Francesco Bei

L^2 [U+2500] de Rham-Hodge and L^2 [U+2500] Atiyah-Bott-Lefschetz theorems on stratified pseudomanifolds

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L^2-de Rham-Hodge and L^2-Atiyah-Bott-Lefschetz theorems on stratified pseudomanifolds

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Introduction

In the last thirty years, starting with the seminal papers of Cheeger and Goresky MacPherson[**23**], [**24**], [**25**] and [**27**], stratified pseudomanifolds turned out to be a very interesting and reach field of interactions betweens analysis and topology. In particular, on stratified pseudomanifolds, the L^2-de Rham-Hodge theory and its relationships with intersection cohomology of Goresky-MacPherson turned out to be a very important topic. Roughly speaking the question is the following:
given a compact and smoothly stratified pseudomanifold X is there a riemannian metric g over its regular part $reg(X)$ such that the L^2 (maximal or minimal) cohomology of $reg(X)$ relative to g is isomorphic to the intersection cohomology of X relative to some perversity? Is it possible to state a Hodge theorem for these groups? In other words is there a self-adjoint extension of $\Delta_i : \Omega^i_c(reg(X)) \rightarrow \Omega^i_c(reg(X))$ such that its kernel is isomorphic to some L^2 cohomology group?
The first results in this direction were obtained by Cheeger in his celebrated papers [**23**], [**24**] and [**25**]. In these papers Cheeger introduced the notion of adapted riemannian metric over the regular part of a stratified pseudomanifold X and he proved that

$$H^i_{2,max}(reg(X), g) \cong I^{\underline{m}}H^i(X, \mathbb{R})$$

that is the L^2 maximal de Rham cohomology of $(reg(X), g)$ is isomorphic the intersection cohomology of X relative to the lower middle perversity. Moreover, when X is a Witt space, Cheeger also proved that

$$\Delta_i : \Omega^i_c(reg(X)) \rightarrow \Omega^i_c(reg(X))$$

as unbounded and densely defined operator on $L^2\Omega^i(reg(X), g)$ is essentially self-adjoint. Subsequently many authors have dealt with these problems; we can cite for example the work of Nagase [**57**] and [**58**]. In these papers the author showed that if p is a perversity such that $p \leq \underline{m}$ then on $reg(X)$ there is a riemannian metric g such that its L^2 de Rham maximal cohomology is isomorphic to the intersection cohomology of X associated to the perversity p. Other examples are provided by the work of Hsiang and Pati [**43**]. In this paper the authors proved the Cheeger-Goresky-MacPherson's conjecture for a class of complex projective surfaces endowed with the Fubini-Study metric. Saper's paper [**63**] which is devoted to the L^2 cohomology of the Weill-Peterson metric, Saper and Stern's paper [**64**] in which the authors proved the Zucker conjecture (see [**72**]), the works of Hunsicker [**44**] and Hunsicker and Mazzeo [**45**].

The first part of our thesis is devoted to a problem of this kind. More precisely we considerer a compact and oriented smoothly stratified pseudomanifold X. Over its regular part, $reg(X)$, we introduce a class of riemannian metrics which we call quasi-edge metrics with weights and which generalize the metrics used by Cheeger in [**23**]. Our goal is to prove an L^2-de Rham-Hodge theorem for these metrics.
The first part of the thesis is structured in the following way: in the first chapter we recall the background, that is Hilbert complexes and L^2 cohomology, stratified pseudomanifolds, intersection cohomology, Thom-Mather stratifications and we introduce the particular class of riemannian metrics we will use in the second and in the third chapter and that we call quasi-edge metrics with weights. The first two sections of the second chapter are devoted to the calculation of the L^2 maximal cohomology of a cone over a riemannian manifold while in the third section we prove the main theorems of the second chapter: in the first theorem we will show that if X is a compact, oriented and smoothly stratified pseudomanifold with a Thom-Mather stratification and if g is a quasi-edge metric with weights on $reg(X)$ then it is possible to associate two general perversities, p_g and q_g, to the metric g such that the following Hodge-de Rham isomorphisms hold:

$$I^{q_g}H^i(X,\mathcal{R}_0) \cong H^i_{2,max}(reg(X),g) \cong \mathcal{H}^i_{abs}(reg(X),g) \tag{0.1}$$

$$I^{p_g}H^i(X,\mathcal{R}_0) \cong H^i_{2,min}(reg(X),g) \cong \mathcal{H}^i_{rel}(reg(X),g) \tag{0.2}$$

This theorem generalizes the de Rham theorems proved in [**23**], [**44**] and [**45**]. Our next result, the second theorem of section 3, gives a partial answer to the inverse question: given a general perversity p on X is there a riemannian metric g over $reg(X)$ such that the L^2 (maximal or minimal) cohomology of $(reg(X),g)$ is isomorphic to the intersection cohomology of X associated to p? Generalizing the results obtained by Nagase in [**57**] and [**58**] we show that:

(1) if p is a general perversity on X in the sense of Friedman such that $p \geq \overline{m}$, where $\overline{m}$ is the upper middle perversity, and such that $p(Y) = 0$ for each stratum with $cod(Y) = 1$, then it is possible to construct on $reg(X)$ a quasi edge metric with weights g such that (0.2) holds.
(2) if q is a general perversity on X in the sense of Friedman such that $p \leq \underline{m}$, where $\underline{m}$ is the lower middle perversity, and such that $p(Y) = -1$ for each stratum with $cod(Y) = 1$, then it is possible to construct on $reg(X)$ a quasi edge metric with weights g such that (0.1) holds.

These results were obtained in [**6**].

The third chapter of the thesis is devoted to the study of the following groups:

$$H^i_{2,m\to M}(M,g),\ \overline{H}^i_{2,m\to M}(M,g) \tag{0.3}$$

where (M,g) is an open, oriented and incomplete riemannian manifold, the groups are defined as the image

$$H^i_{2,min}(M,g) \longrightarrow H^i_{2,max}(M,g),\ \overline{H}^i_{2,min}(M,g) \longrightarrow \overline{H}^i_{2,max}(M,g)$$

and the maps are the natural maps induced by the inclusion of complexes $(L^2\Omega^i(M,g), d_{min,i}) \subset (L^2\Omega^i(M,g), d_{max,i})$. The reason behind this study is given by the fact that, when (M,g) is an open and incomplete riemannian manifold, then usually Poincaré duality doesn't hold for the groups $H^i_{2,max}(M,g)$ $i = 0,...,n$ and $H^i_{2,min}(M,g)$ $i = 0,...,n$. As we will see this is not true for the groups: $\overline{H}^i_{2,m\rightarrow M}(M,g)$ $i = 0,...,n$. More precisely the main results we obtained about these groups can be summarized in the following way:

1) If each vector space $\overline{H}^i_{2,m\rightarrow M}(M,g)$ is finite dimensional then Poincaré duality holds for the sequence $\overline{H}^i_{2,m\rightarrow M}(M,g),\ i = 0,...,n.$

2) If $d_{min,i}$ has closed range for each i then there exists a Hilbert complex $(L^2\Omega^i(M,g), d_{\mathfrak{m},i})$ such that

$$H^i_{2,\mathfrak{m}}(M,g) = H^i_{2,m\rightarrow M}(M,g)$$

where $H^i_{2,\mathfrak{m}}(M,g)$ $i = 0,...,n$ are the cohomology groups of the complex $(L^2\Omega^i(M,g), d_{\mathfrak{m},i})$.

3) If $(L^2\Omega^i(M,g), d_{min,i})$ is a Fredholm complex then also $(L^2\Omega^i(M,g), d_{\mathfrak{m},i})$ is a Fredholm complex. This implies that for each i there exists a self-adjoint extension of $\Delta_i : \Omega^i_c(M) \rightarrow \Omega^i_c(M)$, the $i-$th Laplacian acting on smooth $i-$forms with compact support, that we label $\Delta_{\mathfrak{m},i}$, such that $\Delta_{\mathfrak{m},i}$ is a Fredholm operator on its domain with the graph norm and

$$Ker(\Delta_{\mathfrak{m},i}) \cong H^i_{2,m\rightarrow M}(M,g).$$

In particular Poincaré duality holds for the sequence:

$$H^i_{2,m\rightarrow M}(M,g),\ i = 0,...,n.$$

Moreover, when (M,g) is an open and oriented riemannian manifold of dimension $4n$ such that $\mathrm{im}(\overline{H}^{2n}_{2,min}(M,g) \rightarrow \overline{H}^{2n}_{2,max}(M,g))$ is finite dimensional, we introduce an L^2-signature defined as the signature of the non degenerate pairing

$$\overline{H}^{2n}_{2,m\rightarrow M}(M,g) \times \overline{H}^{2n}_{2,m\rightarrow M}(M,g) \longrightarrow \mathbb{R} \tag{0.4}$$

$$([\omega],[\eta]) \mapsto \int_M \omega \wedge \eta$$

where $\omega, \eta \in Ker(d_{min,2n})$. Using the fact that, if $\mathrm{im}(\overline{H}^i_{2,min}(M,g) \rightarrow \overline{H}^i_{2,max}(M,g))$ is finite dimensional then also $\mathrm{im}(H^i_c(M) \rightarrow H^i(M))$ is finite dimensional, we show that, if (M,g) admits the L^2-signature defined above, then it admits a topological signature as well, defined as the signature of the following pairing:

$$\mathrm{im}(H^i_c(M) \to H^i(M)) \times \mathrm{im}(H^i_c(M) \to H^i(M)) \longrightarrow \mathbb{R} \tag{0.5}$$
$$([\alpha],[\beta]) \mapsto \int_M \alpha \wedge \beta$$

where α, β are closed forms with compact support.
In the rest of the chapter we describe some geometric and topological applications of the above results. In particular we apply them when M is the regular part of a compact, oriented and smoothly stratified pseudomanifold X and g is a quasi-edge metric with weights on $reg(X)$. In this context, as we will see, the kernel of the operator $\Delta_{\mathfrak{m},i}$, previously introduced, admits a topological interpretation:

$$Ker(\Delta_{\mathfrak{m},i}) \cong \mathrm{im}(I^{q_g}H^i(X,\mathcal{R}_0) \to I^{p_g}H^i(X,\mathcal{R}_0)). \tag{0.6}$$

Moreover we have also the following index theorem:

$$\mathrm{ind}((d_{\mathfrak{m}} + d^*_{\mathfrak{m}})_{ev}) = I^{p_g \to q_g}\chi(X,\mathcal{R}_0) \tag{0.7}$$

where

$$I^{p_g \to q_g}\chi(X,\mathcal{R}_0) = \sum_i (-1)^i dim(\mathrm{im}(I^{p_g}H^i(X,\mathcal{R}_0) \to I^{q_g}H^i(X,\mathcal{R}_0)))$$

and $(d_{\mathfrak{m}} + d^*_{\mathfrak{m}})_{ev}$ is the extension of

$$d + \delta : \bigoplus_i \Omega^{2i}_c(M) \to \bigoplus_i \Omega^{2i+1}_c(M)$$

defined by

$$(d_{\mathfrak{m}} + d^*_{\mathfrak{m}})_{ev}|_{L^2\Omega^{2i}(M,g)} := d_{\mathfrak{m},2i} + d^*_{\mathfrak{m},2i-1}$$

which is a Fredholm operator on its domain endowed with the graph norm.
We remark as well that in this framework the L^2 signature introduced previously admit a topological interpretation because it coincides with the **perverse signature** introduced by Friedman and Hunsicker in [**34**], that is

$$\sigma_2(reg(X), g) = \sigma_{p_g \to q_g}(X).$$

Finally, among the others applications, we get a topological obstruction to existence of a riemannian metric g with finite L^2 cohomology over an open and oriented manifold M and we get some properties of $\Delta^{\mathcal{F}}_i$, the Friedrichs extension of Δ_i.
More precisely we prove that, if (M,g) is an open, oriented and incomplete riemannian manifold such that $(L^2\Omega^i(M,g), d_{max,i})$, or equivalently $(L^2\Omega^i(M,g), d_{min,i})$, is a Fredholm complex, then for each i, $\Delta^{\mathcal{F}}_i$, the Friedrichs extension of $\Delta_i : \Omega^i_c(M) \to \Omega^i_c(M)$, is a **Fredholm** operator on its domain endowed with the graph norm. Moreover it satisfies:

$$Ker(\Delta^{\mathcal{F}}_i) = Ker(\Delta_{min,i}) \text{ and } ran(\Delta^{\mathcal{F}}_i) = ran(\Delta_{max,i}).$$

This last result applies, for example, when M is the regular part of a compact and smoothly stratified pseudomanifold with a Thom-Mather stratification. These results were obtained in [**8**].

The second part of the thesis is devoted to the Atiyah-Bott-Lefschetz theorem over a compact manifold with conical singularities. The Atiyah-Bott-Lefschtz theorem, see [**3**], is a fundamental result of elliptic theory on closed manifolds proved by Atiyah and Bott in 1969. It provides a formula for the Lefschetz number of a geometric endomorphism acting on an elliptic complex over a closed manifold. More precisely let M be a closed manifold and consider an elliptic complex over M:

$$0 \to C^{\infty}_c(M, E_0) \stackrel{P_0}{\to} C^{\infty}_c(M, E_1) \stackrel{P_1}{\to} ... \stackrel{P_{n-1}}{\to} C^{\infty}_c(M, E_n) \stackrel{P_n}{\to} 0 \tag{0.8}$$

Let $T = (T_0, ..., T_n)$ be a geometric endomorphism of the above complex where geometric means that, for each $i = 0, ..., n$:

$$T_i = \phi_i \circ f^*$$

where $f^* : C^{\infty}(M, E_i) \to C^{\infty}(M, f^*E_i)$ is the natural map induced by a smooth map $f : M \to M$ and $\phi_i : f^*E_i \to E_i$ is a bundle homomorphism. Then, assuming that f has only simple fixed points, Atiyah and Bott provided a formula for the Lefschetz number of T, that is

$$L(T) := \sum_i (-1)^i \operatorname{Tr}(T_i^* : H^i(M, E_*) \to H^i(M, E_*))$$

showing that

$$L(T) = \sum_{p=f(p)} \sum_i \frac{(-1)^i \operatorname{Tr}(\phi_i)}{|det(Id - d_p f)|}.$$

Moreover in [**4**] Atiyah and Bott applied their formula to the main complexes arising in differential geometry, that is the de Rham complex, the Dolbeault complex, the signature and the spin complex, obtaining several interesting applications. In particular, for the de Rham complex, they obtained a new proof of the Lefschetz's fixed point theorem for compact and smooth manifolds, that is given a map $f : M \to M$ with only simple fixed points, then its Lefschetz number is given by the formula:

$$L(f) = \sum_{p=f(p)} \operatorname{sgn} det(Id - d_p f).$$

Another important application is the holomorphic Lefschetz formula. Given a complex manifold M and an holomorphic map $f : M \to M$ with only simple fixed points, they proved that :

$$L_{\overline{\partial}}(f) = \sum_{f(p)=p} \frac{1}{det_{\mathbb{C}}(Id - d_p f)}.$$

The results recalled above inspired various works in the last forty years. In particular several papers have been devoted to the applications of the Atiyah-Bott-Lefschetz theorem, to investigate new approaches to its proof and to find some generalizations. For example in [**11**], [**36**], [**49**], [**50**] and [**61**] the heat kernel approach is developed, while in [**10**] an approach using probabilistic methods is employed. In [**15**], [**59**], [**60**] [**66**],[**68**], [**70**] and [**71**] the Atiyah-Bott-Lefschetz theorem is extended to some kind of manifolds that are not closed: for example [**59**] is devoted to the case of elliptic conic operators on manifolds with conical singularities defined on suitable Sobolev spaces, in [**66**] the case of a manifold with cylindrical ends is studied and

[**68**] concerns the case of a complex of Hecke operators over an arithmetic variety. In particular the use of the heat kernel turned out to be a powerful tool in order to get alternative proofs and extensions of the theorem. Since the heat kernel associated to a conic operator has been intensively studied in the last thirty years, e.g. [**18**], [**19**] [**20**], [**21**], [**25**],[**51**] and [**56**], it is interesting to explore its applications in this context as well. This is exactly the goal of the second part of this thesis:
to prove an Atiyah-Bott-Lefschetz theorem for the L^2-Lefschetz numbers (maximal and minimal) associated to a geometric endomorphism of an elliptic complex of differential cone operators using a heat kernel approach.
Also this part is divided in three chapters. The first one is devoted to background material such as differential cone operators, elliptic complexes and heat kernel. In the second chapter we define the class of geometric endomorphisms we consider in the rest of the text, we define the L^2-Lefschetz numbers $L_{2,max/min}(T)$ and we prove several properties about them. In the third chapter some explicit formulas for the contribution given by the singular points to the Lefschetz numbers are proved while the last chapter contains the application of the previous results to the L^2 de Rham complexes. Our geometric framework is the following: given a compact and orientable manifold with isolated conical singularities X, we consider over its regular part, $reg(X)$ (usually labeled M), a complex of elliptic conic differential operators:

$$0 \to C_c^{\infty}(M, E_0) \stackrel{P_0}{\to} C_c^{\infty}(M, E_1) \stackrel{P_1}{\to} ... \stackrel{P_{n-1}}{\to} C_c^{\infty}(M, E_n) \stackrel{P_n}{\to} 0 \tag{0.9}$$

and a geometric endomrphism $T = (T_0, ..., T_n)$ of the complex, that is for each $i = 0, .., n$, $T_i = \phi_i \circ f^*$ where $f : X \to X$ is an isomorphism and $\phi_i : f^*E_i \to E_i$ is a bundle homomorphism. Using a conic metric over M we associate to (0.9) two Hilbert complexes $(L^2(M, E_i), P_{max/min,i})$ and then we prove the following properties:

- The cohomology groups of $(L^2(M, E_i), P_{max/min,i})$ are finite dimensional.
- If f satisfies some conditions (see definition 5.1) then each T_i extends to a bounded map acting on $L^2(M, E_i)$ such that $(T_{i+1} \circ P_{max/min,i})(s) = (P_{max/min} \circ T_i)(s)$ for each $s \in \mathcal{D}(P_{max/min,,i})$.

In this way we can associate to T and (0.9) two L^2-Lefschetz numbers $L_{2,max/min}(T)$ defined as

$$L_{2,max}(T) := \sum_{i=0}^{n} (-1)^i \operatorname{Tr}(T_i^* : H^i_{2,max}(M, E_i) \to H^i_{2,max}(M, E_i)) \tag{0.10}$$

and analogously

$$L_{2,min}(T) := \sum_{i=0}^{n} (-1)^i \operatorname{Tr}(T_i^* : H^i_{2,min}(M, E_i) \to H^i_{2,min}(M, E_i)) \tag{0.11}$$

Subsequently, using the operators $\mathcal{P}_i := P_i^t \circ P_i + P_{i-1} \circ P_{i-1}^t$, their absolute and relative extensions and the fact that the respective heat operators $e^{-t\mathcal{P}_{abs/rel,i}} : L^2(M, E_i) \to L^2(M, E_i)$ are trace-class operators we prove the following results:

$$L_{2,max/min}(T)=\sum_{i=0}^{n}(-1)^i \operatorname{Tr}(T_i\circ e^{-t\mathcal{P}_{abs/rel,i}}).$$

In particular, in the expression above, the term on the right end side does not depend on t. Moreover if $Fix(f)$, the fixed points of f, is made only by simple fixed points (condition which in turn implies that each $p\in Fix(f)$ is an isolated fixed point) then we have:

$$L_{2,max/min}(T)=\sum_{q\in Fix(f)}\sum_{i=0}^{n}(-1)^i\int_{U_q}\operatorname{tr}(\phi_i\circ k_{abs/rel,i}(t,f(x),x))dvol_g$$

where $\phi_i\circ k_{abs/rel,i}(t,f(x),x)$ is the smooth kernel of $T_i\circ e^{-t\mathcal{P}_{abs/rel,i}}$ and U_q is an open neighborhood of q. Under some additional hypothesis, in particular that f takes the form $(rA(p),B(p))$ in a suitable neighborhood of each $q\in sing(X)$ (see theorem 6.3), we have the following formulas:

$$L_{2,max/min}(T)=\sum_{p\in Fix(f)\cap M}\sum_{i=0}^{n}\frac{(-1)^i\operatorname{Tr}(\phi_i)}{|det(Id-d_q(f))|}+ \tag{0.12}$$

$$+\sum_{q\in sing(X)}\sum_{i=0}^{n}(-1)^i\zeta_{T_i,q}(\mathcal{P}_{abs/rel,i})(0)$$

where each $\zeta_{T_i,q}(\mathcal{P}_{abs/rel,i})(0)$ satisfies :

$$\zeta_{T_i,q}(\mathcal{P}_{abs/rel,i})(0)= \tag{0.13}$$

$$=\frac{1}{2\nu}\int_0^\infty\frac{dx}{x}\int_{L_q}\operatorname{tr}(\phi_i\circ e^{-x\mathcal{P}_{abs/rel,i}}(A(p),B(p),1,p))dvol_h.$$

Finally, in the last part of the paper, we apply the previous results to the de Rham complex. We get an analytic construction of the Lefschetz numbers arising in intersection cohomology and a topological interpretation of the contributions given by the singular points to the L^2-Lefschetz numbers. In particular, under suitable conditions, we prove the following formula:

$$I^{\underline{m}}L(f)=L_{2,max}(T)=\sum_{q\in Fix(f)\cap reg(X)}\operatorname{sgn} det(Id-d_qf)+ \tag{0.14}$$

$$+\sum_{q\in sing(X)}\sum_{i<\frac{m+1}{2}}(-1)^i\operatorname{Tr}(B^*:H^i(L_q)\to H^i(L_q)).$$

where $I^{\underline{m}}L(f)$ is the intersection Lefschetz number arising in intersection cohomology, T is the endomorphism of $(L^2\Omega^i(M,g),d_{max,i})$ induced by f and B is a diffeomorphism of the link L_q such that, in a neighborhood of q, f satisfies $f=(rA(p),B(p))$. In particular from (0.14) we get:

$$\sum_{i=0}^{m+1}(-1)^i\zeta_{T_i,q}(\Delta_{abs,i})(0)=\sum_{i<\frac{m+1}{2}}(-1)^i\operatorname{Tr}(B^*:H^i(L_q)\to H^i(L_q)). \tag{0.15}$$

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Part 1

L^2-de Rham-Hodge theorems on stratified pseudomanifolds

CHAPTER 1

Background

This chapter contains the background material. It is dived in three sections: in the first section we recall briefly the notion of Hilbert complex and we prove some properties about it. The second section is devoted to a rapid introduction to intersection cohomology. Finally, in the third section, we introduce the notion of Thom-Mather stratification and the class of riemannian metrics we use in the second and in the third chapter.

1. Hilbert complexes

In this first section we start by recalling the notion of Hilbert complex and how it appears in riemannian geometry. It is a very useful abstract framework to analyze the general properties satisfied by the natural L^2 complexes arising in riemannian geometry. The theory is fully developped in [**16**] and we refer to it for a deeper discussion on this subject and for the proofs.

Definition 1.1. A Hilbert complex is a complex, (H_*, D_*) of the form:

$$0 \rightarrow H_0 \stackrel{D_0}{\rightarrow} H_1 \stackrel{D_1}{\rightarrow} H_2 \stackrel{D_2}{\rightarrow} ... \stackrel{D_{n-1}}{\rightarrow} H_n \rightarrow 0, \tag{1.1}$$

where each H_i is a separable Hilbert space and each map D_i is a closed operator called the differential such that:

(1) $\mathcal{D}(D_i)$, the domain of D_i, is dense in H_i.
(2) $ran(D_i) \subset \mathcal{D}(D_{i+1})$.
(3) $D_{i+1} \circ D_i = 0$ for all i.

The cohomology groups of the complex are $H^i(H_*, D_*) := Ker(D_i)/ran(D_{i-1})$. If the groups $H^i(H_*, D_*)$ are all finite dimensional we say that it is a *Fredholm complex*.

Given a Hilbert complex there is a dual Hilbert complex

$$0 \leftarrow H_0 \stackrel{D_0^*}{\leftarrow} H_1 \stackrel{D_1^*}{\leftarrow} H_2 \stackrel{D_2^*}{\leftarrow} ... \stackrel{D_{n-1}^*}{\leftarrow} H_n \leftarrow 0, \tag{1.2}$$

defined using $D_i^* : H_{i+1} \rightarrow H_i$, the Hilbert space adjoints of the differentials $D_i : H_i \rightarrow H_{i+1}$. The cohomology groups of $(H_j, (D_j)^*)$, the dual Hilbert complex, are

$$H^i(H_j, (D_j)^*) := Ker(D_{n-i-1}^*)/ran(D_{n-i}^*).$$

For all i there is also a laplacian $\Delta_i = D_i^* D_i + D_{i-1} D_{i-1}^*$ which is a self-adjoint operator on H_i with domain

$$\mathcal{D}(\Delta_i) = \{v \in \mathcal{D}(D_i) \cap \mathcal{D}(D_{i-1}^*) : D_i v \in \mathcal{D}(D_i^*), D_{i-1}^* v \in \mathcal{D}(D_{i-1})\} \tag{1.3}$$

and nullspace:

$$\mathcal{H}^i(H_*, D_*) := ker(\Delta_i) = Ker(D_i) \cap Ker(D_{i-1}^*). \tag{1.4}$$

The following propositions are standard results for these complexes. The first result is a weak Kodaira decomposition:

Proposition 1.2. *[[16], Lemma 2.1] Let (H_i, D_i) be a Hilbert complex and $(H_i, (D_i)^*)$ its dual complex, then:*

$$H_i = \mathcal{H}^i \oplus \overline{ran(D_{i-1})} \oplus \overline{ran(D_i^*)}.$$

The reduced cohomology groups of the complex are:

$$\overline{H}^i(H_*, D_*) := Ker(D_i)/(\overline{ran(D_{i-1})}).$$

By the above proposition there is a pair of weak de Rham isomorphism theorems:

$$\begin{cases} \mathcal{H}^i(H_*, D_*) \cong \overline{H}^i(H_*, D_*) \\ \mathcal{H}^i(H_*, D_*) \cong \overline{H}^{n-i}(H_*, (D_*)^*) \end{cases} \tag{1.5}$$

where in the second case we mean the cohomology of the dual Hilbert complex.
The complex (H_*, D_*) is said *weak Fredholm* if $\mathcal{H}_i(H_*, D_*)$ is finite dimensional for each i. By the next propositions it follows immediately that each Fredholm complex is a weak Fredholm complex.

Proposition 1.3. *[[16], corollary 2.5] If the cohomology of a Hilbert complex (H_*, D_*) is finite dimensional then, for all i, $ran(D_{i-1})$ is closed and $H^i(H_*, D_*) \cong \mathcal{H}^i(H_*, D_*)$.*

Proposition 1.4 ([16], corollary 2.6). *A Hilbert complex (H_j, D_j), $j = 0, ..., n$ is a Fredholm complex (weak Fredholm) if and only if its dual complex, (H_j, D_j^*), is Fredholm (weak Fredholm). If it is Fredholm then*

$$\mathcal{H}_i(H_j, D_j) \cong H_i(H_j, D_j) \cong H_{n-i}(H_j, (D_j)^*) \cong \mathcal{H}_{n-i}(H_j, (D_j)^*). \tag{1.6}$$

Analogously in the the weak Fredholm case we have:

$$\mathcal{H}_i(H_j, D_j) \cong \overline{H}_i(H_j, D_j) \cong \overline{H}_{n-i}(H_j, (D_j)^*) \cong \mathcal{H}_{n-i}(H_j, (D_j)^*). \tag{1.7}$$

Proposition 1.5. *A Hilbert complex (H_j, D_j), $j = 0, ..., n$ is a Fredholm complex if and only if for each i the operator Δ_i defined in (1.3) is a Fredholm operator on its domain endowed with the graph norm.*

Proof. See [65], lemma 1 pag 203. □

Now we recall another result which shows that it is possible to compute the cohomology groups of an Hilbert complex using a core subcomplex

$$\mathcal{D}^\infty(H_i) \subset H_i.$$

For all i we define $\mathcal{D}^\infty(H_i)$ as consisting of all elements η that are in the domain of Δ_i^l for all $l \geq 0$.

Proposition 1.6 ([16], Theorem 2.12). *The complex $(\mathcal{D}^\infty(H_i), D_i)$ is a subcomplex quasi-isomorphic to the complex (H_i, D_i)*

As it is well known, riemannian geometry offers a framework in which Hilbert and (sometimes) Fredholm complexes can be built in a natural way. The rest of this subsection is devoted to recall these constructions.
Let (M,g) be an open and oriented riemannian manifold of dimension m and let $E_0, ..., E_n$ be vector bundles over M. For each $i = 0, ..., n$ let $C_c^\infty(M, E_i)$ be the space of smooth section with compact support. If we put on each vector bundle a metric h_i $i = 0, ..., n$ the we can construct in a natural way a sequences of Hilbert space $L^2(M, E_i)$, $i = 0, ..., n$ as the completion of $C_c^\infty(M, E_i)$. Now suppose that we have a complex of differential operators :

$$0 \to C_c^\infty(M, E_0) \stackrel{P_0}{\to} C_c^\infty(M, E_1) \stackrel{P_1}{\to} ... \stackrel{P_{n-1}}{\to} C_c^\infty(M, E_n) \to 0, \tag{1.8}$$

To turn this complex into a Hilbert complex we must specify a closed extension of P_* that is an operator between $L^2(M, E_*)$ and $L^2(M, E_{*+1})$ with closed graph which is an extension of P_*. We start recalling the two canonical closed extensions of P.

Definition 1.7. The maximal extension P_{max}; this is the operator acting on the domain:

$$\mathcal{D}(P_{max,i}) = \{\omega \in L^2(M, E_i) : \exists\ \eta \in L^2(M, E_{i+1}) \tag{1.9}$$

$$s.t. < \omega, P_i^t \zeta >_{L^2(M,E_i)} = < \eta, \zeta >_{L^2(M,E_{i+1})}\ \ \forall\ \zeta \in C_0^\infty(M, E_{i+1})\}$$

where P_i^t is the formal adjoint of P_i.
In this case $P_{max,i}\omega = \eta$. In other words $\mathcal{D}(P_{max,i})$ is the largest set of forms $\omega \in L^2(M, E_i)$ such that $P_i\omega$, computed distributionally, is also in $L^2(M, E_{i+1})$.

Definition 1.8. The minimal extension $P_{min,i}$; this is given by the graph closure of P_i on $C_0^\infty(M, E_i)$ respect to the norm of $L^2(M, E_i)$, that is,

$$\mathcal{D}(P_{min,i}) = \{\omega \in L^2(M, E_i) : \exists\ \{\omega_j\}_{j\in J} \subset C_0^\infty(M, E_i),\ \omega_j \to \omega, \tag{1.10}$$

$$P_i\omega_j \to \eta \in L^2(M, E_{i+1})\}$$

and in this case $P_{min,i}\omega = \eta$

Obviously $\mathcal{D}(P_{min,i}) \subset \mathcal{D}(P_{max,i})$. Furthermore, from these definitions, it follows immediately that

$$P_{min,i}(\mathcal{D}(P_{min,i})) \subset \mathcal{D}(P_{min,i+1}),\ P_{min,i+1} \circ P_{min,i} = 0$$

and that

$$P_{max,i}(\mathcal{D}(P_{max,i})) \subset \mathcal{D}(P_{max,i+1}),\ P_{max,i+1} \circ P_{max,i} = 0.$$

Therefore $(L^2(M, E_*), P_{max/min,*})$ are both Hilbert complexes and their cohomology groups, reduced cohomology groups, are denoted respectively by $H^i_{2,max/min}(M, E_*)$ and $\overline{H}^i_{2,max/min}(M, E_*)$.

Another straightforward but important fact is that the Hilbert complex adjoint of
$(L^2(M,E_*), P_{max/min,*})$ is $(L^2(M,E_*), P^t_{min/max,*})$, that is

$$(P_{max,i})^* = P^t_{min,i},\ (P_{min,i})^* = P^t_{max,i}. \tag{1.11}$$

Using proposition 1.2 we obtain two weak Kodaira decompositions:

$$L^2(M,E_i) = \mathcal{H}^i_{abs/rel}(M,E_i) \oplus \overline{ran(P_{max/min,i-1})} \oplus \overline{ran(P^t_{min/max,i})} \tag{1.12}$$

with summands mutually orthogonal in each case. For the first summand in the right, called the absolute or relative Hodge cohomology, we have by (1.4):

$$\mathcal{H}^i_{abs/rel}(M,E_*) = Ker(P_{max/min,i}) \cap Ker(P^t_{min/max,i-1}). \tag{1.13}$$

We can also consider the natural laplacians associated to the complex 1.8:

$$\mathcal{P}_i := P^t_i \circ P_i + P_{i-1} \circ P^t_{i-1} \tag{1.14}$$

where we recall that P^t_i is the formal adjoint of P_i. Using the Hilbert complexes $(L^2(M,E_i), P_{max/min,i})$ we can construct for each i two self-adjoint extensions of $\mathcal{P}_i$:

$$\mathcal{P}_{abs,i} := P^t_{min,i} \circ P_{max,i} + P^t_{min,i-1} \circ P_{max,i-1} \tag{1.15}$$

and

$$\mathcal{P}_{rel,i} := P^t_{max,i} \circ P_{min,i} + P^t_{max,i-1} \circ P_{min,i-1} \tag{1.16}$$

with domain described in (1.3). Using (1.4) and (1.5) it follows that the nullspace of (1.15) is the absolute Hodge cohomology which is in turn isomorphic to the reduced cohomology of the Hilbert complex $(L^2(M,E_*), P_{max,*})$. Analogously, using again (1.4) and (1.5), it follows that the nullspace of (1.16) is the relative Hodge cohomology which is in turn isomorphic to the reduced cohomology of the Hilbert complex $(L^2(M,E_*), P_{min,*})$.
Moreover we can define other two Hodge cohomology groups $\mathcal{H}^i_{max/min}(M,E_*)$ and other two closed extension of $\mathcal{P}_i$ defined as:

$$\mathcal{H}^i_{max/min}(M,E_*) = Ker(P_{max/min,i}) \cap Ker(P^t_{max/min,i-1}). \tag{1.17}$$

$$\mathcal{P}_{max,i} : L^2(M,E_i) \to L^2(M,E_i) \tag{1.18}$$

and

$$\mathcal{P}_{min,i} : L^2(M,E_i) \to L^2(M,E_i). \tag{1.19}$$

$\mathcal{P}_{max,i}$ is defined as the maximal closure of $\mathcal{P}_i : C^\infty_c(M,E_i) \to C^\infty_c(M,E_i)$ that is $u \in \mathcal{D}(\mathcal{P}_{max,i})$ and $v = \mathcal{P}_{max,i}(u)$ if

$$< u, \mathcal{P}^t_i(\phi) >_{L^2(M,E_i)} = < v, \phi >_{L^2(M,E_i)} \text{ for each } \phi \in C^\infty_c(M,E_i).$$

$\mathcal{P}_{min,i}$ is the minimal closure of $\mathcal{P}_i : C^\infty_c(M,E_i) \to C^\infty_c(M,E_i)$ that is $u \in \mathcal{D}(\mathcal{P}_{min,i})$ and $v = \mathcal{P}_{min,i}(u)$ if there is a sequence $\{\phi\}_{i\in\mathbb{N}} \subset C^\infty_c(M,E_i)$ such that

$$\phi_i \to u \text{ in } L^2(M,E_i) \text{ and } \mathcal{P}_i(\phi) \to u \text{ in } L^2(M,E_i).$$

Proposition 1.9. *The operators $\mathcal{P}_{max,i}$ and $\mathcal{P}_{min,i}$ satisfy the following properties:*

(1) $(\mathcal{P}_{max,i})^* = \mathcal{P}_{min,i}$, $(\mathcal{P}_{min,i})^* = \mathcal{P}_{max,i}$.

(2) $Ker(\mathcal{P}_{min,i}) = \mathcal{H}^i_{min}(M, g)$.
(3) $Ker(\mathcal{P}_{max,i}) = \mathcal{H}^i_{max}(M, E_i)$.
(4) $\overline{ran(\mathcal{P}_{min,i})} = \overline{ran(P_{min,i-1})} \oplus \overline{ran(P^t_{min,i})}$.
(5) $\overline{ran(\mathcal{P}_{max,i})} = \overline{ran(P_{max,i-1}) + ran(P^t_{max,i})}$.

PROOF. The first property is immediate. For the second property consider the following operator:

$$P_{max,i-1} \circ P^t_{min,i-1} + P^t_{max,i} \circ P_{min,i} : L^2(M, E_i) \to L^2(M, E_i).$$

We label it $\mathcal{P}_{m,i}$. This is a symmetric operator and it is clear that $\mathcal{P}_{m,i}$ extends $\mathcal{P}_{min,i}$ that is $\mathcal{D}(\mathcal{P}_{min,i}) \subset \mathcal{D}(\mathcal{P}_{m,i})$ and $\mathcal{P}_{min,i}(u) = \mathcal{P}_{m,i}(u)$ for each $u \in \mathcal{D}(\mathcal{P}_{min,i})$. From this it follows that $Ker(\mathcal{P}_{min,i}) \subset \mathcal{H}^i_{min}(M, E_i)$ because $Ker(\mathcal{P}_{min,i}) \subset Ker(\mathcal{P}_{m,i})$ and $Ker(\mathcal{P}_{m,i}) = \mathcal{H}^i_{min}(M, E_i)$. By the fact that $ran(\mathcal{P}_{max,i}) \subset \overline{ran(P_{max,i-1}) + ran(P^t_{max,i})}$ and by the first property it follows that $Ker(\mathcal{P}_{min,i}) = (ran(\mathcal{P}_{max,i}))^{\perp} \supset (\overline{ran(P_{max,i-1}) + ran(P^t_{max,i})})^{\perp}$ $= \mathcal{H}^i_{min}(M, E_i)$. Therefore $Ker(\mathcal{P}_{min,i}) = \mathcal{H}^i_{min}(M, E_i)$.
For the third property consider the following operator:

$$P_{min,i-1} \circ P^t_{max,i-1} + P^t_{min,i} \circ P_{max,i} : L^2(M, E_i) \to L^2(M, E_i).$$

We label it $\mathcal{P}_{M,i}$. Also $\mathcal{P}_{M,i}$ is a symmetric operator and it is clear that $\mathcal{P}_{max,i}$ extends $\mathcal{P}_{M,i}$. Therefore $Ker(\mathcal{P}_{max,i}) \supset \mathcal{H}^i_{max}(M, E_i)$ because $Ker(\mathcal{P}_{max,i}) \supset Ker(\mathcal{P}_{M,i})$ and $Ker(\mathcal{P}_{M,i}) = \mathcal{H}^i_{max}(M, E_i)$. By the fact that $ran(\mathcal{P}_{min,i}) \subset \overline{ran(P_{min,i-1}) + ran(P^t_{min,i})}$ and by the first property it follows that

$$(ran(\mathcal{P}_{min,i}))^{\perp} \supset (\overline{ran(P_{min,i-1}) + ran(P^t_{min,i})})^{\perp} = \mathcal{H}^i_{max}(M, E_i).$$

In this way we can conclude that $Ker(\mathcal{P}_{max,i}) = \mathcal{H}^i_{max}(M, E_i)$ because $Ker(\mathcal{P}_{max,i}) = (ran(\mathcal{P}_{min,i}))^{\perp}$.
For the fourth property we can observe that $\overline{ran(\mathcal{P}_{min,i})} \subset \overline{ran(\overline{P}_{m,i})} \subset \overline{ran(P_{min,i-1})} \oplus \overline{ran(P^t_{min,i})}$. But, by the third point, $(\overline{ran(P_{min,i-1})} \oplus \overline{ran(P^t_{min,i})})^{\perp} = Ker(\mathcal{P}_{max,i})$ and $(Ker(\mathcal{P}_{max,i}))^{\perp} = \overline{ran(\mathcal{P}_{min,i})}$; therefore the fourth point is proved.
For the fifth property we can observe that

$$\overline{ran(\mathcal{P}_{max,i})} \subset \overline{ran(P_{max,i-1}) + ran(P^t_{max,i})}.$$

But, by the second point, $(\overline{ran(P_{max,i-1}) + ran(P^t_{max,i})})^{\perp} = Ker(\mathcal{P}_{min,i})$ and $(Ker(\mathcal{P}_{min,i}))^{\perp} = \overline{ran(\mathcal{P}_{max,i})}$ and therefore the fifth point is proved. □

From proposition 1.9 follows immediately the following Kodaira decomposition for the Hilbert space $L^2(M, E_i)$. It was proved for the de Rham complex in [**48**].

PROPOSITION 1.10. *In the same assumptions of proposition 1.9 we have the following Kodaira decomposition :*

$$L^2(M, E_i) = \mathcal{H}^i_{max}(M; E_i) \oplus \overline{ran(P_{min,i-1})} \oplus \overline{ran(P^t_{min,i})}. \tag{1.20}$$

PROOF. It is well known that, by the fact that $(\mathcal{P}_{max,i})^* = \mathcal{P}_{min,i}$ and $(\mathcal{P}_{min,i})^* = \mathcal{P}_{max,i}$, the following L^2 decomposition holds for $L^2(M, E_i)$:

$$L^2(M, E_i) = Ker(\mathcal{P}_{max,i}) \oplus \overline{ran(\mathcal{P}_{min,i})}.$$

By proposition 1.9 we know that $Ker(\mathcal{P}_{max,i}) = \mathcal{H}^i_{max}(M; E_i)$ and that $\overline{ran(\mathcal{P}_{min,i})} = \overline{ran(P_{min,i-1})} \oplus \overline{ran(P^t_{min,i})}$ and this complete the proof. $\square$

Another useful application of the abstract theory of Hilbert complexes is given in the next proposition:

PROPOSITION 1.11. *Consider a complex as in* (1.8)*; suppose moreover that it is an elliptic complex. Consider now the following complex*

$$0 \rightarrow \mathcal{D}_2(P_0) \stackrel{P_0}{\rightarrow} \mathcal{D}_2(P_1) \stackrel{P_1}{\rightarrow} ... \stackrel{P_{n-1}}{\rightarrow} \mathcal{D}_2(P_n) \rightarrow 0, \tag{1.21}$$

where for each $i = 0, ..., n$ we have:

$$\mathcal{D}_2(P_i) := \{s \in C^\infty(M, E_i) \cap L^2(M, E_i) : P_i(s) \in L^2(M, E_i)\}.$$

Then (6.8) *is a subcomplex quasi-isomorphic to the Hilbert complex* $(L^2(M, E_i), P_{max,i})$.

PROOF. Clearly (6.8) is a subcomplex of $(L^2(M, E_i), P_{max,i})$. To show that the inclusion induces an isomorphism between cohomology groups consider proposition 1.6. By the fact that (1.8) is an elliptic complex it follows that for each $i = 0, ..., n$, $\mathcal{P}_i$ is an elliptic operator. In this way, using elliptic regularity, it follows that the complex $(\mathcal{D}^\infty(L^2(M, E_i)), P_{max,i})$ is a subcomplex of (6.8) and therefore the statement follows. $\square$

Obviously, a particular and fundamental case, which satisfies all the previous results is the de Rham complex:

$$0 \rightarrow \Omega^0_c(M) \stackrel{d_0}{\rightarrow} \Omega^1_c(M) \stackrel{d_1}{\rightarrow} ... \stackrel{d_{n-1}}{\rightarrow} \Omega^n_c(M) \rightarrow 0, \tag{1.22}$$

where $\Omega^i_c(M)$ is the space of smooth $i-$forms with compact support. Subsequently, when we will deal with the Hilbert complexes associated to (1.22), we will use the notations $\Delta_i, \Delta_{abs,i}$ and $\Delta_{rel,i}$ instead of $\mathcal{P}_i, \mathcal{P}_{abs,i}$ and $\mathcal{P}_{rel,i}$ we will label $(\Omega^i_2(M), g_c), d_i)$ the subcomplex of $(L^2\Omega^i(M, g), d_{max,i})$ described in proposition 6.8.

2. Stratified pseudomanifolds and intersection homology

We begin the section by recalling the concept of stratified pseudomanifold. The definition is given by induction on the dimension.

DEFINITION 1.12. A $0-$dimensional stratified space is a countable set with the discrete topology. For $m > 0$ a $m-$dimensional topologically stratified space is paracompact Hausdorff topological space X equipped with a filtration

$$X = X_m \supset X_{m-1} \supset ... \supset X_1 \supset X_0 \tag{1.23}$$

of X by closed subsets X_j such that if $x \in X_j - X_{j-1}$ there exists a neighbourhood N_x of x in X, a compact $(m - j - 1)-$dimensional topologically stratified space L with a filtration

$$L = L_{m-j-1} \supset ... \supset L_1 \supset L_0 \tag{1.24}$$

and a homeomorphism

$$\phi : N_x \to \mathbb{R}^j \times C(L) \tag{1.25}$$

where $C(L) = L \times [0,1)/L \times \{0\}$ is the open cone on L, such that ϕ takes $N_x \cap X_{j+i+1}$ homeomorphically onto

$$\mathbb{R}^j \times C(L_i) \subset \mathbb{R}^j \times C(L) \tag{1.26}$$

for $m - j - 1 \geq i \geq 0$ and ϕ takes $N_x \cap X_j$ homeomorphically onto

$$\mathbb{R}^j \times \{vertex\ of\ C(L)\} \tag{1.27}$$

This definition guaranties that, for each j, the subset $X_j - X_{j-1}$ is a topological manifold of dimension j. The **strata** of X are the connected components of these manifolds. If a stratum Y is a subset of $X - X_{n-1}$ it is called a **regular stratum**; otherwise it is called a **singular stratum**. The space L is referred as to the **link** of the stratum. In general it is not uniquely determined up to homeomorphism, though if X is a stratified pseudomanifold it is unique up to stratum preserving homotopy equivalence (see[**32**] pag 108).

Definition 1.13. A topological pseudomanifold of dimension m is a paracompact Hausdorff topological space X which posses a topological stratification such that

$$X_{m-1} = X_{m-2} \tag{1.28}$$

and $X - X_{m-2}$ is dense in X.(For more details see [**5**] or [**47**]).

Over these spaces, at the end of the seventies, Mark Goresky and Robert MacPherson have defined a new homological theory known as intersection homology. On of the reasons that led Goresky and MacPherson to introduce this new theory was the idea to extend Poincaré duality to these kind of singular spaces. Here we recall briefly the main definitions and we refer to [**5**], [**12**], [**38**], [**39**] and [**47**] for a complete development of the theory.

Definition 1.14. A perversity is a function $p : \{2, 3, 4, ..., n\} \to \mathbb{N}$ such that

$$p(2) = 0 \ and \ p(i) \leq p(i+1) \leq p(i) + 1. \tag{1.29}$$

Example 1.15. Some example of perversities are the following:

1. The lower middle perversity: $\underline{m}(k) = \lfloor \frac{k}{2} \rfloor - 1$ where $\lfloor x \rfloor$ is the integer part of x.
2. The upper middle perversity: $\overline{m}(k) = \lceil \frac{k}{2} \rceil - 1$ where $\lceil x \rceil$ is the smallest integer bigger or equal than x.
3. The top perversity: $t(k) = k - 2$

Finally it is immediate verify that, given a perversity p, than also $t - p$ is a perversity. It is called the dual is the perversity of p.

Remark 1.1. $\overline{m}$ is the dual perversity of $\underline{m}$. The trivial perversity is the dual perversity of t.

Let $\Delta_i \subset \mathbb{R}^{i+1}$ the standard $i-$simplex. The $j-$**skeleton** are of Δ_i is the set of $j-$subsimplices. We say a singular $i-$simplex in X, i.e. a continuous map $\sigma : \Delta_i \to X$, is $p-$**allowable** if for all $k \geq 2$

$$\sigma^{-1}(X_{m-k} - X_{m-k-1}) \subset \{(i - k + p(k)) - skeleton\ of\ \Delta_i\}. \tag{1.30}$$

The elements of the space $I^p S_i(X)$ are the finite linear combinations of singular $i-$simplex $\sigma : \Delta_i \to X$ such that σ and $\partial\sigma$ are $p-$allowable. Clearly $(I^p S_i(X), \partial_i)$ is a complex, more precisely a subcomplex of $(S_i(X), \partial_i)$, and the **perversity p singular intersection homology groups**, $I^p H_i(X)$, are the homology groups of this complex.

Remark 1.2. The above definition is not the original definition given by Goresky and MacPherson in [**38**]. In fact in their paper Goresky and MacPherson use a simplicial point of view and in particular the notion of p-allowable simplicial chains. The definition that we have recalled here was given in [**46**] by H. King. Over a PL-stratified pseudomanifold it is equivalent to the Goresky and MacPherson's definition but the advantage is that it holds even if X is only a stratified pseudomanifold.

Some of the fundamental results proved by Goresky and MacPherson are the following (see [**38**], [**39**], [**5**], [**12**] and [**47**]):

Let X a stratified pseudomanifold, $\mathfrak{X}$ a fixed stratification on X, p a perversity on X, $\mathcal{G}$ a local system on $X - X_{n-2}$ and $\mathcal{O}$ the orientation sheaf on $X - X_{n-2}$.

Consider now the following **set of axioms** $(AX1)_{p,\mathfrak{X},\mathcal{G}\otimes\mathcal{O}}$ for a complex of sheaves $(\mathcal{S}^*, d_*)$:

(1) $\mathcal{S}^*$ is bounded, $\mathcal{S}^i = 0$ for $i < 0$ and $\mathcal{S}^*|_{X-X_{n-2}}$ is quasi-isomorphic to $\mathcal{G} \otimes \mathcal{O}$.
(2) If $x \in Z$ for a stratum Z, then $H_i(\mathcal{S}^*_x) = 0$ for $i > p(k)$ where k is the codimension of Z.
(3) Let $U_k = X - X_{n-k}$ and let $i_k : U_k \to U_{k+1}$ the natural inclusion. Then for $x \in Z \subset U_{k+1}$ the attachment map

$$\alpha_k : \mathcal{S}^*|_{U_{k+1}} \to Ri_{k*}i_k^*\mathcal{S}^*|_{U_{k+1}}$$

given by the composition of natural morphism

$$\mathcal{S}^*|_{U_{k+1}} \to i_{k*}i_k^*\mathcal{S}^*|_{U_{k+1}} \to Ri_{k*}i_k^*\mathcal{S}^*|_{U_{k+1}}$$

is a quasi-isomorphism at x up to $p(k)$.

In almost all references the previous axioms are formulated in the derived category of sheaves on X. In that case the term quasi-isomorphism should be replaced with the term isomorphism.

Theorem 1.16. *Let X a compact stratified pseudomanifold of dimension n, p a perversity on X and $(\mathcal{S}^*, d_*)$ a complex of sheaves that satisfies the set of axioms $(AX1)_{p,\mathfrak{X},\mathcal{G}\otimes\mathcal{O}}$. Then the following isomorphism holds:*

$$\mathbb{H}^i(X, \mathcal{S}^*) \cong I^p H_{n-i}(X, \mathcal{G}) \tag{1.31}$$

that is the $i-$th hypercohomology group of the complex $(\mathcal{S}^*, d_*)$ is isomorphic to the $(n-i)-$th intersection homology group with coefficient in the local system $\mathcal{G}$ and relative to the perversity p.

COROLLARY 1.17. *In the same hypothesis of the previous theorem if* $(\mathcal{S}^*, d_*)$ *is a complex of fine or flabby or soft sheaves then the following isomorphism holds:*

$$H^i(\mathcal{S}^*(X), d_*) \cong I^p H_{n-i}(X, \mathcal{G}) \tag{1.32}$$

where $H^i(\mathcal{S}^*(X), d_*)$ *are the cohomology groups of the complex*

$$0 ... \stackrel{d_{i-1}}{\to} \mathcal{S}^i(X) \stackrel{d_i}{\to} \mathcal{S}^{i+1}(X) \stackrel{d_{i+1}}{\to} \mathcal{S}^{i+2}(X) \stackrel{d_{i+2}}{\to} ...$$

THEOREM 1.18. *Let* F *a field,* X *a compact and* $F-$*oriented stratified pseudomanifold of dimension* n, p, q *perversities on* X *such that* $p + q = t$ *and* $\mathcal{F}$ *a local system over* X *that is* $(X - X_{n-2}) \times F$ *over* $X - X_{n-2}$ *where the fibers* F *have the discrete topology. Then the following isomorphism holds:*

$$I^p H_i(X, \mathcal{F}) \cong Hom(I^q H_{n-i}(X, \mathcal{F}), F). \tag{1.33}$$

Before to recall the last result we give the following definition:

DEFINITION 1.19. Let X be a stratified pseudomanifold. Then X is called a Witt space if the following property is satisfied: let Y be a singular stratum of X and let L_Y be its link. Suppose that Y has odd codimension $2f + 1$. Then

$$I^{\underline{m}} H_{\frac{f}{2}}(L_Y, \mathbb{Q}) = 0$$

THEOREM 1.20. *Let* X *be a compact and orientable Witt space. Then the natural inclusion of complexes* $(I^{\underline{m}} S_i(X), \partial_i) \subset (I^{\overline{m}} S_i(X), \partial_i)$ *induces an isomorphism between the homology groups* $I^{\underline{m}} H_i(X, \mathbb{Q}) \cong I^{\overline{m}} H_i(X, \mathbb{Q})$. *In this case we have Poincaré duality:*

$$I^{\underline{m}} H_i(X, \mathbb{Q}) \cong Hom(I^{\overline{m}} H_{n-i}(X, \mathbb{Q}), Q). \tag{1.34}$$

However, for our goals we need a more general notion of perversity and associated intersection homology. A generalization of the theory of Goresky and MacPherson that is suited for our needs was made by Greg Friedman. As in the previous case we recall only the main definitions and results and we refer to the [**31**], [**32**] and [**33**] for a complete development of the theory. First, we remember that the theory proposed by Friedman applies to a **wider class of spaces**: from now on a **stratified pseudomanifold will be simply a paracompact Hausdorff topological space** X **which posses a topological stratification and such that** $X - X_{n-1}$ **is dense in** X. **That is, we do not require that the condition** $X_{m-1} = X_{m-2}$ **apply**. In the following propositions each stratified pseudomanifolds will have a **fixed stratification**. We start by introducing the notion of **general perversity**:

DEFINITION 1.21. A general perversity on a stratified pseudomanifold X is any function

$$p : \{Singular\ Strata\ of\ X\} \to \mathbb{Z}. \tag{1.35}$$

The notion of $p-$**allowable** singular simplex is modified in the following way: a singular $i-$simplex in X, i.e. a continuous map $\sigma : \Delta_i \to X$, is $p-$**allowable** if

$$\sigma^{-1}(Y) \subset \{(i - cod(Y) + p(Y)) - skeleton\ of\ \Delta_i\} \qquad (1.36)$$

$$for\ any\ singular\ stratum\ Y\ of\ X.$$

A key ingredient in this new theory is the notion of **homology with stratified coefficient system**. (The definition uses the notion homology with local coefficient system; for the definition of local coefficient system see [**28**], [**42**] and[**67**])

Definition 1.22. Let X stratified pseudomanifold and let $\mathcal{G}$ a local system on $X - X_{n-1}$. Then the stratified coefficient sistem $\mathcal{G}_0$ is defined to consist of the pair of coefficient systems given by $\mathcal{G}$ on $X - X_{n-1}$ and the constant 0 system on X_{n-1} i.e. we think of $\mathcal{G}_0$ as consisting of a locally constant fiber bundle $\mathcal{G}_{X-X_{n-1}}$ over $X - X_{n-1}$ with fiber G with the discrete topology together with the trivial bundle on X_{n-1} with the stalk 0.

