v}(q) = P_{u^{-1},v^{-1}}(q).$$

$\boldsymbol{(P_{u,v}(q) = P_{w_0uw_0,w_0vw_0}(q))}$**:** We have that:

$$q^{l(u,v)} P_{u,v}(\frac{1}{q}) - P_{u,v}(q) = \sum_{u<a\leq v} R_{u,a}(q) P_{a,v}(q) =$$

by Proposition 1.5.5:

$$= \sum_{w_0v\leq a<w_0u} R_{u,w_0a}(q) P_{w_0a,v}(q) = \sum_{w_0vw_0\leq aw_0<w_0uw_0} R_{w_0uw_0,aw_0}(q) P_{aw_0,w_0vw_0}(q) =$$

$$= q^{l(u,v)} P_{w_0uw_0,w_0vw_0}(q)(\frac{1}{q}) - P_{w_0uw_0,w_0vw_0}(q)$$

and, as before, we conclude:

$$P_{u,v}(q) = P_{w_0uw_0,w_0vw_0}(q).$$

□

One of the main problems about Kazhdan-Lusztig polynomial is to find a combinatorial interpretation for them. Even for the symmetric group only some partial results are known. Here we list some of these results:

Theorem 1.5.10 (B. Shapiro - M. Shapiro - A. Vainshtein)**.** *Let $v \in S_n$ be such that $D_R(v) \subseteq \{1, n-1\}$. Then:*

$$P_{u,v}(q) = (1+q)^r,$$

where $r := |\{j \in [v(n)+1, v(1)-2] : \sum_{i=1}^{j} u(i) = \binom{j+1}{2}\}|$.

Example 1.5.11. *Given $u = 321465$, $v = 623451$ we have that:*

$$r = |\{3,4\}| = 2$$

and so by Theorem 1.5.10 $P_{u,v}(q) = 1 + 2q + q^2$.

Theorem 1.5.12. *Let $u, v \in S_n$, $u \leq v$, be such that $[2, n-2] \subseteq D_R(v)$ then:*

$$P_{u,v}(q) = \begin{cases} 1 + q^{v(1)-v(n)}, & \text{if } u(n) > v(1) \geq v(n) > u(1), \\ 1, & \text{otherwise.} \end{cases}$$

Proposition 1.5.13. *Let $u, v \in W$, $u \leq v$. If $s \in D_R(v)$ then:*

$$P_{u,v}(q) = P_{us,v}(q).$$

Theorem 1.5.14. *Let $v \in S_n$. Then the following are equivalent:*

- $P_{e,v}(q) = 1$.
- $P_{u,v}(q) = 1$ *for all $u \leq v$.*

- *v is 3412-avoiding and 4231-avoiding.*

the proof of the first two Theorems appears in [23], while the last appears in [16].
M. Dyer in 1987 [9] and G.Lusztig in 1980 (see [2] Section 5.6) independently posed the following:

Conjecture 1.5.15 (M. Dyer - G. Lusztig)**.** *Let $u, v \in S_n$ and $x, y \in S_m$ then:*

$$[u, v] \cong [x, y] \Rightarrow P_{u,v}(q) = P_{x,y}(q).$$

This Conjecture states that the value of the Kazhdan-Lusztig polynomials depend only on the poset structure. In recent years the previous Conjecture has been proved for particular cases, when $[u, v]$ is a lattice (see [3]), when $u = x = e$ (see [8]) and for element $u, v \in W$ such that $l(u, v) \leq 5$ (see [4] and [5]). We define now another class of polynomials:

Theorem 1.5.16 (Definition-Theorem)**.** *Let $u, v \in W$, then there exists a unique polynomial $\widetilde{R}_{u,v}(q) \in \mathbb{N}[q]$ such that:*

$$R_{u,v}(q) = q^{\frac{l(u,v)}{2}} \widetilde{R}_{u,v}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}).$$

The advantage of the $\widetilde{R}$-polynomials over the R-polynomials is that the coefficients are all natural numbers.

Observation 1.5.17. *the previous Definitions-Theorems say that computing the Kazhdan-Lusztig polynomials is equalent to computing the R or $\widetilde{R}$ polynomials.*

We now list some classical results for $\widetilde{R}$-polynomials.

Proposition 1.5.18. *Given $u, v \in W$, $u \leq v$ and $s \in D_R(v)$, then:*

$$\widetilde{R}_{u,v}(q) = \begin{cases} \widetilde{R}_{us,vs}(q), & \text{if } s \in D_R(u), \\ \widetilde{R}_{us,vs}(q) + q\widetilde{R}_{u,vs}(q), & \text{if } s \notin D_R(u). \end{cases}$$

Proposition 1.5.19. *Given $u, v \in W$, $u \leq v$. Then $\widetilde{R}_{u,v}(q)$ is a monic polynomial of degree $l(u, v)$.*

We now define a function that will be used in Conjecture 2.1.5.

Definition 1.5.20. *Let W be a Coxeter group, $u, v \in W$ with $u \leq v$, we define:*

$$\overline{\mu}(u, v) := \begin{cases} [q^{\frac{l(u,v)-1}{2}}]P_{u,v}(q), & \text{if } l(u, v) \equiv 1 mod 2, \\ 0, & \text{otherwise,} \end{cases}$$

where with $[q^i]P_{u,v}(q)$ we denote the coefficient of q^i in $P_{u,v}(q)$.

We recall here, some classical results on $\overline{\mu}(u, v)$ (for proof see [2] and [11]).

Proposition 1.5.21. *Let W a finite Coxeter group. Then:*

$$\overline{\mu}(u, v) = \overline{\mu}(w_0 v, w_0 u),$$

for all $u, v \in W$.

Proposition 1.5.22. *Let $u, v \in W$, $u \leq v$, be such that $\overline{\mu}(u, v) \neq 0$ and $l(u, v) > 1$.Then $D_R(v) \subseteq D_R(u)$.*

Theorem 1.5.23. *Given $u, v \in W$, $u \leq v$ and $s \in D_R(v)$. Then:*

$$P_{u,v}(q) = q^{1-c}P_{us,vs}(q) + q^c P_{u,vs}(q) - \sum_{z:s \in D_R(z)} q^{\frac{l(z,v)}{2}} \overline{\mu}(z, vs)P_{u,z}(q),$$

where $c = 1$ if $s \in D_R(u)$, and $c = 0$ otherwise.

Proposition 1.5.24. *Let $u, v \in W$, $u \leq v$, be such that $\overline{\mu}(u, v) \neq 0$ and $l(u, v) > 1$ then $D_R(v) \subseteq D_R(u)$.*

Proof. Given $s \in D_R(v)$ suppose by contradiction that $s \notin D_R(u)$. Using Proposition 1.5.13 then:

$$P_{u,v}(q) = P_{us,v}(q)$$

and so:

$$\overline{\mu}(u, v) = [q^{\frac{l(u,v)-1}{2}}]P_{u,v}(q) = [q^{\frac{l(u,v)-1}{2}}]P_{us,v}(q),$$

but we have also that (recall Definition-Theorem 1.5.6):

$$deg(P_{us,v}) \leq \frac{l(us,v)}{2} = \frac{l(u,v)-2}{2},$$

so $\overline{\mu}(u,v) = 0$ and this contradiction shows that $s \in D_R(u)$. $\square$

We will use in the sequel the following Theorem (due to Brenti in [5])

Theorem 1.5.25. *Let* $u, v \in S_n^{S\setminus\{s_i\}}$ *then:*

$$\overline{\mu}(u,v) = \begin{cases} 1, & \textit{if } \Lambda(v) - \Lambda(u) \textit{ is a Dyck cbs,} \\ 0, & \textit{otherwise.} \end{cases}$$

The last result will be used in Section 3.3 to prove Conjecture 2.1.8 in the Grassmannian case. There are many others results about the $\overline{\mu}$ function, we refer for interested reader to [18],[25],[22],[14] and [21].

1.6 Linear Coxeter system and Boolean elements

For more informations and proof about Boolean elements, we refer to [20].

Definition 1.6.1. *A Coxeter system* $(W, \{s_1, \ldots, s_n\})$ *is called linear if:*

- $(s_i s_j)^r = e$, *for* $r \geq 3$ *if* $|i - j| = 1$,
- $s_i s_j = s_j s_i$, *if* $1 < |i - j| < n - 1$.

W *is called strictly linear if also* $s_1 s_n = s_n s_1$.

Observation 1.6.2. *The diagram of a Coxter linear group not strictly and strictly are shows in the following figure:*

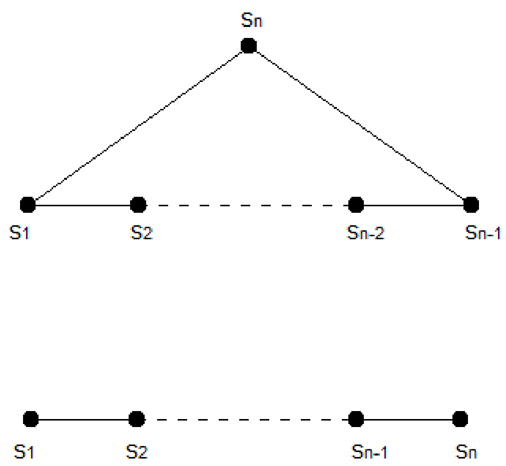

Figure 1.6.1.

Let $(W, \{s_1, \ldots, s_n\})$ be any Coxeter system and let t be a reflection in W we call t a Boolean reflection if it admits a reduced expression:

$$t = s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_1.$$

Every element $w \in W$ such that $w \leq t$ is called a Boolean element.

Observation 1.6.3. *It's easy to see that any Boolean permutations is covexillary i.e.* 3412*-avoiding.*

Example 1.6.4. *In the symmetric group, the Boolean reflections are the reflection and the Boolean elements are the permutation* $v \leq t = (a, b)$ *(for some* $a, b \in [n]^2$*).*

Given $v \in W$, Boolean element, we denote by $\overline{v}_h$ the number of times that s_h appears in a reduced expression of v ($h \in [n]$).
There is an interesting version of Lifting Lemma about Boolean elements.

Lemma 1.6.5. *Given a Coxeter system* (W, S)*, a Boolean element* $u \in W$ *and* u^* *a reduced expression of* u *that is a subword of a Boolean reflection* t*. Then:*

- *any other reduced expression* u' *of* u *such that* $u' \leq t$ *is linked by* u^* *by a sequence of short braid move of the type* $\alpha_{s,s'} = ss'$ *(*$s, s' \in S$*).*
- *any other expression* u' *of* u *such that* $u' \leq t$ *is linked by* u^* *by a sequence of short braid move of the type* $\alpha_{s,s'} = ss'$ *(*$s, s' \in S$*) and null move.*

We list now some useful result about Boolean elements and Kazhdan-Lusztig polynomials.

Theorem 1.6.6 (M. Marietti)**.** *Let* u, v *Boolean elements in* S_{n+1} *with* $u \leq v$*. Then:*

$$P_{u,v}(q) = (1 + q)^b,$$

where:

$$b = |\{k \in [n] : \overline{v}_k = \overline{v}_{k+1} = 2, \overline{u}_{k+1} = 0\}|.$$

Example 1.6.7. *In* S_n *given* $u = e$ *and* $v = (1, 2)(2, 3)(3, 4)(2, 3)(1, 2)$ *then* $b = 1$ *in fact:*

- $\overline{v}_1 = 2, \overline{v}_2 = 2, \overline{v}_3 = 1,$

- $\overline{u}_1 = \overline{u}_2 = \overline{u}_3 = 0$

and then $P_{u,v}(q) = 1 + q$.

This Theorem can be extended to any linear Coxeter group W with a set of generators $\{s_1, \ldots, s_n\}$. Exists, in fact, an isomorphism ϕ (see [20]) such that for any $u, v \in W$:

$$P_{u,v}(q) = P_{\phi(u),\phi(v)}(q),$$

where $\phi(u), \phi(v) \in S_{n+1}$ and $\overline{u}_k = \overline{\phi(u)}_k, \overline{v}_k = \overline{\phi(v)}_k$ for all $k \in [n+1]$. So the last Theorem become the following:

Theorem 1.6.8 (M. Marietti)**.** *Let a linear Coxeter group W and u, v Boolean elements in W with $u \leq v$. Then:*

$$P_{u,v}(q) = (1+q)^b,$$

where:

$$b = |\{k \in [n] : \overline{v}_k = \overline{v}_{k+1} = 2, \overline{u}_{k+1} = 0\}|.$$

Observation 1.6.9. *The last Theorem can't be applied to any Coxeter system. Recalling an example in [20], in fact, if we take the Coxeter system (W, S) such that S contains s_1, s_2, s_3 and r with:*

$$m(s_i, s_j) = 2 \text{ for all } i \neq j,$$

$$m(s_i, r) \geq 3 \text{ for all } i.$$

Then choosing $u = s_1 s_2 r s_3 r s_2 s_1$ and $v = s_3 s_2 s_1$ we have that:

$$P_{u,v}(q) = 1 + 2q.$$

The following result play a central role to prove the Conjecture for Boolean case.

Theorem 1.6.10 (M. Marietti). *Given $u, v \in S_n$ be Boolean elements such that $l(u,v) > 1$. Then:*

$$\overline{\mu}(u,v) \neq 0 \Leftrightarrow \begin{cases} \overline{v}_t = \overline{u}_t, & \text{if } 1 \leq t < a, \\ \overline{v}_t = 2 \wedge \overline{u}_t = 1, & \text{if } t = a, \\ \overline{v}_t = 2 \wedge \overline{u}_t = 0, & \text{if } a < t \leq b, \\ \overline{v}_t = \overline{y}_t, & \text{if } b < t < n-1. \end{cases}$$

Observation 1.6.11. *As before by ϕ we can extend the previous Theorem to Boolean elements in a linear Coxeter group.*

Example 1.6.12. *It's easy to see that if we take the following Boolean reflection:*

$$v = s_i \cdots s_{j-2} s_{j-1} s_{j-2} \cdots s_i,$$

then, by the previous Theorem, $\overline{\mu}(u,v) \neq 0$ if and only if:

$$u = s_i \cdots s_{k-1} \widehat{s}_k \widehat{s}_{k+1} \cdots \widehat{s}_{k+r} s_{k+r+1} \cdots s_{j-1} \cdots \widehat{s}_{k+r} \cdots \widehat{s}_{k+1} s_k \cdots s_i.$$

So if we take in S_7:

- $u = (1,2)(5,6)(2,3)(1,2)$,
- $v = (1,2)(2,3)(3,4)(4,5)(5,6)(4,5)(3,4)(2,3)(1,2)$.

Then by the previous Theorems:

$$P_{u,v}(q) = (1+q)^2 \text{ and } \overline{\mu}(u,v) \neq 0.$$

1.7 Special matchings

Let P be a poset and $G := (V, E)$ its Hasse diagram. Following [2] (Section 5.6) we say that a function $M : V \to V$ is a special matching if:

- M is an involution such that $\{v, M(v)\} \in E$ for all $v \in V$,
- $x \lhd y \Rightarrow M(x) \leq M(y)$ for all $x, y \in V$ such that $M(x) \neq y$.

<u>Example</u> 1.7.1. *The dotted line in Figure 1.7.1 represents a special matching of* $[41256378, 41562738]$.

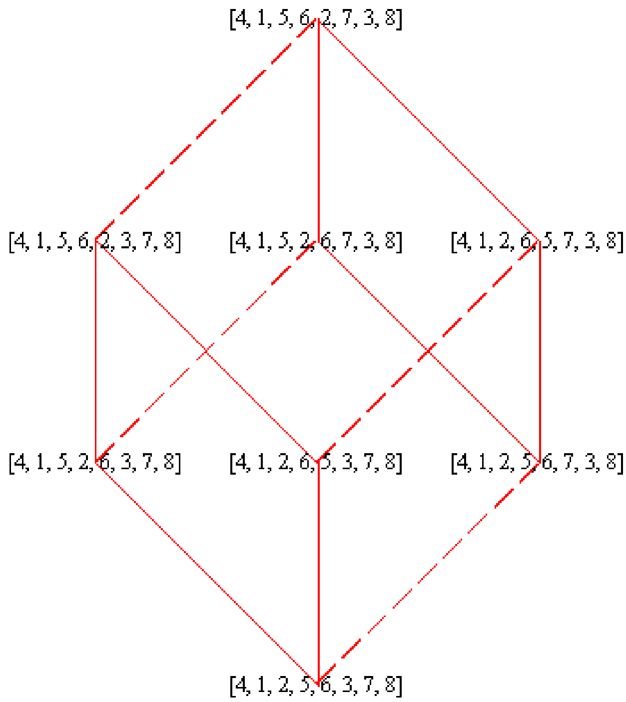

Figure 1.7.1: $[41256378, 41562738]$.

In the sequel, we use the following observations:

<u>Observation</u> 1.7.2. *if $x \lhd y$ and $M(x) \rhd x$ then $M(y) \rhd y$ and $M(x) \lhd M(y)$ (seen Figure 1.7.2).*

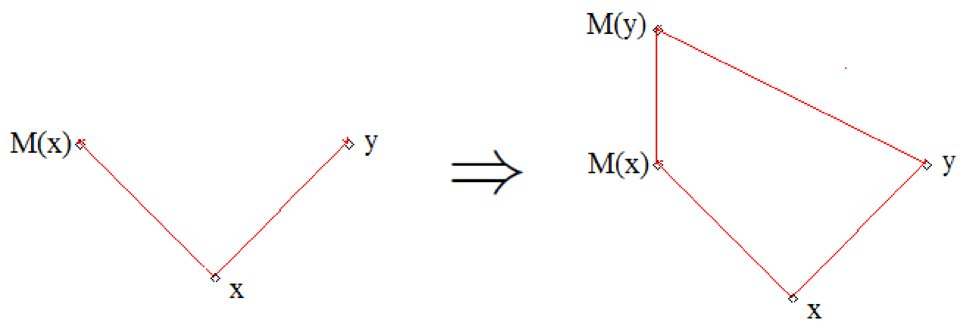

Figure 1.7.2.

Observation 1.7.3. *Dually if $x \lhd y$ and $M(y) \lhd y$ imply $M(x) \lhd x$ and $M(x) \lhd M(y)$ (see Figure 1.7.3).*

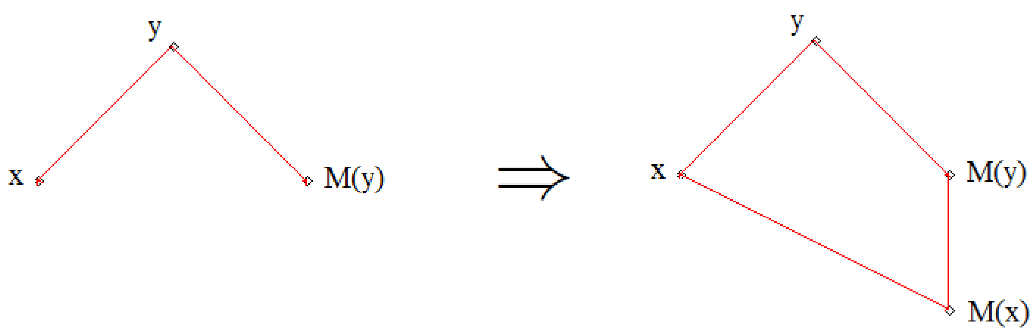

Figure 1.7.3.

In general, it's difficult to know if a poset has or not a special matching but it's true that:

Proposition 1.7.4. *Let* (W, S) *be a Coxeter system,* $u, v \in W$*,* $u \leq v$ *and* $s \in D_R(v) \setminus D_R(u)$*. Then:*

$$M(z) = zs,$$

for all $z \in [u, v]$ *is a special matching of* $[u, v]$*.*

Proof. This follows easily by the Definition of special matching and the Lifting Lemma. ☐

By the previous Proposition, it's easy see that:

Corollary 1.7.5. *Every Bruhat interval* $[e, v]$ *has a special matching.*

There are some connections about Kazhdan-Lusztig polynomials and special matchings.

Theorem 1.7.6 (F. Brenti, F. Caselli, M. Marietti)**.** *Let* (W, S) *be a Coxeter system,* $v \in W$ *and* M *a special matching of* $[e, v]$*. Then:*

$$R_{u,v}(q) = q^c R_{M(u),M(v)}(q) + (q^c - 1) R_{u,M(v)}(q),$$

for all $u \leq v$ *where* $c = 1$ *if* $M(u) \triangleright u$ *and* $c = 0$ *otherwise.*

Theorem 1.7.7 (F. Brenti, F. Caselli, M. Marietti)**.** *Let* $u, v \in W$*,* $u < v$ *and* M *be a special matching of* $[e, v]$*. Then:*

$$P_{u,v}(q) = q^{1-c} P_{M(u),M(v)}(q) + q^c P_{u,M(v)}(q) - \sum_{z : M(z) \triangleleft z} \overline{\mu}(z, M(v)) q^{\frac{l(z,v)}{2}} P_{u,z}(q),$$

where $c = 1$ *if* $M(u) \triangleright u$ *and* $c = 0$ *otherwise.*

2 Chapter 2: Main Conjecture

We now state the Conjecture, due to Brenti in [4], that we study in this work. The Conjecture is motivated by the following questions:

- *When does a Bruhat interval have a special matching?*
- *Is there some connection between special matchings and Kazhdan-Lusztig polynomials for general Bruaht interval?*
- *Can we use the connection between special matchings and Kazhdan-Lusztig polynomials to prove the combinatorial invariance?*

we will show that the Conjecture 2.1.5 answer to this question in a partial way.