Then a **coefficient** n of a singular simplex σ can be described by a lift of $\sigma|_{\sigma^{-1}(X-X_{n-1})}$ to $\mathcal{G}$ over $X - X_{n-1}$ together with the trivial lift of $\sigma|_{\sigma^{-1}(X_{n-1})}$ to the 0 system on X_{n-1}. A coefficient of a simplex σ is considered to be the 0 coefficient if it maps each points of Δ to the 0 section of one of the coefficient systems. Note that if $\sigma^{-1}(X - X_{n-1})$ is path-connected then a coefficient lift of σ to $\mathcal{G}_0$ is completely determined by the lift at a single point of $\sigma^{-1}(X - X_{n-1})$ by the lifting extension property for $\mathcal{G}$. The intersection homology chain complex $(I^pS_*(X, \mathcal{G}_0), \partial_*)$ are defined in the same way as $I^pS_*(X, G)$, where G is any field, but replacing the coefficient of simplices with coefficient in $\mathcal{G}_0$. If $n\sigma$ is a simplex σ with its coefficient n, its boundary is given by the usual formula $\partial(n\sigma) = \sum_j (-1)^j (n \circ i_j)(\sigma \circ i_j)$ where $i_j : \Delta_{i-1} \to \Delta_i$ is the $j-$face inclusion map. Here $n \circ i_j$ should be interpreted as the restriction of n to the jth face of σ, restricting the lift to $\mathcal{G}$ where possible and restricting to 0 otherwise. The basic idea behind the definition is that when we consider if a chain is allowable with respect to a perversity, simplices with support entirely in X_{n-1} should vanish and thus not be counted for admissibility considerations. (For more details see [**31**], [**32**] and [**33**]).

The next proposition shows that Friedman's theory is an extension of the classical theory made by Goresky and MacPherson.

Proposition 1.23. *(see [**32**] pag. 110, [**33**] pag. 1985) If p is a traditional perversity, that is a perversity like those defined in definition 1.14, and $X_{n-1} = X_{n-2}$ then*

$$I^pS_*(X, \mathcal{G}) = I^pS_*(X, \mathcal{G}_0).$$

Example 1.24. Let X be a stratified pseudomanifold and p a general perversity on X. Consider as stratified coefficient system $\mathcal{R}_0$, that is the pair of coefficient systems given by $(X - X_{n-1}) \times \mathbb{R}$ over $X - X_{n-1}$ where the fibers $\mathbb{R}$ have the discrete topology and the constant 0 system on X_{n-1}. Now suppose that X and p satisfy the assumptions of proposition 1.23; then

$$I^pS_*(X,\mathbb{R}) = I^pS_*(X,\mathcal{R}_0)$$

where $I^pS_*(X,\mathbb{R})$ is the usual intersection homology chain complex with coefficient in the field $\mathbb{R}$.

We conclude this section recalling the generalizations, obtained by Friedman, of the previous results obtained by Goresky and MacPherson.

Again let X be a stratified pseudomanifold, $\mathfrak{X}$ a fixed stratification on X, p a generalized perversity on X, $\mathcal{G}$ a local system on $X - X_{n-1}$ and $\mathcal{O}$ the orientation sheaf on $X - X_{n-1}$.
Consider now the following **set of axioms** $(AX1)_{p,\mathfrak{X},\mathcal{G}\otimes\mathcal{O}}$ for a complex of sheaves $(\mathcal{S}^*, d_*)$:

(1) $\mathcal{S}^*$ is bounded, $\mathcal{S}^i = 0$ for $i < 0$ and $\mathcal{S}^*|_{X-X_{n-1}}$ is quasi-isomorphic to $\mathcal{G}\otimes\mathcal{O}$.
(2) If $x \in Z$ for a stratum Z, then $H_i(\mathcal{S}^*_x) = 0$ for $i > p(Z)$.
(3) Let $U_k = X - X_{n-k}$ and let $i_k : U_k \to U_{k+1}$ the natural inclusion. Then for $x \in Z \subset U_{k+1}$ the attachment map

$$\alpha_k : \mathcal{S}^*|_{U_{k+1}} \to Ri_{k*}i_k^*\mathcal{S}^*|_{U_{k+1}}$$

given by the composition of natural morphism

$$\mathcal{S}^*|_{U_{k+1}} \to i_{k*}i_k^*\mathcal{S}^*|_{U_{k+1}} \to Ri_{k*}i_k^*\mathcal{S}^*|_{U_{k+1}}$$

is a quasi-isomorphism at x up to $p(Z)$.

We recall again for the benefit of the reader that in almost all references the previous axioms are formulated in the derived category of sheaves on X. In that case the term quasi-isomorphism should be replaced with the term isomorphism.

THEOREM 1.25. *(see [**31**] pag 116) Let X a compact stratified pseudomanifold of dimension n, p a general perversity on X and $(\mathcal{S}^*, d_*)$ a complex of sheaves that satisfies the set of axioms $(AX1)_{p,\mathfrak{X},\mathcal{G}\otimes\mathcal{O}}$. Then the following isomorphism holds:*

$$\mathbb{H}^i(X,\mathcal{S}^*) \cong I^pH_{n-i}(X,\mathcal{G}_0) \tag{1.37}$$

that is the $i-th$ hypercohomology group of the complex $(\mathcal{S}^*, d_*)$ is isomorphic to the $(n-i)-th$ intersection homology group with coefficient in the stratified system $\mathcal{G}_0$ and relative to the perversity p.

COROLLARY 1.26. *In the same hypothesis of the previous theorem if $(\mathcal{S}^*, d_*)$ is a complex of fine or flabby or soft sheaves then the following isomorphism holds:*

$$H^i(\mathcal{S}^*(X), d_*) \cong I^pH_{n-i}(X,\mathcal{G}_0) \tag{1.38}$$

where $H^i(\mathcal{S}^*(X), d_*)$ are the cohomology groups of the complex

$$0 ... \overset{d_{i-1}}{\to} \mathcal{S}^i(X) \overset{d_i}{\to} \mathcal{S}^{i+1}(X) \overset{d_{i+1}}{\to} \mathcal{S}^{i+2}(X) \overset{d_{i+2}}{\to} ...$$

THEOREM 1.27. *(see* [**31**] *pag 122 or* [**32**] *pag 25.) Let F a field, X a compact and $F-$oriented stratified pseudomanifold of dimension n, p, q general perversities on X such that $p+q=t$ (that is for each stratum $Z \subset X$ $p(Z)+q(Z)=codim(Z)-2$) and $\mathcal{F}_0$ a stratified coefficient system over X, consisting of the pair of coefficient systems given by $(X-X_{n-1}) \times F$ over $X-X_{n-1}$ where the fibers F have the discrete topology and the constant* 0 *system on X_{n-1}. Then the following isomorphism holds:*

$$I^pH_i(X,\mathcal{F}_0) \cong Hom(I^qH_{n-i}(X,\mathcal{F}_0),F). \tag{1.39}$$

REMARK 1.3. In this paper with the symbol $I^pH^i(X,\mathcal{G}_0)$ we mean the cohomology of the complex

$$(Hom(I^pS_i(X,\mathcal{G}_0),G),(\partial_i)^*).$$

We call it the $i-th$ intersection cohomology group of X with respect to the perversity p and the stratified coefficient system $\mathcal{G}_0$. When $G=F$ is a field then

$$I^pH^i(X,\mathcal{F}_0) \cong Hom(I^pH_i(X,\mathcal{F}_0),F).$$

REMARK 1.4. Summarizing, by theorems 1.25 and 1.27, it follows that if $(\mathcal{S}^*,d_*)$ is a complex of sheaves that satisfies the set of axioms $(AX1)_{p,\mathfrak{X},\mathcal{F}\otimes\mathcal{O}}$ then

$$\mathbb{H}^i(X,\mathcal{S}^*) \cong I^qH^i(X,\mathcal{F}_0) \tag{1.40}$$

where $p+q=t$ and if $(\mathcal{S}^*,d_*)$ is a complex of fine or flabby or soft sheaves then, by corollary 1.26,

$$H^i(\mathcal{S}^*(X),d_*) \cong I^qH^i(X,\mathcal{F}_0) \tag{1.41}$$

3. Thom-Mather stratification and quasi edge metrics with weights

In this section we introduce stratified pseudomanifolds with a Thom-Mather stratification and quasi-edge metrics wight weights. Following [**1**], we start recalling the definition of a smoothly stratified pseudomanifold with a Thom-Mather stratification.

DEFINITION 1.28. A smoothly stratified pseudomanifold X with a Thom-Mather stratification is a metrizable, locally compact, second countable space which admits a locally finite decomposition into a union of locally closed strata $\mathfrak{G}=\{Y_\alpha\}$, where each Y_α is a smooth, open and connected manifold, with dimension depending on the index α. We assume the following:

(1) If Y_α, $Y_\beta \in \mathfrak{G}$ and $Y_\alpha \cap \overline{Y}_\beta \neq \emptyset$ then $Y_\alpha \subset \overline{Y}_\beta$
(2) Each stratum Y is endowed with a set of control data T_Y, π_Y and ρ_Y ; here T_Y is a neighbourhood of Y in X which retracts onto Y, $\pi_Y : T_Y \to Y$ is a fixed continuous retraction and $\rho_Y : T_Y \to [0,2)$ is a proper radial function in this tubular neighbourhood such that $\rho_Y^{-1}(0)=Y$. Furthermore, we require that if $Z \in \mathfrak{G}$ and $Z \cap T_Y \neq \emptyset$ then $(\pi_Y,\rho_Y) : T_Y \cap Z \to Y \times [0,2)$ is a proper differentiable submersion.

(3) If $W, Y, Z \in \mathfrak{S}$, and if $p \in T_Y \cap T_Z \cap W$ and $\pi_Z(p) \in T_Y \cap Z$ then $\pi_Y(\pi_Z(p)) = \pi_Y(p)$ and $\rho_Y(\pi_Z(p)) = \rho_Y(p)$.
(4) If $Y, Z \in \mathfrak{S}$, then $Y \cap \overline{Z} \neq \emptyset \Leftrightarrow T_Y \cap Z \neq \emptyset$, $T_Y \cap T_Z \neq \emptyset \Leftrightarrow Y \subset \overline{Z}, Y = Z$ *or* $Z \subset \overline{Y}$.
(5) For each $Y \in \mathfrak{S}$, the restriction $\pi_Y : T_Y \to Y$ is a locally trivial fibration with fibre the cone $C(L_Y)$ over some other stratified space L_Y (called the link over Y), with atlas $\mathcal{U}_Y = \{(\phi, \mathcal{U})\}$ where each ϕ is a trivialization $\pi_Y^{-1}(U) \to U \times C(L_Y)$, and the transition functions are stratified isomorphisms which preserve the rays of each conic fibre as well as the radial variable ρ_Y itself, hence are suspensions of isomorphisms of each link L_Y which vary smoothly with the variable $y \in U$.
(6) For each j let X_j be the union of all strata of dimension less or equal than j, then

$$X - X_{n-1} \text{ is dense in } X$$

We make a few comments to the previous definition (for more details we refer to [**1**]):

(1) The previous definition is more general than that given in [**1**]. In [**1**] a space that satisfies the definition 1.28 is only a smoothly stratified spaces (with a Thom-Mather stratification). To be a smoothly stratified pseudomanifold (with a Thom-Mather stratification) there is another requirement to satisfy: let Xj be the union of all strata of dimensions less or equal than j, then

$$X = X_n \supset X_{n-1} = X_{n-2} \supset X_{n-3} \supset ... \supset X_0 \tag{1.42}$$

and $X - X_{n-2}$ is dense in X. For our goals, thanks to the results of Friedman, we can waive the requirement $X_{n-1} = X_{n-2}$ and therefore we will call smoothly stratified pseudomanifold with a Thom-Mather stratification each space X that satisfies the definition 1.28.
(2) The link L_Y is uniquely determined, up to isomorphism (see point number 5 below for the notion of isomorphism), by the stratum Y.
(3) The depth of a stratum Y is largest integer k such that there is a chain of strata $Y = Y_k, ..., Y_0$ such that $Y_j \subset \overline{Y_{j-1}}$ for $i \leq j \leq k$. A stratum of maximal depth is always a closed subset of X. The maximal depth of any stratum in X is called the depth of X as stratified spaces.
(4) Consider the filtration

$$X = X_n \supset X_{n-1} \supset X_{n-2} \supset X_{n-3} \supset ... \supset X_0 \tag{1.43}$$

We refer to the open subset $X - X_{n-1}$ of a stratified pseudomanifold X as its regular set, and the union of all other strata as the singular set,

$$reg(X) := X - sing(X) \text{ where } sing(X) := \bigcup_{Y \in \mathfrak{S}, depth Y > 0} Y.$$

(5) If X, X' are two stratified spaces a stratified isomorphism between them is a homeorphism $F : X \to X'$ which carries the strata of X

to the strata of X' diffeomorphically, and such that $\pi'_{F(Y)} \circ F = F \circ \pi_Y$, $\rho_Y = \rho'_{(F(Y))} \circ F$ for all $Y \in \mathcal{G}(X)$.

Summarizing a smoothly stratified pseudomanifold with Thom-Mather stratification is a stratified pseudomanifold with a richer structure from a differentiable and topological point of view.

Now we introduce an important class of riemannian metrics on the regular part of a smoothly stratified pseudomanifold with a Thom-Mather stratification. Before giving the definition we recall that two riemannian metrics g, h on a smooth manifold M are **quasi-isometric** if there are constants c_1, c_2 such that $c_1 h \leq g \leq c_2 h$.

Definition 1.29. Let X be a smoothly stratified pseudomanifold with a Thom-Mather stratification and let g a riemannian metric on $reg(X)$. We call g a **quasi edge metric with weights** if it satisfies the following properties:

(1) Take any stratum Y of X; by definition 1.28 for each $q \in Y$ there exist an open neighbourhood U of q in Y such that $\phi : \pi_Y^{-1}(U) \to U \times C(L_Y)$ is a stratified isomorphism; in particular $\phi : \pi_Y^{-1}(U) \cap reg(X) \to U \times reg(C(L_Y))$ is a diffeomorphism. Then, for each $q \in Y$, there exists one of these trivializations (ϕ, U) such that g restricted on $\pi_Y^{-1}(U) \cap reg(X)$ satisfies the following properties:
$$(\phi^{-1})^*(g|_{\pi_Y^{-1}(U)\cap reg(X)}) \cong dr \otimes dr + h_U + r^{2c} g_{L_Y} \tag{1.44}$$
where h_U is a riemannian metric defined over U, $c \in \mathbb{R}$ and $c > 0$, g_{L_Y} is a riemannian metric on $reg(L_Y)$, $dr \otimes dr + h_U + r^{2c} g_{L_Y}$ is a riemannian metric of product type on $U \times reg(C(L_Y))$ and with $\cong$ we mean **quasi-isometric**.

(2) If p and q lie in the same stratum Y then in (1.44) there is the same weight. We label it c_Y.

Before continuing we make some **remarks**:

(1) Obviously if the codimension of Y is 1 then L_Y is just a point and therefore by the previous definition
$$(\phi^{-1})^*(g|_{\pi_Y^{-1}(U)\cap reg(X)}) \cong dr \otimes dr + h_U.$$

(2) In the first point of the previous definition the metric g_{L_Y} depends also on the open neighborhood U and the stratified isomorphism ϕ. However we prefer to use the notation g_{L_Y} instead of $g_{L_Y,U,\phi}$ for the sake of simplicity.

(3) Let g and U be like in the first point of the previous definition and let $\psi : \pi_Y^{-1}(U) \to U \times C(L_Y)$ another stratified isomorphism that satisfies the requirements of definition 1.28. From the fifth point of definition 1.28 it follows that $\psi \circ \phi^{-1} : U \times C(L_Y) \to U \times C(L_Y)$ acts in this way: given $p = (y, [r, x]) \in U \times C(L_Y)$ $(\psi \circ \phi^{-1})(p) = (y, [r, f(y, x)])$ where the maps $x \mapsto f(y, x)$ are a family of smooth stratified isomorphisms of L_Y which vary smoothly with the variable $y \in U$. From this it follows immediately that if we fix a point

$y_0 \in U$ and if we put $h_{L_Y} = (f(y_0, x)^{-1})^*(g_{L_Y})$ then there exists an open subset $V \subset U, y_0 \in V$ such that $(\psi^{-1})^*(g|_{\pi_Y^{-1}(V)\cap reg(X)}) \cong dr \otimes dr + h_U|_V + r^{2c_Y} h_{L_Y}$ where $h_U|_V$ is the metric h_U restricted to V. Therefore the weight c_Y does not depend from the particular trivialization ϕ that it is chosen.

Now we give a definition which is a more refined version of the previous one; it is also a slight generalization of the definition of the adapted metric given by Brasselet, Hector and Saralegi in [**13**]. This definition is given by induction on $depth(X)$.

Definition 1.30. Let X be a stratified pseudomanifold with a Thom-Mather stratification and let g a riemannian metric on $reg(X)$. If $depth(X) = 0$, that is X is a closed manifold, a **quasi rigid iterated edge metric with weights** is any riemannian metric on X. Suppose now that $depth(X) = k$ and that the definition of quasi rigid iterated edge metric with weights is given in the case $depth(X) \leq k-1$; then we call a riemannian metric g on $reg(X)$ a **quasi rigid iterated edge metric with weights** if it satisfies the following properties:

(1) Take any stratum Y of X; by definition 1.28 for each $q \in Y$ there exist an open neighbourhood U of q in Y such that $\phi : \pi_Y^{-1}(U) \to U \times C(L_Y)$ is a stratified isomorphism; in particular $\phi : \pi_Y^{-1}(U) \cap reg(X) \to U \times reg(C(L_Y))$ is a diffeomorphism. Then, for each $q \in Y$, there exists one of these trivializations (ϕ, U) such that g restricted on $\pi_Y^{-1}(U) \cap reg(X)$ satisfies the following properties:
$$(\phi^{-1})^*(g|_{\pi_Y^{-1}(U)\cap reg(X)}) \cong dr \otimes dr + h_U + r^{2c} g_{L_Y} \qquad (1.45)$$
where h_U is a riemannian metric defined over U, $c \in \mathbb{R}$ and $c > 0$, g_{L_Y} is a **quasi rigid iterated edge metric with weights** on $reg(L_Y)$, $dr \otimes dr + h_U + r^{2c} g_{L_Y}$ is a riemannian metric of product type on $U \times reg(C(L_Y))$ and with $\cong$ we mean **quasi-isometric**.

(2) If p and q lie in the same stratum Y then in (1.45) there is the same weight. We label it c_Y.

Also in this case a remark to the previous definition is in order. Let $\psi : \pi_Y^{-1}(U) \to U \times C(L_Y)$ another stratified isomorphism that satisfies the requirements of definition 1.28. Using the same observations and notations of the second remark of definition 1.29 we can conclude that there exists an open subset $V \subset U$ and a quasi rigid iterated edge metric with weights h_{L_Y} on $reg(L_Y)$ such that $(\psi^{-1})^*(g|_{\pi_Y^{-1}(V)\cap reg(X)}) \cong dr \otimes dr + h_U|_V + r^{2c_Y} h_{L_Y}$. Furthermore, by the fact that $f(y_0, x)$ is a smooth stratified isomorphism between L_Y and L_Y such that $(f(y_0, x))^*(h_{L_Y}) = g_{L_Y}$, it follows that g_{L_Y} and h_{L_Y} have the same weights and therefore, by proposition 1.32 below, g_{L_Y} and h_{L_Y} are quasi-isometric on $reg(L_Y)$ when L_Y is compact.

Proposition 1.31. *Let X be a smoothly stratified pseudomanifold with a Thom-Mather stratification $\mathfrak{X}$. For any stratum $Y \subset X$ fix a positive real number c_Y. Then there exists a quasi rigid iterated edge metric with weights g on $reg(X)$ having the numbers $\{c_Y\}_{Y\in\mathfrak{X}}$ as weights.*

PROOF. In [**1**] is defined a class of riemannian metric called **rigid iterated edge metric** and in prop. 3.1 of the same paper is proved the existence of such metrics. Using the same notation of definition 1.30 a riemannian metric g on $reg(X)$ is a rigid iterated edge metric if $(\phi^{-1})^*(g|_{\pi_Y^{-1}(U)\cap reg(X)}) = dr\otimes dr + h_U + r^2 g_{L_Y}(u,y)$, with $u \in U$, $y \in L_Y$, and for any fixed u, $g_{L_Y}(u,y)$ is a rigid iterated edge metric on $reg(L_Y)$. In [**1**] proposition 3.1 is proved in the case $X_{n-1} = X_{n-2}$ but it is easy to see that it holds also in our case that is when $X_{n-1} \neq X_{n-2}$ and $c_Y \neq 1$. Therefore on $reg(X)$ there is a rigid iterated edge metric g having the numbers $\{c_Y\}_{Y\in\mathfrak{X}}$ as weights. Using again the notation of definition 1.30 this means that for each stratum Y and for any point $q \in Y$ $(\phi^{-1})^*(g|_{\pi_Y^{-1}(U)\cap reg(X)}) = dr\otimes dr + h_U + r^{2c_Y} g_{L_Y}(u,y)$, with $u \in U$, $y \in L_Y$, and for any fixed u, $g_{L_Y}(u,y)$ is a rigid iterated edge metric with weights on $reg(L_Y)$. Now it is clear that g is a quasi rigid iterated edge metric on $reg(X)$ having the numbers $\{c_Y\}_{Y\in\mathfrak{X}}$ as weights. Alternatively the existence of such metrics follows using the same arguments used by Brasselet, Hector and Saralegi in [**13**].

∎

PROPOSITION 1.32. *Let X be a compact smoothly stratified pseudomanifold with a Thom-Mather stratification. For any stratum $Y \subset X$ fix a positive real number c_Y. Let g, g' two quasi edge metrics with weights on $reg(X)$ having both the numbers $\{c_Y\}_{Y\in\mathfrak{X}}$ as weights. Then g and g' are quasi-isometric.*

PROOF. Let K be a compact subset of X such that $K \subset reg(X)$. Obviously $g|_K$ is quasi-isometric to $g'|_K$. Now let Y be a stratum such that $Y \subset X_{n-1} - X_{n-2}$. Let $x \in Y$; consider $\pi_Y^{-1}(x)$ and let $V_{Y,x} := \pi_Y^{-1}(x) \cap \rho_Y^{-1}(1)$. Then there exists a compact subset of X, K such that $K \subset reg(X)$ and $reg(V_{Y,x}) \subset K$. Therefore $g|_{reg(V_{Y,x})}$ is quasi-isometric to $g'|_{reg(V_{Y,x})}$ and from this it follows that, given an open neighbourhood U of x in Y sufficiently small such that $\pi_Y^{-1}(U) \cong U \times C(L_Y)$, $g|_{reg(\pi_Y^{-1}(U))}$ is quasi-isometric to $g'|_{reg(\pi_Y^{-1}(U))}$. This last assertion is a consequence of the fact that, by definition 1.29 and remarks following it, there is an isomorphism $\phi : \pi_Y^{-1}(U) \to U \times C(L_Y)$ such that, by definition 1.29, $(\phi^{-1})^*(g|_{reg(\pi_Y^{-1}(U))})$ is quasi isometric to $h + dr^2 + r^{2c_Y} g_{L_Y}$ and analogously $(\phi^{-1})^*(g'|_{reg(\pi_Y^{-1}(U))})$ is quasi isometric to $h' + dr^2 + r^{2c_Y} g'_{L_Y}$. But from the fact that $g|_{reg(V_{Y,x})}$ is quasi-isometric to $g'|_{reg(V_{Y,x})}$ it follows that g_{L_Y} is quasi-isometric to g'_{L_Y} and therefore for a sufficiently small U we get $g|_{reg(\pi_Y^{-1}(U))}$ is quasi-isometric to $g'|_{reg(\pi_Y^{-1}(U))}$. So we can conclude that if $K \subset (X - X_{n-2})$ is a compact subset then $g|_{reg(K)}$ is quasi-isometric to $g'|_{reg(K)}$. Now consider a stratum $Z \subset X_{n-2} - X_{n-3}$ and let $x \in Z$. As before consider $\pi_Z^{-1}(x)$ and let $V_{Z,x} = \pi_Z^{-1}(x) \cap \rho_Z^{-1}(1)$. Then there exists a compact subset $K \subset (X - X_{n-2})$ such that $V_{Z,x} \subset K$. From this it follows that $g|_{reg(V_{Z,x})}$ is quasi-isometric to $g'|_{reg(V_{Z,x})}$ and now, as before, we can conclude that given an open neighbourhood U of x in Z sufficiently small such that $g|_{\pi_Z^{-1}(U)} \cong U \times C(L_Z)$, $g|_{reg(\pi_Z^{-1}(U))}$ is quasi-isometric to $g'|_{reg(\pi_Z^{-1}(U))}$. As before from this it follows that if $K \subset (X - X_{n-3})$ is a compact subset then $g|_{reg(K)}$ is quasi-isometric to $g'|_{reg(K)}$. Now it is obvious that iterating this procedure we obtain what was asserted. $\square$

COROLLARY 1.33. *Let X be a compact smoothly stratified pseudomanifold with a Thom-Mather stratification and let g a quasi edge metric with weights on $reg(X)$. Then there exist g', a* ***quasi rigid iterated edge metric with weights*** *on $reg(X)$, that is quasi-isometric to g.*

We conclude this section introducing the notion of general perversity associated to a quasi edge metric with weights.

DEFINITION 1.34. Let X be a smoothly stratified pseudomanifold with a Thom-Mather stratification and let g a quasi edge metric with weights on $reg(X)$. Then the general perversity p_g associated to g is:

$$p_g(Y) := Y \longmapsto [[\frac{l_Y}{2} + \frac{1}{2c_Y}]] = \begin{cases} 0 & l_Y = 0 \\ \frac{l_Y}{2} + [[\frac{1}{2c_Y}]] & l_Y \textit{ even and } l_Y \neq 0 \\ \frac{l_Y - 1}{2} + [[\frac{1}{2} + \frac{1}{2c_Y}]] & l_Y \textit{ odd} \end{cases} \tag{1.46}$$

where $l_Y = dimL_Y$ and, given any real and positive number x, $[[x]]$ is the greatest integer strictly less than x.

CHAPTER 2

L^2-cohomology and L^2-de Rham-Hodge theorems

In this chapter we prove an L^2-de Rham-Hodge theorem for a stratified pseudomanifold endowed with a quasi edge metric with weights. The chapter is divided in three sections: in the first section some technical proposition we need in order to prove the theorems are proved. In the second sections we calculate the L^2 de Rham maximal cohomology groups of a cone over a riemannian manifold. Finally, in the last section, the L^2-de Rham-Hodge theorems are stated and proved.

1. Preliminary propositions

In this section we establish the necessary tools to calculate the L^2 maximal cohomology of a cone over a riemannian manifold. We follow, with some modifications, [**23**]. Given an oriented riemannian manifold (F,g) of dimension f, $C^*(F)$ will be the regular part of $C(F)$, that is $C(F)-\{v\}$, and g_c will be the riemannian metric on $C^*(F)$

$$g_c = dr\otimes dr + r^{2c}\pi^*g \tag{2.1}$$

where $\pi : C^*(F)\rightarrow F$ is the projection over F and $c\in\mathbb{R},\ c>0$.
With the symbol $d_F : \Omega^i(C^*(F))\rightarrow\Omega^{i+1}(C^*(F))$ we mean the exterior differential obtained by ignoring the variable r.

Proposition 2.1. *Let $\phi\in L^2\Omega^i(F,g), \phi\neq 0$ and let $\pi : C^*(F)\longrightarrow F$ be the projection. Then $\pi^*(\phi)\in L^2\Omega^i(C^*(F),g_c)$ if and only if $i<\frac{f}{2}+\frac{1}{2c}$. In this case the pullback map is also bounded.*

Proof. If $\phi\in L^2\Omega^i(F,g)$ then

$$\|\pi^*(\phi)\|^2_{L^2(C^*(F),g_c)} = \int_{C^*(F)}\|\pi^*(\phi)\|^2_{C^*(F)}dvol_{C^*(F)} =$$

$$\int_0^1\int_F r^{c(f-2i)}\|\phi\|^2_F dvol_F dr = \|\phi\|^2_{L^2(F,g)}\int_0^1 r^{c(f-2i)}dr<\infty$$

if and only if $i<\frac{f}{2}+\frac{1}{2c}$. Since $\int_0^1 r^{c(f-2i)}dr$ is independent of ϕ, the pullback map is bounded. □

Proposition 2.2. *There exists a constant $K>0$ such that for all $\alpha = \phi + dr\wedge\omega\in L^2\Omega^i(C^*(F),g_c)$ and for any null set $S\subset(1/2,1)$ there is an $a\in(1/2,1)-S$ such that*

$$\|\phi(a)\|^2_{L^2(F,g)}\leq K\|\phi\|^2_{L^2(C^*(F),g_c)}\leq K\|\alpha\|^2_{L^2(C^*(F),g_c)}.$$

Proof. Suppose that this proposition is false. Then for any $K > 0$ there is a form $\phi \in L^2\Omega^i(C^*(F), g_c)$ such that

$$\|\phi\|^2_{L^2(C^*(F),g_c)} \geq \int_{1/2}^1 \int_F r^{c(f-2i)} \|\phi\|^2_F dvol_F dr =$$

$$\int_{1/2}^1 r^{c(f-2i)} \|\phi(r)\|^2_{L^2(F,g)} dr > K\|\phi\|^2_{L^2(C^*(F),g_c)} \int_{(1/2,1)-S} r^{c(f-2i)} dr$$

$$= K\|\phi\|^2_{L^2(C^*(F),g_c)} \int_{(1/2,1)} r^{c(f-2i)} dr.$$

In this way by choosing $K > (\int_{(1/2,1)} r^{c(f-2i)} dr)^{-1}$ we obtain a contradiction. $\square$

Proposition 2.3. *If* $i < \frac{f}{2} + \frac{1}{2c} + 1$ *and* $\alpha = \phi + dr \wedge \omega \in L^2\Omega^i(C^*(F), g_c)$, *then for any* $a \in (1/2, 1)$

$$K_a(\alpha) = \int_a^r \omega(s) ds \in L^2\Omega^{i-1}(C^*(F), g_c)$$

and K_a *is a bounded operator uniformly in* $a \in (1/2, 1)$.

Proof. By definition

$$\|K_a(\alpha)\|^2_{L^2(C^*(F),g_c)} = \| \int_a^r \omega(s) ds\|^2_{L^2(C^*(F),g_c)} =$$

$$= \int_0^1 \int_F \| \int_a^r \omega(s) ds\|^2_F r^{c(f-2i+2)} dvol_F dr.$$

We consider the term $\| \int_a^r \omega(s) ds\|^2_F$. The following inequality holds :

$$\| \int_a^r \omega(s) ds\|^2_F \leq (\int_a^r \|\omega(s) ds\|_F)^2$$

and using the Schwartz inequalities the right side of this becomes:

$$(\int_a^r \|\omega(s)\|_F ds)^2 \leq \int_a^r ds \int_a^r \|\omega(s)\|^2_F ds$$

$$\leq \int_a^1 ds \int_a^r \|\omega(s)\|^2_F ds = (1-a) \int_a^r \|\omega(s)\|^2_F ds \leq (1-a) \int_a^1 \|\omega(s)\|^2_F ds.$$

So we have obtained that

$$\|K_a(\alpha)\|^2_{L^2(C^*(F),g_c)} \leq (1-a) \int_0^1 \int_F \int_a^1 \|\omega(s)\|^2_F ds r^{c(f-2i+2)} dvol_F dr.$$

Now consider the term $\int_0^1 \int_F \int_a^1 \|\omega(s)\|^2_F ds r^{c(f-2i+2)} dvol_F dr$

$$= \int_0^1 \int_F \int_a^1 \|\omega(s)\|^2_F (s^{c(f-2i+2)} + 1 - s^{c(f-2i+2)}) ds r^{c(f-2i+2)} dvol_F dr.$$

We can bound the term $\int_a^1 \|\omega(s)\|^2_F s^{c(f-2i+2)} ds$ in the following way

$$\int_a^1 \|\omega(s)\|^2_F s^{c(f-2i+2)} ds \leq \int_0^1 \|\omega(s)\|^2_F s^{c(f-2i+2)} ds$$

and therefore

$$\int_F \int_a^1 \|\omega(s)\|^2_F s^{c(f-2i+2)} ds dvol_F \leq \int_F \int_0^1 \|\omega(s)\|^2_F s^{c(f-2i+2)} ds dvol_F =$$

$$= \|\omega\|^2_{L^2(C^*(F),g_c)}$$

while for the term $\int_a^1 \|\omega(s)\|^2_F(1 - s^{c(f-2i+2)})ds$ we can use the following observation: there exist $l > 0$ such that $1 - s^{c(f-2i+2)} \leq |1 - s^{c(f-2i+2)}| \leq ls^{c(f-2i+2)}$ for any $s \in (\frac{1}{2}, 1]$. Therefore:

$$\int_a^1 \|\omega(s)\|^2_F(1 - s^{c(f-2i+2)})ds \leq \int_a^1 \|\omega(s)\|^2_F|(1 - s^{c(f-2i+2)})|ds \leq$$

$$l\int_a^1 \|\omega(s)\|^2_F s^{c(f-2i+2)}ds \leq l\int_0^1 \|\omega(s)\|^2_F s^{c(f-2i+2)}ds$$

and similarly to the previous case we get

$$\int_F\int_a^1 \|\omega(s)\|^2_F(1 - s^{c(f-2i+2)})dsdvol_F \leq l\|\omega\|^2_{L^2(C^*(F),g_c)}$$

and the constant l is independent of the choice of the form ω and of the choice of a. The fact that $i < \frac{f}{2} + \frac{1}{2c} + 1$ implies that $\int_0^1 r^{c(f-2i+2)}dr = \frac{1}{1+c(f-2i+2)} < \infty$ and so the following inequalities hold:

$$\|K_a(\alpha)\|^2_{L^2(C^*(F),g_c)} \leq (1-a)\int_0^1\int_F\int_a^1 \|\omega(s)\|^2_F ds r^{c(f-2i+2)}dvol_F dr$$

$$\leq \int_0^1 r^{c(f-2i+2)}dr(1-a)(1+l)\|\omega\|^2_{L^2(C^*(F,g_c))} \leq$$

$$\frac{1}{2}\frac{1+l}{1+c(f-2i+2)}\|\alpha\|^2_{L^2(C^*(F),g_c)}.$$

Therefore we can conclude that for $i < \frac{f}{2} + \frac{1}{2c} + 1$

$$K_a : L^2\Omega^i(C^*(F), g_c) \longrightarrow L^2\Omega^{i-1}(C^*(F), g_c)$$

is a bounded operator uniformly in $a \in (\frac{1}{2}, 1)$. □

PROPOSITION 2.4. *Let $0 < \rho < 1$ and endow $(\rho, 1)\times F$ with the metric g_c restricted from $C^*(F)$. Let $\alpha = \phi + dr \wedge \omega \in L^2\Omega^i(C^*(F), g_c)$. If $i \geq \frac{f}{2} + \frac{1}{2c}$ then there exists a sequences $\epsilon_s \to 0$ such that*

$$\lim_{\epsilon_s \to 0} \|\phi(\epsilon_s)\|^2_{L^2((\rho,1)\times F,g_c)} = 0$$

PROOF. By the fact that $\alpha \in L^2\Omega^i(C^*(F), g_c)$ follows that $\phi \in L^2\Omega^i(C^*(F), g_c)$, so we know that $\int_0^1\int_F \|\phi(r)\|^2_F r^{c(f-2i)}dvol_F dr < \infty$. This means that

$$\int_F \|\phi(r)\|^2_F r^{c(f-2i)}dvol_F \in L^1(0, 1).$$

Thus by [**23**] lemma 1.2 there is a sequences $\epsilon_s \to 0$ for wich

$$\left|\int_F \|\phi(\epsilon_s)\|^2_F \epsilon_s^{c(f-2i)}dvol_F\right| < \frac{C}{\epsilon_s|ln(\epsilon_s)|}$$

for some constant $C > 0$. In this way we obtain

$$\left|\int_F \|\phi(\epsilon_s)\|^2_F dvol_F\right| < \frac{C\epsilon_s^{c(f-2i)-1}}{|ln(\epsilon_s)|}.$$

Since $i \geq \frac{f}{2} + \frac{1}{2c}$ the right side tends to zero as $\epsilon_s \to 0$. Thus we obtain:

$$\|\phi(\epsilon_s)\|^2_{L^2((\rho,1)\times F, g_c)} = \int_\rho^1 \int_F \|\phi(\epsilon_s)\|^2_F \epsilon_s^{c(f-2i)} dvol_F dr$$
$$= \|\phi(\epsilon_s)\|^2_{L^2(F,g)} \int_\rho^1 r^{c(f-2i)} dr \longrightarrow 0$$

when $\epsilon_s \to 0$. □

Proposition 2.5. *If $i > \frac{f}{2} - \frac{1}{2c} + 1$ and $\alpha = \phi + dr \wedge \omega \in L^2\Omega^i(C^*(F), g_c)$, then*

$$K_0(\alpha) = \int_0^r \omega(s) ds \in L^2\Omega^{i-1}(C^*(F), g_c)$$

and $K_0 : L^2\Omega^i(C^*(F), g_c) \longrightarrow L^2\Omega^{i-1}(C^*(F), g_c)$ is a bounded operator.

Proof. By definition

$$\|K_0(\alpha)\|^2_{L^2(C^*(F),g_c)} = \int_0^1 \int_F \| \int_0^r \omega(s) ds\|^2_F r^{c(f-2i+2)} dvol_F dr.$$

We consider the term $\| \int_0^r \omega(s)ds\|^2_F$. Then:

$$\| \int_0^r \omega(s) ds\|^2_F \leq (\int_0^r \|\omega(s)\|_F ds)^2 = (\int_0^r s^{\frac{c}{2}(f-2i+2)} s^{\frac{c}{2}(2i-f-2)} \|\omega(s)\|_F ds)^2$$

and applying the Schwartz inequality we get that

$$\leq \int_0^r s^{c(2i-f-2)} ds \int_0^r s^{c(f-2i+2)} \|\omega(s)\|^2_F ds =$$
$$= \frac{r^{1+c(f-2i+2)}}{1+c(f-2i+2)} \int_0^r s^{c(f-2i+2)} \|\omega(s)\|^2_F ds.$$

The last equality is a consequence of the fact that $i > \frac{f}{2} - \frac{1}{2c} + 1$. Substituting the previous inequality in the definition of $\|K_0(\alpha)\|^2_{L^2(C^*(F),g_c)}$ we get:

$$\|K_0(\alpha)\|^2_{L^2(C^*(F),g_c)} \leq$$
$$\int_0^1 \int_F \int_0^1 s^{c(2i-f-2)} ds \int_0^r s^{c(f-2i+2)} \|\omega(s)\|^2_F ds dvol_F r^{c(f-2i+2)} dr$$
$$\leq \int_0^1 \frac{r}{1+c(2i-f-2)} dr \int_F \int_0^1 s^{c(f-2i+2)} \|\omega(s)\|^2_F ds dvol_F$$
$$= \frac{1}{2+2c(2i-f-2)} \|\omega\|^2_{L^2(C^*(F),g_c)} \leq \frac{1}{2+2c(2i-f-2)} \|\alpha\|_{L^2(C^*(F),g_c)}.$$

Thus

$$K_0 : L^2\Omega^i(C^*(F), g_c) \longrightarrow L^2\Omega^{i-1}(C^*(F), g_c)$$

is a bounded operator. □

Proposition 2.6. *Let*

$$K_\epsilon(\alpha) = \int_\epsilon^r \omega(s) ds$$

and let $0 < \rho < 1$. If $i > \frac{f}{2} - \frac{1}{2c} + 1$ then on $(\rho, 1) \times F$ with the restricted metric g_c,

$$K_\epsilon(\alpha) \longrightarrow K_0(\alpha)$$

in the $\| \ \|_{L^2((\rho,1)\times F,g_c)}$ *norm when* $\epsilon \to 0$.

PROOF. We have

$$\|K_\epsilon(\alpha) - K_0(\alpha)\| = \int_\rho^1 \int_F \| \int_0^\epsilon \omega(s)ds\|_F^2 r^{c(f-2i+2)} dvol_F dr.$$

Using the same techniques of the previous proof we obtain that the right hand side is at most

$$\frac{\epsilon^{1+c(2i-f-2)}}{1+c(2i-f-2)} (\int_\rho^1 r^{c(f-2i+2)} dr) \|\omega\|^2_{L^2(C^*(F),g_c)}.$$

Since $i > \frac{f}{2} - \frac{1}{2c} + 1$ the whole expression tends to 0 as $\epsilon \to 0$. $\square$

PROPOSITION 2.7. *Let* (F, g) *be an oriented riemannian manifold. Let* $\phi \in \mathcal{D}(d_{max,i-1}) \subset L^2\Omega^{i-1}(F, g)$, $\eta \in L^2\Omega^i(F, g)$ *such that* $d_{max,i-1}\phi = \eta$. *Then for all* $\rho \in (0, 1)$ *on* $(\rho, 1) \times F$ *with the restricted metric* g_c*:*

1. $\pi^*\phi \in L^2\Omega^{i-1}((\rho, 1) \times F)$
2. $\pi^*\eta \in L^2\Omega^i((\rho, 1) \times F)$
3. *For all* $\beta \in C_0^\infty\Omega^i((\rho, 1) \times F)$ *we have*

$$< \pi^*\phi, \delta_{i-1}\beta >_{L^2((\rho,1)\times F)} =< \pi^*\eta, \beta >_{L^2((\rho,1)\times F)}$$

that is on $(\rho, 1) \times F$ *with the restricted metric* g_c

$$d_{max,i-1}\pi^*\phi = \pi^*\eta.$$

PROOF.

$$\|\pi^*\phi\|^2_{L^2((\rho,1)\times F)} = \int_\rho^1 r^{c(f-2i+2)} dr \int_F \|\phi\|_F^2 dvol_F =$$

$$= \int_\rho^1 r^{c(f-2i+2)} dr \|\phi\|^2_{L^2(F,g)} < \infty$$

so $\pi^*\phi \in L^2\Omega^{i-1}((1, \rho) \times F)$;

$$\|\pi^*\eta\|^2_{L^2((\rho,1)\times F)} = \int_\rho^1 r^{c(f-2i)} dr \int_F \|\eta\|_F^2 dvol_F =$$

$$= \int_\rho^1 r^{c(f-2i)} dr \|\eta\|^2_{L^2(F,g)} < \infty$$

so $\pi^*\eta \in L^2\Omega^i((1, \rho) \times F)$.
By a Cheeger's result, [**23**] pag 93,

$$< \pi^*\phi, \delta_i\beta >_{L^2((\rho,1)\times F)} =< \pi^*\eta, \beta >_{L^2((\rho,1)\times F)} \quad \text{for all } \beta \in C_0^\infty\Omega^i((\rho, 1) \times F)$$

if and only if there is a sequence of smooth forms $\alpha_j \in L^2\Omega^{i-1}((\rho, 1) \times F)$ such that $d_{i-1}\alpha_j \in L^2\Omega^i((\rho, 1) \times F)$,

$$\|\pi^*\phi - \alpha_j\|_{L^2((\rho,1)\times F)} \to 0, \ \|\pi^*\eta - d_{i-1}\alpha_j\|_{L^2((\rho,1)\times F)} \to 0$$

for $j \to \infty$.
Using this Cheeger's result , from the fact that $\phi \in Dom(d_{i-1,max})$, it follows that there is a sequences of smooth forms $\phi_j \in L^2\Omega^{i-1}(F, g)$ such that $d_{i-1}\phi_j \in L^2\Omega^i(F, g)$, $\|\phi - \phi_j\|_{L^2(F,g)} \to 0, \|\eta - d_{i-1}\phi_j\|_{L^2(F,g)} \to 0$ for $j \to \infty$. Now if we put $\alpha_j = \pi^*(\phi_j)$ we obtain a sequence of smooth forms in

$L^2\Omega^{i-1}((\rho,1)\times F)$ satisfying the assumptions of the same Cheeger's result cited above. Indeed for each j

$$d_i\alpha_j \in L^2\Omega^i((\rho,1)\times F)$$

$$\|\alpha_j-\pi^*\phi\|_{L^2((\rho,1)\times F)}=\int_\rho^1 r^{c(f-2i+2)}dr\int_F \|\phi-\alpha_j\|_F^2 dvol_F \to 0$$

for $j\to\infty$ and similarly

$$\|d\alpha_j-\pi^*\eta\|_{L^2((\rho,1)\times F)}\to 0$$

for $j\to\infty$. Therefore we can conclude that for all $\beta\in C_0^\infty\Omega^i((\rho,1)\times F)$

$$<\pi^*\phi,\delta_i\beta>_{L^2((\rho,1)\times F)}=<\pi^*\eta,\beta>_{L^2((\rho,1)\times F)}.$$

□

PROPOSITION 2.8. *Let (F,g) be an oriented odd dimensional riemannian manifold such that* $d_{max,i-1}:\mathcal{D}(d_{max,i-1})\longrightarrow L^2\Omega^i(F,g)$ *has closed range, where* $i=\frac{f+1}{2}$ *and* $f=dimF$. *Let* $\alpha\in L^2\Omega^i(C^*(F),g_c)$ *a smooth* $i-$*form such that* $d_i\alpha\in L^2\Omega^{i+1}(C^*(F),g_c)$. *Then:*

(1) *For almost all* $b\in(0,1)$ *there is an exact* $i-$*form* $\eta_b\in\mathcal{D}(d_{max,i})\subset L^2\Omega^i(F,g)$, $\eta_b=d_{max,i-1}\psi_b$, $\psi_b\in\mathcal{D}(d_{max,i-1})\subset L^2\Omega^{i-1}(F,g)$, *such that for all* $0<\rho<1$ *on* $(\rho,1)\times F$ *with the restricted metric* g_c
$$\|d_{i-1}(K_b\alpha)-(\alpha-K_0(d_i\alpha)-\pi^*(\eta_b))\|_{L^2((\rho,1)\times F)}=0$$
(2) *On* $L^2\Omega^{i-1}(C^*(F),g_c)$ *we have:*
$$d_{max,i-1}(K_b\alpha+\pi^*(\psi_b))+K_0(d_i\alpha)=\alpha$$

PROOF. 1) Let $\alpha=\phi+dr\wedge\omega$. Consider $K_\epsilon(d_i\alpha)=\phi-\pi^*\phi(\epsilon)-\int_\epsilon^r d_F\omega ds$. Obviously for each $0<\rho<1$ $K_\epsilon(d_i\alpha)\in L^2\Omega^i((\rho,1)\times F)$ with the restricted metric g_c. From the fact that α is an $i-$ form and that $i+1=\frac{f+1}{2}+1>\frac{f}{2}+1-\frac{1}{2c}$ follows that we can use prop. 2.6 to conclude that

$$K_0(d_i\alpha)\in L^2\Omega^i(C^*(F),g_c)$$

and

$$\|K_\epsilon(d_i\alpha)-K_0(d_i\alpha)\|_{L^2((\rho,1)\times F)}\to 0$$

for $\epsilon\to 0$. For the same reasons we can use prop. 2.4 to say that there is a sequence $\epsilon_j\to 0$ such that, on $(\rho,1)\times F$ with the restricted metric g_c,

$$\lim_{\epsilon_j\to 0}\|\pi^*\phi(\epsilon_j)\|^2_{L^2((\rho,1)\times F,g_c)}=0.$$

Therefore using these facts we can conclude that

$$\lim_{\epsilon_j\to 0}\int_{\epsilon_j}^r d_F\omega ds \text{ exists in } L^2\Omega^i((\rho,1)\times F)$$

and, if we call this limit γ, we have

$$K_0(d_i(\alpha))=\phi-\gamma$$

in $L^2\Omega^i((\rho,1)\times F)$ with the restricted metric g_c. From this fact it follows that for almost all $b\in(0,1)$ $\int_{\epsilon_j}^b d_F\omega ds\to\gamma(b)$ in $L^2\Omega^i(F,g)$ for $\epsilon_j\to 0$.

But $\int_{\epsilon_j}^b d_F\omega ds$ is a smooth form in $L^2\Omega^i(F,g)$; $\int_{\epsilon_j}^b \omega ds$ is a smooth form in $L^2\Omega^{i-1}(F,g)$ and $d_{i-1}(\int_{\epsilon_j}^b \omega ds) = \int_{\epsilon_j}^b d_F\omega ds$. So we can conclude that $\int_{\epsilon_j}^b d_F\omega ds = d_{max,i-1}(\int_{\epsilon_j}^b \omega ds)$ with $d_{max,i-1} : \mathcal{D}(d_{max,i-1}) \to L^2\Omega^i(F,g)$. From this it follows that $\gamma(b)$ is in the closure of the image of $d_{max,i-1} : \mathcal{D}(d_{max,i-1}) \to L^2\Omega^i(F,g)$ and so it follows from the assumptions that there is $\psi_b \in \mathcal{D}(d_{max,i-1}) \subset L^2\Omega^{i-1}(F,g)$ such that

$$d_{max,i-1}\psi_b = \gamma(b).$$

We choose one of these b and ϵ such that $b > \epsilon$.
Now we consider $d_{i-1}(K_b(\alpha)) = dr \wedge \omega + \int_b^r d_F\omega$. Adding $d_{i-1}(K_b(\alpha))$ and $K_\epsilon(d_i\alpha)$ we obtain

$$d_{i-1}(K_b(\alpha)) = \alpha - K_\epsilon(d_i\alpha) - \pi^*\phi(\epsilon) - \pi^*(\int_\epsilon^b d_F\omega ds) \in L^2\Omega^i((\rho,1)\times F))$$

with the restricted metric g_c for all $\rho \in (0,1)$.
We analyze in detail the terms on the right of equality. As noted above from the prop. 2.4 we know that there is a sequence $\epsilon_j \to 0$ such that

$$\lim_{\epsilon_j\to 0} \|\pi^*\phi(\epsilon_j)\|^2_{L^2((\rho,1)\times F,g_c)} = 0.$$

Similarly from the proposition 2.6 we know that

$$\|K_{\epsilon_j}(d_i\alpha) - K_0(d_i\alpha)\|_{L^2((\rho,1)\times F)} \longrightarrow 0$$

for $\epsilon_j \to 0$. For the term $\pi^*(\int_{\epsilon_j}^b d_F\omega ds)$ we know, by the observations made at the beginning of the proof and prop. 2.7, that there is an $(i-1)-$form $\psi_b \in Dom(d_{max,i-1}) \subset L^2\Omega^{i-1}(F,g)$ such that

$$\|\pi^*(\int_{\epsilon_j}^b d_F\omega ds) - \pi^*(d_{max,i-1}(\psi_b))\|_{L^2((\rho,1)\times F)} \longrightarrow 0$$

for $\epsilon_j \to 0$. Summarizing, for all $\rho \in (0,1)$, we have on $(\rho,1)\times F$ with the restricted metric g_c

$$\lim_{\epsilon_j\to 0} \|\alpha - K_{\epsilon_j}(d_i\alpha) - \phi(\epsilon_j) +$$

$$-\pi^*(\int_{\epsilon_j}^b d_F\omega ds) - (\alpha - K_0(d_i\alpha) - \pi^*(d_{i-1,max}(\psi_b)))\|_{L^2((\rho,1)\times F)} = 0.$$

Therefore, if we put $\eta_b = \gamma(b)$, by the fact that

$$d_{i-1}(K_b(\alpha)) = \alpha - K_{\epsilon_j}(d_i\alpha) - \pi^*\phi(\epsilon_j) - \pi^*(\int_{\epsilon_j}^b d_F\omega ds)$$

for all j, we can conclude that

$$\|d_{i-1}(K_b\alpha) - (\alpha - K_0(d_i\alpha) - \pi^*(\eta_b))\|_{L^2((\rho,1)\times F)} = 0$$

2) Before proving the statement we observe that from that fact that $i = \frac{f+1}{2}$ it follows that we can use prop 2.3 to conclude that $K_b\alpha \in$

$L^2\Omega^{i-1}(C^*(F), g_c)$. Analogously we can use prop 2.1 to conclude that $\pi^*\psi_b \in L^2\Omega^{i-1}(C^*(F), g_c)$. Let $\phi \in C_0^\infty\Omega^i(C^*(F))$. Then there is $\rho \in (0,1)$ such that $supp(\phi) \subset (\rho, 1) \times F$.
We consider now:

$$< K_b\alpha, \delta_{i-1}\phi >_{L^2(C^*(F),g_c)} =< K_b\alpha, \delta_{i-1}\phi >_{L^2((\rho,1)\times F)} .$$

By the fact that $K_b(\alpha)$ is a smooth $(i-1)-$form such that

$$\|K_b(\alpha)\|_{L^2((1,\rho)\times F)} < \infty, \ \|d_{i-1}(K_b\alpha)\|_{L^2((1,\rho)\times F)} < \infty$$

and that ϕ is a smooth form with compact support it follows that:

$$\begin{gathered} < K_b\alpha, \delta_{i-1}\phi >_{L^2((\rho,1)\times F)} =< d_{i-1}(K_b(\alpha), \phi >_{L^2((\rho,1)\times F)} = \\ =< \alpha - K_0(d_i\alpha) - \pi^*(\eta_b), \phi >_{L^2((\rho,1)\times F)} = \\ =< \alpha, \phi >_{L^2((\rho,1)\times F))} - < K_0(d_i\alpha), \phi >_{L^2((\rho,1)\times F)} + \\ - < \pi^*(\eta_b), \phi >_{L^2((\rho,1)\times F)} = \\ =< \alpha, \phi >_{L^2((\rho,1)\times F)} - < K_0(d_i\alpha), \phi >_{L^2((\rho,1)\times F)} + \\ - < \pi^*(\psi_b), \delta_{i-1}\phi >_{L^2((\rho,1)\times F)} = \\ =< \alpha, \phi >_{L^2(C^*(F),g_c)} - < K_0(d_i\alpha), \phi >_{L^2(C^*(F),g_c)} + \\ - < \pi^*(\psi_b), \delta_{i-1}\phi >_{L^2(C^*(F),g_c)} . \end{gathered}$$

In particular the equality:

$$< \pi^*(\psi_b), \delta_{i-1}\phi >_{L^2((\rho,1)\times F)} =< \pi^*(\eta_b), \phi >_{L^2((\rho,1)\times F)}$$

follows from prop. 2.7. We have obtained that for all $\phi \in C_0^\infty\Omega^i(C^*(F))$

$$< K_b\alpha + \pi^*\psi_b, \delta_{i-1}\phi >_{L^2(C^*(F),g_c)} =< \alpha - K_0(d_i\alpha), \phi >_{L^2(C^*(F),g_c)} .$$

So we can conclude that

$$d_{max,i-1}(K_b\alpha + \pi^*(\psi_b)) + K_0(d_i\alpha) = \alpha.$$

□

2. L^2 cohomology of a cone over a riemannian manifold

In this section we continue to use the notations of the previous section.