2.1 Main Conjecture and some considerations

In the sequel, we use the following Proposition to such that a poset doesn't have a special matching:

Proposition 2.1.1 (Coatom's and atom's condition)**.** *Let* $[u, v]$ *then:*

- *if* $|c(u, v)| - 1 > |c(u, v')|$ *for all* $v' \in \{z \in [u, v] : z \lhd v\}$ *then* $[u, v]$ *doesn't have a special matchings,*
- *if* $|a(u, v)| - 1 > |a(u', v)|$ *for all* $u' \in \{z \in [u, v] : u \lhd z\}$ *then* $[u, v]$ *doesn't have a special matchings.*

Proof. Suppose by contradiction that there exists a special matching M of $[u, v]$ and let $M(v) = w_1$ where:

$$c(u, v) = \{w_1, \ldots w_s\},$$

by Observation 1.7.3 we must have that:

$$M(w_2) \lhd M(v), \ldots, M(w_s) \lhd M(v),$$

so $w_1 = M(v)$ must cover in the poset $|c(u,v)| - 1$ different verties but:

$$|c(u,v')| < |c(u,v)| - 1 \text{ for all } v' \in \{z \in [u,v] : z \lhd v\},$$

so w_1 can't satisfy the previous cover relation, so $[u,v]$ doesn't have a special matching.

In a similar way using Observation 1.7.2 we can conclude the second point of the Proposition. $\square$

Observation 2.1.2. *notice that the converse of Proposition 2.1.1 is not true in general. For example the following figure doesn't have a special matching but there is a* $v' \in \{z \in [u,v] : z \lhd v\}$ *such that:*

$$|c(u,v)| - 1 \not> |c(u,v')|$$

and dually there is a $u' \in \{z \in [u,v] : z \rhd u\}$ *such that:*

$$|a(u,v)| - 1 \not> |a(u,v')|.$$

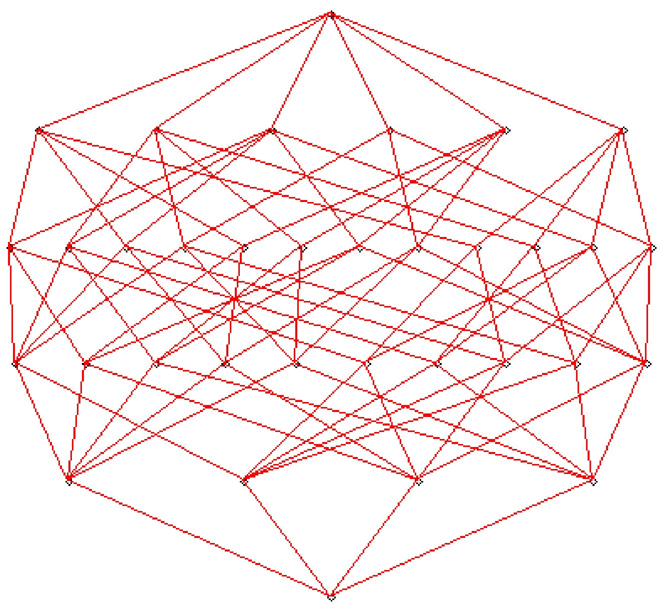

Figure 2.1.1.

Another useful criterion to decide if a poset has or not a special matching is the following:

Lemma 2.1.3. *Let P be a graded poset, M be a special matching of P, and $u, v \in P$ be such that $M(v) \lhd v$ and $u \lhd M(u)$. Then M restricts to a special matching of $[u, v]$.*

Example 2.1.4. *Let P be the poset in Figure 2.1.2 then the above results can't say directly if P has or not a special matching.*

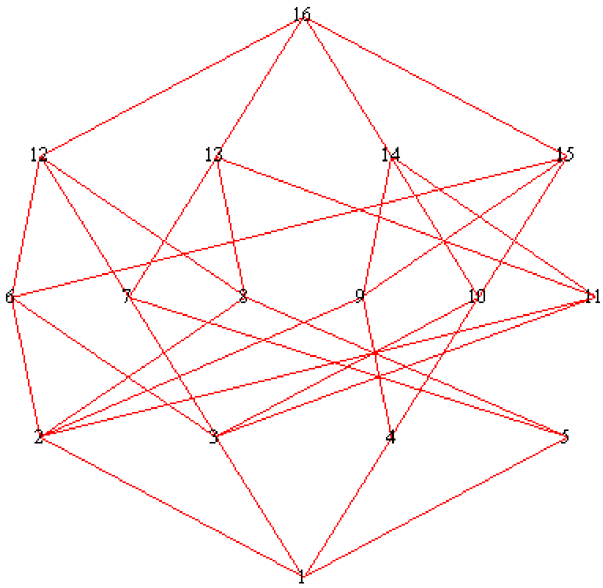

Figure 2.1.2.

Using Proposition 2.1.1 we can say that:

- *the coatoms* $12, 13, 14, 15$ *are a possible choice for* $M(16)$*,*
- *the atoms* $2, 3$ *are a possible choice for* $M(1)$*.*

So suppose that $M(1) = 2$ *then using Lemma 2.1.3 the subposet* $[2, 16]$ *has a special matching, but by Figure 2.1.3 this is not true by the coatom's condition and by the atom's condition, so* $M(1) \neq 2$*.*

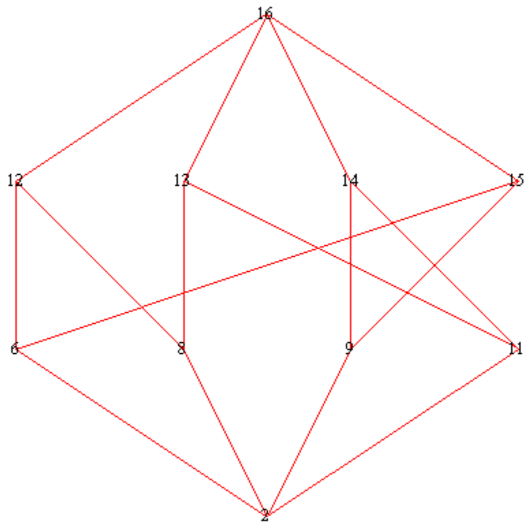

Figure 2.1.3: no special matching by Proposition 2.1.1.

Suppose then that $M(1) = 3$. *Then using lemma 2.1.3 the subposet* $[3, 16]$ *has a special matching, but by Figure 2.1.4 this is not true again by the coatom's condition and by the atom's condition's, so* $M(1) \neq 3$.

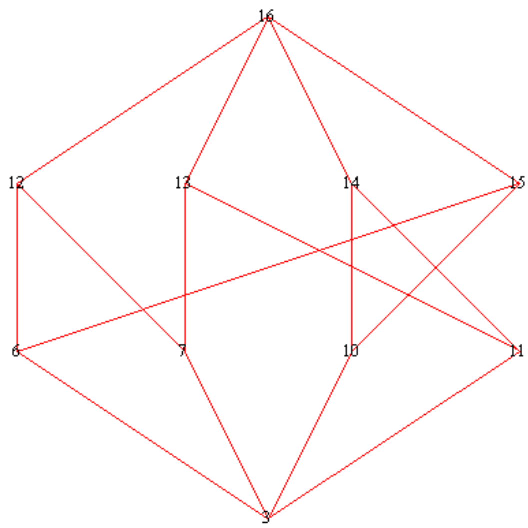

Figure 2.1.4: no special matching by Proposition 2.1.1.

Hence by Proposition 2.1.1 and Lemma 2.1.3 we can say that the poset P

doesn't have a special matching.

Here's the Conjecture that is the starting point of my work.

Conjecture 2.1.5 (F. Brenti)**.** *Let $u, v \in S_n$ with $[u, v]$ irreducible, $l(u, v) > 1$ and $l(u, v)$ odd then:*

$$[u, v] \ \textit{has a special matching} \Leftrightarrow \overline{\mu}(u, v) = 0.$$

The Conjecture appears in [4] and has been verified by Brenti for $1 \leq l(u, v) \leq 5$.
In this first part of my work, I give a counterexample to the direction "$\Leftarrow$".

<u>Counterexaple</u> 2.1.6. *Consider in S_6 the permutations:*

$$u = 231564 \ \textit{and} \ v = 562341,$$

for this pair of elements we have that:

- $P_{u,v}(q) = 1 + 4q + 4q^2$,
- $l(u, v) = 7$ *and so* $\overline{\mu}(u, v) = 0$.

We now show that the poset $[u, v]$ is irreducible. Recalling that given a poset $P = [u, v]$ the rank generating function is:

$$r(u, v) = r([u, v]) = \sum_{i \geq 0} P_i q^i,$$

where P_i are the rank level, then for the irreducibility, we use the following fact (due to Stanley in [24]):

Proposition 2.1.7. *Let $u, v \in S_n$ if there doesn't exist $x, y \in S_m$ $z, t \in S_p$ (with $m, p \leq n$) such that:*

$$r(u, v) = r(x, y)r(z, t),$$

then the poset $[u, v]$ is irreducible.

The rank function of the poset $[u, v]$ *(see figure 2.1.5) is:*

$$r(u, v) = 1 + 6q + 18q^2 + 33q^3 + 39q^4 + 27q^5 + 9q^6 + q^7$$

and factoring this polnomial:

$$r(u, v) = (1 + q)(1 + 5q + 13q^2 + 20q^3 + 19q^4 + 8q^5 + q^6).$$

Using [13], that lists all the possible rank generating functions for $u, v \in S_n$ *with* $6 \leq l(u, v) \leq 7$*, we see that there doesn't exixt a pair of* $z, t \in S_p$ *(with* $p \leq 6$*) such that:*

$$r(z, t) = 1 + 5q + 13q^2 + 20q^3 + 19q^4 + 8q^5 + q^6$$

and so $[u, v]$ *is irreducible.*
We now show, in contradiction with the left direction of Conjecture 2.1.5, that $[u, v]$ *doesn't have a special matching. In fact, by the following figure we can see that:*

$$|c(u, v)| - 1 = 8 > |c(u, v')| \text{ for all } v' \in c(u, v)$$

and so by Proposition 2.1.1 $[u, v]$ *doesn't have a special matching.*

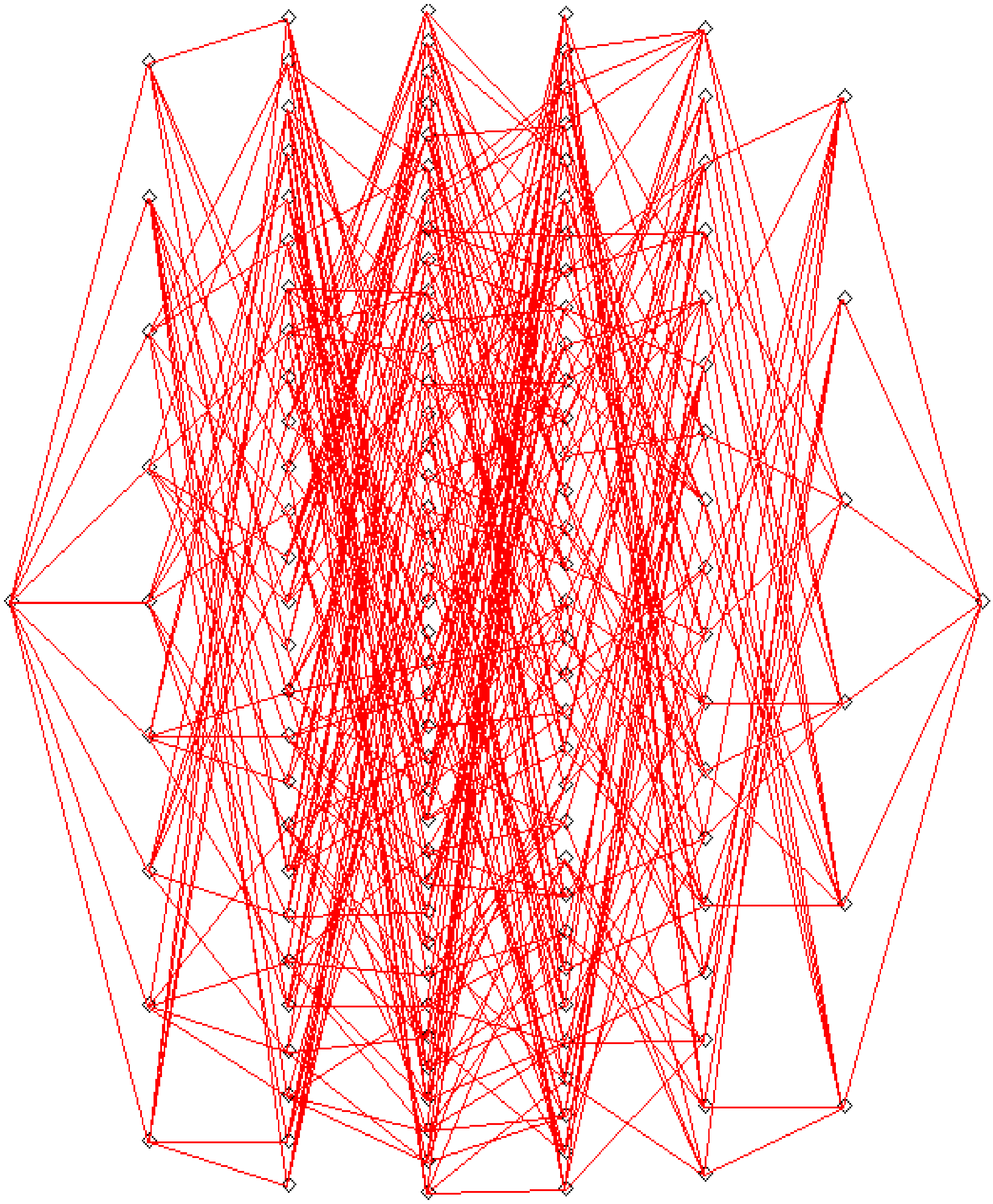

Figure 2.1.5: [231564, 562341].

In this work, we propose and study a new Conjecture, which generalizes one direction of Brenti's Conjecture 2.1.5.

Conjecture 2.1.8 (C.Bosca)**.** *Let W be a Coxeter group and $u, v \in W$ with $l(u,v) > 1$ then:*

$$[u,v] \text{ has a special matching} \Rightarrow \overline{\mu}(u,v) = 0.$$

We have verified the previous Conjecture for all $u, v \in S_n$ with $1 \leq l(u,v) \leq 7$ using the result in [10],[12],[13]. In the rest of this work we prove Conjecture 2.1.8 for:

- $u, v \in S_n$ such that $u \leq v$, $D_R(v) \subseteq \{1, n-1\}$,
- u, v Grassmanian permutation,
- u, v Boolean elements in a Coxeter linear group.

3 Chapter 3: Main results

3.1 Elements in S_n with $D(v) \subseteq \{1, n-1\}$

In this Section, we prove Conjecture 2.1.8 for $u, v \in S_n$ with $u \leq v$ and $D(v) \subseteq \{1, n-1\}$. For this purpose, we use Theorem 1.5.10 with this observation:

Observation 3.1.1. *Let $u, v \in S_n$ $u \leq v$, then the following two numbers are equal:*

$$|\{j \in [v(n)+1, v(1)-2] : \{u(1), \dots, u(j)\} = [j]\}|,$$
$$|\{j \in [v(n)+1, v(1)-2] : \sum_{i=1}^{j} u(i) = \binom{j+1}{2}\}|.$$

We begin with a construction:

Theorem 3.1.2. *Let $u, v \in S_n$, $u \leq v$, $D_R(v) \subseteq \{1, n-1\}$ and $d = u^{-1}(v(1)) - u^{-1}(v(n))$.*
Then $\overline{\mu}(u, v) \neq 0$ if and only if:

$$u = h, 1 \dots, \widehat{j}, \dots, h-1, j, h+1, \dots, h+d-1, i, h+d+1, \dots, \widehat{i}, \dots, n, h+d \quad (5)$$

where $i = v(1)$ and $j = v(n)$ for some $h \in [j+1, i-d-1]$.

Proof. We begin with the case $D_R(v) = \{1\}$:

$$v = k, 1, \dots, \widehat{k}, \dots, n.$$

In this case $r = 0$ because $v(n)+1 > v(1)-2$ so by Theorem 1.5.10, $P_{u,v}(q) = 1$ for all $u \leq v$, so $\overline{\mu}(u, v) = 0$ for all $u \leq v$, and the result follows.
Similary, if $D_R(v) = \{n-1\}$ then:

$$v = 1, \dots, \widehat{k}, \dots, n-1, n, k.$$

So as before, $r = 0$ because $v(n) + 1 > v(1) - 2$.
So the only important cases are the v with $D_R(v) = \{1, n-1\}$.
Let $i := v(1)$ and $j := v(n)$:

$$v = i, 1, 2, \dots, \widehat{j}, \dots, \widehat{i}, \dots, n, j.$$

Observation 3.1.3. *Note that if $j > i$, then $r = 0$ for all $u \leq v$ and so by Theorem 1.5.10 $P_{u,v}(q) = 1$ for all $u \leq v$, and the result again follows.*

We may therefore assume that $j < i$.
We now give two observations:

Observation 3.1.4. *let $u \leq v$. If $u^{-1}(j) > u^{-1}(i)$ then $r = 0$ for either $i \in \{u(1), \ldots, u(h)\}$ or $j \notin \{u(1), \ldots, u(h)\}$ for all $h \in [j+1, i-2]$.*

Observation 3.1.5. *if $u^{-1}(i) - u^{-1}(j) = 1$ and $j + 2 \leq u^{-1}(i) \leq i - 2$ then $r = 1$.*

Denoting by:

$$d := u^{-1}(i) - u^{-1}(j).$$

We have to distinguish two cases:
(*Case I:* $\boldsymbol{r < d}$): We begin with the case $1 = r < d$. The plan of this proof is to start from v and construct a $u \leq v$ such that $\overline{\mu}(u, v) \neq 0$.

Observation 3.1.6. *note that with Theorem 1.5.10 the condition $\overline{\mu}(u, v) \neq 0$ is equal to $l(u, v) = 2r + 1$.*

So first of all we move the integers j and i in v to have:

$$u^{-1}(i) - u^{-1}(j) = d.$$

Then we obtain:

$$v' = h + d, \ldots, h - 1, j, h + 1, \ldots, h + d - 1, i, h + d + 1, \ldots, h$$

where $h \in [j + 1, i - d - 1]$.
It's easy to observe that $l(v', v) = 2$ and that it is impossible, with only one transposition, to obtain a permutation $u \leq v$ from v', such that $\overline{\mu}(u, v) \neq 0$ ($l(u, v) = 3$).
We now suppose by contradiction that there exists a permutation $u \leq v$ with $r < u^{-1}(i) - u^{-1}(j)$ and $\overline{\mu}(u, v) \neq 0$ ($l(u, v) = 2r + 1$). Let:

$$\{s_1, \ldots, s_r\}_< := \{k \in [j + 1, i - 2] : \{u(1), \ldots, u(k)\} = [k]\},$$

<u>**Observation**</u> **3.1.7.** *$u^{-1}(j) \leq s_1$, and $s_r < u^{-1}(i)$ so:*

$$u = \dots, j, \dots, u(s_1), \dots, u(s_2), \dots, u(s_{r-1}) \dots, u(s_r), \dots, i, \dots$$

We will show that such a u can't exist. We have to distinguish four cases:
(*Case I.a*): Suppose that $u^{-1}(i) - s_r > 1$ and $s_r - s_{r-1} > 1$:

$$u = \dots, u(s_{r-1}), \underbrace{\dots}_{>1}, u(s_r) \underbrace{\dots}_{>1}, i, \dots$$

Then recalling that:

$$\{u(1), \dots, u(s_{r-1})\} = [s_{r-1}],$$

$$\{u(1), \dots, u(s_r)\} = [s_r],$$

we have that $u(s_r + 1) > s_r$ and hence $u(s_r + 1) > u(s_r), u(s_r - 1)$, so:

- If $u(s_r) > u(s_r - 1)$ we consider $w_1 = u(s_r + 1, s_r - 1)(s_r, s_r + 1)$,
- if $u(s_r) < u(s_r - 1)$ we consider $w_2 = u(s_r, s_r + 1)(s_r, u^{-1}(s_r))$.

In each cases we have that:

$$r(w_i) = r - 1, \;\; l(w_i) = l(u) + 2 \;\; (i = 1, 2)$$

and $w_i \leq v$ in fact by Theorem 1.3.10 it's easy to see that:

$$v[z, t] \geq w_i[z, t] \;\; \text{for all } z, t \in [n],$$

where $i = 1, 2$.
(*Case I.b*): Suppose that $u^{-1}(i) - s_r > 1$ and $s_r - s_{r-1} = 1$:

$$u = \dots, u(s_{r-1}), u(s_r) \underbrace{\dots}_{>1}, i, \dots$$

Then recalling that:

$$\{u(1), \dots, u(s_{r-1})\} = [s_{r-1}]$$

and

$$\{u(1), \dots, u(s_r)\} = [s_r],$$

we have that $u(s_r) = s_r$. In this case:

- if $u(s_{r-1}) \neq s_r - 1$ then we let $w_3 = u(s_r, s_r - 1)(u^{-1}(s_r - 1), s_r - 1)$,
- if $u(s_{r-1}) = s_r - 1$ then we let $w_4 = u(s_r, s_r + 1)(s_r - t, s_r - t - 1)$ where $t < r - 1$ is the maximal integer such that $u(s_r - t - 1) \neq s_r - t - 1$.