THEOREM 2.9. *Let (F, g) be an oriented riemannian manifold. Then for the riemannian manifold $(C^*(F), g_c)$, with g_c as in (2.1) the following isomorphism holds:*

$$H^i_{2,max}(C^*(F), g_c) = \begin{cases} H^i_{2,max}(F, g) & i < \frac{f}{2} + \frac{1}{2c} \\ 0 & i > \frac{f}{2} + 1 - \frac{1}{2c} \end{cases} \tag{2.2}$$

PROOF. In the first part of the proof we use the complex

$$(\Omega_2^*(C^*(F), g_c), d_*)$$

of prop. 1.11. Let $\alpha \in \Omega_2^i(C^*(F), g_c)$, $\alpha = \phi + dr \wedge \omega$, $i = 0, ..., f+1$. Let $a \in (\frac{1}{2}, 1)$. Consider the following map

$$v_a : \Omega_2^i(C^*(F), g_c) \to \Omega_2^i(F, g), \ v_a(\alpha) = \phi(a). \tag{2.3}$$

By prop.2.2 $v_a(\alpha) \in L^2\Omega^i(F, g)$. Furthermore this map satisfies $v_a \circ d_i = d_i \circ v_a$ where on the left of the equality d_i is the $i-$th differential of the complex $(\Omega_2^*(C^*(F), g_c), d_*)$ while on the right of the equality the operator d_i is the

$i-$th differential of the complex $(\Omega_2^*(F,g),d_*)$. Therefore v_a is a morphism between the complex $(\Omega_2^*(C^*(F),g_c),d_*)$ and the complex $(\Omega_2^*(F,g),d_*)$ so it induces a map between the cohomology groups

$$v_a^*: H_2^i(C^*(F),g_c) \to H_2^i(F,g) \tag{2.4}$$

where $H_2^i(F,g)$ is the $i-th$ cohomology group of the complex $(\Omega_2^*(F,g),d_*)$. Now in the case $i < \frac{f}{2}+\frac{1}{2c}$, by proposition 2.3, we know that $K_a(\alpha)$ and $K_a(d_i\alpha)$ are two smooth form such that $\|K_a(d_i\alpha)\|_{L^2(C^*(F),g_c)} < \infty$ and $\|K_a\alpha\|_{L^2(C^*(F),g_c)} < \infty$. If we add the two following terms, $d_{i-1}(K_a(\alpha))$ and $K_a(d_i(\alpha))$ we obtain:

$$d_{i-1}(K_a\alpha) + K_a(d_i(\alpha)) = \tag{2.5}$$

$$= dr \wedge \omega(s)ds + \int_a^r d_F(s)ds\omega + \phi - \phi(a) - \int_a^r d_F(s)ds\omega = \alpha - \pi^*(v_a(\alpha)).$$

So we have obtained that $\|d_{i-1}(K_a\alpha)\|_{L^2(C^*(F),g_c)} < \infty$ and from this and (2.5) it follows that

$$(\pi^*)^* \circ v_a^*: H_2^i(C^*(F),g_c) \to H_2^i(C^*(F),g_c)$$

is an isomorphism for $i < \frac{f}{2}+\frac{1}{2c}$. Now from this fact it follows that for the same i:

$$v_a^*: H_2^i(C^*(F),g_c) \to H_2^i(F,g)$$

is injective and that

$$(\pi^*)^*: H_2^i(F,g) \to H_2^i(C^*(F),g_c)$$

is surjective. But from prop. 2.1 we know that $v_a^*: H_2^i(C^*(F),g_c) \to H_2^i(F,g)$ is surjective. So for $i < \frac{f}{2}+\frac{1}{2c}$ $H_2^i(C^*(F),g_c)$ and $H_2^i(F,g)$ are isomorphic and therefore by proposition 1.11 for the same i we have

$$H_{2,max}^i(C^*(F),g_c) \cong H_{2,max}^i(F,g).$$

Now we start the second part of the proof. We know that for each i every cohomology class $[\alpha] \in H_{2,max}^i(C^*(F))$ has a smooth representative. So let $\alpha \in L^2\Omega^i(C^*(F),g_c)$, $i > \frac{f}{2}+1-\frac{1}{2c}$, a smooth form such that $d_i\alpha = 0$. Observe that from the fact that α is closed follows that $\phi' = d_F\omega$ and therefore, given $\epsilon \in (0,1)$ we have:

$$d_{i-1}(K_\epsilon\alpha) = d_{i-1}(\int_\epsilon^r \omega(s)ds) = dr\wedge\omega + \int_\epsilon^r d_F\omega(s)ds = dr\wedge\omega + \int_\epsilon^r \phi'(s)ds =$$

$$= dr \wedge \omega + \phi - \phi(\epsilon) = \alpha - \phi(\epsilon).$$

Consider $K_0(\alpha)$; by proposition 2.5 we know that $K_0(\alpha) \in L^2\Omega^i(C^*(F),g_c)$. We want to show that $d_{max,i-1}(K_0(\alpha)) = \alpha$.
Let $\beta \in C_0^\infty\Omega^i(C^*(F))$. Then there is $\rho > 0$ such that $supp(\beta) \subset (\rho,1)\times F$. Therefore:

$$< K_0\alpha, \delta_{i-1}\beta >_{L^2(C^*(F))} = < K_0\alpha, \delta_{i-1}\beta >_{L^2((\rho,1)\times F)} = (by\ prop\ 2.6)$$

$$= \lim_{\epsilon\to 0} < K_\epsilon\alpha, \delta_{i-1}\beta >_{L^2((\rho,1)\times F)}.$$

By the fact that $K_\epsilon(\alpha)$ is a smooth form such that

$$\|K_\epsilon(\alpha)\|_{L^2((1,\rho)\times F)} < \infty,\ \|d_{i-1}(K_\epsilon\alpha)\|_{L^2((1,\rho)\times F)} < \infty$$

and that ϕ is a smooth form with compact support it follows that:

$$\lim_{\epsilon\rightarrow 0} < K_\epsilon\alpha, \delta_{i-1}\beta >_{L^2((\rho,1)\times F)} = \lim_{\epsilon\rightarrow 0} < d_{i-1}(K_\epsilon\alpha), \beta >_{L^2((\rho,1)\times F)} =$$
$$= \lim_{\epsilon\rightarrow 0} < \alpha - \phi(\epsilon), \beta >_{L^2((\rho,1)\times F)} =$$
$$= < \alpha, \beta >_{L^2((\rho,1)\times F)} - \lim_{\epsilon\rightarrow 0} < \phi(\epsilon), \beta >_{L^2((\rho,1)\times F)} .$$

In particular the limit

$$\lim_{\epsilon\rightarrow 0} < \phi(\epsilon), \beta >_{L^2((\rho,1)\times F)}$$

exist. But from prop. 2.4 we know that there is a sequence $\epsilon_j \rightarrow 0$ such that

$$\lim_{\epsilon_j\rightarrow 0} < \phi(\epsilon_j), \beta >_{L^2((\rho,1)\times F)} = 0.$$

Therefore

$$< K_0\alpha, \delta_{i-1}\beta >_{L^2((\rho,1)\times F)} = < \alpha, \delta_{i-1}\beta >_{L^2((\rho,1)\times F)} = < \alpha, \delta_{i-1}\beta >_{L^2(C^*(F),g_c)} .$$

Thus we can conclude that $d_{max,i-1}(K_0(\alpha)) = 0$ and hence that

$$H^i_{2,max}(C^*(F), g_c) = 0 \ \text{ for } i > \frac{f}{2} + 1 - \frac{1}{2c}.$$

□

COROLLARY 2.10. *Suppose that one of three following hypotheses applies:*

(1) $0 < c < 1$.
(2) $c \geq 1$ *and* $f = dimF$ *is even.*
(3) $c \geq 1$, f *is odd and* $d_{max,i-1} : \mathcal{D}(d_{max,i-1}) \rightarrow L^2\Omega^i(F, g)$ *has close range where* $i = \frac{f+1}{2}$. *(By prop 1.3 this happen for example when* $H^i_{2,max}(F, g)$ *is finite dimensional.)*

Then for the riemannian manifold $(C^*(F), g_c)$ *the following isomorphism holds:*

$$H^i_{2,max}(C^*(F), g_c) = \begin{cases} H^i_{2,max}(F, g) & i < \frac{f}{2} + \frac{1}{2c} \\ 0 & i \geq \frac{f}{2} + \frac{1}{2c} \end{cases} \tag{2.6}$$

PROOF. If $0 < 1 < c$ then $\frac{f}{2} + \frac{1}{2c} > \frac{f}{2} + 1 - \frac{1}{2c}$.
If $c \geq 1$ and f is even then $i > \frac{f}{2} + 1 - \frac{1}{2c}$ if and only if $i \geq \frac{f}{2} + \frac{1}{2c}$.
Finally if $c \geq 1$, f is odd and $d_{max,i-1} : Dom(d_{max,i-1}) \rightarrow L^2\Omega^i(F, g)$ has close range then the thesis immediately follows from prop. 2.8. □

REMARK 2.1. Now we make a simple remark; theorem 2.9 also holds in the following two cases:

1) If we replace $C(F)$ with $C_\epsilon(F)$ where $C_\epsilon(F) = F \times [0, \epsilon)/F \times \{0\}$ and where ϵ is any real positive number. In this case we have only to modify prop. 2.2 and prop. 2.3 choosing $a \in (\gamma, \epsilon)$ where γ is a fixed and positive real number strictly smaller than ϵ. Furthermore if $\epsilon < \delta$

$$i^* : (L^2\Omega^*(C^*_\delta(F), g_c), d_{max,*}) \rightarrow (L^2\Omega^*(C^*_\epsilon(F), g_c), d_{max,*})$$

where i^* is the morphism of complexes induced by the inclusion $i : C_\epsilon(F) \rightarrow C_\delta(F)$, induces an isomorphism between the cohomology groups

$$H^i_{2,max}(C^*_\epsilon(F), g_c) \text{ and } H^i_{2,max}(C^*_\delta(F), g_c)$$

for each $i < \frac{f}{2} + \frac{1}{2c}$ or $i > \frac{f}{2} + 1 - \frac{1}{2c}$. This last assertion is easy to see. When $i > \frac{f}{2} + 1 - \frac{1}{2c}$ it is obvious because the cohomology groups are both null; when $i < \frac{f}{2} + \frac{1}{2c}$ it follows by the fact that given $a \in (\gamma, \epsilon)$ and given v_a, which is the evaluation map defined like in (2.3), we have $v_a = v_a \circ i^*$ where at the left of the equality v_a is between $\Omega_2^i(C_\delta^*(F), g_c)$ and $\Omega_2^i(F, g)$ and at the right of the equality it is between $\Omega_2^i(C_\epsilon^*(F), g_c)$ and $\Omega_2^i(F, g)$. Finally if the hypotheses of corollary 2.10 holds then the same corollary holds for $C_\epsilon^*(F)$ and in this case i^* induces an isomorphism between $H^i_{2,max}(C_\epsilon^*(F), g_c)$ and $H^i_{2,max}(C_\delta^*(F), g_c)$ for all i.

2) When (F, g) is a disconnected riemannian manifold made of a finite number of connected components all having the same dimension, that is $(F, g) = \bigcup_{j \in J}(F_j, g_j)$, $dimF_i = dimF_j$ for each $i, j \in J$ and J is finite. Indeed in this case:

$$H^i_{2,max}(C^*(F), g_c) = H^i_{2,max}(C^*(\bigcup_{j \in J} F_j), g_c) = \tag{2.7}$$

$$= \bigoplus_{j \in J} H^i_{2,max}(C^*(F_j), g_{c,j}) = \tag{2.8}$$

$$= \bigoplus_{j \in J} \begin{cases} H^i_{2,max}(F_j, g_j) & i < \frac{f}{2} + \frac{1}{2c} \\ 0 & i > \frac{f}{2} + 1 - \frac{1}{2c} \end{cases} = \begin{cases} H^i_{2,max}(F, g) & i < \frac{f}{2} + \frac{1}{2c} \\ 0 & i > \frac{f}{2} + 1 - \frac{1}{2c} \end{cases}$$

Obviously if each (F_j, g_j) satisfies the assumptions of corollary 2.10 then also corollary 2.10 holds for $(C^*(F), g_c)$. This situation could happen in theorem 2.12 of the next section. In that case the manifold F will be the regular part of a link and it could happen that it is disconnected.

We conclude the section recalling a result from [**23**] that we will use in the proof of theorem 2.12.

PROPOSITION 2.11. *Let (M, g) be a Riemannian manifold. Then for the riemannian manifold $((0,1) \times M, dr \otimes dr + g)$ the following isomorphism holds:*

$$H^i_{2,max}((0,1) \times M, dr \otimes dr + g) \cong H^i_{2,max}(M, g) \ for\ all\ i = 0, ..., dimM + 1 \tag{2.9}$$

PROOF. See [**23**] pag 115. □

3. L^2 de Rham and Hodge theorems

In this section we prove the mail results of the chapter. The first one is an L^2-de Rham-Hodge theorem for $(reg(X), g)$ where X is a compact and oriented smoothly stratified pseudomanifol with a Thom-Mather stratification and g is a quasi-edge metric with weights over $reg(X)$. We will show that the absolute and relative Hodge cohomology groups are respectively isomorphic to the maximal and minimal L^2 de Rham cohomology which are

in turn isomorphic respectively to the intersection cohomology groups associated to $t - p_g$ and p_g . Moreover we give a partial answer to the inverse question: given a general perversity p on X is there a riemannian metric over $reg(X)$ such that its maximal (or minimal)L^2 de Rham cohomology is isomorphic to the intersection cohomology relative to p? Under some assumptions we will show that the answer is positive.
Before starting we make a remark about the notation. Given an open subset $U \subset X$ with $\mathcal{D}(U, d_{max/min,i})$ we mean the domain of $d_{max/min,i}$ in $L^2\Omega^i(reg(U), g|_{reg(U)})$ Given a complex of sheaves $(\mathcal{L}^*, d_*)$ over X and an open subset U of X with the symbol $H^i(\mathcal{L}^*(U), d_*)$ we mean the i−th cohomology group of the complex

$$... \stackrel{d_{i-2}}{\rightarrow} \mathcal{L}^{i-1}(U) \stackrel{d_{i-1}}{\rightarrow} \mathcal{L}^i(U) \stackrel{d_i}{\rightarrow} \mathcal{L}^{i+i}(U) \stackrel{d_{i+1}}{\rightarrow} ...$$

Finally with $\mathbb{H}^i(\mathcal{L}^*, d_*)$ we mean the i−th cohomology sheaf associated to the complex $(\mathcal{L}^*, d_*)$.

Theorem 2.12. *Let X be a compact and oriented smoothly stratified pseudomanifold of dimension n with a Thom-Mather stratification $\mathfrak{X}$. Let g be a quasi edge metric with weights on $reg(X)$, see definition 1.29. Let $\mathcal{R}_0$ be the stratified coefficient system made of the pair of coefficient systems given by $(X - X_{n-1}) \times \mathbb{R}$ over $X - X_{n-1}$ where the fibers $\mathbb{R}$ have the discrete topology and the constant 0 system on X_{n-1}. Let p_g be the general perversity associated to the metric g, see definition 1.34. Then, for all $i = 0, ..., n$, the following isomorphisms holds:*

$$I^{q_g}H^i(X, \mathcal{R}_0) \cong H^i_{2,max}(reg(X), g) \cong \mathcal{H}^i_{abs}(reg(X), g) \tag{2.10}$$

$$I^{p_g}H^i(X, \mathcal{R}_0) \cong H^i_{2,min}(reg(X), g) \cong \mathcal{H}^i_{rel}(reg(X), g) \tag{2.11}$$

where q_g is the complementary perversity of p_g, that is, $q_g = t - p_g$ and t is the usual top perversity. In particular, for all $i = 0, ..., n$ the groups

$$H^i_{2,max}(reg(X), g),\ H^i_{2,min}(reg(X), g),\ \mathcal{H}^i_{abs}(reg(X), g),\ \mathcal{H}^i_{rel}(reg(X), g)$$

are all finite dimensional.

Theorem 2.13. *Let X be as in the previous theorem. Let p a general perversity in the sense of Friedman on X. If p satisfies the following conditions:*

$$\begin{cases} p \geq \overline{m} \\ p(Y) = 0 \quad if\ cod(Y) = 1 \end{cases} \tag{2.12}$$

then there exists g, a quasi edge edge metric with weights on $reg(X)$, such that

$$I^pH^i(X, \mathcal{R}_0) \cong H^i_{2,min}(reg(X), g) \cong \mathcal{H}^i_{rel}(reg(X), g). \tag{2.13}$$

Conversely if p satisfies:

$$\begin{cases} p \leq \underline{m} \\ p(Y) = -1 \quad if\ cod(Y) = 1 \end{cases} \tag{2.14}$$

then, also in this case, there exists a quasi edge metric with weights h on $reg(X)$ such that

$$I^pH^i(X, \mathcal{R}_0) \cong H^i_{2,max}(reg(X), h) \cong \mathcal{H}^i_{abs}(reg(X), h). \tag{2.15}$$

Before proving these theorems we need some preliminary results.

PROPOSITION 2.14. *Let X be an oriented smoothly stratified pseudomanifold of dimension n with a Thom-Mather stratification and let g a riemannian metric on $reg(X)$. Consider, for every $i = 0, ..., n$, the following presheaf:*

$$U \longmapsto \mathcal{D}(U, d_{max,i}) = \begin{cases} \mathcal{D}(U, d_{max,i}) & U \cap X_{n-1} = \emptyset \\ \mathcal{D}(U - (U \cap X_{n-1}), d_{max,i}) & U \cap X_{n-1} \neq \emptyset \end{cases} \tag{2.16}$$

or

$$U \longmapsto \begin{cases} \omega \in \Omega^i_2(U, g|_U) & U \cap X_{n-1} = \emptyset \\ \omega \in \Omega^i_2(reg(U), g|_{reg(U)}) & U \cap X_{n-1} \neq \emptyset \end{cases} \tag{2.17}$$

Let $\mathcal{L}^i_{2,max}$ and $\mathcal{L}^i_2$ be the sheaves associated to the previous presheaves; then for these sheaves we have the following explicit descriptions:

(1) *let U an open subset of X then:*

$$\mathcal{L}^i_{2,max}(U) \cong \{\omega \in L^2_{Loc}\Omega^i(reg(U), g|_{reg(U)}) : \forall\, p \in U \exists\, V \text{ open}$$

$$\text{neighbourhood of} p \text{ in } U \text{ such that } \omega|_{reg(V)} \in \mathcal{D}(reg(V), d_{max,i})\}.$$

(2) $\mathcal{L}^i_2(U) \cong \{\omega \in \Omega^i(reg(U), g|_{reg(U)}) : \forall\, p \in U\ \exists\, V$ *open neighbourhood*

$$\text{of } p \text{ in } U \text{ such that } \omega|_{reg(V)} \in \Omega^i_2(reg(V), g|_{reg(V)})\}.$$

(3) *If X is compact $\mathcal{L}^i_{2,max}(X) = \mathcal{D}(reg(X), d_{max,i})$.*

(4) $\mathcal{L}^i_2(X) = \{\omega \in \Omega^i(reg(X)) : \omega \in L^2\Omega^i(reg(X), g),$

$$d_i\omega \in L^2\Omega^i(reg(X), g)\}.$$

(5) *The complexes $\mathcal{L}^i_{2,max}$ and $\mathcal{L}^i_2$ are quasi isomorphic.*

PROOF. The first and the second statement follow from the fact that the sheaves $\mathcal{L}^i_{2,max}$, $\mathcal{L}^i_2$ and the respective sheaves at the right of $\cong$ have isomorphic stalks. The third and fourth statement are an immediate consequences of the compactness of X. The fifth statement follows immediately from proposition 1.11. $\square$

PROPOSITION 2.15. *Let X be an oriented smoothly stratified pseudomanifold with a Thom-Mather stratification of dimension n such that for each stratum Y the link L_Y is compact and let g be a quasi rigid iterated edge metric with weights on $reg(X)$. Then, for each $i = 0, ..., n$, $\mathcal{L}^i_{2,max}$ and $\mathcal{L}^i_2$ are fine sheaves.*

PROOF. From the description of the sheaves $\mathcal{L}^i_{2,max}$, $\mathcal{L}^i_2$ given in prop. 2.14 it follows that in order to prove this proposition it is sufficient to show that on X, given an open cover $\mathcal{U}_A = \{U_\alpha\}_{\alpha \in A}$, there is a bounded partition of unity with bounded differential subordinate to $\mathcal{U}_A$, that is a family of functions $\lambda_\alpha : X \to [0, 1], \alpha \in A$ such that

(1) Each λ_α is continuous and $\lambda_\alpha|_{reg(X)}$ is smooth.
(2) $supp(\lambda_\alpha) \subset U_\alpha$ for some $\alpha \in A$.
(3) $\{supp(\lambda_\alpha)\}_{\alpha \in A}$ is a locally finite cover of X.
(4) For each $x \in X$ $\sum_{\alpha \in A} \lambda_\alpha(x) = 1$.

(5) There are constants $C_\alpha > 0$ such that each λ_α satisfies
$$\|d(\lambda_\alpha|_{reg(X)})\|_{L^2(reg(X),g)} \leq C_\alpha.$$

The proof is given by induction on the depth of X. If $depth(X) = 0$ the statement is immediate because in this case X is a differentiable manifold. Suppose now that the statement is true if $depth(X) \leq k-1$ and that $depth(X) = k$. Let $\mathcal{U}_J = \{U_j\}_{j\in J}$ be a locally finite refinement of $\mathcal{U}_A$ such that for each U_J there is a diffeomorphism $\phi_j : U_j \to \mathbb{R}^n$ if $U_j \cap X_{n-1} = \emptyset$ or, in the case $U_j \cap X_{n-1} \neq \emptyset$, an isomorphism $\phi_j : U_j \to W_j \subset \mathbb{R}^k \times C(L_j)$ between U_j and an open subset, W_j, of the product $\mathbb{R}^k \times C(L_j)$ for some $k < n$ and stratified space L_j.
Let $\mathcal{V}_J = \{V_j\}_{j\in J}$ a shrinking of $\mathcal{U}_J$; this means that $\mathcal{V}_J$ is a refinement of $\mathcal{U}_J$ such that if $V_j \subset U_j$ then $\overline{V_j} \subset U_j$. Now let $V_j \in \mathcal{V}_J$, $U_j \in \mathcal{U}_J$ such that $V_j \subset U_j$ and $U_j \cap X_{n-1} = \emptyset$. Let $\psi_j : \mathbb{R}^n \to [0,1]$ be a smooth function such that $\psi_j|_{\overline{\phi_j(V_j)}} = 1$ and $supp(\psi_j) \subset \phi_j(U_j)$. Define $\lambda_j : X \to [0,1], \lambda_j := \psi_j \circ \phi_j$. Now let $V_j \in \mathcal{V}_J$, $U_j \in \mathcal{U}_J$ such that $V_j \subset U_j$ and $U_j \cap X_{n-1} \neq \emptyset$. We can take two functions $\eta : \mathbb{R}^k \to [0,1]$, $\xi : [0,1) \to [0,1]$ and, using the inductive hypothesis and the fact that L_Y is compact, a third function $\tau_j : L_j \to [0,1]$ smooth on $reg(L_j)$ and with bounded differential such that $\psi_j := \eta_j \xi_j \tau_j$ is a a continuous function on $\mathbb{R}^k \times C(L_j) \to [0,1]$ smooth on the regular part and with bounded differential such that $\psi_j|_{\overline{\phi_j(V_j)}} = 1$ and $supp(\psi_j) \subset \phi_j(U_j)$. Also in this case define $\lambda_j : X \to [0,1],\ \lambda_j := \psi_j \circ \phi_j$. Finally define
$$\mu_j : X \to [0,1], \mu_j = \frac{\lambda_j}{\sum_{j\in J} \lambda_j} \tag{2.18}$$
$\{\mu_j\}_{\in J}$ is a partition of unity with bounded differential subordinated to the cover $\mathcal{U}_J$ and therefore from this follows immediately that there exist a partition of unity with bounded differential subordinated to the cover $\mathcal{U}_A$. Now the statement of the proposition is an immediate consequence. □

Now we state the last proposition that we will use in the proof of theorem 2.12.

PROPOSITION 2.16. *Let L be a compact smoothly stratified pseudomanifold with a Thom-Mather stratification and let g_L be a riemannian metric on $reg(L)$. Let $C(L)$ be the cone over L and on $reg(C(L))$ consider the metric $dr \otimes dr + r^{2c} g_L$. Finally consider on $C(L)$ the complex of sheaves $(\mathcal{L}^*_{2,max}, d_{max,*})$ associated to the metric $dr \otimes dr + r^{2c} g_L$. Then the canonical inclusion*
$$i_v : C(L) - \{v\} \longrightarrow C(L),$$
where v is the vertex of the cone, induces a quasi-isomorphism between the complexes
$$(\mathcal{L}^*_{2,max}, d_{max,*}) \text{ and } (i_{v*} i_v^* \mathcal{L}^*_{2,max}, d_{max,*})$$
for $i \leq [[\frac{dimL}{2} + \frac{1}{2c}]]$.

PROOF. We start the proof showing that the complexes $(\mathcal{L}^*_{2,max}, d_{max,*})$ and $(i_{v*}i^*_v\mathcal{L}^*_{2,max}, d_{max,*})$ are quasi isomorphic for $i \leq [[\frac{dimL}{2} + \frac{1}{2c}]]$. This is equivalent to show that for each $x \in C(L)$

$$(\mathbb{H}^i(\mathcal{L}^*_{2,max}, d_{max,*}))_x \cong (\mathbb{H}^i(i_{v*}i^*_v\mathcal{L}^*_{2,max}, d_{max,*}))_x$$

where each term in the previous isomorphism is the stalk at the point x of the $i-$th cohomology sheaf associated to $(\mathcal{L}^*_{2,max}, d_{max,*})$ and $(i_{v*}i^*_v\mathcal{L}^*_{2,max}, d_{max,*})$ respectively. For every $i = 0, ..., dimL+1$ the sheaf $i_{v*}i^*_v\mathcal{L}^i_{2,max}$ is isomorphic to the following sheaf; let $U \subset C(L)$ be an open subset then:

$i_{v*}i^*_v\mathcal{L}^i_{2,max}(U) \cong \{\omega \in L^2_{Loc}\Omega^i(reg(U), dr\otimes dr+r^{2c}g_L|_{reg(U)}) : \forall\, p \in U-\{v\}$ $\exists\, V$ *open neighbourhood of* p *in* U *such that* $\omega|_{reg(V)} \in \mathcal{D}(reg(V), d_{max,i})\}$.

From this fact and prop. 2.14 it follows that for every $x \in C(L) - \{v\}$

$$(\mathbb{H}^i(\mathcal{L}^*_{2,max}, d_{max,*}))_x \cong (\mathbb{H}^i(i_{v*}i^*_v\mathcal{L}^*_{2,max}, d_{max,*}))_x. \qquad (2.19)$$

Now by theorem 2.9 and remark 2.1 we know that for $i \leq [[\frac{dimL}{2} + \frac{1}{2c}]]$

$$(\mathbb{H}^i(\mathcal{L}^*_{2,max}, d_{max,*}))_v \cong H^i(\mathcal{L}^*_{2,max}(C(L)), d_{max,*}) \cong H^i_{2,max}(reg(L), g_L).$$

Using the same techniques it is easy to show that for each i

$$(\mathbb{H}^i(i_{v*}i^*_v\mathcal{L}^*_{2,max}, d_{max,*}))_v \cong H^i(i_{v*}i^*_v\mathcal{L}^*_{2,max}(C(L)), d_{max,*}).$$

Therefore we have to show that for $i \leq [[\frac{dimL}{2} + \frac{1}{2c}]]$

$$H^i(i_{v*}i^*_v\mathcal{L}^*_{2,max}(C(L)), d_{max,*}) \cong H^i_{2,max}(reg(L), d_{max,*}).$$

On the whole cone $C(L)$ the main difference between the complexes

$$(\mathcal{L}^*_{2,max}, d_{max,*}) \text{ and } (i_{v*}i^*_v\mathcal{L}^*_{2,max}, d_{max,*})$$

is that for each $\omega \in \mathcal{L}^i_{2,max}(L)$, by prop. 2.1,

$$\pi^*\omega \in \mathcal{L}^i_{2,max}(C(L)) \text{ if and only if } i < \frac{dimL}{2} + \frac{1}{2c}.$$

Instead

$$\pi^*\omega \in i_{v*}i^*_v\mathcal{L}^i_{2,max}(C(L)) \text{ for every } i = 0, ..., dimL.$$

Therefore by the proof of the first part of theorem 2.9 and in particular from (2.5) follows that

$$H^i(i_{v*}i^*_v\mathcal{L}^*_{2,max}(C(L)), d_{max,*}) \cong H^i_{2,max}(reg(L), g_L) \qquad (2.20)$$

for every$i = 0, ..., dimL + 1$.

But from theorem 2.10 we know that for $i \leq [[\frac{dimL}{2} + \frac{1}{2c}]]$

$$H^i(\mathcal{L}^*_{2,max}(C(L)), d_{max,*}) \cong H^i_{2,max}(reg(L), g_L). \qquad (2.21)$$

So for $i \leq [[\frac{dimL}{2} + \frac{1}{2c}]]$

$$(\mathbb{H}^i(i_{v*}i^*_v\mathcal{L}^*_{2,max}, d_{max,*}))_v \cong (\mathbb{H}^i(\mathcal{L}^*_{2,max}, d_{max,*}))_v$$

and therefore we can conclude that for the same i the complexes

$$(\mathcal{L}^*_{2,max}, d_{max,*}) \text{ and } (i_{v*}i^*_v\mathcal{L}^*_{2,max}, d_{max,*})$$

are quasi-isomorphic.

Now let j be the morphism between $(\mathcal{L}^*_{2,max}, d_{max,*})$ and $(i_{v*}i^*_v\mathcal{L}^*_{2,max}, d_{max,*})$ induced from $i_v : C(L) - \{v\} \to C(L)$. It is immediate to note that for each open subset $U \subset C(L)$ j_U is just the inclusion of $\mathcal{L}^*_{2,max}(U)$ in $i_{v*}i^*_v\mathcal{L}^*_{2,max}(U)$. Therefore if we call j^* the morphism induced from j between the cohomology sheaves $H^i(\mathcal{L}^*_{2,max}, d_{max,*})$ and $H^i(i_{v*}i^*_v\mathcal{L}^*_{2,max}, d_{max,*})$ it is immediate to note that j^* induces the isomorphism (2.19). Finally if we call ϕ and ψ respectively the isomorphisms (2.20) and (2.21) we have that for $i \leq [[\frac{dimL}{2} + \frac{1}{2c}]]$

$$\phi \circ j^* = \psi.$$

Therefore we can conclude that

$$j : (\mathcal{L}^*_{2,max}, d_{max,*}) \to (i_{v*}i^*_v\mathcal{L}^*_{2,max}, d_{max,*})$$

is a quasi-isomorphism for $i \leq [[\frac{dimL}{2} + \frac{1}{2c}]]$. □

Corollary 2.17. *Let (M, h) be an oriented riemannian manifold, let L be a compact smoothly stratified pseudomanifold with a Thom-Mather stratification and let g_L be a riemannian metric on $reg(L)$. Consider now $M \times C(L)$ and on $reg(M \times C(L))$ consider the metric $h + dr \otimes dr + r^{2c}g_L$. Let $i_M : M \times C(L) - (M \times \{v\}) \to M \times C(L)$ be the canonical inclusion where v is the vertex of the cone. Finally consider over $M \times C(L)$ the complex of sheaves $(\mathcal{L}^*_{2,max}, d_{max,*})$. Then the canonical inclusion*

$$i_M : M \times C(L) - (M \times \{v\}) \longrightarrow M \times C(L)$$

induces a quasi-isomorphism between the complexes

$$(\mathcal{L}^*_{2,max}, d_{max,*}) \text{ and } (i_{M*}i^*_M\mathcal{L}^*_{2,max}, d_{max,*})$$

for $i \leq [[\frac{dimL}{2} + \frac{1}{2c}]]$.

Proof. The proof is completely analogous to the proof of proposition 2.16. For every $i = 0, ..., dimM + dimL + 1$ the sheaf $i_{M*}i^*_M\mathcal{L}^i_{2,max}$ is isomorphic to the following sheaf; let $U \subset M \times C(L)$ an open subset then:

$$i_{M*}i^*_M\mathcal{L}^i_{2,max}(U) \cong \{\omega \in L^2_{Loc}\Omega^i(reg(U), h + dr \otimes dr + r^{2c}g_L|_{reg(U)}) :$$
$$\forall\, p \in U - (U \cap (M \times \{v\}) \;\exists\, V \text{ open neighbourhood of } p \text{ in } U$$
$$\text{such that } \omega|_{reg(V)} \in \mathcal{D}(reg(V), d_{max,i})\}.$$

From this it follows that for every $x \in M \times C(L) - (M \times \{v\})$

$$(\mathbb{H}^i(\mathcal{L}^*_{2,max}, d_{max,*}))_x \cong (\mathbb{H}^i(i_{M*}i^*_M\mathcal{L}^*_{2,max}, d_{max,*}))_x.$$

Now let $p = (m, v) \in M \times \{v\}$. By theorem 2.9, remark 2.1 and proposition 2.11 we know that:

$$(\mathbb{H}^i(\mathcal{L}^*_{2,max}, d_{max,*}))_p \cong H^i(\mathcal{L}^*_{2,max}(U \times C(L)), d_{max,*}) \cong \tag{2.22}$$
$$\cong H^i_{2,max}(reg(L), g_L)$$

for $i \leq [[\frac{dimL}{2} + \frac{1}{2c}]]$ where U is an open neighborhood of m in M diffeomorphic to an open ball in $\mathbb{R}^s$ where $s = dimM$. Moreover, like in the proof of the previous proposition, it is easy to show that

$$(\mathbb{H}^i(i_{v*}i^*_v\mathcal{L}^*_{2,max}, d_{max,*}))_p \cong H^i(i_{v*}i^*_v\mathcal{L}^*_{2,max}(U \times C(L)), d_{max,*}) \tag{2.23}$$

where U is as in (2.22). Therefore in order to show that

$$(\mathbb{H}^i(i_{M*}i^*_M\mathcal{L}^*_{2,max}, d_{max,*}))_p \cong (\mathbb{H}^i(\mathcal{L}^*_{2,max}, d_{max,*}))_p$$

for $i \leq [[\frac{dimL}{2} + \frac{1}{2c}]]$ it is sufficient to show that for the same i

$$H^i(i_{M*}i_M^*\mathcal{L}^*_{2,max}(U \times C(L)), d_{max,*}) \cong H^i(\mathcal{L}^*_{2,max}(U \times C(L)), d_{max,*})$$

where U is as in (2.22). But from the same observations of the proof of prop. 2.16 and prop. 2.11 follows immediately that

$$H^i(i_{M*}i_M^*\mathcal{L}^*_{2,max}(U \times C(L)), d_{max,*}) \cong H^i_{2,max}(reg(L), g_L) \text{ for each } i$$

and that

$$H^i(\mathcal{L}^*_{2,max}(U \times C(L)), d_{max,*}) \cong H^i_{2,max}(reg(L), g_L) \text{ for } i \leq [[\frac{dimL}{2} + \frac{1}{2c}]].$$

So for $i \leq [[\frac{dimL}{2} + \frac{1}{2c}]]$

$$(\mathbb{H}^i(i_{M*}i_M^*\mathcal{L}^*_{2,max}, d_{max,*}))_p \cong (\mathbb{H}^i(\mathcal{L}^*_{2,max}, d_{max,*}))_p$$

and therefore we can conclude that for the same i the complexes

$$(\mathcal{L}^*_{2,max}, d_{max,*}) \text{ and } (i_{M*}i_M^*\mathcal{L}^*_{2,max}, d_{max,*})$$

are quasi-isomorphic. Now using the same final considerations of the previous proof we get the conclusion. □

Finally we can give the proof of the theorem announced at the beginning of the section:

Proof. (of theorem 2.12). Using corollary 1.33 we know that there is a quasi rigid iterated edge metric on reg(X), g', that is quasi-isometric to g. So, without loss of generality, we can suppose that g is a quasi rigid iterated edge metric with weights. We start by proving the isomorphism 2.10. The proof is given by induction on the depth of X. If $depth(X) = 0$ there is nothing to show because, in this case, X is a closed manifold and therefore the isomorphisms 2.10 are the well know theorems of Hodge and de Rham. Suppose now that the theorem is true if $depth(X) \leq k-1$ and that $depth(X) = k$. We will show that the theorem is also true in this case. We begin showing the first isomorphism, $H^i_{2,max}(reg(X), g) \cong I^{q_g}H^i(X, \mathcal{R}_0)$; to do this we will use theorem 1.25, corollary 1.26 and remark 1.4. More precisely we will show that the complex $(\mathcal{L}^i_{2,max}, d_{max,i})$ satisfies the three axioms of theorem 1.25 respect to the perversity p_g, the stratification $\mathfrak{X}$ and the local system over $reg(X)$ given by $\mathcal{R} \otimes \mathcal{O}$ where $\mathcal{R}$ is $(X - X_{n-1}) \times \mathbb{R}$ with $\mathbb{R}$ endowed of the discrete topology and $\mathcal{O}$ is the orientation sheaf (see example 1.24). By proposition 2.15 we know that $(\mathcal{L}^i_{2,max}, d_{max,i})$ is a complex of fine sheaves. The first two requirements of axiom 1 are clearly satisfied. The third requirement of the same axiom follows by proposition 2.11 wich implies that for each $x \in reg(X)$ $(\mathbb{H}^i(\mathcal{L}^*_{2,max}, d_{max,*}))_x$, that is the stalk at the point x of the $i-$th cohomology sheaf associated to the complex $(\mathcal{L}^*_{2,max}, d_{max,*})$, satisfies:

$$(\mathbb{H}^i(\mathcal{L}^*_{2,max}, d_{max,*}))_x = \begin{cases} \mathbb{R} & i = 0 \\ 0 & i > 0 \end{cases} \tag{2.24}$$

Consider now a stratum $Y \subset X$ and a point $x \in Y$. Let $l = dimY$. If $l = n-1$, that is if the codimension of Y is 1, then it is clear from proposition 2.11 that for all $x \in Y$ the second axiom of theorem 1.25 is satisfied. So we can suppose that $l \leq n-2$. By definition 1.30 we know that there exists an open subset $V \subset Y$ such that $\pi_Y^{-1}(V) \cong V \times C(L_Y)$and such that

$$\phi : (\pi_Y^{-1}(V) \cap reg(X), g|_{\pi_Y^{-1}(V)\cap reg(X)}) \to (V \times reg(C(L_Y)), dr^2 + h_V + r^{2c_Y} g_{L_Y})$$

is a quasi-isometry. Therefore by the invariance of L^2-cohomology under quasi-isometry we can use $(V \times reg(C(L_Y)), dr^2 + h_V + r^{2c_Y} g_{L_Y})$ to calculate the L^2-cohomology of $\pi_Y^{-1}(V) \cap reg(X)$. Choosing V diffeomorphic to $(0,\epsilon)^l$ with ϵ sufficiently small we have that

$$(V \times reg(C(L_Y)), dr^2 + h_V + r^{2c_Y} g_{L_Y}) \tag{2.25}$$

is quasi-isometric to

$$((0,\epsilon)^l \times reg(C(L_Y)), ds_1^2 + ... + ds_l^2 + dr^2 + r^{2c_Y} g_Y).$$

Therefore from proposition 2.11 and the invariance of L^2-cohomology under quasi-isometry it follows that:

$$H^i_{2,max}(V \times reg(C(L_Y)), dr^2 + h_V + r^{2c_Y} g_{L_Y}) \cong \tag{2.26}$$
$$\cong H^i_{2,max}(reg(C(L_Y)), dr^2 + r^{2c_Y} g_{L_Y}).$$

In this way we have obtained that:

$$H^i_{2,max}(reg(\pi_Y^{-1}(V)), g|_{reg(\pi_Y^{-1}(V))}) \cong \tag{2.27}$$
$$\cong H^i_{2,max}(reg(C(L_Y)), dr^2 + r^{2c_Y} g_{L_Y}).$$

As we have already observed in the proof of corollary 2.17 we know that

$$(\mathbb{H}^i(\mathcal{L}^*_{2,max}, d_{max,*}))_x \cong H^i_{2,max}(reg(\pi_Y^{-1}(V)), g|_{reg(\pi_Y^{-1}(V))})$$

where V is as in 2.25. Therefore from this and (2.27) we get that

$$(\mathbb{H}^i(\mathcal{L}^*_{2,max}, d_{max,*}))_x \cong H^i_{2,max}(reg(C(L_Y)), dr \otimes dr + r^{2c_Y} g_{L_Y}) \tag{2.28}$$

Now, using the inductive hypothesis we know that this theorem is true for (L_Y, g_{L_Y}) that is $H^i_{2,max}(reg(L_Y), g_{L_Y}) \cong I^{q_{g_{L_Y}}} H^i(L_Y, \mathcal{R}_0)$ where $q_{g_{L_Y}} = t - p_{g_{L_Y}}$ and $p_{g_{L_Y}}$ is the general perversity associated to g_{L_Y} on L_Y. This implies that $dim H^i_{2,max}(reg(L_Y), g_{L_Y}) < \infty$ for each $i = 0, ..., dim L_Y$. From this it follows that at least one of the three hypotheses of corollary 2.10 is always satisfied. So we can use the same corollary to get:

$$H^i_{2,max}(reg(C(L_Y)), g_c) = \begin{cases} H^i_{2,max}(reg(L_Y), g_{L_Y}) & i < \frac{dim L_Y}{2} + \frac{1}{2c_Y} \\ 0 & i \geq \frac{dim L_Y}{2} + \frac{1}{2c_Y} \end{cases} \tag{2.29}$$

In this way we can conclude that for each $x \in Y$

$$(\mathbb{H}^i(\mathcal{L}^*_{2,max}, d_{max,*}))_x = 0 \ for \ i > p_g(Y)$$

and therefore the complex $(\mathcal{L}^*_{2,max}, d_{max,*})$ satisfies the second axiom of theorem 1.25.

To conclude the first part of the proof we have to show that given any stratum $Z \subset X_{n-k} - X_{n-k-1}$ and any point $x \in Z$ the attaching map, that is the morphism given by the composition of

$$\mathcal{L}^*_{2,max}|_{U_{k+1}} \to i_{k*}\mathcal{L}^*_{2,max}|_{U_k} \to Ri_{k*}\mathcal{L}^*_{2,max}|_{U_k}$$

where the first morphism is induced by the inclusion $i_k : U_k \to U_{k+1}$, is a quasi-isomorphism at x up to $p_g(Z)$. By the fact that $(\mathcal{L}^*_{2,max}, d_{max,*})$ is a complex of fine sheaves it follows that $i_{k*}\mathcal{L}^*_{2,max}|_{U_k} \to Ri_{k*}\mathcal{L}^*_{2,max}|_{U_k}$ is a quasi-isomorphism (for example see [**12**] pag. 32 or [**16**] pag. 222). Therefore, to conclude, we have only to show that the morphism $\mathcal{L}^*_{2,max}|_{U_{k+1}} \to i_{k*}\mathcal{L}^*_{2,max}|_{U_k}$ is a quasi-isomorphism at x up to $p_g(Z)$, that is, for each $x \in Z$ it induces an isomorphism

$$(\mathbb{H}^i(\mathcal{L}^*_{2,max}|_{U_{k+1}}, d_{max,*}))_x \cong (\mathbb{H}^i(i_{k*}\mathcal{L}^*_{2,max}|_{U_k}, d_{max,*}))_x \tag{2.30}$$

for $i \leq p_g(Z)$. Now, like in the previous case to prove the validity of the second axiom, to show that for each $x \in Z$

$$(\mathbb{H}^i(\mathcal{L}^*_{2,max}|_{U_{k+1}}, d_{max,*}))_x \cong (\mathbb{H}^i(i_{k*}\mathcal{L}^*_{2,max}|_{U_k}, d_{max,*}))_x \text{ for } i \leq p_g(Z)$$

it is sufficient to show that there exists an open neighbourhood U of $x \in Z$ such that $\pi_Z^{-1}(U) \cong U \times C(L_Z)$ and such that

$$H^i(\mathcal{L}^*_{2,max}|_{U_{k+1}}(\pi_Z^{-1}(U)), d_{max,*}) \cong H^i(i_{k*}\mathcal{L}^*_{2,max}|_{U_k}(\pi_Z^{-1}(U)), d_{max,*})$$

for $i \leq p_g(Z)$, where the isomorphism is induced by the inclusion $i_k : U_k \to U_{k+1}$. Finally this last statement follows from corollary 2.17. So given a stratum $Z \subset X_{n-k} - X_{n-k-1}$ and a point $x \in Z$ we can conclude that for $i \leq p_g(Z)$ the natural maps induced by the inclusion of U_k in U_{k+1} induces a quasi isomorphism between

$$\mathcal{L}^*_{2,max}|_{U_{k+1}} \to i_{k*}\mathcal{L}^*_{2,max}|_{U_k}.$$

So also the third axiom of theorem 1.25 is satisfied.
Therefore for all $i = 0, ..., n$ $H^i(\mathcal{L}_{2,max}(reg(X)), d_{max,*}) \cong I^{q_g}H^i(X, \mathcal{R}_0)$. Finally by the compactness of X, see the third point of proposition 2.14, we get, for each $i = 0, ..., n$, the desired isomorphisms:

$$H^i_{2,max}(reg(X), g) \cong I^{q_g}H^i(X, \mathcal{R}_0).$$

From the isomorphism $H^i_{2,max}(reg(X), g) \cong I^{q_g}H^i(X, \mathcal{R}_0)$ it follows that $H^i_{2,max}(reg(X), g)$ is finite dimensional and then the isomorphism

$$\mathcal{H}^i_{abs}(reg(X)) \cong H^i_{2,max}(reg(X), g)$$

is an immediate consequence of proposition 1.3 and formula 1.12. The first part of the proof is completed.

To prove the second part of the theorem it is sufficient observe that the finite dimension of $H^i_{2,max}(reg(X), g)$ for all $i = 0, ..., n$ implies that the complex $(L^2\Omega^*(reg(X), g), d_{max,*})$ is a Fredholm complex.
Now, using the isomorphism induced by the Hodge star operator $*$ between the Hilbert complexes $(L^2\Omega^*(reg(X), g), d_{min,*})$ and the adjoint complex of $(L^2\Omega^*(reg(X), g), d_{max,*})$ and proposition 1.4, it follows that

$$H^i_{2,max}(reg(X), g) \cong H^{n-i}_{2,min}(reg(X), g).$$

Finally, using Poincaré duality for intersection homology, that is theorem 1.27, we get the isomorphism

$$H^i_{2,min}(reg(X), g) \cong I^{p_g}H^i(X, \mathcal{R}_0).$$

Now, like in the previous case, we know that $H^i_{2,min}(reg(X), g)$ is finite dimensional and then the isomorphism $\mathcal{H}^i_{rel}(reg(X)) \cong H^i_{2,min}(reg(X), g)$ is an immediate consequences of proposition 1.3 and formula 1.12. □

PROOF. (of theorem 2.13). Suppose that p is a general perversity in the sense of Friedman on X such that $p \geq \overline{m}$ and $p(Y) = 0$ for each one codimensional stratum Y of X. We recall that $\overline{m}$ is defined in the following way: if $Y \subset X$ is a stratum of X and if L_Y is the link relative to Y with $l_Y = dimL_Y$ then

$$\overline{m}(Y) = \begin{cases} \frac{l_Y}{2} & l_Y \ even \\ \frac{l_Y-1}{2} & l_Y \ odd \end{cases}$$

Therefore it follows that for each stratum Y there is a non negative integer n_Y such that

$$p(Y) = \begin{cases} 0 & l_Y = 0 \\ \frac{l_Y}{2} + n_Y & l_Y \ even, \ l_Y \neq 0 \\ \frac{l_Y-1}{2} + n_Y & l_Y \ odd \end{cases}$$

Now we can choose some non negative real numbers $\{c_Y\}_{Y\in\mathfrak{X}}$ such that $n_Y = [[\frac{1}{2c_Y}]]$ if l_Y is even and $n_Y = [[\frac{1}{2} + \frac{1}{2c_Y}]]$ if l_Y is odd. By proposition 1.31 we know that there is a quasi rigid iterated edge metric g on $reg(X)$ having the numbers $\{c_Y\}_{Y\in\mathfrak{X}}$ like weights. In this way $p = p_g$, the general perversity associated to g, and therefore by theorem 2.12 we can get the isomorphism (2.13) .
Conversely if p satisfies $p \leq \underline{m}$ and $p(Y) = -1$ for each one codimensional stratum Y of X, then $q := t - p$, where t is top perversity, satisfies $q \geq \overline{m}$ and $q(Y) = 0$ for each one codimensional stratum Y of X. Therefore by the previous point there exists a quasi edge metric with weights h on $reg(X)$ such that $p_h = q$. Finally using again theorem 2.12 we can get the isomorphism (2.15). □

In the same hypothesis of the theorem 2.12 we have the following corollaries:

COROLLARY 2.18. *For each $i = 0, ..., n$ on $L^2\Omega^i(reg(X), g)$ we have the following decompositions:*

$$L^2\Omega^i(reg(X), g) = \mathcal{H}^i_{abs} \oplus ran(d_{max,i-1}) \oplus ran(\delta_{min,i}) \quad (2.31)$$

$$L^2\Omega^i(reg(X), g) = \mathcal{H}^i_{rel} \oplus ran(d_{min,i-1}) \oplus ran(\delta_{max,i}) \quad (2.32)$$

and

$$L^2\Omega^i(reg(X), g) = \mathcal{H}^i_{max} \oplus ran(d_{min,i-1}) \oplus ran(\delta_{min,i}) \quad (2.33)$$

PROOF. By theorem 2.12 we know that $H^i_{2,max}(reg(X), g)$ and $H^i_{2,min}(reg(X), g)$ are finite dimensional. Therefore by prop. 1.3, the fact that $(L^2\Omega^*(M, g), \delta_{min,*})$ is the dual complex of $(L^2\Omega^*(M, g), d_{max,*})$, $(L^2\Omega^*(M, g), \delta_{max,*})$ is the dual complex of $(L^2\Omega^*(M, g), d_{min,*})$ and proposition 1.4 it follows that, for each i, $ran(d_{max,i}), ran(d_{min,i}), ran(\delta_{max,i})$ and $ran(\delta_{min,i})$ are closed. Now applying (1.12) we can get (2.31) and (2.32) and applying (1.20) we can get (2.33). □

COROLLARY 2.19.

$$d_{max} + \delta_{min},\ d_{min} + \delta_{max} : L^2\Omega^*(reg(X), g) \to L^2\Omega^*(reg(X), g)$$

and for each i

$$\Delta_{abs,i},\ \Delta_{rel,i} : L^2\Omega^i(reg(X), g) \to L^2\Omega^i(reg(X), g)$$

are Fredholm operators. Moreover also

$$d_{max} + \delta_{min},\ d_{min} + \delta_{max} : L^2\Omega^{even}(reg(X), g) \to L^2\Omega^{odd}(reg(X), g)$$

are Fredholm operators and their indexes satisfy:

$$\text{ind}(d_{max} + \delta_{min}) = \sum_{i=0}^{n}(I^{q_g}b_{2i}(X) - I^{p_g}b_{2i+1}(X))$$

$$\text{ind}(d_{min} + \delta_{max}) = \sum_{i=0}^{n}(I^{p_g}b_{2i}(X) - I^{q_g}b_{2i+1}(X))$$

where $I^{p_g}b_{2i}(X) = dim(I^{p_g}H^i(X, \mathcal{R}))$ *and* $I^{q_g}b_{2i}(X) = dim(I^{q_g}H^i(X, \mathcal{R}))$. *Finally*

$$\Delta_{max,i} : L^2\Omega^i(reg(X), g) \to L^2\Omega^i(reg(X), g)$$

has closed range and its orthogonal complement is finite dimensional while

$$\Delta_{min,i} : L^2\Omega^i(reg(X), g) \to L^2\Omega^i(reg(X), g)$$

has closed range and finite dimensional nullspace; in other words $\Delta_{max,i}$ *is essentially surjective and* $\Delta_{min,i}$ *is essentially injective.*

PROOF. The first three assertions follow immediately from theorem 2.12. For the last two we know that $ran(\Delta_{abs,i}) \subset ran(\Delta_{max})$. This implies that there exists a surjective map from

$$\frac{L^2\Omega^i(M, g)}{ran(\Delta_{abs,i})} \longrightarrow \frac{L^2\Omega^i(M, g)}{ran(\Delta_{max,i})}.$$

But we know that Δ_{abs} is Fredholm; this implies that the term on the left in the above equality is finite dimensional and therefore also the term on the right is finite dimensional. So $\Delta_{max,i}$ from its natural domain endowed with the graph norm to $L^2\Omega^i(M, g)$ is a continuous operator with finite dimensional cokernel and this implies the statement of the corollary about $\Delta_{max,i}$. For $\Delta_{min,i}$ we know, see prop. 1.9, that $Ker(\Delta_{min,i}) = Ker(d_{min,i}) \cap Ker(\delta_{min,i-1})$ and therefore by theorem 2.12 it follows that $Ker(\Delta_{min,i})$ is finite dimensional. Using again proposition 1.9 we know that $(\Delta_{max,i})^* = \Delta_{min,i}$ and therefore by the fact that $\Delta_{max,i}$ has closed range it follows that also $\Delta_{min,i}$ has closed range. $\square$

Finally the remaining corollaries follow immediately from theorem 2.12 and from the definition of intersection cohomology with general perversity.