We have that:

$$r(w_i) = r - 1,\ \ l(w_i) = l(u) + 2,\ \ (i = 3, 4)$$

and $w_i \leq v$ in fact using Theorem 1.3.10 it's easy to see that:

$$v[z, t] \geq w_i[z, t] \text{ for all } z, t \in [n],$$

where $i = 3, 4$.
(*Case I.c*): Suppose that $s_r - s_{r-1} > 1$ and $u^{-1}(i) - s_r = 1$:

$$u = \ldots u(s_{r-2}), \ldots, u(s_{r-1}), \underbrace{\ldots}_{>1}, u(s_r), i, \ldots$$

This case is analogous to the two previous one (where we consider $u(s_r)$ in place of i, s_{r-1} in place of s_r and s_{r-2} in place of s_{r-1}).
(*Case I.d*): Suppose that $s_r - s_{r-1} = 1$ and $u^{-1}(i) - s_r = 1$:

$$u = \ldots, u(s_k), \ldots, u(s_{k+1}), u(s_{k+2}), \ldots, u(s_{r-1}), u(s_r), i, \ldots$$

where $1 \leq k \leq r - 2$ is the maximal integer such that $s_{k+1} - s_k > 1$ (if this integer doesn't exists then we have the u of the theorem and this arguments are inapplicable).
In this case we consider:

$$w_5 = u(s_{k+1}, s_{k+1} - 1)(u^{-1}(h + d), s_r).$$

For this permutation we have that:

$$r(w_5) = r - 1,\ \ l(w_5) = l(u) + 2$$

and $w_5 \leq v$ in fact using Theorem 1.3.10 it's easy to see that:

$$v[z, t] \geq w_5[z, t] \text{ for all } z, t \in [n].$$

In each cases (I.a-I.b-I.c-I.d) we can decrease r and maintain $l(w,v) = 2r(w) + 1$.
But this is a contradiction because I can iterate this procedure and find a u with $r = 1 < u^{-1}(i) - u^{-1}(j)$.
(***Case II: r = d***): We start with an observation:

Observation 3.1.8. *if u is given by 5 then by Theorem 1.5.10, $deg(P_{u,v}(q)) = 2d+1$ and so $\overline{\mu}(u,v) \neq 0$. We want to prove that is the unique permutation with this proprieties.*

Suppose, by contradiction, that exist another $w \leq v$ with $\overline{\mu}(w,v) \neq 0$ and $deg(P_{w,v}) = w^{-1}(i) - w^{-1}(j)$. First of all, let $h := w^{-1}(j)$, we observe that $j < h < i - d - 1$ otherwise by Theorem 1.5.10 we have $r < d$. So we can see that:

$$w = \underbrace{\ldots}_{[h]-\{j\}}, j, h+1, \ldots, h+d-1, i, \underbrace{\ldots}_{[h+d,n]-\{i\}},$$

where:

- in the left dots we have the integers $[h] - \{j\}$ permutated,
- $u(k) = w(k)$ for $k \in [h, h+d]$ (they are fixed for have $r = d$),
- in the right dots we have the integer $[h+d, n] - \{i\}$ permutated.

We have to distinguish three cases:
(***Case II.a***): Suppose to fix $u(k) = w(k)$ for $k \in [n] - [h-1]$, we will prove that if exist at least one $k \in [h-1]$ such that $u(k) \neq w(k)$ then $l(w) < l(u)$. We prove this by induction on $w^{-1}(h)$. If $w^{-1}(h) = 2$ then:

- If $j = 1$:

$$w = 2, h, 3, \ldots, j-1, j+1, \ldots, h-1, j, \ldots$$

- Otherwise:

$$w = 1, h, 2, 3 \ldots, j-1, j+1, \ldots, h-1, j, \ldots$$

In both cases $l(w) < l(u)$ and the assertion is true for $w^{-1}(h) = 2$. Therefore assume it be correct for $w^{-1}(h) = s - 1$ with $s \leq h - 1$. Consider $w^{-1}(h) = s$, we know that h is the greater integer in the first $h-1$ position. If we consider:

$$w' = w(s, w^{-1}(s)).$$

By Theorem 1.3.10, for have $w \leq v$, we have $w^{-1}(h) < w^{-1}(s)$. Then $l(w') = l(w) + 1$ and according to the induction hypothesis we can conclude:

$$l(w) = l(w') - 1 < l(w') < l(u).$$

(*Case II.b*):Suppose to fix $u(k) = w(k)$ for $k \in [n] - [h + d + 1, n]$, we will prove that if exist at least one $k \in [h+d+1, n]-\{i\}$ such that $u(k) \neq w(k)$ then $l(w) < l(u)$. We prove this by induction on $w^{-1}(h + d)$. If $w^{-1}(h + d) = n - 1$ then:

- If $i = n$:
$$\ldots, i, h + d + 1, \ldots, h + d, n - 1. \tag{6}$$

- Otherwise:
$$\ldots, i, h + d + 1, \ldots, h + d, n. \tag{7}$$

In both cases $l(w) < l(v)$ and the assertion is true for $w^{-1}(h + d) = n - 1$. Therefore assume it be correct for $w^{-1}(h+d) = n-s+1$. Consider $w^{-1}(h+d) = n - s$, we know that $h + d$ is the smallest integer in the position $[h + d + 1, n]$. If we consider:

$$w' = w(n - s, w^{-1}(n - s)).$$

By Theorem 1.3.10, for have $w \leq v$, we have $w^{-1}(n - s) > w^{-1}(h + d)$. Then $l(w') = l(w) + 1$ and according to the induction hypothesis we can conclude:

$$l(w) = l(w') - 1 < l(w') < l(u).$$

(*Case II.c*): Suppose to fix only $u(k) = w(k)$ for $k \in [h, h + d]$. Then using the previous two cases we can conclude that for each $w \neq u$ we have that $l(w) < l(v)$. $\square$

Example 3.1.9. *Here we list all the* $u, v \in S_7$ *such that* $u \leq v$, $D_R(v) \subseteq \{1, n-1\}$ *and* $\overline{\mu}(u, v) \neq 0$*:*

- $u, v \in S_7$ *such that:* $P_{u,v}(q) = 1 + q$ *and* $\overline{\mu}(u, v) \neq 0$.

$$(3125674, 5134672); (4132675, 6134572); (3126574, 6134572)$$

$$(5124376, 7124563); (4123765, 7124563); (2145673, 4235671)$$

$$(2145673, 4235671); (5123476, 7123564); (4123675, 6124573)$$

$$(4132765, 7134562); (5134276, 7134562); (3127564, 7134562)$$

$$(2154673, 5234671); (3215674, 5234671); (5234176, 7234561)$$

$$(4231765, 7234561); (3217564, 7234561); (2174563, 7234561)$$

$$(3216574, 6234571); (4231675, 6234571); (2164573, 6234571)$$

- $u, v \in S_7$ *such that:* $P_{u,v}(q) = 1 + 2q + q^2$ *and* $\overline{\mu}(u, v) \neq 0$.

$$(3124675, 6134572); (4123576, 7124563); (4132576, 7134562)$$

$$(3124765, 7134562); (2135674, 5234671); (2137564, 7234561)$$

$$(4231576, 7234561); (3214765, 7234561); (3214675, 6234571); (2136574, 6234571)$$

- $u, v \in S_7$ *such that:* $P_{u,v}(q) = 1 + 3q + 3q^2 + q^3$ *and* $\overline{\mu}(u, v) \neq 0$.

$$(3124576, 7134562); (3214576, 7234561); (2134765, 7234561); (2134675, 6234571)$$

- $u, v \in S_7$ *such that:* $P_{u,v}(q) = 1 + 4q + 6q^2 + 4q^3 + q^4$ *and* $\overline{\mu}(u, v) \neq 0$.

$$(2134576, 7234561)$$

We now use the previous Theorem to prove Conjecture 2.1.8 for $u, v \in S_n$ such that $D_R(v) \subseteq \{1, n-1\}$.

Theorem 3.1.10. *Let $u, v \in S_n$, $u \leq v$, $D_R(v) \subseteq \{1, n-1\}$, be such that $\overline{\mu}(u,v) \neq 0$. Then $[u,v]$ doesn't have a special matching.*

Proof. By the previous Theorem we know that:

$$u = h, 1, \ldots, \widehat{j}, \ldots, h-1, j, h+1, \ldots, h+d-1, i, h+d+1, \ldots, \widehat{i}, \ldots, n, h+d$$

for some $h \in [j+1, i-d-1]$. We will show that:

$$|c(u,v')| < |c(u,v)| - 1, \tag{8}$$

for all $v' \in c(u,v)$, and by Proposition 2.1.1 we will conclude that $[u,v]$ doesn't have a special matching.
We start by counting $|c(u,v)|$. The coatoms of v are all of the form $(i,t)v$ or $(t,j)v$ for some $2 \leq t \leq n-1$. However, it is easy to see (using Theorem 1.3.10) that $u \leq (i,t)v$ if and only if $h \leq t \leq h+d$ if and only if $u \leq (t,j)v$. Therefore:

$$|c(u,v)| = 2(d+1).$$

It remains to count $|c(u,v')|$ where $v' \in c(u,v)$. By the previous argument either $v' = (i,t)v$ or $v' = (t,j)v$ for some $h \leq t \leq h+d$.
Then we have to distinguish two cases:

Say $v' = (i,t)v$ for some $h \leq t \leq h+d$ and let w be such that $u \leq w \lhd v'$. Then either $w = (t,a)v'$ for some $2 \leq a \leq t-1$ or $w = (i,a)v'$ for some $t+1 \leq a \leq i-1$ or $w = (a,j)v'$ for some $2 \leq a \leq i-1, a \neq t$. By Theorem 1.3.10 we have that $(t,a)v' \geq u$ if and only if $h \leq a \leq t-1$ while $(i,a)v' \geq u$ if and only if $t+1 \leq a \leq h+d$ and $(a,j)v' \geq u$ if and only if either $h \leq a \leq t-1$ or $t+1 \leq a \leq h+d$.

Say $v' = (t,j)v$ for some $h \leq t \leq h+d$ and let w be such that $u \leq w \lhd v'$. Then either $w = (j,a)v'$ for some $2 \leq a \leq t-1$ or $w = (t,a)v'$ for some $t+1 \leq a \leq i-1$ or $w = (i,a)v'$ for some $2 \leq a \leq i-1, a \neq t$. By Theorem

1.3.10 we have that $(j,a)v' \geq u$ if and only if $h \leq a \leq t-1$ while $(t,a)v' \geq u$ if and only if $t+1 \leq a \leq h+d$ and $(i,a)v' \geq u$ if and only if either $h \leq a \leq t-1$ or $t+1 \leq a \leq h+d$. So in all cases:

$$|c(u,v')| = 2d$$

and therefore:

$$|c(u,v')| < |c(u,v)| - 1 \text{ for all } v' \in c(u,v),$$

as claimed. □

We illustrate the previous Theorem on an example.

Example 3.1.11. *Let $u = 4123576$ and $v = 7124563$ then by Theorem 1.5.10 we have that:*

$$r = 2 \Rightarrow P_{u,v}(q) = 1 + 2q + q^2.$$

Since $l(u,v) = 5$ we have that $\overline{\mu}(u,v) \neq 0$, and $[u,v]$ doesn't have a special matching. Indeed:

$$5 = |c(u,v)| - 1 > |c(u,v')| = 4,$$

for all $v' \in \{z \in [u,v] : z \lhd v\}$. We show this by the following figure:

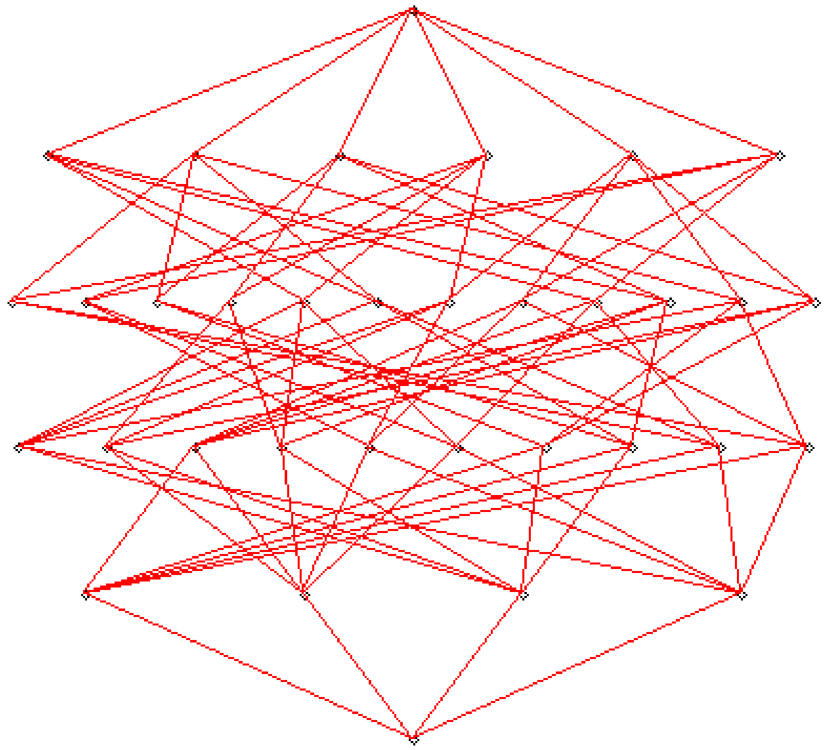

Figure 3.1.1: $[4123576, 7124563]$.

3.2 Boolean elements

In this Section we prove Conjecture 2.1.8 for Boolean elements of linear Coxeter systems. We begin with the following result.

Corollary 3.2.1. *Let $(W, \{s_1, \ldots, s_n\})$ be a linear Coxeter system and $u, v \in W$ be two Boolean elements. Then $\overline{\mu}(u, v) \neq 0$ if and only if:*

$$v = * * * s_k s_{k+1} \cdots s_{k+r} * * * s_{k+r} \cdots s_k * **,$$

$$u = * * * s_k \widehat{s}_{k+1} \cdots \widehat{s}_{k+r} * * * \widehat{s}_{k+r} \cdots \widehat{s}_k * **, \tag{9}$$

for some $1 \leq k < k + r < n$, where by "$*$" we denote the parts where u and v are equal.

Proof. This follows easily from Theorem 1.6.10. □

We now prove Conjecture 2.1.8 for Boolean elements of linear Coxeter systems.

Theorem 3.2.2. *Let $(W, \{s_1, \ldots, s_n\})$ be a linear Coxeter system, u, v be boolean elements, $u \leq v$, such that $\overline{\mu}(u, v) \neq 0$. Then $[u, v]$ doesn't have special matching.*

Proof. We know, by the preceding theorem, that:

$$v = * * * s_k s_{k+1} \cdots s_{k+r} * * * s_{k+r} \cdots s_k * **,$$

$$u = * * * s_k \widehat{s}_{k+1} \cdots \widehat{s}_{k+r} * * * \widehat{s}_{k+r} \cdots \widehat{s}_k * **, \tag{10}$$

for some $1 \leq k < k + r < n$ where by "$*$" we denote the parts where u and v are equal.

We start by counting $|c(u, v)|$. We obtain $2r$ coatoms by deleting s_{k+i} (with $i = 1, \ldots, r$) on the right and on the left , and two other coatoms by deleting s_k on the right and on the left. It's clear that these are the only coatoms so $|c(u, v)| = 2r + 2$.

It remains to count $c(u,v')$ where $v' \in \{z \in [u,v] : z \lhd v\}$. This is simple because if v' is obtained from v by deleting one s_{k+i} (with $i = 1,\dots,r$) there are $2r-2$ possibilities, by deleting the other $s_{k+j} \neq s_{k+1}$ on the right and on the left and two other by deleting s_k on the left and on the right, so in total $2r$ possibilities.
If instead v' is obtained from v by deleting one s_k there are only $2r$ coatoms in $c(u,v')$ obtained by cancelling s_{k+i} (with $i = 1,\dots,r$) on the right and on the left. So in all cases:

$$|c(u,v')| = 2r < 2r+1 = |c(u,v)| - 1,$$

for all $v' \in \{z \in [u,v] : z \lhd v\}$. Hence by Theorem 2.1.1 $[u,v]$ doesn't have a special matching. □

It's easy to see that this result implies the following one for S_n (note that all reflections are Boolean in S_n).

Corollary 3.2.3. *Let $u,v \in S_n$, with $u,v \leq (1,n)$ $u \leq v$, be such that $\overline{\mu}(u,v) \neq 0$. Then $[u,v]$ doesn't have a special matching.*

Example 3.2.4. *Let $(W,\{s_1,\dots,s_5\})$ be a linear Coxeter system and:*

$$u = s_1 s_5 s_2 s_1,$$

$$v = s_1 s_2 s_3 s_4 s_5 s_4 s_3 s_2 s_1,$$

we have that $u \leq v$ and by Theorem 1.6.10:

$$P_{u,v}(q) = 1 + 2q + q^2.$$

Since $l(u,v) = 5$ then $\overline{\mu}(u,v) \neq 0$, and $[u,v]$ doesn't have a special matching by Proposition 2.1.1. Indeed (see Figure 3.2.1) $c(u,v) = 6$ and $c(u,v') = 4$ for all $v' \in \{z \in [u,v] : z \lhd v\}$.

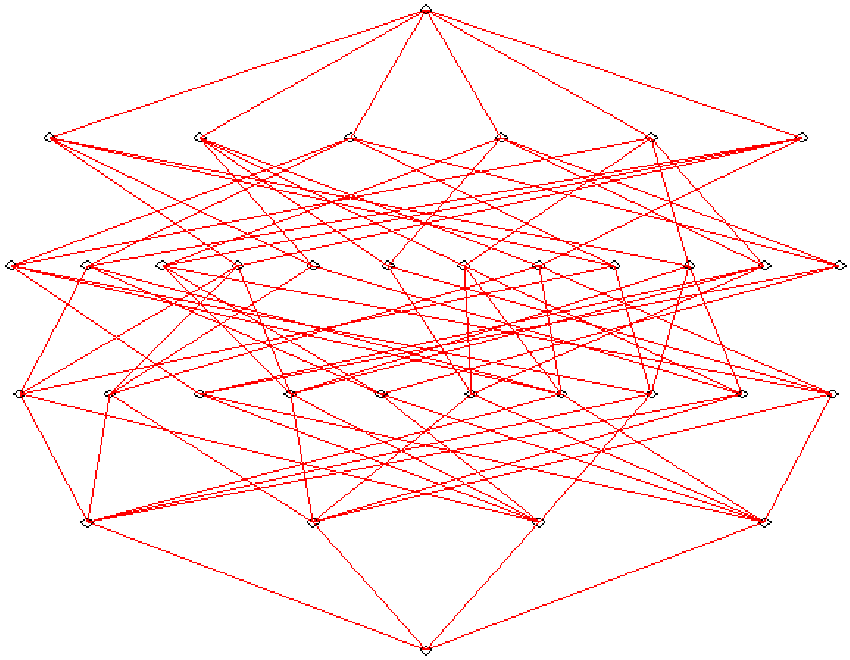

Figure 3.2.1: $[s_1s_5s_2s_1, s_1s_2s_3s_4s_5s_4s_3s_2s_1]$.

Example 3.2.5. *In the symmetric group let:*

$$v = 82345671,$$

$$u = 42318675,$$

then we have that $u \leq v$ *and by Theorem 1.6.10:*

$$P_{u,v}(q) = 1 + q.$$

Since $l(u, v) = 3$ *then* $\overline{\mu}(u, v) \neq 0$*, and* $[u, v]$ *doesn't have a special matching* *by Proposition 2.1.1. Indeed (see Figure 3.2.2)* $c(u, v) = 4$ *and* $c(u, v') = 2$ *for all* $v' \in \{z \in [u, v] : z \triangleleft v\}$.

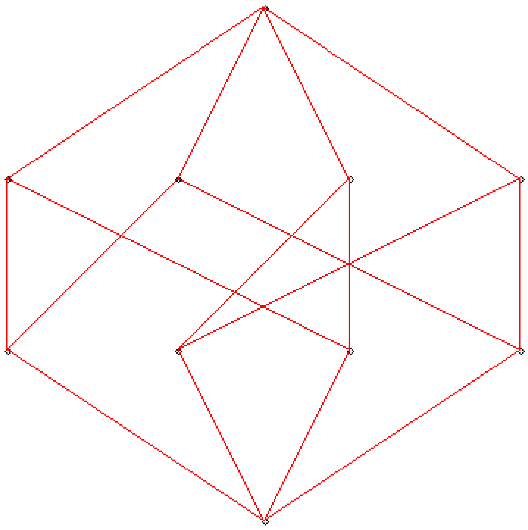

Figure 3.2.2: $[42318675, 82345671]$.

3.3 Grasmannian permutations

In this Section we prove Conjecture 2.1.8 for permutations in S_n^J where $J = S \setminus \{(i, i+1)\}$ for some $i \in [n-1]$. We recall that in this case:

$$S_n^J = \{x \in S_n : x(1) < \ldots < x(i), x(i+1) < \ldots < x(n)\}.$$

We take $u, v \in S_n^J$, with $u \leq v$ such that:

$$\Lambda := \Lambda(v) \setminus \Lambda(u),$$

is a Dyck cbs. As in the previous Section, we start by constructing u, v. In this Section given a permutation $u \in S_n$ we denote by u'_j (where $j \in [n]$) the permutation in S_{n-1}:

$$u'_j = u'_j(1), \ldots, u'_j(j-1), u'_j(j+1), \ldots, u'_j(n)$$

where:

$$u(k)'_j = \begin{cases} u(k), & \text{if } u(k) < u(j), \\ u(k) - 1, & \text{otherwise.} \end{cases}$$

Example 3.3.1. *Let* $u = 461235798$ *then:*

$$u'_1 = 35124687 \;\; u'_2 = 35124687 \;\; u'_3 = 35124687$$

$$u'_4 = 51234687 \;\; u'_5 = 45123687 \;\; u'_6 = 41235687$$

$$u'_7 = 46123587 \;\; u'_8 = 46123578 \;\; u'_9 = 46123678$$

We will use the following function.

Definition 3.3.2. *Let* $u, v \in S_n^J$ *be such that there exists a* $k \in [i]$ *with* $u(k) = v(k)$ *or* $u^{-1}(n) = v^{-1}(n)$ *or* $u^{-1}(1) = v^{-1}(1)$. *Let* $P := [u, v]$ *we define a function* $f : P \to S_{n-1}$ *by letting:*

$$f(x) = \begin{cases} x'_k, & \text{if there exists a } k \in [i] \text{ such that } u(k) = v(k), \\ x'_{u^{-1}(n)}, & \text{if the position of } n \text{ in } u \text{ and } v \text{ is the same,} \\ x'_{u^{-1}(1)}, & \text{if the position of } 1 \text{ in } u \text{ and } v \text{ is the same,} \end{cases}$$

for each $x \in P$.