COROLLARY 2.20. *Consider the following complex* $(C_0^{\infty}\Omega^i(reg(X)), d_i)$. *Then a necessary condition to have the minimal exstension equal to the maximal one is that the perversities* p_g *and* q_g *gives isomorphic intersection cohomology groups.*

COROLLARY 2.21. *If every weight is greater or equal than* 1*, that is for every stratum* Y $c_Y \geq 1$*, then, for all* i*, we obtain the following isomorphisms:*

$$\mathcal{H}^i_{abs}(reg(X),g) \cong H^i_{2,max}(reg(X),g) \cong I^{\underline{m}}H^i(X,\mathcal{R}_0) \tag{2.34}$$

$$\mathcal{H}^i_{rel}(reg(X),g) \cong H^i_{2,min}(reg(X),g) \cong I^{\overline{m}}H^i(X,\mathcal{R}_0) \tag{2.35}$$

where $\underline{m}$ *is the lower middle perversity and* $\overline{m}$ *is the upper middle perversity.*

COROLLARY 2.22. *Suppose that the general perversity associated to the quasi edge metric with weights* g *satisfies* $p_g(Z) \geq cod(Z) - 1$ *for each singular stratum* Z*. Then, for all* i*, we have the following isomorphisms:*

$$\mathcal{H}^i_{abs}(reg(X),g) \cong H^i_{2,max}(reg(X),g) \cong H^i(X - X_{n-1},\mathbb{R}) \tag{2.36}$$

$$\mathcal{H}^i_{rel}(reg(X),g) \cong H^i_{2,min}(reg(X),g) \cong H^i(X,\mathcal{R}_0). \tag{2.37}$$

COROLLARY 2.23. *If* p_g *is classical perversity in the sense of Goresky-MacPherson and* $X_{n-1} = X_{n-2}$ *then, for all* i*, we have the following isomorphisms:*

$$\mathcal{H}^i_{abs}(reg(X),g) \cong H^i_{2,max}(reg(X),g) \cong I^{q_g}H^i(X,\mathbb{R}) \tag{2.38}$$

$$\mathcal{H}^i_{rel}(reg(X),g) \cong H^i_{2,min}(reg(X),g) \cong I^{p_g}H^i(X,\mathbb{R}) \tag{2.39}$$

COROLLARY 2.24. *Let* g,h *be two quasi edge metrics with weights on* $reg(X)$ *such that* $p_g = p_h$*. Then for all* i*:*

$$\mathcal{H}^i_{abs}(reg(X),g) \cong H^i_{2,max}(reg(X),g) \cong H^i_{2,max}(reg(X),h) \cong \tag{2.40}$$
$$\cong \mathcal{H}^i_{abs}(reg(X),h)$$

and

$$\mathcal{H}^i_{rel}(reg(X),g) \cong H^i_{2,min}(reg(X),g) \cong H^i_{2,min}(reg(X),h) \cong \tag{2.41}$$
$$\cong \mathcal{H}^i_{rel}(reg(X),h).$$

In particular a necessary condition for two quasi edge metrics with weights are quasi-isometric is that they induce perversities with isomorphic intersection cohomology groups.

COROLLARY 2.25. *Let* X' *be another compact and oriented smoothly stratified pseudomanifold with a Thom-Mather stratification and* h *a quasi edge metric with weights on* $reg(X')$*. Let* $f : X \to X'$ *a stratum preserving homotopy equivalence, see* [**47**] *pag 62 for the definition. Suppose that both* p_g *and* p_h *depend only on the codimension of the strata and that* $p_g = p_h$*. Then for all* i

$$\mathcal{H}^i_{abs}(reg(x),g) \cong H^i_{2,max}(reg(X),g) \cong H^i_{2,max}(reg(X'),h) \tag{2.42}$$
$$\cong \mathcal{H}^i_{abs}(reg(X'),h)$$

and

$$\mathcal{H}^i_{rel}(reg(x),g) \cong H^i_{2,min}(reg(X),g) \cong H^i_{2,min}(reg(X'),h) \tag{2.43}$$
$$\cong \mathcal{H}^i_{rel}(reg(X'),h).$$

CHAPTER 3

$\overline{H}^i_{2,m\to M}(M,g)$ and $H^i_{2,m\to M}(M,g)$: Poincaré duality and Hodge theorem.

This chapter is devoted to the study of the following groups:

$$\overline{H}^i_{2,m\to M}(M,g) \text{ and } H^i_{2,m\to M}(M,g)$$

defined respectively as the image of $\operatorname{im}(\overline{H}^i_{2,min}(M,g)\to\overline{H}^i_{2,max}(M,g))$ and $\operatorname{im}(H^i_{2,min}(M,g)\to H^i_{2,max}(M,g))$. The reason behind this study is given by the fact that usually, when (M,g) is an open oriented and incomplete riemannian manifold, Poincaré duality does not hold for the complexes $(L^2\Omega^i(M,g),d_{max/min,i})$. Conversely, as we will see, this is not true for the groups: $\overline{H}^i_{2,m\to M}(M,g)$, $i=0,...,n$, $n=dim(M)$. Therefore, when $\overline{H}^i_{2,m\to M}(M,g)=H^i_{2,m\to M}(M,g)$, it is interesting to investigate the existence of a Hilbert complex having the groups $H^i_{2,m\to M}(M,g)$ as cohomology groups. The chapter is structured in the following way: in the first section two abstract theorems on Hilbert complexes are proved. They provide the necessary tools to investigate the groups $\overline{H}^i_{2,m\to M}(M,g)$ and $H^i_{2,m\to M}(M,g)$. The second section contains the main results about these groups. We show that Poincaré duality holds for $\overline{H}^i_{2,m\to M}(M,g)$, $i=0,...,n$ and, under suitable hypothesis, we show the existence of a Hilbert complex having $H^i_{2,m\to M}(M,g)$ as cohomology groups. In particular this lead us to prove a Hodge theorem for the groups $\operatorname{im}(H^i_{2,min}(M,g)\to H^i_{2,max}(M,g))$. Finally the subsequent sections contain several applications of the previous results. In particular in the fourth section an L^2-signature in defined and in fifth section the previous results are applied to the case of a compact and oriented smoothly stratified pseudomanifold X such that $reg(X)$ is endowed with a quasi edge metric with weights.

1. Two theorems on Hilbert complexes

In this section we continue the theory of Hilbert complexes introducing the notion of pair of complementary Hilbert complexes and proving two theorem about it. We will use these theorems subsequently to investigate the groups $\overline{H}^i_{2,m\to M}(M,g)$ and $H^i_{2,m\to M}(M,g)$.

Given a pair of Hilbert complexes (H_j,D_j) and (H_j,D'_j) we will write $(H_j,D_j)\subseteq(H_j,D'_j)$ if for each j one of the two following properties is satisfied:

(1) $D'_j:H_i\to H_{j+1}$ is equal to $D_j:H_j\to H_{j+1}$

(2) $D'_j : H_j \to H_{j+1}$ is a proper closed extension of $D_j : H_i \to H_{j+1}$

We will write $(H_j, D_j) \subset (H_j, D'_j)$ when the second of the above properties is satisfied. For each j let $i_j : \mathcal{D}(D_j) \to \mathcal{D}(L_j)$ denote the natural inclusion of the domain of D_j into the domain of L_j. Obviously i_j induces a maps between $H^j(H_*, D_*)$ and $H^j(H_*, L_*)$ and between $\overline{H}^j(H_*, D_*)$ and $\overline{H}^j(H_*, L_*)$. We will label the first as

$$i^*_j : H^j(H_*, D_*) \to H^j(H_*, L_*) \tag{3.1}$$

and the second as

$$i^*_{r,j} : \overline{H}^j(H_*, D_*) \to \overline{H}^j(H_*, L_*) \tag{3.2}$$

Consider again a pair of Hilbert complexes (H_i, D_i) and (H_i, L_i) with $i = 0, ...n$.

DEFINITION 3.1. The pair (H_i, D_i) and (H_i, L_i) is said to be **related** if the following property is satisfied

- for each i there exist a linear, continuous and bijective map $\phi_i : H_i \to H_{n-i}$ such that $\phi_i(\mathcal{D}(D_i)) = \mathcal{D}(L^*_{n-i-1})$ and $L^*_{n-i-1} \circ \phi_i = C_i(\phi_{i+1} \circ D_i)$ on $\mathcal{D}(D_i)$ where $L^*_{n-i-1} : H_{n-i} \to H_{n-i-1}$ is the adjoint of $L_{n-i-1} : H_{n-i-1} \to H_{n-i}$ and $C_i \neq 0$ is a constant which depends only on i.

Furthermore we call the maps ϕ_i link maps.

- We call the complexes **complementary** if each ϕ_i is an isometry between H_i and H_{n-i}.

We have the following propositions:

PROPOSITION 3.2. *Let (H_i, D_i) and (H_i, L_i) be related Hilbert complexes. Then:*

1. *Also (H_i, L_i) and (H_i, D_i) are related Hilbert complexes. Moreover if $\{\phi_i\}$ are the link maps which make (H_i, D_i) and (H_i, L_i) related then $\{\phi^*_i\}$, the family of respective adjoint maps, are the link maps which make (H_i, L_i) and (H_i, D_i) related.*
2. *The complexes (H_i, D_i) and (H_i, L^*_i) have isomorphic cohomology groups and isomorphic reduced cohomology groups. In the same way the complexes (H_i, L_i) and (H_i, D^*_i) have isomorphic cohomology groups and isomorphic reduced cohomology groups.*
3. *The following isomorphisms hold:*

   $\mathcal{H}^j(H_i, D_i) \cong \mathcal{H}^{n-j}(H_i, L_i)$, $\overline{H}^j(H_i, D_i) \cong \overline{H}^{n-j}(H_i, L_i)$.
4. *If the complexes (H_i, D_i) and (H_i, L^*_i) are complementary then each ϕ_j induces an isomorphism between $\mathcal{H}^j(H_i, D_i)$ and $\mathcal{H}^{n-j}(H_i, L_i)$.*

PROOF. By definition 3.1 we know that $\phi^*_i : H_{n-i} \to H_i$, the adjoint of $\phi_i : H_i \to H_{n-i}$, is a family of linear continuous and bijective maps. In this way if we look at $L^*_{n-i-1} \circ \phi_i$ as an unbounded linear map between H_i and H_{n-i-1} with domain $\mathcal{D}(L^*_{n-i-1} \circ \phi_i) = \phi_i^{-1}(\mathcal{D}(L^*_{n-i-1})) = \mathcal{D}(D_i)$ we have that $(L^*_{n-i-1} \circ \phi_i)^* = \phi^*_i \circ L_{n-i-1}$ that is the adjoint of $L^*_{n-i-1} \circ \phi_i$ is $\phi^*_i \circ L_{n-i-1}$ with $\mathcal{D}(\phi^*_i \circ L_{n-i-1}) = \mathcal{D}(L_{n-i-1})$.
In the same way we have $(\phi_{i+1} \circ D_i)^* = (D^*_i \circ \phi^*_{i+1})$ where $\mathcal{D}(\phi_{i+1} \circ D_i) =$

$\mathcal{D}(D_i)$ and $\mathcal{D}(D_i^* \circ \phi_{i+1}^*) = (\phi_{i+1}^*)^{-1}(\mathcal{D}(D_i^*))$. In this way it follows that, for each i, $\mathcal{D}(D_i^* \circ \phi_{i+1}^*) = \mathcal{D}(\phi_i^* \circ L_{n-i-1})$, $C_i(D_i^* \circ \phi_{i+1}^*) = \phi_i^* \circ L_{n-i-1}$ on $\mathcal{D}(L_{n-i-1})$ and that $\phi_{i+1}^*(\mathcal{D}(L_{n-i-1})) = \mathcal{D}(D_i^*)$. So we can conclude that the complexes (H_i, L_i) and (H_i, D_i) are related with $\{\phi_i^*\}$ as link maps.
The second property is an immediate consequences of definition 3.1 and the first point of the proposition . Now if we compose the isomorphisms of the second point with the isomorphisms of (1.5) we can get the isomorphisms of the third point. Finally if each ϕ_i is an isometry then $\phi_i^* = \phi_i^{-1}$. By definition 3.1 we know that ϕ_i induces an isomorphism between $Ker(D_i)$ and $Ker(L_{n-i-1}^*)$. In the same way by the first point of the proposition we know that ϕ_i^* induces an isomorphism between $Ker(L_{n-i})$ and $Ker(D_{i-1}^*)$. But now we know that $\phi_i^* = \phi_i^{-1}$ and so we can conclude that for each i ϕ_i induces an isomorphism between $Ker(D_i) \cap Ker(D_{i-1}^*)$ and $Ker(L_{n-i}) \cap Ker(L_{n-i-1})^*$, that is an isomorphism between $\mathcal{H}^i(H_*, D_*)$ and $\mathcal{H}^{n-i}(H_*, L_*)$. □

PROPOSITION 3.3. *Let (H_i, D_i), $i = 0, ..., n$ a Hilbert complex and suppose that for each i there exists $\phi_i : H_i \to H_{n-i}$ that is linear, continuous and bijective. Then there exist a Hilbert complex (H_i, L_i) such that the complexes (H_i, D_i) and (H_i, L_i) are related with $\{\phi_i\}$ as link maps. Moreover if each ϕ_j is an isometry then the complexes (H_i, D_i) and (H_i, L_i) are complementary with $\{\phi_i\}$ as link maps.*

PROOF. Consider the following complexes (H_i, L_i) where each L_i is the adjoint of the closed and densely defined operator $(\phi_{n-i} \circ D_{n-i-1} \circ \phi_{n-i-1}^{-1}) : H_{i+1} \to H_i$. It clear that (H_i, L_i) is a Hilbert complex and by its construction it follows immediately that (H_i, D_i) and (H_i, L_i) are a pair of related Hilbert complexes having the maps $\{\phi_i\}$ as link maps. Finally it is clear that if each ϕ_j is an isometry then the complexes (H_i, D_i) and (H_i, L_i) are complementary with $\{\phi_i\}$ as link maps. □

Now we give the following definition which we will use later.

DEFINITION 3.4. Let $V_0, V_1, ..., V_n$ be a finite sequence of finite dimensional vector spaces. We will say that it is a finite sequence of finite dimensional vector spaces with Poincaré duality if for each i:

$$V_i \cong V_{n-i}.$$

We are now in position to state the first of the two main results of this section.

THEOREM 3.5. *Let $(H_j, D_j) \subseteq (H_j, L_j)$ be a pair of complementary Hilbert complexes. Let $i_{r,j}^*$ be the map defined in (3.2). Suppose that for each j*

$$\operatorname{im}(\overline{H}^j(H_*, D_*) \xrightarrow{i_{r,j}^*} \overline{H}^j(H_*, L_*)) \tag{3.3}$$

is finite dimensional. Then

$$\operatorname{im}(\overline{H}^j(H_*, D_*) \xrightarrow{i_{r,j}^*} \overline{H}^j(H_*, L_*)),\ j = 0, ..., n \tag{3.4}$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality.

Now we state some propositions which we will use in the proof of theorem 3.5.

PROPOSITION 3.6. *Let H, K be two Hilbert spaces and let $T : H \to K$ be a linear and continuous map. Let $T^* : K \to L$ the adjoint of T. Suppose that $ran(T)$ is closed. Then $T : Ker(T)^{\perp} \to Ker(T^*)^{\perp}$ is continuous, bijective with bounded inverse.*

PROOF. We have $K = Ker(T^*) \oplus Ker(T^*)^{\perp}$ and $Ker(T^*)^{\perp} = \overline{ran(T)}$. Therefore by the fact that $ran(T)$ is closed it follows that T is a bijection between $Ker(T)^{\perp}$ and $Ker(T^*)^{\perp}$. Now from the fact that $Ker(T)^{\perp}$ and $(Ker(T^*))^{\perp}$ are closed subspace of H and K respectively it follows we can look at them as Hilbert spaces with the products induced by the products of H and K respectively. In this way we can use the closed graph theorem to conclude that $T|_{Ker(T)^{\perp}}$ has a continuous inverse. □

PROPOSITION 3.7. *Let H be a Hilbert space and let M, N be two closed subspace of it. Let π_M, π_N be the orthogonal projection on M and N respectively. Consider M and N as Hilbert spaces with the scalar product induced by the one of H. Then*

$$\pi_M|_N = (\pi_N|_M)^*$$

that is if we look at $\pi_N|_M$ as a linear and continuous map from the Hilbert space M to the Hilbert space N then $\pi_M|_N$ is its adjoint.

PROOF. During the proof we use $<,>_H$ to indicate the scalar product of H and $<,>_M, <,>_N$ to indicate the scalar product induced by $<,>_H$ on M and N respectively. For each $u \in M,\ v \in N$ we have $< \pi_N(u), v >_N =< \pi_N(u) + \pi_{N^{\perp}}(u), v >_H =< u, v >_H =< u, \pi_M(v) + \pi_{M^{\perp}}(v) >_H =< u, \pi_M >_M$ and so we get the assertion. □

Now we are in position to prove theorem 3.5 .

PROOF. From proposition 1.2 we know that

$$H_j = \mathcal{H}^j(H_*, D_*) \bigoplus \overline{ran(D_{i-1})} \bigoplus \overline{ran(D_i^*)}$$

and that

$$H_j = \mathcal{H}^j(H_*, L_*) \bigoplus \overline{ran(L_{i-1})} \bigoplus \overline{ran(L_i^*)}.$$

So for each j we can define π_{D_j} as the orthogonal projection of H_j on $\mathcal{H}^j(H_*, D_*)$ and π_{L_j} as the orthogonal projection of H_j on $\mathcal{H}^j(H_*, L_*)$. In the same way we can define $\pi_{\overline{ran(D_{j-1})}}$, $\pi_{\overline{ran(L_{j-1})}}$, $\pi_{\overline{ran(D_j^*)}}$ and $\pi_{\overline{ran(L_j^*)}}$. Finally we define:

$$\pi_{1,j} := (\pi_{L_j})|_{\mathcal{H}^j(H_*,D_*)},\ \pi_{2,j} := (\pi_{\overline{ran(L_{j-1})}})|_{\mathcal{H}^j(H_*,D_*)},$$

$$\pi_{3,j} := (\pi_{\overline{ran(L_j^*)}})|_{\mathcal{H}^j(H_*,D_*)}.$$

Analogously, but now projecting from $\mathcal{H}^j(H_*, L_*)$ on the orthogonal components of the sum $H_j = \mathcal{H}^j(H_*, D_*) \bigoplus \overline{ran(D_{i-1})} \bigoplus \overline{ran(D_i^*)}$, we define $\pi_{4,j}, \pi_{5,j}, \pi_{6,j}$.
Our first claim is tho show that for each j

$$\pi_{1,j}(\mathcal{H}^j(H_*, D_*)) \cong \operatorname{im}(\overline{H}^j(H_*, D_*) \xrightarrow{i^*_{r,j}} \overline{H}^j(H_*, L_*)) \tag{3.5}$$

Let $[h] \in \overline{H}^j(H_*, D_*)$ a cohomology class. By (1.5) we know that there exists a unique representative of $[h]$ in $\mathcal{H}^j(H_*, D_*)$. We call it ω. Every other representative of $[h]$ differs from ω by an element in $\overline{ran(D_{j-1})}$; therefore $i^*_{r,j}([h]) = [i_j(\omega)]$. Now we can decompose ω as $\omega = \pi_{1,j}(\omega) + \pi_{2,j}(\omega) + \pi_{3,j}(\omega)$. Clearly $[i_j(\omega)] = [\pi_{1,j}(\omega)] + [\pi_{3,j}(\omega)]$. So if we show that $\pi_{3,j}|_{\mathcal{H}^j(H_*,D_*)} \equiv 0$ we get the claim. Now let $\eta \in \mathcal{H}^j(H_*, D_*)$. Then $\pi_{3,j}(\eta) \in \overline{ran(L^*_j)} \cap Ker(L_j)$ because $\pi_{3,j}(\eta) = \eta - \pi_{1,j}(\eta) - \pi_{2,j}(\eta)$ and each term on the right hand side of the equality lies in $Ker(L_j)$. But $(Ker(L_j))^{\perp} = \overline{ran(L^*_j)}$ and therefore $\pi_{3,j}(\eta) = 0$. So for each $\eta \in \mathcal{H}^j(H_*, D_*)$ we have $\pi_{3,j}(\eta) = 0$. Therefore the claim is proved.
In this way we know that $\pi_{1,j}$ has closed range and that $Ker(\pi_{1,j}) = \overline{ran(L_{j-1})} \cap \mathcal{H}^j(H_*, D_*)$. Analogously it follows that $Ker(\pi_{4,j}) = \overline{ran(D^*_j)} \cap \mathcal{H}^j(H_*, L_*)$. Finally from the observations above and from propositions 3.6 and 3.7 we get:

(1) $\mathcal{H}^j(H_*, D_*) = ran(\pi_{4,j}) \oplus (\overline{ran(L_{j-1})} \cap \mathcal{H}^j(H_*, D_*)) =$
$= ran(\pi_{4,j}) \oplus Ker(\pi_{1,j})$ for each j.
(2) $\mathcal{H}^j(H_*, L_*) = ran(\pi_{1,j}) \oplus (\overline{ran(D^*_j)} \cap \mathcal{H}^j(H_*, L_*)) =$
$= ran(\pi_{1,j}) \oplus Ker(\pi_{4,j})$ for each j.
(3) $(\pi_{1,j})^* = \pi_{4,j}$ and both induce an isomorphism between $ran(\pi_{4,j})$ and $ran(\pi_{1,j})$.

By the fourth point of proposition 3.2 it follows that each ϕ_j induces an isomorphism between $\mathcal{H}^j(H_*, D_*)$ and $\mathcal{H}^{n-j}(H_*, L_*)$. For the same reason ϕ_j induces an isomorphism between $\overline{ran(L_{j-1})}$ and $\overline{ran(D^*_{n-j})}$ and between $\overline{ran(D_{j-1})}$ and $\overline{ran(L^*_{n-j})}$. This implies that each ϕ_j induces an isomorphism between $\mathcal{H}^j(H_*, D_*) \cap \overline{ran(L_{j-1})}$ and $\mathcal{H}^{n-j}(H_*, L_*) \cap \overline{ran(D^*_{n-j})}$ that is an isomorphism between $Ker(\pi_{1,j})$ and $Ker(\pi_{4,n-j})$. In this way we can conclude that each ϕ_j induces an isomorphism between

$$\frac{\mathcal{H}^j(H_*, D_*)}{Ker(\pi_{1,j})} \quad \text{and} \quad \frac{\mathcal{H}^{n-j}(H_*, L_*)}{Ker(\pi_{4,n-j})}.$$

But

$$\frac{\mathcal{H}^j(H_*, D_*)}{Ker(\pi_{1,j})} \cong ran(\pi_{4,j}) \cong ran(\pi_{1,j}) \cong \mathrm{im}(\overline{H}^j(H_*, D_*) \stackrel{i^*_{r,j}}{\longrightarrow} \overline{H}^j(H_*, L_*))$$

and similarly

$$\frac{\mathcal{H}^{n-j}(H_*, L_*)}{Ker(\pi_{4,n-j})} \cong ran(\pi_{1,n-j}) \cong \mathrm{im}(\overline{H}^{n-j}(H_*, D_*) \stackrel{i^*_{r,n-j}}{\longrightarrow} \overline{H}^{n-j}(H_*, L_*)).$$

The composition of the above isomorphisms gives

$$\mathrm{im}(\overline{H}^j(H_*, D_*) \stackrel{i^*_{r,j}}{\longrightarrow} \overline{H}^j(H_*, L_*)) \cong \mathrm{im}(\overline{H}^{n-j}(H_*, D_*) \stackrel{i^*_{r,n-j}}{\longrightarrow} \overline{H}^{n-j}(H_*, L_*))$$

and this complete the proof. □

Remark 3.1. By the above proof it follows that given a pair of Hilbert complexes $(H_*, D_*) \subseteq (H_*, L_*)$, without any other assumption, the following isomorphism holds for each j :

$$ran(\pi_{1,j}) \cong \mathrm{im}(\overline{H}^j(H_*, D_*) \stackrel{i^*_{r,j}}{\longrightarrow} \overline{H}^j(H_*, L_*)). \tag{3.6}$$

Moreover when the sequences of vector spaces on the right hand side of the above equality isomorphism is finite dimensional we have

$$\mathcal{H}^j(H_*,D_*)\cap(\mathcal{H}^j(H_*,D_*)\cap\overline{ran(L_{j-1})})^{\perp}\cong$$
$$\cong(\mathcal{H}^j(H_*,L_*)\cap\overline{ran(D_j^*)})^{\perp}\cap\mathcal{H}^j(H_*,L_*)$$

that is

$$ran(\pi_{1,j})\cong ran(\pi_{4,j}).$$

The following statements are immediate consequences of theorem 3.5.

COROLLARY 3.8. *Suppose that one of the two complexes of theorem 3.5 is Fredholm; then also the other complex is Fredholm and*

$$\operatorname{im}(H^j(H_*,D_*)\longrightarrow H^j(H_*,L_*)),\ j=0,...,n \tag{3.7}$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality. Moreover

$$ran(\pi_{1,j})\cong\operatorname{im}(H^j(H_*,D_*)\longrightarrow H^j(H_*,L_*)). \tag{3.8}$$

and

$$\mathcal{H}^j(H_*,D_*)\cap(\mathcal{H}^j(H_*,D_*)\cap ran(L_{j-1}))^{\perp}\cong \tag{3.9}$$

$$\cong(\mathcal{H}^j(H_*,L_*)\cap ran(D_j^*))^{\perp}\cap\mathcal{H}^j(H_*,L_*).$$

PROPOSITION 3.9. *Let $(H_*,D_*)\subseteq(H_*,L_*)$ be a couple of complementary Hilbert complexes. Furthermore suppose that there is a third Hilbert complex (H_*,P_*) with the following properties:*

1. *$(H_*,D_*)\subseteq(H_*,P_*)\subseteq(H_*,L_*)$.*
2. *The reduced cohomology of (H_*,P_*) is finite dimensional.*

Then

$$\operatorname{im}(\overline{H}^j(H_*,D_*)\xrightarrow{i^*_{r,j}}\overline{H}^j(H_*,L_*)),\ j=0,...,n$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality.

PROOF. The assertion is an immediate consequence of the following, simple fact. Let $i_{1,j}$ be the natural inclusion of (H_*,D_*) in (H_*,P_*), let $i_{2,j}$ be the natural inclusion of (H_*,P_*) in (H_*,L_*) and finally let $i_{3,j}$ be the natural inclusion of (H_*,D_*) in (H_*,L_*). Obviously we have $i_{3,j}=i_{2,j}\circ i_{1,j}$. This implies that also the respective maps induced between the reduced cohomology groups commute. So we have $i^*_{r,3,j}=i^*_{r,2,j}\circ i^*_{r,1,j}$ and therefore

$$\operatorname{im}(\overline{H}^j(H_*,D_*)\xrightarrow{i^*_{r,3,j}}\overline{H}^j(H_*,L_*))\subseteq\operatorname{im}(\overline{H}^j(H_*,P_*)\xrightarrow{i^*_{r,2,j}}\overline{H}^j(H_*,L_*)).$$

In this way, by the second hypothesis, we know that

$$\operatorname{im}(\overline{H}^j(H_*,D_*)\xrightarrow{i^*_{r,3,j}}\overline{H}^j(H_*,L_*))$$

is a finite sequence of finite dimensional vector spaces. Now we are in position to apply theorem 3.5 and so the proposition follows. □

Now we state the following theorem:

THEOREM 3.10. *Let $(H_j,D_j)\subseteq(H_j,L_j)$, $j=0,...,n$, be a pair of Hilbert complexes. Suppose that for each j $ran(D_j)$ is closed in H_{j+1}. Then there exists a third Hilbert complex (H_j,P_j) such that*

(1) $(H_j, D_j) \subseteq (H_j, P_j) \subseteq (H_j, L_j)$.
(2) $H^i(H_*, P_*) = \mathrm{im}(H^i(H_*, D_*) \to H^i(H_*, L_*))$.

Moreover if $(H_j, D_j) \subseteq (H_j, L_j)$ are complementary and (H_j, D_j), or equivalently (H_j, L_j), is Fredholm then (H_j, P_j) is a Fredholm complex with Poincaré duality.

Proof. It is immediate that

$$\mathrm{im}(H^i(H_*, D_*) \to H^i(H_*, L_*)) = \frac{Ker(D_i)}{ran(L_{i-1}) \cap \mathcal{D}(D_i)}.$$

Therefore for each $i = 0, ..., n$ we have to construct a closed extension of D_i, that we call P_i, such that $Ker(P_i) = Ker(D_i)$ and $ran(P_{i-1}) = ran(L_{i-1}) \cap \mathcal{D}(D_i)$. To do this, from now on we will consider the following Hilbert space $(\mathcal{D}(L_i), < \, , >_{\mathcal{G}})$, which is by definition the domain of L_i endowed with the graph scalar product. Therefore all the direct sum that will appear and all the assertions of topological type are referred to this Hilbert space $(\mathcal{D}(L_i), < \, , >_{\mathcal{G}})$. We can decompose $(\mathcal{D}(L_i), < \, , >_{\mathcal{G}})$ in the following way:

$$(\mathcal{D}(L_i), < \, , >_{\mathcal{G}}) = Ker(L_i) \oplus V_i \tag{3.10}$$

where $V_i = \{\alpha \in \mathcal{D}(L_i) \cap \overline{ran(L_i^*)}\}$ and obviously these subspaces are both closed in $(\mathcal{D}(L_i), < \, , >_{\mathcal{G}})$.
Consider now $(\mathcal{D}(D_i), < \, , >_{\mathcal{G}})$; it is a closed subspace of $(\mathcal{D}(L_i), < \, , >_{\mathcal{G}})$ and we can decompose it as

$$(\mathcal{D}(D_i), <>_{\mathcal{G}}) = Ker(D_i) \oplus A_i. \tag{3.11}$$

Analogously to the previous case $A_i = \{\alpha \in \mathcal{D}(D_i) \cap \overline{ran(D_i^*)}\}$ and obviously these subspaces are both closed in $(\mathcal{D}(D_i), <>_{\mathcal{G}})$. Now let $C_i = \{\alpha \in \mathcal{D}(L_i) : L_i(\alpha) \in \mathcal{D}(D_{i+1})\}$. C_i is closed in $(\mathcal{D}(D_i), <>_{\mathcal{G}})$ because it is the preimage of a closed subspace under a continuous map. Finally let $W_i = C_i \cap V_i$. Then it is clear that

$$C_i = Ker(L_i) \oplus W_i. \tag{3.12}$$

Obviously if $Ker(D_i) = Ker(L_i)$ then it enough to define $P_i := L_i|_{C_i}$. So we can suppose that $Ker(D_i)$ is properly contained in $Ker(L_i)$. Let π_1 be the orthogonal projection of A_i onto $Ker(L_i)$ and analogously let π_2 be the orthogonal projection of A_i onto V_i. We have the following properties:

(1) π_2 is injective
(2) $ran(\pi_2) \subseteq W_i$
(3) $ran(\pi_2)$ is closed.

The first property follows from the fact that $Ker(\pi_2) = A_i \cap Ker(L_i)$. But L_i is an extension of D_i; therefore if an element α lies in $A_i \cap Ker(L_i)$ then it lies also in $Ker(D_i)$ and so $\alpha = 0$. For the second property, given $\alpha \in A_i$, we have $D_i(\alpha) = L_i(\alpha) = L_i(\pi_1(\alpha) + \pi_2(\alpha)) = L_i(\pi_2(\alpha))$ and therefore $\pi_2(\alpha) \in W_i$. Finally, for the third property, consider a sequence $\{\gamma_m\}_{m \in \mathbb{N}} \subset A_i$ such that $\pi_2(a_m)$ converges to $\gamma \in W_i$. Then

$$\lim_{m\to\infty} D_j(a_m) = \lim_{m\to\infty} L_j(a_m) = \lim_{m\to\infty} L_j(\pi_2(a_m)) = L_j(\gamma).$$

This implies that

$$\lim_{m\to\infty} D_j(a_m) = L_j(\gamma)$$

and therefore the limit exists. So by the assumptions about the range of D_j it follows that there exists an element $\eta \in A_i$ such that

$$\lim_{m\to\infty} D_j(a_m) = D_j(\eta).$$

Moreover $L_j(\gamma) = D_j(\eta) = L_j(\eta) = L_j(\pi_2(\eta))$. This implies that $L_j(\pi_2(\eta) - \gamma) = 0$ and therefore $\pi_2(\eta) = \gamma$ because $\pi_2(\eta), \gamma \in W_i$ and L_i is injective on W_i. In this way we showed that π_2 is closed.
Now define N_i as the orthogonal complement of $ran(\pi_2)$ in W_i. Then for each $\alpha \in A_i$ and for each $\beta \in N_i$ we have $< \alpha, \beta >_{\mathcal{G}} = < \pi_1(\alpha) + \pi_2(\alpha), \beta >= 0$. This last property, joined with the fact that both A_i and N_i are closed, implies that the vector space generated by A_i and N_i is closed and, if we call it M_i, then we have $M_i = A_i \oplus N_i$. Again for each $\alpha \in Ker(D_i)$ and for each $\beta \in M_i$ we have $< \alpha, \beta >_{\mathcal{G}} = 0$. This is because for each $\beta \in M_i$ there exist unique $\beta_1 \in A_i,\ \beta_2 \in N_i$ such that $\beta = \beta_1 \oplus \beta_2$. Now it is clear that $< \alpha, \beta_1 >_{\mathcal{G}} = 0 =< \alpha, \beta_2 >_{\mathcal{G}}$ because $Ker(D_i) \subset Ker(L_i)$, $N_i \subset W_i$, W_i and $Ker(L_i)$ are orthogonal and $Ker(D_i)$ and A_i are orthogonal. Therefore, also in this case, if we call B_i the vector space generated by $Ker(D_i)$ and M_i we have that $B_i = Ker(D_i) \oplus M_i = \mathcal{D}(D_i) \oplus N_i$ and therefore B_i is closed. Finally define P_i as

$$P_i := L_i|_{B_i} \tag{3.13}$$

By the construction it is clear that for each $\alpha \in B_i$ then $P_i(\alpha) \in \mathcal{D}(D_{i+1}) \cap ran(L_i)$ and that $\mathcal{D}(D_i) \subset B_i$. Therefore this implies that the composition $P_{i+1} \circ P_i$ is defined on the whole B_i and that $P_{i+1} \circ P_i \equiv 0$. Moreover, if we look at P_i as an unbounded operator from H_i to H_{i+1}, then it clear that it is densely defined and closed.
To conclude the proof we have to check that $Ker(P_i) = Ker(D_i)$ and that $ran(P_i) = ran(L_i) \cap \mathcal{D}(D_{i+1})$. Let $\alpha \in Ker(P_i)$. We can decompose α in a unique way as $\alpha_1 + \alpha_2 + \alpha_3$ where $\alpha_1 \in Ker(D_i)$, $\alpha_2 \in A_i$ and $\alpha_3 \in N_i$. The assumption on α implies that $\alpha_2 + \alpha_3 \in Ker(P_i)$. We can decompose α_2 in a unique way as $\alpha_2 = \beta_1 + \beta_2$ where $\beta_1 \in ran(\pi_1)$ and $\beta_2 \in ran(\pi_2)$. Again from the assumption on α it follows that $L_i(\beta_2 + \alpha_3) = 0$. This implies that $\beta_2 + \alpha_3 \in W_i \cap Ker(L_i)$ and therefore from (3.12) we can conclude that $\beta_2 + \alpha_3 = 0$. But $\beta_2 + \alpha_3 \in ran(\pi_2) \oplus N_i$, $\beta_2 \in ran(\pi_2)$, $\alpha_3 \in N_i$ and so we got $0 = \beta_2 = \alpha_3$. Now we have $\alpha_2 = \beta_1$ that is $\alpha_2 \in A_i \cap Ker(L_i) = Ker(\pi_2)$. By the injectivity of π_2 it follows that $\alpha_2 = 0$ and therefore $\alpha = \alpha_1 \in Ker(D_j)$. So we got $Ker(P_i) \subseteq Ker(D_i)$; the other inclusion is trivial and therefore we have $Ker(P_i) = Ker(D_i)$. Now we have to check that $ran(P_i) = ran(L_i) \cap \mathcal{D}(D_{i+1})$. Clearly, as observed above, the inclusion $\subseteq$ follows immediately by the construction of P_i. So we have to prove the converse. Let $\gamma \in W_i$. Then there exist and are unique $\gamma_1 \in ran(\pi_2)$ and $\gamma_2 \in N_i$ such that $\gamma = \gamma_1 + \gamma_2$. Now let $\theta \in A_i$ be the unique element in A_i such that $\pi_2(\theta) = \gamma_1$. Finally consider $\theta + \gamma_2$. Then $\theta + \gamma_2 \in B_i$ and $P_i(\theta + \gamma_2) = L_i(\theta + \gamma_2) = L_i(\pi_1(\theta) + \pi_2(\theta) + \gamma_2) = L_i(\gamma_1 + \gamma_2) = L_i(\gamma)$.
In this way we showed that $ran(L_i) \cap \mathcal{D}(D_{i+1}) = ran(P_i)$.
Finally if (H_j, D_j) or equivalently (H_j, L_j) is Fredholm then $H^i(H_*, D_*)$ is finite dimensional for each i and therefore $ran(D_j)$ is closed in H_{j+1} for each

j. We have the following natural and surjective map

$$\frac{Ker(D_{i+1})}{ran(D_i)} \longrightarrow \frac{Ker(D_{i+1})}{ran(P_i)}.$$

This implies that also $H^i(H_*, P_*)$ is finite dimensional, that is (H_j, P_j) is a Fredholm complex, and now using theorem 3.5 it follows that Poincaré duality holds for it. This complete the proof. □

Finally we conclude the section stating the following proposition which assures, under some conditions, that an operator is self-adjoint.

PROPOSITION 3.11. *Let $T : H \to H$ be a closed and densely defined operator such that $ran(T) \subset \mathcal{D}(T)$ and $T^2(u) = 0$ for each $u \in \mathcal{D}(T)$. Then*

$$T + T^* : H \to H$$

with domain given by $\mathcal{D}(T + T^*) = \{u : u \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$ is self-adjoint.

PROOF. Clearly $T + T^*$ is a symmetric operator. Therefore, to prove the statement, we have to show that $(T + T^*)^*$ is extended by $T + T^*$. Let $u \in \mathcal{D}((T + T^*)^*)$. This means that for each $v \in \mathcal{D}(T + T^*)$ we have that

$$v \longmapsto < T(v) + T^*(v), u >$$

is bounded. Now, by the fact that T is closed and densely defined it follows that $H = Ker(T) \oplus \overline{ran(T^*)}$. Therefore $\mathcal{D}(T) = Ker(T) \oplus \mathcal{D}(T) \cap \overline{ran(T^*)}$ and this implies that

$$ran(T) = ran(T|_{\mathcal{D}(T) \cap \overline{ran(T^*)}}).$$

Now consider again an element $u \in \mathcal{D}((T + T^*)^*)$ and let $\alpha \in \mathcal{D}(T)$. Then $< T(\alpha), u >=< T(\alpha_1) + T(\alpha_2), u >=< T(\alpha_2), u >$ where $\alpha = \alpha_1 \oplus \alpha_2$ with $\alpha_1 \in Ker(T)$ and $\alpha_2 \in \mathcal{D}(T) \cap \overline{ran(T^*)}$. But $\alpha_2 \in \mathcal{D}(T + T^*)$ and this implies that for each $\alpha \in \mathcal{D}(T)$ the linear application

$$\alpha \longmapsto < T(\alpha), u >$$

is bounded.
By the properties of T it follows that T^* is closed, densely defined and that $ran(T^*) \subset Ker(T^*)$. Therefore we can applying the same argumentations to T^* getting that for each $\beta \in \mathcal{D}(T^*)$ the linear application

$$\beta \longmapsto < T^*(\beta), u >$$

is bounded. This implies that $u \in \mathcal{D}(T^*) \cap \mathcal{D}(T)$ and therefore the proposition is proved. □

2. Poincaré duality and Hodge theorem.

In this section we apply, to the pairs of complementary Hilbert complexes arising in riemannian geometry, the results stated in the previous section.

THEOREM 3.12. *Let (M, g) be an open and oriented riemannian manifold of dimension m and let $E_0, ..., E_n$ be vector bundles over M endowed with metrics h_i $i = 0, ..., n$. Suppose that we have a complex of differential operator :*

$$0 \to C_c^\infty(M, E_0) \overset{P_0}{\to} C_c^\infty(M, E_1) \overset{P_1}{\to} ... \overset{P_{n-1}}{\to} C_c^\infty(M, E_n) \to 0, \quad (3.14)$$

and let

$$0 \rightarrow L^2(M,E_0) \stackrel{P_{max,0}}{\rightarrow} L^2(M,E_1) \stackrel{P_{max,1}}{\rightarrow} ... \stackrel{P_{max,n-1}}{\rightarrow} L^2(M,E_n) \rightarrow 0, \quad (3.15)$$

and

$$0 \rightarrow L^2(M,E_0) \stackrel{P_{min,0}}{\rightarrow} L^2(M,E_1) \stackrel{P_{min,1}}{\rightarrow} ... \stackrel{P_{min,n-1}}{\rightarrow} L^2(M,E_n) \rightarrow 0, \quad (3.16)$$

the two natural Hilbert complexes associated to (3.14) *as described above. Suppose that for each* $i = 0,...,n$ *there exists an isometry* $\phi_i : (E_i,h_i) \rightarrow (E_{n-i},h_{n-i})$*; with a little abuse of notation let still* ϕ_i *denotes the induced isometry from* $L^2(M,E_i)$ *to* $L^2(M,E_{n-i})$*. Finally suppose that* $P^t_{n-i-1} \circ \phi_i = c_i(\phi_{i+1} \circ P_i)$*, where* c_i *is a constant which depends only on* i*.*
If $\mathrm{im}(\overline{H}^i_{2,min}(M,E_*) \stackrel{i^*_{r,i}}{\longrightarrow} \overline{H}^i_{2,max}(M,E_*))$ *is finite dimensional for each* i *then*

$$\mathrm{im}(\overline{H}^i_{2,min}(M,E_*) \stackrel{i^*_{r,i}}{\longrightarrow} \overline{H}^i_{2,max}(M,E_*))$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality.

PROOF. From the hypothesis we know that for each $i = 0,...,n$ there exists an isometry $\phi_i : (E_i,h_i) \rightarrow (E_{n-i},h_{n-i})$ such that $P^t_{n-i-1} \circ \phi_i = c_i(\phi_{i+1} \circ P_i)$, where c_i is a constant which depends just by i. This isometries of vector bundles induces isometries from $L^2(M,E_i)$ to $L^2(M,E_{n-i})$, that with a little abuse of notation we still label ϕ_i, such that $\phi_i(\mathcal{D}(P_{min,i})) = \mathcal{D}(P^t_{min,n-i-1})$ and $P^t_{min,n-i-1} \circ \phi_i = c_i(\phi_{i+1} \circ P_{min,i})$. So we showed that the complexes $(L^2(M,E_*),P_{min,*}) \subseteq (L^2(M,E_*),P_{max,*})$ are a pair of complementary Hilbert complexes. Now, applying theorem 3.5, we can get the conclusion. □

THEOREM 3.13. *In the same hypothesis of the previous theorem, suppose furthermore that for each* $i = 0,...,n$ $ran(P_{min,i})$ *is closed in* $L^2(M,E_{i+1})$*. Then there exists a Hilbert complex* $(L^2(M,E_i),P_{\mathfrak{m},i})$ *such that for each* $i = 0,...,n$

$$\mathcal{D}(P_{min,i}) \subset \mathcal{D}(P_{\mathfrak{m},i}) \subset \mathcal{D}(P_{max,i}),$$

$P_{max,i}$ *is an extension of* $P_{\mathfrak{m},i}$ *which is an extension of* $P_{min,i}$ *and*

$$H^i_{2,\mathfrak{m}}(M,E_i) = \mathrm{im}(H^i_{2,min}(M,E_*) \stackrel{i^*_i}{\longrightarrow} H^i_{2,max}(M,E_*))$$

where $H^i_{2,\mathfrak{m}}(M,E_i)$ *is the cohomology of the Hilbert complex* $(L^2(M,E_i),P_{\mathfrak{m},i})$*. Finally if* $(L^2(M,E_i),P_{max,i})$ *or equivalently* $(L^2(M,E_i),P_{min,i})$ *is Fredholm then* $(L^2(M,E_i),P_{\mathfrak{m},i})$ *is a Fredholm complex with Poincaré duality.*

PROOF. It follows immediately from the previous theorem and from theorem 3.10. □

As a particular and important case we have the following two theorems:

THEOREM 3.14. *Let* (M,g) *be an open, oriented and incomplete riemannian manifold of dimension* m*. Then the complexes*

$$(L^2\Omega^*(M,g),d_{max,*}) \text{ and } (L^2\Omega^*(M,g),d_{min,*})$$

are a pair of complementary Hilbert complexes.
In particular if $\operatorname{im}(\overline{H}^i_{2,min}(M,g) \xrightarrow{i^*_{r,i}} \overline{H}^i_{2,max}(M,g))$ *is finite dimensional for each* i *then*

$$\operatorname{im}(\overline{H}^i_{2,min}(M,g) \xrightarrow{i^*_{r,i}} \overline{H}^i_{2,max}(M,g))$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality.

Proof. Let $* : \Lambda^i(M) \to \Lambda^{n-i}(M)$ the Hodge star operator. Then $*$ induces a map between $\Omega^i_c(M)$ and $\Omega^{n-i}_c(M)$ such that for $\eta, \omega \in \Omega^i_c(M)$ we have:

$$< *\eta, *\omega >_{L^2(M,g)} = \int_M < *\eta, *\omega >_M dvol_M = \int_M *\eta \wedge **\omega = \int_M \omega \wedge *\eta =$$

$$= < \omega, \eta >_{L^2(M,g)} = < \eta, \omega >_{L^2(M,g)}$$

that is $*$ is an isometry between $\Omega^i_c(M)$ and $\Omega^{n-i}_c(M)$. This implies that $*$ extends to an isometry between $L^2\Omega^i(M,g)$ and $L^2\Omega^{n-i}(M,g)$. Now it is an immediate consequence of definition 1.7 and definition 1.8 that

$$*d_{min,i} = \pm\delta_{min,n-i-1} * \quad \text{and that} \quad * d_{max,i} = \pm\delta_{max,n-i-1}*$$

and the sign depends only on the parity of the degree i. So we can apply theorem 3.5 and the assertion follows. □

Remark 3.2. The previous theorem shows that pair of complementary Hilbert complexes appear naturally in riemannian geometry. In fact the Hodge star operator provides naturally a family of link maps and so we do not need to assume their existence.

Theorem 3.15. *Let* (M,g) *be an open, oriented and incomplete riemannian manifold of dimension* n. *Suppose that for each* $i = 0, ..., n$ $ran(d_{min,i})$ *is closed in* $L^2\Omega^{i+1}(M,g)$. *Then there exists a Hilbert complex:*

$$(L^2\Omega^i(M,g)), d_{\mathfrak{m},i})$$

such that for each $i = 0, ...n$

$$\mathcal{D}(d_{min,i}) \subset \mathcal{D}(d_{\mathfrak{m},i}) \subset \mathcal{D}(d_{max,i}),$$

$d_{max,i}$ *is an extension of* $d_{\mathfrak{m},i}$ *which is an extension of* $d_{min,i}$ *and*

$$H^i_{2,\mathfrak{m}}(M,g) = \operatorname{im}(H^i_{2,min}(M,g) \xrightarrow{i^*_i} H^i_{2,max}(M,g))$$

where $H^i_{2,\mathfrak{m}}(M,g)$ *is the cohomology of the Hilbert complex* $(L^2\Omega^i(M,g), d_{\mathfrak{m},i})$. *Finally, if* $(L^2\Omega^i(M,g), d_{max,i})$ *or equivalently* $(L^2\Omega^i(M,g), d_{min,i})$ *is Fredholm, then* $(L^2\Omega^i(M,g), d_{\mathfrak{m},i})$ *is a Fredholm complex with Poincaré duality.*

Proof. Also in this case it follows immediately from the previous theorem and from theorem 3.10. □

We have the following corollary which is a **Hodge theorem** for the L^2 cohomology groups $\operatorname{im}(H^i_{2,min}(M,g) \xrightarrow{i^*_i} H^i_{2,max}(M,g))$:

COROLLARY 3.16. *In the same assumptions of theorem 3.15; Let $\Delta_i : \Omega^i_c(M) \to \Omega^i_c(M)$ be the Laplacian acting on the space of smooth compactly supported forms. Then there exists a self-adjoint extension $\Delta_{\mathfrak{m},i} : L^2\Omega^i(M,g) \to L^2\Omega^i(M,g)$ with closed range such that*

$$Ker(\Delta_{\mathfrak{m},i}) \cong \operatorname{im}(H^i_{2,min}(M,g) \xrightarrow{i^*_i} H^i_{2,max}(M,g)).$$

Moreover, if $(L^2\Omega^i(M,g), d_{max,i})$ or equivalently $(L^2\Omega^i(M,g), d_{min,i})$ is Fredholm, then $\Delta_{\mathfrak{m},i}$ is a Fredholm operator on its domain endowed with the graph norm.

PROOF. Consider the Hilbert complex $(L^2\Omega^i(M,g), d_{\mathfrak{m},i})$. For each $i = 0,...,n$ define

$$\Delta_{\mathfrak{m},i} := d^*_{\mathfrak{m},i} \circ d_{\mathfrak{m},i} + d_{\mathfrak{m},i-1} \circ d^*_{\mathfrak{m},i-1} \tag{3.17}$$

with domain given by

$$\mathcal{D}(\Delta_{\mathfrak{m},i}) = \tag{3.18}$$

$$\{\omega \in \mathcal{D}(d_{\mathfrak{m},i}) \cap \mathcal{D}(d^*_{\mathfrak{m},i-1}) :\ d_{\mathfrak{m},i}(\omega) \in \mathcal{D}(d^*_{\mathfrak{m},i}) \text{ and } d^*_{\mathfrak{m},i-1}(\omega) \in \mathcal{D}(d_{\mathfrak{m},i-1})\}.$$

In other words, for each $i = 0,...,n$, $\Delta_{\mathfrak{m},i}$ is the $i-th$ Laplacian associated to the Hilbert complex $(L^2\Omega^i(M,g), d_{\mathfrak{m},i})$. So, as recalled in the first section, it follows that (3.17) is a self-adjoint operator. Moreover, by the fact that $d_{min,i}$ has closed range for each $i = 0,...,n$ it follows that also $\delta_{min,i}$ has closed range for each i. Finally this implies that also $d_{max,i}$ has closed range because $d_{max,i} = \delta^*_{min,i}$. This means that for the Hilbert complex $(L^2\Omega^i(M,g), d_{\mathfrak{m},i})$ the L^2 cohomology and the reduced L^2 cohomology are exactly the same. The reason is that $\overline{ran(d_{\mathfrak{m},i})} = \overline{ran(d_{max,i}) \cap Ker(d_{min,i+1})}$ $= ran(d_{max,i}) \cap Ker(d_{min,i+1})$ because they are both closed in $L^2\Omega^{i+1}(M,g)$ and clearly $ran(d_{max,i}) \cap Ker(d_{min,i+1}) = ran(d_{\mathfrak{m},i})$. So we can apply (1.5) to get the first conclusion. Moreover by the fact that $ran(\Delta_{\mathfrak{m},i}) = ran(d_{\mathfrak{m},i-1}) \oplus ran(\delta_{\mathfrak{m},i})$ it follows that $\Delta_{\mathfrak{m},i}$ is an operator with closed range. Finally, using the fact that $(L^2\Omega^i(M,g), d_{\mathfrak{m},i})$ is Fredholm, we get that $\Delta_{\mathfrak{m},i}$ is self-adjoint, with finite dimensional nullspace and with closed range and therefore it is a Fredholm operator on its domain endowed with the graph norm. □

REMARK 3.3. We remark that from the previous proof it follows that, under the assumptions of theorem 3.15, the operator $d_{\mathfrak{m},i}$ has closed range for each i and therefore for the Hilbert complex $(L^2\Omega^i(M,g), d_{\mathfrak{m},i})$ the L^2 cohomology coincides with the reduced L^2 cohomology.

From now on we will focus our attention exclusively on the vector spaces $\operatorname{im}(\overline{H}^i_{2,min}(M,g) \xrightarrow{i^*_{r,i}} \overline{H}^i_{2,max}(M,g))$ because, using these, we will get some geometric and topological applications concerning the manifold M.
Anyway it will be clear that all the following corollaries of the remaining part of this subsection apply also for the vector spaces $\operatorname{im}(\overline{H}^i_{2,min}(M,E_*) \xrightarrow{i^*_{r,i}} \overline{H}^i_{2,max}(M,E_*))$ under the hypothesis of theorem 3.12.

Now, to get a lighter notation, we label the vector spaces

$$\mathrm{im}(\overline{H}^i_{2,min}(M,g) \xrightarrow{i^*_{r,i}} \overline{H}^i_{2,max}(M,g)) := \overline{H}^i_{2,m\to M}(M,g) \text{ and } H^i_{2,m\to M}(M,g)$$

in the non reduced case. Moreover, when it makes sense, we define

$$\overline{\chi}_{2,m\to M}(M,g) := \sum_{i=0}^{m}(-1)^i dim(\overline{H}^i_{2,m\to M}(M,g)) \tag{3.19}$$

and in the non reduced case :

$$\chi_{2,m\to M}(M,g) := \sum_{i=0}^{m}(-1)^i dim(H^i_{2,m\to M}(M,g)) \tag{3.20}$$

We have the following propositions:

PROPOSITION 3.17. *In the hypothesis of theorem 3.14, if m is odd then*

$$\overline{\chi}_{2,m\to M}(M,g) = 0. \tag{3.21}$$

If m is even then

$$\overline{\chi}_{2,m\to M}(M,g) = 2dim(\overline{H}^0_{2,m\to M}) + dim(\overline{H}^{\frac{m}{2}}_{2,m\to M}(M,g) \tag{3.22}$$

when $\frac{m}{2}$ is still even while if $\frac{m}{2}$ is odd then

$$\overline{\chi}_{2,m\to M}(M,g) = 2dim(\overline{H}^0_{2,m\to M}) - dim(\overline{H}^{\frac{m}{2}}_{2,m\to M}(M,g). \tag{3.23}$$

Finally if the complex $(L^2\Omega^i(M,g), d_{max,i})$ is Fredholm, or equivalently if $(L^2\Omega^i(M,g), d_{min,i})$ is Fredholm, then we have: if m is odd

$$\chi_{2,m\to M}(M,g) = 0. \tag{3.24}$$

If m is even then

$$\chi_{2,m\to M}(M,g) = 2dim(H^0_{2,m\to M}) + dim(H^{\frac{m}{2}}_{2,m\to M}(M,g) \tag{3.25}$$

when $\frac{m}{2}$ is still even while if $\frac{m}{2}$ is odd then

$$\chi_{2,m\to M}(M,g) = 2dim(H^0_{2,m\to M}) - dim(H^{\frac{m}{2}}_{2,m\to M}(M,g). \tag{3.26}$$

PROOF. The equalities (3.21), (3.22) and (3.23) are an immediate consequence of theorem 3.14. Finally, if for example $(L^2\Omega^i(M,g), d_{max,i})$ is Fredholm then $H^i_{2,max}(M,g) \cong \overline{H}^i_{2,max}(M,g) \cong \overline{H}^{n-i}_{2,min} \cong H^{n-i}_{2,min}(M,g)$ and so also $(L^2\Omega^i(M,g), d_{min,i})$ is Fredholm. Obviously the same arguments show that, if $(L^2\Omega^i(M,g), d_{min,i})$ is Fredholm, then also $(L^2\Omega^i(M,g), d_{max,i})$ is Fredholm and therefore in (3.24), (3.25) and (3.26) follow immediately because the L^2 cohomology coincides with the reduced L^2 cohomology. □

PROPOSITION 3.18. *In the hypothesis of theorem 3.5. Suppose that one of the two following properties is satisfied*

(1) $i^*_{r,i} : \overline{H}^i_{2,min}(M,g) \longrightarrow \overline{H}^i_{2,max}(M,g)$ *is injective,*
(2) $i^*_{r,i} : \overline{H}^i_{2,min}(M,g) \longrightarrow \overline{H}^i_{2,max}(M,g)$ *is surjective.*

Then

$$\overline{H}^i_{2,min}(M,g),\ \overline{H}^i_{2,max}(M,g)\ i=0,...,n \tag{3.27}$$

are both a finite sequences of finite dimensional vector spaces with Poincaré duality. Finally, under the same hypothesis, if one of the two complexes $(L^2\Omega^i(M,g), d_{max/min,i})$ *is Fredholm then the same conclusion holds for*

$$H^i_{2,min}(M,g),\ H^i_{2,max}(M,g)\ i=0,...,n$$

Proof. If $i^*_{r,i}:\overline{H}^i_{2,min}(M,g)\longrightarrow \overline{H}^i_{2,max}(M,g)$ is injective then

$$\overline{H}^i_{2,min}(M,g)\cong \overline{H}^i_{2,m\rightarrow M}(M,g).$$

This implies that each $\overline{H}^i_{2,min}(M,g)$ is finite dimensional and therefore, using theorem 3.14, we get $\overline{H}^i_{2,min}(M,g)\cong \overline{H}^{n-i}_{2,min}(M,g)$. Finally by the fact that the Hodge star operator induces an isomorphism between $\overline{H}^i_{2,min}(M,g)$ and $\overline{H}^{n-i}_{2,max}(M,g)$ it follows that $\overline{H}^i_{2,max}(M,g)$ is a finite sequences of finite dimensional vector spaces with Poincaré duality. In the same way if $i^*_{r,i}:\overline{H}^i_{2,min}(M,g)\longrightarrow \overline{H}^i_{2,max}(M,g)$ is surjective then $\overline{H}^i_{2,max}(M,g)\cong \overline{H}^i_{2,m\rightarrow M}(M,g)$. Now the same arguments used in the injective case shows that $\overline{H}^i_{2,max}(M,g)$ is a finite sequence of finite dimensional spaces with Poincaré duality. Finally, using again the isomorphism induced by the Hodge star operator between $\overline{H}^i_{2,min}(M,g)$ and $\overline{H}^{n-i}_{2,max}(M,g)$ we get the same conclusions for $\overline{H}^i_{2,min}(M,g)$. □

Finally we conclude the section with the following proposition; before stating it we give some definitions: let

$$d_{\mathfrak{m}}+d^*_{\mathfrak{m}}:\bigoplus_{i=0}^{n}L^2\Omega^i(M,g)\longrightarrow \bigoplus_{i=0}^{n}L^2\Omega^i(M,g) \tag{3.28}$$

the operator defined as $d_{\mathfrak{m}}+d^*_{\mathfrak{m}}|_{L^2\Omega^i(M,g)}=d_{\mathfrak{m},i}+d^*_{\mathfrak{m},i-1}$ where $d_{\mathfrak{m},i}$ is defined in theorem 3.15 and the domain of (3.28) is

$$\mathcal{D}(d_{\mathfrak{m}}+d^*_{\mathfrak{m}})=\bigoplus_{i=0}^{n}\mathcal{D}(d_{\mathfrak{m},i}+d^*_{\mathfrak{m},i-1})$$

and $\mathcal{D}(d_{\mathfrak{m},i}+d^*_{\mathfrak{m},i-1})=\mathcal{D}(d_{\mathfrak{m},i})\cap\mathcal{D}(d^*_{\mathfrak{m},i-1})$.

Proposition 3.19. *Let* (M,g) *be an open oriented and incomplete riemannian manifold of dimension n. Suppose that for each* $i=0,...,n$ $ran(d_{min,i})$ *is closed in* $L^2\Omega^{i+1}(M,g)$ *and that* $(L^2\Omega^i(M,g),d_{\mathfrak{m},i})$ *is a Fredholm complex. Then the operator* $(d_{\mathfrak{m}}+d^*_{\mathfrak{m}})_{ev}$ *defined as*

$$d_{\mathfrak{m}}+d^*_{\mathfrak{m}}:\bigoplus_{i=0}^{n}L^2\Omega^{2i}(M,g)\longrightarrow \bigoplus_{i=0}^{n}L^2\Omega^{2i+1}(M,g)$$

with domain given by

$$\mathcal{D}((d_{\mathfrak{m},i}+d^*_{\mathfrak{m},i-1})_{ev}):=\mathcal{D}(d_{\mathfrak{m}}+d^*_{\mathfrak{m}})\cap\bigoplus_{i=0}^{n}L^2\Omega^{2i}(M,g)$$

is a Fredholm operator on its domain endowed with the graph norm and its index satisfies

$$\operatorname{ind}((d_{\mathfrak{m}} + d^*_{\mathfrak{m}})_{ev}) = \chi_{m\to M}(M,g) \tag{3.29}$$

PROOF. By the fact that $(L^2\Omega^i(M,g), d_{\mathfrak{m},i})$ is a Fredholm complex it follows that $d_{\mathfrak{m}} + d^*_{\mathfrak{m}}$ is a Fredholm operator on its domain endowed with graph norm. Now if we define $(d_{\mathfrak{m}} + d^*_{\mathfrak{m}})_{odd}$ analogously to $(d_{\mathfrak{m}} + d^*_{\mathfrak{m}})_{ev}$, then it is clear that $\mathcal{D}(d_{\mathfrak{m}} + d^*_{\mathfrak{m}}) = \mathcal{D}((d_{\mathfrak{m}} + d^*_{\mathfrak{m}})_{ev}) \oplus \mathcal{D}((d_{\mathfrak{m}} + d^*_{\mathfrak{m}})_{odd})$, that $Ker(d_{\mathfrak{m}}+d^*_{\mathfrak{m}}) = Ker((d_{\mathfrak{m}}+d^*_{\mathfrak{m}})_{ev}) \oplus Ker((d_{\mathfrak{m}}+d^*_{\mathfrak{m}})_{odd})$ and that $ran(d_{\mathfrak{m}}+d^*_{\mathfrak{m}})$ $= ran((d_{\mathfrak{m}}+d^*_{\mathfrak{m}})_{ev}) \oplus ran((d_{\mathfrak{m}}+d^*_{\mathfrak{m}})_{odd})$. This implies immediately that also $(d_{\mathfrak{m}} + d^*_{\mathfrak{m}})_{ev}$ is a Fredholm operator on its domain endowed with the graph norm. Finally (3.29) is an easy consequence of the Hodge theorem stated in corollary 3.16. □

3. Some geometric applications

The aim of this section is show some geometric application of the groups $\overline{H}^i_{2,m\to M}(M,g)$ and $H^i_{2,m\to M}(M,g)$. In particular we will show that, using them, we can deduce the presence of a topological obstruction to the existence of a riemannian metric (complete ore incomplete) with finite L^2 cohomology.

Consider again the complex $(\Omega^*_c(M), d_*)$. We will call a **closed extension** of $(\Omega^*_c(M), d_*)$ any Hilbert complex $(L^2\Omega^i(M,g), D_i)$ where

$$D_i : L^2\Omega^i(M,g) \to L^2\Omega^{i+1}(M,g)$$

is a densely defined, closed operator which extends $d_i : \Omega^i_c(M,g) \to \Omega^{i+1}_c(M,g)$ and such that the action of D_i on $\mathcal{D}(D_i)$, its domain, coincides with the action of d_i on $\mathcal{D}_i$ in a distributional way. Obviously for every closed extension of $(\Omega^*_c(M), d_*)$ we have $(L^2\Omega^*(M,g), d_{min,*}) \subseteq (L^2\Omega^*(M,g), D_i) \subseteq (L^2\Omega^*(M,g), d_{max,*})$. We will label with $\overline{H}^i_{2,D_*}(M,g)$, $H^i_{2,D_*}(M,g)$ respectively the reduced cohomology and the cohomology groups of $(L^2\Omega^i(M,g), D_i)$ and with $\mathcal{H}^i_{D_*}(M,g)$ its Hodge cohomology groups.

Moreover if $(L^2\Omega^*(M,g), D'_i)$ is another closed extension of $(\Omega^*_c(M), d_*)$ such that $(L^2\Omega^*(M,g), D_i) \subseteq (L^2\Omega^*(M,g), D'_i)$ we will label with

$$H^i_{2,D\to D'}(M,g),\ \overline{H}^i_{2,D\to D'}(M,g)$$

respectively the image of the cohomology groups, reduced cohomology groups, of the complex $(L^2\Omega^*(M,g), D_i)$ into the cohomology groups, reduced cohomology groups, of the complex $(L^2\Omega^*(M,g), D'_i)$ induced by the natural inclusion of complexes.

Before we proceed we need the following propositions.

PROPOSITION 3.20. *Let (M,g) be an incomplete and oriented riemannian manifold of dimension m. For each $i = 0, ..., m$ consider $\mathcal{D}(d_{max,i})$. Let $\omega \in \mathcal{D}(d_{max,i})$. Then there exists a sequence of smooth forms $\{\omega_j\}_{j\in\mathbb{N}} \subset \Omega^i(M) \cap L^2\Omega^i(M,g)$ such that :*

1. *$d_i\omega_j \in L^2\Omega^{i+1}(M,g)$.*
2. *$\omega_j \to \omega$ in $L^2\Omega^i(M,g)$.*
3. *$d_i\omega_j \to d_{max,i}\omega$ in $L^2\Omega^{i+1}(M,g)$.*

PROOF. See [**23**] pag 93. $\square$

The next proposition is a variation of a result of de Rham, see [**29**] theorem 24.

PROPOSITION 3.21. *Let (M,g) be an incomplete and oriented riemannian manifold of dimension m. For each $i = 0, ..., m$ consider $\mathcal{D}(d_{max,i})$. Let $\omega \in \overline{ran(d_{max,i})}$ such that ω is smooth. Then there exist $\eta \in \Omega^i(M)$ such that $d_i\eta = \omega$.*

PROOF. By Poincaré duality between de Rham cohomology and compactly supported de Rham cohomology on an open and oriented manifold we know that it sufficient to show that

$$\int_M \omega \wedge \phi = 0$$

for each closed and compactly supported $n-i-$form ϕ to get that ω is an exact $i-$form in the smooth de Rham complex. Now, by proposition 3.20, we know that there exists a sequence of smooth $i-$forms $\{\omega_j\}_{j\in\mathbb{N}}$ such that $d_i\omega_j \to \omega$ in $L^2\Omega^{i+1}(M,g)$. Then:

$$\int_M \omega\wedge\phi = \int_M (\lim_{j\to\infty} d_i\omega_j)\wedge\phi = \lim_{j\to\infty}\int_M d_i\omega_j\wedge\phi = \pm\lim_{j\to\infty}\int_M d_i\omega_j\wedge(**\phi) =$$

$$= \pm\lim_{j\to\infty} < d_i\omega_j, *\phi >_{L^2(M,g)} = \pm\lim_{j\to\infty} < \omega_j, \delta_{i-1}(*\phi) >_{L^2(M,g)} = 0.$$

So the proposition is proved. $\square$

PROPOSITION 3.22. *Let $(L^2\Omega^i(M,g), D_i)$ be any closed extension of $(\Omega^*_c(M), d_*)$ where (M,g) is an incomplete oriented riemannian manifold. Then every cohomology class in $\overline{H}^i_{2,D_*}(M,g)$ has a smooth representative. The same conclusion holds for every cohomology class in $H^i_{2,D_*}(M,g)$.*

PROOF. By (1.5) we know that every cohomology class in $\overline{H}^i_{2,D_*}(M,g)$ has a representative in $\mathcal{H}^i_{D_*}(M,g)$. Now, by elliptic regularity (see for example de Rham book [**29**]), it follows that every element in $\mathcal{H}^i_{D_*}(M,g)$ is smooth. Now if we look at proposition 1.6, elliptic regularity tells us again that every element in $\mathcal{D}^\infty(L^2\Omega^i(M,g))$ is smooth. Therefore form this it follows immediately the statement for $H^i_{2,D_*}(M,g)$. $\square$

From the above propositions 3.21 and 3.22 it follows that that there exists a well defined map from $\overline{H}^i_{2,D_*}(M,g)$, respectively from $H^i_{2,D_*}(M,g)$, to the ordinary de Rham cohomology of M which assigns to each cohomology class $[\omega] \in \overline{H}^i_{2,D_*}(M,g)$, respectively $[\omega] \in H^i_{2,D_*}(M,g)$, the cohomology class in $H^i_{dR}(M)$ given by the smooth representatives of $[\omega]$. By proposition 3.21 this cohomology class in $H^i_{dR}(M)$ does not depend from the choice of the smooth representative of $[\omega]$ and therefore this map is well defined.
We will label these maps:

$$s^*_{D,i} : H^i_{2,D_*}(M,g) \longrightarrow H^i_{dR}(M) \text{ in the non reduced case} \tag{3.30}$$

and

$$s^*_{r,D,i} : \overline{H}^i_{2,D_*}(M,g) \longrightarrow H^i_{dR}(M) \text{ in the reduced case} \tag{3.31}$$

In particular for the maximal and minimal extension we will label these maps:

$$s^*_{M,i} : H^i_{2,max}(M,g) \longrightarrow H^i_{dR}(M) \text{ in the non reduced case} \tag{3.32}$$

and

$$s^*_{r,M,i} : \overline{H}^i_{2,max}(M,g) \longrightarrow H^i_{dR}(M) \text{ in the reduced case} \tag{3.33}$$

and analogously for the minimal extension

$$s^*_{m,i} : H^i_{2,min}(M,g) \longrightarrow H^i_{dR}(M) \text{ in the non reduced case} \tag{3.34}$$

and

$$s^*_{r,m,i} : \overline{H}^i_{2,min}(M,g) \longrightarrow H^i_{dR}(M) \text{ in the reduced case} \tag{3.35}$$

Now we are ready to state the following proposition:

PROPOSITION 3.23. *Let (M,g) be an open, oriented and incomplete riemannian manifold. Let $(L^2\Omega^i(M,g), D_{a,*}), (L^2\Omega^i(M,g), D_{b,*})$ be two closed extension of $(\Omega^*_c(M), d_*)$ such that*

$$(L^2\Omega^*(M,g), d_{min,*}) \subseteq (L^2\Omega^*(M,g), D_{a,*}) \subseteq (L^2\Omega^*(M,g), D_{b,*}) \subseteq \tag{3.36}$$
$$\subseteq (L^2\Omega^*(M,g), d_{max,*}).$$

Then the two following diagrams commute:

$$\begin{tikzcd}
H^i_c(M) \arrow[r] \arrow[d] & H^i_{dR}(M) \\
H^i_{2,min}(M,g) \arrow[r] \arrow[d] & H^i_{2,max}(M,g) \arrow[u, "s^*_{M,i}"'] \\
H^i_{2,D_{a,*}}(M,g) \arrow[r] & H^i_{2,D_{B,*}}(M,g) \arrow[u]
\end{tikzcd}
\quad
\begin{tikzcd}
H^i_c(M) \arrow[r] \arrow[d] & H^i_{dR}(M) \\
\overline{H}^i_{2,min}(M,g) \arrow[r] \arrow[d] & \overline{H}^i_{2,max}(M,g) \arrow[u, "s^*_{r,M,i}"'] \\
\overline{H}^i_{2,D_{a,*}}(M,g) \arrow[r] & \overline{H}^i_{2,D_{B,*}}(M,g) \arrow[u]
\end{tikzcd} \tag{3.37}$$

where all the above arrows without label are the natural maps between cohomology, respectively reduced cohomology groups, induced by the natural inclusion of the relative complexes.

PROOF. It is clear that both the two following diagrams commute:

$$\begin{tikzcd}
H^i_c(M) \arrow[d] \arrow[dr] & \\
H^i_{2,min}(M,g) \arrow[r] \arrow[d] & H^i_{2,max}(M,g) \\
H^i_{2,D_{a,*}}(M,g) \arrow[r] & H^i_{2,D_{B,*}}(M,g) \arrow[u]
\end{tikzcd}
\quad
\begin{tikzcd}
H^i_c(M) \arrow[d] \arrow[dr] & \\
\overline{H}^i_{2,min}(M,g) \arrow[r] \arrow[d] & \overline{H}^i_{2,max}(M,g) \\
\overline{H}^i_{2,D_{a,*}}(M,g) \arrow[r] & \overline{H}^i_{2,D_{B,*}}(M,g) \arrow[u]
\end{tikzcd}$$

So, to complete the proof, we have to show that the two following diagrams are both commutative:

$$\begin{tikzcd} H^i_c(M) \arrow[d] \arrow[dr] & \\ H^i_{2,max}(M,g) \arrow[r, "s^*_{M,i}"] & H^i_{dR}(M) \end{tikzcd} \quad \begin{tikzcd} H^i_c(M) \arrow[d] \arrow[dr] & \\ \overline{H}^i_{2,max}(M,g) \arrow[r, "s^*_{r,M,i}"] & H^i_{dR}(M) \end{tikzcd}$$

To prove this it is enough to show that given a cohomology class $[\omega] \in H^i_{2,max}(M,g)$, respectively $[\omega] \in \overline{H}^i_{2,max}(M,g)$, such that ω is closed, smooth and with compact support, if $[\omega] = 0$ in $H^i_{2,max}(M,g)$ or in $\overline{H}^i_{2,max}(M,g)$ then also $s^*_{M,i}(\omega) = 0$, respectively $s^*_{r,M,i}(\omega) = 0$, that is the cohomology class of ω in $H^i_{dR}(M)$ is null. This last statement follows immediately from proposition 3.21. □

Using the previous proposition we get the following corollary in which the first statement extends a result of Anderson, see [**2**], to the case of an incomplete riemannian metric both for the reduced and the unreduced L^2 cohomology groups.

Corollary 3.24. *Let (M,g) be as in the previous proposition. Then, for each $j = 0, ..., dimM$, there are injective maps:*

$$\operatorname{im}(H^j_c(M) \to H^j_{dR}(M)) \longrightarrow \overline{H}^j_{2,m\to M}(M,g) \longrightarrow \overline{H}^j_{2,D_a\to D_b}(M,g) \quad (3.38)$$

$$\operatorname{im}(H^j_c(M) \to H^j_{dR}(M)) \longrightarrow H^j_{2,m\to M}(M,g) \longrightarrow H^j_{2,D_a\to D_b}(M,g) \quad (3.39)$$

Moreover if $H^i_c(M) \to H^i_{dR}(M)$ is injective then

$$H^i_c(M) \to H^i_{2,m\to M}(M,g),\ H^i_c(M) \to \overline{H}^i_{2,m\to M}(M,g) \quad (3.40)$$

are injective and therefore for each closed extension $(L^2\Omega^*(M,g), D_*)$ also the following maps are injective:

$$H^i_c(M) \to H^i_{2,D}(M,g),\ H^i_c(M) \to \overline{H}^i_{2,D}(M,g) \quad (3.41)$$

Proof. It is an immediate consequence of the previous proposition. □

Now we give other three corollaries of proposition 3.23. In particular the third corollary shows that it could exist a **topological obstruction** to the existence of a riemannian metric on g with certain analytic properties.

Corollary 3.25. *Let M be an open manifold such that for some j $im(H^j_c(M) \stackrel{i^*_j}{\to} H^j_{dR}(M))$ is non trivial. Then for every riemmannian metric g on M and for every pair of closed extensions $(L^2\Omega^*(M,g), D_{a,*})$, $(L^2\Omega^*(M,g), D_{b,*})$ such that $(L^2\Omega^*(M,g), D_{a,*}) \subseteq (L^2\Omega^*(M,g), D_{b,*})$ we have that for the same j both vector spaces*

$$H^j_{2,D_a\to D_b}(M,g),\ \overline{H}^j_{2,D_a\to D_b}(M,g)$$

are non trivial. In particular this implies that for the same j the following four vector spaces are non trivial:

$$H^j_{2,D_a}(M,g),\ H^j_{2,D_b}(M,g),\ \overline{H}^j_{2,D_a}(M,g),\ \overline{H}^j_{2,D_b}(M,g).$$

Corollary 3.26. *Let (M, g) be an open, oriented and incomplete riemannian manifold. Suppose that there exists a pair of closed extension $(L^2\Omega^*(M,g), D_{a,*}), (L^2\Omega^*(M,g), D_{b,*})$ of $(\Omega_c^*(M), d_*)$ such that are both weak Fredholm and $(L^2\Omega^*(M,g), D_{a,*}) \subseteq (L^2\Omega^*(M,g), D_{b,*})$.*

Then $im(H_c^j(M) \xrightarrow{i_j^*} H_{dR}^j(M))$ is finite dimensional and we have

$$dim(\mathrm{im}(H_c^j(M) \xrightarrow{i_j^*} H_{dR}^j(M))) \leq dim\overline{H}_{2,D_a}^j(M,g) \tag{3.42}$$

$$dim(\mathrm{im}(H_c^j(M) \xrightarrow{i_j^*} H_{dR}^j(M))) \leq dim\overline{H}_{2,D_b}^j(M,g) \tag{3.43}$$

In particular if one of the two complexes $(L^2\Omega^*(M,g), d_{max/min,*})$ is weak Fredholm then also the other one is weak Fredholm and for each $j = 0, ..., m$ we have:

$$dim(\mathrm{im}(H_c^j(M) \xrightarrow{i_j^*} H_{dR}^j(M))) \leq dim\overline{H}_{2,max}^j(M,g) \tag{3.44}$$

$$dim(\mathrm{im}(H_c^j(M) \xrightarrow{i_j^*} H_{dR}^j(M))) \leq dim\overline{H}_{2,min}^j(M,g). \tag{3.45}$$

Finally if one of the two complexes $(L^2\Omega^*(M,g), d_{max/min,*})$ is Fredholm then for each $j = 0, ..., m$ we have:

$$dim(\mathrm{im}(H_c^j(M) \xrightarrow{i_j^*} H_{dR}^j(M))) \leq dimH_{2,max}^j(M,g) \tag{3.46}$$

$$dim(\mathrm{im}(H_c^j(M) \xrightarrow{i_j^*} H_{dR}^j(M))) \leq dimH_{2,min}^j(M,g). \tag{3.47}$$

Proof. It is an immediate consequence of corollary 3.24. □

Corollary 3.27. *Let M be an open, oriented and incomplete riemannian manifold where $m = dim(M)$. Suppose that there exists an $j \in \{0, ..., m\}$ such that $\mathrm{im}(H_c^j(M) \xrightarrow{i_j^*} H_{dR}^j(M))$ is infinite dimensional. Then M does not admit any riemannian metrics g (complete or incomplete) such that g implies the existence of a closed extension $(L^2\Omega^*(M,g), D_*)$ of $(\Omega_c^*(M), d_*)$ with one of the following properties for the same j:*

1. *$\overline{H}_{2,D_*}^j(M,g)$ or $\overline{H}_{2,D_*}^{m-j}(M,g)$ is finite dimensional.*
2. *$H_{2,D_*}^j(M,g)$ or $H_{2,D_*}^{m-j}(M,g)$ is finite dimensional.*
3. *$D_j^* \circ D_j + D_{j-1} \circ D_{j-1}^*$ on its domain (as defined in (1.3)) endowed with the graph norm is a Fredholm operator.*

Moreover M does not admit any riemannian metric g such that:

1. *$\Delta_{max,j}$, the maximal closed extension of $\Delta_j : \Omega_c^j(M) \to \Omega_c^j(M)$, has finite dimensional nullspace.*
2. *$\Delta_{min,j}$, the minimal closed extension of $\Delta_j : \Omega_c^j(M) \to \Omega_c^j(M)$, satisfies $dim(ran(\Delta_{min,j})^\perp) < \infty$.*

Proof. The first two points are immediate consequence of corollary 3.24 and theorem 3.14. The third point follows immediately by (1.4) and (1.5). Finally, for the last two points , if $Ker(\Delta_{max,j})$ is finite dimensional then all the other closed extensions of $\Delta_j : \Omega_c^j(M) \to \Omega_c^j(M)$ have finite dimensional nullspace. So we can apply the third point to get the conclusion. Finally if we consider $\Delta_{min,j}$ then we have $\Delta^*_{min,j} = \Delta_{max,j}$. So if $dim(ran(\Delta_{min,j})^{\perp}) < \infty$ then $Ker(\Delta_{max,j})$ is finite dimensional. Now by the previous point we can get the conclusion. □

4. L^2 and topological signature for an incomplete manifold.

The aim of this subsection is to show that if (M,g) is an open oriented and incomplete riemannian manifold such that for $i = 2k$ $\overline{H}^i_{2,m\to M}(M,g)$ is finite dimensional, where $4k = dimM$, then we can define over M an L^2 signature and a topological signature. The first step is to show that using the wedge product we can construct a well defined and non degenerate pairing between $\overline{H}^i_{2,m\to M}(M,g)$ and $\overline{H}^{n-i}_{2,m\to M}(M,g)$ where $n = dimM$. In fact any cohomology class $[\omega] \in \overline{H}^i_{2,m\to M}(M,g)$ is a cohomology class in $\overline{H}^i_{2,max}(M,g)$ which admits a representative in $Ker(d_{min,i})$. So we can define:

$$\overline{H}^i_{2,m\to M}(M,g) \times \overline{H}^{n-i}_{2,m\to M}(M,g) \longrightarrow \mathbb{R},\ ([\eta],[\omega]) \mapsto \int_M \eta\wedge\omega \qquad (3.48)$$

where $\omega \in Ker(d_{min,i})$ and $\eta \in Ker(d_{min,n-i})$

Proposition 3.28. *Let (M,g) be an open, oriented and incomplete riemannian manifold of dimension n. Then* (3.48) *is a well defined and non degenerate pairing and therefore when the vector spaces $\overline{H}^i_{2,m\to M}(M,g)$ $i = 0,...,n$ are finite dimensional it induces an isomorphism between*

$$\overline{H}^i_{2,m\to M}(M,g) \text{ and } (\overline{H}^{n-i}_{2,m\to M}(M,g))^*.$$

Proof. The first step is to show that (3.48) is well defined.
Let η', ω' other two forms such that $[\eta] = [\eta']$ in $\overline{H}^i_{2,m\to M}(M,g)$, $[\omega] = [\omega']$ in $\overline{H}^{n-i}_{2,m\to M}(M,g)$ and that $\omega' \in Ker(d_{min,i})$, $\eta' \in Ker(d_{min,n-i})$. Then there exist $\alpha \in \overline{d_{max,i-1}} \cap \mathcal{D}(d_{min,i})$ and $\beta \in \overline{d_{max,n-i-1}} \cap \mathcal{D}(d_{min,n-i})$ such that $\eta = \eta' + \alpha$ and $\omega = \omega' + \beta$. Therefore

$$\int_M \eta\wedge\omega = \int_M (\eta'+\alpha)\wedge(\omega'+\beta) = \int_M \eta'\wedge\omega' + \int_M \eta'\wedge\beta + \int_M \alpha\wedge\omega' + \int_M \alpha\wedge\beta$$

Now

$$\int_M \eta'\wedge\beta = \pm\int_M <\eta', *\beta> dvol_M = <\eta', *\beta>_{L^2(M,g)} = 0$$

because $Ker(d_{min,i})^{\perp} = \overline{ran(\delta_{max,i})}$. In the same way:

$$\int_M \alpha\wedge\beta = \pm\int_M <\alpha, *\beta> dvol_M = <\alpha, *\beta>_{L^2(M,g)} = 0.$$

Finally

$$\int_M \alpha\wedge\omega' = \pm\int_M <\alpha, *\omega'> dvol_M = <\alpha, *\omega'>_{L^2(M,g)} = 0$$

because $Ker(\delta_{min,i-1})^{\perp} = \overline{ran(d_{max,i-1})}$. So we can conclude that (3.48) is well defined. Now fix $[\eta] \in \overline{H}^i_{2,m\to M}(M,g)$ and suppose that for each $[\omega] \in \overline{H}^{n-i}_{2,m\to M}(M,g)$ the pairing (3.48) vanishes. Then this means that

for each $\omega \in Ker(d_{min,n-i})$ we have $\int_M \eta \wedge \omega = 0$. We also know that $\int_M \eta \wedge \omega =< \eta, *\omega >_{L^2(M,g)}$ and that $*(\overline{Ker(d_{min,n-i})}) = Ker(\delta_{min,i-1})$. So by the fact that $(Ker(\delta_{min,i-1}))^{\perp} = \overline{ran(d_{max,i-1})}$ we obtain that $[\eta] = 0$. In the same way if $[\omega] \in \overline{H}^{n-i}_{2,m\rightarrow M}(M,g)$ is such that for each $[\eta] \in \overline{H}^{i}_{2,m\rightarrow M}(M,g)$ the pairing (3.48) vanishes then we know that for each $\eta \in Ker(d_{max,i})$ we have $\int_M \eta \wedge \omega = 0$. But we know that $\int_M \eta \wedge \omega =< \eta, *\omega >_{L^2(M,g)}$. So by the fact that $*(\overline{ran(d_{max,n-i-1}})) = \overline{ran(\delta_{max,i})}$ and that $(Ker(d_{min,i}))^{\perp} = \overline{ran(\delta_{max,i})}$ we obtain that $[\omega] = 0$.
So we can conclude that the pairing (3.48) is well defined and non degenerate and therefore when the vector spaces $\overline{H}^{i}_{2,m\rightarrow M}(M,g)$ $i = 0, ..., n$ are finite dimensional it induces an isomorphism between $\overline{H}^{i}_{2,m\rightarrow M}(M,g)$ and $(\overline{H}^{n-i}_{2,m\rightarrow M}(M,g))^*$. □

REMARK 3.4. We can look at this proposition as an alternative statement (and proof) of theorem 3.14.

We have the following immediate corollary:

COROLLARY 3.29. *Let (M,g) be an open, oriented and incomplete riemannian manifold of dimension $4n$. Then on $\overline{H}^{2n}_{2,m\rightarrow M}(M,g)$ the pairing* (3.48) *is a symmetric bilinear form.*

We can now state the following definition:

DEFINITION 3.30. Let (M,g) be an open and oriented riemannian manifold of dimension $4n$ such that, for $i = 2n$, $\overline{H}^{2n}_{2,m\rightarrow M}(M,g)$ is finite dimensional. Then we define the L^2 signature of (M,g) and we label it $\sigma_2(M,g)$ as the signature of the pairing (3.48) on $\overline{H}^{2n}_{2,m\rightarrow M}(M,g)$.

Consider now the sequence of vector spaces $\text{im}(H^i_c(M) \rightarrow H^i_{dR}(M))$. A cohomology class in $\text{im}(H^i_c(M) \rightarrow H^i_{dR}(M))$ is a cohomology class in $H^i_{dR}(M)$ which admits as representative a smooth and closed form with compact support. So in a similar way to the previous case we can define:

$$\text{im}(H^i_c(M) \rightarrow H^i_{dR}(M)) \times \text{im}(H^{n-i}_c(M) \rightarrow H^{n-i}_{dR}(M)) \longrightarrow \mathbb{R}, \tag{3.49}$$

$$([\eta], [\omega]) \mapsto \int_M \eta \wedge \omega$$

where ω is a $i-$form closed with compact support and in the same way η is a closed $n-i-$form with compact support. Now by Poincaré duality for open and oriented manifolds it follows easily that this pairing is well defined and non degenerate. So we can conclude that, if for each $i = 0, ..., dimM$ $\text{im}(H^i_c(M) \rightarrow H^i_{dR}(M))$ is finite dimesional, then 3.49 induces an isomorphism between $\text{im}(H^i_c(M) \rightarrow H^i_{dR}(M))$ and $\text{im}(H^{n-i}_c(M) \rightarrow H^{n-i}_{dR}(M))^*$. Moreover it is clear that when $dimM = 4n$ then, for $i = 2n$, (3.49) is a symmetric bilinear form. This implies that when $dimM = 4n$ it is possible to define a signature on M, which is topological by de Rham isomorphism theorem, taking the signature of the pairing (3.49) for $i = 2n$. This leads us to state the next proposition:

Proposition 3.31. *Let (M,g) be an open, oriented and incomplete riemannian manifold of dimension $4n$. If (M,g) admits the L^2 signature $\sigma_2(M,g)$ of definition 3.30 then it admits also a topological signature defined as the signature of the pairing* (3.49) *on* $\mathrm{im}(H^{2n}_c(M) \to H^{2n}_{dR}(M))$.

Proof. If M admits the signature $\sigma_2(M,g)$ then, by definition 3.30, we know that $\overline{H}^{2n}_{2,m\to M}(M,g)$ is finite dimensional. Now, by corollary 3.24, we know that also $\mathrm{im}(H^{2n}_c(M) \to H^{2n}(M))$ is finite dimensional and so the pairing 3.49 admits a signature. □

Moreover in the next section we will see that, on a class of open, incomplete and oriented riemannian manifold, the L^2 signature of definition 3.30 has a topological meaning.

5. Application to stratified pseudomanifolds

The aim of this section is to exhibit some applications of the previous results to stratified pseudomanifolds and intersection cohomology.

Proposition 3.32. *Let X be a compact and oriented smoothly stratified pseudomanifold of dimension n with a Thom-Mather stratification $\mathfrak{X}$. Let g be a quasi edge metric with weights on $reg(X)$. Then*

$$H^i_{2,m\to M}(reg(X),g),\ i=0,...,n$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality. Moreover proposition 3.17 and proposition 3.18 apply to this kind of riemannian manifolds.

Proof. By theorem 2.12 we know that both cohomology groups

$$H^i_{2,max}(reg(X),g) \text{ and } H^i_{2,min}(reg(X),g)$$

are finite dimensional. This implies that in the following sequence

$$H^i_{2,m\to M}(reg(X),g),\ i=0,...,n$$

each vector space is finite dimensional. In this way we are in position to apply theorem 3.14, proposition 3.17, proposition 3.18 and therefore the thesis follows. □

Now consider two general perversities p, q such that $q \le p$. Then the complex associated to q is a subcomplex of that associated to p and therefore the inclusion i induces a maps between the intersection cohomology groups $I^qH^i(X,\mathcal{R}_0)$ and $I^pH^i(X,\mathcal{R}_0)$ that we call i^*. In analogy to the previous section we define for each $j=0,...,n$

$$I^{q\to p}H^j(X,\mathcal{R}_0) := \mathrm{im}(I^qH^j(X,\mathcal{R}_0) \xrightarrow{i*} I^pH^j(X,\mathcal{R}_0)) \tag{3.50}$$

and

$$I^{q\to p}\chi(X,\mathcal{R}_0) := \sum_{i=0}^{n}(-1)^i dim(I^{q\to p}H^j(X,\mathcal{R}_0)) \tag{3.51}$$

Now we are ready to state the following proposition:

PROPOSITION 3.33. *Let X be a compact and oriented smoothly stratified pseudomanifold of dimension n with a Thom-Mather stratification $\mathfrak{X}$. Let*

$$p : \{Singular\ Strata\ of\ X\} \to \mathbb{N}$$

a general perversity such that

$$p(Y) = -1$$

for each stratum Y of X with $cod(Y) = 1$. Suppose that, if we call q its dual, then we have

$$p \leq q.$$

Then

$$I^{q\to p}H^j(X, \mathcal{R}_0),\ \ j = 0, ..., n$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality. Analogously if

$$p(Y) = 0$$

for each stratum Y of X with $cod(Y) = 1$ and

$$p \geq q$$

where q denote again the dual perversity of p, then

$$I^{p\to q}H^j(X, \mathcal{R}_0),\ \ j = 0, ..., n$$

is again a finite sequence of finite dimensional vector spaces with Poincaré duality.

PROOF. We know that $p \leq q$. This means that for each singular stratum of codimension i we have $p \leq i - 2 - p$ that is $p \leq \frac{i-2}{2}$. But p takes values in $\mathbb{N}$ and therefore $p \leq \frac{i-2}{2}$ if and only if $p \leq [\frac{i-2}{2}]$ that is $p \leq \underline{m}$. This implies that p satisfies the assumptions of theorem 2.13 that is there exist a quasi edge metric g on $reg(X)$ such that $p_g = p$. In this way we can use proposition 3.32 to get the conclusion.
In the same way if $p \geq q$ then we get $p \geq \overline{m}$. So we can use again theorem 2.13 and proposition 3.32 to get the assertion. □

We have the following four immediate corollaries:

COROLLARY 3.34. *In the hypothesis of proposition 3.33, if n is odd then*

$$I^{q\to p}\chi(X, \mathcal{R}_0) = 0 \tag{3.52}$$

If n is even then

$$I^{q\to p}\chi(X, \mathcal{R}_0) = 2dim(I^{q\to p}H^0(X, \mathcal{R}_0)) + dim(I^{q\to p}H^{\frac{n}{2}}(X, \mathcal{R}_0)) \tag{3.53}$$

when $\frac{n}{2}$ is still even while if $\frac{n}{2}$ is odd then

$$I^{q\to p}\chi(X, \mathcal{R}_0) = 2dim(I^{q\to p}H^0(X, \mathcal{R}_0)) - dim(I^{q\to p}H^{\frac{n}{2}}(X, \mathcal{R}_0)) \tag{3.54}$$

COROLLARY 3.35. *In the same hypothesis of proposition 3.33 suppose that one of the two following properties is satisfied*

1. $i_j^* : I^qH^j(X, \mathcal{R}_0) \longrightarrow I^pH^j(X, \mathcal{R}_0)$ *is injective,*
2. $i_j^* : I^qH^j(X, \mathcal{R}_0) \longrightarrow I^pH^j(X, \mathcal{R}_0)$ *is surjective.*

Then

$$I^qH^j(X,\mathcal{R}_0),\ I^pH^j(X,\mathcal{R}_0)\ j=0,...,n \tag{3.55}$$

are a finite sequences of finite dimensional vector spaces with Poincaré duality.

COROLLARY 3.36. *In the hypothesis of proposition 3.33 we have the following inequalities:*

$$dim(\mathrm{im}(H^j_c(reg(X)) \stackrel{i^*}{\longrightarrow} H^j_{dR}(reg(X)))) \leq dim I^pH^j(X,\mathcal{R}_0) \tag{3.56}$$

$$dim(\mathrm{im}(H^j_c(reg(X)) \stackrel{i^*}{\longrightarrow} H^j_{dR}(reg(X)))) \leq dim I^qH^j(X,\mathcal{R}_0). \tag{3.57}$$

Moreover if on $reg(X)$ we have that $\mathrm{im}(H^j_c(reg(X)) \stackrel{i^*_j}{\longrightarrow} H^j_{dR}(reg(X)))$ is not trivial for some j then on X $I^pH^j(X,\mathcal{R}_0)$ and $I^qH^j(X,\mathcal{R}_0)$ are always non trivial for each general perversity p such that $p \leq \underline{m}$ or $p \geq \overline{m}$. Finally, if on $reg(X)$ we have that $H^i_c(reg(X)) \to H^i_{dR}(reg(X))$ is injective, then we can improve the inequalities (3.56) *and* (3.57) *in the following way:*

$$dim(H^j_c(reg(X))) \leq dim I^pH^j(X,\mathcal{R}_0) \tag{3.58}$$

$$dim(H^j_c(reg(X))) \leq dim I^qH^j(X,\mathcal{R}_0) \tag{3.59}$$

$$b_{n-j}(reg(X)) \leq dim I^pH^{n-j}(X,\mathcal{R}_0) \tag{3.60}$$

$$b_{n-j}(reg(X)) \leq dim I^qH^{n-j}(X,\mathcal{R}_0) \tag{3.61}$$

PROOF. All the previous inequalities from (3.56) to (3.59) are immediate consequences of the previous results. For the last two inequalities we observe that by Poincaré duality, we know that $dim(H^j_c(reg(X))) = dim(H^{n-j}_{dR}(reg(X))) = b_{n-j}(reg(X))$. Moreover, from theorem 3.14, we know that

$$H^j_{2,m\to M}(reg(X),g) \cong H^{n-j}_{2,m\to M}(reg(X),g).$$

Therefore using corollary 3.24 we get

$$b_{n-j}(reg(X)) \leq dim(H^{n-j}_{2,m\to M}(reg(X),g)) \leq dim(H^{n-j}_{2,max}(reg(X),g)) =$$

$$= dim(I^qH^{n-j}(X,\mathcal{R}_0)),$$

$$b_{n-j}(reg(X)) \leq dim(H^{n-j}_{2,m\to M}(reg(X),g)) \leq dim(H^{n-j}_{2,min}(reg(X),g)) =$$

$$= dim(I^pH^{n-j}(X,\mathcal{R}_0))$$

and so the statement follows. □

Gluing together some of the previous results, now we can state the main result of this section. The first part is a **Hodge theorem** for $\mathrm{im}(I^{q_g}H^i(X,\mathcal{R}_0) \to I^{p_g}H^i(X,\mathcal{R}_0))$, that is we will show the existence of a self-adjoint extension of $\Delta_i : \Omega^i_c(reg(X)) \to \Omega^i_c(reg(X))$ having the nullspace isomorphic to $\mathrm{im}(I^{q_g}H^i(X,\mathcal{R}_0) \to I^{p_g}H^i(X,\mathcal{R}_0))$. In the second part we will show that $(d+\delta)_{ev}$, that is the Gauss-Bonnet operator having as domain the space of the smooth forms of even degree with compact support, admits a Fredholm extension such that its index has a topological meaning.

THEOREM 3.37. *In the same hypothesis or theorem 2.12; Let* $\Delta_{\mathfrak{m},i}$ *and* $(d_{\mathfrak{m}} + d_{\mathfrak{m}}^*)_{ev}$ *be the operators, as defined respectively in corollary 3.16 and proposition 3.19, associated to the riemannian manifold* $(reg(X), g)$*. Then we have the following results:*

$$Ker(\Delta_{\mathfrak{m},i}) \cong \mathrm{im}(I^{q_g}H^i(X, \mathcal{R}_0) \to I^{p_g}H^i(X, \mathcal{R}_0)) \tag{3.62}$$

$$\mathrm{ind}((d_{\mathfrak{m}} + d_{\mathfrak{m}}^*)_{ev}) = I^{p_g \to q_g}\chi(X, \mathcal{R}_0). \tag{3.63}$$

PROOF. (3.62) follows by theorem 2.12 and corollary 3.16; analogously (3.63) follows from theorem 2.12 and from proposition 3.19. ☐

Now suppose that $dimX = 4n$ where X is as in proposition 3.33. Let g be a quasi edge metric with weights on $reg(X)$. Then, by theorem 2.12, it follows that $(L^2\Omega^i(Reg(X), g), d_{max/min,i})$ are Fredholm complexes and so $(Reg(X), g)$ admits the L^2 signature $\sigma_2(reg(X), g)$ as defined in definition 3.30. Moreover, using again theorem 2.12, it follows that in this case the L^2 signature $\sigma_2(reg(X), g)$ is just the analytic version of the **perverse signature** introduced by Hunsicker in [**44**] in the case of $depth(X) = 1$ and reintroduced in a purely topological way and generalized to any compact topological pseudomanifolds by Friedman and Hunsicker in [**34**]. In other words, if p_g is the general perversity of definition 1.34 and q_g it is its dual, then

$$\sigma_2(reg(X), g) = \sigma_{q_g \to p_g}(X) \tag{3.64}$$

and we provided an analytic way to construct $\sigma_{q_g \to p_g}(X)$ when X is a smoothly stratified pseudomanifold with a Thom-Mather stratification wich generalize the construction given by Hunsicker in [**44**] in the particular case of $depth(X) = 1$. (For the definition of $\sigma_{q_g \to p_g}(X)$ see [**34**] pag. 15).
We have the following corollaries:

COROLLARY 3.38. *Let* X *be as in theorem 2.12 and let* g *and* h *two quasi edge metrics with weights on* $reg(X)$*. If* $p_g = p_h$ *then*

$$\sigma_2(reg(X), g) = \sigma_2(reg(X), h).$$

PROOF. It follows immediately from theorem 2.12. ☐

COROLLARY 3.39. *Let* X *and* X' *be as in theorem 2.12. Let* g *and* h *two a quasi edge metric with weights respectively on* $reg(X)$ *and* $reg(X')$*. Let* $f : X \to X'$ *be a stratum preserving homotopy equivalence which preserves also the orientations of* X *and* X'*, see* [**47**] *pag 62 for the definition. Suppose that both* p_g *and* p_h *depend only on the codimension of the strata and that* $p_g = p_h$*. Then*

$$\sigma_2(reg(X), g) = \sigma_2(reg(X'), h).$$

PROOF. As remarked above, by theorem 2.12, it follows that $\sigma_2(reg(X), g)$ is the perverse signature of Friedman and Hunsicker associated to the general perversities p_g and $t - p_g$. Analogously $\sigma_2(reg(X'), h)$ is the perverse signature of Friedman and Hunsicker associated to the general perversities

p_h and $t-p_h$. So the statement follows by the invariance of the perverse signature under the action of stratum preserving homotopy equivalences which preserve also the orientations. $\square$

6. Manifolds without finite L^2 cohomology groups

In this section we exhibit some example of manifolds which satisfy the assumptions of corollary 3.27 and that therefore they do not admit any riemannian metric with finite L^2 cohomology groups (reduced ore not.) Finally in the last part we show some other applications to stratified pseudomanifolds. We start with the following definition:

DEFINITION 3.40. Let M be a smooth manifold and let $A \subset M$.

(1) We will say that A is bounded if its closure, $\overline{A}$, is compact.
(2) We will say that M has only one end if for each compact subset $K \subset M$ $M-K$ has only one unbounded connected component.
(3) We will say that M has k ends (where $k \geq 2$) if there is a compact set $K_0 \subset M$ such that for every compact set $K \subset M$ containing K_0, $M-K$ has exactly k unbounded connected components.

The following proposition is a modified version of lemma 2.3 in [**22**]:

PROPOSITION 3.41. *Let M be a manifold with only one end. Then the natural map*

$$H^1_c(M) \to H^1_{dR}(M)$$

is injective.

PROOF. Let $\alpha \in \Omega^1_c(M)$ closed and let $f : M \to \mathbb{R}$ be a smooth function such that $df = \alpha$. This implies the existance of a costant c such that $f|_{M-supp(\alpha)} = c$. Therefore, by the fact that M has only one end, it follows that $f-c$ has compact support. $\square$

Now using Poincaré duality for open and oriented manifolds we know that the de Rham cohomology with compact support is infinite dimensional if and only if the de Rham cohomology is infinite dimensional. From this it follows that if M is a smooth and oriented manifold with only one end and such that $H^1_{dR}(M)$ is infinite dimensional then also $\operatorname{im}(H^i_c(M) \to H^i_{dR}(M))$ is infinite dimensional. So we can state the following proposition:

PROPOSITION 3.42. *Let M be an open and oriented surface with infinite genus and with only one end. Then $\operatorname{im}(H^i_c(M) \to H^i_{dR}(M))$ is infinite dimensional and therefore M does not admit a riemannian metric g (complete or incomplete) such that g implies one of the properties listed in the corollary 3.27.*

The rest of this subsection is devoted to show another example of an open manifold which satisfies corollary 3.27 but that it is not contemplate in the previous proposition. To do this we state the following lemma which gives another sufficient condition to have $\operatorname{im}(H^i_c \to H_{dR}(M))$ infinite dimensional for some i.

LEMMA 3.43. *Let M be an open and oriented smooth manifold of dimension n. Let $\{A_j\}_{j\in J}$ a sequence of open subset such that*

(1) $\partial\overline{A_j}$ *is smooth for each* j.
(2) $\lim_{j\to\infty} dim(\mathrm{im}(H^i_c(A_j) \to H^i_{dR}(A_j))) = \infty$.

Then for the same i $\mathrm{im}(H^i_c(M) \to H^i_{dR}(M))$ *is infinite dimensional.*

PROOF. It is an immediate consequence of the next proposition. $\square$

PROPOSITION 3.44. *Let* M *be an open and oriented smooth manifold of dimension* n. *Let* $A \subset M$ *an open subset with smooth boundary. Then there a natural injective maps*

$$\mathrm{im}(H^{n-i}_c(A) \to H^{n-i}_{dR}(A)) \longrightarrow (\mathrm{im}(H^i_c(M) \to H^i_{dR}(M))^*$$

PROOF. Consider the following pairing:

$$\mathrm{im}(H^i_c(M) \to H^i_{dR}(M)) \times \mathrm{im}(H^{n-i}_c(A) \to H^{n-i}_{dR}(A)) \longrightarrow \mathbb{R}, \qquad (3.65)$$

$$([\eta],[\omega]) \mapsto \int_M \eta \wedge \omega$$

where ω is a $i-$form closed with compact support in M and η is a closed $(n-i)-$form with compact support in A. As observed at the end of subsection 2.3 this pairing makes sense because a cohomology class in $\mathrm{im}(H^i_c(M) \to H^i_{dR}(M))$, or in $\mathrm{im}(H^i_c(A) \to H^i_{dR}(A))$, is just a cohomology class in $H^i_{dR}(M)$, or in $H^i_{dR}(A)$, such that it admits a representative with compact support respectively in M or A. Moreover from Poincaré duality for open and oriented manifold it follows immediately that this pairing is well defined. Now let $[\omega] \in \mathrm{im}(H^{n-i}_c(A) \to H^{n-i}_{dR}(A))$ such that for each class $[\eta] \in \mathrm{im}(H^i_c(M) \to H^i_{dR}(M))$ the pairing (3.65) is zero. This implies that for each smooth and closed i forms ϕ with compact support in M we have

$$\int_M \phi \wedge \omega = 0.$$

In particular this is true for each smooth and closed i forms ϕ with compact support in A and therefore, using again the Poincaré duality for open and oriented manifold, we get that there exists $\beta \in \Omega^{n-i-1}(A)$ such that $d\beta = \omega$. So we can conclude that $[\omega] = 0$ in $\mathrm{im}(H^i_c(A) \to H^i_{dR}(A))$ and this implies the statement. $\square$

Using the previous lemma we have the following corollary that was suggested to the author by Pierre Albin:

COROLLARY 3.45. *Let* M *be an open and oriented surface obtained gluing an infinite but countable family of tori. Suppose that* M *has a finite number of ends. Then* $\mathrm{im}(H^i_c(M) \to H^1_{dR}(M))$ *is infinite dimensional and therefore* M *does not admit a riemannian metric* g *such that* g *implies one of the properties listed in the corollary 3.27.*

PROOF. The idea is to show that this is a situation in which the previous lemma applies. By the assumptions for each $j \in \mathbb{N}$ big enough, we can find an open subset A_j with the following properties:

(1) $M - A_j$ is disconnected, made of k unbounded components, where k is the number of ends of M.
(2) ∂A_j is smooth, and made of k compact connected components.

(3) By the compactness of ∂A_j it follows that each of its connected components is a compact smooth one dimensional manifold and therefore it is diffeomorphic to S^1. So we can glue to A_j k copies of $\overline{\mathbb{B}}$, the unit ball in $\mathbb{R}^2$ with boundary, to get a closed and oriented surfaces Σ_j of genus j.

Now, recalling that $2-2j = \chi(\Sigma_j) = b_0(\Sigma_j) - b_1(\Sigma_j) + b_2(\Sigma_j)$ and using the Mayer-Vietoris sequence, it is not hard to see that $dim(H^1(A_j)) \geq 2j - k$ where k is the number of ends of M and therefore it is fixed. By the assumptions this implies that on M we can find a sequence of open subsets A_j such that

$$\lim_{j\to\infty} dim(H^1_{dR}(A_j)) = \infty. \tag{3.66}$$

Now recall the fact that, on a compact and oriented manifold with boundary $\overline{M}$, we have $H^i(\overline{M}, \partial\overline{M}) \cong H^i_c(M)$ where M is the interior of $\overline{M}$. So, from the long exact sequence for the relative de Rham cohomology on a compact manifold with boundary, it is easy to show that $dim(H^1(A_j)) = dim(\text{im}(H^1_c(A_j) \to H^1_{dR}(A_j))) + \lambda_{A_j}$ where $\lambda_{A_j} \in \{0, ..., k\}$. This means that the correction term λ_{A_j} could depends from A_j but in any case it lies in $\{0, ..., k\}$ which is a bounded set being k fixed. Therefore, from this equality and from (3.66), it follows that if we take a sequence of open subsets $\{A_j\}$ such that each A_j satisfies the properties listed above then

$$\lim_{j\to\infty} dim(\text{im}(H^1_c(A_j) \to H^1_{dR}(A_j))) = \infty.$$

This implies that we can apply lemma 3.43 and therefore the statement follows. $\square$

Finally, using the notions introduced in definition 3.40 and proposition 3.41, we conclude the section giving another application to the stratified pseudomanifolds and intersection cohomology.