Example 3.3.3. *Given* $u = 1234657$, $v = 1236745$ *then:*

$$f^3(1234657) = 1324,$$

$$f^3(1236745) = 3412.$$

We use the previous function on the Grasmannian permutations. The following Proposition is useful in the sequel.

Proposition 3.3.4. *Let* $u, v \in S_n^J$, $u \leq v$. *If there exist a* $k \in [i]$ *with* $u(k) = v(k)$ *or* $u^{-1}(n) = v^{-1}(n)$ *or* $u^{-1}(1) = v^{-1}(1)$, *then* $f(u), f(v)$ *are Grasmannian permutations.*

Proposition 3.3.5. *Let* $u, v \in S_n^J$ *with* $u \leq v$ *and such that exits a* $k \in [i]$ *with* $u(k) = v(k)$ *or* $u^{-1}(n) = v^{-1}(n)$ *or* $u^{-1}(1) = v^{-1}(1)$. *Then:*

$$[u, v] \cong f([u, v]) = [f(u), f(v)].$$

Proof. This is easy to see noting that if u and v are such that $H = u(k) = v(k)$ for some $k \in [i]$ then all $x \in [u, v]$ are such that $x(k) = H$ since of $u \leq x \leq v$ then (by Proposition 1.3.12):

$$u(j) \leq x(j) \leq v(j),$$

for all $j \in [i]$. $\square$

Example 3.3.6. *Consider* $[2461357, 2671345]$. *Using the previous functions (step by step) we can see that:*

$$[2461357, 2671345] \cong [351246, 561234] \cong [24135, 45123] \cong [1324, 3412],$$

see Figure 3.3.1.

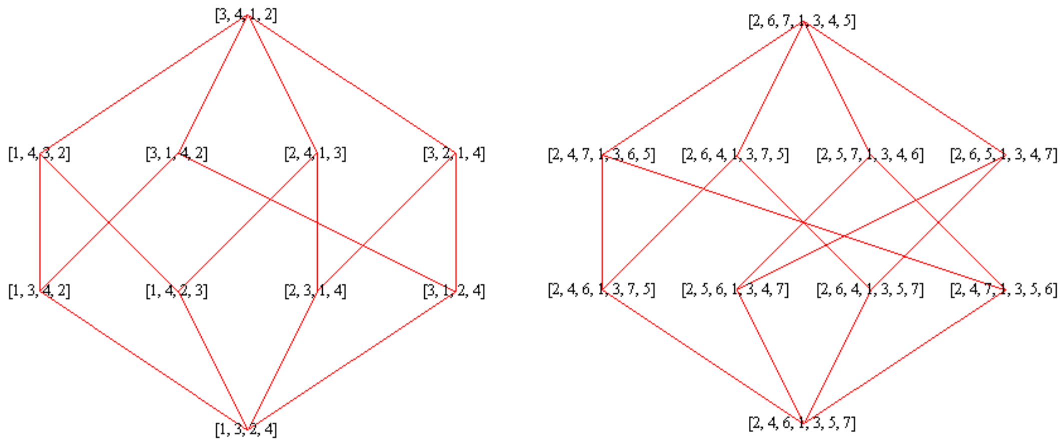

Figure 3.3.1: The two intervals are isomorphic.

We use the function f to simplify the study of Conjecture 2.1.8 for the Grasmannian permutations. So we must prove the following Proposition.

Proposition 3.3.7. *Let $u, v \in S_n^J, u \leq v$ such that exists $k \in [i]$ with $u(k) = v(k)$ or $u^{-1}(n) = v^{-1}(n)$ or $u^{-1}(1) = v^{-1}(1)$. Then:*

$$\Lambda(v) \setminus \Lambda(u) = \Lambda(f(v)) \setminus \Lambda(f(u)).$$

In particular, $\Lambda(v) \setminus \Lambda(u)$ is a Dyck cbs if and only if $\Lambda(f(v)) \setminus \Lambda(f(u))$ is.

Proof. Suppose that exist an integer $j \in [i]$ such that $u(j) = v(j)$.
In that case we have:

$$\Lambda(v) \setminus \Lambda(u) = (v(i) - u(i), \ldots, v(j+1) - u(j+1), 0, \ldots, 0),$$

where $v(k) - u(k) = 0$ for all $k < j$ because Λ is a Dyck cbs by hypothesis or $\Lambda(v) \setminus \Lambda(u) = (0, \ldots, 0, v(j-1) - u(j-1), \ldots, v(1) - u(1))$.
If we apply the function f we obtain:

$$f(u) = u(1), \ldots, u(j-1), u(j+1) - 1, \ldots, u(i) - 1, u(i+1)'_j, \ldots, u(n)'_j$$

$$f(v) = v(1), \dots, v(j-1), v(j+1)-1, \dots, v(i)-1, v(i)'_j \dots, v(n)'_j$$

and so:

$$\Lambda(f(v))\backslash\Lambda(f(u)) = (v(i)-u(i), \dots, v(j+1)-u(j+1), v(j-1)-u(j-1), \dots, v(1)-u(1))$$

$$= (v(i)-u(i), \dots, v(j+1)-u(i+1), 0, \dots, 0)$$

and this is equal to Λ.
Suppose now that $u^{-1}(1) = v^{-1}(1)$ or $u^{-1}(n) = v^{-1}(n)$ it's then easy to see that f conserves the Dyck cbs property because:

- In the first case when u, v have the 1 in the same position the function f deletes 1 and

  $$\Lambda_j(f(v))\backslash\Lambda_j(f(u)) = (v(j)-1)-(u(j)-1) = v(j)-u(j) = \Lambda_j(v)\backslash\Lambda_j(u),$$

  for all $j \in [i]$.

- In the second case when u, v have n in the same position the function f deletes n and doesn't rescale the other integers in u and v. Then also in that case:

  $$\Lambda(f(v)) \setminus \Lambda(f(u)) = \Lambda(v) \setminus \Lambda(u).$$

$\square$

<u>**Example**</u> **3.3.8.** *In S_5 there are 4 different pairs $u, v \in S_n^J$ such that $\overline{\mu}(u, v) \neq 0$:*

$$(12435, 14523); (13245, 34125)$$

$$(13524, 34512); (24135, 45123)$$

Applying f we can see that:

$$f([12435, 14523]) \cong f([13245, 34125]) \cong f([13524, 34512]) \cong f([24135, 45123])$$

$$\cong$$

$$[1324, 3412]$$

and by the previous Proposition:

$$\Lambda(12435, 14523) = \Lambda(13245, 34125) = \Lambda(13524, 34512) = \Lambda(24135, 45123)$$

$$=$$

$$\Lambda(1324, 3412) = (1, 2).$$

Using the previous Proposition we can study Conjecture 2.1.8 only for $u, v \in S_n^J$ such that $u \leq v$, $u(k) \neq v(k)$ for all $k \in [i]$ and $u^{-1}(1) \neq v^{-1}(1), u^{-1}(n) \neq v^{-1}(n)$. We now construct u, v with the following Theorem.

Theorem 3.3.9. *Let $u, v \in S_n^J$ with $u \leq v$ be such that $u(h) \neq v(h)$ for all $h \in [i]$, $u^{-1}(1) \neq v^{-1}(1)$ and $u^{-1}(n) \neq v^{-1}(n)$. Then:*

$$\overline{\mu}(u, v) \neq 0$$

$$\Updownarrow$$

$$v = v(1), v(2), \dots v(i-1), n, 1, \dots \widehat{v}(1), \dots, \widehat{v}(2), \dots, \widehat{v}(i-1) \dots, n-1$$

$$u = 1, v(1), v(2), \dots n-1, 2, \dots \widehat{v}(1), \dots, \widehat{v}(2), \dots, \widehat{v}(i-1), \dots, n.$$

Proof. In the following we use Corollary 1.5.25:

$$\Lambda \text{ is a Dyck cbs} \Leftrightarrow \overline{\mu}(u, v) \neq 0.$$

We start with some considerations:

<u>**Observation**</u> **3.3.10.** *$v(i) = n$ otherwise $v(n) = n$ but to have $u \leq v$ then also $u(n) = n$. This is in contradiction with $u^{-1}(n) \neq v^{-1}(n)$.*

<u>**Observation**</u> **3.3.11.** *$v(i+1) = 1$ otherwise $v(1) = 1$ but to have $u \leq v$ then also $u(1) = 1$. This is in contradiction with $u^{-1}(1) \neq v^{-1}(1)$.*

<u>**Observation**</u> **3.3.12.** *The only choice for $1, n$ that preserves $u^{-1}(1) \neq v^{-1}(1)$ and $u^{-1}(n) \neq v^{-1}(n)$ is the following:*

$$u(1) = 1, u(n) = n, v(i) = n, v(i+1) = 1.$$

We now consider a generic position $j \in [i]$ and will show that $u(j) = v(j-1)$. First of all we know that Λ is a Dyck cbs and so the possibilities are the following:

(*Case I:* $\boldsymbol{\Lambda_{j-1} = 1, \Lambda_j = 1}$): In such case it is easy to see that:

$$\begin{cases} v(j-1) - u(j-1) = 1, \\ \quad u(j) - u(j-1) = 1 \end{cases}$$

and so $u(j) = v(j-1)$.

(*Case II:* $\boldsymbol{\Lambda_{j-1} = 1, \Lambda_j = R}$): In such case:

$$\begin{cases} \quad v(j) - u(j) = R, \\ v(j) - v(j-1) = R \end{cases}$$

and so $u(j) = v(j-1)$.

(*Case III:* $\boldsymbol{\Lambda_{j-1} = R, \Lambda_j = 1}$): In such case:

$$\begin{cases} v(j-1) - u(j-1) = R, \\ \quad u(j) - u(j-1) = R \end{cases}$$

and so $u(j) = v(j-1)$.