Proposition 3.46. *Let X be as in theorem 2.12. Suppose that X is normal, that is for each $p \in sing(X)$ there exists an open neighbourhood U such that $U - (U \cap sing(X))$ is connected. Then, if $sing(X)$ is connected, $reg(X)$ is an open manifold with only one end.*

Proof. Let $K \subset reg(X)$ a compact subset. If $reg(X) - K$ is connected then we have nothing to show. Suppose therefore that it is disconnected and let $A_1, ..., A_l$ the connected components. By the fact that X is normal it follows that there exists an open neighbourhood $sing(X) \subset V \subset X$ such that $V - sing(X)$ is connected. By the fact that $K \subset reg(X)$ it follows that $V = \cup_{i=1}^l (\overline{A}_i \cap V)$ and from this equality it follows that $V - sing(X) = \cup_{i=1}^l (A_i \cap (V - sing(X))$. Every subset $A_i \cap (V - sing(X))$ is an open subset of $V - sing(X)$ and for each $i, j \in \{1, ..., l\}$ we have $(A_i \cap (V - sing(X))) \cap (A_j \cap (V - sing(X))) = \emptyset$. So the fact that $V - sing(X)$ is connected, joined with the fact that $V - sing(X) = \cup_{i=1}^l (A_i \cap (V - sing(X))$, implies that there exists just one index in $\{1, ..., l\}$, which we label γ, such that $A_\gamma \cap (V - sing(X)) \neq \emptyset$. So we can conclude that

(1) $V - sing(X) \subset A_\gamma$.
(2) $A_\gamma \cup sing(X)$ is open in X.

This implies that if we label $\mathfrak{K}$ the closure in X of

$$(\bigcup_{i=1, i\neq\gamma}^{l} A_i) \cup K$$

then we have

$$\mathfrak{K} \subseteq X - (A_\gamma \cup sing(X)\} \tag{3.67}$$

and therefore $\mathfrak{K}$ is a compact subset of X. But from (3.67) it follows that $\mathfrak{K} \subset reg(X)$ and therefore it is a compact subset of $reg(X)$. This allow us to conclude that for each $i \in \{1, ..., l\}$, $i \neq \gamma$ we have that $\overline{A_i}$ is a compact subset of $reg(X)$ and so we got the statement. □

We have the following corollary:

Corollary 3.47. *Let X be as in theorem 2.12 such that X is normal and $sing(X)$ is connected. Let p be a general perversity as in the statement of theorem 2.13 and let q be its dual. Then we have the following inequalities:*

1. $dim(H^1_c(reg(X))) \leq I^p b_1(X, \mathcal{R}_0)$, $dim(H^1_c(reg(X))) \leq I^q b_1(X, \mathcal{R}_0)$
2. $b_{n-1}(reg(X)) \leq I^p b_{n-1}(X, \mathcal{R}_0)$, $b_{n-1}(reg(X)) \leq I^q b_{n-1}(X, \mathcal{R}_0)$

where $I^p b_i(X, \mathcal{R}_0)$ is the dimension of $I^p H^i(X, \mathcal{R}_0)$ and $I^q b_i(X, \mathcal{R}_0)$ is the dimension of $I^q H^i(X, \mathcal{R}_0)$. Finally if $dimX = 2$ and $cod(sing(X)) = 0$ then

$$I^{\underline{m}}\chi(X) \leq \chi(reg(X)) \tag{3.68}$$

where $I^{\underline{m}}\chi(X) = \sum_{i=0}^{2}(-1)^i I^{\underline{m}} b_i(X)$.

Proof. From proposition 3.46 we know that $reg(X)$ has only one end. Therefore from proposition 3.40 it follows that the maps $H^1_c(M) \rightarrow H^1_{dR}(M)$ is injective and so the thesis follows by corollary 3.36. Before to prove the second part of the corollary we do the following observation: by the assumption it follows that $H^1_c(reg(X))$ is finite dimensional; using Poincaré duality for open and oriented manifolds this implies that $b_i(reg(X))$ is finite dimensional for each $i = 0, ..., 2$ and therefore $\chi(reg(X))$ makes sense. Now by the assumptions on X it follows that $sing(X) = \{p\}$ and X is a Witt space (For the definition of Witt space see for example [**47**] pag 75). It is well known that, over a Witt space, the intersection cohomology associated to the lower middle perversity satisfies has the Poincaré duality, that is we have $I^{\underline{m}}H^i(X) \cong I^{\underline{m}}H^{2-i}(X)$. Poincaré duality for open and oriented manifolds implies that $b_2(reg(X)) = dim(H^0_c(reg(X))) = 0$. So, using the previous statements of this corollary, we have $I^{\underline{m}}\chi(X) = -I^{\underline{m}}b_1(X) \leq -b_1(reg(X)) \leq 1 - b_1(reg(X)) = \chi(reg(X))$. □

7. Some application to the Friedrichs extension

This last section is devote to show some properties of the Friedrichs extension of $\Delta_i : \Omega^i_c(M) \rightarrow \Omega^i_c(M)$.
The main result is to show that if (M, g) is an open and oriented riemannian manifold such that $(L^2\Omega^*(M, g), d_{max/min,*})$ are Fredholm complexes then, for each $i = 0, ..., dimM$, the Friedrichs extension of $\Delta_i : \Omega^i_c(M) \rightarrow \Omega^i_c(M)$ is a Fredholm operator. In particular this applies when M is the regular part of a compact and smoothly stratified pseudomanifold with a Thom-Mather

stratification and g is a quasi edge metric with weights on $reg(X)$. We start recalling the definition of the Friedrichs extension:

DEFINITION 3.48. Let H be an Hilbert space and $B : H \to H$ a densely defined operator. Suppose that B is positive, that is for each $u \in \mathcal{D}(B)$ we have $< Bu, u >\geq 0$. The Friedrichs extension of B, usually labeled $B^{\mathcal{F}}$, is the operator defined in the following way:

$$\mathcal{D}(B^{\mathcal{F}}) = \{u \in \mathcal{D}(B^*) : \text{ there exists } \{u_n\} \subset \mathcal{D}(B) \text{ such that}$$
$$< u - u_n, u - u_n > \to 0 \text{ and}$$
$$< B(u_n - u_m), u_n - u_m > \to 0 \ for\ n, m \to \infty\};\ \text{we put } B^{\mathcal{F}}(u) = B^*(u).$$

PROPOSITION 3.49. *In the same assumptions of the previous definition $B^{\mathcal{F}}$ is a positive self-adjoint extension of B.*

PROOF. See [**53**] appendix C. □

LEMMA 3.50. *Let $A_j : H_j \to H_j$, $j = 1, 2$, be two positive and densely defined operators. Then on $H_1 \oplus H_2$, with the natural Hilbert space structure of a direct sum, we have*

$$(A_1 \oplus A_2)^{\mathcal{F}} = A_1^{\mathcal{F}} \oplus A_2^{\mathcal{F}}.$$

PROOF. It follows from the assumptions of the lemma that $A_1 \oplus A_1 : H_1 \oplus H_2 \to H_1 \oplus H_2$ is densely defined and positive. Moreover it clear that $(A_1 \oplus A_2)^* = A_1^* \oplus A_2^*$.
Now let $(a, b) \in \mathcal{D}((A_1 \oplus A_2)^{\mathcal{F}})$. From definition 3.48 it follows that $(a, b) \in \mathcal{D}((A_1 \oplus A_2)^*)$ and there exists a sequence $\{(a_n, b_n)\} \subset \mathcal{D}(A_1 \oplus A_2)$ such that:

$$(a_n, b_n) \to (a, b) \text{ and } < A\oplus B((a_n, b_n)-(a_m, b_m)), (a_n, b_n)-(a_m, b_m) > \to 0.$$

Furthermore from the same definition we know that $(A_1 \oplus A_2)^{\mathcal{F}}(a, b) = (A_1 \oplus A_2)^*(a, b)$. But from these requirements it follows immediately that $a \in \mathcal{D}(A_1^*)$, $b \in \mathcal{D}(A_2^*)$, $\{a_n\} \subset \mathcal{D}(A_1)$, $\{b_n\} \subset \mathcal{D}(A_2)$, $a_n \to a$, $< A_1(a_n - a_m), a_n - a_m > \to 0$ and analogously that $b_n \to b$ and that $< A_2(b_n - b_m), b_n - b_m > \to 0$. So it follows that $a \in \mathcal{D}(A_1^{\mathcal{F}})$, $b \in \mathcal{D}(A_2^{\mathcal{F}})$ and $(A_1 \oplus A_2)^{\mathcal{F}}(a, b) = A_1^{\mathcal{F}}(a) \oplus A_2^{\mathcal{F}}(b)$. In this way we know that $A_1^{\mathcal{F}} \oplus A_2^{\mathcal{F}}$ is an extension of $(A_1 \oplus A_2)^{\mathcal{F}}$. Moreover it is clear that also $A_1^{\mathcal{F}} \oplus A_2^{\mathcal{F}}$ it is a self-adjoint operator because it is a direct sum of two self-adjoint operators acting on H_1 and H_2 respectively. Finally, by the fact that both $A_1^{\mathcal{F}} \oplus A_2^{\mathcal{F}}$ and $(A_1 \oplus A_2)^{\mathcal{F}}$ are self-adjoint operators, it follows that $A_1^{\mathcal{F}} \oplus A_2^{\mathcal{F}} = (A_1 \oplus A_2)^{\mathcal{F}}$. □

REMARK 3.5. It clear that the previous proposition generalizes to the case of a finite sum, that is if we have $A_j : H_j \to H_j$ $j = 1, ..., n$ such that for each j A_j is positive and densely defined then:

$$(A_1 \oplus ... \oplus A_n)^{\mathcal{F}} : \bigoplus_{j=1}^{n} H_j \to \bigoplus_{j=1}^{n} H_j = A_1^{\mathcal{F}} \oplus ... \oplus A_n^{\mathcal{F}} : \bigoplus_{j=1}^{n} H_j \to \bigoplus_{j=1}^{n} H_j \quad (3.69)$$

LEMMA 3.51. *Let E, F be two vector bundles over an open, incomplete and oriented riemannian manifold (M, g). Let g and h be two metrics on E and F respectively. Let $d : C_c^\infty(M, E) \to C_c^\infty(M, F)$ an unbounded an* *densely defined differential operator. Let $d^t : C^\infty_c(M,F) \to C^\infty_c(M,E)$ its formal adjoint. Then for $d^t \circ d : L^2(M,E) \to L^2(M,E)$ we have:*

$$(d^t \circ d)^{\mathcal{F}} = d_{max} \circ d_{min}.$$

PROOF. See [**17**], lemma 3.1 pag. 447. □

From lemma 3.51 we get, as it is showed in [**17**] pag. 448, the following useful corollary:

COROLLARY 3.52. *Let (M,g) be an open, oriented and incomplete riemannian manifold of dimension n. Consider the Laplacian acting on the space of smooth forms with compact support:*

$$\Delta : \bigoplus_{i=0}^{n} \Omega^i_c(M) \longrightarrow \bigoplus_{i=0}^{n} \Omega^i_c(M).$$

Then for

$$\Delta^{\mathcal{F}} : \bigoplus_{i=0}^{n} L^2\Omega^i(M,g) \longrightarrow \bigoplus_{i=0}^{n} L^2\Omega^i(M,g)$$

we have

$$\Delta^{\mathcal{F}} = (d+\delta)_{max} \circ (d+\delta)_{min}.$$

Now we are in positions to state the following result:

THEOREM 3.53. *Let (M,g) be an open, oriented and incomplete riemannian manifold of dimension n. Then for each $i = 0, ..., n$ we have the following properties:*

1. *$Ker(\Delta^{\mathcal{F}}_i) = \mathcal{H}^i_{min}(M,g) = Ker(\Delta_{min,i})$, $\overline{ran(\Delta^{\mathcal{F}}_i)} = \overline{ran(\Delta_{max,i})}$.*
2. *If $\overline{H}^i_{2,m\to M}(M,g)$ is finite dimensional then $Ker(\Delta^{\mathcal{F}}_i)$ is finite dimensional.*
3. *If $(L^2\Omega^*(M,g), d_{max,*})$ is a Fredholm complex, or equivalently if $(L^2\Omega^*(M,g), d_{min,*})$ is a Fredholm complex, then for each i $\Delta^{\mathcal{F}}_i$ is a Fredholm operator on its domain endowed with graph norm and $ran(\Delta^{\mathcal{F}}_i) = ran(\Delta_{max,i})$.*

PROOF. In the first point the equality $\mathcal{H}^i_{min}(M,g) = Ker(\Delta_{min,i})$ is showed in [**6**] prop. 5. For the other equality, from lemma 3.50 and corollary 3.52, we know that

$$(d+\delta)_{max} \circ (d+\delta)_{min} = \Delta^{\mathcal{F}} = \oplus_{i=0}^n \Delta^{\mathcal{F}}_i$$

and therefore

$$Ker((d+\delta)_{max} \circ (d+\delta)_{min}) = Ker(\oplus_{i=0}^n \Delta^{\mathcal{F}}_i) = \oplus_{i=0}^n Ker(\Delta^{\mathcal{F}}_i).$$

But for $Ker((d+\delta)_{max} \circ (d+\delta)_{min})$ we have:

$$Ker((d+\delta)_{max} \circ (d+\delta)_{min}) = Ker(d+\delta)_{min}) = Ker(d_{min}) \cap Ker(\delta_{min}) =$$

$$= \bigoplus_{i=0}^{n} Ker(d_{min,i}) \cap Ker(\delta_{min,i-1}).$$

The first equality follows by the fact that for each

$\eta \in \mathcal{D}((d+\delta)_{max} \circ (d+\delta)_{min})$ $< ((d+\delta)_{max} \circ (d+\delta)_{min})(\eta), \eta >_{L^2\Omega(M,g)}=$

$$=< (d+\delta)_{min})(\eta),(d+\delta)_{min})(\eta) >_{L^2\Omega(M,g)} .$$

For the second equality it is clear that $Ker(d+\delta)_{min}) \subseteq Ker(d_{min}) \cap Ker(\delta_{min})$. But $Ker(d_{min}) \cap Ker(\delta_{min}) = (\overline{ran(d_{max}) + ran(\delta_{max})})^{\perp}$ and $Ker(d+\delta)_{min}) = (\overline{ran((d+\delta)_{max})})^{\perp}$. By the fact that $\overline{(ran((d+\delta)_{max}))} \subseteq \overline{(ran(d_{max}) + ran(\delta_{max}))}$ it follows that $Ker(d_{min}) \cap Ker(\delta_{min}) \subseteq Ker((d+\delta)_{min})$ and so we have obtained the second equality. The last equality follows because $\overline{(ran(d_{max}) + ran(\delta_{max}))} = \oplus_{i=0}^n \overline{(ran(d_{max,i-1}) + ran(\delta_{max,i}))}$ that is

$$(Ker(d_{min}) \cap Ker(\delta_{min}))^{\perp} = \bigoplus_{i=0}^{n} (Ker(d_{min,i}) \cap Ker(\delta_{min,i-1}))^{\perp}$$

and both $Ker(d_{min}) \cap Ker(\delta_{min})$ and $\oplus_{i=0}^n Ker(d_{min,i}) \cap Ker(\delta_{min,i-1})$ are closed.

In this way can conclude that

$$\bigoplus_{i=0}^{n} Ker(\Delta_i^{\mathcal{F}}) = \bigoplus_{i=0}^{n} \mathcal{H}^i_{min}(M,g)$$

and therefore that

$$\mathcal{H}^i_{min}(M,g) = Ker(\Delta_i^{\mathcal{F}}).$$

Finally, using the fact that $\Delta_i^{\mathcal{F}}$ is self-adjoint and that $\Delta_{min,i} = (\Delta_{max,i})^*$ it follows that

$$\overline{ran(\Delta_i^{\mathcal{F}})} = \overline{ran(\Delta_{max,i})}.$$

For the second point, if we call $\pi_{abs,i} : L^2\Omega^i(M,g) \to \mathcal{H}^i_{abs}(M,g)$ the projection on $\mathcal{H}^i_{abs}(M,g)$, we know that $\pi_{abs,i}(\mathcal{H}^i_{rel}) \cong \overline{H}^i_{2,m\to M}(M,g)$. This property is showed in a more general context in the proof of theorem 3.5 and remarked in remark 3.1. But $\mathcal{H}^i_{min}(M,g) = Ker(d_{min,i}) \cap Ker(\delta_{min,i-1}) = \mathcal{H}^i_{abs}(M,g) \cap \mathcal{H}^i_{rel}(M,g)$. So $\mathcal{H}^i_{min}(M,g) \subseteq \pi_{abs,i}(\mathcal{H}^i_{rel}(M,g))$ and therefore the second statement follows.

Now consider the third point; we want to show that if $(L^2\Omega^*(M,g), d_{max,*})$ is a Fredholm complex then also $(d+\delta)_{max} \circ (d+\delta)_{min} : \oplus_{i=0}^n L^2\Omega^i(M,g) \to \oplus_{i=0}^n L^2\Omega^i(M,g)$ is a Fredholm operator. By the previous point, we already know that the nullspace of $(d+\delta)_{max} \circ (d+\delta)_{min}$ is finite dimensional. So we have to show that its range it is closed with finite dimensional orthogonal complement. To do this is equivalent to show that the cokernel of $(d+\delta)_{max} \circ (d+\delta)_{min}$ is finite dimensional. We will do this showing that $ran((d+\delta)_{max} \circ (d+\delta)_{min}) = ran((d+\delta)_{max})$ and that $(d+\delta)_{max}$ has finite dimensional cokernel. To do this we observe that, by the fact that $(d+\delta)^*_{min} = (d+\delta)_{max}$, it follows that

$$ran((d+\delta)_{max}) = \tag{3.70}$$

$$= \{(d+\delta)_{max}(u) : \ u \in \overline{ran((d+\delta)_{min})} \cap \mathcal{D}((d+\delta)_{max})\}.$$

Now, as we showed in corollary 6 of [**6**], if $(L^2\Omega^*(M,g), d_{max,*})$ is a Fredholm complex then $d_{max} + \delta_{min}$ is a Fredholm operator. But the fact that $ran(d_{max} + \delta_{min}) \subset ran((d+\delta)_{max})$ implies that there is a surjective map

$$\frac{(\oplus_{i=0}^n L^2\Omega^i(M,g))}{ran((d+\delta)_{max})} \longrightarrow \frac{(\oplus_{i=0}^n L^2\Omega^i(M,g))}{ran(d_{max} + \delta_{min})}.$$

So $(d+\delta)_{max}$ on its domain with the graph norm is a bounded linear operator with finite dimensional cokernel and this implies that the range of $(d+\delta)_{max}$ is closed with finite dimensional orthogonal complement. But $((d+\delta)_{max})^* = (d+\delta)_{min}$ and therefore also $(d+\delta)_{min}$ has closed range. In this way (3.70) becomes:

$$ran((d+\delta)_{max}) = \{(d+\delta)_{max}(u) : \ u \in ran((d+\delta)_{min}) \cap \mathcal{D}((d+\delta)_{max})\}.$$

So we can conclude that $ran((d+\delta)_{max} \circ (d+\delta)_{min}) = ran((d+\delta)_{max})$ and therefore $(d+\delta)_{max} \circ (d+\delta)_{min}$ is a Fredholm operator.
Now, by the equality $(d+\delta)_{max} \circ (d+\delta)_{min} = \oplus_{i=0}^{n} \Delta_i^{\mathcal{F}}$, we get, for each $i = 0, ..., n$, that also $\Delta_i^{\mathcal{F}}$ has closed range. Moreover we already know that its nullspace of $\Delta_i^{\mathcal{F}}$ is finite dimensional and so, because it is self-adjoint and with closed range, we can conclude that it is Fredholm. Finally, as we showed in [**6**] corollary 6, we know that $\Delta_{max,i}$ has finite dimensional cokernel and so we can conclude that $ran(\Delta_{max,i}) = ran(\Delta_i^{\mathcal{F}})$. $\square$

As mentioned at the beginning of the section the following corollary is an application of the previous theorem; it already known when X is a compact manifold with isolated singularities for any positive conic operator (see [**51**]) and also for $\Delta_i^{\mathcal{F}}$ when (M, g) is a manifold with incomplete edges, see [**54**].

Corollary 3.54. *Let X be a compact smoothly and oriented stratified pseudomanifold of dimension n with a Thom Mather stratification. Let g be a quasi-edge metric with weights on $reg(X)$. Then on $L^2\Omega^i(reg(X), g)$, for each $i = 0, ..., n$, $\Delta_i^{\mathcal{F}}$ is a Fredholm operator; moreover $ran(\Delta_i^{\mathcal{F}}) = ran(\Delta_{max,i})$ and $Ker(\Delta_i^{\mathcal{F}}) = Ker(\Delta_{min,i}) = \mathcal{H}^i_{min}(M, g)$.*

8. Additional remarks

Consider again an open, oriented and incomplete riemannian manifold (M, g) of dimension n. By corollary 3.24 we now that that there is a copy of $\operatorname{im}(\overline{H}^i_{2,min}(M, g) \rightarrow \overline{H}^i_{2,max}(M, g))$ in each $i-th$ reduced cohomology group $\overline{H}^i_{2,D_*}(M, g)$ of each closed extension $(L^2\Omega^*(M, g), D_*)$ of $(\Omega_c^*(M), d_*)$. In the same way, using again corollary 3.24, we know that there is a copy of $\operatorname{im}(H^i_{2,min}(M, g) \rightarrow H^i_{2,max}(M, g))$ in each $i-th$ cohomology group $H^i_{2,D_*}(M, g)$ of each closed extension $(L^2\Omega^*(M, g), D_*)$ of $(\Omega_c^*(M), d_*)$. In particular, by theorem 3.15, we know that when $d_{min,i}$ has closed range for each i then the groups $\operatorname{im}(H^i_{2,min}(M, g) \rightarrow H^i_{2,max}(M, g))$ are really the cohomology groups of an Hilbert complex that we labeled $(L^2\Omega^i(M, g), d_{\mathfrak{m},i})$. Therefore we can look at $\operatorname{im}(H^i_{2,min}(M, g) \rightarrow H^i_{2,max}(M, g))$ as the smallest possible L^2-cohomology groups for (M, g).
From the Hodge point of view the smallest Hodge cohomology groups are $\mathcal{H}^i_{min}(M, g)$ defined, for each $i = 0, ..., n$, as $Ker(d_{min,i}) \cap Ker(\delta_{min,i-1})$ or equivalently, see proposition 1.9, as the nullspace of $\Delta_{min,i}$, where $\Delta_{min,i}$ is the minimal closed extension of $\Delta_i : \Omega_c^i(M) \rightarrow \Omega_c^i(M)$. Therefore a natural question is:

- Is there any relations between

$$\mathcal{H}^i_{min}(M, g) \text{ and } \operatorname{im}(\overline{H}^i_{2,min}(M, g) \rightarrow \overline{H}^i_{2,max}(M, g))$$

or between

$$\mathcal{H}^i_{min}(M,g) \text{ and } \operatorname{im}(H^i_{2,min}(M,g) \to H^i_{2,max}(M,g))?$$

In [**45**] theorem 4.8, using techniques arising from Mazzeo's edge calculus, the author showed that if (M,g) is an incomplete manifold with edge then we have the following isomorphism:

$$\mathcal{H}^i_{min}(M,g) \cong \operatorname{im}(H^i_{2,min}(M,g) \to H^i_{2,max}(M,g)). \tag{3.71}$$

Therefore, using corollary 3.16 and theorem 3.53, we get the following immediate consequences:

COROLLARY 3.55. *Let (M,g) be an incomplete manifold with edge. Then, for each $i = 0,...,n$*

(1) $Ker(\Delta_{\mathfrak{m},i}) = Ker(\Delta_{min,i}) = Ker(\Delta^{\mathcal{F}}_i)$
(2) $ran(\Delta_{\mathfrak{m},i}) = ran(\Delta_{max,i}) = ran(\Delta^{\mathcal{F}}_i)$

Finally we conclude the section showing that the isomorphism (3.71) is equivalent to require that the Hilbert space $L^2\Omega^i(M,g)$ satisfies some geometric properties.

PROPOSITION 3.56. *Let (M,g) an open oriented and incomplete riemannian manifold. Suppose that, for each $i = 0,...,n$, $\operatorname{im}(\overline{H}^i_{2,min}(M,g) \to \overline{H}^i_{2,max}(M,g))$ is finite dimensional. Then there exists alway an injective map*

$$\mathcal{H}^i_{min}(M,g) \to \operatorname{im}(\overline{H}^i_{2,min}(M,g) \to \overline{H}^i_{2,max}(M,g)).$$

Moreover the following properties are equivalent:

(1) $\mathcal{H}^i_{min}(M,g) \cong \operatorname{im}(\overline{H}^i_{2,min}(M,g) \to \overline{H}^i_{2,max}(M,g))$
(2) $\mathcal{H}^i_{abs}(M,g) = \mathcal{H}^i_{min}(M,g) \oplus (\overline{ran(\delta_{max,i})} \cap \mathcal{H}^i_{abs}(M,g))$
(3) *Let $\pi_{abs/rel/min,i} : L^2\Omega^i(M,g) \to \mathcal{H}^i_{abs/rel/min}(M,g)$ be the orthogonal projections of $L^2\Omega^i(M,g)$ respectively on $\mathcal{H}^i_{abs}(M,g)$, $\mathcal{H}^i_{rel}(M,g)$ and $\mathcal{H}^i_{min}(M,g)$. Then:*
$\pi_{rel,i} \circ \pi_{abs,i} = \pi_{min,i} = \pi_{abs,i} \circ \pi_{rel,i}$.
(4) $\mathcal{H}^i_{rel}(M,g) = \mathcal{H}^i_{min}(M,g) \oplus (\overline{ran(d_{max,i})} \cap \mathcal{H}^i_{rel}(M,g))$
(5) $\overline{ran(d_{max,i})} = (\overline{ran(d_{max,i})} \cap \mathcal{H}^i_{rel}(M,g)) \oplus \overline{ran(d_{min,i})} \oplus (\overline{ran(d_{max,i})} \cap \overline{ran(\delta_{max,i})})$

Finally, if $(L^2\Omega^i(M,g), d_{max,i})$ or equivalently $(L^2\Omega^i(M,g), d_{min,i})$ is a Fredholm complex then there exists always an injective map

$$\mathcal{H}^i_{min}(M,g) \to \operatorname{im}(H^i_{2,min}(M,g) \to H^i_{2,max}(M,g)).$$

Moreover the previous four equivalent conditions become:

(1) $\mathcal{H}^i_{min}(M,g) \cong \operatorname{im}(H^i_{2,min}(M,g) \to H^i_{2,max}(M,g))$
(2) $\mathcal{H}^i_{abs}(M,g) = \mathcal{H}^i_{min}(M,g) \oplus (ran(\delta_{max,i}) \cap \mathcal{H}^i_{abs}(M,g))$
(3) *Let $\pi_{abs/rel/min,i} : L^2\Omega^i(M,g) \to \mathcal{H}^i_{abs/rel/min}(M,g)$ be the orthogonal projections of $L^2\Omega^i(M,g)$ respectively on $\mathcal{H}^i_{abs}(M,g)$, $\mathcal{H}^i_{rel}(M,g)$ and $\mathcal{H}^i_{min}(M,g)$. Then:*
$\pi_{rel,i} \circ \pi_{abs,i} = \pi_{min,i} = \pi_{abs,i} \circ \pi_{rel,i}$.
(4) $\mathcal{H}^i_{rel}(M,g) = \mathcal{H}^i_{min}(M,g) \oplus (ran(d_{max,i}) \cap \mathcal{H}^i_{rel}(M,g))$

(5) $ran(d_{max,i}) = (ran(d_{max,i}) \cap \mathcal{H}^i_{rel}(M,g)) \oplus ran(d_{min,i}) \oplus (\overline{ran(d_{max,i})} \cap \overline{ran(\delta_{max,i})})$

Proof. Clearly it is enough to prove just the first part of the proposition. The second part follows by the first part of the proposition and by the fact that if $(L^2\Omega^i(M,g), d_{max/min,i})$ is a Fredholm complex then $d_{max/min,i}$ has closed range. Let $\pi_{1,i} : \mathcal{H}^i_{rel}(M,g) \to \mathcal{H}^i_{abs}(M,g)$, $\pi_{4,i} : \mathcal{H}^i_{abs}(M,g) \to \mathcal{H}^i_{rel}(M,g)$ as defined in the proof of theorem 3.5. Moreover, by proposition 3.7, we know that $(\pi_{1,i})^* = \pi_{4,i}$ and analogously $(\pi_{1,i})^* = \pi_{4,i}$. By the proof of theorem 3.5 we know that $\pi_{1,i}(\mathcal{H}^i_{rel}(M,g)) \cong \operatorname{im}(\overline{H}^i_{2,min}(M,g) \to \overline{H}^i_{2,max}(M,g))$. Clearly, by the fact that

$$\mathcal{H}^i_{min}(M,g) = \mathcal{H}^i_{abs}(M,g) \cap \mathcal{H}^i_{rel}(M,g)$$

it follows that $\mathcal{H}^i_{min}(M,g) \subset \pi_{1,i}(\mathcal{H}^i_{rel}(M,g))$ and so we got the first assertion.

Now we pass to show that 1) $\Rightarrow$ 2). As recalled above we know that $\pi_{1,i}(\mathcal{H}^i_{rel}(M,g)) \cong \operatorname{im}(\overline{H}^i_{2,min}(M,g) \to \overline{H}^i_{2,max}(M,g))$ and that $\mathcal{H}^i_{min}(M,g) = \mathcal{H}^i_{abs}(M,g) \cap \mathcal{H}^i_{rel}(M,g)$; therefore using 1) it follows that $\mathcal{H}^i_{min}(M,g) = \pi_{1,i}(\mathcal{H}^i_{rel}(M,g))$. This implies that

$$(\mathcal{H}^i_{min}(M,g))^{\perp} \cap \mathcal{H}^i_{abs}(M,g) = (\pi_{1,i}(\mathcal{H}^i_{rel}(M,g)))^{\perp} \cap \mathcal{H}^i_{abs}(M,g) =$$
$$= Ker(\pi_{4,i}) = (\overline{ran(\delta_{max,i})} \cap \mathcal{H}^i_{abs}(M,g))$$

and this complete the proof of the first implication.

Now suppose that 2) is satisfied. Then it is immediate that $\pi_{rel,i} \circ \pi_{abs,i} = \pi_{min,i}$ and therefore it is an easy consequence that also $\pi_{abs,i} \circ \pi_{rel,i} = \pi_{min,i}$. Moreover it is still immediate that 3) $\Rightarrow$ 4) because in this case $\pi_{4,i}(\mathcal{H}^i_{abs}(M,g)) = \mathcal{H}^i_{min}(M,g)$. Now we want to show that 4) $\Rightarrow$ 5). Clearly $\mathcal{H}^i_{min}(M,g)$ is orthogonal to $\overline{ran(\delta_{max,i})}$ and to $\overline{ran(d_{max,i})}$. This implies that the range of the orthogonal projection of $\overline{ran(d_{max,i})}$ onto $\mathcal{H}^i_{rel}(M,g)$ is just the intersection $\mathcal{H}^i_{rel}(M,g) \cap \overline{ran(d_{max,i})}$. From this it follows that also the range of the orthogonal of projection of $\overline{ran(d_{max,i})}$ onto $\overline{ran(\delta_{max,i})}$ is just the intersection $\overline{ran(d_{max,i})} \cap \overline{ran(\delta_{max,i})}$ and therefore the implication 3) $\Rightarrow$ 4) is proved. Finally, if 5) holds, it is immediate to show that $\pi_{1,i}(\mathcal{H}^i_{rel}(M,g)) = \mathcal{H}^i_{min}(M,g)$ and this , using the fact that

$$\pi_{1,i}(\mathcal{H}^i_{rel}(M,g)) \cong \operatorname{im}(\overline{H}^i_{2,min}(M,g) \to \overline{H}^i_{2,max}(M,g))$$

implies 1). This complete the proof of the proposition. $\square$

Part 2

The L^2-Atiyah-Bott-Lefschetz theorem on manifolds with conical singularities.

CHAPTER 4

Background

This chapter contains the background material we need in order to define the L^2-Lefschetz numbers of a geometric endomorphism acting on an elliptic complex of differential cone operators. In the first section the notion of differential cone operator and the relative notion of ellipticity are given. In the second section the notion of elliptic complex of differential cone operators is introduced. Finally the last section contains a brief remainder on heat kernel.

As recalled during the introduction also [**59**] is devoted to the Atiyah-Bott-Lefschetz theorem on manifolds with conical singularities. Anyway there are some substantial differences between our paper and [**59**]: the notion of ellipticity used there, which is taken from [**65**], is stronger than that one used in this paper; in particular the de Rham complex is not elliptic for the definition given in [**65**]. Moreover the complexes considered in [**59**] are complexes of weighted Sobolev space while our complexes are Hilbert complexes of unbounded operator defined on some natural extensions of their core domain; finally also the techniques used are different because we use the heat kernel while in [**59**] the existence of a parametrix of an elliptic cone operator is used. Some results of this paper are also close to results proved in [**51**]: indeed in [**51**] the heat kernel is studied in an equivariant situation and an equivariant index theorem is proved (see corollary 2.4.7). Also in this case there are some relevant differences: the Lie group G acting in [**51**] is a *compact* Lie group of isometry, while in our work we just require that the map f is a diffeomorphism. Moreover the non degeneracy conditions that we require on the fixed point of f led us to different formulas to those stated in [**51**]. On the other hand, for the geometric endomorphisms considered in [**51**], that is those induced by isometries g lying in a compact Lie group G, the formula obtained by Lesch applies to a more general case than the ours because in his work there are not assumptions on the fixed points set while in our work there are.

Moreover the last part of this work contains several applications to the de Rham complex which are not mentioned in the other papers.

1. Differential cone operators

DEFINITION 4.1. Let M an open manifold. The cone over M, usually labeled $C(M)$, is the topological space defined as

$$M \times [0, \infty)/(\{0\} \times M). \tag{4.1}$$

The truncated cone, usually labeled $C_a(M)$, is defined as

$$M \times [0, a)/(\{0\} \times M). \tag{4.2}$$

Finally with $\overline{C_a(M)}$ we mean

$$M \times [0,a]/(\{0\} \times M). \tag{4.3}$$

In both the above cases, with v, we will label the vertex of the cone or the truncated cone, that is $C(M) - (M \times (0,\infty))$, $C_a(M) - (M \times (0,a))$ and $C_a(M) - (M \times (0,a])$ respectively.

Definition 4.2. A manifold with conical singularities X is a metrizable, locally compact, Hausdorff space such that there exists a sequence of points $\{p_1, ..., p_n, ...\} \subset X$ which satisfies the following properties:

(1) $M - \{p_1, ..., p_n, ...\}$ is a smooth open manifold.
(2) For each p_i there exist an open neighbourhood U_{p_i}, a closed manifold L_{p_i} and a map $\phi_{p_i} : U_{p_i} \to C_2(L_{p_i})$ such that $\phi_{p_i}(p_i) = v$ and $\phi_{p_i}|_{U_{p_i} - \{p_i\}} : U_{p_i} - \{p_i\} \to M \times (0,2)$ is a diffeomorphism.

The regular and the singular part of X are defined as

$$sing(X) = \{p_1, ..., p_n, ...\},\ reg(X) := X - sing(X) = X - \{p_1, ..., p_n, ...\}.$$

The singular points p_i are usually called conical points and the smooth closed manifold L_{p_i} is usually called the link relative to the point p_i. If X is compact then it is clear, from the above definition, that the sequences of conical points $\{p_1, ..., p_n, ...\}$ is made of isolated points and therefore on X there are just a finite number of conical points.
A manifold with conical singularities is a particular case of a compact smoothly stratified pseudomanifold; more precisely it is a compact smoothly stratified pseudomanifold with depth 1 and with the singular set made of a sequence of isolated points. Since in this paper we will work exclusively with compact manifolds with conical singularities we prefer to omit the definition of smoothly compact stratified pseudomanifold and the notions related to it and refer to [**1**] for a thorough discussion on this subject.

Remark 4.1. Let X be a compact manifold with one conical singularity p and let L_p its link; it follows from definition 4.2 that we can decompose X as

$$X \cong \overline{Y} \cup_{L_p} \overline{C_1(L_p)}$$

where $\overline{Y}$ is a compact manifold with boundary defined as $X - \phi_p^{-1}(C_1(L_p))$. Obviously this decomposition generalizes in a natural way when X has several conical points. As we will see in one of the following sections this decomposition is the starting point to study the heat kernel on X and we will use it to calculate the contribution given by the conical points to the Lefschetz number of some geometric endomorphisms.

Now we recall from [**1**] a particular case, which is suitable for our purpose, of an important result which describe a blowup process to resolve the singularities of a compact smoothly stratified pseudomanifold.

Proposition 4.3. *Let X be a compact manifold with conical singularities. The there exists a manifold with boundary $\overline{M}$ and a blow-down map $\beta : \overline{M} \to X$ which has the following properties:*

(1) *$\beta|_M : M \to reg(X)$, where M is the interior of $\overline{M}$, is a diffeomorphsim.*
(2) *There is a bijective correspondence between the conical points of X and the (possibly disconnected) boundary hypersurfaces of $\overline{M}$ which blow down to these conical points through β;*
(3) *If for each conical point p_i the relative link L_{p_i} is connected, then there is a bijection between the conical points of X and the connected components of $\partial\overline{M}$.*

PROOF. See [**1**], proposition 2.5. □

Now we introduce a class of natural riemannian metrics on these spaces.

DEFINITION 4.4. Let X be a manifold with conical singularities. A conic metric g on $reg(X)$ is riemannian metric with the following property: for each conical point p_i there exists a map ϕ_{p_i}, as defined in definition 4.2, such that

$$(\phi_{p_i}^{-1})^*(g|_{U_{p_i}}) = dr^2 + r^2 h_{Lp_i}(r) \tag{4.4}$$

where $h_{Lp_i}(r)$ depends smoothly on r up to 0 and for each fixed $r \in [0,1)$ it is a riemannian metric on L_{p_i}. Analogously, if $\overline{M}$ is manifold with boundary and M is its interior part, then g is a conic metric on M if it is a smooth, symmetric section of $T^*\overline{M} \otimes T^*\overline{M}$, degenerate over the boundary, such that over a collar neighborhood U of $\partial\overline{M}$, g satisfies (4.4) with respect to some diffeomorphism $\phi : U \to [0,1) \times \partial\overline{M}$.

The next step is to recall the notion of differential cone operator and its main properties. Before to proceed we introduce some notations that we will use steadily through the paper.
Given an open manifold M and two vector bundles E, F over it, with $\mathrm{Diff}^n(M,E,F), n \in \mathbb{N}$, we will label the space of differential operator $P : C_c^\infty(M,E) \to C_c^\infty(M,F)$ of order n. Given $\overline{M}$, a manifold with boundary, we will label with N the boundary of $\overline{M}$ and with M the interior part of $\overline{M}$. Given a vector bundle E over $\overline{M}$, with E_N we mean the restriction of E on N. Finally each metric ρ over E (riemannian if E is real or hermitian if E is complex) is assumed to be a non degenerate metric up to the boundary. The next definition is taken from [**51**]:

DEFINITION 4.5. Let $\overline{M}$ be a manifold with boundary $N = \partial\overline{M}$. Let E, F be two vector bundles on $\overline{M}$. Let $\overline{U}_N$ be a collar neighborhood of N, $\overline{U}_N \cong [0,\epsilon) \times N$ and let $U_N = \overline{U}_N - N$. A differential cone operator of order $\mu \in \mathbb{N}$ and weight $\nu > 0$ is a differential operator $P : C_c^\infty(M,E) \to C_c^\infty(M,F)$ such that on U_N it takes the form:

$$P|_{U_N} = x^{-\nu}\sum_{i=0}^{\mu} A_k(-x\frac{\partial}{\partial x})^k \tag{4.5}$$

where $A_k \in C^\infty([0,\epsilon), \mathrm{Diff}^{\mu-k})(E_N, F_N))$ and x is a boundary defining function. As in [**51**] we will label with $\mathrm{Diff}_0^{\mu,\nu}(M,E,F)$ the space of differential cone operators between the bundles E and F.

Now we explain what we mean by *differential cone operator* on a manifold X with conical singularities. In the previous definition we recalled the notion of differential cone operator acting on the smooth sections with compact support of two vector bundles E, F defined on a manifold $\overline{M}$ with boundary. In proposition 4.3, given a manifold with conical singularities X, we stated the existence of a manifold with boundary $\overline{M}$ endowed with a blow down map $\beta : \overline{M} \to X$ which desingularize X. Therefore given two vector bundles E, F on $reg(X)$ and $P \in \operatorname{Diff}(reg(X), E, F)$ we will say that P is a differential cone operators if the following properties are satisfied:

(1) $\beta^*(E), \beta^*(F)$ that are vector bundles on M, the interior of $\overline{M}$, extend as smooth vector bundles over the whole $\overline{M}$. In the same way, if E and F are endowed with metrics ρ_1 and ρ_2 then $\beta^*\rho_1$ and $\beta^*\rho_2$ extend as non degenerate metric up to the boundary of $\overline{M}$.
(2) The differential operator induced by P through β acting on $C^\infty_c(M, \beta^*E, \beta^*F)$ is a differential cone operator in the sense of definition 4.5.

In the rest of the paper, with a slight abuse of notation, we will identify M with $reg(X)$, E with β^*E, F with β^*F and P with the operator that it induces through β between $C^\infty_c(M, \beta^*E, \beta^*F)$.

Remark 4.2. We can reformulate definition 4.5 in the following way: P is differential cone operator of order μ and weight ν if and only if $x^\nu P$ is a $b-$differential operator of order μ in the sense of Melrose. For the definition of $b-$operator and the full development of this subject we refer to the monograph [**55**]. Using this approach we have $\operatorname{Diff}^{\mu,\nu}_0(M, E, F) = x^{-\nu}\operatorname{Diff}^\mu_b(M, E, F)$. This last point of view is used for example in [**35**] .

Now we introduce the notion of ellipticity:

Definition 4.6. Let $\overline{M}$ be a manifold with boundary and let E, F be two vector bundles over $\overline{M}$. Let $P \in \operatorname{Diff}^{\mu,\nu}_0(\overline{M}, E, F)$ and let $\sigma^\mu(P)$ its principal symbol. Then P is called elliptic if it is elliptic on M in the usual sense and if

$$x^\nu\sigma^\mu(P)(x, p, x^{-1}\tau, \xi) \tag{4.6}$$

is invertible for $(x, p) \in [0, \epsilon) \times N$ and $(\tau, \xi) \in T^*\overline{M} - \{0\}$.

In the above definition there is implicit the natural identification of $T^*\overline{M}|_{[0,\epsilon)\times N}$ with $\mathbb{R} \times T^*N$.

Definition 4.7. Let $\overline{M}, E, F$ and P be as in the previous definition. The conormal symbol of P, as defined in [**51**], is the family of differential operators, acting between $C^\infty(N, E_N, F_N)$, defined as

$$\sigma^{\mu,\nu}_M(P)(z) := \sum_{k=0}^{\mu} A_k(0)z^k \tag{4.7}$$

Now we make some further comments about the notion of ellipticity introduced in definition 4.6. The requirement (4.6) in definition 4.6 means that

$$\sum_{k=0}^{\mu}\sigma^{\mu-k}(A_k(x))(\xi)\sigma^k((-x\frac{\partial}{\partial x})^k)(x, x^{-1}\tau) = \sum_{k=0}^{\mu}\sigma^{\mu-k}(A_k(x))(\xi)(-i\tau)^k$$

is invertible. On M this is covered by classical ellipticity and for $x = 0$ it is equivalent to require that (4.7) is a parameter dependent elliptic family of differential operators with parameters in $i\mathbb{R}$.
Using again the b framework of Melrose, definition 4.6 is equivalent to say that the $b-$principal symbol of $P' := x^{\nu}P$, that is

$$\sigma_b^{\mu}(P') := \sigma^{\mu}(P')(x, p, x^{-1}\tau, \xi)$$

as an object lying in $C^{\infty}(T_b^*\overline{M}, \mathrm{Hom}(\pi_b^*E, \pi_b^*F))$, where $\pi_b : T_b^*\overline{M} \to \overline{M}$ is the $b-$cotangent bundle of $\overline{M}$, is an isomorphism on $T_b^*\overline{M} - \{0\}$. For further details on these approach see [**35**] and the relative bibliography.
Finally we remark that in definition 4.6 we followed [**51**] and [**35**]. This is slightly different from those given, for example, in [**59**], [**60**] and [**65**]. The definition given in these papers, in fact, requires the invertibility of the conormal symbol on a certain weight line (for more details see the above papers). By the fact that we are interested to study the operators on their natural domains, that is the maximal and the minimal one, we can waive this requirement (see [**51**] pag. 13 for more comments about this).

Finally we conclude this subsection stating an important proposition on the theory of differential cone operators:

THEOREM 4.8. *Let $(\overline{M}, g)$ be a compact and oriented manifold of dimension m with boundary where g is a conic metric over M; let E, F be two hermitian vector bundles over $\overline{M}$ and let $P \in \mathrm{Diff}_0^{\mu,\nu}(M, E, F)$ be an elliptic differential cone operator.*

1. *Each closed extension $\overline{P} : L^2(M, E) \to L^2(M, F)$ of P is a Fredholm operator on its domain, $\mathcal{D}(\overline{P})$, endowed with the graph norm.*
2. *Suppose that $E = F$ and that P is positive. Suppose, in addition, that on a collar neighborhood of $\partial\overline{M}$ the metric ρ on E does not depend on r and that the conic metric g satisfies $g = dr^2 + r^2h$ where h is any riemannian metric over $\partial\overline{M}$ which does not depend on r. Then, for each positive self-adjoint extension $\overline{P}$ of P, the heat operator $e^{-t\overline{P}} : L^2(M, E) \to L^2(M, E)$ is a trace-class operator. Moreover $\overline{P}$ is discrete and the sequences of eigenvalues of $\overline{P}$ satisfies $\lambda_j \sim Cj^{\frac{\mu}{m}}$.*

PROOF. For the first statement see [**51**] prop. 1.3.16 or [**35**] prop. 3.14. For the second one see [**51**] theorem 2.4.1 and corollary 2.4.3. □

2. Elliptic complex on manifolds with conical singularities

The aim of this subsection is to define the notion of elliptic complex on a manifold with conical singularities. As for the notion of ellipticity, the definition of elliptic complex on a manifold with conical singularities was introduced in [**65**], pag. 205, but our definition is slightly different because we waive some requirements about the sequence of conormal symbols on a certain weight line. The reason is still given by the fact that we are interested on the minimal and maximal extension of a complex differential cone operators.

Let $\overline{M}$ be a manifold with boundary, $E_0, ..., E_n$ a sequence of vector bundle over $\overline{M}$ and consider $P_i \in \mathrm{Diff}_0^{\mu,\nu}(M, E_i, E_{i+1})$ such that

$$0 \to C_c^\infty(M, E_0) \stackrel{P_0}{\to} C_c^\infty(M, E_1) \stackrel{P_1}{\to} ... \stackrel{P_{n-1}}{\to} C_c^\infty(M, E_n) \stackrel{P_n}{\to} 0 \tag{4.8}$$

is a complex. We have the following definition:

Definition 4.9. The complex (4.8) is an elliptic complex if it is an elliptic complex in the usual sense on M and if the sequence

$$0 \to \pi^* E_0 \to \pi^* E_1 \to ... \to \pi^* E_n \to 0 \tag{4.9}$$

where the maps are given by $x^\nu \sigma^\mu(P_i)(x, p, x^{-1}\tau, \xi) : \pi_i^* E_i \to \pi_{i+1}^* E_{i+1}$ is an exact sequence up to $x = 0$ over $T^*\overline{M} - \{0\}$.

With the help of Melrose's b framework we can reformulate the previous definition in the following way: (4.8) is an elliptic complex if and only if the following sequence is exact over $T_b^*(\overline{M}) - \{0\}$:

$$0 \to \pi_b^* E_0 \stackrel{\sigma_b^\mu(P'_0)}{\to} \pi_b^* E_1 \stackrel{\sigma_b^\mu(P'_1)}{\to} ... \stackrel{\sigma_b^\mu(P'_{n-1})}{\to} \pi_b^* E_n \stackrel{\sigma_b^\mu(P'_n)}{\to} 0 \tag{4.10}$$

where $P' = x^\nu P$, that is the $b-$operator naturally associated to P, $\pi_b : T_b^*\overline{M} \to \overline{M}$ is the $b-$cotangent bundle and

$$\sigma_b^\mu(P'_i) \in C^\infty(\overline{M}, \mathrm{Hom}(\pi_b^* E_i, \pi_b^* E_{i+1}))$$

is the $b-$principal symbol of P'_i.
We have the following proposition:

Proposition 4.10. *Consider a complex of differential cone operators as in* (4.8)*. Suppose moreover that M is endowed with a conic metric g. Then the complex is an elliptic complex if and only if for each $i = 0, ..., n$*

$$P_i^t \circ P_i + P_{i-1} \circ P_{i-1}^t : C_c^\infty(M, E_i) \to C_c^\infty(M, E_i)$$

is an elliptic differential cone operator.

Proof. It is clear that if $P \in \mathrm{Diff}_0^{\mu,\nu}(M, E_i, E_{i+1})$ then also $P^t \in \mathrm{Diff}_0^{\mu,\nu}(M, E_{i+1}, E_i)$ where $P_t : C_c^\infty(M, E_{i+1}) \to C_c^\infty(M, E_i)$ is the formal adjoint of P. Now, as in the previous comment, let $P'_i = x^\nu P$ be the $b-$operator that is naturally associated to P. It is well known that $\sigma_b^\mu(P'_{i+1} \circ P'_i) = \sigma_b^\mu(P'_{i+1}) \circ \sigma_b^\mu(P'_i)$ and that $\sigma_b^\mu((P'_i)^t) = (\sigma_b^\mu(P'_i))^t$. The proof follows now by standard arguments of linear algebra, in complete analogy with the case of an elliptic complex on a closed manifold. □

From the above proposition it follows the following useful corollary:

Corollary 4.11. *In the same hypothesis of the previous proposition. The Hilbert complexes $(L^2(M, E_*), P_{max/min,*})$ are both Fredholm complexes. Moreover each Hilbert complex that extends $(L^2(M, E_*), P_{min,*})$ and that is extended by $(L^2(M, E_*), P_{max,*})$ is still an Fredholm complex.*

Proof. From theorem 4.8 it follows that $P_{min,i}^t \circ P_{max,i} + P_{max,i-1} \circ P_{min,i-1}^t$ and $P_{max,i}^t \circ P_{min,i} + P_{min,i-1} \circ P_{max,i-1}^t$ are both Fredholm operators on their natural domain endowed with the graph norm. Now the statement follows from prop. 1.5 □

We remark the fact that we gave the definition of an elliptic complex of differential cone operators on a manifold with boundary $\overline{M}$. Following the remark after definition 4.5 the notion of elliptic complex of differential cone operators is naturally extended on a manifold X with conical singularities.

3. A brief reminder on the heat kernel

The aim of this subsection is to recall briefly the main local properties of the heat kernel on an open and oriented riemannian manifold (M, g).
Let (M, g) be an open and oriented riemannian manifold, E a vector bundle over M, $P_0 : C^\infty_c(M, E) \to C^\infty_c(M, E)$ a non-negative symmetric differential operator and $P : \mathcal{D}(P) \subset L^2(M, E) \to L^2(M, E)$ a non-negative, self-adjoint extension of P_0. It is well know that, using the spectral theorem for unbounded self-adjoint operators and its associated functional calculus (see [**30**], chap. XXII), it is possible to construct the operator e^{-tP}. The next result we are going to recall summarizes the main local properties of e^{-tP} that we will use in the rest of the paper. We start with the following definitions:

Definition 4.12. A cut-off function is a smooth function $\eta : [0, \infty) \to [0, 1]$ which admits a $\epsilon > 0$ such that $\eta(x) = 1$ for $x \leq \frac{\epsilon}{4}$ and $\eta = 0$ for $x \geq \epsilon$.

Definition 4.13. Let M be an open manifold, E a vector bundle over M and $P_0 : C^\infty_c(M, E) \to C^\infty_c(M, E)$ a differential operator of second order. Then P_0 is a generalized Laplacian if its principal symbol satisfies:

$$\sigma^2(P_0)(x, \xi) = \|\xi\|^2.$$

An operator of this type is clearly elliptic. We refer to [**9**] for a comprehensive discussion on this class of operators.

Theorem 4.14. *Let (M, g) be an open and oriented riemannian manifold, E a vector bundle over M, $P_0 : C^\infty_c(M, E) \to C^\infty_c(M, E)$ a non-negative symmetric differential operator of order d and*

$$P : \mathcal{D}(P) \subset L^2(M, E) \to L^2(M, E)$$

a non-negative, self-adjoint extension of P. Then e^{-tP} satisfies the following properties:

- *e^{-tP} has a $C^\infty-$kernel, that is usually labeled $e^{-tP}(s, q)$ or $k_P(t, s, q)$, which lies in $C^\infty((0, \infty) \times M \times M, E \boxtimes E^*)$.*
- *If K_1, K_2 are compact subset of M such that $K_1 \cap K_2 = \emptyset$ then*
$$\|k_P(t, s, q)\|_{C^k(K_1 \times K_2, E \boxtimes E^*)} = O(t^n),\ t \to 0$$
for all $k, n \in \mathbb{N}$.
- *Let $\phi, \chi \in C^\infty_c(M)$; then the operator $\phi e^{-tP}\chi$ is a trace-class operator and we have, on $C^l(K_1 \times K_2, E \boxtimes E^*|_{K_1 \times K_2})$ for each $l \in \mathbb{N}$,*
$$(\phi e^{-tP}\chi)(q, q) \sim_{t \to 0} \sum_{n=0}^{\infty} \phi(q)\chi(q)\Phi_n(q)t^{\frac{n-m}{d}}$$
and
$$\mathrm{Tr}((\phi e^{-tP}\chi)(q, q)) \sim_{t \to 0} \sum_{n=0}^{\infty} (\int_M \phi(q)\chi(q)\,\mathrm{tr}(\Phi(q)) dvol_g) t^{\frac{n-m}{d}}$$

where $q \in M$, $\{\Phi_1, ..., \Phi_n, ...,\}$ *is a suitable sequence of sections in* $C^{\infty}(M, End(E))$, $K_1 = supp(\phi)$ *and* $K_2 = supp(\chi)$.

Finally if P_0 *is a generalized Laplacian then the last property above modifies in the following way:*

- *Let* $\phi, \chi \in C^{\infty}_c(M)$; *then the operator* $\phi e^{-tP}\chi$ *is a trace-class operator and we have*
$$\phi(s)e^{-tP}(s,q)\chi(q) \sim_{t\to 0} h_t(s,q)\sum_{n=0}^{\infty}\phi(s)\chi(q)\Phi_n(s,q)t^n$$
where $(s,q) \in M\times M$, $\{\Phi_1, ..., \Phi_n, ...,\}$ *is a suitable sequence of sections in* $C^{\infty}(M\times M, E\boxtimes E^*)$ *and* $h_t(s,q) = (4\pi t)^{\frac{-n}{2}}e^{\frac{-d(s,q)^2}{4t}}\eta(d(s,q)^2)$ *with* η *a cut-off function. As in the previous case the above expansion holds in* $C^l(K_1 \times K_2, E \boxtimes E^*|_{K_1\times K_2})$ *for each* $l \in \mathbb{N}$, *where* $K_1 = supp(\phi)$ *and* $K_2 = supp(\chi)$.