So we know the structure of $u([i])$ starting from the structure of $v([i])$. It's easy to see that, by the conditions on the first i integers, we can also conclude that:

$$v(j) = u(j-1) \text{ for } j \in [i+2, n].$$

□

<u>Observation</u> 3.3.13. *Note that in the previous Proposition $u(k) \neq v(k)$ for all $k \in [n]$.*

We can now use the following Corollary:

Corollary 3.3.14. *Let $u, v \in S_n^J$ be such that $\Lambda(v) - \Lambda(u) = \Lambda(f(v)) - \Lambda(f(u))$ and $u(k) \neq v(k)$ for all $k \leq i$ then:*

- *$u(j) \neq v(j)$ for all $j \in [n]$,*

- *$u(i) = n-1, u(i+1) = 2$ and so $v(i-1) = n-1, v(i+2) = 2$.*

Example 3.3.15. *In the following we list, up to isomorphism induced by f, the different pairs of Grasmannian permutation $u, v \in S_n$, with $n \leq 7$, such that Λ is a Dyck cbs.*

- *In S_3 there isn't any such pair while in S_4 the only pair of Grasmannian permutation with Λ a Dick cbs is:*

$$[1324, 3412].$$

- *In S_5 there are 4 cases but all are isomorphic by f to the poset $[1324, 3412]$.*

- *In S_6 there are 14 cases but up to isomorphism by f we have only three cases:*

$$[1324, 3412]; [135246, 356124]; [145236, 456123].$$

- *In S_7 there are 40 cases but up to isomorphism by f we reduce to three previous cases:*

$$[1324, 3412]; [135246, 356124]; [145236, 456123].$$

So note that to study the Conjecture 2.1.8 for S_n^J and $1 \leq l(u, v) \leq 7$ we must only consider three different posets. This is a very useful simplification. Here we list these three posets:

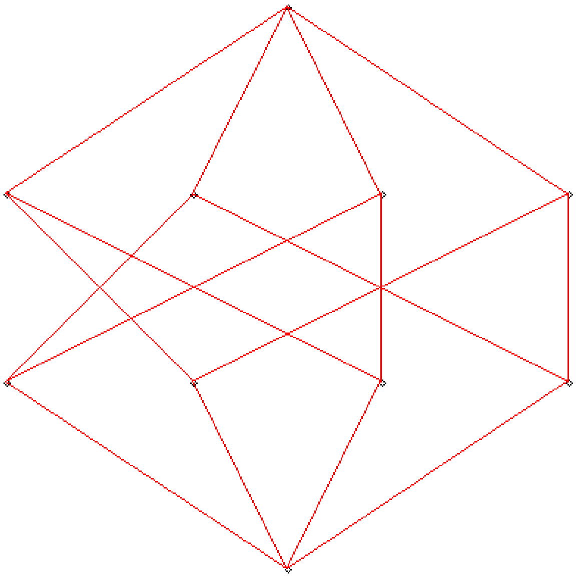

Figure 3.3.2: $[1324, 3412]$.

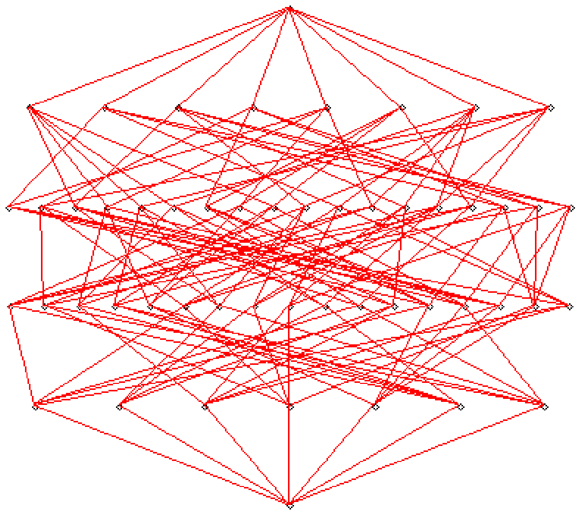

Figure 3.3.3: $[135246, 356124]$.

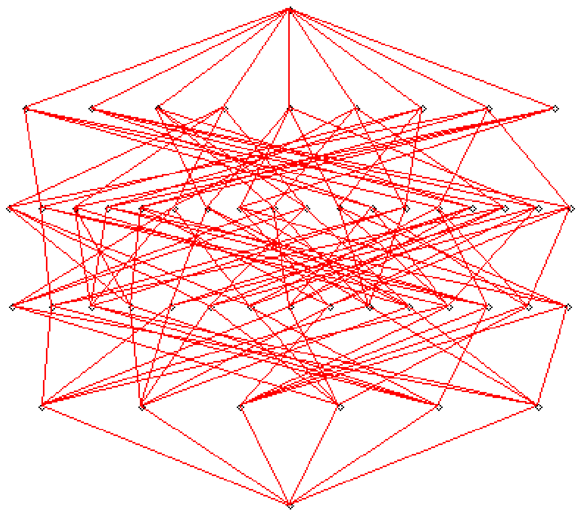

Figure 3.3.4: $[145236, 456123]$.

We can now prove the main result of this Section.

Theorem 3.3.16. *Let* $u, v \in S_n^J$ *be such that* $\overline{\mu}(u, v) \neq 0$ *then* $[u, v]$ *doesn't have a special matching*

Proof. We begin by counting the number of coatoms in $[u, v]$. Recall that:

$$v = v(1), v(2), \ldots, n-1, n, 1, 2 \ldots v(n)$$

and

$$u = 1, v(1), v(2), \ldots, v(i-2), n-1, 2 \ldots n.$$

It's easy to see that to have $v' \triangleleft v$ we can exchange on v:

- $v(1)$ with the elements of the set $\{2, \ldots v(1)-1\}$,
- $v(2)$ with $\{2, \ldots v(2)-1\} \setminus \{v(1)\}$,
- $\vdots$

- $v(i-1) = n-1$ with $\{2, \ldots, n-1\} \setminus \{v(1), \ldots v(i-2)\}$,
- n with all the $v(j)$ for $j \in [i+2, n]$,
- 1 with $v(j)$ for all $j \in [i]$

and so:

$$c(u, v) = (v(1)-2)+(v(2)-3)+\ldots+(v(i-2)-(i-1))+(v(i-1)-i)(n-i-1)+i =$$
$$= \sum_{j=1}^{i-1}(v(j)) - (\sum_{k=2}^{i} k) + n - 1.$$

Observation 3.3.17. *For each coatom $v' \lhd v$ we can say that:*

$$c(u, v') \leq c(u, v) - 1. \tag{11}$$

This is because the only way to have a $v'' \lhd v'$ is to change an element $v(h)$ with $h < i$ with another $v(s) < v(h)$ as before. So each $v(h)$ can be exchanged at most with:

$$\{2, \ldots v(h) - 1\} \setminus \{v(1), \ldots, v(h-1)\},$$

as before. We lose at least the change for pass from v to v' and so we have disequation 11.

So if we take a coatom:

$$v' = v(1), \ldots v(j-1), v(k), v(j+1), \ldots, v(i-1), n, 1, \ldots, v(k-1), v(j), v(k+1), \ldots$$

in that case for have $v'' \lhd v'$ we have the same change of previous minus at least the change between the elements of the set $\{v(j+1), \ldots, n\}$ with $\{v(s) : s \in [k+1, n], v(s) < v(j)\}$. So we have that:

$$c(u, v') \leq c(u, v) - 1 - (i-j)|\{v(s) : s \in [k+1, n], v(s) < v(j)\}|,$$

now for conclude the proof we must consider the follow particular case:
(*Case I.a:* $\boldsymbol{i - j = 0}$):
This is the case where we move $v(i) = n$. In such case we have:

$$v_a = v(1), \ldots, n-1, v(k), 1, \ldots, v(k-1), n, v(k+1), \ldots$$

For have $v' \lhd v_a$ we can change $n-1$ with $v(k)$ but no change $n-1$ with the elements of the set $\{1, n-2\} \setminus \{v(1), \dots, v(i-2)\}$. So we have that:

$$c(u, v_a) \leq c(u, v) - 1 - (n - 2 - i + 2) = c(u, v) - n + i - 1.$$

So for no have $c(u,v)-1 > c(u, v_a)$ we must $i = n$ but by hypothesis $i \in [n-1]$.

(*Case I.b:* $|\{v(s) : s \in [k+1, n], v(s) < v(j)\}| = 0$):
This is the case where:

$$v_b = v(1), \dots v(j-1), v(k), v(j+1), \dots, v(i-1), n, 1, \dots, v(k-1), v(j), v(k+1), \dots$$

and $v(k) < h < v(j)$ are all in the first $j-1$ position of v.
We can see that for have $v' \lhd v_b$ we can't change elements of the set $\{v(j+1), \dots, n\}$ with elements of the set $\{1, \dots, v(k-1)\}$ otherwise for Theorem 1.3.10 we have $v' \ngeq u$. In such case we have:

$$c(u, v_b) \leq c(u, v) - 1 - (i - j)|\{1, \dots, v(k-1)\}|.$$

The particular case $i - j = 0$ is the previous so we can only consider:

$$|\{1, \dots, v(k-1)\}| = 0.$$

In such situation v_b is on the form:

$$v_b = v(1), \dots v(j-1), 1, v(j+1), \dots, v(i-1), n, v(j), \dots$$

we can see that the elements of the set $\{v(j+1), \dots, v(i-1)\}$ can't be moved otherwise we obtain $v' \lhd v_b$ such that $v' \ngeq u$. So:

$$c(u, v_b) \leq c(u, v) - 1 - (i - j).$$

We can again consider the particular case $i - j = 0$ but this is again the case I.a. $\square$

We now show Theorem 3.3.16 with an example.

Example 3.3.18. *Let* $u = 145236$ *and* $v = 456123$ *in* $S_6^{(S \setminus \{s_3\})}$ *following the proof of theorem we have that:*

$$v(1) = 4, v(2) = 5, n = 6$$

and so:

$$\sum_{j=1}^{2}(v(j)) - (\sum_{k=2}^{3} k) + n - 1 = 4 + 5 + -(2 + 3) + 5 = 9.$$

And in the following figure we can see that $|c(u, v)| - 1 > |c(u, v')|$ *for all* $v' \in c(u, v)$

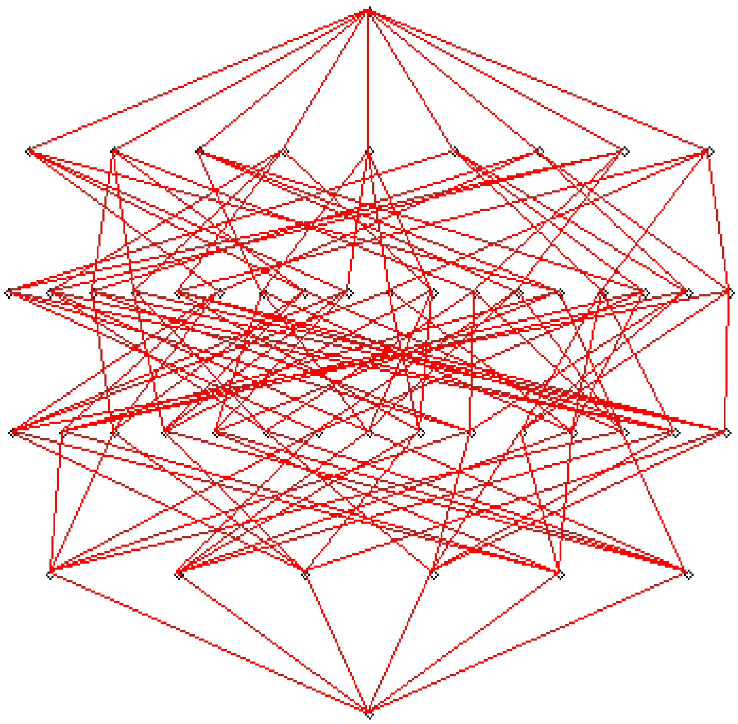

Figure 3.3.4.

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