Proof. For the first three properties we refer to [**51**], theorem 1.1.18. As explained there these properties are proved globally, for example in [**37**], when M is a closed manifold. A careful examination of those proofs shows that the same properties remain true locally when M is an open manifold. The same argumentation applies to the last property which is proved globally, on a closed manifold, in [**9**] prop. 2.46 or in [**62**] theorem 7.15. □

The rest of the subsection is a brief reminder about the heat kernel of a differential cone operator. For more details and for the proof we refer to [**51**]. As already recalled in theorem 4.8 we know that, if $\overline{M}$ is a compact and oriented manifold with boundary, M its interior part, $P_0 \in \text{Diff}_0(M, E; E)$ is a positive operator and g is a conic metric over M, then for each positive self-adjoint extension P of P_0, $e^{-tP} : L^2(M, g) \to L^2(M, g)$ is a trace-class operator. Now we want to recall an important property named **scaling property**. Before doing this we need to introduce some notations:
Let N be a compact manifold; consider $C(N)$ and endow it with a product metric $g = dr^2 + h$ where h is a riemannian metric over N. Finally let E be a vector bundle over $reg(C(N))$.
Define $U_t : L^2(reg(C(N)), E) \to L^2(reg(C(N)), E)$ as $s(r,p) \mapsto t^{\frac{1}{2}}s(tr,p)$. It is immediate to show that $U_t : L^2(reg(C(N)), E) \to L^2(reg(C(N)), E)$ is an isometry and that $U_{t_1} \circ U_{t_2} = U_{t_1t_2}$.

Proposition 4.15. *Let* N *be a compact manifold,* E *a vector bundle over* $reg(C(N))$, *let* $P_0 \in \text{Diff}^{\mu,\nu}_0(reg(C(N)), E, E)$ *be a symmetric differential cone operator and let* P *be a self-adjoint extension of* P_0. *Endow* $reg(C(N))$ *with a product metric* g, *that is* $g = dr^2 + h$ *where* h *is a riemannian metric over* N. *Finally let* $P_t = t^{\nu}U_tPU_t^*$ *and let* $f : \mathbb{R} \to \mathbb{R}$ *a function such that* $f(P)$ *has a measurable kernel. Then for each* $\lambda > 0$
$$f(P)(r,p,s,q) = \frac{1}{\lambda}f(\lambda^{-\nu}P_{\lambda})(\frac{r}{\lambda}, p, \frac{s}{\lambda}, q),\ \lambda > 0 \tag{4.11}$$
As particular case, given $P_0 \in \text{Diff}_0(reg(C(N)), E, E)$ *positive and* P *a positive self-adjoint extension then*
$$e^{-tP}(r,p,r,q) = \frac{1}{r}e^{-tr^{-\nu}P_r}(1,p,1,q) \tag{4.12}$$

PROOF. See [**51**] lemma 2.2.3. □

Now we modify the above proposition for the heat operator in the case that g is a conic metric over M. As we will see, we are interested to the study of the L^2-Lefschetz numbers where the L^2 space are built using a conic metric. The reason is that when the considered complex is the L^2 de Rham complex (built using a conic metric) then its L^2-cohomology has a topological meaning. More precisely, as showed by Cheeger in [**24**], we have the following theorem:

THEOREM 4.16. *Let (F, h) be a compact and oriented riemannian manifold of dimension f. Consider the cone $C_b(F)$ with b a positive real number and endow $C_b(F)$ with the conic metric $g = dr^2 + r^2h$. Then*

$$H^i_{2,max}(C_b(F), g) \cong \begin{cases} H^i(F) & i < \frac{f}{2} + \frac{1}{2} \\ 0 & i \geq \frac{f}{2} + \frac{1}{2} \end{cases} \tag{4.13}$$

If X is a compact and oriented manifold with conical singularities and if g is a conic metric over $reg(X)$ then

$$H^i_{2,max}(reg(X), g) \cong I^{\underline{m}}H^i(X),\ H^i_{2,min}(reg(X), g) \cong I^{\overline{m}}H^i(X). \tag{4.14}$$

PROOF. See [**24**]. □

For the definition and the main properties of intersection cohomology we refer to [**38**] and [**39**]

LEMMA 4.17. *Let N be a compact manifold of dimension n, E a vector bundle over $reg(C(N))$, let $P_0 \in \mathrm{Diff}^{\mu,\nu}_0(reg(C(N)), E, E)$ be a positive differential cone operator and let P be a positive self-adjoint extension of P_0. Endow $reg(C(N))$ with a conic metric g, that is $g = dr^2 + r^2h$ where h is a riemannian metric over N. Then for each $\lambda > 0$*

$$e^{-tP}(r, p, s, q) = \frac{1}{\lambda^{n+1}} e^{-t\lambda^{-\nu}P_\lambda}(\frac{r}{\lambda}, p, \frac{s}{\lambda}, q),\ \lambda > 0 \tag{4.15}$$

In particular we have

$$e^{-tP}(r, p, r, q) = \frac{1}{r^{n+1}} e^{-t\lambda^{-\nu}P_r}(1, p, 1, q),\ \lambda > 0. \tag{4.16}$$

PROOF. The proof is completely analogous to the proof of proposition 4.15. We have just to add the natural modifications caused by the fact that now the Hilbert space $L^2(reg(C(N)), E)$ is built using the conic metric $g = dr^2 + r^2h$ and this means that given $\gamma \in L^2(reg(C(N)), E)$ we have $\|\gamma\|_{L^2(reg(C(N)),E)} = \int_{reg(C(N))} \|\gamma\| r^n dr dvol_h$ where $\|\gamma\|$ is the pointwise norm induced by the metric on E (which is a riemannian metric if E is a real vector bundle and is a Hermitian metric if E is complex.). This implies that now the isometry U_t, introduced above proposition 4.15, is defined as $U_t : L^2(reg(C(N)), E) \to L^2(reg(C(N)), E)$, $U_t(\gamma) = t^{\frac{n+1}{2}}\gamma(tr, p)$. The proof follows now in completely analogy to that one of proposition 4.15 . Moreover, in the case that P is a positive self-adjoint extension of $\Delta_i : \Omega^i_c(reg(C(N))) \to \Omega^i_c(reg(C(N)))$, the Laplacian constructed using a conic metric and acting on the space of smooth $i-$forms with compact support, the proof is given in [**25**], pag. 582. □

Finally we conclude the section with the following proposition; before to state it we introduce some notations. Given $\lambda \in \mathbb{R}$ we define

$$p^+(\lambda) := |\lambda + \frac{1}{2}| \text{ and}$$
$$p^-(\lambda) := \begin{cases} |\lambda - \frac{1}{2}| & |\lambda| \geq \frac{1}{2} \\ \lambda - \frac{1}{2} & |\lambda| < \frac{1}{2} \end{cases} \tag{4.17}$$

Moreover we recall that $I_a(x)$ is the modified Bessel function of order a. For the definition see [**51**] pag. 67.

PROPOSITION 4.18. *Let (N, h) be a compact and oriented riemannian manifold of dimension n. Consider $C(N)$ and let E be a vector bundle over $reg(C(N))$ endowed with a metric ρ (hermitian if it is complex o riemannian if it is real). Suppose that E admits an extension over all $[0,\infty) \times N$ that we denote $\overline{E}$. Let $E_N = \overline{E}|_N$ and suppose that (E, ρ) is isometric to $\pi^*(E_N, \rho|_N)$ where $\pi : (0,\infty) \times N \to N$ is the natural projection. Finally let $P : C_c^\infty(E) \to C_c^\infty(E)$ be an elliptic differential cone operator of order one. Then:*

(1) *On $L^2(reg(C_2(N)), E)$ built with the product metric $g_p = dr^2 + h$, if P satisfies $P = \frac{\partial}{\partial r} + \frac{1}{r}S$, where $S \in \mathrm{Diff}^1(N, E_N)$ is elliptic, we have*
$$e^{-tP^t_{max} \circ P_{min}}(r,p,s,q) = \sum_{\lambda \in \operatorname{spec} S} \frac{1}{2t}(rs)^{\frac{1}{2}} I_{p^+(\lambda)}(\frac{rs}{2t}) e^{-\frac{r^2+s^2}{4t}} \Phi_\lambda(p,q) \tag{4.18}$$
and
$$e^{-tP_{min} \circ P^t_{max}}(r,p,s,q) = \sum_{\lambda \in \operatorname{spec} S} \frac{1}{2t}(rs)^{\frac{1}{2}} I_{p^-(\lambda)}(\frac{rs}{2t}) e^{-\frac{r^2+s^2}{4t}} \Phi_\lambda(p,q)$$
where $\Phi_\lambda(p,q)$ is the smooth kernel of $\Phi_\lambda : L^2(N, E_N) \to V_\lambda$, the orthogonal projection on the eigenspace V_λ.

(2) *On $L^2(reg(C_2(N)), E)$ built with the conic metric $g_c = dr^2 + r^2 h$, if P satisfies*
$P = \frac{n}{2r} + \frac{\partial}{\partial r} + \frac{1}{r}S$, where $S \in \mathrm{Diff}^1(N, E_N)$ is elliptic, we have
$$e^{-tP^t_{max} \circ P_{min}}(r,p,s,q) = \sum_{\lambda \in \operatorname{spec} S} \frac{1}{2t}(rs)^{\frac{1-n}{2}} I_{p^+(\lambda)}(\frac{rs}{2t}) e^{-\frac{r^2+s^2}{4t}} \Phi_\lambda(p,q) \tag{4.19}$$
and
$$e^{-tP_{min} \circ P^t_{max}}(r,p,s,q) = \sum_{\lambda \in \operatorname{spec} S} \frac{1}{2t}(rs)^{\frac{1-n}{2}} I_{p^-(\lambda)}(\frac{rs}{2t}) e^{-\frac{r^2+s^2}{4t}} \Phi_\lambda(p,q)$$
where $\Phi_\lambda(p,q)$ is the smooth kernel of $\Phi_\lambda : L^2(N, E_N) \to V_\lambda$, the orthogonal projection on the eigenspace V_λ.

PROOF. The first assertion is proved in [**51**], see proposition 2.3.11 and pag. 68. The second statement follows using the following argument. Only for the remaining part of this proof let us label $L^2(reg(C_2(N)), E, g_p)$ the L^2 space of sections built using the product metric $g_p = dr^2 + h$ and $L^2(reg(C_2(N)), E, g_c)$ the L^2 space of sections built using the conic metric $g_c = dr^2 + r^2 h$. The measure induced by g_p is $drdvol_h$ while the measure induce by g_c is $r^n drdvol_h$. Therefore it is clear that the map $\tau : L^2(reg(C(N)), E, g_c) \to L^2(reg(C_2(N)), E, g_p)$, $\tau(\gamma) = r^{\frac{n}{2}}\gamma$ is an isometry with inverse given by $\tau^{-1}(\gamma) = r^{\frac{-n}{2}}\gamma$. A simple calculation shows that $\tilde{P} := \tau^{-1} \circ P \circ \tau$ satisfies $\tilde{P} = \frac{\partial}{\partial r} + \frac{1}{r}S$. Therefore $\tilde{P}^t_{max} \circ \tilde{P}_{min} = r^{\frac{n}{2}} P^t_{max} \circ P_{min} r^{\frac{-n}{2}}$ and this implies that

$$e^{-t\tilde{P}^t_{max} \circ \tilde{P}_{min}} = r^{\frac{n}{2}} e^{-tP^t_{max} \circ P_{min}} r^{\frac{-n}{2}}.$$

Therefore if we call $\tilde{k}(t, r, p, s, q)$ the heat kernel relative to $e^{-t\tilde{P}^t_{max} \circ \tilde{P}_{min}}$ and analogously $k(t, r, p, s, q)$ the heat kernel relative to $e^{-tP^t_{max} \circ P_{min}}$ we have, for each $\gamma \in L^2(reg(C_2(N)), E, g_p)$:

$$\int_{reg(C_2(N))} \tilde{k}(t, r, p, s, q)\gamma(s)dsdvol_h =$$

$$= \int_{reg(C_2(N))} r^{\frac{n}{2}} k(t, r, p, s, q)s^{\frac{-n}{2}}\gamma(s)s^n dsdvol_h$$

and therefore $\tilde{k}(t, r, p, s, q) = r^{\frac{n}{2}} k(t, r, p, s, q)s^{\frac{n}{2}}$. Finally, applying this last equality to (4.18), we get (4.19). For the heat kernel of $e^{-tP_{min} \circ P^t_{max}}$ the proof is completely analogous to the previous one. □

CHAPTER 5

L^2-Lefschetz numbers

This chapter is devote to the L^2-Lefschetz numbers. In the first section the notion of geometric endomorphism is developed. The second section contains the definitions of L^2-Lefschetz numbers. Finally in the last part the approach using the heat kernel in order to calculate this numbers is developed.

1. Geometric endomorphism

The goal of this section is to introduce and study the notion of **geometric endomorphism** of an elliptic complex of differential cone operators. Let X be a compact manifold with conical singularities and let M be its regular part that, as explained after definition 4.5, we identify with the interior part of $\overline{M}$ the manifold with boundary which desingularizes X, see prop. 4.3. Finally consider an elliptic complex of differential cone operators as described in definition 4.9:

$$0 \to C^{\infty}_c(M, E_0) \stackrel{P_0}{\to} C^{\infty}_c(M, E_1) \stackrel{P_1}{\to} \ldots \stackrel{P_{n-1}}{\to} C^{\infty}_c(M, E_n) \stackrel{P_n}{\to} 0 \tag{5.1}$$

Definition 5.1. A **geometric endomorphism** T of (5.1) is given by a $n-$tuple of maps $T = (T_1, ..., T_n)$ constructed in the following way: there exists a smooth map $f : \overline{M} \to \overline{M}$ and a $n-$tuples of morphisms of bundles $\phi_i : f^*E_i \to E_i$ such that the following properties hold:

1. $f : \overline{M} \to \overline{M}$ is a diffeomorphism.
2. If $\{N_1, ..., N_k\}$ are the connected components of $\partial\overline{M}$ then $f(N_i) = N_i$ for each $i = 1, ..., k$.
3. $T_i = \phi_i \circ f^*$ where f^* acts naturally between $C^{\infty}(M, E)$ and $C^{\infty}(M, f^*E)$.
4. $P_i \circ T_i = T_{i+1} \circ P_i$.

We make a little comment on the above definition. The second and the third property are exactly the definition of geometric endomorphism of an elliptic complex over a closed manifold given in [**3**]. However our definition is not a complete extension of that one given by Atiyah and Bott in [**3**]. The reason is that in the closed case any smooth map is allowed. For our purposes we need that T_i induce a bounded map from $L^2(M, E_i)$ to itself and clearly this prevents us to allow every smooth map in definition 5.1. As we will see in the following lemma, the property that $f : \overline{M} \to \overline{M}$ is a diffeomorphism is a reasonable sufficient condition in order to get a bounded extension of T_i on $L^2(M, E_i)$.

Lemma 5.2. *In the same hypothesis of the above definition the endomorphism T satisfies that the following properties:*

(1) *For each i and for each $\psi \in C^{\infty}_c(M,E_i)$ we have $T_i(\psi) \in C^{\infty}_c(M,E_i)$.*
(2) *For each i T_i extends as a bounded operator from $L^2(M,E_i)$ to itself; with a small abuse of notation, we denote this again by T_i.*
(3) *Let $T_i^* : L^2(M,E_i) \to L^2(M,E_i)$ be the adjoint of T_i. Then for each $\psi \in C^{\infty}_c(M,E_i)$ we have $T^*_i(\psi) \in C^{\infty}_c(M,E_i)$.*

Proof. The first two properties follow immediately by the fact that $f : \overline{M} \to \overline{M}$ is a diffeomorphism and that $\overline{M}$ is compact. For the third properties, we observe first of all that T_i admits an adjoint because it is densely defined and that T_i^* is bounded and defined over the whole $L^2(M,E_i)$ because T_i is bounded. Now consider the bundle f^*E. The metric over E induces in a natural way through f a metric over f^*E. Therefore it make sense consider the bundle homomorphism $\phi^* : E \to f^*E$ defined in each fiber as the adjoint of ϕ. Now consider the pull-back under f of the volume form $dvol_g$. Then there exists a smooth function τ such that $\tau dvol_g = f^*dvol_g$ and $\tau > 0$ if f preserves the orientation of M, $\tau < 0$ if f reverses the orientation of M. Finally define $S : C^{\infty}_c(M,E_i) \to C^{\infty}_c(M,E_i)$ as

$$S_i(\psi) := \begin{cases} \tau(\phi^*_i \circ (f^{-1})^*)(\psi) & \text{if } f \text{ preserves the orientation} \\ -\tau(\phi^*_i \circ (f^{-1})^*)(\psi) & \text{if } f \text{ reserves the orientation} \end{cases} \tag{5.2}$$

It is immediate to check that for each $\psi_1, \psi_2 \in C^{\infty}_c(M,E_i)$ we have

$$< T_i(\psi_1), \psi_2 >_{L^2(M,E_i)} = < \psi_1, S_i(\psi_2) >_{L^2(M,E_i)} .$$

Therefore, over $C^{\infty}_c(M,E_i)$, T_i^* coincides with S and so from this the third property follows immediately. □

Now we state the following property :

Proposition 5.3. *Let M be an open and oriented riemannian manifold and let g be an incomplete riemannian metric on M. Let $E_0, ..., E_n$ be a sequence of vector bundles over M and consider a complex of differential operators:*

$$0 \to C^{\infty}_c(M,E_0) \stackrel{P_0}{\to} C^{\infty}_c(M,E_1) \stackrel{P_1}{\to} ... \stackrel{P_{n-1}}{\to} C^{\infty}_c(M,E_n) \stackrel{P_n}{\to} 0 \tag{5.3}$$

Let T be an endomorphism of (5.3) *that satisfies the second, the third and the fourth property of definition 5.1. Then we have the following properties:*
(1) *For each $i = 0, ..., n$, for each $s \in \mathcal{D}(P_{min,i})$ we have $T_i(s) \in \mathcal{D}(P_{min,i})$ and $P_{min,i} \circ T_i = T_{i+1} \circ P_{min,i}$.*
(2) *For each $i = 0, ..., n$, for each $s \in \mathcal{D}(P_{max,i})$ we have $T_i(s) \in \mathcal{D}(P_{max,i})$ and $P_{max,i} \circ T_i = T_{i+1} \circ P_{max,i}$.*

Proof. Let $i \in \{0, ..., n\}$ and let $s \in \mathcal{D}(P_{min,i})$. Then there exists a sequence $\{s_j\}_{j\in\mathbb{N}}$ such that $s_j \to s$ in $L^2(M,E_i)$ and $P_i(s_j) \to P_i(s)$ in $L^2(M,E_{i+1})$. Using definition 5.3, we know that $\{T_i(s_j)\}_{j\in\mathbb{N}}$ is a sequence of smooth sections with compact support contained in $C^{\infty}_c(M,E_i)$ such that $T_i(s_j) \to T_i(s)$ in $L^2(M,E_i)$ and $T_{i+1}(P_i(s_j)) \to T_{i+1}(P_i(s))$ in $L^2(M,E_{i+1})$. But $T_{i+1}(P_i(s_j)) = P_i(T_i(s_j))$. Therefore $P_i(T_i(s_j))$ converges in $L^2(M,E_{i+1})$ and this implies that $T_i(s) \in \mathcal{D}(P_{min,i})$ and that $P_{min,i} \circ T_i = T_{i+1} \circ P_{min,i}$.
Now we give the proof of the second statement. From the first part of the proof it follows that, if we look at $T_{i+1} \circ P_{min,i}$, $P_{min,i} \circ T_i$ as unbounded

operator with domain $\mathcal{D}(P_{min,i})$ then $T_{i+1} \circ P_{min,i} = P_{min,i} \circ T_i$ and therefore $(T_{i+1} \circ P_{min,i})^* = (P_{min,i} \circ T_i)^*$. Moreover, by the fact that T_{i+1} is bounded, it follows that $(T_{i+1} \circ P_{min,i})^* = P^*_{min,i} \circ T^*_{i+1}$ with domain given by $(T^*_{i+1})^{-1}(\mathcal{D}(P^*_{min,i}))$. Now let $s \in \mathcal{D}(P_{max,i})$ and let $\phi \in C^\infty_c(M, E_{i+1})$. Then:

$$< T_i(s), P^t_i(\phi) >_{L^2(M,E_i)} = < s, T^*_i(P^t_i(\phi)) >_{L^2(M,E_i)} =$$
$$= < s, (P_{min,i} \circ T_i)^*(\phi) >_{L^2(M,E_i)} =$$
$$= < s, P^*_{min,i}(T^*_{i+1}(\phi)) >_{L^2(M,E_i)} = \text{(because } T^*_{i+1}(\phi) \in C^\infty_c(M, E_{i+1}))$$
$$= < s, P^*_{max,i}(T^*_{i+1}(\phi)) >_{L^2(M,E_i)} = < P_{max,i}(s), (T^*_{i+1}(\phi)) >_{L^2(M,E_i)}$$
$$= < T_{i+1}(P_{max,i}(s)), \phi >_{L^2(M,E_i)} .$$

So we can conclude that $T_i(s) \in \mathcal{D}(P_{max,i})$ and that $T_{i+1} \circ P_{max,i} = P_{max,i} \circ T_i$. □

In the rest of this section we describe the notion of **non degeneracy condition** for a fixed point of a map $f : X \to X$. As we will see, over the regular part of X, this is the same of the one used in [**3**].
Let X be a compact manifold with conical singularities and let $f : X \to X$ a continuous map such that $f(sing(X)) \subset sing(X)$, $f(reg(X)) \subset reg(X)$ and $f|_{reg(X)}$ is a smooth map. Define

$$Fix(f) := \{p \in X : f(p) = p\} \tag{5.4}$$

Definition 5.4. A point $p \in reg(X) \cap Fix(f)$ is said to be simple if $det(Id - d_p f) \neq 0$.

Obviously this definition make sense because, being p a fixed point, it follows that $d_p f$ is an endomorphism of $T_p(reg(X))$. Moreover it is easy to show that definition 5.4 is equivalent to require that, on $reg(X) \times reg(X)$, $\mathcal{G}(f)$ meets transversely $\Delta_{reg(X)}$ on (p, p), where $\mathcal{G}(f)$ is the graph of $f|_{reg(X)}$ and $\Delta_{reg(X)}$ is the diagonal of $reg(X)$. In this way we get the following useful corollary:

Corollary 5.5. *Each simple fixed point in $reg(X) \cap Fix(f)$ is an isolated fixed point.*

Now, following [**59**], [**60**] but with little modifications, we recall what is a simple fixed point $p \in Fix(f) \cap sing(X)$. As we said above, we assumed that $f(sing(X)) \subset sing(X)$ and that $f(reg(X)) \subset reg(X)$. Therefore if $q \in sing(X) \cap Fix(f)$ is a fixed conical point it follows that, on a neighborhood $U_q \cong C_2(L_q)$ of q, f takes the form:

$$f(r, p) = (rA(r, p), B(r, p)) \tag{5.5}$$

We make the additional assumption that $A(r, p)$ and $B(r, p)$ are smooth up to zero, that is

$$A(r, p) : [0, 2) \times L_q \to [0, 2)$$

is smooth up to 0 and analogously

$$B(r, p) : [0, 2) \times L_q \to L_q$$

is smooth up to 0. Moreover, by the fact that $f(sing(X)) \subset sing(X)$ and that $f(reg(X)) \subset reg(X)$ it follows that $A(r, p) \neq 0$ for $r > 0$. Obviously if our starting point is a diffeomorphism $\overline{f}:\overline{M}\to\overline{M}$ as in definition 5.1, then these requirements are automatically satisfied.

DEFINITION 5.6. A point $q\in Fix(f)\cap sing(X)$ is a **simple** fixed point if at least one of the two following conditions is satisfied:

(1) For each $p\in L_q$ $\lim_{r\to 0}A(r,p)\neq 1$.
(2) There exists $\epsilon>0$ such that, for each fixed $r\in[0,\epsilon)$, $B(r,.):L_q\to L_q$ satisfies $B(r,p)\neq p$.

Obviously in the first requirement the limit exists because in (5.5) we required that $A(r,p)$ is smooth up to 0. A natural question follows from definition 5.6: what is the meaning of these requirements? The answer is that if f satisfies one of the two requirements above then a sequence of fixed point converging to q cannot exists and therefore q is an isolated fixed point. We can show this last properties in the following way: suppose that $\{(r_j,p_j)\}$ is a sequence of fixed point of f contained in $U_q\cong C_2(L_q)$. Then $\{p_j\}$ is a sequence of point in L_q which is compact and therefore there exists a subsequence, that with a little abuse of notations we still label $\{p_j\}$, such that p_j converges to some $p\in L_q$. By the assumptions, for each j, $(r_j,p_j)=(r_jA(r_j,p_j),B(r_j,p_j))$. Therefore $A(r_j,p_j)=1=\lim_{j\to\infty}A(r_j,p)$ and $B(r_j,p_j)=p_j$ and this implies that f does not satisfies both the properties of definition 5.6.

So we can state the following useful corollary:

COROLLARY 5.7. *Let X be a compact manifold with conical singularities and let $f:X\to X$ a map such that $f(sing(X))\subset sing(X)$, $f(reg(X))\subset reg(X)$, $f|_{reg(X)}:reg(X)\to reg(X)$ is smooth and, on a neighborhood of a conical point, $A(r,p)$ and $B(r,p)$ are smooth up to 0. Then, if f has only simple fixed point, $Fix(f)$ is made of a finite number of points.*

PROOF. If f has only simple fixed points then we already know that each of this fixed points is an isolated fixed point and this implies that $Fix(f)$ is a sequence without accumulation points. Therefore, by the compactness of X, it follows that $Fix(f)$ is made of a finite number of points. $\square$

Now we state the following definition:

DEFINITION 5.8. Let f be as in the previous corollary. Let $q\in Fix(f)\cap sing(X)$ a simple fixed point for f such that f satisfies the first requirement of definition 5.6. Then if for each $p\in L_q$

$$\lim_{r\to 0}A(r,p)<1 \tag{5.6}$$

q is called **attractive simple fixed point** while if

$$\lim_{r\to 0}A(r,p)>1 \tag{5.7}$$

then q is called **repulsive simple fixed point**.

Clearly if for each $q\in sing(X)$ the relative link L_q is connected then each simple fixed point $q\in sing(X)$ is necessarily attractive or repulsive.

Finally we conclude the section observing that in [**40**], pag. 384, Goresky and MacPherson introduced the notion of contracting fixed point. An elementary check shows that (5.6) is equivalent to the definition given by Goresky and MacPherson.

2. L^2-Lefschetz numbers of a geometric endomorphism

Let X be a compact manifold with conical singularities of dimension $m+1$. Consider an elliptic complex of cone differential operators as defined in definition 4.9:

$$0 \to C_c^\infty(M, E_0) \stackrel{P_0}{\to} C_c^\infty(M, E_1) \stackrel{P_1}{\to} \dots \stackrel{P_{n-1}}{\to} C_c^\infty(M, E_n) \stackrel{P_n}{\to} 0 \tag{5.8}$$

where $P_i \in \mathrm{Diff}_0^{\mu,\nu}(M, E_i, E_{i+1})$ and let $T = \phi \circ f$ be a geometric endomorphism of (5.8) as in definition 5.1. Obviously, with a small abuse of notation, we are using the same notation for the diffeomorphism $f : \overline{M} \to \overline{M}$ and for the isomorphism that it induces on X. Clearly the isomorphism $f : X \to X$ satisfies

1. $f|_{reg(X)} : reg(X) \to reg(X)$ is a diffeomorphism
2. For each $p \in sing(X)$ we have $f(p) = p$
3. $A(r, p)$ and $B(r, p)$ (see (5.5)) are smooth up to 0.

Using corollary 4.11 we know that both the complexes $(L^2(M, E_i), P_{max/min,i})$ are Fredholm complexes, that is the cohomology groups $H^i_{2,max/min}(M, E_i)$ are finite dimensional.
Moreover by proposition 5.3 we know that T is a morphism of both complexes $(L^2(M, E_i), P_{max/min,i})$. Therefore, for each $i = 0, ..., n$, it induces an endomorphism

$$T_i^* : H^i_{2,max}(M, E_i) \to H^i_{2,max}(M, E_i)$$

and analogously

$$T_i^* : H^i_{2,min}(M, E_i) \to H^i_{2,min}(M, E_i).$$

So we are in position to give the following definition:

Definition 5.9. The L^2-Lefschetz numbers of T are defined in the following way:

$$L_{2,max}(T) = \sum_{i=0}^{n} (-1)^i \operatorname{tr}(T_i^* : H^i_{2,max}(M, E_i) \to H^i_{2,max}(M, E_i)) \tag{5.9}$$

and analogously

$$L_{2,min}(T) = \sum_{i=0}^{n} (-1)^i \operatorname{tr}(T_i^* : H^i_{2,min}(M, E_i) \to H^i_{2,min}(M, E_i)) \tag{5.10}$$

The L^2-Lefschetz numbers satisfy the following property:

Proposition 5.10. *$L_{2,max/min}(T)$ do not depend on the conic metric g that we fix on M and on the metrics $\rho_0, ..., \rho_n$ that we fix on $E_0, ..., E_n$*

Proof. By the fact that $\overline{M}$ is compact and that, as explained above definition 4.5, (E_i, ρ_i) are defined over all $\overline{M}$ and ρ_i is non degenerate up to the boundary, it follows that all the metrics we consider on E_i are quasi-isometric. Moreover, using proposition 1.32, it follows that if g and g' are two conic metric over M then they are quasi-isometric, that is there exists a positive real number c such that $g' \leq g \leq g'$. Therefore, for each $i = 0, .., n$, $L^2(M, E_i)$ doesn't depend from the metric that we fix on E_i and from the conic metric that we fix over M. This in turn implies that same conclusion holds for $H^i_{2,max}(M,E_i)$ and for $H^i_{2,min}(M,E_i)$, that is they do not depend from the metric that we fix on E_i and from the conic metric that we fix over M. In this way we can conclude that also the traces of $T_i^* : H^i_{2,max}(M,E_*) \to H^i_{2,max}(M,E_*)$ and $T_i^* : H^i_{2,min}(M,E_*) \to H^i_{2,min}(M,E_*)$ satisfy the same property and so the proposition is proved. □

- From the above proposition it follows that in order to calculate $L_{2,max/min}(T)$ we can use any conic metric g on M and any metrics $\rho_0,...,\rho_n$ over $E_0,...,E_n$. Therefore, in the remaining part of this section, we make the following assumptions: for each singular point q there exists U_q, an open neighborhood of q satisfying $U_q \cong C_2(L_q)$, such that on $reg(C_2(L_q))$ the conic metric g satisfies $g = dr^2 + r^2h$ where h is any riemannian metric over L_q that does not depend on r. Moreover we assume that each metric ρ_i on E_i does not depend on r in a collar neighborhood of $\partial\overline{M}$.

Consider, for each $i = 0,...,n$, the operator

$$\mathcal{P}_i := P_i^t \circ P_i + P_i \circ P_i^t : C_c^\infty(M,E_i) \to C_c^\infty(M,E_i).$$

It is clearly a positive operator. As stated in proposition 4.10, we know that $\mathcal{P}_i$ is an elliptic differential cone operator. Therefore, by theorem 4.8, we know that for each positive self-adjoint extension of $\mathcal{P}_i$, the relative heat operator is a trace-class operator. In particular this is true for $\mathcal{P}_{abs,i}$ that we recall it is defined as $P^t_{min,i} \circ P_{max,i} + P_{max,i-1} \circ P^t_{min,i-1}$ and for $\mathcal{P}_{rel,i}$ that it is defined as $P^t_{max,i} \circ P_{min,i} + P_{min,i-1} \circ P^t_{max,i-1}$. A well known and basic result of operators theory (see [**62**], prop. 8.8) says that, given an Hilbert space H, the space of trace-class operators is a two sided ideal of $\mathcal{B}(H)$, the space of bounded operators of H, and that the trace doesn't depend on the order of composition. In this way we know that for each $i = 0,...,n$

$$T_i \circ e^{-t\mathcal{P}_{abs/rel,i}} : L^2(M,E_i) \to L^2(M,E_i)$$

are trace-class operator and that $\mathrm{Tr}(T_i \circ e^{-t\mathcal{P}_{abs/rel,i}}) = \mathrm{Tr}(e^{-t\mathcal{P}_{abs/rel,i}} \circ T_i)$[1]. Moreover it is clear that $T_i \circ e^{-t\mathcal{P}_{abs/rel,i}}$ are operators with smooth kernel given by

$$\phi_i \circ k_{abs,i}(t,f(x),y) \text{ for } T_i \circ e^{-t\mathcal{P}_{abs,i}} \tag{5.11}$$

and analogously

$$\phi_i \circ k_{rel,i}(t,f(x),y) \text{ for } T_i \circ e^{-t\mathcal{P}_{rel,i}} \tag{5.12}$$

where $k_{abs/rel,i}(t,x,y)$ are respectively the smooth kernel of $e^{-tP_{abs/rel,i}}$. In both the expressions above ϕ_i acts on the x variable of $k_{abs/rel,i}(t,f(x),y)$ because $k_{abs/rel,i}(t,f(x),y)$ is a section of $f^*E_i \boxtimes E_i^*$ and $\phi_i : f^*E_i \to E_i$ is a morphism of bundle. So the kernels $\phi_i \circ k_{abs/rel,i}(t,f(x),y)$ are well defined and they are smooth sections of $E \boxtimes E^*$.
Now we are in position to state the following theorem which is one of the main results of this section:

[1]This is the reason because we need to require that $f : \overline{M} \to \overline{M}$ is a diffeomorphism. In this way each $T_i : L^2(M,E_i) \to L^2(M,E_i)$ is bounded and so we can conclude that $T_i \circ e^{-t\mathcal{P}_{abs/rel,i}}$ is a trace-class operator

THEOREM 5.11. *Consider an elliptic complex of differential cone operators as in* (5.8) *and let T be a geometric endomorphism as in definition 5.1. Then*

$$L_{2,max}(T) = \sum_{i=0}^{n}(-1)^i \operatorname{Tr}(T_i \circ e^{-t\mathcal{P}_{abs,i}}) \tag{5.13}$$

and analogously

$$L_{2,min}(T) = \sum_{i=0}^{n}(-1)^i \operatorname{Tr}(T_i \circ e^{-t\mathcal{P}_{rel,i}}) \tag{5.14}$$

In particular, in both the equalities, the member on the right hand side does not depend on t.

We need to state some propositions in order to prove the above theorem. We give the proof only for the complex $(L^2(M, E_i), P_{max,i})$. The other one is completely analogous.

LEMMA 5.12. *Consider an abstract Fredholm complex as in* (1.1) *and let T be an endomorphism of this complex, that is $T = (T_0, ..., T_n)$, for each $i = 0, ..., n$ $T_i : H_i \to H_i$ is bounded and $D_i \circ T_i = T_{i+1} \circ D_i$ on $\mathcal{D}(D_i)$. Let $\pi_i : H_i \to \mathcal{H}_i(H_*, D_*)$ be the orthogonal projection induced by the Kodaira decomposition of proposition 1.2. Then for each $i = 0, .., n$ we have*

$$\operatorname{Tr}(\pi_i \circ T_i : \mathcal{H}^i(H_*, D_*) \to \mathcal{H}^i(H_*, D_*)) = \operatorname{Tr}(T_i^* : H^i(H_*, D_*) \to H^i(H_*, D_*))$$

PROOF. Let $\gamma : \mathcal{H}^i(H_*, D_*) \to H^i(H_*, D_*)$ the isomorphism of (1.6). Then it is clear that T_i^*, that is the endomorphism of $H^i(H_*, D_*)$ induced by T_i, satisfies $T_i^* = \gamma \circ \pi_i \circ T_i \circ \gamma^{-1}$. Now from this it follows immediately that $\operatorname{Tr}(\pi_i \circ T_i : \mathcal{H}^i(H_*, D_*) \to \mathcal{H}^i(H_*, D_*)) = \operatorname{Tr}(T_i^* : H^i(H_*, D_*) \to H^i(H_*, D_*))$. □

LEMMA 5.13. *We have the following properties.*

1. *For each $i = 0, ..., n$ the operators $\mathcal{P}_{abs,i}$ have the same non zero eigenvalues.*
2. *Let $E_i(\lambda)$ be the eigenspace relative to $\mathcal{P}_{abs,i}$ and the eigenvalue λ. Then $E_i(\lambda)$ is finite dimensional and made of smooth eigensections.*
3. *Finally, for each eigenvalue $\lambda \neq 0$, consider the following complex:*

$$...... \xrightarrow{P^{\lambda}_{max,i-1}} E_i(\lambda) \xrightarrow{P^{\lambda}_{max,i}} E_{i+1}(\lambda) \xrightarrow{P^{\lambda}_{max,i+1}} E_{i+2}(\lambda) \xrightarrow{P^{\lambda}_{max,i+2}} ... \tag{5.15}$$

where $P^{\lambda}_{max,i} := P_{max,i}|_{E_i(\lambda)}$ Then it is an acyclic complex.

PROOF. Let $\lambda \neq 0$ an eigenvalue of $\mathcal{P}_{abs,i}$ and let $s \in \mathcal{D}(\mathcal{P}_{abs,i})$ such that $\mathcal{P}_{abs,i}(s) = \lambda s$. Consider $P_{max,i}(s)$. Then $P_{max,i}(s) \in \mathcal{D}(\mathcal{P}_{abs,i+1})$ if and only if $P^t_{min,i}(P_{max,i}(s)) \in \mathcal{D}(P_{max,i})$. Clearly $P^t_{min,i}(P_{max,i}(s)) \in \mathcal{D}(P_{max,i})$ if and only if $(P^t_{min,i}(P_{max,i}(s)) + P_{max,i-1}(P^t_{min,i-1}(s))) \in \mathcal{D}(P_{max,i})$. But this last condition is satisfied because $P^t_{min,i}(P_{max,i}(s)) + P_{max,i-1}(P^t_{min,i-1}(s)) = \mathcal{P}_{abs,i}(s) = \lambda s$ and this implies that $P_{max,i}(s) \in \mathcal{D}(\mathcal{P}_{abs,i+1})$ and that $\mathcal{P}_{abs,i+1}(P_{max,i}(s)) = \lambda P_{max,i}(s)$. In the same way, if $s \in \mathcal{D}(\mathcal{P}_{abs,i+1})$ satisfies $\mathcal{P}_{abs,i+1}(s) = \lambda s$, then $P^t_{min,i}(s) \in \mathcal{D}(\mathcal{P}_{abs,i})$ and $\mathcal{P}_{abs,i}(P^t_{min,i}(s)) = \lambda P^t_{min,i}(s)$. Therefore we can conclude that for each $i = 0, ..., n$ the operators $\mathcal{P}_{abs,i}$ and $\mathcal{P}_{abs,i+1}$have the same non zero eigenvalues.

Now consider the eigenspaces $E_i(\lambda)$. That is finite dimensional for each $\lambda \neq 0$ follows by the fact that $e^{-t\mathcal{P}_{abs,i}}$ is a trace-class operator while that it is finite dimensional for $\lambda = 0$ follows by the fact that $\mathcal{P}_{abs,i}$ is a Fredholm operator on its domain endowed with the graph norm. Moreover elliptic regularity tells us that $E_i(\lambda)$ is made of smooth eigensections.
Finally consider

$$...... \stackrel{P^{\lambda}_{max,i-1}}{\rightarrow} E_i(\lambda) \stackrel{P^{\lambda}_{max,i}}{\rightarrow} E_{i+1}(\lambda) \stackrel{P^{\lambda}_{max,i+1}}{\rightarrow} E_{i+2}(\lambda) \stackrel{P^{\lambda}_{max,i+2}}{\rightarrow} ... \tag{5.16}$$

where $P^{\lambda}_{max,i} := P_{max,i}|_{E_i(\lambda)}$.
Let $s \in Ker(P_{max,i})$. Then $\mathcal{P}_{abs,i}(s) = \lambda s = P_{max,i-1}(P^t_{min}(s))$. Therefore $s \in ran(P_{max,i-1})$ and this implies that (5.16) is a long exact sequences, or in other words, it is an acyclic complex. □

Now we state the last result we need to prove theorem 5.11. We take it from [**3**].

Lemma 5.14. *Consider a complex of finite dimensional vector space*

$$0 \rightarrow V_0 \stackrel{f_0}{\rightarrow} ... \stackrel{f_{i-1}}{\rightarrow} V_i \stackrel{f_i}{\rightarrow} V_{i+1} \stackrel{f_{,i+1}}{\rightarrow} V_{i+2} \stackrel{f_{i+2}}{\rightarrow} ... \stackrel{f_{n-1}}{\rightarrow} V_n \stackrel{f_n}{\rightarrow} 0. \tag{5.17}$$

and for each i let $G_i : V_i \rightarrow V_i$ an endomorphism such that $f_i \circ G_i = G_{i+1} \circ f_i$. Then

$$\sum_{i=0}^{n} (-1)^i \operatorname{Tr}(G_i) = \sum_{i=0}^{n} (-1)^i \operatorname{Tr}(G_i^*)$$

where G_i^* is the endomorphism of the $i-th$ cohomology group of the complex (5.17) *induced by G_i.*

Proof. See [**3**]. □

Proof. (of theorem 5.11). As said above we give the proof only for (5.13). The proof for (5.14) is completely analogous. Consider the heat operator $e^{-t\mathcal{P}_{abs,i}} : L^2(M, E_i) \rightarrow L^2(M, E_i)$. By the third point of theorem 4.8 it follows that there exists an Hilbert base of $L^2(M, E_i)$, $\{\phi_j\}_{j\in\mathbb{N}}$, made of smooth eigensections of $\mathcal{P}_{abs,i}$, in such way the smooth kernel of $e^{-t\mathcal{P}_{abs,i}}$ satisfies $k(t, x, y) = \sum_j e^{-t\lambda_j}\phi_j(x) \boxtimes \phi_j^*(y)$. Moreover, by the fact that $T_i : L^2(M, E_i) \rightarrow L^2(M, E_i)$ is bounded, we know that $T_i \circ e^{-t\mathcal{P}_{abs,i}}$ and $e^{-t\mathcal{P}_{abs,i}} \circ T_i$ are trace class and that $\operatorname{Tr}(T_i \circ e^{-t\mathcal{P}_{abs,i}}) = \operatorname{Tr}(e^{-t\mathcal{P}_{abs,i}} \circ T_i)$. Now, if we label $\pi(i, \lambda_j)$ the orthogonal projection $\pi(i, \lambda_j) : L^2(M, E_i) \rightarrow E_i(\lambda_j)$, then we can write $e^{-t\mathcal{P}_{abs,i}} = \sum_j e^{-t\lambda_j}\pi(i, \lambda_j)$ and therefore $e^{-t\mathcal{P}_{abs,i}} \circ T_i = (\sum_j e^{-t\lambda_j}\pi(i, \lambda_j)) \circ T_i = \sum_j e^{-t\lambda_j}(\pi(i, \lambda_j) \circ T_i)$. In this way we get

$$\operatorname{Tr}(T_i \circ e^{-t\mathcal{P}_{abs,i}}) = \operatorname{Tr}(e^{-t\mathcal{P}_{abs,i}} \circ T_i) = \sum_j e^{-t\lambda_j} \operatorname{Tr}((\pi(i, \lambda_j) \circ T_i)). \tag{5.18}$$

Consider $\sum_{i=0}^{n}(-1)^i \operatorname{Tr}(T_i \circ e^{-t\mathcal{P}_{abs,i}})$. Then $\sum_{i=0}^{n}(-1)^i \operatorname{Tr}(T_i \circ e^{-t\mathcal{P}_{abs,i}}) =$

$$= \sum_{i=0}^{n}(-1)^i \sum_j e^{-t\lambda_j} \operatorname{Tr}((\pi(i, \lambda_j) \circ T_i)) = \sum_j e^{-t\lambda_j} \sum_{i=0}^{n}(-1)^i \operatorname{Tr}((\pi(i, \lambda_j) \circ T_i)). \tag{5.19}$$

Now examine carefully this last expression. Both $\pi(i,\lambda_j)\circ T_i : L^2(M,E_i) \to E_i(\lambda_j)$ and $\pi(i,\lambda_j) : L^2(M,E_i) \to E_i(\lambda_j)$ are trace-class operators. This implies that $\mathrm{Tr}(\pi(i,\lambda_j)\circ T_i) = \mathrm{Tr}(\pi(i,\lambda_j)\circ \pi(i,\lambda_j)\circ T_i) = \mathrm{Tr}(\pi(i,\lambda_j)\circ T_i \circ \pi(i,\lambda_j))$ and this last one is equal to the trace of $\pi(i,\lambda_j)\circ T_i : E_i(\lambda_j) \to E_i(\lambda_j)$. But if we take the following complex for $\lambda_j \neq 0$

$$......\overset{P^{\lambda}_{max,i-1}}{\rightarrow} E_i(\lambda_j) \overset{P^{\lambda}_{max,i}}{\rightarrow} E_{i+1}(\lambda_j) \overset{P^{\lambda}_{max,i+1}}{\rightarrow} E_{i+2}(\lambda_j) \overset{P^{\lambda}_{max,i+2}}{\rightarrow} ... \tag{5.20}$$

we know that (5.20) is an acyclic complex. Moreover it is immediate to check that $\pi(i,\lambda_j)\circ T_i$ is an endomorphism of (5.20) and therefore, applying lemma 5.17, we can conclude that $\sum_{i=0}^{n}(-1)^i \mathrm{Tr}(\pi(i,\lambda_j)\circ T_i) = 0$ for $\lambda_j \neq 0$. This leads to a relevant simplification of (5.19):

$$\sum_{i=0}^{n}(-1)^i \mathrm{Tr}(T_i e^{-t\mathcal{P}_{abs,i}}) = \sum_j e^{-t\lambda_j}\sum_{i=0}^{n}(-1)^i \mathrm{Tr}(\pi(i,\lambda_j)\circ T_i) = \tag{5.21}$$

$$\sum_{i=0}^{n}(-1)^i \mathrm{Tr}(\pi(i,0)\circ T_i).$$

Finally, using lemma 5.12, it follows that $\mathrm{Tr}(\pi(i,0)\circ T_i) = \mathrm{Tr}(T_i^*)$ and therefore the theorem is proved. □

As an immediate consequence of theorem 5.11 we have the following corollary

COROLLARY 5.15. *In the same assumptions of theorem 5.11 then*

$$L_{2,max}(T) = \lim_{t\to 0}\sum_{i=0}^{n}(-1)^i \mathrm{Tr}(T_i \circ e^{-t\mathcal{P}_{abs,i}}) \tag{5.22}$$

and analogously

$$L_{2,min}(T) = \lim_{t\to 0}\sum_{i=0}^{n}(-1)^i \mathrm{Tr}(T_i \circ e^{-t\mathcal{P}_{rel,i}}) \tag{5.23}$$

Before to go ahead we add some comments to theorem 5.11.

REMARK 5.1. In the statement of theorem 5.11 we assume that the endomorphism T satisfies definition 5.1. But from the proof it is clear that the particular structure of the endomorphism, that is $T_i = \phi_i \circ f^*$ doesn't play any role. It is just a sufficient condition to assure that each T_i induces a bounded map acting on $L^2(M,E_i)$ and that T is an endomorphism of $(L^2(M,E_i), P_{max/min,i})$. Therefore if we have a $n-$ tuple of map $T = (T_1, ..., T_n)$ such that, for each $i = 0, ..., n$, $T_i : L^2(M,E_i) \to L^2(M,E_i)$ is bounded and $T_{i+1}\circ P_{max/min,i} = P_{max/min,i}\circ T_i$ on $\mathcal{D}(P_{max/min,i})$ then we can state and prove theorem 5.11 in the same way.

REMARK 5.2. We stated theorem 5.11 in the case of an elliptic complex of differential cone operators over a compact manifold with conical singularities. This is because, using the result coming from the theory of elliptic differential cone operators, we know that $(L^2(M,E_i), P_{max/min,i})$ are Fredholm complexes and that $e^{-t\mathcal{P}_{abs/rel,i}}$ are trace-class operators. Therefore it is possible to define maximal and minimal L^2-Lefschetz numbers and to prove theorem 5.11. A priori it is not possible to do the same for an arbitrary elliptic complex of differential operators over a (possible incomplete) riemannian manifold (M,g). But it is clear that if we know that the maximal and the minimal extension of our complex are Fredholm complexes and that for each i the heat operator constructed from the i-th laplacian associated to the maximal/minimal complex is a trace-class operator, then it is possible to state and prove in the same way formulas (5.13) and (5.14) for the L^2-Lefschetz numbers associated to the maximal and minimal extension of our complex.

We conclude the section with the following theorems:

THEOREM 5.16. *Let X be a compact manifold with conical singularities of dimension $m+1$ and let g be a conic metric on $reg(X)=M$. Consider an elliptic complex of differential cone operators as in* (5.8) *and let $T=\phi\circ f^*$ be a geometric endomorphism of* (5.8) *as in definition 5.1. Finally suppose that f has only simple fixed points. Then we have:*

$$L_{2,max/min}(T)=\lim_{t\to 0}(\sum_{q\in Fix(f)}\sum_{i=0}^{n}(-1)^i\int_{U_q}\operatorname{tr}(T\circ e^{-t\mathcal{P}_{abs/rel,i}})dvol_g) \quad (5.24)$$

where U_q is an open neighborhood of $q\in Fix(f)$.

PROOF. We know, by the assumptions, that f has only simple fixed points. For each of these point, that we label q, let U_q be an open neighborhood of q. Then, using again corollary 5.15, we know that $L_{2,max/min}(T)=\lim_{t\to 0}\sum_i(-1)^i\int_M\operatorname{tr}(T_i\circ e^{-t\mathcal{P}_{abs/rel,i}})$. Obviously we can break the member on the right as

$$\sum_{q\in Fix(f)}\sum_{i=0}^{n}(-1)^i\int_{U_q}\operatorname{tr}(T_i\circ e^{-t\mathcal{P}_{abs/rel,i}})dvol_g+$$

$$+\sum_{i=0}^{n}(-1)^i\int_{V}\operatorname{tr}(T_i\circ e^{-t\mathcal{P}_{abs/rel,i}})dvol_g$$

where $V=M-\cup_{q\in Fix(f)}U_q$. Clearly, in the term on the left we mean the regular part of U_q when $q\in Fix(f)\cap sing(X)$. Now, as remarked previously, we know that $f(q)=q$ for each $q\in sing(X)$. This implies $\{(f(q),q):\ q\in V\}$ is a compact subset of $M\times M$ disjoint from Δ_M. So we can use the second property of theorem 4.14 to conclude that

$$\lim_{t\to 0}\int_V\operatorname{tr}(\phi_i\circ e^{-t\mathcal{P}_{abs/rel,i}}(f(q),q))dvol_g=$$

$$=\int_V\lim_{t\to 0}\operatorname{tr}(\phi_i\circ e^{-t\mathcal{P}_{abs/rel,i}}(f(q),q))dvol_g=0.$$

This complete the proof. □

The second point in the above theorem suggests to break the Lefschetz numbers as a contribution of two terms, that is

$$L_{2,max/min}(T)=\mathcal{L}_{max/min}(T,\mathcal{R})+\mathcal{L}_{max/min}(T,\mathcal{S}) \quad (5.25)$$

where $\mathcal{L}_{max/min}(T,\mathcal{R})$ is the contribution given by the simple fixed point lying in $reg(X)$, that is

$$\mathcal{L}_{max/min}(T,\mathcal{R}) = \lim_{t\to 0}(\sum_{q\in Fix(f)\cap reg(X)}\sum_{i=0}^{n}(-1)^i\int_{U_q}\operatorname{tr}(T_i\circ e^{-t\mathcal{P}_{abs/rel,i}})dvol_g)$$

and analogously $\mathcal{L}_{max/min}(T,\mathcal{S})$ is the contribution given by the simple fixed point lying in $Fix(f)\cap sing(X)$, that is

$$\mathcal{L}_{max/min}(T,\mathcal{S}) = \lim_{t\to 0}(\sum_{q\in Fix(f)\cap sing(X)}\sum_{i=0}^{n}(-1)^i\int_{U_q}\operatorname{tr}(T_i\circ e^{-t\mathcal{P}_{abs/rel,i}})dvol_g).$$

THEOREM 5.17. *In the hypothesis of the previous theorem, suppose furthermore that for each $i = 0,...,n$*

$$P_i^t\circ P_i + P_{i-1}\circ P_{i-1}^t : C_c^\infty(M,E_i)\to C_c^\infty(M,E_i)$$

is a generalized Laplacian (see definition 4.13). Then we get :

$$L_{2,max}(T) = \sum_{q\in Fix(f)\cap M}\sum_{i=0}^{n}\frac{(-1)^i\operatorname{Tr}(\phi_i)}{|det(Id-d_qf)|} + \mathcal{L}_{2,max}(T,\mathcal{S}).$$

Analogously for $L_{2,min}(T)$ we have

$$L_{2,min}(T) = \sum_{q\in Fix(f)\cap M}\sum_{i=0}^{n}\frac{(-1)^i\operatorname{Tr}(\phi_i)}{|det(Id-d_qf)|} + \mathcal{L}_{2,min}(T,\mathcal{S}).$$

PROOF. By theorem 5.16, we know that the L^2–Lefschetz numbers depend only on the simple fixed point of f and that we can localize their contribution, that is,

$$L_{2,max/min}(T) = \lim_{t\to 0}(\sum_{q\in Fix(f)}\sum_{i=0}^{n}(-1)^i\int_{U_q}\operatorname{tr}(T\circ e^{-t\mathcal{P}_{abs/rel,i}})dvol_g)$$

where U_q is an arbitrary open neighborhood of q. Now if $q\in reg(X)\cap Fix(f)$, by the assumptions, we can use the local asymptotic expansion recalled in the last point of theorem 4.14. Now, to get the conclusion, the proof is exactly the same as in the closed case; see for example [**9**] theorem 6.6 or [**62**] theorem theorem 10.12. □

We have the following immediate corollary:

COROLLARY 5.18. *In the same hypothesis of theorem 5.17; Then:*

1. $\mathcal{L}_{max}(T,\mathcal{R}) = \mathcal{L}_{min}(T,\mathcal{R})$ *that is, the simple fixed points in M give the same contributions for both the Lefschetz numbers $L_{2,max/min}(T)$.*
2. $\mathcal{L}_{max/min}(T,\mathcal{S})$ *do not depend on the particular conic metric fixed on M and on the metrics $\rho_0,...,\rho_n$ respectively fixed on $E_0,...,E_n$.*

PROOF. The first assertion is an immediate consequence of the second point of theorem 5.17. For the second statement, by proposition 5.10, we know that $L_{2,max/min}(T)$ are independent on the conic metric we put over M and on the metric $\rho_0,...,\rho_n$ respectively on $E_0,...,E_n$. Again, by the second point of theorem 5.17, we know that also $\mathcal{L}_{max/min}(T,\mathcal{R})$ are independent from the conic metrics and on the metric $\rho_0,...,\rho_n$ respectively on $E_0,...,E_n$.

Therefore the same conclusion holds for $\mathcal{L}_{max/min}(T, \mathcal{S})$. The corollary is proved. $\square$

CHAPTER 6

The contribution of the singular points

In this chapter the main formulas for the L^2-Lefschetz numbers are proved. In the fist section, under suitable hypothesis, we describe the contribution of the singular points using a sort of "modified zeta function" which involves in its definition the action of the endomorphism T. Moreover a geometric interpretation for the contribution given by the singular points of attractive or repulsive type is provided. Finally the second section concerns the case of a short elliptic complex of differential operators.

1. First case

The aim of this section is to give, in some particular cases, an explicit formula for $\mathcal{L}_{max/min}(T, \mathcal{S})$, that is for the contribution given by the singular points to the Lefschetz numbers $L_{2,max/min}(T)$.
Consider the same situation described in theorem 5.16. Suppose moreover that the following properties hold:

(1) For each $q \in sing(X)$ there exists an isomorphism $\chi_q : U_q \to C_2(L_q)$ such that on $[0, 2) \times L_q$, using (4.5), each operator A_k is **constant** in x and, using the decomposition (5.5), the map f takes the form:
$$f = (rA(p), B(p)). \tag{6.1}$$
(2) On $reg(C_2(L_q))$, using again the isomorphism $\chi_q : U_q \to C_2(L_q)$, the conic metric g satisfies $g = dr^2 + r^2h$ with h that does not depend on r and each metric ρ_i on E_i does not depend on r in a collar neighborhood of $\partial\overline{M}$.

Before stating the next theorem we recall a definition from [**51**].

Definition 6.1. Let $U_t : L^2(reg(C(N)), E) \to L^2(reg(C(N)), E)$ be the isometry as defined in the proof of lemma (4.17), that is $U_t(\gamma) = t^{\frac{n+1}{2}}\gamma(tr, p)$. Consider an operator $P_0 \in \mathrm{Diff}_0^{\mu,\nu}(reg(C(N)))$ such that, using the expression (4.5), each A_k is constant in x. Then a closed extension P of P_0 is said scalable if $U_t^* P U_t = t^\nu P$.

Lemma 6.2. *Given $P_0 \in \mathrm{Diff}_0^{\mu,\nu}(reg(C(N)))$ as in definition 6.1 then $P_{0,max}$ and $P_{0,min}$ are always scalable. If we take P_0^t, the formal adjoint of P_0, then also $P_{0,min}^t \circ P_{0,max}$, $P_{0,max}^t \circ P_{0,min}$, $P_{0,min} \circ P_{0,max}^t$ and $P_{0,max} \circ P_{0,min}^t$ are scalable extensions of $P_0^t \circ P_0$ and $P_0 \circ P_0^t$ respectively. Finally, if in a complex we consider $\mathcal{P}_i := P_i^t \circ P_i + P_{i-1} \circ P_{i-1}^t$ (see the statement of theorem 5.17) then also the closed extension $\mathcal{P}_{abs,i}$ and $\mathcal{P}_{rel,i}$ (see* (1.15) *and* (1.16)*) are scalable extensions.*

PROOF. For the first assertion see [**51**] pag. 58. The others assertions are an immediate consequence of the previous one and of the definition of scalable extension. □

Now we are ready to state the following theorem:

THEOREM 6.3. *In the same hypothesis of theorem 5.16. Suppose moreover that the two properties described above definition 6.1 hold. Then we have:*

$$\mathcal{L}_{max/min}(T, \mathcal{S}) = \tag{6.2}$$

$$\sum_{q \in sing(X)} \sum_{i=0}^{n} (-1)^i \frac{1}{2\nu} \int_0^\infty \frac{dx}{x} \int_{L_q} \mathrm{tr}(\phi_i \circ e^{-x\mathcal{P}_{abs/rel,i}}(A(p), B(p), 1, p)) dvol_h.$$

PROOF. Let $q \in sing(X)$. By the hypothesis we know that there exists an open neighborhood U_q and an isomorphism $\chi_q : U_q \to C_2(L_q)$ such that, on $C_2(L_q)$, f takes the form (6.1) and each A_k is constant in x. Using the properties stated in [**51**] pag. 42-43, we get that the limit

$$\lim_{t \to 0} \int_{reg(U_q)} \mathrm{tr}(\phi_i \circ e^{-t\mathcal{P}_{abs/rel,i}}(rA(p), B(p), r, p)) dvol_g$$

is equal to

$$\lim_{t \to 0} \int_{reg(C_2(L_q))} \mathrm{tr}(\phi_i \circ e^{-t\mathcal{P}_{abs/rel,i}}(rA(p), B(p), r, p)) r^m dvol_h dr$$

where, with a little abuse of notation, in the second expression we mean the heat kernel associated to the absolute and relative extension of the operator, induced by $\mathcal{P}_i|_{U_q}$ through χ_q, acting on $C_c^\infty(reg(C_2(L_q)), (\chi_q^{-1})^* E_i)$. So, for each $i = 0, ..., n$, we have to calculate

$$\lim_{t \to 0} \int_{reg(C_2(L_q))} \mathrm{tr}(\phi_i \circ e^{-t\mathcal{P}_{abs/rel,i}}(rA(p), B(p), r, p)) r^m dr dvol_h.$$

Moreover, we assumed that, on $reg(C_2(L_q))$, the conic metric g satisfies $g = dr^2 + r^2 h$ with h that does not depend on r and that each metric ρ_i on E_i does not depend on r in a neighborhood of $\partial \overline{M}$. This implies that, for each $i = 0, ..., n$, the operator $\mathcal{P}_i$ satisfies the assumption at the beginning of the subsection, that is each A_k does not depend on x. Therefore, using lemma 6.2, we get that $\mathcal{P}_{abs/rel,i}$ are scalable extensions of $\mathcal{P}_i$. Now, after these observations, we can go on to calculate

$$\lim_{t \to 0} \int_{reg(C_2(L_q))} \mathrm{tr}(\phi_i \circ e^{-t\mathcal{P}_{abs/rel}}(rA(p), B(p), r, p)) dvol_g.$$

Using lemma 4.17 and the fact that $\mathcal{P}_{abs/rel,i}$ are scalable extensions of $\mathcal{P}_i$ we get

$$\int_{reg(C_2(L_q))} \mathrm{tr}(\phi_i \circ e^{-t\mathcal{P}_{abs/rel,i}}(rA(p), B(p), r, p)) r^m dr dvol_h =$$

$$= \int_0^2 \int_{L_q} \frac{1}{r} \mathrm{tr}(\phi_i \circ e^{-tr^{-2\nu}\mathcal{P}_{abs/rel,i}}(A(p), B(p), 1, p)) dvol_h h dr.$$

Now if we put $\frac{t}{r^{2\nu}} = x$ we get $\frac{-2\nu t dr}{r^{2\nu+1}} = dx$ which implies that $\frac{dr}{r} = \frac{dx}{x} = \frac{-2\nu t dr}{r^{2\nu+1}}\frac{r^{2\nu}}{t}$ and in conclusion $\frac{dr}{r} = \frac{-1}{2\nu}\frac{dx}{x}$. Moreover when r goes to 0 then x goes to ∞ and when r goes to 2 then x goes to $\frac{t}{4}$. So we get

$$\int_0^2 \int_{L_q} \frac{1}{r} \operatorname{tr}(\phi_i \circ e^{-tr^{-2\nu}\mathcal{P}_{abs/rel,i}}(A(p), B(p), 1, p))dvol_h h dr =$$

$$= \frac{1}{2\nu}\int_{t/4}^{\infty} \frac{dx}{x} \int_{L_q} \operatorname{tr}(\phi_i \circ e^{-x\mathcal{P}_{abs/rel,i}}(A(p), B(p), 1, p))dvol_h. \tag{6.3}$$

Therefore to conclude we have to evaluate the limit

$$\lim_{t\to 0} \frac{1}{2\nu}\int_{t/4}^{\infty} \frac{dx}{x} \int_{L_q} \operatorname{tr}(\phi_i \circ e^{-x\mathcal{P}_{abs/rel,i}}(A(p), B(p), 1, p))dvol_h \tag{6.4}$$

To do this consider the term $\int_{L_q} \operatorname{tr}(\phi_i \circ e^{-x\mathcal{P}_{abs/rel,i}}(A(p), B(p), 1, p))dvol_h$. We know, by the hypothesis, that f has only simple fixed points. In particular each $q \in sing(X)$ is a simple fixed point. The conditions described in definition 5.6 together with (6.1) implies that either $A(p) \neq 1$ for all $p \in L_q$ or $B : L_q \to L_q$ has not fixed points. Anyway each of these conditions implies that when p runs over L_q then $\{(A(p), B(p), 1, p)\}$ is a compact subset of $reg(C_2(L_q)) \times reg(C_2(L_q))$ that doesn't intersect the diagonal. Therefore we can use the second property stated in theorem 4.14 to conclude that, when $x \to 0$,

$$\int_{L_q} \operatorname{tr}(\phi_i \circ e^{-x\mathcal{P}_{abs/rel,i}}(A(p), B(p), 1, p))dvol_h = O(x^N) \text{ for each } N > 0. \tag{6.5}$$

In this way we can conclude that the limit (6.4) exists and we have

$$\lim_{t\to 0} \frac{1}{2\nu}\int_{t/4}^{\infty} \frac{dx}{x} \int_{L_q} \operatorname{tr}(\phi_i \circ e^{-x\mathcal{P}_{abs/rel,i}}(A(p), B(p), 1, p))dvol_h =$$

$$= \frac{1}{2\nu}\int_{0}^{\infty} \frac{dx}{x} \int_{L_q} \operatorname{tr}(\phi_i \circ e^{-x\mathcal{P}_{abs/rel,i}}(A(p), B(p), 1, p))dvol_h. \tag{6.6}$$

Finally it is also clear that (6.6) converges because, given a sufficient small $\epsilon > 0$ we have

$$(6.6) = \int_0^{\epsilon} O(x^N)dx + \int_{\epsilon}^{\infty} x^{-1}dx \int_{L_q} \operatorname{tr}(\phi_i \circ e^{-x\mathcal{P}_{abs/rel,i}}(A(p), B(p), 1, p))dvol_h.$$

The first term is clearly finite and the second one is finite because, by (6.3), it is the trace of $T_i \circ e^{-t\mathcal{P}_{abs/rel,i}}$ valued in ϵ and $T_i \circ e^{-t\mathcal{P}_{abs/rel,i}}$ are trace-class. This completes the proof. □

Now, for each $i = 0, .., n$, using again the hypothesis and the notations of theorem 6.3, and assuming still that q is a simple fixed point for f, define the following "modified version" of the classical $\zeta-$function:

$$\zeta_{T_i,q}(\mathcal{P}_{abs/rel,i})(s) := \tag{6.7}$$

$$= \frac{1}{2\nu}\int_0^{\infty} x^{s-1}dx \int_{L_q} \operatorname{tr}(\phi_i \circ e^{-x\mathcal{P}_{abs/rel,i}}(A(p), B(p), 1, p))dvol_h.$$

The definition makes sense for each $s \in \mathbb{C}$ because, as observed in the proof of theorem 6.3, $\{(A(p), B(p), 1, p)\}$ is a compact subset of $reg(X) \times reg(X)$ that is disjoint from the diagonal $\Delta_{reg(X)}$. Therefore we can apply the second point of theorem 4.14 to conclude that, when $x \to 0$,

$$\int_{L_q} \operatorname{tr}(\phi_i \circ e^{-x\mathcal{P}_{abs/rel,i}}(A(p), B(p), 1, p))dvol_h = O(x^N) \text{ for each } N > 0. \tag{6.8}$$

and this implies that $\zeta_{T_i,q}(\mathcal{P}_{abs/rel,i})(s)$ is a holomorphic function over the whole complex plane. The reason behind (6.6) is that if we compare (6.6) with the definitions of the zeta functions for a generalized Laplacian, see for example [**9**] pag. 300, then it natural to think at (6.6) as a sort of zeta function for the operators $\mathcal{P}_{abs/rel,i}$ valued in 0, which takes account of the action of T_i in its definition. In this way, using (6.7), we can reformulate theorem 6.3 in a more concise way:

$$\mathcal{L}_{max/min}(T, \mathcal{S}) = \sum_{q \in sing(X)} \sum_{i=0}^{n} (-1)^i \zeta_{T_i,q}(\mathcal{P}_{abs/rel,i})(0). \tag{6.9}$$

Before to conclude the section we make the following remarks.
In the same hypothesis of theorem 5.16 consider a point $q \in sing(X)$ such that q is an attractive simple fixed point. We recall that over a neighborhood $U_q \cong [0, 2) \times L_q$ of q we can look at f as a map given by $(rA(r, p), B(r, p))$: $[0, 2) \times L_q \to [0, 2) \times L_q$ with A and B smooth up to 0. From definition 5.8 we know that q is attractive if $\lim_{r \to 0} A(r, p) < 1$ for each fixed $p \in L_q$. Clearly this implies that $f(U_q) \subset U_q$. Therefore it follows that, if we consider the complex

$$0 \to C^\infty_c(U_q, E_0|_{U_q}) \xrightarrow{P_0} C^\infty_c(U_q, E_1|_{U_q}) \xrightarrow{P_1} \ldots \xrightarrow{P_{n-1}} C^\infty_c(U_q, E_n|_{U_q}) \xrightarrow{P_n} 0 \tag{6.10}$$

then T is also a geometric endomorphism of (6.10) and, using proposition 5.3, we get that T extends as a bounded endomorphism of the complexes $(L^2(U_q, E_i|_{U_q}), (P|_{U_q})_{max/min,i})$.
Moreover, by the results proved in the first and the second chapter of [**51**], it follows that $(L^2(U_q, E_i|_{U_q}), (P|_{U_q})_{max/min,i})$ are both Fredholm complexes and that, the respective heat operators , $e^{-t(\mathcal{P}|_{U_q})_{abs/rel,i}} : L^2(U_q, E_i|_{U_q}) \to L^2(U_q, E_i|_{U_q})$, are trace-class operators.
Using again the properties stated in [**51**] at pag. 42-43, it follows that for each open neighborhood V_q of q, such that $\overline{V_q}$ is a subset of U_q, we have

$$\lim_{t \to 0} \int_{V_q} \operatorname{tr}(\phi_i \circ e^{-t\mathcal{P}_{abs/rel,i}}(rA(r, p), B(r, p), r, p)dvol_g =$$

$$= \lim_{t \to 0} \int_{V_q} \operatorname{tr}(\phi_i \circ e^{-t(\mathcal{P}|_{U_q})_{abs/rel,i}}(rA(r, p), B(r, p), r, p)dvol_g.$$

Suppose now that we are in the hypothesis of theorem 6.3. By the proof of the same theorem, it follows that for each $0 < b \leq 2$

$$\lim_{t \to 0} \int_0^b \int_{L_q} \operatorname{tr}(\phi_i \circ e^{-t(\mathcal{P}|_{U_q})_{abs/rel,i}})(rA(p), B(p), r, p)r^m dvol_h dr =$$

$$\int_0^{\infty} x^{-1}dx \int_{L_q} \mathrm{tr}(\phi_i \circ e^{-x(\mathcal{P}|_{U_q})_{abs/rel,i}})(A(p), B(p), 1, p)dvol_h$$

that is it does not depend on the particular b we fixed. Therefore we can conclude that

$$\lim_{t\to 0}\int_{U_q} \mathrm{tr}(\phi_i \circ e^{-t\mathcal{P}_{abs/rel,i}}(rA(p), B(p), r, p)dvol_g = \tag{6.11}$$

$$= \lim_{t\to 0}\int_{U_q} \mathrm{tr}(\phi_i \circ e^{-t(\mathcal{P}|_{U_q})_{abs/rel,i}}(rA(p), B(p), r, p)dvol_g.$$

Summarizing we obtained that it makes sense to define, for an attractive simple fixed point, $L_{2,max/min}(T|_{U_q})$ as the L^2-Lefschetz numbers of T acting on the maximal/minimal extension of (6.10) and that, under the hypothesis of theorem 6.3, it satisfies

$$L_{2,max/min}(T|_{U_q}) = \lim_{t\to 0}\sum_{i=0}^{n}(-1)^i \int_{U_q} \mathrm{tr}(\phi_i \circ e^{-t\mathcal{P}_{abs/rel,i}}(rA(p), B(p), r, p)dvol_g. \tag{6.12}$$

Now we proceed making another remark before the conclusion.
As showed in the second section, T_i^*, the adjoint of T_i, has the following form:

$$T_i^* = \theta_i \circ (f^{-1})^* \tag{6.13}$$

where $\theta_i = \tau\phi_i^*$ with τ positive or negative function respectively if f preserves or reverses the orientation.
Moreover, a simple computation, shows that T^* is an endomorphism of the following Fredholm complexes: $(L^2(M, E_i), P^t_{max/min,i})$. By the fact that, if $Q : H \to H$ is a trace-class operator acting on the Hilbert space H then also Q^* is trace-class and $\mathrm{Tr}(Q) = \mathrm{Tr}(Q^*)$, it follows that

$$\mathrm{Tr}(T_i \circ e^{-t\mathcal{P}_{abs/rel,i}}) = \mathrm{Tr}(e^{-t\mathcal{P}_{abs/rel,i}} \circ T_i^*) = \mathrm{Tr}(T_i^* \circ e^{-t\mathcal{P}_{abs/rel,i}}). \tag{6.14}$$

In other words we proved that:

$$L_{2,max/min}(T) = L_{2,min/max}(T^*) \tag{6.15}$$

where T acts on $(L^2(M, E_i), P_{max/min,i})$ and T^* acts on $(L^2(M, E_i), P^t_{min/max,i})$.
A second consequence is the following: consider a point $q \in sing(X)$ such that q is a repulsive simple fixed point. Clearly, by the fact that f on $U_q \cong C_2(L_q)$ takes the form $f = (rA(p), B(p))$ it follows that $f^{-1} = (rG(p), B^{-1}(p))$ where $G = \frac{1}{A\circ B^{-1}}$. The fact that q is repulsive means that $A > 1$. Therefore it follows that q is an **attractive** simple fixed point for T^*.
Finally we are in positions to conclude with the following results:

Corollary 6.4. *In the same hypothesis of theorem 6.3; Suppose moreover that $q \in sing(X)$ is an attractive fixed point. Then*

$$\sum_{i=0}^{n}(-1)^i\zeta_{T_i,q}(\mathcal{P}_{abs/rel,i})(0) = L_{2,max/min}(T|_{U_q}).$$

In particular this tells us that $\sum_{i=0}^{n}(-1)^i\zeta_{T_i,q}(\mathcal{P}_{abs/rel,i})(0)$ has a geometric meaning itself.

PROOF. It follows immediately from theorem 6.3 and (6.12). □

THEOREM 6.5. *In the same hypothesis of theorem 5.17. Suppose moreover that the first property stated at the beginning of the section holds. Then we have:*

$$L_{2,max/min}(T) = \tag{6.16}$$

$$= \sum_{p\in Fix(f)\cap M} \sum_{i=0}^{n} \frac{(-1)^i \operatorname{Tr}(\phi_i)}{|det(Id-d_qf)|} + \sum_{q\in sing(X)} \sum_{i=0}^{n} (-1)^i \zeta_{T_i,q}(\mathcal{P}_{abs/rel,i})(0)$$

where in (6.16) *the contribution given by the singular points is calculated fixing any conic metric g on $reg(X)$ and any metrics $\rho_0,...,\rho_n$ on $E_0,...,E_n$ which satisfy the hypothesis of theorem 6.3.*
Moreover if each point $q \in sing(X)$ is an attractive fixed point we have:

$$L_{2,max/min}(T) = \tag{6.17}$$

$$= \sum_{p\in Fix(f)\cap M} \sum_{i=0}^{n} \frac{(-1)^i \operatorname{Tr}(\theta_i)}{|det(Id-d_q(f^{-1}))|} + \sum_{q\in sing(X)} L_{2,min/max}(T^*|_{U_q})$$

while if each $q \in sing(X)$ is a repulsive fixed point then we have :

$$L_{2,max/min}(T) = \tag{6.18}$$

$$= \sum_{p\in Fix(f)\cap M} \sum_{i=0}^{n} \frac{(-1)^i \operatorname{Tr}(\theta_i)}{|det(Id-d_q(f^{-1}))|} + \sum_{q\in sing(X)} L_{2,min/max}(T^*|_{U_q}).$$

Finally we remark again that, when $\mathcal{P}_i$ is a generalized Laplacian, the contribution given by the singular simplex fixed points, that is

$$\mathcal{L}_{max/min}(T,\mathcal{S}) = \sum_{q\in sing(X)} \sum_{i=0}^{n} (-1)^i \zeta_{T_i,q}(\mathcal{P}_{abs/rel,i})(0)$$

does not depend on the particular conic metric that we fix on $reg(X)$ and on the metrics $\rho_0,...,\rho_n$ that we fix on $E_0,...,E_n$.

PROOF. As showed in corollary 5.18, when each $\mathcal{P}_i$ is a generalized Laplacian, then $L_{2,max/min}(T)$, $\mathcal{L}(T,\mathcal{R})$ and $\mathcal{L}_{max/min}(T,\mathcal{S})$ do not depend on the conic metric we fix on $reg(X)$ and do not depend on the metrics we fix $\rho_0,...,\rho_n$ on $E_0,...,E_n$. Therefore, without loss of generality, we can assume that for each $q \in sing(X)$, using the isomorphism $\chi_q : U_q \to C_2(L_q)$ of (6.1), the conic metric g satisfies $g = dr^2 + r^2h$ with h that does not depend on r and that each metric ρ_i on E_i does not depend on r in a neighborhood of $\partial\overline{M}$. In this way we are in position to apply theorem 6.3 and so (6.16) follows combining the theorems 5.17 and 6.3. Moreover this tell us that, in (6.16), the contribution of the singular points is well defined and does not depend on the metrics $g, \rho_0, ..., \rho_n$ (satisfying the assumptions of theorem 6.3) used to calculate it. The second assertion follows from corollary 6.4 while the last assertion follows from (6.13) and (6.15). □

REMARK 6.1. We stress on the fact that, unlike theorem 6.3, in theorem 6.5 there are not assumptions about the conic metric g on $reg(X)$ and about the metrics $\rho_0,...,\rho_n$ on $E_0,...,E_n$ respectively.

Finally we conclude the section with the following comment.
The condition that we required at the beginning of the subsection for each operator P_i, that each A_k does not depend from x, might appear as to be too strong at first right. Obviously this is indeed a strong assumption but it is at the same time quite natural because the most natural complex arising in differential geometry, the de Rham complex, satisfies this assumption.
The requirement (6.1), about the behavior of f near the point p, is justified by the idea to evaluate $\mathcal{L}_{max/min}(T,\mathcal{S})$ using the scaling invariance of the heat kernel, see lemma 4.17. In fact if $f = (rA(r,p), B(r,p))$ then, after the scaling invariance is used, we get in our expression the term $\mathrm{tr}(\phi_i \circ e^{-tr^{-2\nu}\mathcal{P}_{abs/rel,i}}(A(r,p), B(r,p), 1, p))$. To have that this last expression make sense we need that $(A(r,p), B(r,p), 1, p) \in \mathcal{G}(f)$ and therefore this leads us to assume (6.1).

2. The case of a short complex

The aim of this subsection is to give a formula for the L^2-Lefschetz numbers in the particular case of a short complex, that is is an elliptic conic operator $P : C_c^\infty(M,E) \to C_c^\infty(M,E)$, using the result stated in proposition 4.18. To do this we start describing our geometric situation which is the same of the previous results with some additional requirements: let X be a compact and oriented manifold with conical singularities of dimension $m+1$. Let M be its regular part and let $\overline{M}$ be the compact manifold with boundary which desingularize X. Endow M with a conic metric g. Let (E,ρ) be a vector bundle endowed with a metric (riemannian or hermitian) according if E is complex or real. Let $(\overline{E},\rho)$ be the extension of (E,ρ) over $\overline{M}$. Let $T = (T_1, T_2)$ be a geometric endomorphism where, as we already know, $T_i = \phi_i \circ f^*$ with $f : \overline{M} \to \overline{M}$ is a diffeomorphism as described in definition 5.1 and $\phi : f^*E \to E$ a bundle homorphism. Suppose that $Fix(f)$ is made only by simple fixed points. Finally, suppose that in each neighborhood $U_q \cong C_2(L_q)$ of $q \in sing(X)$ the operator P take the form

$$P = \frac{n}{2r} + \frac{\partial}{\partial r} + \frac{1}{r}S \tag{6.19}$$

where $S \in \mathrm{Diff}^1(N, E_N)$ is an elliptic operator and the map f take the form

$$f = (rc, B(p)),\ \ c \neq 1 \tag{6.20}$$

where $c > 0$ and depends only on q.

Theorem 6.6. *In the same hypothesis of theorem 6.3; suppose moreover that the properties described above hold. Then for each $q \in sing(X)$ we have:*

$$\zeta_{T_0,q}(P^t_{max} \circ P_{min})(0) = \frac{c^{\frac{1-n}{2}}}{4}\int_0^\infty e^{-\frac{u(c^2+1)}{4}} \sum_{\lambda \in \mathrm{spec}\, S} I_{p^+(\lambda)}(\frac{uc}{2}) du\, \mathrm{Tr}(\tilde{\Phi}_{0,\lambda,q}) \tag{6.21}$$

and analogously

$$\zeta_{T_1,q}(P_{min} \circ P^t_{max})(0) = \frac{c^{\frac{1-n}{2}}}{4}\int_0^\infty e^{-\frac{u(c^2+1)}{4}} \sum_{\lambda \in \mathrm{spec}\, S} I_{p^-(\lambda)}(\frac{uc}{2}) du\, \mathrm{Tr}(\tilde{\Phi}_{1,\lambda,q}) \tag{6.22}$$

where

$$\mathrm{Tr}(\tilde{\Phi}_{j,\lambda,q}) = \int_{L_q} \mathrm{tr}(\phi_j \Phi_{\lambda,q}(B(p),p))dvol_h, \ j = 0, 1.$$

PROOF. We give the proof only for (6.21) because for (6.22) is completely analogous. To prove the assertion we have to calculate

$$\lim_{t\to 0} \int_{reg(C_2(L_q))} \mathrm{tr}(T_0 \circ e^{-P^t_{max}\circ P_{min}})dvol_g.$$

By the assumptions we are in position to use the second statement of proposition 4.18 and therefore it is clear that the smooth kernel of $T_0 \circ e^{-P^t_{max}\circ P_{min}}$ is

$$\sum_{\lambda\in\mathrm{spec}\, S} \frac{1}{2t}(crs)^{\frac{1-n}{2}} I_{p^+(\lambda)}(\frac{crs}{2t})e^{-\frac{c^2r^2+s^2}{4t}} \phi_0\Phi_\lambda(B(p),q) \qquad (6.23)$$

In this way we have to calculate

$$\lim_{t\to 0} \int_0^2 \sum_{\lambda\in\mathrm{spec}\, S} \frac{1}{2t}(cr^2)^{\frac{1-n}{2}} I_{p^+(\lambda)}(\frac{cr^2}{2t})e^{-\frac{r^2(c^2+1)}{4t}} r^m dr \int_{L_q} \mathrm{tr}(\phi_0\Phi_\lambda(B(p),q))dvol_h.$$

Clearly $\int_{L_q} \mathrm{tr}(\phi_0\Phi_\lambda(B(p),q))dvol_h$ does not depend on t and so, if we label it $\mathrm{Tr}(\tilde{\Phi}_{0,\lambda,q})$, our task now is to calculate

$$\lim_{t\to 0} \int_0^2 \sum_{\lambda\in\mathrm{spec}\, S} \frac{1}{2t}(cr^2)^{\frac{1-n}{2}} I_{p^+(\lambda)}(\frac{cr^2}{2t})e^{-\frac{r^2(c^2+1)}{4t}} r^m dr.$$

To do this put $\frac{r^2}{t} = u$. Then $rdr = \frac{tdu}{2}$. Moreover when r goes to 2 u goes to $\frac{4}{t}$ while when r goes to 0 u goes to zero. So, applying this change of variable, we get

$$\lim_{t\to 0} \frac{c^{\frac{1-n}{2}}}{4} \int_0^{\frac{4}{t}} e^{-\frac{u(c^2+1)}{4}} \sum_{\lambda\in\mathrm{spec}\, S} I_{p^+(\lambda)}(\frac{uc}{2})du.$$

Now, by the asymptotic behavior of the integrand, we know that this limit exists and is equal to

$$\frac{c^{\frac{1-n}{2}}}{4} \int_0^{\infty} e^{-\frac{u(c^2+1)}{4}} \sum_{\lambda\in\mathrm{spec}\, S} I_{p^+(\lambda)}(\frac{uc}{2})du.$$

So we proved the statement. □

From theorem 6.6 we have the following immediate corollary:

COROLLARY 6.7. *In the same hypothesis of theorem 6.6 but without any assumptions about the conic metric g on $reg(X)$ and the metric ρ on E. Suppose moreover that $P^t \circ P : C^\infty_c(M,E) \to C^\infty_c(M,E)$ is a generalized Laplacian. Then we have the following formula:*

$$L_{2,min}(T) = \sum_{q\in M\cap Fix(f)} \sum_{j=0}^{1} \frac{(-1)^j\,\mathrm{Tr}(\phi_j)}{|det(Id - d_q f)|} + \qquad (6.24)$$

$$+ \sum_{q\in sing(X)} \frac{c^{\frac{1-n}{2}}}{4} \int_0^{\infty} e^{-\frac{u(c^2+1)}{4}} \sum_{\lambda\in\mathrm{spec}\, S} I_{p^+(\lambda)}(\frac{uc}{2})du\, \mathrm{Tr}(\tilde{\Phi}_{0,\lambda,q}) +$$

$$-\sum_{q\in sing(X)}\int_0^\infty e^{-\frac{u(c^2+1)}{4}}\sum_{\lambda\in\operatorname{spec} S} I_{p^-(\lambda)}(\frac{uc}{2})du\,\operatorname{Tr}(\tilde{\Phi}_{1,\lambda,q})$$

where the contribution of the singular points is calculated fixing any conic metric g on $reg(X)$ and any metric ρ on E which satisfy the assumptions of theorem 6.6.

Proof. As observed in the proof of theorem 6.5, by the fact that $P^t \circ P$ is a generalized Laplacian, it follows that $\mathcal{L}(T, \mathcal{S})$ does not depend on the conic metric we fix on $reg(X)$ and does not depend on the metric ρ we fix on E. Therefore, without loss of generality, we can assume that for each $q \in sing(X)$, using the isomorphism $\chi_q : U_q \to C_2(L_q)$ of (6.1), the conic metric g satisfies $g = dr^2 + r^2 h$ with h that does not depend on r and that each metric ρ_i on E_i does not depend on r in a neighborhood of $\partial\overline{M}$. In this way we are in position to apply theorem 6.6 and therefore (6.24) follows. $\square$

CHAPTER 7

A thorough analysis of the de Rham case

In this chapter we deal with the L^2-de Rham complexes over a compact manifold with conical singularities endowed with a conic metric over its regular part. The first section contains the applications of the previous results. In particular we give some formulas in the case that each $q \in sing(X)$ is an attractive fixed point. Finally the last section contains some consequences arising from Cheeger's work on the heat kernel.

1. Applications of the previous results

As remarked previously, theorems 5.17 and 6.5, corollary 6.4 and in particular (6.16) hold for the Hilbert complexes $(L^2\Omega^i(M,g), d_{max/min,i})$. More explicitly, we have the following result:

THEOREM 7.1. *Let X be a compact and oriented manifold with isolated conical singularities and of dimension $m+1$. Let g be a conic metric over its regular part $reg(X)$. Let $f : X \to X$ a map induced by a diffeomorphism $f : \overline{M} \to \overline{M}$ which fixes each connected component of ∂M. Consider $T := (df)^* \circ f^*$, the natural endomorphism of the de Rham complex induced by f. Finally suppose that f has only simple fixed points. Then we have:*

$$L_{2,max/min}(T) = \sum_{q \in Fix(f) \cap reg(X)} \operatorname{sgn} det(Id - d_q f) + \mathcal{L}_{max/min}(T, \mathcal{S}). \quad (7.1)$$

If in a neighborhood of each simple fixed point q f satisfies the condition described in (6.1), *then we have:* $L_{2,max/min}(T) =$

$$= \sum_{q \in Fix(f) \cap reg(X)} \operatorname{sgn} det(Id - d_q f) + \sum_{q \in sing(X)} \sum_{i=0}^{m+1} (-1)^i \zeta_{T_i,q}(\Delta_{abs/rel,i})(0) \quad (7.2)$$

where in (7.2) *the contribution of the singular points is calculated using any conic metric g on $reg(x)$ such that, again through the isomorphism $\chi_q : U_q \to C_2(L_q)$ of* (6.1), *g takes the form $dr^2 + r^2 h$ and h does not depend on r.*
In particular if each $q \in sing(X)$ is an attractive simple fixed point then we have:

$$L_{2,max/min}(T) = \sum_{q \in Fix(f) \cap reg(X)} \operatorname{sgn} det(Id - d_q f) + \sum_{q \in sing(X)} L_{2,max/min}(T|_{U_q}). \quad (7.3)$$

while if each $q \in sing(X)$ is a repulsive simple fixed point then we have:

$$L_{2,max/min}(T) = \quad (7.4)$$

$$= \sum_{q \in Fix(f) \cap reg(X)} \operatorname{sgn} det(Id - d_q(f^{-1})) + \sum_{q \in sing(X)} L_{2,min/man}(T^*|_{U_q}).$$

Moreover in (7.2) *the member on the right, that is* $\mathcal{L}_{max/min}(T, \mathcal{S})$*, does not depend on the particular conic metric that we fix on* $reg(X)$*.*

Proof. (7.1) follows immediately from theorem 5.17. In particular the expression for $\mathcal{L}_{max/min}(T, \mathcal{R})$ follows by a standard argument of linear algebra; see for example [**4**] or [**62**]. (7.2) follows as in the proof of theorem (6.5); in particular, as remarked in the proof of lemma 4.17, the scaling invariance property for the heat operator associated to positive self-adjoint extension of Δ_i, was proved by Cheeger in [**25**]. Finally (7.3) and (7.4) follows again from theorem 6.5. □

By the fact that $f : X \to X$ is induced by a diffeomorphism of $\overline{M}$ it follows that the map f satisfies $f(sing(X)) = sing(X)$ and $f(reg(X)) = reg(X)$. This implies, see for example [**38**], that if we fix a perversity p then f induces a well defined map, f^*, between the intersection cohomology groups respect to the perversity p. In particular we have $f^* : I^{\overline{m}}H(X) \to I^{\overline{m}}H(X)$ and $f^* : I^{\underline{m}}H(X) \to I^{\underline{m}}H(X)$. Therefore it is natural to define in this context, as it is showed in [**40**], the **intersection Lefschetz number** respects to a given perversity p as

$$I^pL(f) = \sum_{i=0}^{n} \operatorname{tr}(f^* : I^pH^i(X) \to I^pH^i(X)). \tag{7.5}$$

$I^pL(f)$ is deeply studied, from a topological point of view, in [**40**] and [**41**] in the more general context of a stratified pseudomanifold; our goal in the next corollaries is to give an analytic description of $I^{\overline{m}}L(f)$ and $I^{\underline{m}}L(f)$ when X is a compact manifold with conical singularities. In particular in (7.10) we will give an analytic proof of a formula already proved in [**40**]. So, using theorem 7.1 and theorem 4.13, we get the following results:

Proposition 7.2. *In the same hypothesis of theorem 7.1; let* $q \in sing(X)$ *be an attractive fixed point . Let* U_q *be an open neighborhood of* q *isomorphic to* $C_2(L_q)$ *and suppose that* f *satisfies* (6.1) *and* g *takes the form* $g = dr^2 + r^2h$ *where* h *does not depend on* r*. Then, for* $i < \frac{m+1}{2}$*, we have:*

$$\operatorname{Tr}((f|_{U_q})^* : H^i_{2,max}(U_q, g|_{U_q}) \to H^i_{2,max}(U_q, g|_{U_q})) = \tag{7.6}$$

$$= \operatorname{Tr}(B^* : H^i(L_q) \to H^i(L_q)).$$

Proof. As it is showed in [**24**], in (4.13) the isomorphism between $H^i_{2,max}(reg(C_2(L_q)), g)$ and $H^i(L_q)$, for $i < \frac{m}{2} + \frac{1}{2}$, is given by the pull-back π^* where $\pi : (0, b) \times F \to F$ is the projection on the second factor and inverse is given by v_a, the evaluation map in a, where a is any point $(0, 2)$. Now by the hypothesis, over U_q f can be written as $(rA(p), B(p))$. An immediate check shows that $\pi^* \circ B^* = B^* \circ \pi^*$ and therefore $\operatorname{Tr}((f|_{U_q})^*) = \operatorname{Tr}(B^*)$. □

COROLLARY 7.3. *In the same hypothesis of theorem 7.1, suppose moreover that near each point $q \in sing(X)$ f satisfies* (6.1). *Then we have:*

$$I^{\underline{m}}L(f) = \sum_{q \in Fix(f) \cap reg(X)} \operatorname{sgn} det(Id - d_q f) + \sum_{q \in sing(X)} \sum_{i=0}^{m+1} (-1)^i \zeta_{T_i,q}(\Delta_{abs,i})(0) \tag{7.7}$$

and analogously

$$I^{\overline{m}}L(f) = \sum_{q \in Fix(f) \cap reg(X)} \operatorname{sgn} det(Id - d_q f) + \sum_{q \in sing(X)} \sum_{i=0}^{m+1} (-1)^i \zeta_{T_i,q}(\Delta_{rel,i})(0) \tag{7.8}$$

Finally, if $q \in sing(X)$ is an attractive fixed point, then we have

$$\sum_{i=0}^{m+1} (-1)^i \zeta_{T_i,q}(\Delta_{abs,i})(0) = \sum_{i < \frac{m}{2} + \frac{1}{2}} (-1)^i \operatorname{tr}(B^* : H^i(L_q) \to H^i(L_q)) \tag{7.9}$$

and therefore from (7.7) *we get:*

$$I^{\underline{m}}L(f) = L_{2,max}(T) = \sum_{q \in Fix(f) \cap reg(X)} \operatorname{sgn} det(Id - d_q f) + \tag{7.10}$$

$$+ \sum_{q \in sing(X)} \sum_{i < \frac{m+1}{2}} (-1)^i \operatorname{Tr}(B^* : H^i(L_q) \to H^i(L_q)).$$

PROOF. As in theorem 7.1, to get the Lefschetz numbers, we can use a conic metric g such that, in each neighborhood U_q of $q \in sing(X)$, using the isomorphism $\chi_q : U_q \to C_2(L_q)$, g takes the form $g = dr^2 + r^2 h$ where h does not depend on r. Now (7.7) and (7.8) follow immediately by the previously theorems. Finally (7.9) and (7.10) follow immediately from proposition 7.2. □

Finally we have this last corollary; before stating it we recall that a manifold with conical singularities of dimension $m+1$ is a **Witt** space if $m+1$ is even or, when it is odd, if $H^{\frac{m}{2}}(L_q) = 0$ for each link L_q. For more details see, for example, [**38**] .

COROLLARY 7.4. *In the same hypothesis of corollary 7.3. Suppose moreover that X is a Witt space. Then we get:*

$$L_{2,max}(T) = L_{2,min}(T), \ \mathcal{L}_{max}(T, \mathcal{S}) = \mathcal{L}_{min}(T, \mathcal{S}) \tag{7.11}$$

and, if each $q \in sing(X)$ is an attractive fixed point then

$$\mathcal{L}_{max}(T, \mathcal{S}) = \mathcal{L}_{min}(T, \mathcal{S}) = \sum_{q \in sing(X)} L_{2,max}(T|_{U_q}) = \sum_{q \in sing(X)} L_{2,min}(T|_{U_q}) = \tag{7.12}$$

$$= \sum_{q \in sing(X)} \sum_{i < \frac{m+1}{2}} (-1)^i \operatorname{Tr}(B_a^* : H^i(L_q) \to H^i(L_q)).$$

Finally if each $q \in sing(X)$ is repulsive then we have:

$$\mathcal{L}_{max}(T, \mathcal{S}) = \mathcal{L}_{min}(T, \mathcal{S}) = \sum_{q \in sing(X)} L_{2,max}(T^*|_{U_q}) = \sum_{q \in sing(X)} L_{2,min}(T^*|_{U_q}). \tag{7.13}$$

PROOF. (7.11) follows by the fact that, as it is showed in [**24**], if X is a Witt space then for each i, $\Delta_i : \Omega_c^i(reg(X)) \to \Omega_c^i(reg(X))$ is essentially self-adjoint as unbounded operator acting on $L^2\Omega(reg(X), g)$ and this implies that $d_{max,i} = d_{min,i}$ for $i = 0, ..., m+1$. (7.12) follows by (7.11) combined with (7.3) and (7.10). Finally (7.13) follows from the fact that X is Witt and from theorem 6.5. $\square$

2. Some further results arising from Cheeger's work on the heat kernel

The aim of this section is to approach the L^2-Lefschetz numbers of the L^2-de Rham complex using the results of Cheeger stated in [**24**] and in [**25**]. For simplicity assume that X is a Witt space. As recalled previously, if X is a Witt space and if over $reg(X)$ we put a conic metric, then $\Delta_i : L^2\Omega^*(reg(X), g) \to L^2\Omega^*(reg(X), g)$ is essentially self-adjoint for each $i = 0, ..., m+1$, with core domain given by the smooth compactly supported forms. In particular this implies that, if $dimX = m+1$, then for each $i = 0, ..., m+1$, $d_{max,i} = d_{min,i}$. Therefore, for each map $f : X \to X$ that induces a geometric endomorphism T as in theorem 7.1, we have just one L^2-Lefschetz number that we label $L_2(T)$.
Now we recall briefly the results we need and we refer to [**24**] and in particular to [**25**], section 3, for the complete details and for the proofs. Let N be an oriented closed manifold of dimension m and let $C(N)$ be the cone over N. Endow $reg(C(N))$ with a conic metric $g = dr^2 + r^2h$ where h is a riemannian metric over N. In the mentioned papers Cheeger introduce four types of differential forms over $reg(C(N))$, called forms of type 1, 2, 3 and 4, such that each eigenform of Δ_i, the Laplacian acting on the $i-$forms over $reg(C(N))$, can be expressed as convergent sum of these forms. For the definition of these forms see [**25**] pag. 586-588. The main reason to introduce these four types of forms is that now we can break the heat operator in four pieces, see [**25**] pag. 90-92:

$$e^{-t\Delta_i} = \ {}_1e^{-t\Delta_i} + \ {}_2e^{-t\Delta_i} + \ {}_3e^{-t\Delta_i} + \ {}_4e^{-t\Delta_i}$$

where, for each $l = 1, ..., 4$, ${}_le^{-t\Delta_i}$ is the heat operator built using the $i-$forms of type l. As it is showed in [**25**], pag. 590-592, it is possible to give an explicit expression for ${}_le^{-t\Delta_i}$. In particular for type 1 forms we have:

$${}_1e^{-t\Delta_i} = (r_1r_2)^{a(i)} \sum_j \int_0^\infty e^{-t\lambda^2} J_{\nu_j(i)}(\lambda r_1) J_{\nu_j(i)}(\lambda r_2) \lambda d\lambda \phi_j^i(p_1) \otimes \phi_j^i(p_2) = \tag{7.14}$$

$$= (r_1r_2)^{a(i)} \sum_j \frac{1}{2t} e^{-\frac{r_1^2+r_2^2}{4t}} I_{\nu_j(i)}(\frac{r_1r_2}{2t}) \phi_j^i(p_1) \otimes \phi_j^i(p_2) \tag{7.15}$$

where $I_{\nu_j(i)}$ is the modified Bessel function (see [**51**] pag. 67), $a(i) = \frac{1}{2}(1 + 2i - m)$, $\nu_j(i) = (\mu_j + a^2(i))^{\frac{1}{2}}$ and $a_j^{\pm}(i) = a(i) \pm \nu_j(i)$. The corresponding expression for type 2 forms is

$${}_2e^{-t\Delta_i} = \sum_j d_1d_2((r_1r_2)^{a(i-1)} \int_0^\infty e^{-t\lambda^2} J_{\nu_j(i-1)}(\lambda r_1) \tag{7.16}$$

$$J_{\nu_j(i-1)}(\lambda r_2)\lambda^{-1}d\lambda\phi_j^{i-1}(p_1)\otimes\phi_j^{i-1}(p_2)).$$

The expression for forms of type 3 is:

$$_3e^{-t\Delta_i}=\sum_j\int_0^{\infty}e^{-t\lambda^2}((-a(i-1)r_1^{a(i-1)}J_{\nu_j(i-1)}(\lambda r_1)+ \tag{7.17}$$

$$+r_1^{a(i-1)+1}J'_{\nu_j(i-1)}(\lambda r_1)\lambda)\frac{d\phi_j^{i-1}(p_1)}{\sqrt{\mu_j}}+$$

$$+r_1^{a(i-1)-1}J_{\nu_j(i-1)}(\lambda r_1)dr_1\wedge\sqrt{\mu_j}\phi_j^{i-1}(p_1))\otimes((-a(i-1)r_2^{a(i-1)}J_{\nu_j}(\lambda r_2)+$$

$$+r_2^{a(i-1)+1}J'_{\nu_j(i-1)}(\lambda r_2)\lambda)\frac{d\phi_j^{i-1}(p_2)}{\sqrt{\mu_j}}+r_2^{a(i-1)-1}J_{\nu_j}(\lambda r_2)dr_2\wedge\sqrt{\mu_j}\phi_j^{i-1}(p_2))\lambda^{-1}d\lambda$$

Finally for forms of type 4 we have:

$$_4e^{-t\Delta_i}=(r_1r_2)^{a(i-1)}\sum_j\int_0^{\infty}e^{-t\lambda^2}J_{\nu_j(i-2)}(\lambda r_1)$$

$$J_{\nu_j(i-2)}(\lambda r_2)\lambda d\lambda dr_1\wedge\frac{d\phi_j^{i-2}(p_1)}{\sqrt{\mu_j}}\otimes dr_2\wedge\frac{d\phi_j^{i-2}(p_2)}{\sqrt{\mu_j}}=$$

$$=(r_1r_2)^{a(i-2)}\sum_j\frac{1}{2t}e^{-\frac{r_1^2+r_2^2}{4t}}I_{\nu_j(i-2)}(\frac{r_1r_2}{2t})dr_1\wedge\frac{d\phi_j^{i-2}(p_1)}{\sqrt{\mu_j}}\otimes dr_2\wedge\frac{d\phi_j^{i-2}(p_2)}{\sqrt{\mu_j}} \tag{7.18}$$

Now suppose that for each point $q\in sing(X)$, over a neighborhood $U_q\cong C_2(L_q)$, f satisfies (6.20).

Using Cheeger's results recalled above, it make sense to break $T\circ e^{-t\Delta_i}$, over $C_2(L_q)$, as a sum of four pieces such that:

$$\lim_{t\to 0}\mathrm{Tr}(T\circ e^{-t\Delta_i})=\lim_{t\to 0}\mathrm{Tr}(T\circ\ _1e^{-t\Delta_i}+T\circ\ _2e^{-t\Delta_i}+T\circ\ _3e^{-t\Delta_i}+T\circ\ _4e^{-t\Delta_i}). \tag{7.19}$$

Moreover, using (5.11), (6.20), (7.15) and (7.18) it is clear that on $reg(C_2(L_q))$ we have:

$$\mathrm{tr}(T\circ\ _1e^{-t\Delta_i})(r,p)=(cr^2)^{a(i)}\sum_j\frac{1}{2t}e^{-\frac{r^2(c^2+1)}{4t}}I_{\nu_j(i)}(\frac{cr^2}{2t})\,\mathrm{tr}(B^*\phi_j^i\otimes B^*\phi_j^i) \tag{7.20}$$

and analogously

$$\mathrm{tr}(T\circ\ _4e^{-t\Delta_i})(r,p)= \tag{7.21}$$

$$(cr^2)^{a(i-2)}\sum_j\frac{1}{2t}e^{-\frac{r^2(c^2+1)}{4t}}I_{\nu_j(i-2)}(\frac{cr^2}{2t})\,\mathrm{tr}(dr\wedge\frac{d(B^*\phi_j^{i-2})}{\sqrt{\mu_j}}\otimes dr\wedge\frac{d(B^*\phi_j^{i-2})}{\sqrt{\mu_j}}).$$

Now we are in position to state the following result:

THEOREM 7.5. *Let X, g and f be as in theorem 7.1 such that $dimX=m+1$. Suppose moreover that X is a Witt space and that, on each neighborhood $U_q\cong C_2(L_q)$ of each point $q\in sing(X)$, f satisfies (6.20) and g takes the form $g=dr^2+r^2h$ where h does not depend on r. Then, for each $q\in sing(X)$, we have:*

(1) *The forms of type 1 give a contribution only in degree 0.*

(2) *The contribution given by q in degree zero depends only on the forms of type 1 and we have*

$$\zeta_{T_0,q}(\Delta_0)(0) = \frac{c^{\frac{1-m}{2}}}{4}\left(\int_0^\infty e^{-u(c^2+1)} \sum_j I_{\nu_j(0)}\left(\frac{cu}{2}\right)du\right)(\mathrm{Tr}(B^*\phi_j^i \otimes B^*\phi_j^i)) \tag{7.22}$$

(3) *The forms of type 4 give a contribution only in degree 2 and this contribution is*

$$\mathrm{Tr}(T_2 \circ \ {}_4e^{-t\Delta_2}) = \tag{7.23}$$

$$\frac{c^{\frac{1-m}{2}}}{4}\left(\int_0^\infty e^{-\frac{u(c^2+1)}{4}} \sum_j I_{\nu_j(0)}\left(\frac{cu}{2}\right)du\right)\left(\mathrm{Tr}\left(dr \wedge \frac{d(B^*\phi_j^{i-2})}{\sqrt{\mu_j}} \otimes dr \wedge \frac{d(B^*\phi_j^{i-2})}{\sqrt{\mu_j}}\right)\right).$$

where $\mathrm{Tr}(T_2 \circ \ {}_4e^{-t\Delta_2})$ *is taken over* $reg(C_2(L_q))$.

(4) *The contribution given by q in the others degrees, that is* $i \neq 0, 2$, *depends only on the forms of type 2 and 3.*

Proof. First of all we note that from (7.15), (7.16), (7.17) and (7.18) it follows that ${}_1e^{-t\Delta_i} = e^{-t\Delta_i}$ for $i = 0$ and that ${}_4e^{-t\Delta_i}$ occurs only for $i \geq 2$. Now, using (7.15) and (7.20) we know that, over $reg(C_2(L_q))$,

$$\lim_{t\to 0} \mathrm{Tr}(T_i \circ \ {}_1e^{-t\Delta_i}) =$$

$$= \lim_{t\to 0} \int_0^2 \int_{L_q} (cr^2)^{a(i)} \sum_j \frac{1}{2t} e^{-\frac{r^2(c^2+1)}{4t}} I_{\nu_j(i)}\left(\frac{cr^2}{2t}\right) \mathrm{tr}(B^*\phi_j^i \otimes B^*\phi_j^i) r^m dr dvol_h.$$

Clearly this last term it is in turn equal to

$$\lim_{t\to 0}\left(\left(\int_0^2 (cr^2)^{a(i)} \frac{1}{2t} e^{-\frac{r^2(c^2+1)}{4t}} \sum_j I_{\nu_j(i)}\left(\frac{cr^2}{2t}\right) r^m dr\right)(\mathrm{Tr}(B^*\phi_j^i \otimes B^*\phi_j^i))\right) \tag{7.24}$$

and therefore, to get the first two points we have to calculate

$$\lim_{t\to 0} \int_0^2 (cr^2)^{a(i)} \frac{1}{2t} e^{-\frac{r^2(c^2+1)}{4t}} \sum_j I_{\nu_j(i)}\left(\frac{cr^2}{2t}\right) r^m dr \tag{7.25}$$

First of all remember that $a(i) = \frac{1}{2}(1-m+2i)$; therefore $r^{2a(i)}r^m = r^{2i+1}$. Now put $\frac{r^2}{t} = u$. It follows immediately that $dr = \frac{tdu}{2r}$. Now, by the fact that $r^2 = tu$ it follows that $r^{2i+1} = t^i u^i r$ and therefore we also get $r^{2i+1}dr = \frac{t^{i+1}u^i du}{2}$. Moreover when r goes to 2 then u goes to $\frac{2}{t}$ and when r goes to 0 then u goes to 0. In this way we have

$$\lim_{t\to 0} \frac{c^{a(i)}}{4} t^i \int_0^{\frac{2}{t}} e^{-u(c^2+1)} \sum_j I_{\nu_j(i)}\left(\frac{c^2 u}{2}\right) u^i du \tag{7.26}$$

Now, by the asymptotic behavior of the integrand it follows that

$$\lim_{t\to 0} \frac{c^{a(i)}}{4} \int_0^{\frac{2}{t}} e^{-u(c^2+1)} \sum_j I_{\nu_j(i)}\left(\frac{c^2 u}{2}\right) u^i du =$$

$$= \frac{c^{a(i)}}{4} \int_0^\infty e^{-u(c^2+1)} \sum_j I_{\nu_j(i)}\left(\frac{c^2 u}{2}\right) u^i du.$$

Therefore we can conclude that

$$(7.26) = \begin{cases} \frac{c^{\frac{1-m}{2}}}{4}\int_0^\infty e^{-u(c^2+1)}\sum_j I_{\nu_j(0)}(\frac{c^2u}{2})du & i=0 \\ 0 & i>0 \end{cases} \tag{7.27}$$

In this way we proved the first and the second assertion. For the third statement the proof is completely analogous to the previous one. Also in this case it is clear that in order to establish the assertion we have to calculate:

$$\lim_{t\to 0} c^{a(i-2)+1}\int_0^2 \frac{1}{2t}e^{-\frac{r^2(c^2+1)}{4t}}\sum_j I_{\nu_j(i-2)}(\frac{cr^2}{2t})r^{2i-3}dr.$$

Now if we put again $\frac{r^2}{t} = u$ the remaining part of the proof is completely analogous to that one of the first two points.
Finally the last point follows from the first three points. $\square$

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